Dummit & Foote Abstract Algebra 3rd Ed

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Chapter 0

0.1.1

This exercise is contained within 0.1.4

0.1.2

$(Q+P)X = QX + PX = XQ + XP = X(Q+P)$ Thus $(Q+P) \in B$

0.1.3

$(QP)X = Q(PX) = Q(XP) = (QX)P = (XQ)P = X(QP)$ Thus $(QP) \in B$

0.1.4

Take $p, q, r, s \in \Re$ s.t.

$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $AX = XA \implies \begin{pmatrix} p & p+q \\ r & r+s \end{pmatrix} = \begin{pmatrix} p+r & q+s \\ r & s \end{pmatrix}$

From here, we compare the entries on either side; we see $p = p + r \implies r = 0$,
and $p + q = q + s \implies p = s$

Thus the general form for $A \in B$ is $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

As a sanity check, we check $AX = XA$ and get $\begin{pmatrix} a & a+b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & a+b \\ 0 & a \end{pmatrix}$

0.1.5

(a) No, take $f(1/2) = 1$ and $f(2/4) = 2$, thus $a = a \not\Rightarrow f(a) = f(a)$
(b) Yes, since there is no information lost in this map, it must be well defined (i.e. you aren't throwing away any piece of the input)

0.1.6

This is a well defined map; each real number has a unique decimal representation, thus there is no way to change the first digit after the decimal point.

0.1.7

This relation is predicated on the $=$ relation under the image of f, so this is clearly a equivalence relation, but we will show the properties nonetheless:

Reflexive - $a \sim a \implies f(a) = f(a)$ ✓

Symmetric - $a \sim b \implies f(a) = f(b) \implies f(b) = f(a) \implies b \sim a$ ✓

Transitive - $a \sim b,\ b \sim c \implies f(a) = f(b),\ f(b) = f(c) \implies f(a) = f(c) \implies a \sim c$ ✓

This relation is the definition of a fiber, as it relates all elements of the domain with the same value under f. If f were not surjective, we could find some element $b \in B$ such that $f(a) \neq b\ \forall a \in A$, and the fiber of f over b is the empty set. This empty set breaks our equivalence partitioning for our relation; however if we restrict f to surjection, the relation partitions A nicely into equivalence classes!

0.2.1

Syntax: $ax + by = gcd; lcm = (xy)/gcd$

(a) $2 * 20 + (-3) * 13 = 1; lcm = (20 * 13)$

(b) $27 * 69 + (-5) * 372 = 3; lcm = (23 * 372)$

(c) $8 * 792 + (-23) * 275 = 11; lcm = (792 * 25)$

(d) $(-126) * 11391 + 253 * 5673 = 3; lcm = (3797 * 5673)$

(e) $(-105) * 1761 + 118 * 1567 = 1; lcm = (1761 * 1567)$

(f) $(-17) * 507885 + 142 * 60808 = 691; lcm = (735 * 60808)$

0.2.2

We have for $a, b, n, m \in \mathbb{Z}; a = nk;\ b = mk \implies as + bt = (nk)s + (mk)t = k(ns + mt)$ and k divides $as + bt\ \forall s, t \in \mathbb{Z}$

0.2.3

We have $n = mk$ for some $m, k \in \mathbb{Z}$. Take $a = mq$ and $b = kp$ where $k \not| q$ and $m \not| p$, thus $n \not| a$ and $n \not| b$. Consider $ab = mqkp = (mk)qp = n(qp) \implies n|ab$

0.2.4

$ax + by = a(x_0 + \frac{b}{d}t) + b(y_0 - \frac{a}{d}t) = ax_0 + \frac{ab}{d}t + by_0 - \frac{ab}{d}t = ax_0 + by_0 + (\frac{ab}{d} - \frac{ab}{d})t = ax_0 + by_0 + 0t;$ this is clearly invariant under choice of t and represents a valid solution space.

