"Extended-Cycle Integrals of the j-function for Badly Approximable Numbers"

Yuya Murakami

Math Inst. Tohoku Univ., JAPAN

June 1st, 2021

Abstract

- Kaneko (2009) $\cdots$ $\mathrm{val}(w) \in \mathbb{C}$ for a real quadratic number w as the cycle integral of $j(z)$.
- Today $\cdots$ $\mathrm{val}(x) \in \mathbb{C}$ for a badly approximable number x as the limit along a geodsic.

Contents

§1 val(w) & Motivation

$\mathbb{H} := \{ z = x + y\sqrt{-1} \in \mathbb{C} \mid y > 0 \}$.

$j : \mathbb{H} \to \mathbb{C}$: the j-function

$\rightsquigarrow SL_2(\mathbb{Z})$-invariant

<u>Fix</u> a real quadratic number w.

<u>Def</u> • $SL_2(\mathbb{Z})_w$: the stabilizer of w

as $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R} \setminus \mathbb{Q}$.

• $\exists! \, \gamma_w = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})_w$ s.t.

$$\begin{cases} \triangleright \; SL_2(\mathbb{Z})_w = \{ \pm \gamma_w^n \mid n \in \mathbb{Z} \}, \\ \triangleright \; \varepsilon_w := cw + d > 1. \end{cases}$$

• w' : the non-trivial

Galois conjugate of w.

Def (Kaneko, 2009)

$\forall z_0 \in \mathbb{H}$.

- $\widetilde{\mathrm{val}}(w) := \int_{z_0}^{\gamma_w z_0} j(z) \left(\frac{1}{z-w'} - \frac{1}{z-w} \right) dz,$

- $\tilde{I}(w) := \int_{z_0}^{\gamma_w z_0} \left(\frac{1}{z-w'} - \frac{1}{z-w} \right) dz$

$$= 2 \log \varepsilon_w,$$

- $\mathrm{val}(w) := \dfrac{\widetilde{\mathrm{val}}(w)}{\tilde{I}(w)}.$

Zagier (2002)

$j(z)$ for $z \in \mathbb{H}$: imaginary quadratic

$\rightsquigarrow$ a modular form.

Duke-Imamoglu-Tóth (2011)

$\widetilde{\mathrm{val}}(w) \rightsquigarrow$ a mock modular form.

Q $\exists ?\, H$: a topological space s.t.

- $H \supset \{\text{real quadratic numbers}\}$,
- $SL_2(\mathbb{Z}) \curvearrowright H$,
- $\mathrm{val} : H \longrightarrow \mathbb{C}$: continuous, $SL_2(\mathbb{Z})$-invariant

$$\left(\begin{array}{l}\text{A real quadratic analog of} \\ j : \mathbb{H} \longrightarrow \mathbb{C}.\end{array}\right)$$

Today We can choose

$H = \{\text{badly approximable numbers}\}$

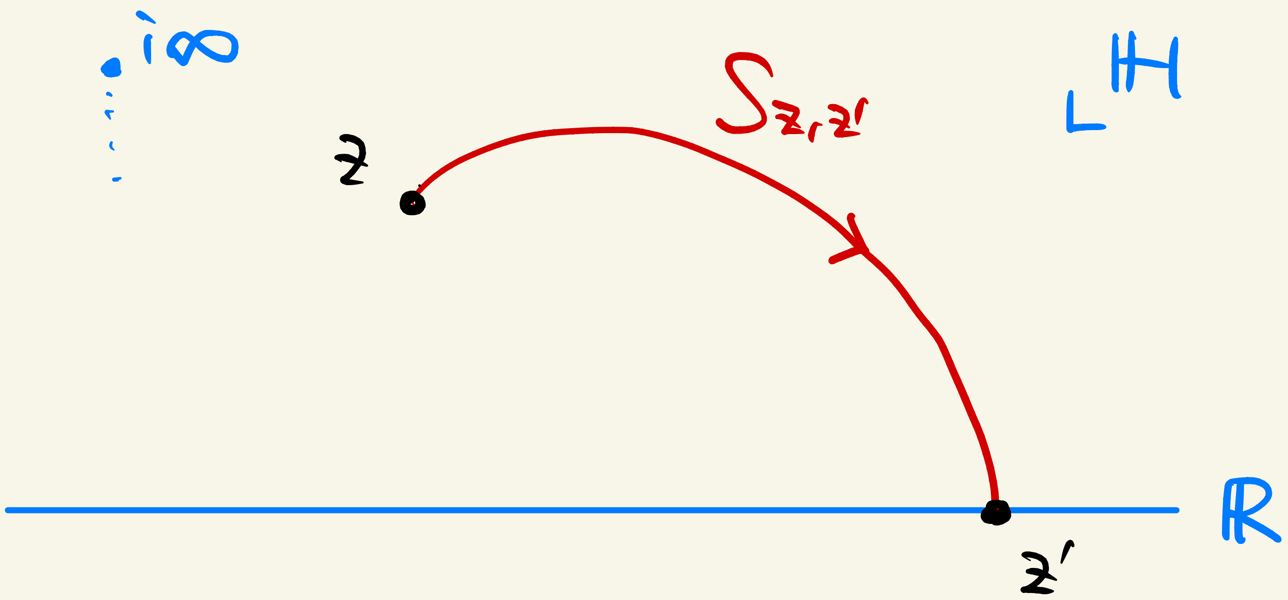
$\subset \mathbb{R} \setminus \mathbb{Q}$.

