Continued Fractions and Hardy Sums

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D MATH

Overview

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- Hardy sums are the integer-valued analogs of Dedekind sums. The latter arises from the transformation of $\log \eta(z)$ under the modular group $\mathrm{SL}_2(\mathbb{Z})$, where $\eta(z)$ is the Dedekind η-function

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- Hardy sums can be used in the theory of Gaussian sums (see Sczech 1995) and in the representation of integers as sums of squares.

Continued Fractions

A continued fraction is defined to be

$$[a_0; a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}},$$

where $a_0 \in \mathbb{Z}$ and $a_1, ..., a_n \geq 1$.

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Similarly, we may define the negative continued fraction by

$$[\![c_0; c_1, c_2, ..., c_n]\!] = c_0 - \frac{1}{c_1 - \frac{1}{c_2 - \frac{1}{\dots - \frac{1}{c_n}}}},$$

where $c_0 \in \mathbb{Z}$ and $c_1, ..., c_n \geq 2$.

Continued Fractions

The continued fraction expansions (CFE) of a rational number

$$\frac{a}{c} = [a_0; a_1, ..., a_n], \ c > 0 \text{ and } (a, c) = 1,$$

is connected to words in the modular group $\mathrm{SL}_2(\mathbb{Z})$ under a set of generators.

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The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ or by T and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Continued Fractions

The word $T^{a_0} V^{a_1} \cdots V^{a_n} \in \mathrm{SL}_2(\mathbb{Z})$ gives rise to the CFE of $\frac{a}{c}$ by acting on the cusp ∞ by linear fractional transformations:

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We have $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = T^{a_0}V^{a_1}\cdots V^{a_n} \in \mathrm{SL}_2(\mathbb{Z})$ with $0 \leq d < c$ and $0 \leq a < b$.

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Similarly, the negative CFE of $\frac{a}{c}$ gives rise to a word in $\mathrm{SL}_2(\mathbb{Z})$ in terms of T and S.

Continued Fractions

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- Pick $a_0 \in \mathbb{Z}$ such that $r_0 = r - a_0 \in [0, 1)$. If $r_0 = 0$, we are done.

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Dedekind Sums

Let $s(d, c)$ be the classical Dedekind sum

$$s(d,c) = \frac{1}{4c} \sum_{k=1}^{|c|-1} \cot \frac{\pi k}{c} \cot \frac{\pi k d}{c}, \ (d,c) = 1.$$

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It exhibits many properties, e.g.

- the reciprocity law

$$s(d, c) + s(c, d) = \frac{c^2 + d^2 + 1}{cd} - \frac{1}{4}, \ (d, c) = 1.$$

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Theorem 1 (Hickerson 1977)

The set $\{(d/c,\ s(d, c)) : (d, c) = 1\}$ *is dense in* $\mathbb{R} \times \mathbb{R}$.

Dedekind Sums

To prove the density of Dedekind sums, Hickerson provided an expression for $s(d, c)$ in terms of the CFE of the rational

$$\frac{d}{c} = [0; a_1, ..., a_{2n+1}] = \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_{2n+1}}}} \neq 1, \ n > 0,$$

namely

$$s(d, c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a+d}{c} - \sum_{k=1}^{2n+1}(-1)^k a_k\right), \ 0 < a < c, \ ad \equiv 1 \pmod{c}.$$

Dedekind Sums and the η-Function

Dedekind sums first arose in the context of the transformation of $\log \eta(z)$ under the modular group, where $\eta(z) = e^{\frac{\pi i}{12} z} \prod_{n=1}^{\infty}(1 - e^{2\pi i n z})$.

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For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$, the transformation of $\log \eta(z)$ is:

$$\log \eta(A.z) - \log \eta(z) = \frac{1}{2} \log\left(\frac{cz + d}{i}\right) + \frac{\pi i}{12}\left(\frac{a + d}{c} - 12\, s(d, c)\right).$$

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Reciprocity law for Dedekind sums: Compare the transformation of $\log \eta(z)$ under the matrices AS and A.

Dedekind Sums and the η-Function

Differentiating $\log \eta(A.z) - \log \eta(z)$ with respect to z, yields

$$\begin{aligned}
\partial_z(\log \eta(A.z) - \log \eta(z)) &= \partial_z \left(\frac{1}{2} \log \left(\frac{cz+d}{i} \right) + \frac{\pi i}{12} \left(\frac{a+d}{c} - 12\, s(d,c) \right) \right) \\
&= \frac{1}{2} \frac{c}{cz+d} \\
&= \frac{\pi i}{12} \left(E_2(z)|_2 A - E_2(z) \right),
\end{aligned}$$

where

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}, \ \sigma_1(n) = \sum_{d|n} d$$

is the Eisenstein series of weight 2.

