Weierstrass mock modular forms and vertex operator algebras

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joint work with Lea Beneish

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The beginning of Monstrous Moonshine

Observation (McKay, 1978)

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Conjecture (Thompson; Conway-Norton (1979))

There is a graded representation of $\mathbb{M}$ whose graded dimensions (characters) agree with the j-function (other Hauptmoduln).

Moonshine and VOAs

Theorem (Frenkel-Lepowsky-Meurman (1985); Borcherds (1992))

There is such a graded representation $V^\natural$ of $\mathbb{M}$. It carries the structure of a vertex operator algebra (VOA). $\mathbb{M}$ acts on $V^\natural$ as VOA-automorphisms.

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New(ish) development: Connection of Moonshine (e.g. for ON) to arithmetic of elliptic curves (Duncan-M.-Ono, 2017).

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New(ish) development: Connection of Moonshine (e.g. for ON) to arithmetic of elliptic curves (Duncan-M.-Ono, 2017).
Goal: Show a different kind of connection between VOAs and elliptic curves.

Classification of VOAs

Fact: Given an integral lattice $L \subseteq \mathbb{R}^n$, there is an associated VOA V_L of central charge $c = n$ (Frenkel-Lepowsky-Meurman).

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- Construction of all candidates by van Ekeren, Höhn, Möller, Scheithauer ($\sim$2016–2020)

A dimension formula

Theorem (Möller, 2016)

Let $V = \bigoplus_{n\geq 0} V_n$ be a “nice” VOA of central charge 24 and $G = \langle g\rangle$ a cyclic group of automorphisms of V of order $N \in \{2,3,5,7,13\}$.

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$$\begin{aligned}\dim V_1 + \dim V_1^{\mathrm{orb}(g)} &= 24 + (N+1)\dim V_1^G \\ &\quad - \frac{24}{N-1}\sum_{k=1}^{N-1}\sigma(N-k)\sum_{i\in\mathbb{Z}/N\mathbb{Z}\setminus\{0\}}\dim V(g^i)_{k/N}\end{aligned}$$

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- Extension by van Ekeren, Möller, Scheithauer to $N \in \{2, ..., 10, 12, 13, 16, 18, 25\}$ (i.e. such that $X_0(N)$ has genus 0)
- Extension to all N by Möller and Scheithauer

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- Grading: $V_n = \{v \in V \ : \ L(0)v = nv\}$, $n \in \mathbb{Z}$.

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- strongly rational: rational, C_2-cofinite, of CFT-type, self-dual

Orbifold VOAs

V a nice VOA, $G = \langle g \rangle$ group of automorphisms, order N.

- $V(g^i)$ (unique) g^i-twisted module. Decomposition $V(g^i) = \bigoplus_{j \pmod N} W^{(i,j)}$ as $\mathbb{C}[G]$-modules

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Let V be a nice VOA of central charge c with $24 \mid c$. The characters $\mathrm{ch}_{W^{(i,j)}}(\tau) = \mathrm{tr}_{W^{(i,j)}}\, q^{L(0)-c/24}$ form a vector-valued modular form of weight 0 for $\mathrm{SL}_2(\mathbb{Z})$.

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Definition

A harmonic Maaß form of weight $k \in \mathbb{Z}$ for the group $\Gamma_0(N)$ is a smooth function $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following three conditions:

1. f is invariant under the weight k slash operator,

$$f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau), \quad \tau \in \mathfrak{H} \text{ and } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N$$

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Fact: $f = f^+ + f^-$ splits into a holomorphic part (called a mock modular form) and a non-holomorphic part

The Bruinier-Funke pairing I

Proposition (Bruinier-Funke)

The operator $\xi_k = 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}}$ defines a surjective $\mathbb{C}$-antilinear map

$$H_k(N) \twoheadrightarrow S_{2-k}(N)$$

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$$\{\cdot,\cdot\} : M_k(N) \times H_{2-k}(N) \to \mathbb{C},\ \{g,f\} := \langle g, \xi_{2-k} f\rangle,$$

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Let $g \in M_k(N)$ and $f \in H_{2-k}(N)$ with Fourier expansions

$$(g|\gamma)(\tau) = \sum_{n=0}^{\infty} a_{\mathfrak{a}}(n) q^{n/h} \quad \text{and} \quad (f|\gamma)^{+}(\tau) = \sum_{m \gg -\infty} b_{\mathfrak{a}}(n) q^{n/h}$$

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$$\{g, f\} = \sum_{\mathfrak{a}} \sum_{n \leq 0} a_{\mathfrak{a}}(-n) b_{\mathfrak{a}}(n).$$

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Transforms like an elliptic function but is no longer holomorphic.

