Evaluating the wild Brauer group

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Local-global principles

This talk is about joint work with **Martin Bright**.

Set-up:

- L number field
- X/L smooth, projective, geometrically irreducible variety

$$X(L) \subset X(\mathbb{A}_L) = \prod_{v \text{ place of } L} X(L_v)$$

so

$$X(L) \neq \emptyset \Longrightarrow X(\mathbb{A}_L) \neq \emptyset.$$

- If ' $\Longleftarrow$ ' holds in a family of varieties, we say the Hasse principle holds for that family.
- If $\overline{X(L)} = X(\mathbb{A}_L)$, we say weak approximation holds.

Brauer–Manin obstruction

Definition 1

The Brauer group of X is $\mathrm{Br}\, X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.

Let $P_v \in X(L_v)$. Then evaluation at P_v gives a map

$$\begin{aligned} \mathrm{Br}\, X &\to \mathrm{Br}\, L_v \\ \mathcal{A} &\mapsto \mathcal{A}(P_v). \end{aligned}$$

For finite v, the Hasse invariant gives a canonical isomorphism

$$\mathrm{inv}_v : \mathrm{Br}\, L_v \to \mathbb{Q}/\mathbb{Z}.$$

We also have $\mathrm{Br}\, \mathbb{C} = 0$ and $\mathrm{Br}\, \mathbb{R} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Brauer–Manin obstruction

Theorem 2 (Manin, 1970)

Summing up the invariants gives a pairing

$$X(\mathbb{A}_L) \times \operatorname{Br} X \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$$((P_v)_v \ , \ \mathcal{A}) \longmapsto \textstyle\sum_v \operatorname{inv}_v \mathcal{A}(P_v)$$

such that $\overline{X(L)} \subset X(\mathbb{A}_L)^{\operatorname{Br} X}$ *:= adelic points orthogonal to* $\operatorname{Br} X$*.*

- If $X(\mathbb{A}_L) \neq \emptyset$ but $X(\mathbb{A}_L)^{\operatorname{Br} X} = \emptyset$ then $X(L) = \emptyset$ and we say there's a Brauer–Manin obstruction to the Hasse principle.
- If $X(\mathbb{A}_L)^{\operatorname{Br} X} \neq X(\mathbb{A}_L)$ then $\overline{X(L)} \neq X(\mathbb{A}_L)$ and we say there's a Brauer–Manin obstruction to weak approximation.

Brauer–Manin obstruction

Suppose $X(\mathbb{A}_L) \neq \emptyset$.

Let $X(\mathbb{A}_L)^{\mathcal{A}}$ denote the set of adelic points orthogonal to $\mathcal{A} \in \operatorname{Br} X[n]$.

Observations:

- If $|\mathcal{A}| : X(L_v) \to \operatorname{Br} L_v[n]$ is non-constant for some v then $X(\mathbb{A}_L)^{\mathcal{A}} \neq X(\mathbb{A}_L)$, i.e. **$\mathcal{A}$ obstructs weak approximation.**

  Proof: Let $(P_w)_w \in X(\mathbb{A}_L)$. If $\sum_w \operatorname{inv}_w \mathcal{A}(P_w) = 0$ then replace P_v with some Q_v such that $\operatorname{inv}_v \mathcal{A}(Q_v) \neq \operatorname{inv}_v \mathcal{A}(P_v)$. $\square$

- If $|\mathcal{A}| : X(L_v) \to \operatorname{Br} L_v[n] = \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is surjective for some finite v then $X(\mathbb{A}_L)^{\mathcal{A}} \neq \emptyset$, i.e. **$\mathcal{A}$ does not obstruct the Hasse principle.**

Which primes can be involved in the Brauer–Manin obstruction?

Notation and assumptions:

- S set containing the Archimedean primes of L and the primes of bad reduction for X
- $\operatorname{Pic} \bar{X}$ torsion-free

Question (Swinnerton-Dyer)

Is there an open and closed set $Z \subset \prod_{v \in S} X(L_v)$ such that

$$X(\mathbb{A}_L)^{\operatorname{Br} X} = Z \times \prod_{v \notin S} X(L_v)?$$

Are the evaluation maps $|\mathcal{A}| : X(L_v) \to \operatorname{Br} L_v$ constant for all $v \notin S$?

