Non-vanishing of Hilbert Poincaré series

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Outline

- Brief introduction of modular forms
- History of the problem
- Hilbert modular forms
- Main results and idea of the proof

- Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half-plane and

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\}.$$

- $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ by the fractional linear transformation

$$\gamma z := \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \;\; z \in \mathbb{H}.$$

- For $N \geq 1$, the congruence subgroup of level N is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Modular forms

Definition

A modular form of weight $k \in \mathbb{Z}$ for $\Gamma_0(N)$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies the following properties:

- $f(\gamma z) = (cz+d)^k f(z), \;$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$,
- f is holomorphic at each cusps of $\Gamma_0(N)$, in particular

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

Moreover, if f also vanishes at each cusps of $\Gamma_0(N)$, we say f is a cusp form.

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$M_k(N)$:= The space of modular forms of weight k for $\Gamma_0(N)$.

$S_k(N)$:= The space of cusp forms of weight k for $\Gamma_0(N)$.

Examples of modular forms

- **Eisenstein Series:** Let $k > 2$ be an even integer. Then

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z} \in M_k(1),$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is the k-th Bernoulli number.

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- **Ramanujan delta function**

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Lehmer's conjecture:

$$\tau(n) \neq 0, \text{ for all } n \geq 1.$$

Poincaré series

Let $m \geq 1$ and k be integers.

Definition (Poincaré series)

$$\mathcal{P}_{m,k,N}(z) = \sum_{M \in \Gamma_\infty \backslash \Gamma_0(N)} (cz+d)^{-k} e^{2\pi i m M z},$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma_0(N) \right\}$ and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

- $\mathcal{P}_{m,k,N}(z)$ is a cusp form of weight k for the group $\Gamma_0(N)$.
- If $p_{m,k,1}(n)$ is the n-th coefficient of $\mathcal{P}_{m,k,1}$, then

$$p_{m,k,1}(n) = \delta(m,n) + 2\pi(-1)^{k/2} \sum_{q=1}^{\infty} \frac{S(n,m,q)}{q} J_{k-1}\Big(\frac{4\pi\sqrt{nm}}{q}\Big).$$

Petersson inner product

- Let f and g be modular forms of weight k for $\Gamma_0(N)$ and assume that one of them is a cusp form. Then inner product of f and g is defined

$$\langle f, g\rangle = \frac{1}{[\Gamma_0(1) : \Gamma_0(N)]} \int_{\mathcal{F}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

- If $f \in S_k(N)$, then

$$\langle f, \mathcal{P}_{m,k,N}\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_m.$$

- For fixed k and N, $S_k(N) = \text{span}\ \langle \mathcal{P}_{m,k,N} : m \in \mathbb{N}\rangle$.

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- $\dim S_k(N) < \infty$.
- A finite dimensional vector space is spanned by infinitely many elements.

Questions

1. Find all the linear relations between $\mathcal{P}_{m,k,N}$.

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Questions

1. Find all the linear relations between $\mathcal{P}_{m,k,N}$.
2. Find all the basis of $S_k(N)$ consisting of $\mathcal{P}_{m,k,N}$.
3. When these functions are non-zero?

Conjecture

For $k = 12$ or $k > 14$ and $m \geq 1$,

$$\mathcal{P}_{m,k,1}(z) \neq 0.$$

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- For $k = 12$, it is equivalent to the famous Lehmer's conjecture which says

$$\tau(m) \neq 0, \qquad \text{for all } m \geq 1,$$

where τ is the Ramanujan τ-function.

$$\mathcal{P}_{m,12,1}(z) = \frac{2\pi\Gamma(11)\tau(m)}{(2m)^{11}\|\Delta\|^2}\ \Delta(z).$$

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Facts:

- $\dim S_k(1) = 0$ for $k = 4, 6, 8, 10, 14$ implies that for all $m \geq 1$

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$$\mathcal{P}_{m,k,1}(z) \equiv 0.$$

- $\mathcal{P}_{m,k,1}(z)$ are linearly independent for all $m \leq \dim S_k(1)$. In particular, $\mathcal{P}_{m,k,1}(z) \neq 0$ for all $m \leq \dim S_k(1) \sim k$.

