Mixed mock modularity of Special divisors.

Jt with
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Motivation:

$\mathcal{X} \xrightarrow{\pi} S$ smooth proj morphism. (1-param.)

$b \in S$: $\mathcal{X}_b$ has extra (algebraic cycles) → Hodge cycles.

Elliptic curves: $\mathcal{X}_b$ CM elliptic fiber.

Abelian surfaces: $\mathcal{X}_b \sim \mathcal{E}_1 \times \mathcal{E}_2$

K3 surface: $L = PH^2(\mathcal{X}_b, \mathbb{Z})$ ← primitive cohomology

$\boxed{(L, Q)}$ Signature: $(n, 2)$ $n \geq 1$

$n \leq 19$

has a Hodge structure of wt 2 (K3 type):

① $L_{\mathbb{C}} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}$

$\uparrow$ 1 $\quad$ $\uparrow$ n

② $L^{2,0} = \overline{L^{0,2}}$ $\qquad L^{1,1} = (L^{2,0} \oplus L^{0,2})^{\perp}$

Noether-Lefschetz locus: $\{ b \in S, \mathrm{rk}_{\mathbb{Z}} L \cap L^{1,1} > 0 \}$

We introduce the period domain:

$$D = \{ L^{2,0} \in \mathbb{P}(L_{\mathbb{C}}), \ \underset{\mathbb{Q}}{(L^{2,0} \cdot L^{2,0})} = 0, \ (L^{2,0} \cdot \overline{L^{2,0}}) < 0 \}$$

Hermitian Symmetric domain.

$\Gamma \subseteq O(L)$

$\Gamma \supseteq \mathrm{Ker}(O(L) \to O(L^{\vee}/L))$

$\rightsquigarrow$ $\Gamma \backslash D$: orthogonal Shimura Variety.

= algebraic Variety by Baily_Borel.

$$\Gamma \backslash D \hookrightarrow (\Gamma \backslash D)^{BB}$$

$\rightsquigarrow$ period maps: $S \longrightarrow \Gamma \backslash D$ alg morphism

Special divisors: $L^\vee$: dual lattice

$\beta \in L^\vee / L \qquad m \in Q(\beta) + \mathbb{Z}, \qquad m > 0$

$$Z(\beta, m) = \Gamma \backslash \left(\bigcup_{\substack{\lambda \in \beta + L \\ Q(\lambda) = m}} \{ w \in D, \ (w . \lambda) = 0 \} \right)$$

$Z(\beta, m) \longrightarrow \Gamma \backslash D$: Special divisor.

$m=0,\ \beta=0$: $Z(0,0)$: class is equal to dual of Hodge bundle.

$$\Phi_L = \sum_{(\beta,m)} Z(\beta,m)\, q^m e_\beta \in \mathrm{Pic}(\mathcal{P}_T)_{\mathbb{Q}} \otimes \mathbb{C}[L^\vee/L][[q]]$$

$(e_\beta)_\beta$ basis of $\mathbb{C}[L^\vee/L]$ $\qquad q = e^{2\pi i \tau}$

$\tau \in \mathbb{H}$.

Hirzebruch - Zagier (Hilbert modular surfaces)

Kudla - Millson.

Thm: (Borcherds) :

Φ is a vector-valued modular form in $\mathrm{Pic}(\mathcal{P}_T)$ of weight $1+\frac{17}{2}$, w.r.t $\rho_L : Mp_2(\mathbb{Z}) \to \mathrm{Aut}(\mathbb{C}[L^\vee/L])$

$\leftarrow$ Weil representation

In particular: S Compact

$$\sum (S.Z(\beta,m))\, q^m e_\beta$$

Vector Valued modular form.

Question: what happens if S is not compact?

$n=1$: (L,Q) isotropic, $\mathcal{P}/\Gamma \simeq Y(1) \hookrightarrow X(1)$
is
$\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$

Special divisors: CM elliptic curves.

in $\mathbb{A}^1$: O modular.

in $\mathbb{P}^1$: $\displaystyle\sum_{N=0}^{\infty} H(N)\, q^N$

↑ class number of ≥ 0 binary forms of disc. $-N$

Zagier:

$$\hat{F} = \sum_{N=0}^{\infty} H(N) q^N + \sqrt{y} \sum_{n\in\mathbb{Z}} \beta(4\pi n^2 y) e^{-2\pi i n^2 \tau}$$

hol

non-hol.

transforms like a modular form of wt $3/2$ w.r.t $\Gamma_0(4)$.

Zwegers (Ramanujan): Mock modular forms

$n=2$: $(L,Q) = U \oplus U$

$$\mathcal{N}^D = Y(1) \times Y(1)$$

$$Z(N) = T_N \quad \text{Hecke Corresp.}$$

$$\sum T_N q^N \qquad \text{in } \mathbb{P}^1 \times \mathbb{P}^1$$

$$= E_2 \otimes T_0$$

↑

Eisenstein Series of wt 2.

$$= 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n .$$

$$E_2^* = E_2 - \frac{3}{\pi y}$$ transforms like a modular form.

↑ mock modular form.

<u>In general</u>: (Zagier - Folsom ...... |

a mixed mock modular form.

$$\sum_i (\text{mock modular form}) \times \text{holomorphic modular form.}$$

Thm (EGT-23): $X_\Gamma = \Gamma\backslash D \hookrightarrow X_\Gamma^{\Sigma}$

smooth.

In X_Γ^{Σ}, Φ is a mixed mock modular form.

× Bruinier-Zemel.

× Garcia.