Dedekind Sums and the η-Function

Hence, the function

$$\nu(A) = \log \eta(A.z) - \log \eta(z) - \frac{\pi i}{12} \int_z^{A.z} E_2(w)dw$$

is constant in z and a group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to (\mathbb{C}, +)$:

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For $A, B \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}\nu(AB) &= \log \eta(AB.z) - \log \eta(z) - \frac{\pi i}{12} \int_z^{AB.z} E_2(w) dw \\ &= \eta(A.(B.z)) - \log \eta(B.z) + \log \eta(B.z) - \log \eta(z) \\ &- \frac{\pi i}{12} \int_{B.z}^{AB.z} E_2(w) dw - \frac{\pi i}{12} \int_z^{B.z} E_2(w) dw \\ &= \nu(A) + \nu(B).\end{aligned}$$

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is constant in z and a group homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to (\mathbb{C}, +)$.

But as $\mathrm{SL}_2(\mathbb{Z})$ is generated by the torsion-elements S (with $S^4 = I$) and $U = ST = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ (with $U^6 = I$), every group-homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to (\mathbb{C}, +)$ is trivial.

Dedekind Sums and the Eta-Function

We have hence shown that $\nu \equiv 0$, which leads to the well-known formula

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$$\log \eta(A.z) - \log \eta(z) = \frac{\pi i}{12} \int_z^{A.z} E_2(w) dw.$$

This formula can be used to prove Hickerson's representation of $s(d, c)$ by continued fractions.

Idea of the Proof:

Continued Fractions and Dedekind Sums

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$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = V^{a_{2n+1}}T^{a_{2n}}\cdots T^{a_2}V^{a_1}\in \mathrm{SL}_2(\mathbb{Z}).$$

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Write $A = A_1 A_2 \cdots A_N$ with $A_k \in \{T, V\}$ for $k = 1, ..., N$ (i.e. all coefficients of the continued fraction expansion are positive). Write $A_1 \cdots A_k = \begin{pmatrix} * & * \\ c_k & d_k \end{pmatrix}$.

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Set

$$t = |\{k : A_k = T\}| = a_2 + a_4 + ... + a_{2n} \textit{ and}$$
$$v = |\{k : A_k = V\}| = a_1 + a_3 + ... + a_{2n+1}.$$

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With this notation, Hickerson's formula is

$$s(d,c)=-\frac{1}{4}+\frac{1}{12}\left(\frac{a+d}{c}-(t-v)\right).$$

Continued Fractions and Dedekind Sums

Idea of the Proof:

Specialize to $z = \rho^2 = e^{\frac{2\pi i}{3}}$ in the formula

$$\log \eta(A.z) - \log \eta(z) = \frac{\pi i}{12} \int_z^{A.z} E_2(w) dw.$$

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\begin{aligned}
\int_{\rho^2}^{A.\rho^2} E_2(w)dw &= \sum_{k=1}^{N} \int_{A_1\cdots A_{k-1}.\rho^2}^{A_1\cdots A_k.\rho^2} E_2(w)dw \\
&= \sum_{k=1}^{N} \int_{\rho^2}^{A_k.\rho^2} E_2(w)|_2(A_1\cdots A_{k-1})dw \\
&= \sum_{k=1}^{N} \left(\int_{\rho^2}^{\rho^2+1} E_2(w)dw + \frac{6}{\pi i} \int_{\rho^2}^{\rho^2+1} \frac{c_{k-1}}{c_{k-1}w + d_{k-1}} dw \right).
\end{aligned}
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We may now split the integral $\int_{\rho^2}^{A.\rho^2} E_2(w)dw$ as follows:

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$$= \underbrace{t+v}_{=N} + \frac{6}{\pi i} \sum_{k=1}^{N} \left(\log(c_{k-1}\rho^2 + (c_{k-1} + d_{k-1})) - \log(c_{k-1}\rho^2 + d_{k-1}) \right),$$

as $\int_{\rho^2}^{\rho} E_2(w)dw = 1$.

