Resonances of Schottky surfaces

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joint with

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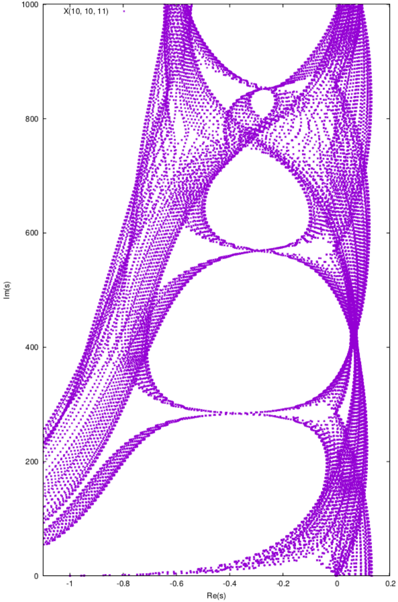
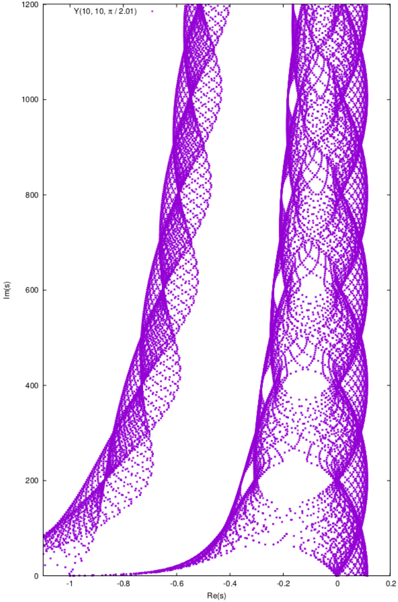
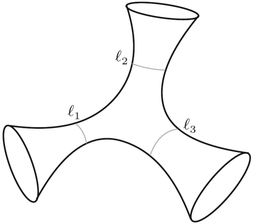
International Seminar on Automorphic Forms, 2023

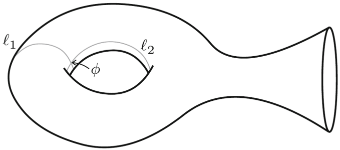

X(10, 10, 11)
1000
800
600
400
200
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
Im(s)
Re(s)


Y(10, 10, π / 2.01)
1200
1000
800
600
400
200
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
Im(s)
Re(s)

Schottky surfaces

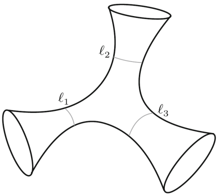
- convex co-compact hyperbolic surfaces of infinite area

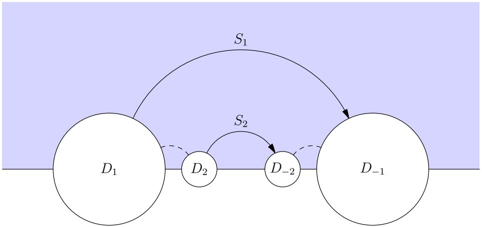




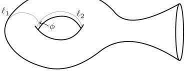
- hyperbolic surfaces $\Gamma\backslash\mathbb{H}$, where Γ is a Fuchsian group, non-cofinite, torsion-free, without parabolic elements