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The function

$$\mathfrak{Z}_E(\tau) = \zeta(\Lambda_E; \mathcal{E}_E(\tau)) - G_2^*(\Lambda_E)\mathcal{E}_E(\tau),$$

called the Weierstrass mock modular form is a polar mock modular form of weight 0 for the group $\Gamma_0(N)$.

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called the Weierstrass mock modular form is a polar mock modular form of weight 0 for the group $\Gamma_0(N)$. To be more precise, there exists a meromorphic modular function M_E for $\Gamma_0(N)$ such that the function

$$\widehat{\mathfrak{Z}}_E(\tau) = \widehat{\zeta}(\Lambda_E; \mathcal{E}_E(\tau)) - M_E(\tau)$$

is a harmonic Maaß form of weight 0 for $\Gamma_0(N)$.

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called the Weierstrass mock modular form is a polar mock modular form of weight 0 for the group $\Gamma_0(N)$. To be more precise, there exists a meromorphic modular function M_E for $\Gamma_0(N)$ such that the function

$$\widehat{\mathfrak{Z}}_E(\tau) = \widehat{\zeta}(\Lambda_E; \mathcal{E}_E(\tau)) - M_E(\tau)$$

is a harmonic Maaß form of weight 0 for $\Gamma_0(N)$.

Note: If $E = X_0(N)$ is a strong Weil curve, we can choose $M_E(\tau) = 0$.

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Generating harmonic Maaß forms

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Let E denote the strong Weil curve of conductor

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Idea of the proof: $\widehat{\mathfrak{Z}}_E$ in the cases considered has a simple pole at ∞ and nowhere else. Move the pole to another cusp using W_Q, and increase the pole order using T_m (action on Poincaré series).

Another dimension formula

Theorem 2 (Beneish-M., 2020)

Let V be a nice VOA of central charge 24, $G = \langle g \rangle$ be a cyclic group of automorphisms of V of order $p \in \{11, 17, 19\}$. Further let $E = X_0(p)$ be the $\Gamma_0(p)$-optimal elliptic curve of conductor p.

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$$C_E := -\frac{3 - \#E(\mathbb{F}_2)}{2} - \widehat{\zeta}\left(\Lambda_E; L(E,1)\right).$$

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Theorem 3 (Beneish-M.; indep. Möller-Scheithauer, 2020)

Assume the hypotheses and notation from Theorem 2, except that p may now denote any prime number, and let $f(\tau) = \sum_{n=1}^{\infty} a(n)e^{2\pi in\tau} \in S_2(p)$ be a newform with Atkin-Lehner eigenvalue $\varepsilon \in \{\pm 1\}$.

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Idea of the proof: Since ch_{V^G} is a modular function, we must have $\{\mathrm{ch}_{V^G}, f\} = 0$. The theorem follows from the formula for the Bruinier-Funke pairing.

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An algebraicity result

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Corollary (Beneish-M., 2020)

Assume the notations as in Theorem 2. If we have

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \sigma(p-j)\dim V(g^i)_{j/p} \neq p-1$$

for some VOA V as in Theorem 2, then the value $\widehat{\zeta}(\Lambda_E; L(E,1))$ is rational.

Some remarks

- Möller and Scheithauer showed that one always has the inequality

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sigma(p-j)\dim V(g^i)_{j/p}\leq p-1$$

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- Numerically one finds $\widehat{\zeta}(\Lambda_E;L(E,1))=17/5,\ 2,\ 4/3$ for $p=11,\ 17,\ 19$, respecitvely, yielding $C_E=-\frac{24}{p-1}$ ins Theorem 2.

Thank you for your attention.