# Distinguishing Siegel eigenforms from Hecke eigenvalues


Arvind Kumar

(Indian Institute of Technology Jammu, India)


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# Outline

1. Modular forms
   - Multiplicity one results
2. Siegel modular forms
   - Satake parameters
   - Galois representations
3. Main results

   Determining Siegel eigenforms by
   - the eigenvalues
   - the signs of the eigenvalues

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex upper half-plane $\mathbb{H}$ via

$$\gamma z := \frac{az+b}{cz+d}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Consider a congruence subgroup of level $N$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

and a Dirichlet character $\chi$ modulo $N$.

**Definition** (Modular forms)

Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. Then $f \in M_k(N, \chi)$ if

- $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) =: f\mid_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \chi(d) f(z)$, for all $\gamma \in \Gamma_0(N)$.
- $f$ is holomorphic at the cusps of $\Gamma_0(N)$. In particular, $f(z) = \sum_{n \geq 0} a_f(n) q^n$.

Moreover, if $f$ also vanishes at the cusps of $\Gamma_0(N)$, we say $f \in S_k(N, \chi)$.

- $S_k(N, \chi)$ is a Hilbert space with respect to Petersson inner product.

**Examples:**

- Eisenstein series:

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \in M_k(1) \text{ for } k \geq 4 \qquad (q = e^{2\pi i z}).$$

- Ramanujan delta function:

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n \in S_{12}(1).$$

**Definition (Hecke algebra)**

The Hecke algebra $\mathcal{T}$ is generated by the $p$-th Hecke operator $T_p$ for all prime $p$, where

- $T_p f(z) = \dfrac{1}{p}\displaystyle\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + \chi(p)p^{k-1}f(pz);$
- $T_m T_n = \displaystyle\sum_{d|(m,n)} \chi(d)d^{k-1}T_{\frac{mn}{d^2}}.$

- $T_p : M_k(N,\chi) \longrightarrow M_k(N,\chi)$ and preserves the space of cusp forms.
- $f \in S_k(N,\chi)$ is called an eigenform if it is an eigenfunction for $T_p$ $(p \nmid N)$.
- $S_k^{\mathrm{new}}(N,\chi) \subset S_k(N,\chi)$ : have basis which are eigenfunctions for all $T_p$ called newforms.
- Let $f$ be a newform. Then
  - $a_f(1) \neq 0$ and so we can normalize the form $f$ so that $a_f(1) = 1$.
  - $n$-th eigenvalue of $f = a_f(n)$, $n$-th Fourier coefficient of $f$.

- For a newform $f$, the eigenvalues

$$\{a_f(n) : n \geq 1\} \text{ depend only on } \{a_f(p) : p \text{ prime}\}.$$

- Let $a_1(p)$ and $a_2(p)$ be the $p$th eigenvalues of newforms $f_1$ and $f_2$. Then

$$a_1(p) = a_2(p) \text{ for all primes } p \implies f_1 = f_2.$$

- $\Delta(z), \Delta(2z) \in S_{12}(2)$ are eigenforms with same $p$th eigenvalues for all $p \neq 2$ but these two are linearly independent.

In particular, eigenvalues indexed by all primes determine the newform.

**Question:** Find smaller sets of eigenvalues which determine a newform?

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**Multiplicity one theorem**

Let $f_1$ and $f_2$ be newforms. Then

$$a_1(p) = a_2(p) \textit{ for all but finitely many } p \implies f_1 = f_2.$$

- The multiplicity one theorem can be made stronger $a_1(p) = a_2(p)$ *for primes $p$ of density* $> 7/8 \implies f_1 = f_2$.

**Theorem** (Rajan, 1998)

Let $f_1$, $f_2$ be newforms and $f_1$ be a non-CM cusp form. Then

$$a_1(p) = a_2(p) \textit{ for primes } p \textit{ of positive density} \implies f_1 = f_2 \otimes \chi,$$

where $\chi$ is a Dirichlet character.