Baily-Borel: (L, Q) $(n, 2)$:

$$\Gamma\backslash D \hookrightarrow \Gamma\backslash\left(D \cup \bigcup_I D_I \cup \bigcup_J D_J\right) = \left(\Gamma\backslash D\right)^{BB}$$

I: $\mathbb{Q}$-rank 1 isotropic space: D_I point

J: $\mathbb{Q}$-rank 2 ———— $D_J = \mathbb{H}$

AMRT: $X_\Gamma^{\Sigma} \longrightarrow X_\Gamma^{BB}$

I : rank 1 isotropic:

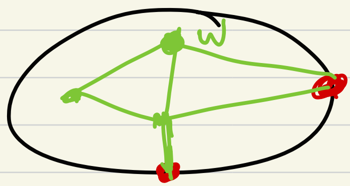
$$K = I^{\perp}/I \qquad (n-1, 1)$$

$$C_I = \{ x \in K_{\mathbb{R}} \quad Q(x) < 0 \}$$

Σ: Cone decomposition of $C_I \cup \{ x \in K_{\mathbb{Q}}, Q(x) = 0 \}$

$\rightsquigarrow$ toric variety

that can be glued to $\Gamma \backslash \mathcal{D}$ $\rightsquigarrow$ $\boxed{X_{\Gamma}^{\Sigma}}$ Compact.



Has boundary divisors:

$J : \longrightarrow \quad B_J$: divisor.

(I, w) $\longrightarrow$ $B_{I,w}$: divisor.

$\uparrow$

inner ray

in C_I

$\rightsquigarrow J$: $\quad J^{\perp}/J \longrightarrow (n-2, 0)$

$\rightsquigarrow \Theta_J$: theta Series

of $J^{\perp}/J$

$\rightsquigarrow I : K = I^{\perp}/I \quad (n-1, 1)$

$w^{\perp} \subseteq K \longrightarrow (n-1, 0)$

$\longrightarrow \Theta_{(I,w)}$ theta Series.

Thm:

$$\boxed{\Phi - \frac{1}{2}\sum_{(I,w)} \left(\Theta_{I,w} \otimes F\right) \otimes B_{I,w} + \frac{1}{12}\sum_{J} \left(E_2 \otimes \Theta_J^{1}/J\right) \otimes B_J}$$

$1+n/2$ $\quad$ $\frac{n-1}{2}$ $\quad$ $+1$ $\quad$ $3/2 = 1+n/2$

Zagier's series

$\frac{n}{2} - 1 = 1 + \frac{n}{2}$ $\quad$ -1 $\quad$ 2

is a holomorphic modular form of wt. $1+n/2$ (ρ_L)

$\boxed{1\ H^2}$

Sketch of the proof:

+ Superrigidity (Margulis)

$\Rightarrow$ $n \geq 3$

$n=2$: isotropy 1

$\longrightarrow$ $\mathrm{Pic}(X_n^{\Sigma})$

1st observation (Greer)

$$H_2(X_n^{\Sigma}) = H_2(X_n) + H_2(\partial X_n^{\Sigma})$$

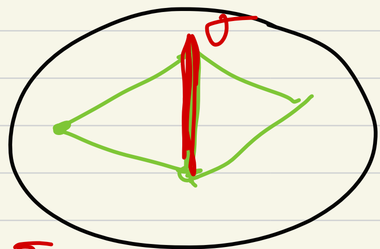
$$[C] = [C_1] + [C_2]$$

$$Z(\beta,m)\cdot[C] = Z(\beta,m)[C_1] + Z(\beta,m)\cdot[C_2]$$

Borcherds

$H_2(\partial X_P^\Sigma)$: generated by some curves in the boundary

$\sigma \subseteq \Sigma$ codim 1 face



$\Rightarrow$ C_σ : curve in ∂X_P^Σ

$\rightsquigarrow$ It is enough to consider.

$$Z(\beta,m).C_\sigma$$

(Fulton)

$$\sum_{(\beta,m)} (Z(\beta,m).C_\sigma)\, q^m e_\beta$$

)

$$= \sum_{\lambda \in K^\vee} P^+(\lambda)\, q^{\varphi(\lambda)}\, e_{\bar\lambda} \qquad \mathbb{C}[K^\vee / K]$$

Piecewise linear function.

$\equiv 0$ outside $\{\lambda, \varphi(\lambda) > 0\}$

Zwegers (2001) : $\boxed{P(\lambda) = P^+(\lambda) + \boxed{P^-(\lambda)}}$

(H/Z 70's) :

Smooth

[Christina - Zwegers]

$|x|$

<u>Vignéras</u>: $\left(E - \frac{1}{4\pi}\Delta\right) P = P$ $\quad \|(\lambda \cdot a_i)\|$

$e^{-\varphi(\lambda)} \cdot P \in L^1 \cap L^2$

$\hookrightarrow$ $\sum P(\lambda)\, q^{\varphi(\lambda)}$: transforms like a modular

Identify the non-holomorphic form p^+ p^-

$= -$ (non-holomorphic part

of $\frac{1}{2}\left(\Theta_{I,\omega} \otimes \hat{F}\right)\left(B_{I,\omega} \cdot C_\sigma\right)$

non holomorphic

$+ \frac{1}{12}\left(\hat{E}_2 \otimes \Theta_J^{1/J}\right)\left(B_J \cdot C_\sigma\right)$

$\rightsquigarrow$ Sum

$\left(\tilde{\Phi} \cdot C_\sigma\right) =$ holomphic part

holomphic modular form

of wt $1 + \frac{n}{2}$. ρ_L.