Meromorphic modular forms and their iterated integrals

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1. Brief review of meromorphic modular forms
2. Iterated integrals and their algebraic structure
3. Geometric interpretation (after Brown–Fonseca)

Introduction

Let

$$f(q) = \sum_{n \gg -\infty} a_n q^n \in \mathbb{Z}((q))$$

be a meromorphic modular form.

Two phenomena

(i) magnetism, $n | a_n$. Broadhurst–Zudilin, Li–Neururer, Pasol–Zudilin, Löbrich–Schwagenscheidt

(ii) algebraic independence of primitives (over the field of quasimodular functions) Pasol–Zudilin

Difference: (i) is *arithmetic*; (ii) is *algebraic geometric* (will see later)

Goal of today's talk:

Establish most general algebraic independence results for primitives

1. Brief review of meromorphic modular forms

2. Iterated integrals and their algebraic structure

3. Geometric interpretation (after Brown–Fonseca)

Basic notation

- $\mathfrak{H}$ = upper half-plane, $q := e^{2\pi iz}$
- $f : \mathfrak{H} \to \mathrm{P}^1(\mathbb{C})$ meromorphic, $k \in \mathbb{Z}$:

$$f[\gamma]_k(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

- $\delta := q\frac{d}{dq}$, q-derivative
- M_k = space of modular forms of weight k, level 1
- S_k = space of cusp forms of weight k, level 1
- Eisenstein series of weight $2k \geq 2$:

$$E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} n^{2k-1} \frac{q^n}{1-q^n}$$

- $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$

Meromorphic modular forms

Definition

A meromorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a function $f : \mathfrak{H} \to \mathrm{P}^1(\mathbb{C})$ such that

(i) f is meromorphic;

(ii) $f[\gamma]_k = f$, for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$;

(iii) f is "meromorphic at ∞": $f(z) = \sum_{n \gg -\infty}^{\infty} a_n q^n$

Remark

Similar for general discrete groups $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ (now f meromorphic at cusps of Γ).

Example in weight 0

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + \dots,$$

The algebra of meromorphic modular forms

- $\mathcal{M}_k = \{$meromorphic modular forms, weight k$\}$

$$\mathcal{M}_* = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k$$

graded $\mathbb{C}$-algebra of meromorphic modular forms

Fact

$\mathcal{M}_k = \mathbb{C}(E_4, E_6)_k \subset \mathbb{C}(E_4, E_6)$, homogeneous rational functions, weight k

Warning

(i) $\dim_{\mathbb{C}} \mathcal{M}_k < \infty \Leftrightarrow \mathcal{M}_k = 0$

(ii) $\mathcal{M}_k = h \cdot \mathcal{M}_0$, for every $h \in \mathcal{M}_k \setminus \{0\}$, and $\mathcal{M}_0 = \mathbb{C}(j)$

Meromorphic quasimodular forms and derivatives

- Let $f \in \mathcal{M}_*$.

Fact

Have $\delta(f) \in \mathcal{M}_*$ if and only if $f \in \mathcal{M}_0$.

Example

$\delta(j) = -E_4^2 E_6/\Delta \in \mathcal{M}_2$

Generalization Bol, 1949

$$\delta^{k-1}(\mathcal{M}_{2-k}) \subset \mathcal{M}_k, \qquad \text{for all } k \geq 2$$

Remark

$\mathcal{QM}_* := \mathcal{M}_*[E_2]$ is closed under δ, algebra of meromorphic quasimodular forms

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Integrals of modular forms I Hecke (1920s), Eichler, Shimura, Manin (1950s–1970s)

- $f = \sum_{n\geq 1} a_n q^n$ cusp form, weight k

Periods of f

$$\int_0^{i\infty} f(z) z^{m+1} = \frac{m!}{(-2\pi i)^{m+1}} L(f,m), \qquad 0 \leq m \leq k-2$$

$L(f,s) = \sum_{n=1}^{\infty} a_n/n^s$, the L-series of f

- Remarkable arithmetic properties (action of Hecke operators, conjecturally transcendental, etc.)
- Generalization: Iterated Eichler–Shimura integrals Manin, Brown (2000s–2010s)

Integrals of modular forms II

- $f(z) = \sum_{n\neq 0} a_n q^n$ meromorphic cusp form, $a_n \in \mathbb{Z}$
- $I(f) := \sum_{n\neq 0} \frac{a_n}{n} q^n \in \mathbb{Q}((q))$ formal integral of f

Question

When is $I(f) \in \mathbb{Z}((q))$? Equivalently, when does n divide a_n, for all n?