- The latter result is also valid for normalized eigenvalues. (Murty-Pujahari (2016), Rajan-Patankar (2018))

The real symplectic unimodular group of degree 2 is defined by

$$\mathrm{Sp}_4(\mathbb{R}) = \{M \in \mathrm{GL}_4(\mathbb{R}) : MJM^t = J\}, \quad J = \begin{pmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{pmatrix}.$$

The action of the group $\mathrm{Sp}_4(\mathbb{R})$ on the Siegel upper-half space

$$\mathbb{H}_2 := \{Z = X + iY \in M_2(\mathbb{C}) : Z = Z^t, Y > 0\}$$

is given by

$$MZ := (AZ+B)(CZ+D)^{-1},\ \textit{where } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}).$$

## Siegel modular forms, $M_k(\Gamma_2)$

A Siegel modular form of degree 2 and weight $k$ on $\Gamma_2 := \mathrm{Sp}_4(\mathbb{Z})$ is a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ such that

$$F(MZ) = \det(CZ+D)^k F(Z), \quad \text{for all } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2.$$

- Any such $F$ has a Fourier series representation of the form
$$F(Z) = \sum_{T \in \mathbb{T}} A(T) e^{2\pi i \mathrm{tr}(TZ)},$$
where $\mathbb{T}$ is the set of half-integral, positive semi-definite, symmetric $2 \times 2$ matrices.
- Moreover, if $A(T) = 0$ whenever $\det(T) = 0$, we say that $F \in S_k(\Gamma_2)$, the space of Siegel cusp forms of weight $k$.
- There is an algebra $\mathcal{T}_2$ of Hecke operators acting on the space $M_k(\Gamma_2)$ which preserves $S_k(\Gamma_2)$ and it is generated by $T_p$ and $T_{p^2}$.

**Siegel eigenform**

$F \in S_k(\Gamma_2)$ is called a *Siegel eigenform* if it is a common eigenfunction of all the Hecke operators.

- For even $k$, the *Saito-Kurokawa* conjecture asserts the existence of a lifting of any form in $M_{2k-2}(1)$ to a Siegel modular form in $M_k(\Gamma_2)$.

Let $\mu_F(n)$ be the $n$th eigenvalue of an eigenform $F \in S_k(\Gamma_2)$.

- The set $\{\mu_F(n) : n \geq 1\}$ is determined by $\{\mu_F(p), \mu_F(p^2) : p\ prime\}$.
- Unlike the case of modular eigenform, the relation between the Fourier coefficients and eigenvalues of a Siegel eigenform is very mysterious.
- In general, the set of Hecke eigenvalues does not determine a Siegel eigenform of higher level, i.e., a multiplicity one result is not true.

**Questions:**

1. Do the eigenvalues determine an eigenform?
2. If yes, what are the smaller sets of eigenvalues determining an eigenform?
3. Do the signs of the eigenvalues determine an eigenform?
4. Are there any relations among (sufficiently many) eigenvalues of two eigenforms?

For each $i = 1, 2$, let $\mu_i(n)$ be the $n$th eigenvalue of a Siegel eigenform $F_i$.

- $\lambda_i(n) = \mu_i(n)/n^{k_i - 3/2}$ : $n$th normalized eigenvalue of $F_i$.

**Theorem** (Schmidt, 2018)

$$\begin{cases} \lambda_1(p) = \lambda_2(p);\ \textit{and} \\ \lambda_1(p^2) = \lambda_2(p^2) \end{cases} \quad \textit{for all but finitely many } p \implies F_1 = cF_2.$$

**Theorem** (K., Meher, Shankhadhar, 2021)

1. $\lambda_1(p) = \lambda_2(p)$ *for positive density of primes* $p \implies F_1 = cF_2$.
2. $\mu_1(p) = \mu_2(p)$ *for positive density of primes* $p \implies F_1 = cF_2$.

Note: The eigenvalues of $T(p)$ are sufficient to determine a Siegel eigenform even though the local Hecke algebra at a prime $p$ is generated by the Hecke operators $T(p)$ and $T(p^2)$.

# Idea of the proof

$$S_k(\Gamma_2) = \text{Space of Saito-Kurokawa lifts} \bigoplus \text{Orthogonal complement}$$

$$F \in S_k(\Gamma_2); \mu_F(p) = a_f(p) + p^{k-1} + p^{k-2}$$

$$\Big\|$$

$$f \in S_{2k-2}(1); a_f(p)$$

(forms coming from modular forms)

**Case I:** Both $F_1$ and $F_2$ are Saito-Kurokawa lifts.

The result follows from a similar result for modular eigenforms due to Rajan.

**Case II:** $F_1$ is not a Saito-Kurokawa lift (we need to use certain properties of associated Galois representations).