Does the Brauer–Manin obstruction involve only Archimedean primes and primes of bad reduction?

Which primes can be involved in the Brauer–Manin obstruction?

Why the assumption on $\operatorname{Pic}\bar{X}$?

Non-example

$E/\mathbb{Q}$ elliptic curve with $\#\text{Ш}(E) < \infty$ and $E(\mathbb{Q}) = \{\mathcal{O}_E\}$. Then

$$E(\mathbb{A}_\mathbb{Q})^{\operatorname{Br} E} = E(\mathbb{R})^0 \times \prod_p \{\mathcal{O}_E\}$$

where $E(\mathbb{R})^0$ is the connected component of the identity in $E(\mathbb{R})$.

Which primes can be involved in the Brauer–Manin obstruction?

Theorem 3 (Colliot-Thélène and Skorobogatov, 2013)

If $\operatorname{Pic}\bar{X}$ *is torsion free and* $\operatorname{Im}(\operatorname{Br}X \to \operatorname{Br}\bar{X})$ *is finite then the only primes that can play a rôle in the Brauer–Manin obstruction are:*

- *Archimedean primes;*
- *primes of bad reduction;*
- *primes dividing* $\#\operatorname{Im}(\operatorname{Br}X \to \operatorname{Br}\bar{X})$.

Which primes can be involved in the Brauer–Manin obstruction?

Theorem 4 (Bright–N., 2023)

If $\mathrm{H}^0(X, \Omega_X^2) \neq 0$ *then every prime of good ordinary reduction is involved in a Brauer–Manin obstruction over some finite extension of* L*.*

More precisely: Suppose $\mathrm{H}^0(X, \Omega_X^2) \neq 0$. Let $\mathfrak{p} \mid p$ be a prime of good ordinary reduction. Then $\exists\ L'/L$ finite, a prime $\mathfrak{p}' \mid \mathfrak{p}$, and $\mathcal{A} \in \operatorname{Br} X_{L'}\{p\}$ such that

$$|\mathcal{A}| : X(L'_{\mathfrak{p}'}) \to \operatorname{Br} L'_{\mathfrak{p}'}$$

is non-constant. In particular, $\mathcal{A}$ obstructs weak approximation on $X_{L'}$.

Which primes can be involved in the Brauer–Manin obstruction?

Theorem 5 (Margherita Pagano, 2022)

Let $X/\mathbb{Q}$ be given by

$$X : x^3y + y^3z + z^3w + w^3x + xyzw = 0$$

and let $\mathcal{A} = \left(\frac{z^3+w^2x+xyz}{x^3}, \frac{-z}{x}\right) \in \operatorname{Br} X$. *Then:*

- *2 is a prime of good reduction for X, and*
- $|\mathcal{A}| : X(\mathbb{Q}_2) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ *is not constant.*

Which primes can be involved in the Brauer–Manin obstruction?

Why assume $\mathrm{H}^0(X, \Omega^2_X) \neq 0$?

Hodge theory:

$$\mathrm{H}^2(X(\mathbb{C}), \mathbb{C}) = \mathrm{H}^{0,2}(X(\mathbb{C})) \oplus \mathrm{H}^{1,1}(X(\mathbb{C})) \oplus \mathrm{H}^{2,0}(X(\mathbb{C}))$$

where $\mathrm{H}^{p,q}(X(\mathbb{C})) \cong \mathrm{H}^q(X_{\mathbb{C}}, \Omega^p)$.

Let $b_2 = \dim \mathrm{H}^2(X(\mathbb{C}), \mathbb{C})$ and $\rho = \operatorname{rk} \mathrm{NS}\, X_{\mathbb{C}} \leq \dim \mathrm{H}^{1,1}(X(\mathbb{C}))$.

Then $\mathrm{H}^0(X, \Omega^2_X) \neq 0 \implies b_2 - \rho > 0$.

Which primes can be involved in the Brauer–Manin obstruction?

Grothendieck: $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \hookrightarrow \operatorname{Br} \bar{X}$.

So $\exists L'/L$ finite such that the transcendental Brauer group $\operatorname{Im}(\operatorname{Br} X_{L'} \to \operatorname{Br} \bar{X})$ is non-trivial.

