L-series values of twists of elliptic curves

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Question

Do we have rational points over $\mathbb{Q}[\sqrt{-3}]$ for the elliptic curve:

$$E_{D,\alpha} : y^2 = x^3 - 432D^2\alpha^3$$

for $\alpha = a + b\sqrt{-3} \in \mathbb{Q}[\sqrt{-3}]$.

- Particular case:

$$E_{D,1} : Y^2 = X^3 - 432D^2$$

is equivalent to

$$x^3 + y^3 = Dz^3$$

by a change of coordinates $X = \frac{12Dz}{(x+y)}$, $Y = \frac{36D(x-y)}{(x+y)}$

Elliptic curve

$$E_{D,\alpha} : y^2 = x^3 - 432D^2\alpha^3$$

- $E_{D,\alpha}$ is a sextic twist over $\mathbb{Q}[\sqrt{-3}]$ of the elliptic curve $E_1 : y^2 = x^3 - 1$
  i.e. we have an isomorphism $E_{D,\alpha}(\mathbb{C}) \cong E_1(\mathbb{C})$ by taking $x \to 6x\alpha\sqrt[3]{3D^2}, y \to 12\alpha y\sqrt{3\alpha}$.

- $E_{D,\alpha}$ has Complex Multiplication(CM) by $\mathcal{O}_K$ for $K = \mathbb{Q}[\sqrt{-3}]$
  i.e. $\mathrm{End}_{\mathbb{C}}(E_{D,\alpha}) \cong \mathcal{O}_K = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}])$

- **Mordell-Weil Theorem**: $E_{D,\alpha}(K)$ is a finitely generated abelian group.

$E_{D,\alpha}$ has non-torsion rational solutions over K **iff** $\operatorname{rank} E_{D,\alpha}(K) \geq 1$.

Birch and Swinnerton-Dyer Conjecture (BSD) for $E = E_{D,\alpha}$

1. $\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{rank}\, E(K)$
2. $L(E/K, 1) \neq 0 \Longrightarrow L(E/K, 1) = \dfrac{\Omega\overline{\Omega} \prod\limits_{v|6D\alpha} c_v}{(\#E(K)_{tor})^2} \#Ш_{E/K}.$

- $\Omega = \dfrac{\Gamma\left(\frac{1}{3}\right)^3 \sqrt{3}}{(2\pi)^2 \sqrt[3]{D}\sqrt{\alpha}}$
- $Ш_E$ = the Tate-Shafarevich group
- c_v Tamagawa numbers
- $E(K)_{tor}$ torsion part of $E(K)$

- Assuming BSD:

$$L(E_{D,\alpha},1) \neq 0 \overset{BSD}{\Longleftrightarrow} E_{D,\alpha} \text{ has no non-torsion points over } K$$

- Coates-Wiles for E with CM (Gross-Zagier-Kolyvagin for all elliptic curves):

$$L(E_{D,\alpha},1) \neq 0 \Longrightarrow E_{D,\alpha} \text{ has no non-torsion points over } K$$

Goal

Compute $L(E_{D,\alpha}/K, 1)$

- Writing $S_{D\alpha} = \frac{(\#E_D(K)_{tor})^2}{\Omega_D \overline{\Omega_D}} L(E_{D,\alpha}, 1)$, we want to be able to check if $S_{D\alpha} \neq 0$

If $S_{D\alpha} \neq 0$, then:

- $E_D(K) = E_D(K)_{tor}$
- $S_{D\alpha}$ should give us the **order of** Ш up to the Tamagawa numbers

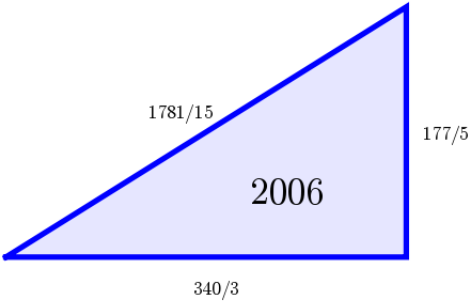
- What we know about Ш:
  - Kolyvagin: $L(E_D/K, 1) \neq 0 \rightarrow \#$Ш is finite
  - Cassels: $\#$Ш is finite $\Longrightarrow$ $\#$Ш is a square

