# Dimensions for the spaces of Siegel cusp forms of level 4

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# Outline

- Available dimensions of spaces of Siegel cusp forms of degree 2.
- Counting cuspidal automorphic representation of $\mathrm{GSp}(4)$.
- New dimensions of spaces of Siegel cusp forms of degree 2 and level 4.

# The group $\mathrm{GSp}(4)$

$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) : {}^t gJg = \lambda(g)J,\ \lambda(g) \in \mathrm{GL}(1)\}$, where

$$J = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}.$$

$\mathrm{Sp}(4) := \{g \in \mathrm{GSp}(4) : \lambda(g) = 1\}$. We consider the congruence subgroups:

$$\mathrm{K}(N) := \mathrm{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

$$\Gamma_0(N) := \mathrm{Sp}(4, \mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

$$\Gamma_0'(N) := \mathrm{Sp}(4, \mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

$$\mathrm{B}(N) := \mathrm{Sp}(4, \mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

# Siegel modular forms (degree 2)

Let $\mathcal{H}_2 = \{Z = X + iY \in M_2(\mathbb{C}) : Z^t = Z \text{ and } Y \text{ is positive definite}\}$.

A holomorphic function $f : \mathcal{H}_2 \to \mathbb{C}$ is called a **Siegel modular form** of degree 2 and weight $k$ if

$$\det(CZ+D)^{-k} f\left((AZ+B)(CZ+D)^{-1}\right) = f(Z) \text{ for } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_N.$$

We consider the following congruence subgroups $\Gamma_N$ of $\mathrm{Sp}(4,\mathbb{Q})$:

| $\Gamma_N$: | $\mathrm{K}(N)$ | $\Gamma_0(N)$ | $\Gamma_0'(N)$ | $\mathrm{B}(N)$ |
|---|---|---|---|---|

- $F$ is called a **Siegel cusp form** if $\lim\limits_{\lambda\to\infty} (F|g)\left(\left[\begin{smallmatrix} i\lambda & \\ & \tau \end{smallmatrix}\right]\right) = 0$ for all $g \in \mathrm{Sp}(4,\mathbb{Q})$ and $\tau \in \mathcal{H}_1$.

$S_k(\Gamma_N)$: the space of Siegel cusp forms of weight $k$ with respect to $\Gamma_N$.

# Dimensions of Siegel cusp forms of level $p$

| Dimensions | $k$ | $p$ | References |
|---|---|---|---|
| $\dim_{\mathbb{C}} S_k(\mathrm{Sp}(4,\mathbb{Z}))$ | | | Igusa 1964, Hashimoto 1983 |
| $\dim_{\mathbb{C}} S_k(\mathrm{K}(p))$ | $\geq 5$ | $\geq 2$ | Ibukiyama 1985 |
| $\dim_{\mathbb{C}} S_k(\Gamma_0(p))$ | $\geq 5$ | $\geq 3$ | Hashimoto 1983 |
| | $\geq 4$ | $\geq 2$ | Tsushima 1997 |
| $\dim_{\mathbb{C}} S_k(\Gamma_0'(p))$ | $\geq 5$ | $\geq 5$ | Hashimoto, Ibukiyama 1985 |
| $\dim_{\mathbb{C}} S_k(\mathrm{B}(p))$ | $\geq 5$ | $\geq 5$ | Hashimoto, Ibukiyama 1985 |
| all | $\geq 5$ | $\geq 2$ | Wakatsuki 2012 |
| all | $3, 4$ | $\geq 2$ | Ibukiyama 2007 |
| all but $\dim_{\mathbb{C}} S_k(\mathrm{B}(3))$ | $2$ | $2, 3$ | Ibukiyama 1984, Ibukiyama 2018 |

# Dimensions of Siegel cusp forms of non-squarefree level

| Dimensions | $k$ | References |
|---|---|---|
| $\dim_{\mathbb{C}} S_k(\Gamma_0(4))$ | $\geq 0$ | Tsushima 2003 |
| $\dim_{\mathbb{C}} S_k(\mathrm{K}(4))$ | $\geq 0$ | Poor, Yuen 2013 |
| $\dim_{\mathbb{C}} S_k(\mathrm{K}(8))$ | $10, 12$ | Poor, Schmidt, Yuen 2018 |
| $\dim_{\mathbb{C}} S_k(\mathrm{K}(16))$ | $\leq 14$ | Poor, Schmidt, Yuen 2018 |

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**Available methods:**

- Riemann-Roch-Hirzebruch theorem for $k \geq 4$ and Selberg trace formula for $k \geq 5$.
- Igusa's theorem to find $\dim_{\mathbb{C}} M_k(\Gamma_N)$ and Satake's theorem to find the codimension formula.