Theorem (R. A. Rankin, 1980)

There exist positive constants $B > 4\log 2$ and k_0 such that, for all $k > k_0$ and for all positive integers

$$m \leq k^2 \exp\left(\frac{-B\log k}{\log\log k}\right),$$

the Poincaré series $\mathcal{P}_{m,k,1}(z)$ does not vanish identically.

- The constant k_0 is effective but difficult to compute explicitly.

Rankin uses

- The Fourier coefficients of elliptic Ponicaré series has an explicit formula as an infinite series involving Bessel functions and certain complicated number-theoretic sums (Kloosterman sums).
- Sharp estimates for the magnitude of Kloosterman sums and Bessel functions.

- J. Lehner and C. J. Mozzochi generalized the Rankin's result for an arbitrary Fuchsian group and for the congruence subgroups $\Gamma_0(N)$, respectively.

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- In 1986 E. Gaigalas, using Weil's estimate for the Kloosterman sum, proved the following.

Theorem (E. Gaigalas, 1986)

For any $m \in \mathbb{N}$ there exist infinitely many $k \in 2\mathbb{N}$ for which the m-th Poincaré series of weight k (with respect to any finite index subgroup of $SL_2(\mathbb{Z})$) is not identically zero.

Sketch of the proof

Let $p_{m,k,1}(n)$ be the n-th Fourier coefficient of the Poincaré series $\mathcal{P}_{m,k,1}(z)$.

- From inner product formula

$$\mathcal{P}_{m,k,1}(z) = 0 \Leftrightarrow p_{m,k,1}(m) = 0.$$

- Consider the series

$$\sum_{k=2}^{\infty} (-1)^{\frac{k}{2}} (k-1) m^{1-k} p_{m,k,1}(m).$$

- Use the explicit expression for $p_{m,k,1}(m)$ in terms of Kloosterman sum and Bessel function.

- Finally, use the Weil's estimate for Kloosterman sum and some identity for Bessel function to prove that the partial sum of the series goes to infinity. In other words, the series diverges.

Theorem (Das and Ganguly, 2013)

Given m and N with $(m, N) = 1$, there is an absolute constant $k_0 > 0$ such that if $K \gg k_0$, then

$$|\{k : k \equiv 0 \pmod 2, K \leq k \leq 2K, \mathcal{P}_{m,k,N} \neq 0\}| \gg \frac{K}{d(m)^2}.$$

They have used the similar tools as Rankin and Gaigalas, i.e., writing the explicit expression for the Fourier coefficients and used bounds for the Kloosterman sums and Bessel functions.

Problems

- Given m and N, does there exist a constant k_0 such that for all $k \geq k_0$

$$\mathcal{P}_{m,k,N}(z) \neq 0?$$

- Given m and k, does there exist N_0 such that for all $N \geq N_0$

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- More generally, analogous questions for Hilbert Poincaré series.

Hilbert modular forms

- Let F be a totally real number field of degree n over $\mathbb{Q}$ and $\mathcal{O}_F$ be its ring of algebraic integers.
- $\sigma_1, \sigma_2, ..., \sigma_n$ are its real embedding.
- Using these embeddings, we embed the field F in $\mathbb{R}^n$ as

$$\alpha \to \big(\sigma_1(\alpha), \sigma_2(\alpha), \cdots, \sigma_n(\alpha)\big).$$

- The Hilbert modular group

$$\Gamma_F = SL_2(\mathcal{O}_F) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathcal{O}_F \ \text{ and } \ \alpha\delta - \beta\gamma = 1 \right\}.$$

- For an integral ideal $\mathcal{I}$ of $\mathcal{O}_F$, the congruence subgroup of level $\mathcal{I}$ is

$$\Gamma_0(\mathcal{I}) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_F : \gamma \in \mathcal{I} \right\}.$$

Group action

- Recall, $GL_2^+(\mathbb{R})$ acts on $\mathbb{H} = \{x + iy : y > 0\}$ via,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

- Using the above action, we define an action of $GL_2^+(\mathbb{R})^n$ on $\mathbb{H}^n$ as follows:

$$gz = (g_1 z_1, g_2 z_2, \cdots, g_n z_n),$$

  where $g \in GL_2^+(\mathbb{R})^n$ and $z \in \mathbb{H}^n$.

- Hence the group $SL_2(\mathcal{O}_F)$ also acts on $\mathbb{H}^n$.