- Never(?), if f is holomorphic
- Pasol–Zudilin (2020): Yes, e.g. for

$$F_{4a} = \frac{\Delta}{E_4^2}, \qquad F_{4b} = \frac{E_4\Delta}{E_6^2}, \qquad F_6 = \frac{E_6\Delta}{E_4^3}$$

$\rightsquigarrow$ Magnetic modular forms Broadhurst–Zudilin, Li–Neururer, Löbrich–Schwagenscheidt, related to Borcherds–Shimura lifts

Integrals of modular forms III

- $K := \mathbb{C}(E_2, E_4, E_6, q) = \mathrm{Frac}(QM_*[q])$, closed under qd/dq

Result Pasol–Zudilin (2020)

$$I(F_{4a}),\ I(F_{4b}),\ I(F_6) \in \mathbb{Q}((q))$$

are algebraically independent over K

Wish to generalize this to arbitrary meromorphic modular forms.

Formal integration

Formal integration

$$\begin{aligned} I : \mathbb{C}[z][[q]] &\to \mathbb{C}[z][[q]] \\ f &\mapsto F - F_0(0), \end{aligned}$$

where $F = \sum_{n\geq 0} F_n(z)q^n$ satisfies $\delta(F) = f$.

Easy fact

$I(f)$ has all the usual properties, e.g.

- $\mathbb{C}$-linear
- shuffle product: $I(f_1)I(f_2) = I(f_1 \cdot I(f_2)) + I(f_2 \cdot I(f_1))$
- integration by parts, e.g. $I(\delta(f_1) \cdot I(f_2)) = f_1 \cdot I(f_2) - I(f_1 \cdot f_2)$

Formal iterated integrals

Definition

For $f_1, \ldots, f_n \in \mathbb{C}[z][[q]]$, define

$$I(f_1, \ldots, f_n) = \begin{cases} 1, & n = 0, \\ I(f_1 \cdot I(f_2, \ldots, f_n)), & n \geq 1. \end{cases}$$

Shuffle product

$$I(f_1, \ldots, f_m) I(f_{m+1}, \ldots, f_{m+n}) = \sum_{\sigma} I(f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(m+n)}),$$

sum over permutations of $\{1, \ldots, m+n\}$, which are strictly increasing on $\{1, \ldots, m\}$ and on $\{m+1, \ldots, m+n\}$

Poles at ∞

- Could extend I to (finite-tailed) Laurent series

$$\mathbb{C}[z]((q)) = \left\{ \sum_{n \gg -\infty} f_n(z) q^n \right\}$$

using the same definition, but bad properties (no shuffle product, etc.)

- Solution: Use renormalization techniques (à la Connes–Kreimer)

Example with renormalization

$$I(q) = q, \qquad I(1/q) = -1/q$$

$$I(q, 1/q) = -z, \qquad I(1/q, q) = z - 1$$

$$\Rightarrow I(q)I(1/q) = I(q, 1/q) + I(1/q, q)$$

Algebra of iterated integrals

Recall: $K = \mathbb{C}(E_2, E_4, E_6, q)$.

Definition

$$\mathcal{I}^{\mathcal{M}} := \mathrm{Span}_K\{I(f_1, \ldots, f_n) \,:\, n \geq 0,\ f_1, \ldots, f_n \in \mathcal{M}_*\} \subset \mathbb{C}[z]((q))$$

differential K-algebra of iterated integrals of meromorphic modular forms

Question

What is the algebraic structure of $\mathcal{I}^{\mathcal{M}}$?

Free/universal shuffle algebras

- $K\langle\mathcal{M}_*\rangle := \bigoplus_{n\geq 0}\mathcal{M}_*^{\otimes n}$
- elements are K-linear combinations of $[f_1|\dots|f_n]$, for $f_i \in \mathcal{M}_*$
- $K\langle\mathcal{M}_*\rangle$ commutative K-algebra w. shuffle product

Fact

Every algebraic relation in $K\langle\mathcal{M}_*\rangle$ is a consequence of shuffle

More precisely:

Theorem Cartier–Milnor–Moore (1960s), Radford (1979)

k = field of characteristic zero, V = k-vector space. Then $k\langle V\rangle$ is a free polynomial algebra + explicit polynomial basis known.

