SIEGEL MODULAR FORMS & HIGHER ALGEBRAIC CYCLES

Motivic action conjectures (Venkatesh, Harris, Prasanna, ...):

$$\left\{\begin{matrix}\text{automorphic}\\ \text{reps of } G(\mathbb{A})\end{matrix}\right\} \xrightarrow{\text{Langlands}} \{\varphi : G_{\mathbb{Q}} \longrightarrow {}^L G\} \longleftarrow \{\text{"motives"}\}$$

an automorphic representation appears in multiple coh. degrees <-- --> higher algebraic cycles (motivic cohomology)

<u>Goal.</u> Two examples:

1. Bianchi modular forms: $G = GL_{2,K}$, $K = \mathbb{Q}(\sqrt{-d})$ [Prasanna-Venkatesh] (more general!)
   $\overset{??}{\longleftrightarrow}$ E/K elliptic curve (motive)
2. Siegel modular forms: $G = GSp_{4,\mathbb{Q}}$ [H.-Prasanna]
   $\overset{??}{\longleftrightarrow}$ $A/\mathbb{Q}$ abelian surface (motive) [related work: Oh]

(Hilbert modular forms: my thesis (H. '23).)

<u>1. Bianchi modular forms</u>: $G = GL_{2,K}$, $K = \mathbb{Q}(\sqrt{-d})$

$\mathfrak{h}_3 := \mathbb{C} \times \mathbb{R}_{>0} \ni (z,t)$, $z = x+iy$

$\circlearrowleft$ <u>hyperbolic 3-space</u> with $ds^2 = \dfrac{dx^2+dy^2+dt^2}{t^2}$

$GL_2(\mathbb{C}) \supseteq GL_2(\mathcal{O}_K) \supseteq \Gamma$

<u>Bianchi modular form of weight 2</u>: $f : \mathfrak{h}_3 \longrightarrow \mathbb{C}^3$, $f = (f_1, f_2, f_3)$
satisfying transf. property under Γ: $f|_2 \gamma = f$;
equivalently:

(1) $\omega_f^1 := f_1 \dfrac{dz}{t} - f_2 \dfrac{dt}{t} + f_3 \dfrac{d\bar{z}}{t} \in H^1(\Gamma \backslash \mathfrak{h}_3, \mathbb{C})$

(2) $\omega_f^2 := f_1 \dfrac{dt \wedge dz}{t^2} - f_2 \dfrac{dz \wedge d\bar{z}}{t^2} + f_3 \dfrac{dt \wedge d\bar{z}}{t^2} \in H^2(\Gamma \backslash \mathfrak{h}_3, \mathbb{C})$.

Actually: (1) $\Leftrightarrow$ (2).

Note: $\Gamma \backslash \mathbb{h}_3$: real threefold $\Rightarrow$ no algebraic structure! But...

$$\left\{ \begin{array}{c} \text{Bianchi mod.} \\ \text{forms of wt } 2 \end{array} \right\} \xrightarrow{\text{Langlands}} \left\{ G_K \to GL_2(\mathbb{Q}_\ell) \right\} \longleftarrow \left\{ \begin{array}{c} \text{elliptic} \\ \text{curves}/K \end{array} \right\}$$

Harris-Soudry-Taylor: transfer to GSp_4

Caraiani-Newton: E/K modular

Assume: $f \longleftrightarrow E/K$.

Goal: understand rationality of ω_f^1, ω_f^2 in terms of E

$$\begin{array}{ccc} \omega_f^1 & \longmapsto & \omega_f^2 \\ \cap & & \cap \\ H^1(\Gamma\backslash\mathbb{h}_3, \mathbb{C})_f & \longrightarrow & H^2(\Gamma\backslash\mathbb{h}_3, \mathbb{C})_f \\ \cup & & \cup \\ \text{(Cremona-Whitley)}\ H^1(X_{\mathbb{C}}, \mathbb{Q})_f & \longrightarrow & H^2(X_{\mathbb{C}}, \mathbb{Q})_f\ \text{(Prasanna-Venkatesh)} \\ \omega\,? & & \omega\,? \end{array}$$