# New dimension formulas of Siegel cusp forms of level 4

**Goal:** Find the dimension of spaces of Siegel cusp forms of degree 2 with respect to

(1) The Klingen congruence subgroup of level 4

$$\Gamma_0'(4) := \mathrm{Sp}(4,\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & 4\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 4\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & 4\mathbb{Z} & 4\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

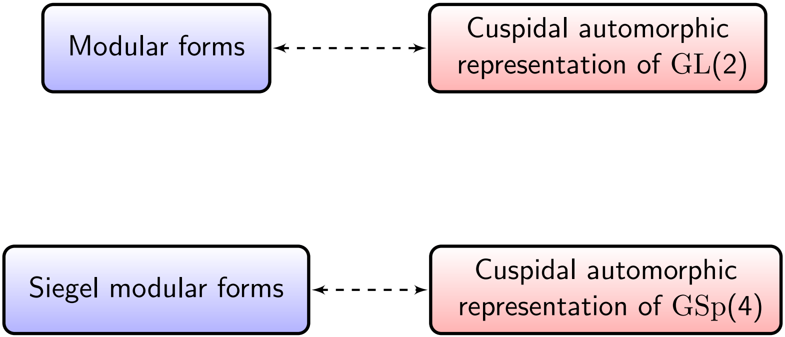
(2) The middle group of level 4

$$\mathrm{M}(4) := \mathrm{Sp}(4,\mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & 4\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 2^{-1}\mathbb{Z} \\ \mathbb{Z} & 4\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & 4\mathbb{Z} & 4\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

Note that, $\Gamma_0'(4) \subset \mathrm{M}(4) \subset \mathrm{K}(4)$.

**Counting certain set of cuspidal automorphic representations of $\mathrm{GSp}(4)$.**

# Classical modular forms and automorphic representations



# Automorphic forms

- $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the $p$-adic norm.
- Consider a global (adelic) group $G(\mathbb{A}_\mathbb{Q}) = \prod'_{p\leq\infty} G(\mathbb{Q}_p)$, which is a restricted direct product of local groups.

A smooth function $\Phi\colon G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ is called an **automorphic form** if

$$\Phi(\gamma g) = \Phi(g) \quad \text{for all } g \in G(\mathbb{A}_\mathbb{Q}) \text{ and } \gamma \in G(\mathbb{Q}),$$

and $\Phi$ satisfies a few other properties. $\Phi$ is called **cusp form** if

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \Phi(ng)\, dn = 0$$

for all unipotent radicals $N$ of parabolic subgroups of $G$.

# Automorphic representations

An **automorphic representation** $(\pi, V)$ of $G(\mathbb{A}_{\mathbb{Q}})$ is an irreducible subquotient of the set of automorphic forms on $G(\mathbb{A}_{\mathbb{Q}})$.

An automorphic rep. $(\pi, V)$ is called **cuspidal** if $V$ consists of cusp forms.

Roughly speaking, an automorphic representation $\pi$ of $G(\mathbb{A}_{\mathbb{Q}})$ is an irreducible, admissible representation $G(\mathbb{A}_{\mathbb{Q}})$ that has the form

$$\pi \cong \bigotimes_{p \leq \infty} \pi_p$$

where $\pi_p$ is an irreducible, admissible representation of $G(\mathbb{Q}_p)$.

# Four types of local irreducible admissible representations

- (Constituents of) Borel-induced representations:

$$\chi_1 \times \chi_2 \rtimes \sigma \qquad B = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} = \underbrace{\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix}}_{\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times} \ltimes \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}$$

- (Constituents of) Siegel-induced representations:

$$\pi \rtimes \sigma \qquad P = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} = \underbrace{\begin{bmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{bmatrix}}_{\mathrm{GL}(2,\mathbb{Q}_p) \times \mathbb{Q}_p^\times} \ltimes \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- (Constituents of) Klingen-induced representations:

$$\chi \rtimes \pi \qquad Q = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix} = \underbrace{\begin{bmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{bmatrix}}_{\mathbb{Q}_p^\times \times \mathrm{GL}(2,\mathbb{Q}_p)} \ltimes \begin{bmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}$$

- Supercuspidal representations

Sally and Tadić (1993) classified the non-supercuspidal representations.

# Borel-induced representations

| $\Omega$ | constituents of | | representation | tempered | $L^2$ | g |
|---|---|---|---|---|---|---|
| I | $\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible) | | | • | | • |
| II | $\nu^{1/2}\chi \times \nu^{-1/2}\chi \rtimes \sigma$ | a | $\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$ | • | | • |
| | $(\chi^2 \neq \nu^{\pm 1}, \chi \neq \nu^{\pm 3/2})$ | b | $\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$ | | | |
| III | $\chi \times \nu \rtimes \nu^{-1/2}\sigma$ | a | $\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$ | • | | • |
| | $(\chi \notin \{1, \nu^{\pm 2}\})$ | b | $\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$ | | | |
| IV | $\nu^2 \times \nu \rtimes \nu^{-3/2}\sigma$ | a | $\sigma\mathrm{St}_{\mathrm{GSp}(4)}$ | • | • | • |
| | | b | $L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$ | | | |
| | | c | $L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$ | | | |
| | | d | $\sigma 1_{\mathrm{GSp}(4)}$ | | | |
| V | $\nu\xi \times \xi \rtimes \nu^{-1/2}\sigma$ | a | $\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$ | • | • | • |
| | $(\xi^2 = 1,\ \xi \neq 1)$ | b | $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ | | | |
| | | c | $L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$ | | | |
| | | d | $L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$ | | | |
| VI | $\nu \times 1_{F^\times} \rtimes \nu^{-1/2}\sigma$ | a | $\tau(S, \nu^{-1/2}\sigma)$ | • | | • |
| | | b | $\tau(T, \nu^{-1/2}\sigma)$ | • | | |
| | | c | $L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$ | | | |
| | | d | $L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$ | | | |