On the other hand, if the transcendental Brauer group is trivial, then Theorem 3 (Colliot-Thélène and Skorobogatov) shows that the answer to Swinnerton-Dyer's question is yes.

Which primes can be involved in the Brauer–Manin obstruction?

Question

Suppose $\operatorname{Pic} \bar{X}$ is torsion-free. Is there a finite set S of primes that can be involved in the Brauer–Manin obstruction for X/L? Can we describe S?

Which primes can be involved in the Brauer–Manin obstruction?

Theorem 6 (Bright–N., 2023)

Assume $\operatorname{Pic}\bar{X}$ *is torsion-free. Then* $\exists$ *a finite set of primes* S *such that, for all* $\mathcal{A} \in \operatorname{Br} X$ *and all* $\mathfrak{p} \notin S$, $|\mathcal{A}| : X(L_\mathfrak{p}) \to \operatorname{Br} L_\mathfrak{p}$ *is constant. The set* S *can be taken to consist of:*

1. *Archimedean primes;*
2. *primes of bad reduction;*
3. *primes* $\mathfrak{p}$ *satisfying* $e_\mathfrak{p} \geq p - 1$, *where* $\mathfrak{p} \mid p$ *and* $e_\mathfrak{p}$ *is the absolute ramification index;*
4. *primes* $\mathfrak{p}$ *for which* $\mathrm{H}^0(\mathcal{X}(\mathfrak{p}), \Omega^1) \neq 0$, *where* $\mathcal{X}(\mathfrak{p})$ *is the special fibre.*

Which primes can be involved in the Brauer–Manin obstruction?

Corollary 7

Let $X/\mathbb{Q}$ be a K3 surface. Let $S = \{\infty, 2\} \cup \{\textit{primes of bad reduction}\}$. Then, for all $\mathcal{A} \in \mathrm{Br}\, X$ and all $\mathfrak{p} \notin S$, $|\mathcal{A}| : X(L_\mathfrak{p}) \to \mathrm{Br}\, L_\mathfrak{p}$ is constant.

Proof.

For all odd primes p, $e_p = 1 < p - 1$.
If p is a good prime then the special fibre $\mathcal{X}(p)$ is also a K3 surface and hence $\mathrm{H}^0(\mathcal{X}(p), \Omega^1) = 0$. $\square$

Which primes can be involved in the Brauer–Manin obstruction?

Note: in Theorem 6 we do not require the transcendental Brauer group to be finite.

Question: Can the transcendental Brauer group of X be infinite?

Answer: No, if X is an abelian variety or K3 surface (Skorobogatov–Zarhin, 2008). Unknown in general.

The local picture

From now on:

- k p-adic field
- π uniformiser
- $\mathbb{F}$ residue field
- X/k smooth geometrically irreducible variety
- $\mathcal{X}/\mathcal{O}_k$ smooth model
- $Y = \mathcal{X} \times_{\mathcal{O}_k} \mathbb{F}$ special fibre, assumed geometrically irreducible

The evaluation filtration

Suppose $Q \in X(k)$ extends to $\mathcal{X}(\mathcal{O}_k)$ and let $Q_0 \in Y(\mathbb{F})$ be its reduction. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{A} & & \\ \downarrow & & \\ \mathcal{A}(Q) & & \end{array} \qquad \begin{tikzcd} \operatorname{Br} X(p') \arrow[r, "\partial"] \arrow[d, "Q^*"'] & \mathrm{H}^1(Y, \mathbb{Q}/\mathbb{Z})(p') \arrow[d, "Q_0^*"] \\ \operatorname{Br} k(p') \arrow[r, "\partial", "\cong"'] & \mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z})(p') \end{tikzcd}$$

where $\operatorname{Br} X(p')$ denotes the prime-to-p torsion and ∂ denotes the residue map.

Key point: If $\mathcal{A} \in \operatorname{Br} X(p')$ then $\mathcal{A}(Q)$ only depends on Q_0.

This is **not true** for $\mathcal{A} \in \operatorname{Br} X[p]$.

The evaluation filtration

Classifying elements of $\operatorname{Br} X$ according to the π-adic accuracy needed to evaluate them yields a filtration on $\operatorname{Br} X$ called the evaluation filtration.