$\sigma : K \hookrightarrow \mathbb{C}$

$\bar\sigma : K \hookrightarrow \mathbb{C}$

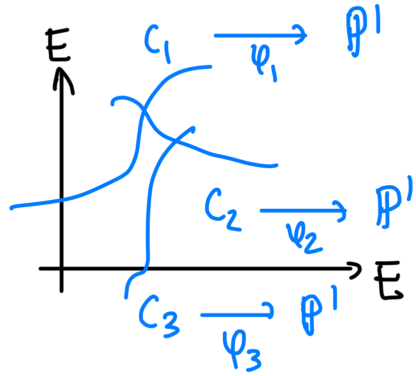
$$\frac{\omega_f^1}{\int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \overline{\omega^\sigma}} \xmapsto{\text{"rational action of } \alpha\text{"}} \frac{\omega_f^2}{\sum_i \int_{C_i(\mathbb{C})} \log|\varphi_i| \cdot \delta}$$

δ = explicit $(1,1)$-form $= \omega^\sigma \wedge \overline{\omega^{\bar\sigma}} + \overline{\omega^\sigma} \wedge \omega^{\bar\sigma}$

$\approx L(\pi \otimes \chi_+, \tfrac{1}{2}) \cdot L(\pi \otimes \chi_-, \tfrac{1}{2})$

χ_+ / χ_- even/odd char.

$\alpha = \{(C_i, \varphi_i)\}$



- $C_i \subseteq E \times E$ curves
- $\varphi_i : C_i \longrightarrow \mathbb{P}^1$
- $\sum \mathrm{div}(\varphi_i) = 0$

motivic cohomology class

$$\boxed{\begin{array}{l} \underline{\text{Conj}}\ \text{(Prasanna-Venkatesh).}\ \Gamma = \text{arithmetic gp}\quad \pi = \text{temp. rep.} \\ \qquad \Lambda^* H^1_{\mathcal{M}/\mathbb{Z}}(\mathrm{Ad}\, M(\pi), \mathbb{Q}(1)) \circlearrowright H^*(\Gamma, \mathbb{Q})_\pi . \end{array}}$$

Note: $\ell_0 = 0$ $\Rightarrow$ conj. is vacuous

2. Siegel modular forms: $G = GSp_{4,\mathbb{Q}}$

($\ell_0 = 0 \Rightarrow$ Prasanna–Venkatesh is vacuous)

$$\mathcal{H}_2 := \left\{ \tau := \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in M^{\mathrm{sym}}_{2\times 2}(\mathbb{C}) \ : \ \mathrm{Im}\,\tau \text{ pos. def.} \right\}$$ Siegel upper half space

$$\circlearrowleft$$

$$GSp_4(\mathbb{R}) \ni \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad \tau \longmapsto (A\tau + B)(C\tau + D)^{-1}$$

Def. holomorphic Siegel modular form of wt 2, level $\Gamma \leq GSp_4(\mathbb{R})$

$f : \mathcal{H}_2 \longrightarrow \mathbb{C}$ holomorphic, $f|_2 \gamma = f \quad \forall \gamma \in \Gamma$,

where $f|_2\gamma(\tau) = \det(C\tau + D)^{-2} \cdot f(\gamma\tau)$.

In this case, $\Gamma \backslash \mathcal{H}_2^* = X(\mathbb{C})$, $\quad X =$ Siegel modular threefold$/\mathbb{Q}$.

However, f is not cohomological, i.e. $H^*(X(\mathbb{C}), \mathbb{C})_f = 0$.

[also with local systems].

But...

$$\left\{ \begin{array}{c} \text{Siegel modular} \\ \text{forms wt 2} \end{array} \right\} \longrightarrow \left\{ G_{\mathbb{Q}} \to GSp_4(\mathbb{Q}_\ell) \right\} \longleftarrow \left\{ \begin{array}{c} \text{abelian} \\ \text{surfaces}/\mathbb{Q} \end{array} \right\}$$

Taylor: using congruences

Boxer–Calegari–Gee–Pilloni: $A/\mathbb{Q}$ potentially modular

Subtlety: $\Pi = \{\pi^{\mathrm{hol}}, \pi^{\mathrm{gen}}\} \longmapsto$ one Galois rep.

A-packet

and have $f \in \pi^{\mathrm{hol}}$ holomorphic SMF,

$f^W \in \pi^{\mathrm{gen}}$ generic SMF, Whittaker-normalized.

Assume: $\Pi = \{ \underset{\substack{\cup \\ f}}{\pi^{\mathrm{hol}}}, \underset{\substack{\cup \\ f^W}}{\pi^{\mathrm{gen}}} \} \longleftrightarrow A/\mathbb{Q}$

Where can we find f & f^W in cohomology?

Fact. $\exists\ \mathcal{E}/X$ line bundle s.t.