What we expect

$$S_{D\alpha} = \begin{cases} 0, \textit{ or} \\ \text{an } \textbf{integer square} \text{ up to the Tamagawa numbers} \end{cases}$$

Other families of twists - quadratic twists

- **Congruent numbers:** $D \in \mathbb{Z}$ is a congruent number if it is the area of a right triangle with rational sides



Credit photo: William Stein

- $D = ab/2$, $a^2 + b^2 = c^2$, $a, b, c \in \mathbb{Q}$ equivalent to:

$$E^{(D)} : Y^2 = X^3 - D^2 X$$

Other families of twists - quadratic twists $E^{(D)}: Y^2 = X^3 - D^2 X$

- Idea for congruent numbers: write a generating series

$$f(z) \sim \sum_{N \geq 1} \sqrt{L(E^{(N)}, 1)} q^N, q = e^{2\pi i z}$$

- **Tunnell:** recognize $f(z)$ as a nice modular form and compute its coefficients explicitly

$$N \text{ odd congruent number} \overset{BSD}{\Longleftrightarrow}$$
$$\#\{x, y, z \in \mathbb{Z} : 2x^2 + y^2 + 32z^2 = N\} = \frac{1}{2}\#\{x, y, z \in \mathbb{Z} : 2x^2 + y^2 + 8z^2 = N\}$$

- Similar condition for N even
- Why it works: **Waldspurger's theorem** for quadratic twists:

$$f(z) \sim \sum_{N \geq 1} \sqrt{L(E \otimes \chi_N, 1)} q^N \text{ is modular.}$$

Families of twists

Congruent numbers	**Sum of two cubes**	Rational points over K
$D = ab/2$, $a^2 + b^2 = c^2$, $a, b, c \in \mathbb{Q}$	$X^3 + Y^3 = D$	
$E^{(D)} : Y^2 = X^3 - D^2 X$	$E_D : Y^2 = X^3 - 432D^2$	$E_{D,\alpha} : Y^2 = X^3 - 432D^2\alpha^3$ $\alpha \in \mathbb{Q}[\sqrt{-3}]$
CM by $\mathbb{Z}[i]$	CM by $\mathbb{Z}[\omega]$	CM by $\mathbb{Z}[\omega]$
quadratic twist	**cubic twist**	**sextic twist**
$L(E^{(D)}, 1) = a(D)^2 \beta \frac{1}{4\sqrt{D}}$ $L(E^{(2D)}, 1) = b(D)^2 \beta \frac{1}{2\sqrt{2D}}$	$L(E_p, 1)$ p prime (Rodriguez-Villegas-Zagier) $L(E_D, 1)$ all D (R.)	$L(E_{D,\alpha}, 1) = ?$

Computing the L-function

- Want to use:

$$E_{D,\alpha} \text{ has CM by } \mathcal{O}_K \text{ for } K = \mathbb{Q}[\sqrt{-3}]$$

CM theory

There is a **Hecke character** $\chi_E : \{\text{Ideals in } \mathcal{O}_K\} \to \mathbb{C}$ such that:

$$L(E_{D,\alpha}/K, s) = L(s, \chi_E) L(s, \overline{\chi}_E)$$

- $\chi_E = \left(\frac{D}{}\right)_3 \left(\frac{\alpha}{}\right)_2 \varphi$
  - $\varphi((\alpha)) = \alpha$, $\alpha \equiv 1(3)$