# Klingen- and Siegel-induced representations

Klingen-induced:

| $\Omega$ | constituents of | | representation | tempered | $L^2$ | g |
|---|---|---|---|---|---|---|
| VII | $\chi \rtimes \pi$ | (irreducible) | | • | | • |
| VIII | $1_{F^\times} \rtimes \pi$ | a | $\tau(S,\pi)$ | • | | • |
| | | b | $\tau(T,\pi)$ | • | | |
| IX | $\nu\xi \rtimes \nu^{-1/2}\pi$ | a | $\delta(\nu\xi, \nu^{-1/2}\pi)$ | • | • | • |
| | $(\xi \neq 1,\ \xi\pi = \pi)$ | b | $L(\nu\xi, \nu^{-1/2}\pi)$ | | | |

Siegel-induced:

| $\Omega$ | constituents of | | representation | tempered | $L^2$ | g |
|---|---|---|---|---|---|---|
| X | $\pi \rtimes \sigma$ | (irreducible) | | • | | • |
| XI | $\nu^{1/2}\pi \rtimes \nu^{-1/2}\sigma$ | a | $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ | • | • | • |
| | $(\omega_\pi = 1)$ | b | $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ | | | |

# Three supercuspidal representations

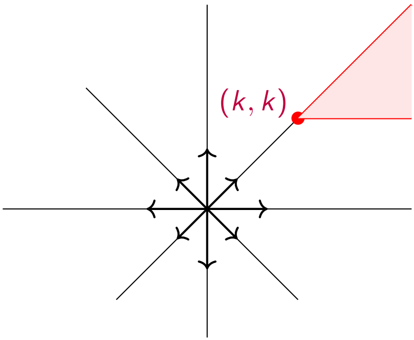
| $\Omega$ | representation | tempered | $L^2$ | g |
|---|---|---|---|---|
| Va* | $\delta^*([\xi, \nu\xi], \nu^{-1/2}\sigma)$ | • | • | |
| XIa* | $\delta^*(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$ | • | • | |
| sc(16) | $\text{c-Ind}_{ZG(\mathbb{Z}_2)}^{G(\mathbb{Q}_2)}([2,2,1,1])$ | • | • | • |

$L$-packets: $\{$VIa, VIb$\}$, $\{$VIIIa, VIIIb$\}$, $\{$Va, Va*$\}$, $\{$XIa, XIa*$\}$

sc(16): The only generic depth-zero supercuspidal of $\mathrm{GSp}(4, \mathbb{Q}_2)$.

# What about $\pi_\infty$?

A representation of $\mathrm{GSp}(4, \mathbb{R})$ can be visualized by the weight structures with a minimal weight.




We consider $\pi_\infty$ as the *lowest weight module* with minimal $K$-type $(k, k)$.

This is related to the Siegel modular forms of weight $k$.

# The family of automorphic representations we study

Let $k \in \mathbb{Z}_{>0}$. Let $S_k(\Omega)$ be the set of cuspidal automorphic representations $\pi \cong \bigotimes_{v \leq \infty} \pi_v$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ with trivial central character (i.e., a rep. of $\mathrm{PGSp}(4, \mathbb{A}_{\mathbb{Q}})$) such that

$\pi_v$ is unramified for all $v \neq 2, \infty$.

$\pi_\infty$ is the lowest weight module with minimal $K$-type $(k, k)$.

$k = 1$: *a non-tempered representation*

$k = 2$: *a holomorphic limit of discrete series representation*

$k \geq 3$: *a holomorphic discrete series representation*

$\pi_2$ is a depth-zero representation of $\mathrm{PGSp}(4, \mathbb{Q}_2)$ of type $\Omega$.

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$$s_k(\Omega) := \# S_k(\Omega)$$

# Dimensions of spaces of Siegel cusp forms and $s_k(\Omega)$

A Siegel modular form of degree 2, level 4, weight $k$

A representation of $\mathrm{PGSp}(4)$ $\pi \cong \bigotimes_{v \leq \infty} \pi_v$ in $S_k(\Omega)$

Global dimensions $\dim_{\mathbb{C}} S_k(\Gamma)$ where $\Gamma$ are congruence subgroups

Dimensions of spaces of fixed vectors of $(\pi_2, V_2)$ under compact open subgroups $C$

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| $\Gamma$ | $\mathrm{Sp}(4,\mathbb{Z})$ | $\mathrm{K}(4)$ | $\Gamma_0(4)$ | $\Gamma_0'(4)$ | $\mathrm{M}(4)$ | |
|---|---|---|---|---|---|---|
| $C$ | $\mathrm{GSp}(4,\mathbb{Z}_2)$ | $\mathrm{K}(\mathfrak{p}^2)$ | $\mathrm{Si}(\mathfrak{p}^2)$ | $\mathrm{Kl}(\mathfrak{p}^2)$ | $\mathrm{M}(\mathfrak{p}^2)$ | $(\mathfrak{p} = 2\mathbb{Z}_2)$ |

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$$\pi \in S_k(\Omega) \longrightarrow \text{cusp form } \Phi \in V \cong \bigotimes_{v\leq\infty} V_v \longrightarrow \text{eigenform } f \in S_k(\Gamma)$$

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$$\pi \in S_k(\Omega) \longrightarrow \text{cusp form } \Phi \in V \cong \bigotimes_{v \leq \infty} V_v \longrightarrow \text{eigenform } f \in S_k(\Gamma)$$

$$\dim_{\mathbb{C}} S_k(\Gamma) = \sum_{\Omega} \sum_{\pi \in S_k(\Omega)} \dim \pi_2^C = \sum_{\substack{\Omega \\ \text{s.t. } \pi \in S_k(\Omega)}} s_k(\Omega) \dim \pi_2^C.$$