0.2.5

$\varphi(1) = 1;\ \varphi(2) = 1;\ \varphi(3) = 2;\ \varphi(4) = 2;\ \varphi(5) = 4;$

$\varphi(6) = 2;\ \varphi(7) = 6;\ \varphi(8) = 4;\ \varphi(9) = 6;\ \varphi(10) = 4;$

$\varphi(11) = 10;\ \varphi(12) = 4;\ \varphi(13) = 12;\ \varphi(14) = 6;\ \varphi(15) = 8;$

$\varphi(16) = 8;\ \varphi(17) = 16;\ \varphi(18) = 6;\ \varphi(19) = 18;\ \varphi(20) = 8;$

$\varphi(21) = 12;\ \varphi(22) = 10;\ \varphi(23) = 22;\ \varphi(24) = 8;\ \varphi(25) = 20;$

$\varphi(26) = 12;\ \varphi(27) = 18;\ \varphi(28) = 12;\ \varphi(29) = 28;\ \varphi(30) = 8;$

0.2.6

Take $S \subset \mathbb{N}$ and P to be the complement of S with $1 \in S$ and $s \in S \implies s+1 \in S$
Now take $p \in P$ such that p is the minimal element in P. We know $p \neq 1$ since $1 \in S$
Thus $p-1$ exists and can't be in P since p is the minimal element of P. $p-1 \notin P \implies p-1 \in S \implies p-1+1 = p \in S$. From here we see p is in S and the complement of S and can not exist, Thus P must be empty and $S = \mathbb{N}$

0.2.7

The power of p in pb^2 is bound to be odd, where the power of p in a^2 is bound to be even.
More explicitly,taking $a, b \in \mathbb{Z}$ we can write $a = k_1^{a_1} \dots k_n^{a_n} p^{a_p}; b = q_1^{b_1} \dots q_n^{b_n} p^{b_p}$ where q_i, k_i are primes. This means $a^2 = k_1^{2a_1} \dots k_n^{2a_n} p^{2a_p}$ and $pb^2 = q_1^{2b_1} \dots q_n^{2b_n} p^{2b_p+1}$, so we need to find a_p, b_p s.t. $2a_p = 2b_p + 1$ though this is impossible.

0.2.8

First we start by counting up to n by multiples of p. Note there are $\left\lfloor \frac{n}{p} \right\rfloor$ such numbers. At this point, we have counted up all single multiples of p, though we have yet to account for the multiples of p^2 (i.e. every p^{th} multiple of p). In order to get the number of p^2 terms, we count up to n over multiples of p^2 for a total number of $\left\lfloor \frac{n}{p^2} \right\rfloor$ (looks familiar). This counting method continues up to the ith power. Now to arrive at the largest power of p that divides into n, we sum up all these terms: $\sum\limits_{i \in \mathbb{N}} \left\lfloor \frac{n}{p^i} \right\rfloor$

0.2.9

Haskell implementation

```
linearGCD :: Integer-> Integer-> (Integer, Integer, Integer)
linearGCD a b = (d, u, v) where
    (d, x, y) = eGCD 0 1 1 0 (abs a) (abs b)
    u | a < 0 = negate x
      | otherwise = x
    v | b < 0 = negate y
      | otherwise = y
    eGCD n1 o1 n2 o2 r s
      |(s == 0) = (r, o1, o2)
      | otherwise = case r `quotRem` s of
        (q, t)-> eGCD (o1 - q * n1) n1 (o2 - q * n2) n2 s t
```

0.2.10

Let p be a prime larger than $N+1$ such that $\varphi(p^k) = (p-1)(p^{k-1}) > N$. Therefore any prime q dividing n is no larger than $N+1$ and there are only finitely many choices for this q. Furthermore, we know $\varphi(n) = \varphi(q^k)\varphi(m)$ for some number m that is not divisible by q. Note that $\varphi(m)$ is constant, so this equation relies on k. This limits $k \le log_q(\frac{N}{m})$. Now we see that both the choice for q and k are of a finite set, thus there are finitely many numbers such that $\varphi(n) = N$.