    Ideal $\mathcal{A} \longleftrightarrow \sigma_{\mathcal{A}}$ Galois action via Artin map:
  - $(\alpha^{1/2})^{\sigma_{\mathcal{A}}^{-1}} = \alpha^{1/2} \left(\frac{\alpha}{\mathcal{A}}\right)_2$,
  - $(D^{1/3})^{\sigma_{\mathcal{A}}^{-1}} = D^{1/3} \overline{\left(\frac{D}{\mathcal{A}}\right)_3}$
- $L(s, \chi_E) = \sum_{k\in\mathcal{O}_K} \chi_E(k)(\mathrm{Nm}\, k)^{-s} = \sum_{k\in\mathcal{O}_K} \left(\frac{D}{k}\right)_3 \left(\frac{\alpha}{k}\right)_2 \overline{k}\, (\mathrm{Nm}\, k)^{-s}$

Result for $D = X^3 + Y^3$

- $K = \mathbb{Q}[\sqrt{-3}]$, $\Theta_K(z) = \sum_{a,b\in\mathbb{Z}} e^{2\pi i(a^2+b^2-ab)z}$ modular form wt 1, level 3
- $\omega = \frac{-1+\sqrt{-3}}{2}$
- H_{3D} = ring class field for $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$

  i.e. $\mathrm{Gal}(H_{3D}/K) \cong I(3D)/P_{\mathbb{Z},3D}$, where

  - $I(3D) = \{$fractional ideals prime to $3D\}$,
  - $P_{\mathbb{Z},3D} = \{$principal ideals (α)s.t. $\alpha \equiv a \mod 3D$, $a \in \mathbb{Z}$, $(a, 3D) = 1\}$

Theorem 0 (R.)

- D integer, $(D, 6) = 1$
- $c_D = \frac{\Gamma\left(\frac{1}{3}\right)^3\sqrt{3}}{(2\pi)^2\sqrt[3]{D}}$,

Denoting $S_D = L(E_{D,1}, 1)/c_D$, then:

$$S_D = \mathrm{Tr}_{H_{3D}/K}\left(\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\sqrt[3]{D}\right), \quad S_D \in \mathbb{Z}$$

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- **Rodriguez-Villegas, Zagier (1991)** proved similar formulas for $X^3 + Y^3 = p$, for primes $p \equiv 1 \mod 9$:

$$L(E_{p,1}, 1) = c_p \,\mathrm{Tr}\left(\frac{\sqrt[3]{p}\,\Theta_K(p\delta)}{54\Theta_K(\delta)}\right), \quad \delta = \frac{-9 + \sqrt{-3}}{18} \qquad (1)$$

- More formulas: showing that $L(E_{p,1}, 1)$ is a square
  - Rodriguez-Villegas, Zagier for p prime, $p \equiv 1(3)$
  - R. for D integer, D split in $\mathcal{O}_K$

Results for $E_{D,\alpha} : y^2 = x^3 - 432D^2\alpha^3$

- $m = \mathrm{Nm}(\alpha)$, $M = \mathbb{Q}[\sqrt{-m^*}]$, $m^* = \begin{cases} m & \text{for } m \equiv 3(4) \\ 4m & \text{for } m \equiv 1(4) \end{cases}$
- $\Theta_M(z) = \sum_{[\mathcal{A}] \in Cl(\mathcal{O}_M)} \sum_{\lambda \in \mathcal{A}} e^{2\pi i \frac{\mathrm{Nm}\,\lambda}{\mathrm{Nm}\,\mathcal{A}} z}$ weight 1, level $3m$
- $\Theta_K(z) = \sum_{x,y \in \mathbb{Z}} e^{2\pi i (x^2 + y^2 - xy) z}$
- H_{3Dm^*} = ring class field of conductor $3Dm^*$

Theorem 1(R.)

- D integer, $(D, 6) = 1$,
- $\alpha \in \mathcal{O}_K$, $(\alpha, 6D) = 1$.