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Hence

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The claim follows by comparing the integral of E_2 with

$$\log\eta(A.\rho^2)-\log\eta(\rho^2)=\frac{1}{2}\log\left(\frac{c\rho^2+d}{i}\right)+\frac{\pi i}{12}\left(\frac{a+d}{c}-12\,s(d,c)\right).$$

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$$\theta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 z} = \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right) \left(1 - e^{\pi i (2n-1) z}\right)^2 .$$

Hardy Sums

Analogously to the logarithm of the η-function, we may look at the transformation behavior of the logarithm of θ-functions:

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right) \left(1 + e^{\pi i (2n-1) z}\right)^2 \text{ and}$$

$$\theta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 z} = \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right) \left(1 - e^{\pi i (2n-1) z}\right)^2 .$$

The functions $\theta(z)$ and $\theta_4(z)$ exhibit modular transformations for the subgroups

$$\Gamma_\theta = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d,\ b \equiv c \pmod 2 \right\}, \text{ resp.}$$
$$\Gamma^0(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : b \equiv 0 \pmod 2 \right\}$$

instead of the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Hardy Sums

The correction factors of the modular transformations of $\log\theta(z)$ and $\log\theta_4(z)$ are called Hardy sums:

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- For $A = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\theta$ with $c > 0$:

$$\log \theta(A.z) - \log \theta(z) = \frac{1}{2} \log \left(\frac{cz + d}{i} \right) + \frac{\pi i}{4} S(d, c).$$

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- For $B = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma^0(2)$ with $c > 0$:

$$\log \theta_4(B.z) - \log \theta_4(z) = \frac{1}{2} \log \left(\frac{cz + d}{i} \right) - \frac{\pi i}{4} S_4(d, c).$$

Hardy Sums

The Hardy sums take the following explicit forms:

- $$S(d, c) = 8\ s(d, 2c) + 8\ s(2d, c) - 20\ s(d, c)$$

  and

  $$S_4(d, c) = -4\ s(d, c) + 8\ s(d, 2c)$$

  (Rademacher 1967; see also Sitaramachandrarao 1987).

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- $$S(d,c) = \sum_{k=1}^{|c|-1} (-1)^{k+1+\left\lfloor \frac{kd}{c} \right\rfloor}$$

  and

  $$S_4(d,c) = \sum_{k=1}^{|c|-1} (-1)^{\left\lfloor \frac{kd}{c} \right\rfloor}$$

  (Berndt 1978).

Hardy Sums

The Hardy sum $S(d,c)$ satisfies a reciprocity law:

$$S(d,c) + S(c,d) = \operatorname{sign}(cd), \qquad (d,c) = 1,\ c+d \text{ odd},$$

as $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma_\theta$ (see Sczech 1995).

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Since $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \notin \Gamma^0(2)$, there is no reciprocity law for $S_4(d,c)$ (some results resembling reciprocity laws for $S_4(d,c)$ can be found in Meyer 1997, 2000).

Continued Fractions for Γ_θ and $\Gamma^0(2)$

Any matrix in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\theta = \langle T^2, S \rangle$ has a representation as a word in T^2, S:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = T^{2c_0} S T^{2c_1} \cdots T^{2c_n} S$$

with $c_0 \in \mathbb{Z}$ and $c_1, ..., c_n \in \mathbb{Z} - \{0\}$.

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Similarly, we get continued fractions associated to the subgroup $\Gamma^0(2) = \langle T^2, V \rangle$.

Continued Fractions for Γ_θ and $\Gamma^0(2)$

Algorithm for the Γ_θ-CFE:

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- ...

Continued Fractions and Hardy Sums

Theorem 2 (L. 2020)

Let d be an integer and $c > 0$ such that $(d, c) = 1$.

1. *For $c + d$ odd, let $\frac{d}{c} = [\![2c_0; 2c_1, ..., 2c_n]\!] = 2c_0 - \cfrac{1}{2c_1 - \cfrac{1}{\cdots - \cfrac{1}{2c_n}}}$ with $c_0 \in \mathbb{Z}$ and $c_1, ..., c_n$ non-zero integers be the negative continued fraction of $\frac{d}{c}$. The Hardy sum $S(d, c)$ takes the form*
$$S(d,c) = -\sum_{k=1}^{n} \operatorname{sign}(c_k).$$
2. *For d odd, let $\frac{d}{c} = [2a_0; a_1, 2a_2, a_3, ..., 2a_{n-1}, a_n]$ with $a_0 \in \mathbb{Z}$ and $a_1, ..., a_n$ non-zero integers such that $|a_k| > 1$ for $k = 1, ..., n-2$ an odd integer. The Hardy sum $S_4(d, c)$ takes the form*
$$S_4(d,c) = (a_1 + a_3 + ... + a_n) + \sum_{k=1}^{n} (-1)^k \operatorname{sign}(a_k).$$

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By the same argument as we used before to prove Hickerson's formula, we can prove the representation of Hardy sums by the Γ_θ-CFE resp. $\Gamma^0(2)$-CFE.