Let's say there is some number M such that $\varphi(n) < M \forall n \in \mathbb{N}$. Since we are mapping an infinite set, $\mathbb{N}$, onto a finite set, we are bound to break the finite ceiling we set in the last previous portion (i.e. at least one of the numbers from $1 \dots M$ will have an infinite amount of numbers mapped to it). Thus we may not incur a maximum M.

0.2.11

Take $d = p_1^{a_1} \dots p_n^{a_n}$ where p_i is a prime that divides d. Since $d|n$, $n = p_1^{b_1} \dots p_n^{b_n} q$ where $a_i \le b_i$.
From here, $\varphi(d) = \varphi(p_1^{a_1}) \dots \varphi(p_n^{a_n}) = p_1^{a_1-1} \dots p_n^{a_n-1}(p_1-1)\dots(p_n-1)$ and
$\varphi(n) = \varphi(p_1^{b_1})\varphi(p_2^{b_2})\dots\varphi(p_n^{b_n})\varphi(q) = p_1^{b_1-1}\dots p_n^{b_n-1}(p_1-1)\dots(p_n-1)\varphi(q)$
Now since $a_i - 1 \le b_i - 1$, $\varphi(d)|\varphi(n)$

0.3.1

$\mathbb{Z}/18\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \overline{10}, \overline{11}, \overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}\}$
Where $\overline{a} = \{x \in \mathbb{Z} | x = 18k + a\}$

0.3.2

For some $a \in \mathbb{Z}$, we have $a = qn + r \implies a \equiv r \ (mod\ n) \implies \overline{a} = \overline{r}$ Since $0 \le r < n$, $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$; if we take $0 \le a, b < n$ where $a \ne b$ and $a - b > 0 \implies n \not| \, a - b \implies \overline{a} \ne \overline{b}$ thus the residue classes are distinct.

0.3.3

First, note that $10 \equiv 1 \ (mod\ 9)$ so $10^n \equiv 1^n \ (mod\ 9)$.
Now we have $a \equiv \overline{a_n 10^n + a_{n-1}10^{n-1} + \dots + a_0} \equiv \overline{a_n 10^n} + \dots + \overline{a_0} \equiv \overline{a_n}\,\overline{10^n} + \dots + \overline{a_0} \equiv \overline{a_n} + \dots + \overline{a_0} \ (mod\ 9)$

0.3.4

$37 \equiv 8 \ (mod\ 29); 8^2 \equiv 6 \ (mod\ 29); 8^4 \equiv 7 \ (mod\ 29); 8^8 \equiv 20 \ (mod\ 29);$
$8^{16} \equiv 23 \ (mod\ 29); 8^{32} \equiv 7 \ (mod\ 29); 8^{64} \equiv 20 \ (mod\ 29)$
$8^{100} = 8^{64}8^{32}8^4 \equiv 20 * 7 * 7 \ (mod\ 29) \equiv 23 \ (mod\ 29)$

0.3.5

$9^5 \equiv 49 \ (mod\ 100); 9^{10} \equiv 1 \ (mod\ 100); (9^{10})^{150} \equiv (1)^{150} \equiv 1 \ (mod\ 100)$

0.3.6

$0 * 0 \equiv 0 \ (mod\ 4); 1 * 1 \equiv 1 \ (mod\ 4); 2 * 2 \equiv 0 \ (mod\ 4); 3 * 3 \equiv 1 \ (mod\ 4)$

0.3.7

As seen in the previous exercise, the addition of any two squares can only equal 0, 1, and 2.

0.3.8

We can see right away the only way $a^2, b^2, c^2 \in \mathbb{Z}/4\mathbb{Z}$ can satisfy this is if $a^2 = b^2 = c^2 = 0$. This means a, b, c are all even. We now see a^2, b^2, c^2 all have a factor of 2^2 which implies there is a smaller solution available.

