Reductions of K3 surfaces via intersections on GSpin Shimura varieties

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Main result for K3 surfaces

Let K be a number field and X/K an (algebraic) K3 surface; i.e., X/K a smooth projective surface such that its canonical bundle is trivial and $H^1(X, \mathcal{O}_X) = 0$.

Consider $\mathcal{X}/\mathcal{O}_K$ an integral model of X/K and let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ at which $\mathcal{X}$ has good reduction.

Fact. $\mathrm{Pic}(X_{\overline{K}}) \hookrightarrow \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}})$;
in particular, $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}}) \leqslant \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}})$.

Theorem (Ananth Shankar, Arul Shankar, T., Salim Tayou)

Assume X/K has potentially good reduction everywhere.
Then there are infinitely many $\mathfrak{p}$ s.t. $\mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(X_{\overline{K}}) < \mathrm{rk}_{\mathbb{Z}} \mathrm{Pic}(\mathcal{X}_{\overline{\mathbb{F}}_{\mathfrak{p}}})$.

Remark. For certain K3 surfaces, by the work of Charles, the set of such primes is of density 0.

Motivation: Abelian surfaces

Let A/K be an abelian surface over a number field K.

Conjecture (Achter–Howe 2017)

Assume A is principally polarizable and $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$, then

$$\#\{\mathfrak{p} \mid \mathrm{Nm}\,\mathfrak{p} < N, A_{\mathbb{F}_\mathfrak{p}} \textit{ not simple } \} \asymp \frac{N^{1/2}}{\log N}, \; N \to \infty.$$

They also expect a similar heuristic replacing not simple by not geometrically simple.

Achter 2009, Zywina 2014: for any abelian variety A/K such that $A_{\overline{K}}$ is simple and $\mathrm{End}(A_{\overline{K}})$ is commutative, then (after a finite extension of K), $\{\mathfrak{p} \mid A_{\overline{\mathbb{F}}_\mathfrak{p}} \text{ not simple } \}$ is of density 0.

Main result for abelian surfaces

Theorem (Ananth Shankar, Arul Shankar, T., Salim Tayou)

For an abelian surface A/K with potentially good reduction everywhere, there are infinitely many primes $\mathfrak{p}$ such that $A_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ is not simple.

Previous work.
(1) Charles 2018: for $E_1, E_2/K$ elliptic curves, there are infinitely many $\mathfrak{p}$ such that $E_{1,\overline{\mathbb{F}}_{\mathfrak{p}}}$ is isogenous to $E_{2,\overline{\mathbb{F}}_{\mathfrak{p}}}$.
(2) Ananth Shankar–T. 2020: for A/K an abelian surface such that $\mathrm{End}(A_{\overline{K}}) \otimes \mathbb{Q}$ contains a real quadratic field, there are infinitely many $\mathfrak{p}$ such that $A_{\overline{\mathbb{F}}_{\mathfrak{p}}}$ is isogenous to the self-product of some elliptic curve (depending on $\mathfrak{p}$).
(No good reduction assumption in these results.)

GSpin Shimura varieties

(V, Q) a non-deg. quad. vector space over $\mathbb{Q}$ with sig. $(b, 2)$;
L a maximal lattice in V with Q being $\mathbb{Z}$-valued;
$C(V), C(L)$ the Clifford algebras of V, L.

$G := \mathrm{GSpin}(V, Q)$, i.e., for a $\mathbb{Q}$-alg. R,
$$G(R) = \{g \in C^+(V_R)^\times \mid gV_Rg^{-1} = V_R\};$$
Hermitian domain $D := \{[z] \in \mathbb{P}(V_\mathbb{C}) \mid [\bar{z}, z] < 0, Q(z) = 0\}$,
where $[x, y] := Q(x + y) - Q(x) - Q(y)$;
Open compact $\mathbb{K} := G(\mathbb{A}_f) \cap C(L \otimes_\mathbb{Z} \widehat{\mathbb{Z}})$;
Shimura variety of Hodge type $M(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/\mathbb{K}$
with canonical model $M/\mathbb{Q}$.

There is a universal family of Kuga–Satake abelian varieties A^{univ} over M such that $H_{1,B}(-, \mathbb{Q})$ for each fiber over $\mathbb{C}$ is isomorphic to $C(V)$.
Examples. Moduli of polarized abelian surfaces; moduli of quasi-polarized K3 surfaces.