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Recall that

$$\nu(A) = \log \eta(A.z) - \log \eta(z) - \frac{\pi i}{12} \int_z^{A.z} E_2(w) dw$$

is a group homomorphism $\nu : \mathrm{SL}_2(\mathbb{Z}) \to (\mathbb{C}, +)$.

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The function

$$\phi(A) = \log \theta(A.z) - \log \theta(z) - \frac{\pi i}{12} \int_z^{A.z} E_2(w)dw$$

is also a group homomorphism $\phi : \Gamma_\theta \to (\mathbb{C}, +)$.

Idea of the Proof:

The group homomorphisms $\Gamma_\theta \to (\mathbb{C}, +)$ form a vector-space, which is one-dimensional and spanned by the function

$$A \mapsto \int_z^{A.z} R(w)dw,$$

where

$$\begin{aligned} R(z) &= 2\, E_2(2z) - E_2\left(\frac{z+1}{2}\right) \\ &= 1 + 24 \sum_{n=1}^{\infty} (-1)^n \sigma_1^{\mathrm{odd}}(n) e^{\pi i n z},\ \sigma_1^{\mathrm{odd}}(n) = \sum_{\substack{d \mid n, \\ d \text{ odd}}} d, \end{aligned}$$

is the unique modular form for Γ_θ of weight 2.

Idea of the Proof:

The evaluation $\phi(T^2) = -\frac{\pi i}{6}$ gives the constant of proportionality of ϕ to $T^2 \mapsto \int_z^{T^2.z} R(w)dw = 2$.

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Instead of

$$\log \eta(A.z) - \log \eta(z) = \frac{\pi i}{12} \int_z^{A.z} E_2(w)dw,\ A \in \mathrm{SL}_2(\mathbb{Z}),$$

we have

$$\log \theta(A.z) - \log \theta(z) = \frac{\pi i}{12} \int_z^{A.z} (E_2(w) - R(w))dw,\ A \in \Gamma_\theta,$$

where $R(z)$ is the unique modular form of weight 2 for Γ_θ

Continued Fractions and Hardy Sums

Idea of the Proof:

With the formula

$$\log\theta(A.z) - \log\theta(z) = \frac{\pi i}{12}\int_z^{A.z}(E_2(w) - R(w))dw,\ A \in \Gamma_\theta,$$

we can prove the formula for $S(d,c)$ in terms of coefficients of the Γ_θ-CFE similarly to how we proved Hickerson's formula for $s(d,c)$.

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we can prove the formula for $S(d,c)$ in terms of coefficients of the Γ_θ-CFE similarly to how we proved Hickerson's formula for $s(d,c)$.

Since we also allow negative coefficients in the Γ_θ-CFE, this is technically more involved.

Density of Hardy Sums

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Hickerson used his formula to prove that $\{(d/c,\ s(d,c)) : (d,c) = 1, c > 0\}$ is dense in $\mathbb{R} \times \mathbb{R}$. What is known for Hardy sums?

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Theorem 3 (Meyer, 1997)

The sets $\{(d/c,\ S(d,c)) : c > 0,\ (d,c) = 1,\ c + d \text{ odd}\}$ *and* $\{(d/c,\ S_4(d,c)) : c > 0,\ (d,c) = 1,\ d \text{ odd}\}$ *are dense in* $\mathbb{R} \times \mathbb{Z}$.

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The set $\{(d/c,\ S(d,c)) : c > 0, (d,c) = 1, c + d \text{ odd}\}$ being dense in $\mathbb{R} \times \mathbb{Z}$ means that

- for all $x \in \mathbb{R}$, $N \in \mathbb{Z}$ and $\varepsilon > 0$ there are coprime integers c, d with $c > 0$ and $c + d$ being odd such that

$$\left| x - \frac{d}{c} \right| < \varepsilon \text{ and } S(d,c) = N.$$

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Numerical example: Take $x = \sqrt{13} - 3 \approx 0.60555127...$, $N = -8$, and $\varepsilon = \frac{1}{1000}$.