Denote $S_{D\alpha} = \frac{1}{c_{D\alpha}} L(E_{D,\alpha}, 1)$, for $c_{D\alpha} = \frac{\Gamma\left(\frac{1}{3}\right)^3 \sqrt{3}}{(2\pi)^2 \sqrt[3]{D}\sqrt{m}}$. Then:

$$S_{D\alpha} = \left| c\,\mathrm{Tr}_{H_{3Dm}/K} \left(\frac{\Theta_M(D\omega)}{\Theta_K(\omega)} \sqrt[3]{D}\sqrt{\alpha} \right) \right|^2, \quad S_{D\alpha} \in \mathbb{Z}$$

Here $|c|^2 = \frac{16|L_\alpha(1,\chi)|^2}{m^*}$, where $L_\alpha(1, \chi) = \prod_{\mathfrak{p}|(\alpha)} L_{\mathfrak{p}}(1, \chi)$.

Results for $E_{D,\alpha} : y^2 = x^3 - 432D^2\alpha^3$

Technical conditions:

- D an integer, $(D, 6) = 1$
- $\alpha \in \mathcal{O}_K$ prime, $\mathrm{Nm}\,\alpha \equiv 1(4)$
- $\begin{cases} D \equiv \pm 1(9), \textit{or} \\ D \equiv \pm 4(9), \alpha \equiv -1(\sqrt{-3}). \end{cases}$

Theorem 2(R.)

Under the technical conditions above:

$$S_{D\alpha} = \left(s_\alpha \,\mathrm{Tr}_{H_{\mathbf{3}Dm}/K} \frac{\Theta_M(D\omega)}{\Theta_K(\omega)} D^{1/3} \alpha^{1/2} \right)^2$$

where $s_\alpha = 4\omega^l \frac{L_\alpha(1,\overline{\chi})}{\alpha}$ for a cubic root of unity ω^l.

When $(D, m/L_\alpha(1,\overline{\chi})) = 1$, we have:

$$S_{D\alpha} = \begin{cases} \text{integer square}, & \textit{if } \left(\frac{D}{m}\right) \left(\frac{\alpha}{\overline{\alpha}}\right)_2 = 1 \\ \text{3 times an integer square}, & \textit{if } \left(\frac{D}{m}\right) \left(\frac{\alpha}{\overline{\alpha}}\right)_2 = -1 \end{cases}.$$

Sketch of proof: Theorem 1

Goal

Compute $L(s, \chi_E)$ for the Hecke character χ_E corresponding to $E_{D,\alpha}$

(1) For the adelic Hecke character $\chi_E : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$:

$$L(s, \chi_E) = \prod_{v \text{ place of } K} L_v(s, \chi_v) = \prod_v (1 - \chi_v(\mathfrak{p}_v) \operatorname{Nm} \mathfrak{p}_v^{-s})^{-1}$$

Tate's thesis

At almost all places v of K:

$$L_v(s, \chi_v) = Z_v(s, \chi_v, \Phi_v)$$

where Φ_v are *Schwartz-Bruhat functions*
E.g $\Phi_v = \mathrm{char}_{\mathcal{O}_{K_v}}$, $\Phi_\infty(x) = e^{-\pi|x|^2}$

Tate's *Zeta*-function:

$$Z_v(s, \chi_v, \Phi_v) := \int_{K^\times} \chi_v(x) |x|^s \Phi_v(x) d^\times x$$

Sketch of proof: Theorem 1

We compute the global integral:

$$Z(s,\chi_f,\Phi_f) = \prod_{v\nmid\infty} Z(s,\chi_v,\Phi_v) = \int_{\mathbb{A}_{K,f}^\times} \chi_f(x)|x|_f^s \Phi_f(x) d^\times x$$

Take quotients:

- $Z(s,\chi_v,\Phi_f) = \displaystyle\int_{\mathbb{A}_{K,f}^\times/K^\times} \sum_{k\in K^\times} \chi_f(kl)|kl|_f^s \Phi_f(kl) d^\times l$
- $Z(s,\chi_v,\Phi_f) = \displaystyle\int_{U(3Dm^*)\backslash\mathbb{A}_{K,f}^\times/K^\times} \left(\sum_{k\in K^\times} \chi_f(k)|k|_f^s \Phi_f(kl)\right) \chi_f(l)|l|_f^s d^\times l$

