Rademacher symbols for Fuchsian groups

Dedekind

$$\eta(z) = q^{1/24} \prod_{n \geq 1} (1-q^n)$$

$$\log \eta(z) = \frac{\pi i z}{12} - \sum_{m,n>0} \frac{q^{mn}}{m}$$

$\Gamma = PSL_2\mathbb{Z}$ $\quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\log\eta(\gamma z) - \log\eta(z) = \frac{1}{2}(\operatorname{sign} c)^2 \log\left(\frac{cz+d}{i \operatorname{sign} c}\right) + \frac{i\pi}{12}\phi(\gamma)$$

Dedekind symbol

1. $\phi : PSL_2\mathbb{Z} \longrightarrow \mathbb{Z}$
2. $\phi(\gamma_1\gamma_2) - \phi(\gamma_1) - \phi(\gamma_2)$
$$= 3 \operatorname{sign}(c_{\gamma_1} c_{\gamma_2} c_{\gamma_1\gamma_2})$$

"quasimorphism"

$$\sup_{\gamma_1,\gamma_2 \in \Gamma} |\phi(\gamma_1\gamma_2) - \phi(\gamma_1) - \phi(\gamma_2)| < \infty$$

Rademacher symbol:

$$\Psi(\gamma) = \lim_{n\to\infty} \frac{\phi(\gamma^n)}{n}$$

(*) $\Psi(\gamma^n) = n \cdot \Psi(\gamma)$

Prop: $\Psi : PSL_2\mathbb{Z} \longrightarrow \mathbb{Z}$ is a homogeneous(*) conjugacy class invariant quasimorphism.

Proof: Let $a, b \in \Gamma$.

$$|\Psi(aba^{-1}) - \Psi(b)| \leq |\cancel{\Psi(a)} + \cancel{\Psi(b)} + \cancel{\Psi(a^{-1})} - \cancel{\Psi(b)}| + 2 \cdot 3$$

$$= \frac{1}{n} |\Psi((aba^{-1})^n) - \Psi(b^n)|$$

$$= \frac{1}{n} |\Psi(ab^n a^{-1}) - \Psi(b^n)| \leq \frac{6}{n}$$

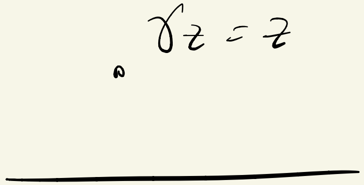
$$\xrightarrow[n\to\infty]{} 0$$

$\square$

What is Ψ?

Let γ be

... elliptic



$$\gamma^m = e$$

$$\Psi(\gamma^m) = m\Psi(\gamma) = \Psi(e) = 0$$

$$\Psi(\gamma) = 0$$

... parabolic

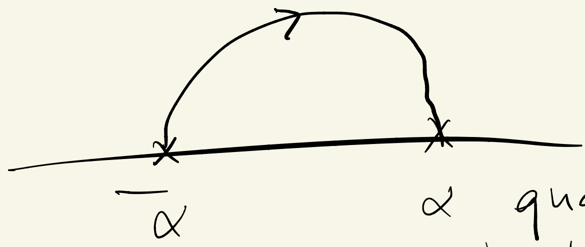
γ is conjugate to $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

$$\frac{\pi i}{12}\Phi\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \log\eta(z+m) - \log\eta(z)$$