⚠ Any suitable topology of H is still unknown.

§2 val(x) as the limit along a geodesic

<u>Def</u> For $z, z' \in \mathbb{H} \cup \mathbb{R} \cup \{i\infty\}$, define

$S_{z,z'}$: the geodesic from z to z'.



<u>Def</u> • $x \in \mathbb{R} \setminus \mathbb{Q}$: badly approximable

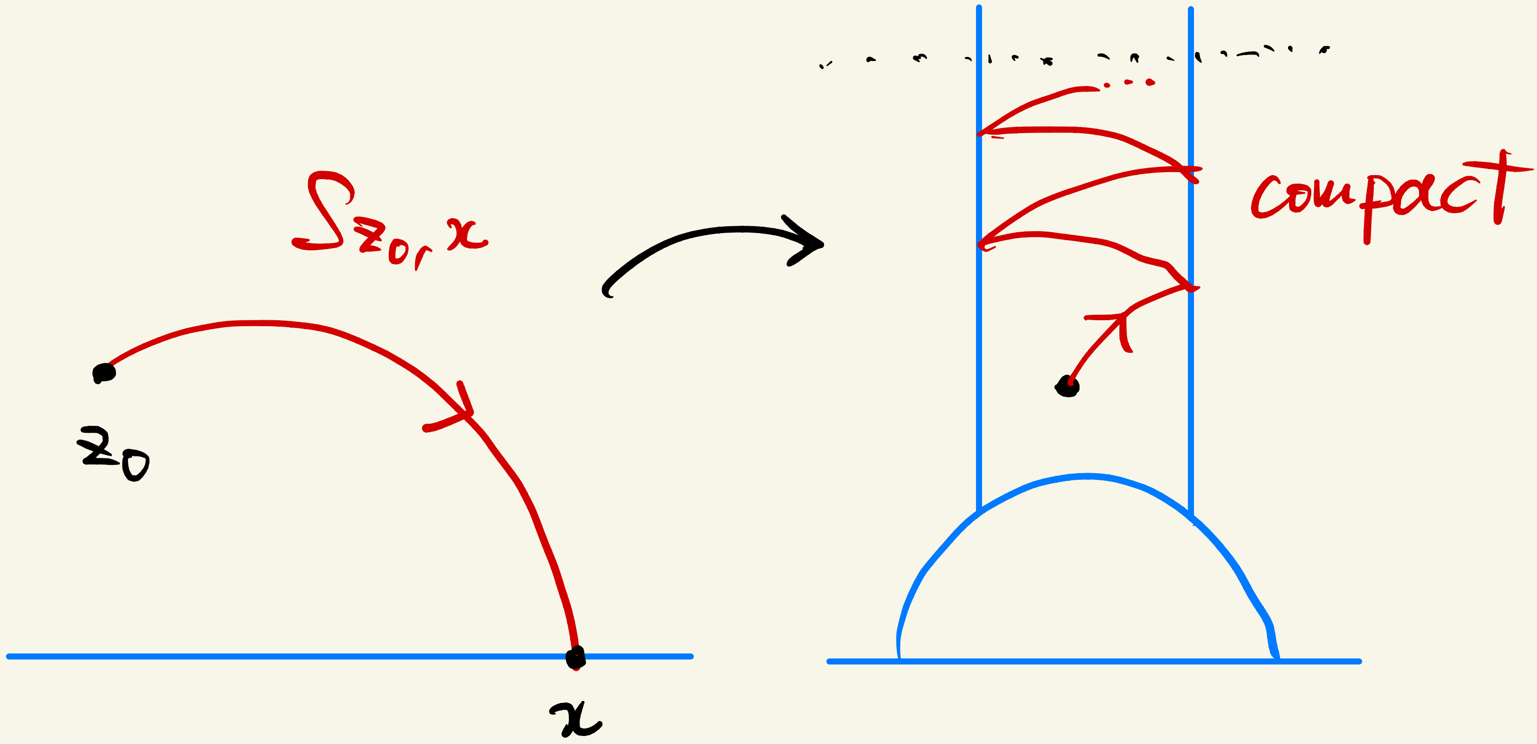
$\overset{\text{def}}{\Longleftrightarrow}$ $\forall z_0 \in \mathbb{H}$,

