Jacobi forms and modular differential equations

Dmitrii Adler

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1 Jacobi forms

Definition. Let $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$. Then *a weak Jacobi form of weight k and index m* is a holomorphic function $\varphi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the following equations:

1) $\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} \varphi(\tau, z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$;

2) $\varphi(\tau, z+\lambda\tau+\mu) = e^{-4\pi i m\lambda z - 2\pi i m\lambda^2\tau} \varphi(\tau, z)$ for $\lambda, \mu \in \mathbb{Z}$;

3) $\varphi(\tau, z)$ has a Fourier expansion of the form

$$\varphi(\tau, z) = \sum_{n \geqslant 0} \sum_{l \in \mathbb{Z}} a(n,l) q^n \zeta^l = \sum_{n \geqslant 0} \sum_{l \in \mathbb{Z}} a(n,l) e^{2\pi i n\tau} e^{2\pi i l z}.$$

The set of all weak Jacobi forms is a bigraded ring

$$J^w_{*,*} = \bigoplus_{k,m} J^w_{k,m}.$$

Holomorphic and cusp Jacobi forms

There are two more types of Jacobi forms depending on their Fourier expansion.

A Jacobi form is a holomorphic Jacobi form $(J_{k,m})$ if

$$a(n,l) \neq 0 \Rightarrow 4nm - l^2 \geqslant 0.$$

A Jacobi form is a cusp Jacobi form $(J_{k,m}^{cusp})$ if

$$a(n,l) \neq 0 \Rightarrow 4nm - l^2 > 0.$$

It is clear that

$$J_{*,*}^{cusp} \subset J_{*,*} \subset J_{*,*}^{w}.$$

Remark. If $\varphi(\tau, z)$ is a Jacobi form of weight k and index m, and $t \in \mathbb{Z}$, then $\varphi(\tau, tz)$ is a Jacobi form of weight k and index $t^2 m$.

2 Examples of Jacobi forms

Let us define the odd Jacobi theta-function

$$\vartheta(\tau,z)=q^{\frac{1}{8}}(\zeta^{\frac{1}{2}}-\zeta^{-\frac{1}{2}})\prod_{n\geqslant 1}(1-q^n\zeta)(1-q^n\zeta^{-1})(1-q^n)=$$

$$=q^{\frac{1}{8}}\zeta^{\frac{1}{2}}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n(n+1)}{2}}\zeta^n.$$

It is a holomorphic Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ with a multiplier system $v_\eta^3\times v_H$. One can also define

$$\varphi_{0,\frac{3}{2}}=\frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)}=$$

$$=(\zeta^{\frac{1}{2}}+\zeta^{-\frac{1}{2}})\prod_{n\geqslant 1}(1+q^n\zeta)(1+q^n\zeta^{-1})(1-q^{2n-1}\zeta^2)(1-q^{2n-1}\zeta^{-2}),$$

$$\varphi_{-1,\frac{1}{2}}=\frac{\vartheta(\tau,z)}{\eta^3(\tau)}=(\zeta^{\frac{1}{2}}-\zeta^{-\frac{1}{2}})\prod_{n\geqslant 1}\frac{(1-q^n\zeta)(1-q^n\zeta^{-1})}{(1-q^n)^2}.$$

3 The structure of $J^w_{*,*}$

Theorem (M. Eichler, D. Zagier 1985).

$$J^w_{2*,*} = \oplus_{k,m} J^w_{k,m} = \mathcal{M}_*[\varphi_{-2,1}, \varphi_{0,1}]$$

where

$$\varphi_{-2,1}(\tau,z) = \varphi^2_{-1,\frac{1}{2}} = \frac{\vartheta(\tau,z)^2}{\eta^6(\tau)} = (\zeta - 2 + \zeta^{-1}) + q\cdot(\ldots) \in J^w_{-2,1},$$

$$\varphi_{0,1}(\tau,z) = -\frac{3}{\pi^2}\wp(\tau,z)\varphi_{-2,1}(\tau,z) = (\zeta + 10 + \zeta^{-1}) + q\cdot(\ldots) \in J^w_{0,1}.$$

Remark (M. Eichler, D. Zagier 1985). For any integral weights and indices

$$J^w_{*,*} = \mathcal{M}_*[\varphi_{-2,1}, \varphi_{-1,2}, \varphi_{0,1}]/\sim.$$