  $U(3Dm^*)$ big compact

Zeta computation

$$Z_f(s,\chi,\Phi) = c_0 \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_{3Dm^*})} E_M(2s-2, Dz_{\mathcal{A}}) \overline{\left(\frac{D}{\mathcal{A}}\right)_3} \left(\frac{\alpha}{\mathcal{A}}\right)_2 \frac{\varphi(\mathcal{A})}{\mathrm{Nm}\,\mathcal{A}}$$

Sketch of proof: Theorem 1

Zeta computation

$$Z_f(s,\chi,\Phi) = c_0 \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_{3Dm^*})} E_M(2s-2, Dz_{\mathcal{A}}) \overline{\left(\frac{D}{\mathcal{A}}\right)_3} \left(\frac{\alpha}{\mathcal{A}}\right)_2 \frac{\varphi(\mathcal{A})}{\operatorname{Nm}\mathcal{A}}$$

(2)

- $\chi_E = \left(\frac{D}{\cdot}\right)_3 \left(\frac{\alpha}{\cdot}\right)_2 \varphi$
- *Eisenstein series*: $E_M(s,z) = \sum_{a,b\in\mathbb{Z}} \frac{\left(\frac{a}{3m^*}\right)}{(3maz+b)|3maz+b|^s}$,
- $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ primary ideal with $\operatorname{Nm}\mathcal{A} = a$, $b^2 \equiv -3 \mod 4a$

$$z_{\mathcal{A}} := \frac{-b+\sqrt{-3}}{2a}$$

Sketch of proof: Theorem 1

Formula Zeta

$$Z_f(s,\chi,\Phi) = c_0 \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_{3Dm^*})} E_M(2s-2, Dz_{\mathcal{A}}) \overline{\left(\frac{D}{\mathcal{A}}\right)_3} \left(\frac{\alpha}{\mathcal{A}}\right)_2 \frac{\varphi(\mathcal{A})}{\operatorname{Nm}\mathcal{A}}$$

(3) We plug in $s = 1$: $Z_f(1,\chi,\Phi) = c_1 L(1,\chi_E)$

- Siegel-Weil type theorem: $E_M(0,z) = \frac{2\pi}{18}\Theta_M(z)$
- $\frac{\Theta_K(\omega)}{\Theta_K(z_{\mathcal{A}})} = \frac{\varphi(\mathcal{A})}{\operatorname{Nm}\mathcal{A}}$

L-function

$$L(1,\chi_E) = c^{\circ}_{D,\alpha} \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_{3Dm^*})} \frac{\Theta_M(Dz_{\mathcal{A}})}{\Theta_K(z_{\mathcal{A}})} D^{1/3}\alpha^{1/2} \overline{\left(\frac{D}{\mathcal{A}}\right)_3} \left(\frac{\alpha}{\mathcal{A}}\right)_2$$

Sketch of proof: Theorem 1

(4) **Shimura reciprocity law**: Computes all the Galois conjugates of modular functions (with rational coefficients in Fourier expansion) evaluated at CM points:

$$f(\tau)^{\sigma_{\mathcal{A}}^{-1}} = f^{g_{\mathcal{A}}}(\tau),$$

- τ CM point i.e. $A\tau^2 + B\tau + C = 0, A, B, C \in \mathbb{Z}$
- ideal $\mathcal{A} \longleftrightarrow \sigma_{\mathcal{A}}$ Galois action coming from the Artin map
- $g_{\mathcal{A}}$ a 2×2 matrix, acting on f

In our case:

$$\left(\frac{\Theta_M(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathcal{A}}^{-1}} = \frac{\Theta_M(Dz_{\mathcal{A}})}{\Theta_K(z_{\mathcal{A}})}$$