# Computing new dimensions of some spaces of Siegel cusp forms of level $4$ with respect to $\Gamma_0'(4)$ and $\mathrm{M}(4)$

# Main ingredients

Suppose $\Gamma = \Gamma_0'(4), \mathrm{M}(4)$. Then we have

$$\dim_{\mathbb{C}} S_k(\Gamma) = \sum_{\substack{\Omega \\ \text{s.t. } \pi \in S_k(\Omega)}} s_k(\Omega) \dim \pi_2^C.$$

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**[Yi, 2019]**: A representation of $\mathrm{GSp}(4, \mathbb{Q}_2)$ which has a non-zero $\mathrm{Kl}(\mathfrak{p}^2)$-invariant vectors is depth-zero. Give the dimensions of spaces of $\mathrm{Kl}(\mathfrak{p}^2)$ and $\mathrm{M}(\mathfrak{p}^2)$-invariant vectors.

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**[R., Schmidt, Yi, 2021]**: We compute $s_k(\Omega)$ explicitly.

# Arthur packets

Six types of discrete automorphic representations $\pi$ of $\mathrm{PGSp}(4, \mathbb{A}_{\mathbb{Q}})$:

**(G)** **General**: $L(s,\pi) = L(s,\Pi)$ with $\Pi \in \mathcal{A}_0(\mathrm{GL}(4,\mathbb{A}_{\mathbb{Q}}))$

**(Y)** **Yoshida**: $L(s,\pi) = L(s,\mu_1)L(s,\mu_2)$ with $\mu_1, \mu_2 \in \mathcal{A}_0(\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}}))$

**(P)** **Saito-Kurokawa, *P*-CAP**: $L(s,\pi) = L(s,\mu)L(s+1/2,\chi)L(s-1/2,\chi)$

**(Q)** **Soudry, *Q*-CAP**: $L(s,\pi) = L(s+1/2,\mu)L(s-1/2,\mu)$

**(B)** **Howe–Piatetski-Shapiro, *B*-CAP**:
$L(s,\pi) = \prod_{i=1}^{2} L(s+\frac{1}{2},\chi_i)L(s-\frac{1}{2},\chi_i)$

**(F)** **Finite** (one-dimensional)**:** Not occur in the cuspidal spectrum!

$$s_k(\Omega) = s_k^{(\mathbf{G})}(\Omega) + s_k^{(\mathbf{Y})}(\Omega) + s_k^{(\mathbf{P})}(\Omega) + s_k^{(\mathbf{Q})}(\Omega) + s_k^{(\mathbf{B})}(\Omega)$$

Note: **(Q)** and **(B)** packets could have only contributed to weight 1 or 2.

# How to compute $s_k(\Omega)$?

**Theorem (R., Schmidt, Yi, 2021)**

$s_k^{(\mathbf{B})}(\Omega) = s_k^{(\mathbf{Q})}(\Omega) = s_k^{(\mathbf{Y})}(\Omega) = 0$ *for all* $k$ *and all* $\Omega$.

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$$s_k(\Omega) = s_k^{(\mathbf{G})}(\Omega) + s_k^{(\mathbf{P})}(\Omega).$$

$$\dim_{\mathbb{C}} S_k(\Gamma) = \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma) + \dim_{\mathbb{C}} S_k^{(\mathbf{P})}(\Gamma).$$

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- groups I-VI: we need $\dim_{\mathbb{C}} S_k(\Gamma)$ for $\Gamma \in \{\mathrm{Sp}(4,\mathbb{Z}), \mathrm{K}(2), \Gamma_0(2), \Gamma_0'(2), I(2)\}$.
- groups VII-XI: we need $\dim_{\mathbb{C}} S_k(\Gamma)$ for $\Gamma \in \{\Gamma(2), \mathrm{K}(4), \Gamma_0(4), \Gamma_0^*(4)\}$.

$$\Gamma_0^*(4) = \left\{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_0(4) : D \equiv \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right] \pmod 2 \right\}$$