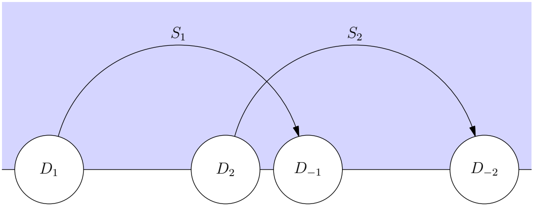
Construction of a three-funnel surface





Construction of a funneled torus surface





Resonances

- pole of resolvent

$$R(s) = \big(\Delta_{\mathrm{hyp}} - s(1-s)\big)^{-1}$$

Resonances

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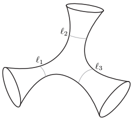
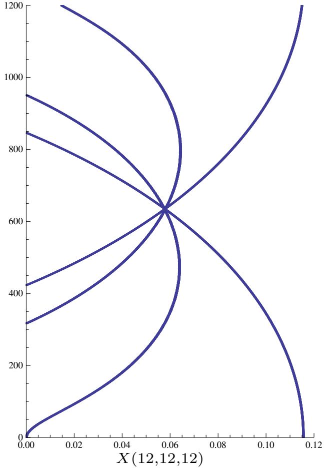
- first resonance is δ, where $\delta = \dim \Lambda(\Gamma)$ (Hausdorff dimension of limit set of Γ)
- no resonances with $\operatorname{Re} s > \delta$
- Weyl-type law (Guillopé–Zworski):

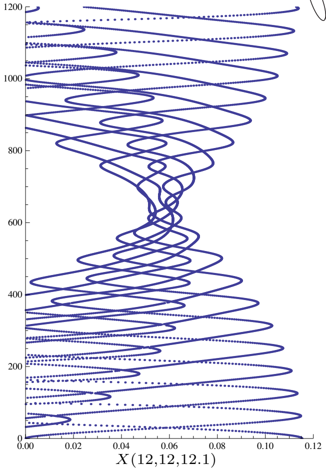
$$c_1 r^2 \leq \#\{s \text{ resonance, } |s| < r\} \leq c_2 r^2$$

- spectral gap, resonance-free zones: Naud, Bourgain–Gamburd–Sarnak (generalization of Selberg's $\frac{3}{16}$ theorem), Oh–Winter, ...
- progress on Zaremba's conjecture: Bourgain–Kontorovich

Borthwick (2013; periodic orbit expansion)







$$R(s) = \left(\Delta_{\mathrm{hyp}} - s(1-s)\right)^{-1}$$

Approach

Let $X = \Gamma\backslash\mathbb{H}$ be a Schottky surface.

Selberg zeta function:

$$Z(s) = \prod_{\substack{\gamma \\ \text{prim. per.} \\ \text{geod on } X}} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right)$$

- converges for $\operatorname{Re} s > \delta = \dim \Lambda(\Gamma)$ (Hausdorff dimension of limit set of Γ)
- has holomorphic extension to all of $\mathbb{C}$

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- has holomorphic extension to all of $\mathbb{C}$
- **Patterson–Perry** (2001):

$$Z(s) = 0 \qquad \text{"} \Leftrightarrow \text{"} \qquad \begin{cases} s \text{ is a resonance, or} \\ s \in -\mathbb{N}_0. \end{cases}$$

We have a transfer operator $\mathcal{L}_s$, deriving from a discretization of the geodesic flow on X, such that

$$Z(s) = \det\left(1 - \mathcal{L}_s\right).$$

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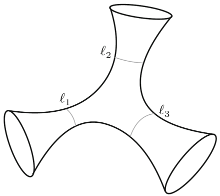
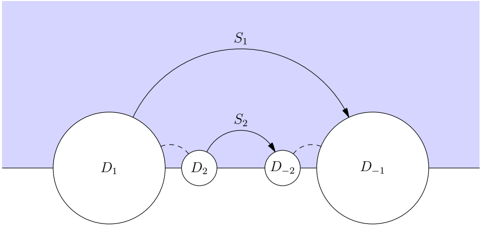




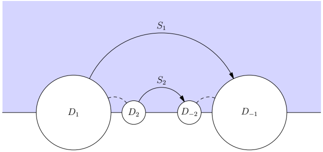
Transfer operator

For Schotty group with 2 generators,

$$\mathcal{L}_s = \begin{pmatrix} \tau_s(S_{-2}) & \tau_s(S_{-1}) & \tau_s(S_1) & 0 \\ \tau_s(S_{-2}) & \tau_s(S_{-1}) & 0 & \tau_s(S_2) \\ \tau_s(S_{-2}) & 0 & \tau_s(S_1) & \tau_s(S_2) \\ 0 & \tau_s(S_{-1}) & \tau_s(S_1) & \tau_s(S_2) \end{pmatrix},$$

acting on function vectors of the form

$$\begin{pmatrix} f_{-2}\colon D_{-2} \to \mathbb{C} \\ f_{-1}\colon D_{-1} \to \mathbb{C} \\ f_1\colon D_1 \to \mathbb{C} \\ f_2\colon D_2 \to \mathbb{C} \end{pmatrix}.$$