Definition 8 (Evaluation filtration)

For $n \geq -1$, $\mathrm{Ev}_n \operatorname{Br} X$ consists of elements whose evaluation factors through $\mathcal{X}(\mathcal{O}_k) \to \mathcal{X}(\mathcal{O}_k/\pi^{n+1})$.

Kato's filtration by Swan conductor

Now let $K = \mathsf{Frac}(\text{henselisation}(\mathcal{O}_{\mathcal{X},Y}))$.

$$\operatorname{Br} X \hookrightarrow \operatorname{Br}(k(X)) \to \operatorname{Br} K$$

Kato defined a filtration fil_n on $\operatorname{Br} K$ called the filtration by Swan conductor.

Kato's filtration by Swan conductor

Proposition 9 (Kato)

$$\mathrm{fil}_0 \operatorname{Br} K = \ker(\operatorname{Br} K \to \operatorname{Br} K^{nr}).$$

There is a residue map $\partial : \mathrm{fil}_0 \operatorname{Br} K[n] \to \mathrm{H}^1_{\text{ét}}(F, \mathbb{Z}/n\mathbb{Z})$, where F is the residue field of K, which is the function field of Y.

The refined Swan conductor

Theorem 10 (Kato)

For $n \geq 1$, $\exists$ an injective homomorphism

$$\begin{aligned} \mathrm{rsw}_n : \frac{\mathrm{fil}_n \operatorname{Br} K}{\mathrm{fil}_{n-1} \operatorname{Br} K} &\hookrightarrow \Omega^2_F \oplus \Omega^1_F \\ \mathcal{A} &\mapsto (\alpha, \beta) =: \mathrm{rsw}_n(\mathcal{A}) \quad \text{"refined Swan conductor".} \end{aligned}$$

The refined Swan conductor

If $\zeta_p \in K$ then $\mathrm{fil}_n \operatorname{Br} K[p]$ is generated by cyclic algebras of the form

$$(1 + x\pi^{e'-n}, y)_p$$

where $x \in \mathcal{O}_K, y \in K^\times$ and $e' = \frac{pe}{p-1}$ where $e = \mathrm{ord}_K(p)$.

In this case for $0 < n < e'$ and $p \nmid n$,

$$(1 + x\pi^{e'-n}, y)_p \longleftarrow\!\shortmid \bar{x}\frac{d\bar{y}}{\bar{y}}$$

$$\frac{\mathrm{fil}_n \operatorname{Br} K}{\mathrm{fil}_{n-1} \operatorname{Br} K} \xleftarrow{\ \cong\ } \Omega^1_F \xrightarrow{\mathrm{rsw}_n} \Omega^2_F \oplus \Omega^1_F$$

$$\gamma \longmapsto (d\gamma, n\gamma).$$

Comparison of filtrations

Recall:

$$\mathrm{Br}\, X \hookrightarrow \mathrm{Br}(k(X)) \to \mathrm{Br}\, K$$

so fil_n gives a filtration on $\mathrm{Br}\, X$.

Theorem 11 (Bright–N., 2023)

- *For* $n \geq 1$, $\mathrm{Ev}_n \mathrm{Br}\, X = \{\mathcal{A} \in \mathrm{fil}_{n+1} \mathrm{Br}\, X \mid \mathrm{rsw}_{n+1}(\mathcal{A}) \in \Omega^2_F \oplus 0\}$*;*
- $\mathrm{Ev}_0 \mathrm{Br}\, X = \mathrm{fil}_0 \mathrm{Br}\, X$*;*
- $\mathrm{Ev}_{-1} \mathrm{Br}\, X = \{\mathcal{A} \in \mathrm{fil}_0 \mathrm{Br}\, X \mid \partial\mathcal{A} \in \mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z})\}$.

Comparison of filtrations

Claim:

$\mathrm{fil}_n \operatorname{Br} X \subset \{\mathcal{A} \in \mathrm{fil}_{n+1} \operatorname{Br} X \mid \mathrm{rsw}_{n+1}(\mathcal{A}) \in \Omega^2_F \oplus 0\}$, with equality if $p \nmid n+1$.