Special endomorphisms

Let A be a Kuga–Satake abelian scheme (i.e., for a M-scheme S, we take A_S^{univ}).
A special endomorphism s of A is an endomorphism of A whose Betti realization lies in $V \subset \mathrm{End}(C(V)) \cong \mathrm{End}(H_{1,B}(A_\sigma(\mathbb{C}), \mathbb{Q}))$, where A_σ is a $\mathbb{C}$-fiber of A and V acts on $C(V)$ by left multiplication.

Examples.
(1) For a polarized abelian surface A, one may consider special endomorphisms as $s \in \mathrm{End}(A)^{\mathrm{tr}=0,\dagger=\mathrm{id}}$;

moreover, if $Q(s) = m^2$ for some $m \in \mathbb{Z}_{>0}$, then $s - [m]$ is not invertible and cut out a non-trivial isogeny factor of A and hence A is not simple.

(2) For a polarized K3 surface X, a special endomorphism of its Kuga–Satake abelian variety gives an element in $\mathrm{Pic}(X_{\overline{K}})$ perpendicular to the given polarization.

Integral models and special divisors

Kisin, Madapusi Pera, Andreatta–Goren–Howard–Madapusi-Pera constructed a normal flat integral model $\mathcal{M}/\mathbb{Z}$ of $M/\mathbb{Q}$ such that $\mathcal{M}_{\mathbb{Z}_p}$ is smooth if L is self-dual at p;
A^{univ} extends to an abelian scheme $\mathcal{A}^{\mathrm{univ}}$ over $\mathcal{M}$ and there is a notion of special endomorphisms over $\mathbb{Z}$ using de Rham, étale, and crystalline realizations.

Example. For abelian surfaces, we may still define special endomorphisms to be $s \in \mathrm{End}(A)^{\mathrm{tr}=0, \dagger=\mathrm{id}}$.

Special divisors $\mathcal{Z}(m)$ param. Kuga–Satake abelian varieties with special endomorphisms s such that $s \circ s = [m]$ for $m \in \mathbb{Z}_{>0}$.

Facts. (1) $\mathcal{Z}(m)$ is étale locally a Cartier divisor on $\mathcal{M}$.
(2) $\mathcal{Z}(m)_{\mathbb{Q}}$ is also a GSpin Shimura variety.

Main theorem for number fields

Theorem (Shankar–Shankar–T.–Tayou)

Let (L, Q) be a max. quad. lattice of signature $(b, 2)$ with $b \geqslant 3$ and let $\mathcal{M}$ denote the integral model of the GSpin Shimura variety attached to (L, Q). Fix $D \in \mathbb{Z}_{>0}$.
For $\mathcal{Y} \in \mathcal{M}(\mathcal{O}_K)$ such that $\mathcal{Y}_K \in \mathcal{M}(K)$ doesn't lie in any $\mathcal{Z}(m)_{\mathbb{Q}}$, there are infinitely many primes $\mathfrak{p}$ such that $\mathcal{Y}_{\overline{\mathbb{F}}_{\mathfrak{p}}} \in \mathcal{Z}(Dm^2)$ for some $m \in \mathbb{Z}_{>0}$.

Remarks.

(1) It recovers the abelian surface theorem by taking $D = 1$.

(2) It recovers the K3 surface theorem by taking L being the transcendence part of $H^2_B(X(\mathbb{C}), \mathbb{Z})$.

(3) A similar theorem for unitary Shimura varieties of signature $(n, 1)$ is a direct consequence of the main theorem.

Main theorem for global function fields

$p > 2$ prime and L self-dual at p hence $\mathcal{M}_{\mathbb{F}_p}$ is smooth.
For simplicity, we assume $\mathrm{rk}\, L \geqslant 5$.

Theorem (Davesh Maulik–Ananth Shankar–T.)

Assume $p \geqslant 5$ *and* $C \subset \mathcal{M}_{\overline{\mathbb{F}}_p}$ *irreducible projective curve such that* $C[\mathrm{ord}] \neq \varnothing$ *and* $C \not\subset \mathcal{Z}(m)_{\overline{\mathbb{F}}_p}, \forall m$. *Then there are infinitely many* $\overline{\mathbb{F}}_p$*-points on* C *such that they lie on* $\bigcup_{p \nmid m} \mathcal{Z}(m)_{\overline{\mathbb{F}}_p}$.
Moreover if $\mathcal{M} = \mathcal{A}_2$*, the moduli space of principally polarized abelian surfaces, then there are infinitely many* $\overline{\mathbb{F}}_p$*-points on* C *which correspond to non-simple abelian surfaces for* $p \geqslant 5$.