Here, for $g \in \Gamma$ and $\varphi\colon D \to \mathbb{C}$,

$$\tau_s(g)\varphi(z) := \left((g^{-1})'(z)\right)^s \varphi(g^{-1}.z)\,.$$

Approach

With a view towards

$$Z(s) = \det\left(1 - \mathcal{L}_s\right)$$

deduce the power series expansion (goes back to Ruelle)

$$\begin{aligned}
Z(s,z) &= \prod_{\substack{\gamma \\ \text{prim. per.} \\ \text{geod on } X}} \prod_{k=0}^{\infty} \left(1 - z^{w(\gamma)} e^{-(s+k)\ell(\gamma)}\right) \\
&= \det\left(1 - z\mathcal{L}_s\right) \\
&= \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr} \mathcal{L}_s^n\right) \\
&= 1 + \sum_{n=1}^{\infty} d_n(s) z^n \,.
\end{aligned}$$

Approach

$$Z(s, z) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr} \mathcal{L}_s^n\right) = 1 + \sum_{n=1}^{\infty} d_n(s) z^n$$

- Each $\operatorname{Tr} \mathcal{L}_s^n$ is a sum of $c(X)^n$ 'elementary traces'.
- Recursion formulas to calculate the $d_n(s)$ from the traces of $\mathcal{L}_s^n$, $n \in \mathbb{N}$.

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For $N \in \mathbb{N}$ set

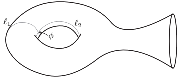
$$Z_N(s) := 1 + \sum_{n=1}^{N} d_n(s).$$

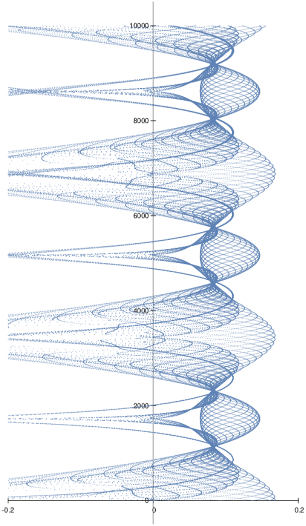
- Find zeros of Z_N and use them as approximation for zeros (=resonances) of Z.
- Approximation error: $|d_n(s)| \le C^n e^{-c_1 n^2 - c_2 n \operatorname{Re} s + c_3 n |\operatorname{Im} s|}$

Borthwick–Weich (2014):

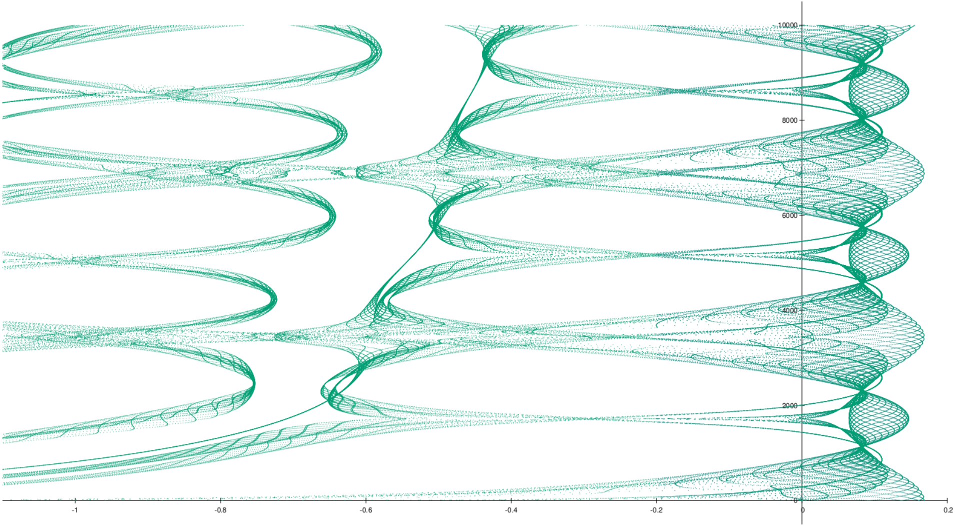
Reduce required expansion order / number of summed terms using Venkov–Zograf factorization

Based on Cvitanović–Eckhardt





Bandtlow–P.–Schick–Weiße:



Our approach

$$Z(s) = \det(1 - \mathcal{L}_s)$$

- Avoid truncation, avoid recursion formulas.

