The fiber bundle method applied to triple product L-functions

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§1 Poisson Summation Conjecture
§2 Integral representation of triple product L-functions
§3 The fiber bundle method

§1 Poisson Summation Conjecture

F : number field. $\mathbb{A}_F$: Adele ring

Poisson Summation formula for vector spaces

- $V(\mathbb{R}) = \mathbb{R}^n$. $V(\mathbb{Z}) = \mathbb{Z}^n$

  $$R : V(\mathbb{R}) \times GL_V(\mathbb{R}) \to V(\mathbb{R})$$
  $$(v, g) \mapsto g^{-1} v$$

- $S(V(\mathbb{R}))$: space of rapidly decreasing functions
- Fourier transform

  $$\mathcal{F}(f)(\gamma) = \int_{\mathbb{R}^n} e^{-2\pi i \gamma \cdot x} f(x)\,dx$$
  $$\mathcal{F}_V \circ R(g) = |\det g| \cdot R(g^{-t}) \cdot \mathcal{F}_V$$

- Poisson summation formula

  $$\sum_{\gamma \in V(\mathbb{Z})} f(\gamma) = \sum_{\gamma \in V(\mathbb{Z})} \mathcal{F}(f)(\gamma)$$

Integral representation for standard L-functions for GL_n

Godement–Jacquet.

- Affine $GL_n \times GL_n$ equivariant embedding $GL_n \hookrightarrow M_n$.
  (generalizes $\mathbb{G}_m$-equivariant embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}_a$ in Tate's thesis)
- Schwartz space

$$\mathcal{S}(M_n(\mathbb{A}_F)) = \mathcal{S}(M_n(F_\infty)) \otimes C_c^\infty(M_n(\mathbb{A}_F^\infty))$$

· Fourier transform

$$\mathcal{F} : \mathcal{S}(M_n(\mathbb{A}_F)) \circlearrowleft$$

· P.S.F.

$$\sum_{\gamma \in M_n(F)} f(\gamma) = \sum_{\gamma \in M_n(F)} \mathcal{F}(f)(\gamma)$$

· Zeta integral : integral representation for $L(s, \pi)$ of GL_n (π : cusp. aut. rep. of GL_n).

$$Z(s, f, \varphi) = \int_{GL_n(\mathbb{A}_F)} f(g)\varphi(g) |\det g|^{s+\frac{n-1}{2}} dg.$$

$f \in \mathcal{S}(M_n(\mathbb{A}_F))$. $\varphi \in C(\pi)$ (matrix coefficient of π)

· Apply P.S.F. $\Rightarrow$ global meromorphic continuation & functional equation of $L(s, \pi)$

Poisson Summation Conjecture for spherical varieties.

- Braverman-Kazhdan, Ngô, L. Lafforgue, Sakellaridis.

$\forall$ affine spherical G-variety X (G-varieties w/ open dense Borel orbit).

$\exists$ Schwartz space

$$\mathcal{S}(X(\mathbb{A}_F)) = \bigotimes_v{}' \mathcal{S}_v(X(F_v))$$

w.r.t. basic functions $b_X \in \mathcal{S}_v(X(F_v))$ at each unramified places v.

Fourier transform

$$\mathcal{F}_X : \mathcal{S}(X(\mathbb{A}_F)) \circlearrowleft$$

P.S.F.

$$\sum_{x \in X^{sm}(F)} f(x) + * = \sum_{x \in X^{sm}(F)} \mathcal{F}_X(f)(x) + **$$

Known cases

· Braverman-Kazhdan spaces (Braverman-Kazhdan, Getz-Liu, Getz-Hsu-Leslie)

$$X^\circ = \frac{G}{[P,P]} \qquad X = \overline{X^\circ}^{\,\mathrm{aff}}$$

G : simply connected, split, simple. P: maximal parabolic.

- Triples of quadratic spaces (Getz-Liu, Getz-Hsu, Getz-Hsu-Leslie).
- Automorphic-twisted summation formula for pairs of quadratic spaces (G)
  - Space of test functions on zero locus of quadratic form
  - functions are built from Whittaker coeff. of cust rep of GL_n.
- Schubert varieties (Choie-Getz.)
- See also special cases by Jiang, Luo, Ngô, Shahidi, Sakellaridis, L. Zhang ...