$$= \frac{\pi i}{12} m$$

$$\Psi(\gamma) = \Psi\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = m$$

... hyperbolic $\qquad \gamma$



α quad. irrational

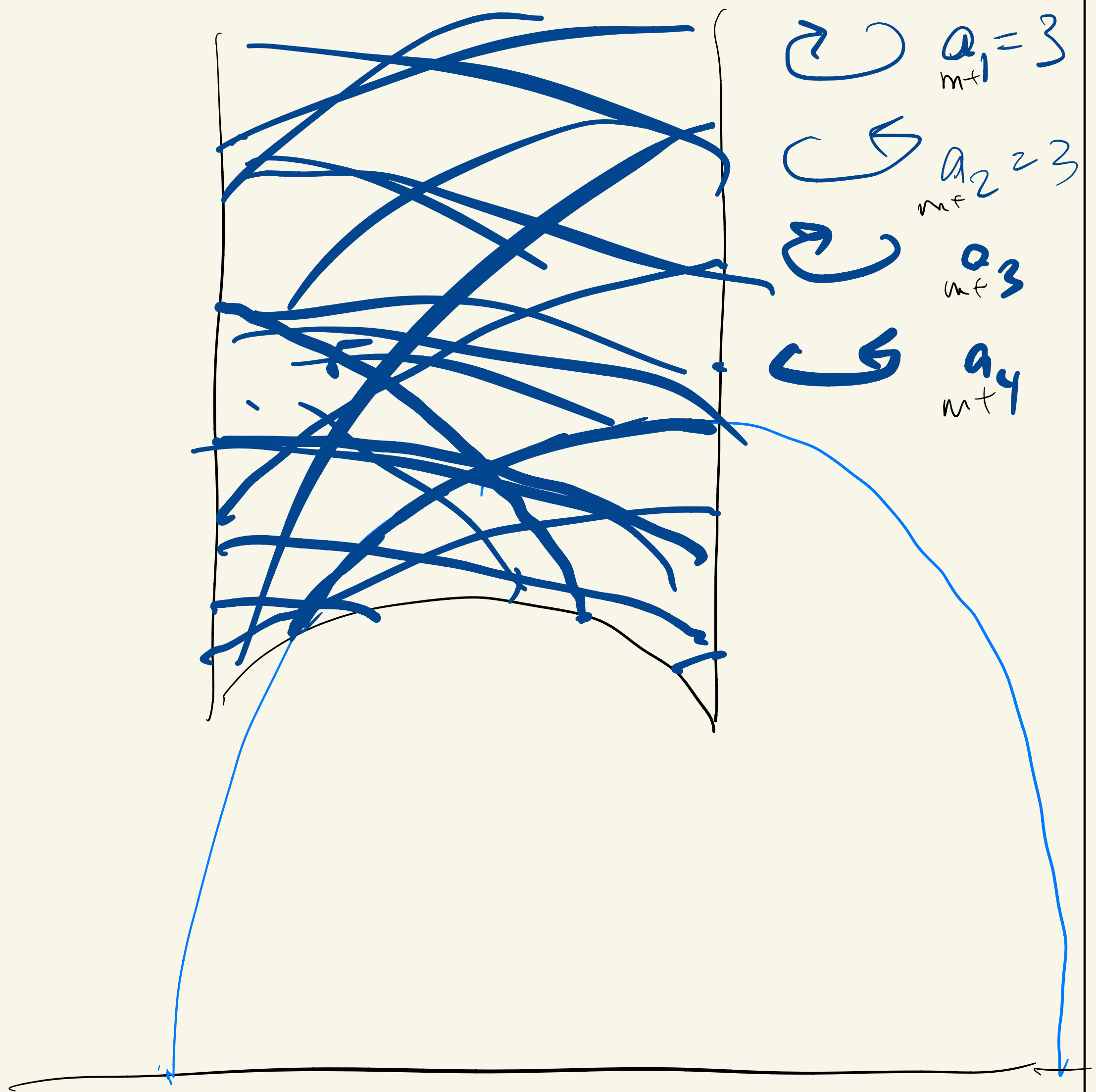
$\rightsquigarrow$ the CFE of α is eventually periodic

$$\alpha = [a_0, \dots, a_m, \overline{a_{m+1}, \dots, a_{m+n}}]$$

Rmk: α is purely periodic iff α is reduced: $\alpha > 1$, $-1 < \bar{\alpha} < 0$