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- Transfer operator has an integral kernel:

$$\big(\mathcal{L}_s f\big)(w) = \int_\Omega K_s(z,w) f(z)\,dz,$$

where $K_s(z,w)$ is a finite size matrix, each entry is either 0 or of the form

$$C^2 \cdot \frac{(cz+d)^{-2s}}{\left(C_1^2 - \left(\frac{az+b}{cz+d} - m\right)(\overline{w} - m)\right)^2}.$$

Here, Ω is a finite union of open sets in $\mathbb{C}$, and $C, C_1, m, a, b, c, d \in \mathbb{R}$ are known (and depend on the open set).

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- Use **Nyström method** to approximate Fredholm determinant: approximate integral kernel with **Gauss–Chebyshev quadrature rule**.

$Z(s) = \det(1 - \mathcal{L}_s)$ gets approximated by

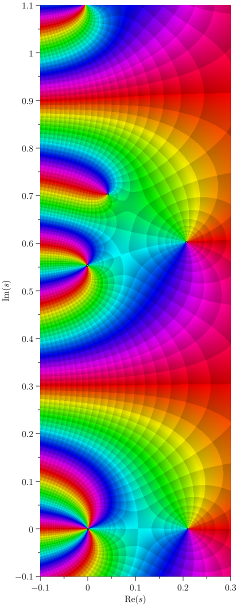
$$D(s) = \det\left(\delta_{i,j} - w_j K_s(x_i, x_j)\right)_{i,j=1}^{N},$$

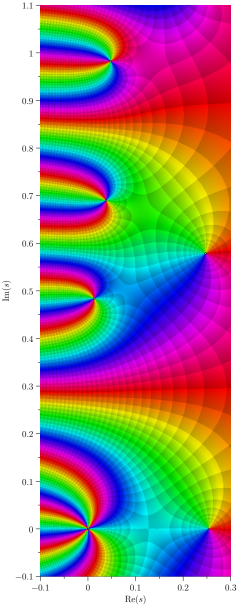
where

- the first N Chebyshev polynomials are used,
- x_k are the collocation points, all in $\mathbb{R}$
  $\left(x_j = \cos\left(\frac{2j-1}{2N}\pi\right)\right.$ for interval $[-1,1]$),
- $w_j = \frac{1}{N}$ are the weights.

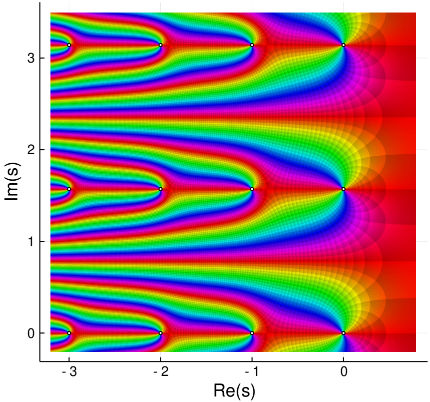
$$K_s(z,w) = \left(C^2 \cdot \frac{(cz+d)^{-2s}}{(C_1^2 - (\frac{az+b}{cz+d} - m)(\overline{w} - m))^2}\right)$$

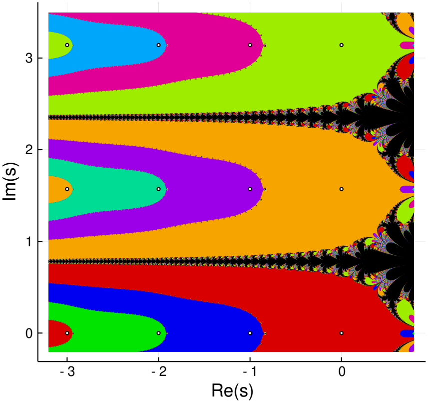
Phase plots, zeros by Newton's method





Phase plots and Newton fractal





$$s_{n+1} = s_n - D(s_n)/D'(s_n)$$

Related

- method has long history (**Bornemann** 2010 survey), Lagrange–Chebyshev approximation, Galerkin method
- recent application in physics, e.g., Dugave–Göhmann–Kozlowski
- recent application to transfer operator numerics, e.g., Nielsen, Wormell, etc.