Proposition (V. Gritsenko 1999). Let m be an integer, then

$$J^w_{2k,m+\frac{1}{2}} = \varphi_{0,\frac{3}{2}} \cdot J^w_{2k,m-1},$$

$$J^w_{2k+1,m+\frac{1}{2}} = \varphi_{-1,\frac{1}{2}} \cdot J^w_{2k+2,m}.$$

4 Elliptic genus

Let M be an (almost) complex compact manifold of (complex) dimension d and T_M be its tangent bundle. Let $\tau \in \mathcal{H}$, $q = e^{2\pi i\tau}$ and $z \in \mathbb{C}$, $\zeta = e^{2\pi iz}$. Define a formal series

$$\mathbb{E}_{q,\zeta} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-\zeta^{-1}q^n} T_M^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-\zeta q^n} T_M \otimes \bigotimes_{n=0}^{\infty} S_{q^n} T_M^* \otimes \bigotimes_{n=0}^{\infty} S_{q^n} T_M$$

where $\wedge^k$ is the k^{th} exterior power, S^k is the k^{th} symmetric product and

$$\bigwedge\nolimits_x E = \sum_{k\geqslant 0} (\wedge^k E) x^k, \quad S_x E = \sum_{k\geqslant 0} (S^k E) x^k.$$

Then the elliptic genus of M is a function of two variables $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$:

$$\chi(M;\tau,z) = \zeta^{\frac{d}{2}} \int_M \mathrm{ch}(\mathbb{E}_{q,\zeta})\mathrm{td}(T_M) = \sum_{n\geq 0,\, l\in\mathbb{Z}} a(n,l)\, q^n \zeta^l =$$

$$= \sum_{p=0}^{d} (-1)^p \chi_p(M) \zeta^{d/2-p} + q(\dots) \quad \text{where} \quad \chi_p(M) = \sum_{q=0}^{d} (-1)^q h^{p,q}(M).$$

5 The elliptic genus and Jacobi forms

Theorem. If M is a compact complex manifold of dimension d with $c_1(M) = 0$ (over $\mathbb{R}$), then its elliptic genus $\chi(M;\tau,z)$ is a weak Jacobi form of weight 0 and index $\frac{d}{2}$ with integral Fourier coefficients.

Corollary. The elliptic genus of any Calabi–Yau manifold is a Jacobi form of weight 0.

6 The Euler characteristics

Note that

$$q^0[\chi(M;\tau,0)] = \sum_{p=0}^{d}(-1)^p\chi_p(M) = e(M)$$

is the Euler characteristics of M.

If $\dim J_{0,m} = 1$ then the elliptic genus of a Calabi–Yau manifold of dimension $2m$ depends only on $e(M)$.

7 The elliptic genus of a $K3$ surface, CY_3, and CY_5

For $m = 1, \frac{3}{2}$, and $\frac{5}{2}$ the spaces $J^w_{0,m}$ are one-dimensional.

It is known that $e(K3) = 24$. Thus,

$$\chi(K3; \tau, z) = 2\varphi_{0,1}(\tau, z) = 2\zeta + 20 + 2\zeta^{-1} + q(\dots).$$

For CY_3 and CY_5

$$\chi(CY_3; \tau, z) = b\,\varphi_{0,\frac{3}{2}} = b\left(\zeta^{-\frac{1}{2}} + \zeta^{\frac{1}{2}} + q(\dots)\right),$$

$$\chi(CY_5; \tau, z) = c\,\varphi_{0,\frac{3}{2}} \cdot \varphi_{0,1} = c\left(\zeta^{\pm\frac{3}{2}} + 11\zeta^{\pm\frac{1}{2}} + q(\dots)\right).$$

Hence,

$$\chi(CY_3; \tau, z) = \frac{e(CY_3)}{2}\,\varphi_{0,\frac{3}{2}} \quad \text{and} \quad \chi(CY_5; \tau, z) = \frac{e(CY_5)}{24}\,\varphi_{0,\frac{3}{2}} \cdot \varphi_{0,1}$$

Corollary (V. Gritsenko 1999). The Euler characteristics of a CY_3 is even. The Euler characteristics of a CY_5 is divisible by 24.