- $z_{\mathcal{A}} = \frac{-b+\sqrt{-3}}{2a}$ CM point for primitive ideal $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$
- $\Theta_M(Dz)/\Theta_K(z)$ modular function

Sketch of proof: Theorem 1

L-function

$$L(1,\chi_E) = c^{\circ}_{D,\alpha} \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_{3Dm^*})} \frac{\Theta_M(Dz_{\mathcal{A}})}{\Theta_K(z_{\mathcal{A}})} D^{1/3}\alpha^{1/2} \overline{\left(\frac{D}{\mathcal{A}}\right)_3} \left(\frac{\alpha}{\mathcal{A}}\right)_2$$

- Shimura reciprocity: $\left(\frac{\Theta_M(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathcal{A}}^{-1}} = \frac{\Theta_M(Dz_{\mathcal{A}})}{\Theta_K(z_{\mathcal{A}})}$
- From definition: $\begin{cases} (\alpha^{1/2})^{\sigma_{\mathcal{A}}^{-1}} = \alpha^{1/2}\left(\frac{\alpha}{\mathcal{A}}\right)_2, \\ (D^{1/3})^{\sigma_{\mathcal{A}}^{-1}} = D^{1/3}\overline{\left(\frac{D}{\mathcal{A}}\right)_3} \end{cases}$

$$\Longrightarrow L(1,\chi_E) = c^{\circ}_{D,\alpha} \sum_{[\mathcal{A}]\in Cl(\mathcal{O}_{3Dm^*})} \left(\frac{\Theta_M(D\omega)}{\Theta_K(\omega)} D^{1/3}\alpha^{1/2}\right)^{\sigma_{\mathcal{A}}^{-1}}$$

L-function

$$L(1,\chi_E) = c^{\circ}_{D,\alpha}\, \mathrm{Tr}_{H_{3Dm^*}/K} \frac{\Theta_M(D\omega)}{\Theta_K(\omega)} D^{1/3}\alpha^{1/2}$$

Sketch of proof: Theorem 1

(6) Recall $L(E_{D,\alpha},1) = L(1,\chi_E)L(1,\overline{\chi}_E)$ and we get:

L-function for $E_{D,\alpha}$

$$L(E_{D,\alpha},1) = c_{D,\alpha}\left|\mathrm{Tr}_{H_{3Dm^*}/K}\frac{\Theta_M(D\omega)}{\Theta_K(\omega)}D^{1/3}\alpha^{1/2}\right|^2$$

Goal for proving Theorem 2:

Show that $X_{D,\alpha} = \mathrm{Tr}_{H_{3Dm^*}/K}\frac{\Theta_M(D\omega)}{\Theta_K(\omega)}D^{1/3}\alpha^{1/2}$ is an integer up to a constant.

Idea: compare $X_{D,\alpha}$ to $\overline{X_{D,\alpha}}$

Sketch of proof: Theorem 2

- **Theorem 1:** $X_{D\alpha} = \mathrm{Tr}_{H_{3Dm}/K} \frac{\Theta_M(D\omega)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3}$,

$$S_{D\alpha} = |s_\alpha X_{D\alpha}|^2,$$

- Want to show:

$$S_{D\alpha} = (X'_{D\alpha})^2, \quad \overline{X'_{D\alpha}} = \pm X'_{D\alpha}$$

where $X'_{D\alpha} = c'_{\alpha,D} X_{D\alpha}$, $|c'_{D,\alpha}| = |s_\alpha|$

$X_{D\alpha} = \mathrm{Tr}_{H_{3Dm}/K} \frac{\Theta_M(D\omega)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3}$