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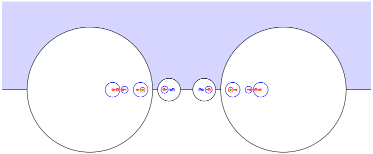
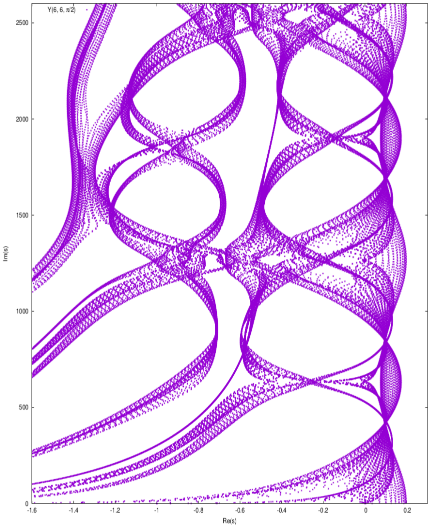
Our contribution:

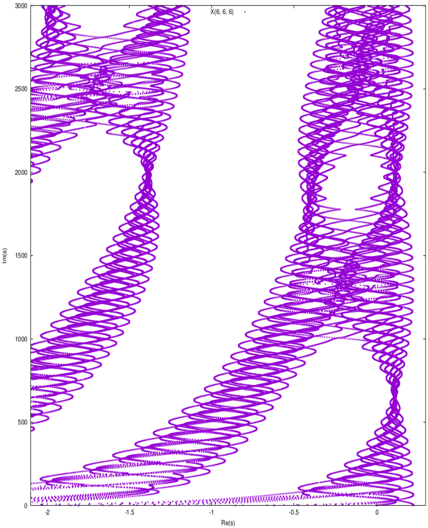
- good spaces
- adaptive domain refinement
- error estimates

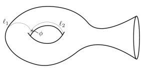


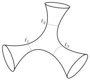
Error estimates

- Approximation error (pure math approximation):
  exponential convergence (**BPSW**; functional analysis methods)
- Numerical error:
  good convergence (no error propagation; **Bornemann** (2010))
- Zeros by Newton's method:
  good convergence

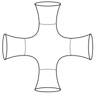

Y(6, 6, π/2)
Im(s)
Re(s)

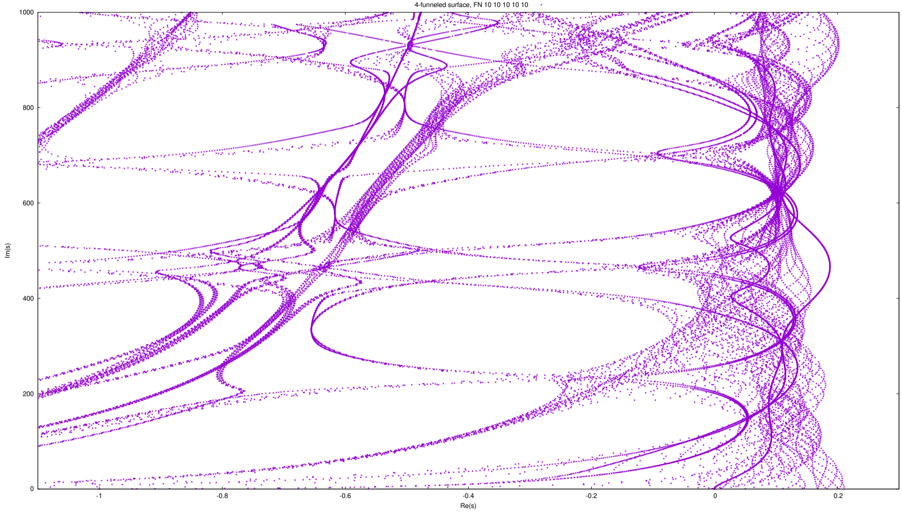

X(6, 6, 6)
Im(s)
Re(s)