Hilbert modular forms

Let $\mathcal{I}$ be a non-zero integral ideal of $\mathcal{O}_F$ and $k = (k_1, k_2, \cdots, k_n) \in \mathbb{N}_0^n$.

Definition

A Hilbert modular form of weight k for the congruence subgroup $\Gamma_0(\mathcal{I})$ is a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ such that

- $f(\gamma z) = \prod_{j=1}^n (c_j z_j + d_j)^{k_j} f(z)$, for all $\gamma = \left(\begin{pmatrix} * & * \\ c_j & d_j \end{pmatrix} \right)_{1 \le j \le n} \in \Gamma_0(\mathcal{I})$.
- For $n = 1$, we also need holomorphicity condition at the cusps of $\Gamma_0(\mathcal{I})$.

- Moreover, if f vanishes at all the cusps of $\Gamma_0(\mathcal{I})$, we call f is a Hilbert cusp form.
- If the field F is $\mathbb{Q}$, then we have elliptic modular forms.

Koecher Principle

Let $f(z)$ is a Hilbert modular form of weight k for the group $\Gamma_0(\mathcal{I})$. By Koecher principle f has a Fourier expansion at the cusp ∞ of the form

$$f(z) = \sum_{\substack{m \in \mathcal{O}_F^* \\ m \succeq 0}} a_m e^{2\pi i \mathrm{tr}(mz)},$$

where $\mathcal{O}_F^* := \left\{\mu \in F : \mathrm{tr}(\mu \mathcal{O}_F) \subseteq \mathbb{Z}\right\}$ is the dual space of $\mathcal{O}_F$.

Hilbert Poincaré series

Let $\mathcal{I} \subseteq \mathcal{O}_F$ be a non-zero integral ideal and $\Gamma_0(\mathcal{I})$ be the associated congruence subgroup. For a totally positive element ν of $\mathcal{O}_F^*$ and weight $k = (k_1, k_2, \cdots, k_n)$ $(k_j > 2,\ j = 1, 2, \cdots, n)$, the ν-th Hilbert Poincaré series is defined as follows:

Definition

$$\mathcal{P}_{\nu,k,\mathcal{I}}(z) = \sum_{M \in \Gamma_\infty \backslash \Gamma_0(\mathcal{I})} \mu(M, z)^{-k} e^{2\pi i \mathrm{tr}(\nu(Mz))},$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} : \mu \in \mathcal{O}_F \right\}$ and for $M = \left(\begin{pmatrix} * & * \\ c_j & d_j \end{pmatrix} \right)_{1 \le j \le n}$

$$\mu(M, z)^k := \prod_{j=1}^{n} (c_j z_j + d_j)^{k_j}.$$

- $\mathcal{P}_{\nu,k,\mathcal{I}}(z)$ is a Hilbert cusp form of weight k for the group $\Gamma_0(\mathcal{I})$.

Non-vanishing of HPS

Theorem (-)

Let $\nu \in \mathcal{O}_F^*$. Let $\mathcal{P}_{\nu,\vec{k},\mathcal{I}}$ be the ν-th Hilbert Poincaré series of (parallel) weight $\vec{k}$ and level $\mathcal{I}$. Then there exists a positive constant k_0 such that for all $k > k_0$, we have

$$\mathcal{P}_{\nu,\vec{k},\mathcal{I}} \neq 0.$$

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Corollary

For any positive integer m, there exists a positive constant k_0 such that for all $k > k_0$, we have

$$\mathcal{P}_{m,k,N} \neq 0,$$

where $\mathcal{P}_{m,k,N}$ be the m-th elliptic Poincaré series of weight k and of level N.

Next, we prove the non-vanishing result with respect to other parameter, the level $\mathcal{I}$. Here we work with any weight $k \in \mathbb{N}_0^n$, not necessarily parallel weight.

Theorem (-)

Let $\nu \in \mathcal{O}_F^*$. Then for a fixed $2 < k \in \mathbb{N}^n$, there exists a positive constant n_0 such that

$$\mathcal{P}_{\nu,k,\mathcal{I}} \neq 0,$$

for all integral ideal $\mathcal{I}$ with $N(\mathcal{I}) \geq n_0$.