$F \in S_k(\Gamma_2)$ : Siegel eigenform with eigenvalues $\mu_F(n)$.

$E$: the number field generated by $\mu_F(n)$ over $\mathbb{Q}$.

**Theorem (Galois representations attached to Siegel eigenforms)**

For any prime $\ell$ and any extension $\lambda$ of $\ell$ to $E$, there exists a continuous semisimple Galois representation

$$\rho_{F,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GSp}_4(\overline{E}_\lambda)$$

such that $\rho_{F,\lambda}$ is unramified outside $\ell$. Moreover, the characteristic polynomial of Frobenius at $p \neq \ell$ is

$$X^4 - \mu_F(p)X^3 + \left(\mu_F(p)^2 - \mu_F(p^2) - p^{2k-4}\right) X^2 - p^{2k-3}\mu_F(p)X + p^{4k-6}.$$

In fact, for all but finitely many primes $\ell$, $\rho_{F,\lambda}$ is valued in $\mathrm{GSp}_4(E_\lambda)$ and hence we can view $\rho_{F,\lambda}$ as a representation valued in $\mathrm{GSp}_4(\mathcal{O}_{E_\lambda})$.

- (Weissauer) If $F$ is a Saito-Kurokawa lift, then $\rho_{F,\lambda}$ is reducible.
- (Ramakrishnan) If $F$ is not a Saito-Kurokawa lift, then $\rho_{F,\lambda}$ is irreducible if $\ell > 4k - 5$.

It follows that, up to conjugation, the maximal possible image for $\rho_{F,\lambda}$ is

$$A_\lambda = \{\gamma \in \mathrm{GSp}_4(\mathcal{O}_{E_\lambda}) : \mathrm{sim}(\gamma) \in \mathbb{Z}_\ell^{\times\, 2k-3}\}.$$

- (Dieulefait, K. *et al.*) Let $F \in S_k(\Gamma_2)$ be a non Saito-Kurokawa Siegel eigenform. Then for all but finitely many $\lambda$, the image of $\rho_{F,\lambda}$ is $A_\lambda$.

$\Longrightarrow$ Zariski closure of the image of $\rho_{F,\lambda} = \mathrm{GSp}_4$, a connected algebraic group.

**Case II:** $F_1$ is not a Saito-Kurokawa lift.

Let $E$ be the number field generated by all the eigenvalues of $F_1$ and $F_2$.

Choose a large prime $\ell$ and $\lambda|\ell$ in $E$ for which the image of $\rho_{1,\lambda}$ is large.

- Our hypotheses + Rajan-Patanakar result $\implies$ $\rho_{1,\lambda} \simeq \rho_{2,\lambda} \otimes \chi$.

  $\chi$ is a Dirichlet character of a Galois group of some finite extension of $\mathbb{Q}$.

- $\rho_{1,\lambda}$ is irreducible $\implies$ $F_2$ is also non Saito-Kurokawa.

- $\rho_{1,\lambda}, \rho_{2,\lambda}$ unramified outside $\ell$ and $\chi$ is a Dirichlet character

$$\implies \chi \text{ is a trivial character and hence } \rho_{1,\lambda} \simeq \rho_{2,\lambda}.$$

Consequently, $k_1 = k_2$ and for all but finitely many primes $p$

$$\mu_1(p) = \mu_2(p) \;\text{ and }\; \mu_1(p^2) = \mu_2(p^2).$$

Invoking the multiplicity one theorem of Schmidt completes the proof.

# Determining eigenforms from the signs of eigenvalues

Let $F$ be an eigenform with $n$th normalized eigenvalue $\lambda_F(n) := \frac{\mu_F(n)}{n^{k-\frac{3}{2}}}$.
As $\lambda_F(n) \in \mathbb{R}$ and $\lambda_F(n) > 0$ if $F$ is a Saito-Kurokawa lift, it is natural to ask:

**Question:** To what extent the signs of the eigenvalues determine a non-Saito-Kurokawa Siegel eigenform uniquely?

For elliptic modular forms, this problem has been studied by

- Kowalski, Lau, Soundararajan and Wu (2010): The proof uses a very deep result of Ramakrishnan about Rankin–Selberg $L$-function.
- Matomäki (2012): refined the above result. She has proved that if $f_1$ and $f_2$ are non-CM elliptic newforms and

$$\mathrm{sign}(a_1(p)) = \mathrm{sign}(a_2(p))$$

for a set of primes $p$ of analytic density $> 19/25$, then $f_1 = f_2$.