0.3.9

This problem reduces to showing that the odd elements of $\mathbb{Z}/8\mathbb{Z}$ square to $\overline{1}$
$1 * 1 \equiv 1\ (mod\ 8); 3 * 3 \equiv 1\ (mod\ 8); 5 * 5 \equiv 1\ (mod\ 8); 7 * 7 \equiv 1\ (mod\ 8)$

0.3.10

The elements in $(\mathbb{Z}/n\mathbb{Z})^*$ are all numbers with multiplicative inverses modulo n. In order for a^{-1} to exist, $(a, n) = 1$. By definition, $\varphi(n)$ is the number of relatively prime numbers less than n. This will be hashed out in more detail in the following exercises!

0.3.11

Take $a, b \in (\mathbb{Z}/n\mathbb{Z})^* \implies \exists a^{-1}, b^{-1}$ such that $a * b * b^{-1} * a^{-1} = a * 1 * a^{-1} = a * a^{-1} = 1$
Thus $a * b \in (\mathbb{Z}/n\mathbb{Z})^*$ where $b^{-1} * a^{-1}$ is the inverse

0.3.12

We can write $a = a_1 * (a, n); n = n_1 * (a, n); a * n_1 = a_1 * n_1 * (a, n) = a_1 * n \equiv 0\ (mod\ n)$. Now assume $\exists c\ s.t.\ ac \equiv 1\ (mod\ n) \implies \exists k\ s.t.\ kn = ac - 1$ Since $(a, n) | ac - kn\ \forall c, k \in \mathbb{Z}/n\mathbb{Z} \implies (a, n)|1 \implies (a, n) = 1$. However, we have taken $(a, n) > 1$ and a contradiction arises!

0.3.13

From Euclid's algorithm, we have $ac + kn = (a, n) = 1$ for some $c, k \in \mathbb{Z}/n\mathbb{Z} \implies ac \equiv 1\ (mod\ n)$

0.3.14

From 12 we see all relatively prime a can't have a multiplicative inverse in $\mathbb{Z}$, and from 13 we see all relatively prime a has an inverse than may be computed with Euclid's algorithm; thus we have $(\mathbb{Z}/n\mathbb{Z})^* = a|(a, n) = 1$ For a concrete example, we can provide elements that either send some a to 0, or 1
$1 * 1 \equiv 1\ (mod\ 12); 2 * 6 \equiv 0\ (mod\ 12); 3 * 4 \equiv 0\ (mod\ 12); 5 * 5 \equiv 1\ (mod\ 12);$
$7 * 7 \equiv 1\ (mod\ 12); 8 * 3 \equiv 0\ (mod\ 12); 9 * 4 \equiv 0\ (mod\ 12); 10 * 6 \equiv 0\ (mod\ 12); 11 * 11 \equiv 1\ (mod\ 12)$

0.3.15

(a) $(13)^{-1} \equiv 17\ (mod\ 20)$
(b) $(69)^{-1} \equiv 40\ (mod\ 89)$
(c) $(1891)^{-1} \equiv 253\ (mod\ 3797)$
(d) $(6003722857)^{-1} \equiv 77695236753\ (mod\ 77695236973)$

0.3.16

Haskell implementation

```
reduceMod :: Integer-> Integer
reduceMod a n | (a < 0)  =  reduceMod (a + n) n
              | (a > n - 1)  =  reduceMod (a - n) n
              | otherwise  =  a

multiMod :: Integer-> Integer-> Integer-> Integer
multiMod a b n  =  reduceMod (a * b) n

addMod :: Integer-> Integer-> Integer-> Integer
addMod a b n  =  reduceMod (a + b) n

getInverse :: Integer-> Integer-> Maybe Integer
getInverse a n  =  relativePrime where
                   relativePrime | (gcd  ==  1)  =  Just (reduceMod inverse n)
                                 | otherwise  =  Nothing
                                 (gcd,  inverse, _)  =  linearGCD a n
```

Chapter 1