We can always find $\gamma' \sim \gamma$ (conjugate) s.t. α' is reduced, and

$$\alpha' = [\overline{a_{m+1}, \dots, a_{m+n}}]$$



real analytic
weight 2
"modified" Eisenstein series

Fact:

$$\varphi(\gamma) = \sum_{i=1}^{n} (-1)^i a_{i+m}$$

= winding of the oriented closed geod. associated to γ around the cusp of $\Gamma \backslash \mathbb{H}$

We can make explicit the relation of φ to the cusp:

$$\varphi(\gamma) = \int_\gamma E_2(z)\,dz$$

$$E_2(z) = \lim_{s\to 1}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right) E(z,s)$$

$$= \lim_{s\to 1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \frac{s\, y^{s-1}}{(cz+d)^2\,|cz+d|^{2s-2}}$$

$\Gamma < PSL_2\mathbb{R}$ Fuchsian cofinite and noncocompact.

To each cusp $\mathfrak{a}$, there is an Eisenstein series $E_{\mathfrak{a}}(z,s)$ and a modified E.S. $E_{2,\mathfrak{a}}(z)$

$$\Psi_{\mathfrak{a}}(\gamma) = \int_{\gamma} E_{2,\mathfrak{a}}(z)\,dz$$

(on positive hyperbolic elements)

Does $\Psi_{\mathfrak{a}}$ once again compute the oriented winding around the cusp at $\mathfrak{a}$?

... a priori, $\Psi_{\mathfrak{a}} : \Gamma \longrightarrow \mathbb{R}$

① If $\Psi_{\mathfrak{a}}$ is $\mathbb{Q}$-valued, then $\Psi_{\mathfrak{a}}$ can again be understood as a winding number

(work in progress w/ F. von Essen)

② Link to arithmetic geometry.

$$X = \overline{\Gamma\backslash\mathbb{H}} \longrightarrow J(X) \text{ Jacobian}$$

$$P \longmapsto \left(\int_0^P \omega_1, \ldots, \int_0^P \omega_g\right) \text{(mod periods)}$$

extend this map to

$$\mathrm{Div}^0(X) = \left\{ \sum n_P P : \begin{array}{l} n_P \in \mathbb{Z} \\ \sum n_P = 0 \end{array} \right\}$$

<u>Abel-Jacobi:</u>

$$\mathrm{Div}^0(X)/_{\text{lin. equiv.}} \cong J(X)$$

Lemma: $D = \sum n_a a$.

$$\Rightarrow \int_\gamma \omega_D = \sum n_a \int_\gamma E_{2,a}\, dz$$
$$= \sum n_a \Psi_a(\gamma)$$

D torsion iff $\int_\gamma \omega_D \in 2\pi i \mathbb{Q}$

Thm: (B. 2021)

If Rademacher symbols on Γ are rational-valued then the subgroup $C(\Gamma) < J(X)^{tor}$ spanned by the cusps is torsion.

Consequences:

1. Rademacher symbols are not always $\mathbb{Q}$-valued.

Rohrlich: $\exists$ noncongruence arithmetic Fuchsian groups for which $|C(\Gamma)| = \infty$

Manin - Mumford: $|J^{tor}(X)| < \infty$

2. Manin-Drinfeld: If Γ is congruence then $C(\Gamma) < J(X)^{tor}$.

New proof of MD (and a slight extension of the statement beyond congruence subgroups):

Sketch of proof:

1– Show that Ψ_∞ for $\Gamma(N)$ is $\mathbb{Q}$-valued.

2– Use grp-theoretic arguments regarding

$$\begin{array}{c} \Gamma(N)\backslash\mathbb{H} \\ \downarrow \\ \Gamma\backslash\mathbb{H} \end{array} \quad (\Gamma \text{ cong.})$$

to extend the result of 1 to any Rademacher symbol on Γ

In fact, this also applies for $\Gamma = \Gamma_0(N)^+$ (the Fuchsian group formed by $\Gamma_0(N)$ together with all Atkin-Lehner involutions.

Helling: Every maximal (under inclusion) arithmetic noncompact Fuchsian group is isomorphic to $\Gamma_0(N)^+$ for N $\square$free.