Corollary

For any positive integer m and for a fixed weight $k \in \mathbb{N}$, there exists a positive constant q_0 such that for all $N > N_0$, we have

$$\mathcal{P}_{m,k,N} \neq 0.$$

Idea of the proof (weight aspect)

- From inner product formula

$$\mathcal{P}_{\nu,\vec{k},\mathcal{I}}(z) \neq 0 \iff p_{\nu,\vec{k},\mathcal{I}}(\nu) \neq 0.$$

- Hence it suffices to show that $p_{\nu,\vec{k},\mathcal{I}}(\nu) \neq 0$ for sufficiently lagre k.

Theorem

$$\lim_{k\to\infty} p_{\nu,\vec{k},\mathcal{I}}(\mu) = \delta(\nu,\mu).$$

For $n = 1$ the analogous result has been proved dy Kowalski, Saha and Tsimerman.

- $$p_{\nu,\vec{k},\mathcal{I}}(\mu) = \mathrm{vol}(\Omega)^{-1} \int_{\Omega} \mathcal{P}_{\nu,\vec{k},\mathcal{I}}(z) e^{-2\pi i \mathrm{tr}(\mu z)} dz,$$

  where $\Omega = \{z = x + iy : x \in \mathcal{O}_F \setminus \mathbb{R}^n\}$, for a fixed but arbitrary $y \in \mathbb{R}^n_+$.

- Use dominated convergence theorem for the sequence $\{\mathcal{P}_{\nu,\vec{k},\mathcal{I}}\}_{k\in\mathbb{N}}$.

  (a) For all $k > 2$ and $z \in \Omega$, there exists an integrable function G on Ω such that

  $$|\mathcal{P}_{\vec{k},\nu,\mathcal{I}}(z)| \leq \sum_{M \in \Gamma_\infty \setminus \Gamma_0(\mathcal{I})} |\mu(M,z)|^{-3}.$$

  (b) For any $z \in \Omega$, we have

  $$\mathcal{P}_{\nu,\vec{k},\mathcal{I}}(z) \to e^{2\pi i \mathrm{tr}(\nu z)} \text{ as } k \to \infty.$$

- To prove the last assertion, we show that as $k \to \infty$ exactly one of the term in the series expansion of $\mathcal{P}_{\vec{k},\nu,\mathcal{I}}(z)$ converges to $e^{2\pi i \mathrm{tr}(\nu z)}$ and others (individually) tend to 0.
- Any term in the series expansion looks like $\mu(M,z)^{-\vec{k}} e^{2\pi i \mathrm{tr}(\nu(Mz))}$, for some $M = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma_\infty \setminus \Gamma_0(\mathcal{I})$.
- If $\gamma = 0$, we can choose $M = I_2$ and hence

$$\mu(M,z)^{-\vec{k}} e^{2\pi i \mathrm{tr}(\nu(Mz))} = e^{2\pi i \mathrm{tr}(\nu z)},$$

for all weight $\vec{k}$ and $z \in \mathbb{H}^n$.

- Let $\gamma \neq 0$. For $z = (z_1, z_2, \cdots, z_n) \in \mathbb{H}^n$ it is easy to see that $\mathrm{Im}(\mathrm{tr}(\nu(Mz))) > 0$, which gives

$$|\mu(M,z)^{-\vec{k}} e^{2\pi i \mathrm{tr}(\nu(Mz))}| \leq |\mu(M,z)^{-\vec{k}}|.$$

- Since $0 \neq \gamma \in \mathcal{O}_F$, i.e., $N(\gamma) \in \mathbb{Z}$, and hence $N(\gamma)^2 \geq 1$. Now

$$|\mu(M,z)|^2 = \prod_{j=1}^{n} |\gamma_j z_j + \delta_j|^2 \geq \prod_{j=1}^{n} (\gamma_j y_j)^2 = N(\gamma)^2 N(y)^2 \geq N(y)^2,$$

where $y_j = \mathrm{Im}(z_j)$ for $1 \leq j \leq n$.

- Combining the last two inequality, we get

$$|\mu(M,z)^{-\vec{k}} e^{2\pi i \mathrm{tr}(\nu(Mz))}| \leq N(y)^{-k}.$$

- Now we choose y such that $N(y) > 1$. Therefore, for any $z \in \Omega$ the above inequality yields

$$|\mu(M,z)^{-\vec{k}} e^{2\pi i \mathrm{tr}(\nu(Mz))}| \to 0 \quad \text{as} \quad k \to \infty.$$

THANK YOU