A quotient of the free shuffle algebra

- Have a surjection of differential K-algebras:

$$K\langle \mathcal{M}_* \rangle \twoheadrightarrow \mathcal{I}^{\mathcal{M}}$$
$$[f_1|\dots|f_n] \mapsto I(f_1,\dots,f_n)$$

- not injective, more relations in $\mathcal{I}^{\mathcal{M}}$, e.g.

$$I(\delta(f)) - f = 0$$

- But:

these are the "only" new relations

A basis for $\mathcal{I}^{\mathcal{M}}$: I

Theorem Chen (1977), Deneufchâtel–Duchamp–Minh–Solomon (2011)

Let $f_1, \ldots, f_n \in \mathcal{M}_*$, s.t., for all $g \in K$:

$$\sum_{i=1}^{n} \alpha_i f_i = \delta(g),\ \alpha_i \in \mathbb{C} \qquad \Rightarrow \alpha_1 = \ldots = \alpha_n = 0.$$

Then

$$I : K\langle W\rangle \to \mathcal{I}^{\mathcal{M}}, \qquad W := \mathrm{Span}_K\{f_1, \ldots, f_n\}$$

is injective.

A basis for $\mathcal{I}^{\mathcal{M}}$: II

- Write $\mathcal{M}_* = (\delta(K) \cap \mathcal{M}_*) \oplus C_*$
- Apply previous theorem "$+\varepsilon$":

Theorem

(i) The natural map

$$K\langle C_* \rangle \to \mathcal{I}^{\mathcal{M}}$$

is an isomorphism of differential K-algebras.

(ii) We have

$$\delta(K) \cap \mathcal{M}_k = \begin{cases} 0, & k < 2, \\ \delta^{k-1}(\mathcal{M}_{2-k}), & k \geq 2. \end{cases}$$

Remark

Similar result for $\mathcal{I}^{\mathcal{QM}} = \mathrm{Span}_K\{I(f_1, \ldots, f_n) : n \geq 0,\ f_1, \ldots, f_n \in \mathcal{QM}_*\}$.

A criterion for algebraic independence “à la Lindemann–Weierstrass”

- $f_1, \ldots, f_n \in \mathcal{M}_k,\ k \geq 2$

Corollary

$$\overline{f}_1, \ldots, \overline{f}_n \in \mathcal{M}_k/\delta^{k-1}(\mathcal{M}_{2-k}),\ \mathbb{C}\text{-linearly independent}$$

$$\Leftrightarrow$$

$$I(f_1), \ldots, I(f_n),\ K\text{-algebraically independent}$$

Remark

This gives a conceptual proof of Pasol–Zudilin’s result:

$$I\left(\frac{\Delta}{E_4^2}\right), \quad I\left(\frac{E_4\Delta}{E_6^2}\right), \quad I\left(\frac{E_6\Delta}{E_4^3}\right)$$

are K-algebraically independent (compare pole orders at $z = i$, $z = \rho$).

Dimension formulas

- $k \geq 2$ integer, $S \subset \mathfrak{H}$, $\mathrm{SL}_2(\mathbb{Z})$-finite set.
- $M_k^!(*S) = \{f \in \mathcal{M}_k \,:\, f \text{ holomorphic outside } S \cup \{\infty\}\}$
- $B_k(*S) := M_k^!(*S)/\delta^{k-1}(M_{2-k}^!(*S)) \cong C_k(*S)$

Theorem (well-known?)