Let $F_1$ and $F_2$ be distinct non Saito-Kurokawa Siegel eigenforms with $n$th normalized eigenvalues $\lambda_1(n)$ and $\lambda_2(n)$ resp.

**Question:** To what extent the signs of the eigenvalues determine a non-Saito-Kurokawa Siegel eigenform uniquely?

The question makes sense if

1. the sequence $\{\lambda_i(n)\}_{n\geq 1}$ has infinitely many sign changes; and
2. there exists $n_0$ such that $\mathrm{sign}(\lambda_1(n_0)) \neq \mathrm{sign}(\lambda_2(n_0))$.

**Theorem** (Kohnen, Pitale-Schmidt, Das-Kohnen): There exists a set of primes $p$ of density 1 such that $\{\lambda_i(p^r)\}_{r\geq 0}$ has infinitely many sign changes.

- There is no unconditional result towards simultaneous sign change in the literature (till now).

**Theorem** (K., Meher, Shankhadhar)

There exists a set of primes of density 1 such that for each prime $p$ in that set, the sequence $\{\lambda_1(p^r)\lambda_2(p^r)\}_{r\geq 0}$ has infinitely many sign changes.

We use properties of associated $L$-functions and Galois representations.

**Satake parameters:**

Let $F \in S_k(\Gamma_2)$ be an eigenform with eigenvalues $\mu_F(n)$ and $\lambda_F(n) := \frac{\mu_F(n)}{n^{k-3/2}}$.

- For any prime $p$, let $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ be the classical Satake $p$-parameters of $F$. Then $\alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p} = p^{2k-3}$.
- $\beta_{1,p} := \frac{\alpha_{0,p}}{p^{k-3/2}}, \quad \beta_{2,p} := \frac{\alpha_{0,p}\alpha_{1,p}}{p^{k-3/2}}, \quad \beta_{3,p} := \frac{\alpha_{0,p}\alpha_{2,p}}{p^{k-3/2}}, \quad \beta_{4,p} := \frac{\alpha_{0,p}\alpha_{1,p}\alpha_{2,p}}{p^{k-3/2}}.$
- $\beta_{1,p} = \beta_{4,p}^{-1}$ and $\beta_{2,p} = \beta_{3,p}^{-1}$.

We call $\beta_{i,p}$'s as the (normalized) Satake $p$-parameters of $F$.

The degree 4 spinor $L$-function attached to $F$ is defined by

$$L(s,F,spin) = \prod_{p \text{ prime}} L_p(s,F,spin) = \prod_{p \text{ prime}} \prod_{1\le i\le 4} (1-\beta_{i,p}p^{-s})^{-1}.$$

$$L(s,F,spin) = \zeta(2s+1)\sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s}.$$

Indeed, one can show that $p^{k-3/2}\beta_{i,p}$'s are the roots of the $p$-th Hecke polynomial

$$X^4 - \mu_F(p)X^3 + \left(\mu_F(p)^2 - \mu_F(p^2) - p^{2k-4}\right)X^2 - \mu_F(p)p^{2k-3}X + p^{4k-6}.$$

We now assume that the eigenform $F$ is a non-Saito-Kurokawa lift.

- The generalized Ramanujan conjecture proved by Weissauer asserts that for any prime $p$
$$|\beta_{i,p}| = 1 \quad \text{for all } 1\le i\le 4.$$
- This gives $|\lambda_F(p)| \le 4$ and $|\lambda_F(n)| \ll_\epsilon n^\epsilon$.

**Theorem** (K., Meher, Shankhadhar)

There exists a set of primes of density 1 such that for each prime $p$ in that set, the sequence $\{\lambda_1(p^r)\lambda_2(p^r)\}_{r\geq 0}$ has infinitely many sign changes.

**Proof:** Assume that the sequence $\{\lambda_1(p^r)\lambda_2(p^r)\}_{r\geq 0}$ has all but finitely many terms non-negative. So by Landau theorem, the Dirichlet series

$$\sum_{r=0}^{\infty} \frac{\lambda_1(p^r)\lambda_2(p^r)}{p^{rs}}$$

which converges on $\mathrm{Re}(s) > 0$ is either entire or has a pole at the real point on its line of convergence.

We show that for a set primes $p$ of density 1, this can not be true.