8 The modular differential operator D

Let $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Then for any modular form $f = \sum a(n)q^n$ of weight k:

$$D(f) = \sum na(n)q^n.$$

Generally speaking, $D(f)$ is not a modular form, but the Serre derivative

$$D_k(f) = D(f) - \frac{k}{12}E_2 f \in M_{k+2}.$$

One can show that for the Dedekind η-function $\eta(\tau) = q^{\frac{1}{24}}\prod(1-q^n)$

$$\frac{D(\eta)}{\eta} = \frac{1}{24}E_2 \quad \Longrightarrow \quad D_{\frac{k}{2}}(\eta^k) = 0.$$

The well-known Ramanujan differential equations also hold

$$D_2(E_2) = -\frac{1}{12}(E_2^2 + E_4), \quad D_4(E_4) = -\frac{1}{3}E_6, \quad D_6(E_6) = -\frac{1}{2}E_4^2.$$

Thus, the ring $\mathbb{C}[E_2, E_4, E_6]$ is invariant w.r.t. the action of D.

9 The modular differential operator H_k

In the case of Jacobi forms an analogue of D is **the heat operator**

$$H = \frac{3}{m}\frac{1}{(2\pi i)^2}\left(8\pi i m\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial z^2}\right) = 12q\frac{d}{dq} - \frac{3}{m}\left(\zeta\frac{d}{d\zeta}\right)^2.$$

It transforms $\varphi_{k,m} = \sum a(n,l)q^n\zeta^l$ of weight k and index m to

$$H\left(\sum a(n,l)q^n\zeta^l\right) = \frac{3}{m}\sum(4nm-l^2)a(n,l)q^n\zeta^l.$$

As in the case of modular forms, $H(\varphi_{k,m})$ is not, generally speaking, a Jacobi form, but

$$H_k(\varphi_{k,m}) = H(\varphi_{k,m}) - \frac{(2k-1)}{2}E_2\cdot\varphi_{k,m} \in J^{w,hol,cusp}_{k+2,m}.$$

Examples. 1) $H_{\frac{1}{2}}(\vartheta) = H(\vartheta) = 0.$

$$2)\quad H_{-2}(\varphi_{-2,1}) = -3\zeta - 3\zeta^{-1} + \frac{5}{2}(\zeta - 2 + \zeta^{-1}) + q(\ldots) =$$

$$= -\frac{1}{2}(\zeta + 10 + \zeta^{-1}) + q(\ldots) = -\frac{1}{2}\varphi_{0,1}.$$

10 The kernel of the modular differential operator

For $f \in M_{k_1}$, $\varphi \in J_{k_2,m}$:

$$H_{k_1+k_2}(f\varphi) =$$

$$= 12D(f\varphi) + \frac{3}{2\pi^2 m}\left(\frac{\partial}{\partial \mathfrak{z}}, \frac{\partial}{\partial \mathfrak{z}}\right)(f\varphi) + \left(\frac{1}{2} - k_1 - k_2\right) E_2 f\varphi =$$

$$= f H_{k_2}(\varphi) + 12 D_{k_1}(f)\varphi.$$

Remark. For any two Jacobi forms of non-zero indices the Leibniz product rule does not hold.

Theorem (D.A., V. Gritsenko 2023). Let $\varphi \in J^w_{k,m}$ be a weak Jacobi form of integral weight and integral or half-integral index. Then $\varphi \in \operatorname{Ker} H_k$ if and only if

$$\varphi = \begin{cases} \varphi_{0,\frac{3}{2}}(\tau, tz) \cdot \Delta(\tau)^n \\ \varphi_{-1,\frac{1}{2}}(\tau, tz) \cdot \Delta(\tau)^n \end{cases}$$

with $t \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geqslant 0}$.

11 Kaneko–Zagier type equations

The Kaneko–Zagier equation is a differential equation of the form

$$f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0,$$

which is equivalent to

$$D_{k+2}D_k f = \frac{k(k+2)}{144}E_4 f.$$

One can notice that $E_4(\tau)$ and $E_6(\tau)$ satisfy this equation. There is a unique non-cusp solution for each $k \equiv 0$ or $4 \mod 6$.

The Kaneko–Zagier type equation is a differential equation of the form

$$H_{k+2}H_k\varphi_{k,m} = \lambda E_4 \varphi_{k,m}.$$

Proposition (D.A. 2026+). There is the following list of solutions of Kaneko–Zagier type equations.