$\overline{\pi(S_{z_0,x})} \subset SL_2(\mathbb{Z})\backslash\mathbb{H}$: compact

where $\pi : \mathbb{H} \to SL_2(\mathbb{Z})\backslash\mathbb{H}$.

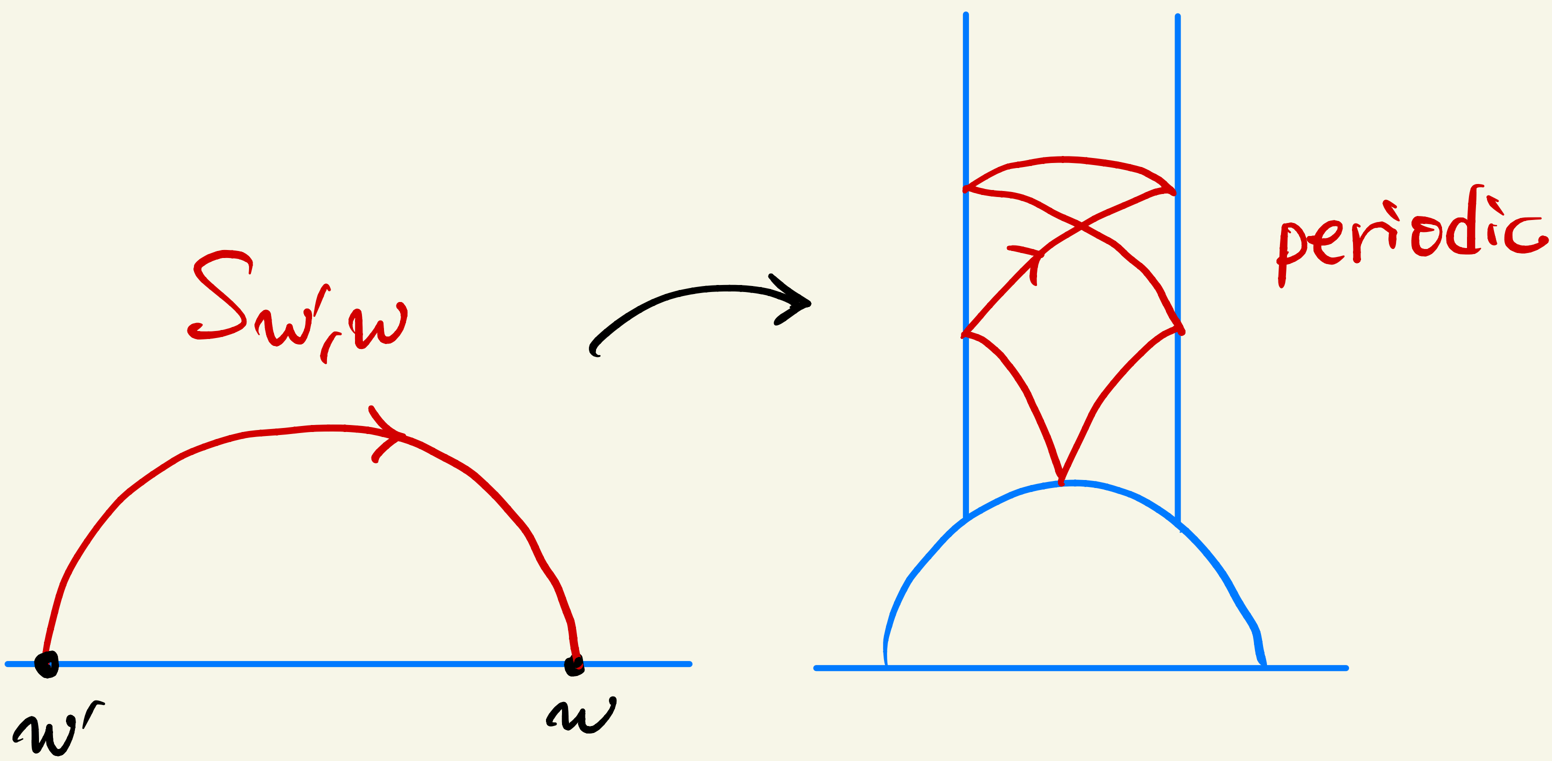
• $\mathbb{R}_{ba} := \{\text{badly approximable numbers}\}$