ℓ_1
ℓ_2
ϕ

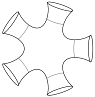

ℓ_1
ℓ_2
ℓ_3

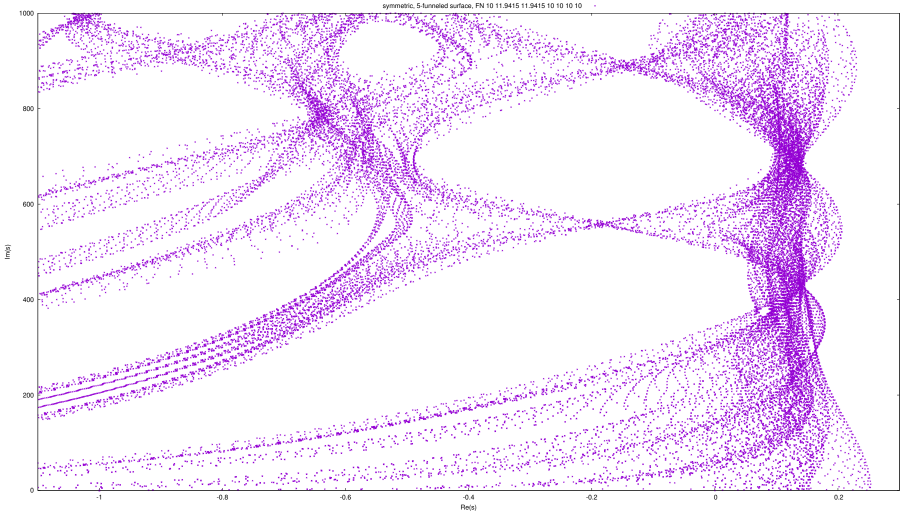
More complicated surfaces: 4 funnels





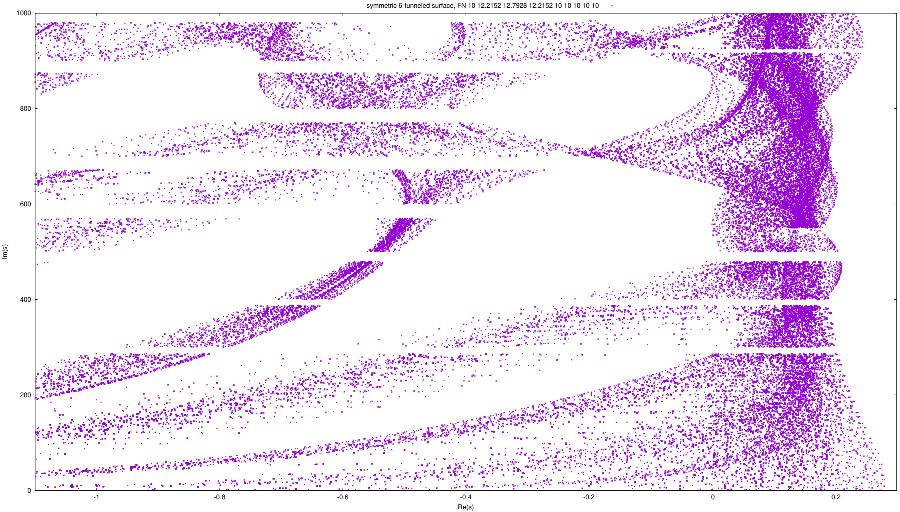
5 funnels



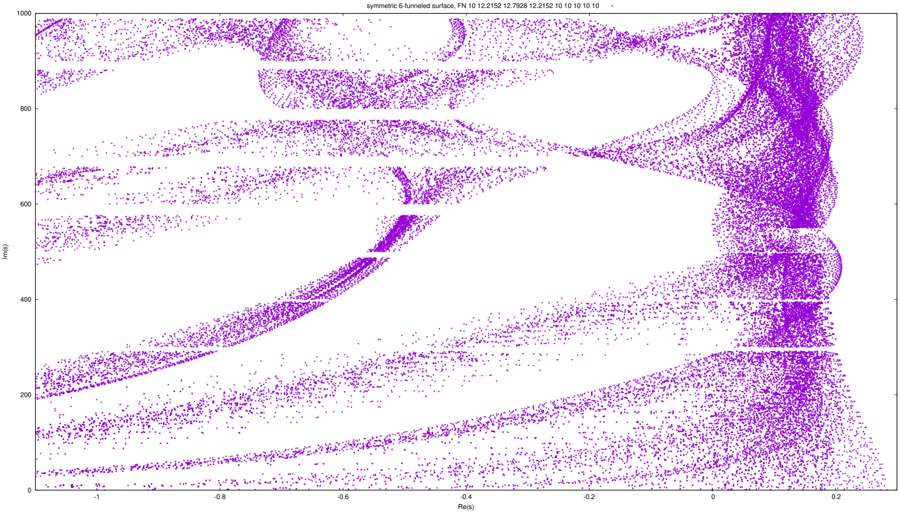


symmetric, 5-funneled surface, FN 10 11.9415 11.9415 10 10 10 10
1000
800
600
400
200
0
Im(s)
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