1) If $f_k \in M_k$ is a solution of the Kaneko–Zagier equation, then $f_k\varphi_{-1,\frac{1}{2}}$ and $f_k\varphi_{0,\frac{3}{2}}$ are solutions of Kaneko–Zagier type equations.

2)

$$\begin{array}{ll} & \varphi_{4,1} \to \varphi_{10,1} \to \varphi_{16,1} \to \ldots \to \varphi_{6k+4,1} \to \ldots \\ \frac{\vartheta^2(\tau,z)}{\eta^6(\tau)} = \varphi_{-2,1} & \begin{array}{l} \nearrow \\ \searrow \end{array} \\ & \psi_{4,1} \to \psi_{10,1} \to \psi_{16,1} \to \ldots \to \psi_{6k+4,1} \to \ldots \end{array}$$

3)

$$\begin{array}{ll} & \varphi_{4,\frac{5}{2}} \to \varphi_{10,\frac{5}{2}} \to \varphi_{16,\frac{5}{2}} \to \ldots \to \varphi_{6k+4,\frac{5}{2}} \to \ldots \\ \frac{\vartheta(\tau,z)\vartheta(\tau,2z)}{\eta^6(\tau)} = \varphi_{-2,\frac{5}{2}} & \begin{array}{l} \nearrow \\ \searrow \end{array} \\ & \psi_{4,\frac{5}{2}} \to \psi_{10,\frac{5}{2}} \to \psi_{16,\frac{5}{2}} \to \ldots \to \psi_{6k+4,\frac{5}{2}} \to \ldots \end{array}$$

4)

$$\begin{tikzcd}[column sep=small] & \varphi_{3,\frac{3}{2}} \arrow[r] & \varphi_{9,\frac{3}{2}} \arrow[r] & \varphi_{15,\frac{3}{2}} \arrow[r] & \ldots \arrow[r] & \varphi_{6k+3,\frac{3}{2}} \arrow[r] & \ldots \\ \frac{\vartheta^3(\tau,z)}{\eta^9(\tau)} = \varphi_{-3,\frac{3}{2}} \arrow[ur] \arrow[dr] & & & & & & \\ & \psi_{3,\frac{3}{2}} \arrow[r] & \psi_{9,\frac{3}{2}} \arrow[r] & \psi_{15,\frac{3}{2}} \arrow[r] & \ldots \arrow[r] & \psi_{6k+3,\frac{3}{2}} \arrow[r] & \ldots \end{tikzcd}$$

5)

$$\begin{tikzcd}[column sep=small] & \varphi_{3,3} \arrow[r] & \varphi_{9,3} \arrow[r] & \varphi_{15,3} \arrow[r] & \ldots \arrow[r] & \varphi_{6k+3,3} \arrow[r] & \ldots \\ \frac{\vartheta^2(\tau,z)\vartheta(\tau,2z)}{\eta^9(\tau)} = \varphi_{-3,3} \arrow[ur] \arrow[dr] & & & & & & \\ & \psi_{3,3} \arrow[r] & \psi_{9,3} \arrow[r] & \psi_{15,3} \arrow[r] & \ldots \arrow[r] & \psi_{6k+3,3} \arrow[r] & \ldots \end{tikzcd}$$

6) Other solutions could be obtained from solutions listed above by scaling $z \mapsto t \cdot z$ and multiplication by $\Delta^n, n \geqslant 0$.

12 Modular differential equations (MDEs) of higher order

Let $\varphi \in J^w_{k,m}$. We can apply modular differential operators to φ and multiply it by modular forms.

- weight k: φ;
- weight $k+2$: $H_k(\varphi)$;
- weight $k+4$: $H_{k+2}(H_k(\varphi))$, $E_4\varphi$;
- weight $k+6$: $H_{k+4}(H_{k+2}(H_k(\varphi)))$, $E_4 H_k(\varphi)$, $E_6\varphi$;
- ...

If for some linear combination in weight $k+2s$ its q^0-coefficient vanishes, then this linear combination is divisible by $\Delta(\tau)$. If $J^w_{k+2s-12,m} = 0$, we get a MDE. If $J^w_{k+2s-12,m} \neq 0$ then we try to vanish both q^0- and q^1-terms, etc.