Remark. We have a similar theorem when $\mathcal{M}$ is the Hilbert modular surface without assuming C being projective.

Previous work

Chai–Oort 2006. For a non-isotrivial pair of elliptic curves E_1, E_2 over $k(C)$, if E_1, E_2 are ordinary, then there are infinitely many places v such that $E_{1,v}$ is geometrically isogenous to $E_{2,v}$.

Question. Why ordinary?

(non-)Example. Consider $C \subset X_0(1) \times X_0(1)$ and $C = X_0(1) \times P$, where $P \in X_0(1)(\overline{\mathbb{F}}_p)$ supersingular.

The points on C corresponding to a pair of geometrically isogenous elliptic curves must be supersingular in the first $X_0(1)$ and hence finitely many.

An application of Thm[MST]

A special case of the Hecke orbit conjecture by Chai–Oort.
Let $x \in \mathcal{M}(\overline{\mathbb{F}}_p)$ be an ordinary point. Let T_x denote the union of all prime-to-p Hecke orbits of x. Then $\overline{T_x}^{\mathrm{Zar}} = \mathcal{M}_{\overline{\mathbb{F}}_p}$.

Strategy. Induction on $\dim \mathcal{M}_{\mathbb{Q}}$.
(1) $\overline{T_x}^{\mathrm{Zar}} \not\subset \mathcal{Z}(m), \forall m$ and $\dim \overline{T_x}^{\mathrm{Zar}} \geqslant 1$, so we may pick $C \subset \overline{T_x}^{\mathrm{Zar}}$ such that $C[\mathrm{ord}] \neq \varnothing$ and $C \not\subset \mathcal{Z}(m), \forall m$.
(2) $\forall y \in C$ ord., $\overline{T_y}^{\mathrm{Zar}} \subset \overline{T_x}^{\mathrm{Zar}}$ b/c $\overline{T_x}^{\mathrm{Zar}}$ is stable under any prime-to-p Hecke translation. So it suffices to prove $\overline{T_y}^{\mathrm{Zar}} = \mathcal{M}_{\overline{\mathbb{F}}_p}$.
(3) If we may choose C to be proj., Thm[MST] $\implies \exists y \in C$ ord. s.t. $y \in \mathcal{Z}(m)(\overline{\mathbb{F}}_p)$ for some $p \nmid m$. The induction hypothesis $\implies \overline{T_y}^{\mathrm{Zar}} \supset \mathcal{Z}(m)_{\overline{\mathbb{F}}_p}$.
(4) Conclude by the fact that the Zariski closure of all prime-to-p Hecke orbits of $\mathcal{Z}(m)_{\overline{\mathbb{F}}_p}$ is $\mathcal{M}_{\overline{\mathbb{F}}_p}$.

On the existence of projective $C \subset \overline{T_x}^{\mathrm{Zar}}$

Let $\mathcal{M}^{\mathrm{BB}}_{\bar{\mathbb{F}}_p}$ denote the Bailey–Borel compactification of $\mathcal{M}_{\bar{\mathbb{F}}_p}$ (constructed by Madapusi Pera). The boundary $\mathcal{M}^{\mathrm{BB}}_{\bar{\mathbb{F}}_p} \backslash \mathcal{M}_{\bar{\mathbb{F}}_p}$ consists of 0-dimensional and 1-dimensional cusps.

1. If the closure of $\overline{T_x}^{\mathrm{Zar}}$ in $\mathcal{M}^{\mathrm{BB}}_{\bar{\mathbb{F}}_p}$ doesn't hit the boundary, then any C we pick is projective.
2. If it hits the 0-dimensional cusp, then an argument along the same line as Chai's proof of the Hecke orbit conjecture for $\mathcal{A}_g$ when the Zariski closure of Hecke orbits hits a totally degenerated point applies to our case.
3. If the closure of $\overline{T_x}^{\mathrm{Zar}}$ hits some 1-dimensional cusp, we use a toroidal compactification and the Hecke action on the formal neighborhood of the 1-dimensional cusp to either prove the theorem directly or construct a projective curve C.

Strategy of the proof of Thm[MST]

Recall that $C \subset \mathcal{M}_{\bar{\mathbb{F}}_p}$ and $Z(m) := \mathcal{Z}(m)_{\mathbb{F}_p}$ special divisors.