Re(s)

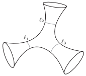
6 funnels (still in progress)



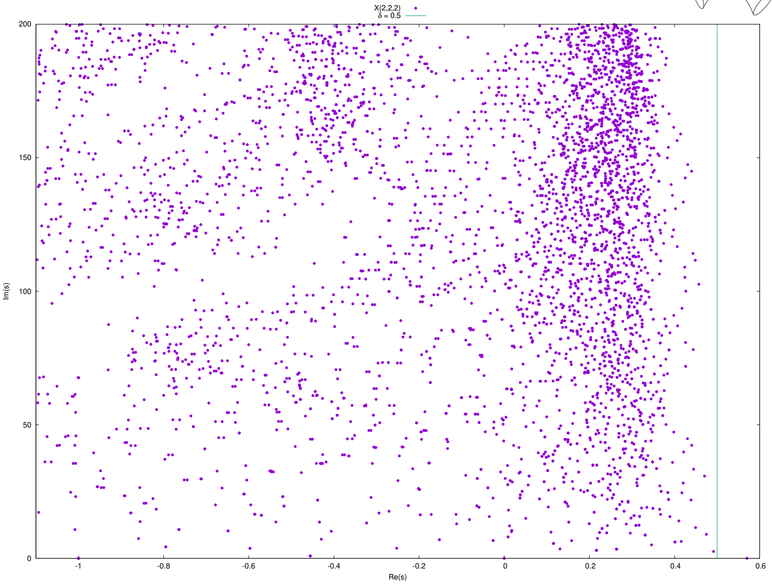
6 funnels (4 hours later)



Large δ

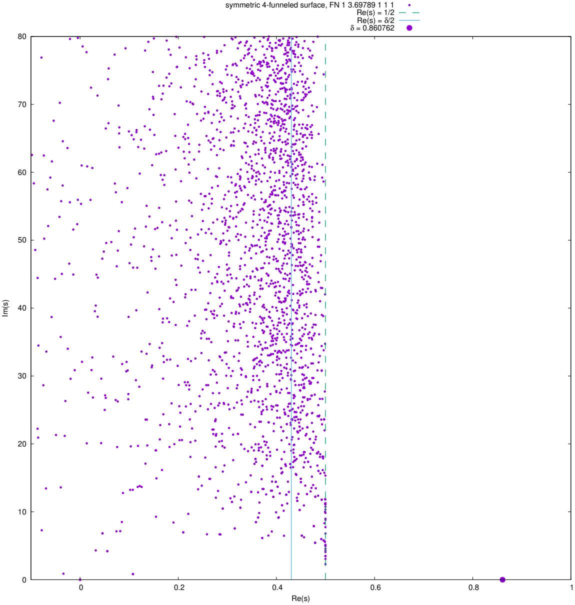


ℓ_2
ℓ_1
ℓ_3

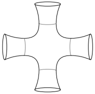


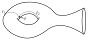
X(2,2,2)
δ = 0.5
Im(s)
Re(s)
0
50
100
150
200
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