(1) $[f] \in H^0(X_{\mathbb{Q}}, \mathcal{E})_\pi$

(2) $[f^W] \in H^1(X_{\mathbb{C}}, \mathcal{E})_\pi$

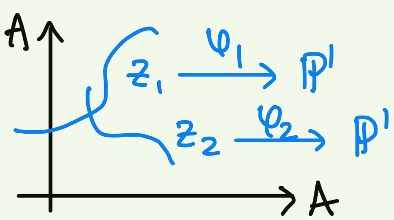
(3) these are all the contr. of π.

Conj (H. Prasanna).

There exists $\alpha = \{(C_i, \varphi_i)\}$ s.t.
$$H^0(X_{\mathbb{Q}}, \mathcal{E})_f \longrightarrow H^1(X_{\mathbb{C}}, \mathcal{E})_f$$
$$[f] \longmapsto \frac{[f^W]}{\sum_i \int_{C_i(\mathbb{C})} \log|\varphi_i|\, \delta}$$

- $C_i \subseteq A \times A$ irred 3-folds
- $\varphi_i : C_i \longrightarrow \mathbb{P}^1$
- $\sum \mathrm{div}(\varphi_i) = 0$

δ = explicit $(3,3)$-class on $A \times A$.



Thm 1 (H.-Prasanna). Beilinson's conj. on special values of L-functions $\Longrightarrow$ our conj. (up to $\mathbb{Q}^{ab,\times}$)

Pf sketch. Goal understand rationality of $[f^W]$.

$$[f^W] \in H^1(X_{\mathbb{C}}, \mathcal{E})_f \xleftrightarrow{\text{Serre duality}} H^2(X_{\mathbb{C}}, \mathcal{E}^\vee \otimes \Omega^3_X) \ni [f^{W,\vee}]$$

$?\ \cup \qquad\qquad \cup\ ?$ (Loeffler–Pilloni–Skinner–Zerbes)

$$\frac{[f^W]}{???} \in H^1(X_{\mathbb{Q}}, \mathcal{E})_f \xleftrightarrow{\text{Serre duality}} H^2(X_{\mathbb{Q}}, \mathcal{E}^\vee \otimes \Omega^3_X) \ni \frac{[f^{W,\vee}]}{L(\frac{1}{2}, \pi \otimes \chi_+) \cdot L(\frac{1}{2}, \pi \otimes \chi_-)}$$

χ_+ / χ_- even/odd Dirichlet chars

Then use:

- Chen–Ichino $\langle [f^W], [f^{W,\vee}] \rangle_{SD} = L(1, \mathrm{Ad}, \pi)$
- Radziwiłł–Yang: $\exists\ \chi_+, \chi_-$ s.t. $L(\frac{1}{2}, \pi \otimes \chi_+) \cdot L(\frac{1}{2}, \pi \otimes \chi_-) \neq 0$.

- explicit calculation using Beilinson's conj.

3. Special cases.

3.1. E = elliptic curve $/F=\mathbb{Q}(\sqrt{d})$ $\longleftrightarrow$ Hilbert mod. form f_F

$\rightsquigarrow$ $A := R_{F/\mathbb{Q}} E = E \times E^{\sigma} / \mathbb{Q}$ $\longleftrightarrow$ Yoshida lift $f := Y(f_F)$

Thm 2 (H.-Prasanna). Conjecture is true in this case.

[Caveat: the statement itself is conditional on $\Pi \leftrightarrow A$,
but we have a statement just about the Siegel mod. form.]

3.2. E = elliptic curve $/K=\mathbb{Q}(\sqrt{-d})$ $\longleftrightarrow$ Bianchi mod. form f_K

$\rightsquigarrow$ $A := R_{K/\mathbb{Q}} E = E \times E^{\sigma} / \mathbb{Q}$ $\longleftrightarrow$ Yoshida lift $f := Y(f_F)$

Thm 3 (H.-Prasanna). Our conj. implies the conj. of Prasanna-Venkatesh.

More precisely:

$$\begin{array}{ccc}
H^1(X_K,\mathbb{Q})_{f_K} & \xrightarrow{\theta_1} & H^0(X_{\mathbb{Q}},\mathcal{E})_f \\
\text{Prasanna-Venkatesh} \downarrow & \circlearrowleft & \downarrow \text{H.-Prasanna} \\
H^2(X_K,\mathbb{Q})_{f_K} & \xrightarrow[\theta_2]{} & H^1(X_{\mathbb{Q}},\mathcal{E})_f
\end{array}$$