**Step 1:** For a prime $p$ and $s$ with $\mathrm{Re}(s) > 0$, we have

$$\sum_{r=0}^{\infty} \frac{\lambda_1(p^r)\lambda_2(p^r)}{p^{rs}} = g_p(p^{-s}) \prod_{1\le i,j\le 4} (1-\beta_{i,p}\delta_{j,p}p^{-s})^{-1}, \qquad (\star)$$

where $g_p(p^{-s})$ is a polynomial in $p^{-s}$ of degree at most 14.

**Step 2:** Since $|\beta_{i,p}| = 1 = |\delta_{j,p}|$ for $1 \le i, j \le 4$,

$$\prod_{1\le i,j\le 4} (1-\beta_{i,p}\delta_{j,p}p^{-s})^{-1} \text{ has 16 poles on } \mathrm{Re}(s) = 0.$$

$\implies$ The series on the left of $(\star)$ has at least two singularities on the line $\mathrm{Re}(s) = 0$ and so by Landau's theorem we deduce that this series has a singularity at $s = 0$. This gives

$$\prod_{1\le i,j\le 4} (1-\beta_{i,p}\delta_{j,p}) = 0 \implies \beta_{i,p} = \delta_{j,p}^{-1} = \delta_{j',p} \text{ for some } 1 \le i, j, j' \le 4.$$

**Step 3:** This can not be true for a set primes $p$ of density 1 because

**Lemma**

There exists a set of primes $\mathcal{P}$ of density one such that for any $p \in \mathcal{P}$, any two elements in $\{\beta_{1,p}, \beta_{2,p}, \beta_{3,p}, \beta_{4,p}, \delta_{1,p}, \delta_{2,p}, \delta_{3,p}, \delta_{4,p}\}$ are distinct.

**A special case of Rajan's result:**

- $K$ : finite extension of $\mathbb{Q}_\ell$; $\mathcal{G}$ : algebraic group.
- $R : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{G}(K)$ unramified outside a finite set of primes.
- $\mathcal{H}$ : Zariski closure of the image of $R$.
- $\mathcal{X}$ : subscheme of $\mathcal{G}$ and stable under the adjoint action of $\mathcal{G}$.

**Theorem** (Rajan, 1998)

Let $\mathcal{H}$ be connected and $\mathcal{H} \not\subset \mathcal{X}$. Then

the set of primes $p$ with $R(\mathrm{Frob}_p) \in \mathcal{X}(K) \cap \mathrm{Im}(R)$ has density 0.

**Preparation for the proof of the lemma:**

- $E$ : the number field generated by the eigenvalues of $F_1$ and $F_2$.
- $\lambda$ be prime in $E$ above a large $\ell$.
- $R_\lambda := \rho_{F,\lambda} \times \rho_{G,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathcal{G}(\mathcal{O}_{E_\lambda}), \quad \mathcal{G} := \mathrm{GSp}_4 \times \mathrm{GSp}_4.$

**Theorem** (K., Kumari, Weiss)

Let $F$ and $G$ be Siegel eigenforms such that $F$ is not a constant multiple of $G$. Then for all but finitely many primes $\ell$, the image of $\rho_{F,\lambda} \times \rho_{G,\lambda}$ is

$$\{(\gamma_1, \gamma_2) \in \mathrm{GSp}_4(\mathcal{O}_{E_\lambda}) \times \mathrm{GSp}_4(\mathcal{O}_{E_\lambda}) : \mathrm{sim}(\gamma_i) = v^{2k_i - 3}, v \in \mathbb{Z}_\ell^\times, 1 \le i \le 2\}.$$

It follows that $\mathcal{H} := \overline{\mathrm{Im}(R_\lambda)} = \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GSp}_4$, a connected algebraic group.

**Lemma**

There exists a set of primes of density one such that for any such prime $p$, any two elements in $\{\beta_{1,p}, \beta_{2,p}, \beta_{3,p}, \beta_{4,p}, \delta_{1,p}, \delta_{2,p}, \delta_{3,p}, \delta_{4,p}\}$ are distinct.