$$1 \to \Gamma_0^*(4) \to \Gamma_0(4) \to \{\pm 1\} \to 1.$$

# Arthur packets for depth-zero representations

| Ω | Tempered | (**G**) | (**P**) | (**Y**) | $\Gamma(\mathfrak{p})$ | $K$ | $\mathrm{K}(\mathfrak{p})$ | $\mathrm{K}(\mathfrak{p}^2)$ | $\mathrm{Si}(\mathfrak{p})$ | $\mathrm{Si}(\mathfrak{p}^2)$ | $\mathrm{Si}^*(\mathfrak{p}^2)$ | $\mathrm{Kl}(\mathfrak{p})$ | $\mathrm{Kl}(\mathfrak{p}^2)$ | $\mathrm{M}(\mathfrak{p}^2)$ | $I$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| I | • | • | | ∘ | 45 | 1 | 2 | 4 | 4 | 12 | 15 | 4 | 11 | 8 | 8 |
| IIa | • | • | | ∘ | 30 | 0 | 1 | 2 | 1 | 5 | 8 | 2 | 7 | 5 | 4 |
| IIb | | | • | | 15 | 1 | 1 | 2 | 3 | 7 | 7 | 2 | 4 | 3 | 4 |
| IIIa | • | • | | | 30 | 0 | 0 | 1 | 2 | 8 | 10 | 1 | 5 | 3 | 4 |
| IIIb | | | | | 15 | 1 | 2 | 3 | 2 | 4 | 5 | 3 | 6 | 5 | 4 |
| IVa | • | • | | | 16 | 0 | 0 | 0 | 0 | 2 | 4 | 0 | 2 | 1 | 1 |
| IVb | never unitary | | | | 14 | 0 | 0 | 1 | 2 | 6 | 6 | 1 | 3 | 2 | 3 |
| IVc | never unitary | | | | 14 | 0 | 1 | 2 | 1 | 3 | 4 | 2 | 5 | 4 | 3 |
| IVd | | | | | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Va | • | • | | ∘ | 21 | 0 | 0 | 1 | 0 | 2 | 5 | 1 | 5 | 3 | 2 |
| Vb | | | • | | 9 | 0 | 1 | 1 | 1 | 3 | 3 | 1 | 2 | 2 | 2 |
| Vc | | | ∘ | | 9 | 0 | 1 | 1 | 1 | 3 | 3 | 1 | 2 | 2 | 2 |
| Vd | | | | | 6 | 1 | 0 | 1 | 2 | 4 | 4 | 1 | 2 | 1 | 2 |
| VIa | • | • | | ∘ | 25 | 0 | 0 | 1 | 1 | 5 | 7 | 1 | 5 | 3 | 3 |
| VIb | • | • | • | • | 5 | 0 | 0 | 0 | 1 | 3 | 3 | 0 | 0 | 0 | 1 |
| VIc | | | • | | 5 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 2 | 1 |
| VId | | | | | 10 | 1 | 1 | 2 | 2 | 4 | 4 | 2 | 4 | 3 | 3 |
| VII | • | • | | | 15 | 0 | 0 | 0 | 0 | 4 | 5 | 0 | 2 | 0 | 0 |
| VIIIa | • | • | | ∘ | 10 | 0 | 0 | 0 | 0 | 3 | 4 | 0 | 2 | 0 | 0 |
| VIIIb | • | • | | • | 5 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| IXa | • | • | | | 10 | 0 | 0 | 0 | 0 | 3 | 4 | 0 | 1 | 0 | 0 |
| IXb | | | | | 5 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| X | • | • | | ∘ | 15 | 0 | 0 | 1 | 0 | 1 | 7 | 0 | 3 | 2 | 0 |
| XIa | • | • | | ∘ | 10 | 0 | 0 | 0 | 0 | 1 | 4 | 0 | 2 | 1 | 0 |
| XIb | | | • | | 5 | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 1 | 1 | 0 |
| Va* | • | • | • | • | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| sc(16) | • | • | | | 9 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 1 | 0 | 0 |

# How to compute $s_k(\Omega)$?

$$
\begin{bmatrix}
\dim_{\mathbb{C}} S_k(\mathrm{Sp}(4,\mathbb{Z})) \\
\dim_{\mathbb{C}} S_k(\mathrm{K}(2)) \\
\dim_{\mathbb{C}} S_k(\Gamma_0'(2)) \\
\dim_{\mathbb{C}} S_k(\Gamma_0(2)) \\
\dim_{\mathbb{C}} S_k(B(2)) \\
\dim_{\mathbb{C}} S_k(\mathrm{K}(4)) \\
\dim_{\mathbb{C}} S_k(\Gamma_0(4)) \\
\dim_{\mathbb{C}} S_k(\Gamma_0^*(4)) \\
\dim_{\mathbb{C}} S_k(\Gamma(2))
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 3 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 4 & 4 & 4 & 1 & 2 & 2 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
12 & 5 & 7 & 8 & 2 & 2 & 3 & 5 & 3 & 0 & 4 & 3 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\
15 & 8 & 7 & 10 & 4 & 5 & 3 & 7 & 3 & 1 & 5 & 4 & 1 & 4 & 7 & 4 & 3 & 1 & 3 \\
45 & 30 & 15 & 30 & 16 & 21 & 9 & 25 & 5 & 5 & 15 & 10 & 5 & 10 & 15 & 10 & 5 & 1 & 9
\end{bmatrix}
\begin{bmatrix}
s_k(\mathrm{I}) \\
s_k(\mathrm{IIa}) \\
s_k(\mathrm{IIb}) \\
s_k(\mathrm{IIIa}) \\
s_k(\mathrm{IVa}) \\
s_k(\mathrm{Va}) \\
s_k(\mathrm{Vb}) \\
s_k(\mathrm{VIa}) \\
s_k(\mathrm{VIb}) \\
s_k(\mathrm{VIc}) \\
s_k(\mathrm{VII}) \\
s_k(\mathrm{VIIIa}) \\
s_k(\mathrm{VIIIb}) \\
s_k(\mathrm{IXa}) \\
s_k(\mathrm{X}) \\
s_k(\mathrm{XIa}) \\
s_k(\mathrm{XIb}) \\
s_k(\mathrm{Va}^*) \\
s_k(\mathrm{sc}(16))
\end{bmatrix}
$$

- $\dim \pi_2^C$ for $C = \mathrm{K}(\mathfrak{p}), \mathrm{Si}(\mathfrak{p}), \mathrm{Kl}(\mathfrak{p}), \mathrm{I}(\mathfrak{p})$ have been computed by Schmidt (2005).
- $\dim \pi_2^C$ for $C = \mathrm{Si}(\mathfrak{p}^2), \mathrm{Si}^*(\mathfrak{p}^2), \Gamma(\mathfrak{p})$ are computed by using hyperspecial parahoric restriction of $\mathrm{PGSp}(4, \mathbb{Q}_2)$.