13 Examples

Theorem (D.A., V. Gritsenko 2023). The following differential equations holds:

$$H_4H_2H_0(\varphi_{0,1}) - \frac{101}{4}E_4H_0(\varphi_{0,1}) + 10E_6\varphi_{0,1} = 0,$$

$$H_0(\varphi_{0,\frac{3}{2}}) = 0,$$

$$H_4H_2H_0(\varphi_{0,\frac{5}{2}}) - \frac{611}{25}E_4H_0(\varphi_{0,\frac{5}{2}}) + \frac{88}{25}E_6\varphi_{0,\frac{5}{2}} = 0.$$

Thus, the elliptic genus of a 3-dimensional Calabi–Yau manifold satisfies the simplest MDE of order 1 w.r.t. the heat operator. The elliptic genus of a $K3$ surface and a Calabi–Yau manifold of dimension 5 satisfy a modular differential equation of order 3.

14 Modular differential equations of weak Jacobi forms of weight 0 and index 2

Theorem (D.A., V. Gritsenko 2023). Any Jacobi form $\Phi_{0,2} \in J^{w}_{0,2} = \mathbb{C}\langle E_4\varphi^2_{-2,1}, \varphi^2_{0,1}\rangle$ satisfies a general MDE of minimal possible order 5, with the following exceptions:

$$\varphi_{0,2}(\tau, z) = \zeta^{\pm 1} + 4 + q(\ldots),$$

$$\psi_{0,2}(\tau, z) = \zeta^{\pm 2} + 22 + q(\ldots),$$

$$\rho_{0,2}(\tau, z) = 2\zeta^{\pm 2} - 11\zeta^{\pm 1} + q(\ldots),$$

which satisfy 3-order MDEs,

$$\xi_{0,2} = 115\zeta^{\pm 2} + 8624\zeta^{\pm 1} + 37026 + q(\ldots),$$

which satisfies a 4-order MDE, and the Jacobi form

$$\sigma_{0,2} = 5\zeta^{\pm 2} - 308\zeta^{\pm 1} - 1122 + q(\ldots),$$

which satisfies only a 6-order MDE.

3-order:

$$H_4H_2H_0(\varphi_{0,2}) - \tfrac{47}{4}E_4H_0(\varphi_{0,2}) + \tfrac{13}{4}E_6\varphi_{0,2} = 0,$$
$$H_4H_2H_0(\psi_{0,2}) - \tfrac{263}{4}E_4H_0(\psi_{0,2}) + \tfrac{121}{4}E_6\psi_{0,2} = 0,$$
$$H_4H_2H_0(\rho_{0,2}) - \tfrac{335}{4}E_4H_0(\rho_{0,2}) - \tfrac{275}{4}E_6\rho_{0,2} = 0.$$

4-order:

$$H_0^{[4]}(\xi_{0,2}) - \tfrac{599}{4}E_4H_0^{[2]}(\xi_{0,2}) - \tfrac{1179}{4}E_6H_0(\xi_{0,2}) + \tfrac{99}{2}E_4^2\xi_{0,2} = 0.$$

General:

$$H_0^{[5]}(\Phi_{0,2}) - \tfrac{5(7931a+167b)}{308a+5b}E_4H_0^{[3]}(\Phi_{0,2}) + \tfrac{5(137060a+389b)}{4(308a+5b)}E_6H_0^{[2]}(\Phi_{0,2}) +$$
$$+ \tfrac{15(723184a+8017b)}{16(308a+5b)}E_4^2H_0(\Phi_{0,2}) - \tfrac{165(13552a+169b)}{16(308a+5b)}E_4E_6\Phi_{0,2} = 0.$$

6-order:

$$H_0^{[6]}(\sigma_{0,2}) - \tfrac{983}{4}E_4H_0^{[4]}(\sigma_{0,2}) + \tfrac{357}{2}E_6H_0^{[3]}(\sigma_{0,2}) + \tfrac{11343}{2}E_4^2H_0^{[2]}(\sigma_{0,2}) +$$
$$+ \tfrac{2069}{16}E_4E_6H_0(\sigma_{0,2}) + \tfrac{19217}{16}E_6^2\sigma_{0,2} = 0.$$

15 MDEs of order 3 and Calabi–Yau fourfolds

Let M_4 be a Calabi–Yau fourfold in the strict sense (i.e., $h^{p,0}(M) = 0$ for every $0 < p < 4$).