$\subset \mathbb{R} \setminus \mathbb{Q}$.



Rem · $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}_{ba}$.

· $\{$real quadratic numbers$\} \subset \mathbb{R}_{ba}$.



<u>Thm</u> (M.)

$\forall x \in \mathbb{R}_{ba}$, $\forall z_0 \in \mathbb{H}$,

$$\mathrm{val}(x) := \lim_{\substack{z \in S_{z_0,x} \\ z \to x}} \frac{1}{\mathrm{length}(S_{z_0,x})} \int_{S_{z_0,x}} j(z) \frac{\sqrt{dx^2+dy^2}}{y}$$

converges,

is independent of z_0,

is $SL_2(\mathbb{Z})$-invariant, &

coincides Kaneko's $\mathrm{val}(w)$

if $x = w$: real quadratic.

§3 $\widehat{\mathrm{val}}(x)$, ε_x as the limits along continued fraction expansion

<u>Def</u> (The continued fraction expansion)

$$[k_1, k_2, k_3, \dots] := k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cfrac{1}{\ddots}}}$$

for $k_1 \in \mathbb{Z}$, $k_2, k_3, \dots \in \mathbb{Z}_{>0}$.

<u>Lem</u> For $x = [k_1, k_2, \dots] \in \mathbb{R} \setminus \mathbb{Q}$,

$$x \in \mathbb{R}_{ba} \iff \#\{k_1, k_2, \dots\} < \infty .$$

<u>Rem</u> x : real quadratic

$$\iff x = [k_1, \dots, k_q, \overline{l_1, \dots, l_r}]$$

$$:= [k_1, \dots, k_q, l_1, \dots, l_r, l_1, \dots, l_r, l_1, \dots, l_r, \dots].$$

Thm (M.)

$\forall x = [k_1, \dots] \in \mathbb{R}_{ba}$, $\forall z_0 \in \mathbb{H}$,

$$\hat{\mathrm{val}}(x) := \lim_{n\to\infty} \frac{1}{2n} \int_{z_0}^{\gamma_{(k_1,\dots,k_{2n})} z_0} j(z) \frac{-dz}{z-x},$$

$$\hat{1}(x) := \lim_{n\to\infty} \frac{1}{2n} \int_{z_0}^{\gamma_{(k_1,\dots,k_{2n})} z_0} \frac{-dz}{z-x}$$

converge,

are independent of z_0,

are $SL_2(\mathbb{Z})$-invariant,

$\hat{1}(x) > 0$, &

$$\mathrm{val}(x) = \frac{\hat{\mathrm{val}}(x)}{\hat{1}(x)}$$

where

$$\gamma_{(k_1,\dots,k_{2n})} := \begin{pmatrix} k_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} k_{2n} & 1 \\ 1 & 0 \end{pmatrix}$$

$$\in SL_2(\mathbb{Z}).$$

Rem We can prove :

$\forall x = [k_1, \dots] \in \mathbb{R}_{ba}$, $\forall z_0 \in \mathbb{H}$,

$\forall x' \in \mathbb{R} \setminus \{x\}$,

$$\lim_{n\to\infty} \frac{1}{2n} \int_{z_0}^{\gamma_{(k_1,\dots,k_{2n})} z_0} j(z) \frac{-dz}{z-x'} = 0.$$

Thm (M.)