# $s_k^{(\mathbf{P})}(\Omega)$ (Saito-Kurokawa types)

$$s_k(\mathrm{IIb}) = \begin{cases} \dim_{\mathbb{C}} S_{2k-2}(\mathrm{SL}(2,\mathbb{Z})) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

$$s_k(\mathrm{Vb}) = \begin{cases} \dim_{\mathbb{C}} S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{VIb}) = \begin{cases} \dim_{\mathbb{C}} S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

$$s_k(\mathrm{VIc}) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dim_{\mathbb{C}} S_{2k-2}^{-,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \text{ is odd.} \end{cases}$$

$$s_k^{(\mathbf{P})}(\mathrm{Va}^*) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ \dim_{\mathbb{C}} S_{2k-2}^{+,\mathrm{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \text{ is odd.} \end{cases}$$

$$s_k(\mathrm{XIb}) = \begin{cases} 0 & \text{if } k \text{ even,} \\ \dim_{\mathbb{C}} S_{2k-2}^{\mathrm{new}}(\Gamma_0^{(1)}(4)) & \text{if } k \text{ odd.} \end{cases}$$

| $\Gamma$ | $\sum_{k\geq 0} \dim_{\mathbb{C}} S_k^{(\mathsf{P})}(\Gamma) t^k$ |
|---|---|
| $\mathrm{Sp}(4,\mathbb{Z})$ | $\dfrac{t^{10}}{(1-t^2)(1-t^6)}$ |
| $\mathrm{K}(2)$ | $\dfrac{t^8(1+t^2+t^3+t^4)}{(1-t^4)(1-t^6)}$ |
| $\Gamma_0(2)$ | $\dfrac{t^6(1+t^2+2t^4)}{(1-t^2)(1-t^6)}$ |
| $\Gamma_0'(2)$ | $\dfrac{t^8(1+2t^2+t^3+2t^4)}{(1-t^4)(1-t^6)}$ |
| $B(2)$ | $\dfrac{t^6(1+3t^2+4t^4+t^5+3t^6)}{(1-t^4)(1-t^6)}$ |
| $\mathrm{K}(4)$ | $\dfrac{t^7(1+t+t^2+2t^3+t^4+2t^5)}{(1-t^4)(1-t^6)}$ |
| $\Gamma_0(4)$ | $\dfrac{t^6(3+3t^2+4t^4)}{(1-t^2)(1-t^6)}$ |
| $\Gamma_0^*(4)$ | $\dfrac{t^5(1+3t+2t^2+3t^3+t^4+4t^5)}{(1-t^2)(1-t^6)}$ |
| $\Gamma(2)$ | $\dfrac{t^5(1+t+t^2)(1+4t+10t^3-5t^4+10t^5)}{(1-t^4)(1-t^6)}$ |

# General type supercuspidal representations

$$\sum_{k\geq 0} s_k^{(\mathbf{G})}(\mathrm{Va}^*)t^k = \frac{t^{15}(1+t^2-t^{12})+t^{30}}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}$$

$$\sum_{k\geq 0} s_k(\mathrm{sc}(16))t^k = \frac{t^9}{(1-t^2)(1-t^4)^2(1-t^5)}$$

- We consider the hyperspecial parahoric restriction of $\mathrm{PGSp}(4,\mathbb{Q}_2)$. This is the equivalence class of the representation of $\mathrm{Sp}(4,\mathbb{F}_2) \cong S_6$ acting on the space of $\Gamma(\mathfrak{p})$-invariant vectors.
- $\mathrm{Sp}(4,\mathbb{Z})/\Gamma(2) \cong \mathrm{Sp}(4,\mathbb{F}_2) \cong S_6$ acts on the space $M_k(\Gamma(2))$ of Siegel modular forms of weight $k$. The characters for the representation of $S_6$ on $M_k(\Gamma(2))$ are given in **[Igusa, 1964]**.

# How to compute $\dim_{\mathbb{C}} S_k(\Gamma_0(4)^*)$?

**[Igusa, 1964]:** One can obtain $\dim_{\mathbb{C}} M_k(\Gamma)$ whenever $\Gamma(2) \subset \Gamma$.

# How to compute $\dim_{\mathbb{C}} S_k(\Gamma_0(4)^*)$?

**[Igusa, 1964]:** One can obtain $\dim_{\mathbb{C}} M_k(\Gamma)$ whenever $\Gamma(2) \subset \Gamma$.

Using **Satake's theorem [1957-1958]**:

**Theorem (R., Schmidt, Yi, 2021)**

*Let $\Gamma$ be a congruence subgroup of $\mathrm{Sp}(4,\mathbb{Q})$. Let $X$ be a fixed set of representatives of $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/P(\mathbb{Q})$ and $Y$ be a fixed set of representatives of $\Gamma\backslash\mathrm{Sp}(4,\mathbb{Q})/Q(\mathbb{Q})$. Let $\omega : \begin{bmatrix} a & 0 & b & * \\ * & * & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & * \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\Gamma_y := \omega\left(y^{-1}\Gamma y \cap Q(\mathbb{Q})\right)$ for $y \in Y$. Then, for even $k \geq 6$, we have*

$$\dim_{\mathbb{C}} M_k(\Gamma) - \dim_{\mathbb{C}} S_k(\Gamma) = |X| + \sum_{y \in Y} \dim_{\mathbb{C}} S_k(\Gamma_y).$$