1. Sum up Galois conjugates of $\frac{\Theta_M(D\omega)}{\Theta_K(\omega)}\alpha^{1/2}D^{1/3}$:

$$\sum_{\substack{\mathcal{A}_i \\ b_i \equiv i(m)}} \frac{\Theta_M\left(\frac{b_i+\sqrt{-3}}{6Dm}\right)}{\Theta_K\left(\frac{b_i+\sqrt{-3}}{6}\right)} \alpha^{1/2} D^{1/3} \left(\frac{D}{\mathcal{A}}\right)_3 \left(\frac{\alpha}{\mathcal{A}}\right)_2$$

   - use Fourier expansion of Θ_M and computing cubic Gauss sums
   - take traces

$$(m-2)\left(\frac{D}{m}\right) DX + T = \frac{X_1}{L_\alpha(1,\overline{\chi})}$$

$T = \mathrm{Tr}_{H_{\mathbf{3}Dm}/K} \frac{\Theta_M\left(\frac{b+\sqrt{-3}}{6D}\right)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3}$	$X_1 = \mathrm{Tr}_{H_{\mathbf{3}Dm}/K} \frac{\Theta_M\left(\frac{b+\sqrt{-3}}{6Dm}\right)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3}$

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- Similar methods: summing up over the Galois conjugates of $\frac{\Theta_M\left(\frac{b+\sqrt{-3}}{6D}\right)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3}$ (terms in the sum T):

$$\sum \frac{\Theta_M\left(\frac{b_i+\sqrt{-3}}{6D}\right)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3} \left(\frac{D}{\mathcal{A}_i}\right)_3 \left(\frac{\alpha}{\mathcal{A}_i}\right)_2, \quad \mathcal{A}_i = \left(\frac{b_i+\sqrt{-3}}{2}\right)$$

$$(m-2)T + \left(\frac{D}{m}\right) DX = \frac{X_1}{L_\alpha(1,\overline{\chi})}$$

Together with $(m-2)\left(\frac{D}{m}\right) DX + T = \frac{X_1}{L_\alpha(1,\overline{\chi})}$, we get:

$$\left(\frac{D}{m}\right)(m-1)DX = \frac{X_1}{L_\alpha(1,\overline{\chi})}.$$

Sketch of proof: Theorem 2

$$X_1 = \mathrm{Tr}_{H_{3Dm}/K} \frac{\Theta_M\left(\frac{b_{0,1}+\sqrt{-3}}{6Dm}\right)}{\Theta_K(\omega)} \alpha^{1/2} D^{1/3}$$

2. Using the transformation:

$$\Theta_M(z) = \frac{i}{\sqrt{3mz}} \Theta_M\left(-\frac{1}{3mz}\right)$$

and Shimura reciprocity law, get a relation:

$$X_1 = -\left(\frac{D}{m}\right) \frac{\varphi(\alpha)}{\overline{\alpha}} \left(\frac{D}{\alpha}\right)_3 \left(\frac{\overline{\alpha}}{\alpha}\right)_2 \left(\frac{\omega}{D}\right)_3 \overline{X_1}$$

Simpler formula

$$\frac{X_1}{\alpha} = \pm \omega^k \frac{\overline{X_1}}{\alpha}$$

Sketch of proof: Theorem 2

- $\left(\frac{D}{m}\right)(m-1)DX = \frac{X_1}{L_\alpha(1,\overline{\chi})}$.
- $\frac{X_1}{\alpha} = \pm\omega^k \overline{\frac{X_1}{\alpha}}$
- $S_{D\alpha} = |s_\alpha X|^2, \quad s_\alpha = \frac{L_\alpha(1,\overline{\chi})}{\alpha}$

3. Rescale everything to get

Final result

For $X' = X\frac{L_\alpha(1,\overline{\chi})}{\alpha}(\pm\omega^k)$ we have

$$\overline{X'} = c_1 X', \quad = \pm X'$$

where $c_1 = \left(\frac{D}{m}\right)\left(\frac{\overline{\alpha}}{\alpha}\right)_2 = \pm 1$ and we have:

$$S_{D\alpha} = \pm(X'^2) = \begin{cases} N^2 & \text{or} \\ -3N^2 \end{cases}$$

Thank you!