Then $\chi(M_4; \tau, z)$ is uniquely defined by its q^0-term and

$$q^0[\chi(M_4; \tau, z)] = \chi_0(M_4)\zeta^2 - \chi_1(M_4)\zeta + \chi_2(M_4) - \chi_3(M_4)\zeta^{-1} + \chi_4(M_4)\zeta^{-2}.$$

It is also known that M_4 has four non-trivial Hodge numbers, $h^{1,1}$, $h^{2,1}$, $h^{3,1}$, and $h^{2,2}$ with an additional linear relation

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}).$$

The order of a MDE for $\chi(M_4; \tau, z)$ is at least 3.

Proposition (D.A., V. Gritsenko 2023). Let M be a Calabi–Yau fourfold in the strict sense. The elliptic genus of M satisfies MDEs of the minimal possible order if and only if $h^{2,1} = h^{1,1} + h^{3,1}$ or $h^{2,2} = 2h^{2,1}$, respectively. The first condition is equivalent to $e(M_4) = 48$, the second to $e(M_4) = -18$.

16 MDEs for hyperkähler manifolds of dimension 4.

Two types of such manifolds are known. The first type is the length 2 Hilbert scheme $\mathrm{Hilb}^2(K3)$ for a K3 surface. The second type is a generalized Kummer variety $\mathrm{Kum}^2(A)$ of a two-dimensional complex torus A.

Let A_4 be a generalized Kummer fourfold. Its Hodge diamond is defined by the following Hodge numbers:

$$h^{0,0} = h^{0,2} = h^{0,4} = 1, \quad h^{0,1} = h^{0,3} = 0,$$

$$h^{1,1} = 5, \quad h^{1,2} = 4, \quad h^{1,3} = 5, \quad h^{2,2} = 96.$$

Therefore, $\chi_0(A_4) = 3$, $\chi_1(A_4) = -6$, $\chi_2(A_4) = 90$ and

$$\chi(A_4; \tau, z) = 3\zeta^{\pm 2} + 6\zeta^{\pm 1} + 90 + q(\dots) = 3(\psi_{0,2} + 2\varphi_{0,2}).$$

Then the elliptic genus $\Psi = \chi(A_4; \tau, z)$ satisfies a MDE of order 5

$$\big(H_0^{[5]} - \tfrac{13775}{106} E_4 H_0^{[3]} + \tfrac{114865}{212} E_6 H_0^{[2]} + \tfrac{1848045}{848} E_4^2 H_0 - \tfrac{381975}{848} E_4 E_6\big)(\Psi) = 0.$$

Let K_4 be deformation equivalent to $\mathrm{Hilb}^2(\mathrm{K3})$. The Hodge diamond of K_4 is defined by the Hodge numbers:

$$h^{0,0} = h^{0,2} = h^{0,4} = 1, \quad h^{0,1} = h^{0,3} = 0,$$

$$h^{1,1} = 5, \quad h^{1,2} = 0, \quad h^{1,3} = 21, \quad h^{2,2} = 232.$$

Thus, $\chi_0(K_4) = 3$, $\chi_1(K_4) = -42$, $\chi_2(K_4) = 234$ and

$$\chi(K_4; \tau, z) = 3\zeta^{\pm 2} + 42\zeta^{\pm 1} + 234 + q(\dots) = 3(\psi_{0,2} + 14\varphi_{0,2}).$$

The elliptic genus $\Phi = \chi(K_4; \tau, z)$ satisfies a MDE of order 5

$$\left(H_0^{[5]} - \tfrac{815}{6} H_0^{[3]} + \tfrac{1885}{4} E_6 H_0^{[2]} + \tfrac{99455}{48} E_4^2 H_0 - \tfrac{20845}{48} E_4 E_6\right)(\Phi) = 0.$$

Also note that for the direct product of two $K3$ surfaces

$$\chi(K3 \times K3; \tau, z) = 4\varphi_{0,1}(\tau, z)^2 = 4(\psi_{0,2} + 20\varphi_{0,2}),$$

and we obtain

$$\left(H_0^{[5]} - \tfrac{1105}{8} E_4 H_0^{[3]} + \tfrac{1775}{4} E_6 H_0^{[2]} + \tfrac{64965}{32} E_4^2 H_0 - \tfrac{13695}{32} E_4 E_6\right)(\varphi_{0,1}^2) = 0.$$