Goal: as $X \to \infty$,

1. For $P \in C(\bar{\mathbb{F}}_p)$ not supersingular,

$$\sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} (C.Z(m))_P = o\Big(\sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} (C.Z(m)) \Big);$$

2. There exists an absolute constant $0 < \alpha < 1$ such that

$$\sum_{\substack{P \in C(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} \sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} (C.Z(m))_P \leqslant \alpha \sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} C.Z(m) + o\Big(\sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} (C.Z(m)) \Big).$$

1. + 2. $\implies$ infinitely many $\bar{\mathbb{F}}_p$-points on $C \cap (\cup_{p \nmid m} Z(m))$.

Asymptotic of $C.Z(m)$

By Borcherds theory (or the arith. version Howard–Madapusi-Pera), the generating series $-(C.\omega) + \sum_{m=1}^{\infty} (C.Z(m))q^m$ is a (part of a vector-valued) non-cuspidal modular form of $\mathrm{Mp}_2(\mathbb{Z})$ of weight $(b+2)/2$, where ω is the line bundle of modular forms of weight 1 (corresponding to $\mathrm{Fil}^1 V \subset V$).

Decompose into Eisenstein + cuspidal;

For $b \geqslant 3$, m-th Fourier coefficients for Eis. (if not zero) $\asymp m^{b/2}$;

trivial bound for m-th Fourier coefficients for cusp. $= O(m^{(b+2)/4})$.

Hence $C.Z(m) \asymp m^{b/2}$, $\displaystyle\sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} (C.Z(m)) \asymp X^{1+b/2}$.

Local intersection number at non-ss points

For any $P \in (C \cap Z(m))(\bar{\mathbb{F}}_p)$,
let t be a local coordinate (i.e., $\widehat{C}_P = \operatorname{Spf} \bar{\mathbb{F}}_p[[t]]$) and
let L_n denote the lattice of special endomorphisms of the pullback of $\mathcal{A}^{\mathrm{univ}}$ to $\bar{\mathbb{F}}_p[t]/t^n$.

These L_n are equipped with positive definite quadratic forms Q compatible with $L_n \subset L_{n-1}$. By the moduli interpretation of $Z(m)$,

$$(C.Z(m))_P = \sum_{n=1}^{\infty} \#\{v \in L_n \colon Q(v) = m\}.$$

This formula + asymptotic of $C.Z(m)$ implies

Lemma. Let $a_n := min_{0 \neq v \in L_n} Q(v)^{1/2}$. Then $a_n \gg n^{1/b}$.

Lemma + a geom-of-numbers argument $\implies$
$\sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} (C.Z(m))_P = o\Big(\sum_{\substack{X \leqslant m \leqslant 2X \\ p \nmid m}} C.Z(m) \Big)$ b/c $\operatorname{rk} L_n \leqslant b$ (nonss).

Local intersection number at supersingular points

Apply the previous Lemma on a_n + a geom-of-numbers argument to supersingular P (equivalently, $\operatorname{rk} L_n = b+2$), we reduce Goal 2 to that there exists an absolute constant $\alpha' < 1$ such that

$$\sum_{\substack{P \in C(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} (C.Z(m))_P^{\mathrm{main}} \leqslant \alpha' C.Z(m) + o(m^{b/2}),$$

where $(C.Z(m))_P^{\mathrm{main}} = \sum_{n=1}^{N} \#\{v \in L_{n,P} \colon Q(v) = m\}$; here $N \gg 1$ is a large constant (only depending on $\alpha, \alpha', (L, Q)$).

Thus the LHS is a finite sum of theta series attached to $L_{n,P}$ for $1 \leqslant n \leqslant N, P \in C(\bar{\mathbb{F}}_p)$ supersingular. Using the trivial bound on Fourier coefficients of cusp forms, we reduce the desired inequality into a comparison of Eisenstein series $E_{n,P}$ attached to these theta series and the Eisenstein series E attached to

$$-(C.\omega) + \sum_{m=1}^{\infty} (C.Z(m)) q^m.$$

Comparing the Eisenstein series

Bruinier–Kuss + Siegel mass formula give explicit formula for the Fourier coefficients of these Eisenstein series in terms of local density of the lattices $L, L_{n,P}$ and also their discriminants.

Howard–Pappas $L_{n,P} \otimes \mathbb{Q}_\ell \cong L \otimes \mathbb{Q}_\ell$ for all $\ell \neq p$ and their work and Ogus's work also give explicit classification of $L_{1,P} \otimes \mathbb{Z}_p$ (the lattice of special endomorphisms of the supersingular point P).