**Proof:** For any $p \neq \ell$, let $R_\lambda(\mathrm{Frob}_p) = (\gamma_1, \gamma_2) \in \mathcal{G}(E_\lambda)$. Then the eigenvalues of $\mathrm{sim}(\gamma_2)\gamma_1^2$ and $\mathrm{sim}(\gamma_1)\gamma_2^2$ are

$$p^{2(k_1+k_2-3)}\beta_{1,p}^2, \quad p^{2(k_1+k_2-3)}\beta_{2,p}^2, \quad p^{2(k_1+k_2-3)}\beta_{3,p}^2, \quad p^{2(k_1+k_2-3)}\beta_{4,p}^2;$$
$$p^{2(k_1+k_2-3)}\delta_{1,p}^2, \quad p^{2(k_1+k_2-3)}\delta_{2,p}^2, \quad p^{2(k_1+k_2-3)}\delta_{3,p}^2, \quad p^{2(k_1+k_2-3)}\delta_{4,p}^2.$$

$$\mathcal{X} = \{(\gamma_1, \gamma_2) \in \mathcal{G} : \mathrm{sim}(\gamma_2)\gamma_1^2 \text{ and } \mathrm{sim}(\gamma_1)\gamma_2^2 \text{ have repeated eigenvalues}\}.$$

**(1)** $\mathcal{X}$ is stable under the conjugate action of $\mathcal{G}$.

**(2)** $\mathcal{X}$ is a closed subscheme of $\mathcal{G}$

because

$$\mathcal{X} = \text{Vanishing set of the discriminant of the product of the charecteristic polynomials of } \mathrm{sim}(\gamma_2)\gamma_1^2 \text{ and } \mathrm{sim}(\gamma_1)\gamma_2^2.$$

Then $\mathcal{H} := \overline{\mathrm{Im}(R_\lambda)}$ is connected and $\mathcal{H} \not\subset \mathcal{X}$. Applying Rajan's result, we

**Question:** To what extent the signs of the eigenvalues determine a non-Saito-Kurokawa Siegel eigenform uniquely?

**Theorem** (K., Meher, Shankhadhar)

$F_1$, $F_2$: non-Saito-Kurokawa Siegel eigenforms with eigenvalues $\lambda_1(n)$, $\lambda_2(n)$.

$$\operatorname{sign}(\lambda_1(n)) = \operatorname{sign}(\lambda_2(n)) \text{ *for density 1 set of integers* } n \implies F_1 = cF_2.$$

**Proof:** Suppose $F_1$ is not a scalar multiple of $F_2$. It is sufficient to show that

$$\#\{n \le x : \lambda_1(n)\lambda_2(n) < 0\} \gg x.$$

Let $p_0$ be a prime such that $\lambda_1(p_0^t)\lambda_2(p_0^t) < 0$. Define

$$\mathcal{B} := \{p_0\} \cup \{p : p \neq p_0, \lambda_1(p)\lambda_2(p) = 0\} \cup \{p^2 : p \neq p_0, \lambda_1(p)\lambda_2(p) \neq 0\}.$$

- $\sum_{b \in \mathcal{B}} \frac{1}{b} < \infty$. Put $M = \prod_{b_i \in \mathcal{B}} \left(1 - \frac{1}{b_i}\right)$.
- $\mathcal{A}$ : The set of all $\mathcal{B}$-free numbers.
- $\#\{n \le x : \lambda_1(n)\lambda_2(n) < 0\} \ge \#\{n \le x/p_0^t : n \in \mathcal{A}\} \sim \frac{M}{p_0^t} x.$ □

# Thank You

The notion of $\mathcal{B}$-free numbers was first introduced by Erdös. These numbers are a certain generalization of squarefree numbers.

Let $\mathcal{B} = \{b_i : 1 < b_1 < b_2 < \cdots\}$ be a sequence of positive integers such that

$$\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty \quad and \quad \gcd(b_i, b_j) = 1 \quad for \quad i \neq j.$$

A positive integer $n$ is said to be $\mathcal{B}$-free if it is not divisible by $b_i$ for any $i \geq 1$.

Let $\mathcal{A}$ be the set of all $\mathcal{B}$-free numbers, then a result of Erdös states that

$$\#\{n \leq x : n \in \mathcal{A}\} \sim \delta x \quad \text{as} \quad x \to \infty, \quad \text{where} \quad \delta = \prod_{b_i \in \mathcal{B}} \left(1 - \frac{1}{b_i}\right).$$

Note that if we take $\mathcal{B}$ to be the sequence of squares of all primes, then the set of $\mathcal{B}$-free numbers is nothing but the set of all squarefree numbers.