For w : real quadratic,

let $w = [k_1, \dots, k_{2q}, \overline{l_1, \dots, l_{2r}}]$

s.t. r is minimum

$$\Longrightarrow \quad \hat{\mathrm{val}}(w) = \frac{1}{r} \widetilde{\mathrm{val}}(w),$$

$$\hat{1}(w) = \frac{1}{r} \hat{1}(w).$$

Thm (M.) $\forall x = [k_1, \dots] \in \mathbb{R}_{ba}$,

$$\frac{a_n}{c_n} := [k_1, \dots, k_{2n}]$$

$$\Rightarrow \quad \varepsilon_x := \lim_{n\to\infty} \sqrt[n]{c_n} \text{ converges } \&$$

$$\hat{1}(x) = 2 \log \varepsilon_x .$$

Rem (Khinchin-Lévy's theorem in ergodic theory)

For almost all $x \in [0,1]$ with respect to $d\mu := \frac{1}{\log 2} \frac{dx}{1+x}$,

$$\varepsilon_x := \lim_{n\to\infty} \sqrt[n]{c_n} \text{ converges } \&$$

$$\log \varepsilon_x = -2 \int_0^1 \log(x)\, d\mu = \frac{\pi^2}{6 \log 2} .$$

Rem $\mathrm{vol}(\mathbb{R}_{ba} \cap [0,1], d\mu) = 0$.

§4 Continuity

We give two formulas of type

$$\boxed{\begin{gathered} \mathrm{val}(x) = \lim_{n\to\infty} \mathrm{val}(w_n), \\ x \in \mathbb{R}_{\mathrm{ba}}, \quad w_n : \text{real quadratic} \end{gathered}}$$

... partial results of continuity of $\mathrm{val} : \mathbb{R}_{\mathrm{ba}} \to \mathbb{C}$.

<u>Def</u>

- We call $W = (k_1, \ldots, k_{2r}) \in \mathbb{Z}_{>0}^{2r}$ an **even word**.
- For even words

$$V = (k_1, \ldots, k_{2r}), \quad W = (l_1, \ldots, l_{2s}),$$

let $VW := (k_1, \ldots, k_{2r}, l_1, \ldots, l_{2s})$

- W : **reduced**

$\overset{\text{def}}{\Longleftrightarrow}$ $\nexists n \geq 2$, $\nexists W_0$: an even word s.t. $W = W_0^n$.

<u>Thm</u> (M.)

- $V_0, \dots, V_k$: even words or ϕ,
- $W_1, \dots, W_k$: reduced even words,
- $w_i := [\overline{W_i}]$: real quadratic,
- $U_n := V_0 W_1^n V_1 W_2^n \cdots V_{k-1} W_k^n V_k$,
- $x := [U_1 U_2 \cdots U_n \cdots] \in \mathbb{R}_{ba}$

$$\Rightarrow \mathrm{val}(x) = \frac{\widetilde{\mathrm{val}}(w_1) + \cdots + \widetilde{\mathrm{val}}(w_k)}{\tilde{I}(w_1) + \cdots + \tilde{I}(w_k)}.$$

<u>Thm</u> (M. (2020))

On the above setting,

let $u_n := [\overline{U_n}]$

$$\Rightarrow \lim_{n\to\infty} \mathrm{val}(u_n)$$

$$= \frac{\widetilde{\mathrm{val}}(w_1) + \cdots + \widetilde{\mathrm{val}}(w_k)}{\tilde{I}(w_1) + \cdots + \tilde{I}(w_k)}.$$

<u>Thm</u> (Bengoechea-Imamoglu (2019), conjectured by Kaneko (2009))

- $V_0, V_1, W_1 \in \langle (1,1), (2,2) \rangle$ s.t. V_0 or $V_1 = \phi$,
- $w := [\overline{W_1}]$,
- $u_n = [\overline{V_0 W_1^n V_1}]$

$$\Longrightarrow \lim_{n\to\infty} \mathrm{val}(u_n) = \mathrm{val}(w).$$

Thm (M.)

- $V \neq W$: even words.
- $\langle V, W \rangle$: the monoid generated by V, W.
- $h : \langle V, W \rangle \longrightarrow \langle V, W \rangle$: a monoid hom.

$$V \longmapsto VW$$
$$W \longmapsto WV$$

- $V_h := \lim\limits_{n\to\infty} h^n(V) \not\ni U^3, \ \forall U \in \langle V, W \rangle.$

$$= VWWVWVVW\cdots$$

: the Thue-Morse word.

- $w_n := [\overline{h^{2n}(V)}]$: real quadratic.
- $x := [V_h] \in \mathbb{R}_{ba}$.

$$\Longrightarrow \mathrm{val}(x) = \lim_{n\to\infty} \mathrm{val}(w_n).$$