*If $\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \in y^{-1}\Gamma y \cap Q(\mathbb{Q})$, then for any odd $k \geq 1$,*

$$\dim_{\mathbb{C}} M_k(\Gamma) - \dim_{\mathbb{C}} S_k(\Gamma) = 0.$$

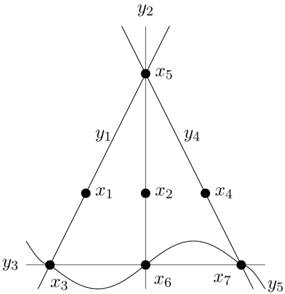
# $\dim_{\mathbb{C}} S_k(\Gamma_0(4)^*)$

$$\sum_{k\geq 0} \dim_{\mathbb{C}} M_k(\Gamma_0(4)^*) t^k = \frac{1+t^4+t^5+t^6+t^9+t^{10}+t^{11}+t^{15}}{\left(1-t^2\right)^3\left(1-t^6\right)}.$$

# $\dim_{\mathbb{C}} S_k(\Gamma_0(4)^*)$

$$\sum_{k\geq 0} \dim_{\mathbb{C}} M_k(\Gamma_0(4)^*) t^k = \frac{1+t^4+t^5+t^6+t^9+t^{10}+t^{11}+t^{15}}{(1-t^2)^3(1-t^6)}.$$

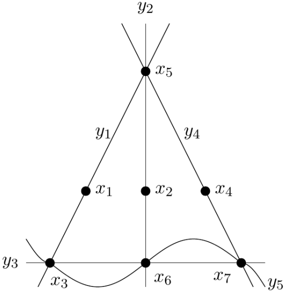
$$\dim_{\mathbb{C}} M_k(\Gamma_0^*(4)) - \dim_{\mathbb{C}} S_k(\Gamma_0^*(4)) = 7 + 5 \dim S_k(\mathrm{SL}(2,\mathbb{Z}) \cap \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right]).$$



# $\dim_{\mathbb{C}} S_k(\Gamma_0(4)^*)$

$$\sum_{k\geq 0} \dim_{\mathbb{C}} M_k(\Gamma_0(4)^*) t^k = \frac{1+t^4+t^5+t^6+t^9+t^{10}+t^{11}+t^{15}}{(1-t^2)^3(1-t^6)}.$$

$$\dim_{\mathbb{C}} M_k(\Gamma_0^*(4)) - \dim_{\mathbb{C}} S_k(\Gamma_0^*(4)) = 7 + 5 \dim S_k(\mathrm{SL}(2,\mathbb{Z}) \cap \left[\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right]).$$




$$\sum_{k\geq 0} \dim_{\mathbb{C}} S_k(\Gamma_0(4)^*) t^k = \frac{t^5(1+3t+t^3+t^4+2t^5+t^6-t^7-t^9+t^{10})}{(1-t^2)^3(1-t^6)}$$

# Other general types

$$s_k(p, \mathrm{VIa/b}) := s_k(p, \mathrm{VIa}) = s_k^{(\mathbf{G})}(p, \mathrm{VIb})$$

$$s_k(p, \mathrm{IIIa} + \mathrm{VIa/b}) := s_k(p, \mathrm{IIIa}) + s_k(p, \mathrm{VIa/b})$$

$$s_k(p, \mathrm{VIIIa/b}) := s_k(p, \mathrm{VIIIa}) = s_k^{(\mathbf{G})}(p, \mathrm{VIIIb})$$

$$s_k(p, \mathrm{VII} + \mathrm{VIIIa/b}) := s_k(p, \mathrm{VII}) + s_k(p, \mathrm{VIIIa/b})$$

$$\begin{bmatrix} \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\mathrm{Sp}(4,\mathbb{Z})) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\mathrm{K}(2)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma_0'(2)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma_0(2)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(B(2)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\mathrm{K}(4)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma_0(4)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma_0^*(4)) - s_k^{(\mathbf{G})}(\mathrm{Va}^*) - 3 s_k(\mathsf{sc}(16)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma(2)) - s_k^{(\mathbf{G})}(\mathrm{Va}^*) - 9 s_k(\mathsf{sc}(16)) \\ -s_k(\mathsf{sc}(16)) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 4 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 12 & 5 & 8 & 2 & 2 & 4 & 3 & 1 & 1 & 0 & 0 \\ 15 & 8 & 10 & 4 & 5 & 5 & 4 & 7 & 4 & 0 & 0 \\ 45 & 30 & 30 & 16 & 21 & 15 & 10 & 15 & 10 & 0 & 0 \\ 11 & 7 & 5 & 2 & 5 & 2 & 1 & 3 & 2 & -1 & 0 \\ 8 & 5 & 3 & 1 & 3 & 0 & 0 & 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} s_k(\mathrm{I}) \\ s_k(\mathrm{IIa}) \\ s_k(\mathrm{IIIa{+}VIa/b}) \\ s_k(\mathrm{IVa}) \\ s_k(\mathrm{Va}) \\ s_k(\mathrm{VII{+}VIIIa/b}) \\ s_k(\mathrm{IXa}) \\ s_k(\mathrm{X}) \\ s_k(\mathrm{XIa}) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\Gamma_0'(4)) \\ \dim_{\mathbb{C}} S_k^{(\mathbf{G})}(\mathrm{M}(4)) \end{bmatrix}$$