Proof of Claim:

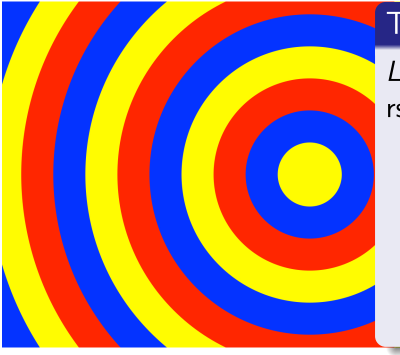
$$\mathrm{rsw}_{n+1} : \frac{\mathrm{fil}_{n+1} \operatorname{Br} X}{\mathrm{fil}_n \operatorname{Br} X} \hookrightarrow \Omega^2_F \oplus \Omega^1_F.$$

- Let $\mathcal{A} \in \mathrm{fil}_{n+1} \operatorname{Br} X$. Then $\mathcal{A} \in \mathrm{fil}_n \operatorname{Br} X \iff \mathrm{rsw}_{n+1}(\mathcal{A}) = (0,0)$. So $\mathrm{fil}_n \operatorname{Br} X \subset \{\mathcal{A} \in \mathrm{fil}_{n+1} \operatorname{Br} X \mid \mathrm{rsw}_{n+1}(\mathcal{A}) \in \Omega^2_F \oplus 0\}$.
- Let $\mathrm{rsw}_{n+1}(\mathcal{A}) = (\alpha, \beta)$. If $p \nmid n+1$ then $\beta = 0 \implies \alpha = 0$. $\square$

Variation of wild evaluation maps on p-adic discs

Definition 12

For $P \in \mathcal{X}(\mathcal{O}_k)$, let $B(P,n) = \{Q \in \mathcal{X}(\mathcal{O}_k) \mid Q \equiv P \pmod{\pi^n}\}$.



Theorem 13 (Bright–N., 2023)

Let $P \in \mathcal{X}(\mathcal{O}_k)$. Let $\mathcal{A} \in \mathrm{fil}_n \operatorname{Br} X$ with $\mathrm{rsw}_n(\mathcal{A}) = (\alpha, \beta) \in \Omega^2_F \oplus \Omega^1_F$. Then:

- *$|\mathcal{A}|$ varies linearly on $B(P,n)$, controlled by β_{P_0};*
- *if $\beta = 0$ then $|\mathcal{A}|$ is constant on $B(P,n)$ and varies quadratically on larger discs, controlled by α_{P_0}.*

Variation of wild evaluation maps on p-adic discs

More precisely, for

$$B(P,n) \ni Q = P + \pi^n \underline{v}$$

write

$$[\overrightarrow{PQ}]_n := \underline{v} \pmod{\pi} \in T_{P_0} Y.$$

Then for $Q \in B(P,n)$,

$$\operatorname{inv} \mathcal{A}(Q) = \operatorname{inv} \mathcal{A}(P) + \operatorname{Tr}_{\mathbb{F}/\mathbb{F}_p} \beta_{P_0}([\overrightarrow{PQ}]_n).$$

In particular, if $\beta_{P_0} \neq 0$, and $\mathcal{A} \in \operatorname{Br} X[p]$ then $|\mathcal{A}|$ maps $B(P,n)$ surjectively to $\operatorname{Br} k[p]$.

Applications to the Brauer–Manin obstruction

Recall:

Theorem 6 (Bright–N., 2023)

Assume $\operatorname{Pic} \bar{X}$ *is finitely generated and torsion-free. Then* $\exists$ *a finite set of places* S *such that, for all* $\mathcal{A} \in \operatorname{Br} X$ *and all* $\mathfrak{p} \notin S$, $|\mathcal{A}| : X(L_{\mathfrak{p}}) \to \operatorname{Br} L_{\mathfrak{p}}$ *is constant. The set* S *can be taken to consist of:*

1. *Archimedean places;*
2. *places of bad reduction ;*
3. *places* $\mathfrak{p}$ *satisfying* $e_{\mathfrak{p}} \geq p - 1$, *where* $\mathfrak{p} \mid p$ *and* $e_{\mathfrak{p}}$ *is the absolute ramification index;*
4. *places* $\mathfrak{p}$ *for which* $\mathrm{H}^0(\mathcal{X}(\mathfrak{p}), \Omega^1) \neq 0$, *where* $\mathcal{X}(\mathfrak{p})$ *is the special fibre.*

Idea of the proof of Theorem 6

Let $\mathfrak{p} \notin S$. Let $X_{\mathfrak{p}} = X \times_L L_{\mathfrak{p}}$.