Poisson Summation Conjecture

$\Downarrow$

F.E. & mero. cont. for fairly general Langlands L-functions.

$\Downarrow$ Converse Theorem (+ local Langlands)

Langlands functoriality in great generality,

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§2. Integral representation for triple L-functions.

Triple product L-functions

- $r_1, r_2, r_3 \in \mathbb{Z}_+$, $\pi = \bigotimes_{i=1}^{3} \pi_i$, π_i : cusp. aut. rep of $GL_{r_i}(\mathbb{A}_F)$.
- $L(s, \pi, \otimes^3) = L(s, \pi_1 \times \pi_2 \times \pi_3)$
  Langlands L-function defined by
  $\otimes^3 : {}^L(GL_{r_1} \times GL_{r_2} \times GL_{r_3}) \to GL_{r_1 r_2 r_3}(\mathbb{C})$
- The case $r_1 = r_2 = r_3 = 3$: smallest case where their global analytic properties are unknown.
- Analytic properties + Converse theorem $\Rightarrow$ tensor product functoriality

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Ingredient of the integral representation

Groups

R : F-algebra.

$H(R) := \{ (h_1, h_2, h_3) \in GL_2^3(R) : \det h_1 = \det h_2 = \det h_3 \}$.

$\mathcal{E}(R) := \{ \left(\begin{smallmatrix} \varepsilon_1 & \\ & \varepsilon_1 \end{smallmatrix} \right), \left(\begin{smallmatrix} \varepsilon_2 & \\ & \varepsilon_2 \end{smallmatrix} \right), \left(\begin{smallmatrix} \varepsilon_3 & \\ & \varepsilon_3 \end{smallmatrix} \right) \} : \varepsilon_i = \{\pm 1\}, \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1 \}$.

$Z := Z_{GL_2} \times Z_{GL_2} \times Z_{GL_2}$

$H^e := \mathbb{Z} \wedge^{\varepsilon} H = (\mathbb{Z} \times H)/\varepsilon$.

$\mathbb{Z}$-extension of H/ε (in the sense of Kottwitz).

$p_1 : H^e \to Gl_2 \times Gl_2 \times Gl_2$. $p_2 : H^e \to H/\varepsilon$.

Assume $r_i \geq 3$ for all i;

$M_i := \{(u_1, u_2) \in M_{r_i-2,2}(R) : u_1 \wedge u_2 = 0\}$

$M_i^\circ = M_i - \{0\}$. $\mathcal{M} = \prod_{i=1}^{3} M_i$ $\quad \mathcal{M}^\circ = \prod_{i=1}^{3} M_i^\circ$.

The space $\mathcal{Y}$

- $V_3 = \mathbb{G}_a^2 \otimes \mathbb{G}_a^2 \otimes \mathbb{G}_a^2$
- $H \curvearrowright V_3$

$$(v_1 \otimes v_2 \otimes v_3).(h_1, h_2, h_3) = (\det h_1)^{-1} v_1 h_1 \otimes v_2 h_2 \otimes v_3 h_3.$$

This action factors through ε and pulls back along p_2 to an action of H^e.

- $Gl_{\underline{r-2}} \times H^e \curvearrowright \mathcal{M} \times V_3$ (preserving $\mathcal{M}^\circ \times V_3$).

  $R: ((m, v), g, h) \mapsto (g^t m\, p_1(h)^{-t}, v.p_2(h))$
- $\mathcal{Y} \subset \mathcal{M} \times V_3$ (preserved under R) s.t. $\mathcal{Y}_{\mathcal{M}^\circ}$ is a vector bundle of rank 4 over $\mathcal{M}^\circ$.