# Final results (dimensions of cusp form spaces)

$$\sum_{k\geq 0}\dim_{\mathbb{C}} S_k(\Gamma_0'(4))t^k =$$

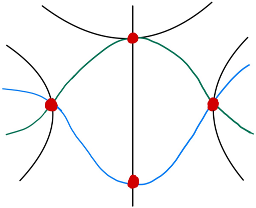
$$=\frac{t^7\left(1+2t^2+5t^4+4t^6+5t^8+4t^{10}+2t^{12}+t^{16}\right)+t^8\left(3+9t^2+13t^4+6t^6-3t^{10}-2t^{12}-2t^{14}\right)}{\left(1-t^4\right)^2\left(1-t^6\right)^2}$$

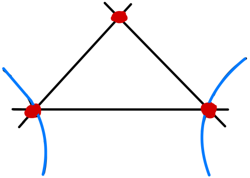
$$\sum_{k\geq 0}\dim_{\mathbb{C}} S_k(\mathrm{M}(4))t^k =$$

$$=\frac{t^7\left(1+3t^4-t^6+4t^8+3t^{12}+2t^{16}-t^{18}+t^{20}\right)+t^8\left(2+2t^2+4t^4+5t^8+2t^{12}-2t^{14}+t^{16}-2t^{18}\right)}{\left(1-t^2\right)\left(1-t^4\right)\left(1-t^6\right)\left(1-t^{12}\right)}$$

# Final results (dimensions of modular form spaces )

$$\mathrm{codim}_k(\Gamma_0'(4)) = 4 + 3\dim_{\mathbb{C}} S_k(\mathrm{SL}(2,\mathbb{Z})) + \dim_{\mathbb{C}} S_k(\Gamma_0^{(1)}(2)) + 2\dim_{\mathbb{C}} S_k(\Gamma_0^{(1)}(4))$$
$$\mathrm{codim}_k(\mathrm{M}(4)) = 3 + 2\dim_{\mathbb{C}} S_k(\mathrm{SL}(2,\mathbb{Z})) + 3\dim_{\mathbb{C}} S_k(\Gamma_0^{(1)}(2))$$





$$\sum_{k\geq 0} \dim_{\mathbb{C}} M_k(\Gamma_0'(4)) t^k =$$
$$= \frac{1 + 2t^4 + 4t^6 + t^7 + 5t^8 + 2t^9 + 4t^{10} + 5t^{11} + 5t^{12} + 4t^{13} + 2t^{14} + 5t^{15} + t^{16} + 4t^{17} + 2t^{19} + t^{23}}{(1-t^4)^2(1-t^6)^2}$$
$$\sum_{k\geq 0} \dim_{\mathbb{C}} M_k(\mathrm{M}(4)) t^k =$$
$$= \frac{1 - t^2 + 2t^4 + t^7 + 3t^8 + 3t^{11} + 4t^{12} - t^{13} - t^{14} + 4t^{15} + 3t^{16} + 3t^{19} + t^{20} + 2t^{23} - t^{25} + t^{27}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{12})}$$

# $s_k(p, \Omega)$ for Iwahori-spherical representations at $p$

At $\pi_p$ consider Iwahori-Spherical representations of type $\Omega$. These are types I-VI with unramified characters.

$$\begin{bmatrix} \dim_{\mathbb{C}} S_k(\mathrm{Sp}(4,\mathbb{Z})) \\ \dim_{\mathbb{C}} S_k(\mathrm{K}(p)) \\ \dim_{\mathbb{C}} S_k(\Gamma_0'(p)) \\ \dim_{\mathbb{C}} S_k(\Gamma_0(p)) \\ \dim_{\mathbb{C}} S_k(B(p)) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 4 & 1 & 3 & 2 & 0 & 0 & 1 & 1 & 1 & 0 \\ 4 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 8 & 4 & 4 & 4 & 1 & 2 & 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} s_k(p,\mathrm{I}) \\ s_k(p,\mathrm{IIa}) \\ s_k(p,\mathrm{IIb}) \\ s_k(p,\mathrm{IIIa}) \\ s_k(p,\mathrm{IVa}) \\ s_k(p,\mathrm{Va}) \\ s_k(p,\mathrm{Vb}) \\ s_k(p,\mathrm{VIa}) \\ s_k(p,\mathrm{VIb}) \\ s_k(p,\mathrm{VIc}) \end{bmatrix}$$

# $s_k(p,\Omega)$ for Iwahori-spherical representations at $p$

**Theorem (R., Schmidt, Yi, 2020)**

- *For $k \geq 3$ and $p \geq 5$, we $s_k(p,\Omega)$ compute them explicitly. Specifically,*

$$s_k(p,\Omega) = a_\Omega \frac{(k-2)(k-1)(2k-3)}{2^7 3^3 5} + O_p(k^2),$$

*where $a_\Omega$ are given as follows.*

| $\Omega$ | I | IIa | IIIa + VIa/b | IVa | Va |
|---|---|---|---|---|---|
| $a_\Omega$ | $1$ | $p^2-1$ | $\frac{(p-1)(p^2+p+2)}{2}$ | $(p-1)(p^3-1)$ | $\frac{p(p-1)^2}{2}$ |

- *For $k \geq 3$ and $p = 2, 3$, we give the corresponding generating functions.*
- $s_1(p,\Omega) = 0$ *for any $p$, and* $s_2(p,\Omega) = 0$ *for $p \in \{2,3\}$.*

# Final remarks

- How far can we extend this explicit method of counting automorphic representations?

- Can we compute new dimension formulas of Siegel cusp form spaces using this method?

Thank You!