Thus by a standard estimate of local density following Hanke, we obtain

$$\sum_{\substack{P \in C(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} \sum_{n=1}^{N} q_{E_{n,P}}(m) \leqslant \alpha' q_E(m),$$

once we have a good enough lower bound of $\operatorname{disc} L_{n,P}$, where $q_G(m)$ denotes the m-th Fourier coefficient of a modular form G.

A lower bound of disc L_n

The number field case. Let $\mathcal{A}$ denote the Kuga–Satake abelian scheme over $\mathcal{Y}$ and let $\mathfrak{p} \mid p$ be a finite place of $\mathcal{O}_K$ unram in $K/\mathbb{Q}$; Let Λ denote the $\mathbb{Z}_p$-lattice of the special endomorphisms of the p-divisible group $\mathcal{A}[p^\infty]$ over $\mathcal{O}_{K_{\mathfrak{p}}^{nr}}$; note that $\operatorname{rk}\Lambda \leqslant b$.

Grothendieck–Messing $\Rightarrow L_{n_0+k} = (\Lambda + p^k L_{n_0} \otimes \mathbb{Z}_p) \cap L_{n_0}, n_0 \gg 1$. In particular, $(\operatorname{disc} L_n)^{1/2} \gg p^{2n}$.

The global function field case.
Decay Lemma. There exists a rank 2 saturated $\mathbb{Z}_p$-submodule of $L_{1,P} \otimes \mathbb{Z}_p$ such that for each primitive w in this submodule, for any $r \geqslant 0$, the special endomorphism $p^r w$ does not lift to an end of $\mathcal{A}[p^\infty]$ mod t^{h_r+1}, where $h_r = [h(p^r + \cdots + 1 + 1/p)]$ and h is the t-adic valuation of the Hasse inv restricted to $k[[t]]$.

Supersingular vs superspecial

Let E_0 denote the Eisenstein series given by $(C.\omega)^{-1}E$ (i.e., constant term -1). Then Bruinier–Kuss implies that $\frac{q_{E_{n,P}}(m)}{q_{E_0}(m)} \approx (\mathrm{disc}_p\, L_{n,P})^{-1/2}$, and for $n = 1$, this ratio is p^{-1} for P superspecial and $\leqslant p^{-2}$ for P supersingular but not superspecial.

By the Decay Lemma above, the first h_P lattices $L_{n,P}$ has the same p-adic disc as $L_{1,P}$ and thus if all non-ordinary points on C are superspecial (worst scenario), then

$$\sum_{\substack{P\in C(\bar{\mathbb{F}}_p)\\ \text{supersingular}}} \sum_{n=1}^{h_P} \frac{q_{E_{n,P}}(m)}{q_{E_0}(m)} \approx \sum_{\substack{P\in C(\bar{\mathbb{F}}_p)\\ \text{supersingular}}} p^{-1}h_P \approx C.\omega,$$

where we use the fact that the global Hasse inv is a section of ω^{p-1}.

A stronger version of Decay Lemma for superspecial point

Decay Lemma. If P is superspecial, then there exists a primitive vector $w \in L_{1,P} \otimes \mathbb{Z}_p$ such that w does not lift to an endomorphism of $\mathcal{A}[p^\infty]$ mod $t^{[a+a/p]+1}$ for some $a \leqslant h_P/2$.

This $1/2$-factor ensure that for $p \gg_\epsilon 1$,

$$\sum_{\substack{P \in C(\bar{\mathbb{F}}_p) \\ \text{supersingular}}} \sum_{n=1}^{N} q_{E_{n,P}}(m) \leqslant (\frac{1}{2} + \epsilon) q_E(m).$$

Heuristic reason: the superspecial points are the singular points in the non-ordinary locus.

Ingredients in the proof of the Decay Lemmas

de Jong: a Dieudonne theory for p-divisible groups over rings in char p; our decay result is amount to study horizontal sections of the F-crystal $\mathbb{L}_{\mathrm{cris}}$, the crystalline realization of $L \subset C(L)$.

Ogus/Howard–Pappas + Kisin: explicit description of $\mathbb{L}_{\mathrm{cris}}$ on $\widehat{\mathcal{M}}_P$ for supersingular points P.

Kisin (Dwork's trick): the horizontal sections are given by $\lim_{n\to\infty} F^n(v_0)$, where $v_0 \in \mathbb{L}_{\mathrm{cris},P}(W)$ is Frob invariant and F is the semi-linear Frobenius on $\mathbb{L}_{\mathrm{cris}}$.

Our result follows from an explicit computation of the product F^n.

Thank you for your attention!