17 MDEs of weak Jacobi forms of weight 0 and index 3

One of the natural bases of $J^w_{0,3}$ consists of Jacobi forms

$$\varphi_{0,3} = \zeta^{\pm 1} + 2 + O(q), \quad \psi_{0,3} = \zeta^{\pm 2} + 14 + O(q), \quad \rho_{0,3} = \zeta^{\pm 3} + 34 + O(q).$$

Thus, any Jacobi form $\Phi_{0,3} = a\varphi_{0,3} + b\psi_{0,3} + c\rho_{0,3}$.

Theorem (D.A., V. Gritsenko 2025). Any Jacobi form $\Phi_{0,3} \in J^w_{0,3}$ satisfies a general MDE of minimal possible order 7, with the following exceptions:

1) there are exactly 10 Jacobi forms satisfying 4-order MDEs;

2) there are exactly 5 Jacobi forms satisfying 5-order MDEs;

3) there are 5 one-parameter families (in $\mathbb{P}^2(\mathbb{C})$) of Jacobi forms satisfying 6-order MDEs;

4) there is a set $S(a, b, c) = 0$ of Jacobi forms satisfying MDEs of order > 7, where

$$S(a, b, c) = 1196694415125a^3 - 819233068500a^2b - 18637542000ab^2 -$$
$$-30184000b^3 - 20559305385a^2c + 1081838520abc + 4504080b^2c +$$
$$+16833519ac^2 - 137844bc^2 - 595c^3.$$

Varieties K_6 of $\mathrm{Hilb}^{[3]}(K3)$-type:

$$\Big(H_0^{[7]} - \tfrac{15121249171}{1747\cdot 21391} E_4 H_0^{[5]} + \tfrac{693022919057}{2^3\cdot 1747\cdot 21391} E_6 H_0^{[4]} + \tfrac{16455397736281}{2^4\cdot 1747\cdot 21391} E_4^2 H_0^{[3]} -$$

$$- \tfrac{112040811677277}{2^4\cdot 1747\cdot 21391} E_4 E_6 H_0^{[2]} - (\tfrac{438449632667109}{2^5\cdot 1747\cdot 21391} E_4^3 + \tfrac{4495401229430736}{1747\cdot 21391} \Delta) H_0 +$$

$$+ \tfrac{172173893177097}{2^6\cdot 1747\cdot 21391} E_4^2 E_6\Big) (\chi(K_6)) = 0.$$

Varieties A_6 of $\mathrm{Kum}_3(A)$-type:

$$\Big(H_0^{[7]} - \tfrac{15426984953}{2^2\cdot 20776739} E_4 H_0^{[5]} + \tfrac{284631287485}{2\cdot 20776739} E_6 H_0^{[4]} + \tfrac{9301863314105}{2^4\cdot 20776739} E_4^2 H_0^{[3]} -$$

$$- \tfrac{30760061368995}{2^2\cdot 20776739} E_4 E_6 H_0^{[2]} - (\tfrac{793296354632355}{2^6\cdot 20776739} E_4^3 + \tfrac{1110984049847088000}{20776739} \Delta) H_0 +$$

$$+ \tfrac{36177330650265}{2^5\cdot 20776739} E_4^2 E_6\Big) (\chi(A_6)) = 0.$$

Varieties of OG_6-type:

$$\Big(H_0^{[7]} - \tfrac{122813480461}{2^2\cdot 11\cdot 7214393} E_4 H_0^{[5]} + \tfrac{432883583489}{2\cdot 11\cdot 7214393} E_6 H_0^{[4]} + \tfrac{36278381836753}{2^4\cdot 11\cdot 72143938} E_4^2 H_0^{[3]} -$$

$$- \tfrac{62444977311759}{2^2\cdot 11\cdot 7214393} E_4 E_6 H_0^{[2]} - (\tfrac{1932747726669879}{2^6\cdot 11\cdot 7214393} E_4^3 + \tfrac{46727205157384128}{11\cdot 7214393} \Delta) H_0 +$$

$$+ \tfrac{94397656019709}{2^5\cdot 11\cdot 7214393} E_4^2 E_6\Big) (\chi(OG_6)) = 0.$$

Thank you!