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We want to find a $\frac{d}{c}$ with $(d, c) = 1$, $c > 0$ and $c + d$ odd such that

$$\left|(\sqrt{13} - 3) - \frac{d}{c}\right| < \frac{1}{1000} \text{ and } S(d, c) = -8.$$

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We will find such a rational number by approximating x with a Γ_θ-type continued fraction $\frac{d}{c} = [\![2c_0; 2c_1, ..., 2c_n]\!]$ with $c_0 \in \mathbb{Z}$ and $c_1, ..., c_n \in \mathbb{Z} - \{0\}$ and the formula

$$S(d, c) = -\sum_{k=1}^{n} \operatorname{sign}(c_k).$$

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- Approximate x by a Γ_θ-type continued fraction $\frac{\varepsilon}{2}$-closely, e.g.

$$|(\sqrt{13} - 3) - [\![0; -2, -2, 2, 2, 2, 2, 2, 2, 2]\!]| < \frac{1}{2000}.$$

Here $\frac{23}{38} = [\![0; -2, -2, 2, 2, 2, 2, 2, 2, 2]\!]$.

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$$|(\sqrt{13} - 3) - [\![0; -2, -2, 2, 2, 2, 2, 2, 2, 2]\!]| < \frac{1}{2000}.$$

  Here $\frac{23}{38} = [\![0; -2, -2, 2, 2, 2, 2, 2, 2, 2]\!]$.
- Compute $S(23, 38)$ to see how far away we are from $N = -8$, i.e. $S(23, 38) = -(2 \cdot \mathrm{sign}(-2) + 7 \cdot \mathrm{sign}(2)) = -5$.

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We want to find a $\frac{d}{c}$ with $(d, c) = 1$, $c > 0$ and $c + d$ odd such that

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- The fraction

$$\frac{d}{c} = [\![0; -2, -2, 2, 2, 2, 2, 2, 2, 2, 2y_1, 2y_2, 2y_3]\!] \text{ with } y_1, y_2, y_3 > 0$$

would yield $S(d, c) = -8$ as

$$S(d, c) = \underbrace{-(2 \cdot \operatorname{sign}(-2) + 7 \cdot \operatorname{sign}(2))}_{=-5} - \operatorname{sign}(y_1) - \operatorname{sign}(y_2) - \operatorname{sign}(y_3).$$

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Since

$$\lim_{y \to \pm\infty} [\![0; 2c_1, ..., 2c_n, y]\!] = -\frac{1}{2c_1 - \frac{1}{\dots - \frac{1}{2c_n - \lim_y \frac{1}{y}}}} = [\![0; 2c_1, ..., 2c_n]\!],$$

we may choose $y_1, y_2, y_3 > 0$ big enough so that $\frac{d}{c}$ is $\frac{\varepsilon}{2}$-close to $\frac{23}{38}$.

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we may choose $y_1, y_2, y_3 > 0$ big enough so that $\frac{d}{c}$ is $\frac{\varepsilon}{2}$-close to $\frac{23}{38}$.

Changing the size of y_1, y_2, y_3 does not change the value of the Hardy sum!

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$$\left|(\sqrt{13} - 3) - \frac{d}{c}\right| < \frac{1}{1000} \text{ and } S(d, c) = -8.$$

In our case, it suffices to pick $y_1 = 2$ and $y_2 = y_3 = 1$, so that

$$\frac{170}{281} = [\![0; -2, -2, 2, 2, 2, 2, 2, 2, 2, 4, 2, 2]\!]$$

differs from $x = \sqrt{13} - 3$ by less than $\frac{1}{1000}$ and $S(170, 281) = -8$.

Density of Hardy Sums

For the Hardy sum $S_4(d, c)$ a similar argument yields that

$$\{(d/c,\ S_4(d, c)) : c > 0, (d, c) = 1, d \text{ odd}\}$$

is dense in $\mathbb{R} \times \mathbb{Z}$.

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In fact, one can even show more, namely that

$$\{(d/c, S(d,c) + S_4(d,c), S_4(d,c)) : c > 0 \text{ even}, (d,c) = 1, d \text{ odd}\}$$

is dense in $\mathbb{R} \times 2\mathbb{Z} \times (2\mathbb{Z} + 1)$.

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Theorem 4 (L. 2020)

The set $\{(d/c,\ S(d,c) + S_4(d,c)) : c > 0 \text{ even},\ (d,c) = 1,\ d \text{ odd}\}$ *is dense in* $\mathbb{R} \times 2\mathbb{Z}$.

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