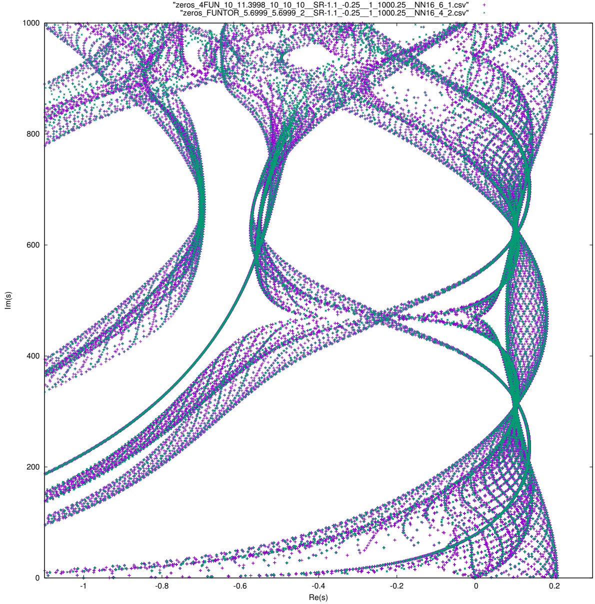
Large δ


symmetric 4-funneled surface, FN 1 3.69789 1 1 1
Re(s) = 1/2
Re(s) = δ/2
δ = 0.860762
80
70
60
50
40
30
20
10
0
Im(s)
0
0.2
0.4
0.6
0.8
1
Re(s)

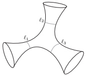
Relations

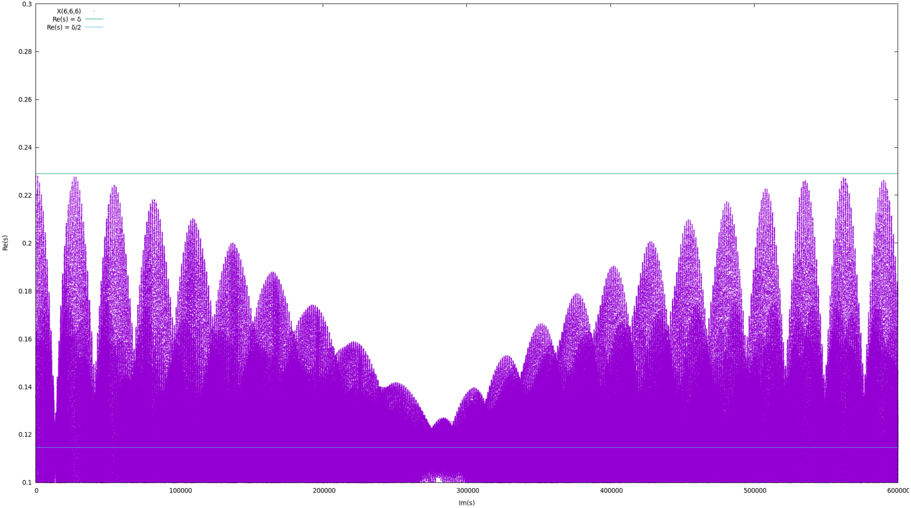





"zeros_4FUN_10_11.3998_10_10_10__SR-1.1_-0.25__1_1000.25__NN16_6_1.csv"
"zeros_FUNTOR_5.6999_5.6999_2__SR-1.1_-0.25__1_1000.25__NN16_4_2.csv"
1000
800
600
400
200
0
Im(s)
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
Re(s)

Large Im s


ℓ_2
ℓ_1
ℓ_3


X(6,6,6)
Re(s) = δ
Re(s) = δ/2
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
Re(s)
0
100000
200000
300000
400000
500000
600000
Im(s)