Whittaker induction

- $X_{r_i} = \dfrac{Gl_{r_i}}{N_{r_i-2,2}}$ $\quad N_{r_i-2,2}$: unipotent radical of parabolic $P_{r_i-2,2}$
- X_{r_i} is a spherical variety w.r.t. an action of $GL_{r_i-2} \times GL_2 \times GL_{r_i}$
- We defined a $\mathbb{G}_{a, M_i^\circ}$-torsor

$$\mathcal{V}_{r_i}^\circ \to X_{r_i} \times M_i^\circ$$

  equipped w/ an action R of $Gl_{r_i-2} \times Gl_2 \times GL_{r_i}$
- $\mathcal{V}_i^\circ$ is a Whittaker induction in the sense of Sakellaridis-Venkatesh.
- We also define certain stack $\mathcal{V}_{r_i}$ over M_i that can be viewed as partial affine closure of $\mathcal{V}_{r_i}^\circ$.

$$\mathcal{V}_{r_i}^\circ = \mathcal{V}_{r_i, M_i^\circ}.$$

Fiber product

$$\begin{array}{ccc} \mathcal{V}\times_{\mathcal{M}}\mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & \mathcal{M} \end{array}$$

$X := \prod_{i=1}^{3} X_{r_i}$

$\mathcal{V}\times_{\mathcal{M}}\mathcal{Y}$, $X\times_{\mathcal{M}}\mathcal{Y}$ admit actions R of $GL_{\underline{r-2}}\times H^e\times GL_{\underline{r}}$

We have equivariant map $\mathcal{V}\times_{\mathcal{M}}\mathcal{Y}\to X\times_{\mathcal{M}}\mathcal{Y}$

- $(X\times_{\mathcal{M}}\mathcal{Y})/(H^e)_{der}$ is $GL_{\underline{r}}\times GL_{\underline{r}}\times H^e/(H^e)_{der}$ spherical.
- $\mathcal{V}\times_{\mathcal{M}}\mathcal{Y}$: generalized Whittaker induction (plays similar role as Whittaker inductions)

The space of functions

$C^\infty((\mathcal{V}\times_{\mathcal{M}}\mathcal{Y})(\mathbb{A}_F),\psi)$: Space of smooth functions $f: GL_{\underline{r}}(\mathbb{A}_F)\times\mathcal{Y}^\circ(\mathbb{A}_F)\to\mathbb{C}$.

s.t. $f\left(\begin{pmatrix} I_{r-2} & z \\ & I_2\end{pmatrix} g, m, v\right) = \psi(\langle m, z\rangle) f(g,m,v)$

We have an action

$R: C^\infty((\mathcal{V}\times_{\mathcal{M}}\mathcal{Y})(\mathbb{A}_F),\psi)\times GL_{\underline{r-2}}\times H^e\times GL_{\underline{r}}(\mathbb{A}_F)$

$\longrightarrow C^\infty((\mathcal{V}\times_{\mathcal{M}}\mathcal{Y})(\mathbb{A}_F),\psi)$

$R(g, g', h) f(g_0, m, v) = f\left(\begin{pmatrix} g' & \\ & p_1(h)\end{pmatrix}^{-1} g_0 g,\ g'^t m p_1(h)^{-t},\ v\cdot p_2(h)\right)$

$\theta_f(g, g', h) := \sum_{(x,m,v)\in X(F)\times\mathcal{Y}^\circ(F)} R(g, g', h) f(x, m, v)$

The integral representation

$$Z(f,\varphi,\varphi',\underline{s},\underline{s}',s_0) := \int_{[GL_{\underline{r-2}}]\times[H^e]\times[GL_{\underline{r}}]} \theta_f(g,g',h)\varphi(g)\varphi'(g') \times \eta_{\underline{s},\underline{s}',s_0}(g,g',h)\, dg\, dg'\, dh$$

φ, φ' : cusp forms on GL_{r-2}, GL_r

$\eta_{\underline{s}, \underline{s}', s_0}$: character defined using $|\det g|$, $|\det g'|$, $|\det h|$.