For $k \geq 2$, we have

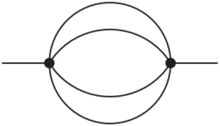
$$\dim B_k(*S) = 2 \dim S_k + 1 + |S'|(k-1) + \left(2\left\lfloor \frac{k-2}{4} \right\rfloor + 1\right)\chi_i(S) + \left(2\left\lfloor \frac{k-2}{6} \right\rfloor + 1\right)\chi_\rho(S)$$

where

- $S' := S \setminus \{i, \rho\}$, $\rho = e^{2\pi i/3}$,
- $\chi_z(S) = \begin{cases} 1 & z \in S \\ 0 & z \notin S \end{cases}$

Generalizations and applications (jt. with J. Broedel and C. Duhr)

- Everything generalizes to arbitrary finite-index subgroups $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ of genus zero
- This case appears naturally in quantum field theory: for $\Gamma = \Gamma_1(6)$, get analytic (rather than just numerical) expressions for the three-loop equal mass banana integral



- Key mathematical point: these integrals satisfy a Picard–Fuchs equation which is a symmetric square of a rank two operator

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The Hodge bundle and modular forms

- $\mathcal{M}_{1,1}$ = moduli space of elliptic curves (Deligne–Mumford stack over $\mathbb{Q}$)
- $\pi : \mathcal{E} \to \mathcal{M}_{1,1}$ universal elliptic curve

Hodge bundle

$$\mathcal{L} := \pi_* \Omega^1_{\mathcal{E}/\mathcal{M}_{1,1}}$$

line bundle over $\mathcal{M}_{1,1}$

- $\mathcal{L}_{[E]} = H^0(E, \Omega^1_E)$, fibre at $[E] \in \mathcal{M}_{1,1}$
- $\Gamma(\mathcal{M}_{1,1}, \mathcal{L}^{\otimes k}) = M^!_k$, "weakly holomorphic modular forms" (=only poles at cusp)

Remark

- $\exists\, \overline{\mathcal{L}}$ extension to $\overline{\mathcal{M}}_{1,1}$
- $\Gamma(\overline{\mathcal{M}}_{1,1}, \overline{\mathcal{L}}^{\otimes k}) = M_k$

The de Rham bundle

The de Rham bundle

$$\mathcal{V} := R^1\pi_*\Omega^\bullet_{\mathcal{E}/\mathcal{M}_{1,1}}$$

- rank two vector bundle over $\mathcal{M}_{1,1}$
- $\mathcal{L} \subset \mathcal{V}$ sub-bundle, "Hodge filtration"
- fibre at $[E]$ is $H^1_{dR}(E)$, algebraic de Rham cohomology of E, Hodge filtration $H^0(E, \Omega^1_E) \subset H^1_{dR}(E)$

Folklore Scholl, Coleman (1980s)

For $S \subset \mathcal{M}_{1,1}$ finite set, have

$$H^1_{dR}(\mathcal{M}_{1,1} \setminus S, \mathrm{Sym}^{k-2}\mathcal{V}) \cong B_k(*S)$$

Reference (for $S = \emptyset$): Brown–Hain, *Algebraic de Rham theory for weakly holomorphic modular forms of level one*, 2018

Interpretation

Upshot 1:

$$
\begin{array}{ll}
 & \text{Gysin sequence in algebraic de Rham cohomology} \\
\Longrightarrow & \text{dimension formulas (should generalize to more general } \Gamma\text{)}
\end{array}
$$

NB:

For genus zero groups, can prove this by elementary means (=no de Rham cohomology needed)

Upshot 2:

Given $f_1, \ldots, f_n \in M_k^!(*S)$:

$$
\begin{array}{ll}
 & \text{Cohomology classes } \overline{f}_1, \ldots, \overline{f}_n \in B_k(*S) \text{ are } \mathbb{C}\text{-linearly independent} \\
\Longleftrightarrow & I(f_1), \ldots, I(f_n)\ K\text{-algebraically independent}
\end{array}
$$

Summary

- Meromorphic modular form: modular form with poles
- Fourier expansion: (finite-tailed) Laurent series
- interesting arithmetic properties of Fourier coefficients ("magnetism")
- iterated integrals of meromorphic (quasi-)modular forms via renormalization (à la Connes–Kreimer)
- main results: (i) complete algebraic description of $\mathcal{I}^{\mathcal{QM}}$ as a shuffle algebra; (ii) dimension formulas for Bol space
- (i) $\Rightarrow$ algebraic independence criterion for integrals of meromorphic (quasi-) modular forms, reproving results of Pasol–Zudilin
- Applications to Feynman integrals in quantum field theory
- extending results to general groups might need more geometric approach via algebraic de Rham cohomology of (Zariski-open subsets of) $\mathcal{M}_{1,1}$

Thank you!