Aim: show that $\operatorname{Br} X_{\mathfrak{p}} = \operatorname{Ev}_{-1} \operatorname{Br} X_{\mathfrak{p}}$.

Step 1: show that $\operatorname{Br} X_{\mathfrak{p}} = \operatorname{fil}_0 \operatorname{Br} X_{\mathfrak{p}}$.

Step 2: show that $\operatorname{fil}_0 \operatorname{Br} X_{\mathfrak{p}} = \operatorname{Ev}_{-1} \operatorname{Br} X_{\mathfrak{p}}$.

Idea of the proof of Theorem 6

Step 1: show that $\mathrm{Br}\, X_{\mathfrak{p}} = \mathrm{fil}_0 \,\mathrm{Br}\, X_{\mathfrak{p}}$.

Sketch of Step 1: Let $\mathcal{A} \in \mathrm{fil}_n \mathrm{Br}\, X_{\mathfrak{p}}$, $n \geq 1$. Write $\mathrm{rsw}_n(\mathcal{A}) = (\alpha, \beta)$. Need to show $\mathrm{rsw}_n(\mathcal{A}) = (0, 0)$.

Fact: $\beta \in \mathrm{H}^0(\mathcal{X}(\mathfrak{p}), \Omega^1)$.

- Now $\mathrm{H}^0(\mathcal{X}(\mathfrak{p}), \Omega^1) = 0$, since $\mathfrak{p} \notin S$. So $\beta = 0$. Remains to show $\alpha = 0$.
- If $p \nmid n$ then $\beta = 0 \implies \alpha = 0$. ☺
- Since $\mathfrak{p} \notin S$, we have $e < p - 1 \implies e' = \frac{pe}{p-1} < p$. Thus, $p \nmid n \;\forall n \leq e'$.
- Remaining case: $n > e'$ and $p \mid n$. In this case, $p\mathcal{A} \in \mathrm{fil}_{n-e} \mathrm{Br}\, X_{\mathfrak{p}}$ and $\mathrm{rsw}_{n-e}(p\mathcal{A}) = (\frac{p}{\pi^e}\alpha, \frac{p}{\pi^e}\beta)$. Since $p \nmid n - e$ and $\beta = 0$, we get $\alpha = 0$. $\square$

Idea of the proof of Theorem 6

Step 2: show that $\mathrm{fil}_0 \operatorname{Br} X_{\mathfrak{p}} = \mathrm{Ev}_{-1} \operatorname{Br} X_{\mathfrak{p}}$.

Sketch of Step 2: Colliot-Thélène–Skorobogatov: $\operatorname{Br} X_{\mathfrak{p}}(p') \subset \mathrm{Ev}_{-1} \operatorname{Br} X_{\mathfrak{p}}$.

Remains to deal with $\mathcal{A} \in \mathrm{fil}_0 \operatorname{Br} X_{\mathfrak{p}}[p^r]$.

$|\mathcal{A}|$ factors through $\partial : \mathrm{fil}_0 \operatorname{Br} X_{\mathfrak{p}} \to \mathrm{H}^1(Y, \mathbb{Z}/p^r)$.

The Hochschild–Serre spectral sequence gives a short exact sequence

$$0 \to \mathrm{H}^1(\mathbb{F}, \mathbb{Z}/p^r) \to \mathrm{H}^1(Y, \mathbb{Z}/p^r) \to \mathrm{H}^1(\bar{Y}, \mathbb{Z}/p^r).$$

Since $\operatorname{Pic} \bar{X}$ is torsion-free, $\mathrm{H}^1(\bar{X}, \mathbb{Z}/p^r) = 0$.

With some work, this implies that $\mathrm{H}^1(\bar{Y}, \mathbb{Z}/p^r) = 0$ and hence $\mathrm{H}^1(\mathbb{F}, \mathbb{Z}/p^r) = \mathrm{H}^1(Y, \mathbb{Z}/p^r)$. $\square$