Thm 1 (Getz – G. – Hsu – Leslie)

For f satisfies certain local conditions, $(\mathrm{Re}(\underline{s}), \mathrm{Re}(\underline{s}'), \mathrm{Re}(s_0)) \in \mathbb{R}^2_{>2} \times \mathbb{R}_{>\frac{3}{2}}$

$Z(f, \varphi, \varphi', \underline{s}, \underline{s}', s_0)$ converges absolutely and is equal to

$$\int_{Z(\mathbb{A}_F) \backslash H^e(\mathbb{A}_F)} \int_{U_{r-2}(\mathbb{A}_F) \backslash GL_{r-2}(\mathbb{A}_F)} \int_{N_{r-2,2}(\mathbb{A}_F) \backslash GL_r(\mathbb{A}_F)} W_\psi^\varphi(g)\, W_{\bar{\psi}}^{\varphi'}(g')$$

$$\times f\left(({}^{g'} p_1(h))^{-1} g,\ g'^{t} m_0 p_1(h)^{-t},\ V_5 \cdot P_2(h) \right) \eta_{\underline{s}, \underline{s}', s_0}(g, g', h)\, dg\, dg'\, dh$$

Assume π, π', ψ unramified, take $b^{nai} \in C^\infty(Y \times \underline{\xi}, \psi)$ as naive basic function (analogue to the characteristic function of the integral point)

Thm 2 (Getz – G. – Hsu – Leslie).

For $(\mathrm{Re}(s), \mathrm{Re}(s'), \mathrm{Re}(s_0)) \in \mathbb{R}^2_{>1} \times \mathbb{R}_{>\frac{3}{2}}$.

$Z(W_0, W_0', b^{nai}, \underline{s}', \underline{s}, s_0)$

$$= \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\ell=0}^{\infty} q^{-\ell s_0} \prod_{i=1}^{3} L(s_i', \pi_i \times \pi_i') \sum_{n=0}^{r_i - 2} (-1)^{n_i} S_{0,0,\dots,0,n-\ell-k_i}(\alpha_i) \operatorname{Tr}(\wedge^n \alpha_i'^{-}) q^{-ns_i' - k_i s_i}$$

Corollary (Getz – G. – Hsu – Leslie)

Assume π, π' tempered. Let $\varepsilon > 0$. $(\mathrm{Re}(\underline{s}), \mathrm{Re}(\underline{s}')) \in \mathbb{R}^2_{>1+\varepsilon}$.

$$\frac{Z(W_0, W_0', \underline{b^{nai}}, \underline{s}, \underline{s}', s_0)}{L(s_0, \pi^\vee, \otimes^3)} = 1 + O_\varepsilon(q^{-1-\frac{\varepsilon}{2}})$$

for $\mathrm{Re}(s_0) > \frac{1}{2} + \varepsilon$

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§3 Fiber bundle method

F : number field

G : reductive group $/F$

X : spherical variety for G

Given

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ B_1 & & B_2 \end{tikzcd}$$

of G-schemes.

<u>Principle</u> : Prove P.S. conjecture for $p_i^{-1}(b_i)$ $\Rightarrow$ P.S. Conjecture for X

<u>Example</u> : basic affine spaces (Braverman–Kazhdan.)

$G = GL_3$

$X = \{(v, v^\vee) \in R^3 \times (R^3)^\vee : v^\vee(v) = 0\}$.

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{G}_a^3 & & (\mathbb{G}_a^3)^\vee \end{tikzcd}$$

- Fibers of p_1 are generically $\mathbb{G}_a^2$-torsors w/ skew symmetric pairing.
- Fiber bundle method $\Rightarrow$ P.S. for $\overline{GL_3/N_3}^{aff}$.
- works for $\overline{N\backslash G}^{aff}$ w/ G : split, simple, simply connected.

The case at hand.

$$\begin{tikzcd} & V \times_M \xi \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ V & & V_3 = \mathbb{G}_a^2 \otimes \mathbb{G}_a^2 \otimes \mathbb{G}_a^2 \end{tikzcd}$$

- General fibers of p_2 : Whittaker induction
- General fibers of p_1 : Vector spaces of rank 4

$\Updownarrow$

Known P.S. formula.

- We are proving P.S.C. for fibers of p_2

<u>Aim</u>

Prove P.S. conjecture for $V \times_M \xi$

Apply fiber bundle method:

- construct Schwartz space
$$\mathcal{S}((V \times_M S)(\mathbb{A}_F)[1,4]) \subset C^\infty((V \times_M S)(\mathbb{A}_F)[1,4])$$
- construct Fourier transforms and prove it has correct equivariance properties
- Prove P.S.F.