Gan-Gross-Prasad p-adic L-functions (jt. with Wei Zhang)

1. Setup

E CM
$|2$
F totally real field , $G = \mathrm{Res}_{F/\mathbb{Q}} GL_n \times GL_{n+1}$

We say that an aut. representation $\Pi_n \otimes \Pi_{n+1}$ of $G(\mathbb{A})$ is *nice* if it is cuspidal, $\Pi_m^\vee \simeq \Pi_m^c$, $\Pi_{m,\infty} = BC(\mathbb{1}_{U(m)/\mathbb{R}})$ $m = n, n+1$

(if $m=2$, this $\Leftrightarrow$ $\Pi_\infty = BC_{GL_{2,\mathbb{C}} \leftarrow GL_{2,\mathbb{R}}}(\text{Disc. series of wt. } 2))$

For $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ finite,

consider
$$\mathcal{L}(s, \Pi, \chi) = \frac{L(s, \Pi_n \times \Pi_{n+1} \otimes \chi_E)}{L(1, \Pi_n, As^{\pm})\, L(1, \Pi_{n+1}, As^{\mp})}$$

adjoint L-function for rep of a unitary group

2. Rationality statement [Clozel]

For $L \supset \mathbb{Q}$, we have a notion of "nice rep. Π

defined over L'', $\forall \iota : L \hookrightarrow \mathbb{C} \rightsquigarrow \Pi^{\iota}$ nice rep of Π over $\mathbb{C}$.

Let $$Y(m) = \overbrace{\left(F^\times \backslash \mathbb{A}^\times / F_{\mathbb{R}}^\times (1 + m \hat{\mathcal{O}}_F \cap \hat{\mathcal{O}}_F^\times) \right)}$$ finite scheme $/\mathbb{Q}$

$$Y(\infty) = \bigcup_{m \geq 1} Y(m)$$

<u>Thm</u> . Let π nice representation over L.

Then $\exists\ \mathcal{L}(\pi) \in \mathcal{O}(Y(\infty)_L)$ s.t. $\forall \chi \in Y(\infty)_L(\mathbb{C})$

$(\leftrightarrow Y(\infty) \xrightarrow{\chi_0} L(\chi_0)$

$\iota : L(\chi_0) \hookrightarrow \mathbb{C})$

we have

$$\mathcal{L}(\pi)(\chi) = \mathcal{L}(1/2, \pi^{\iota}, \chi)$$

in $\iota L(\chi_0) \subseteq \mathbb{C}$

<u>Rmk</u> . for $n = 1$: Shimura $\left(\mathcal{L}(1/2, \pi_{/\mathbb{C}})^{\sigma} \in L_{\pi}^{\sigma} \right.$

$\qquad \qquad \qquad \qquad \parallel$

$\left. \mathcal{L}(1/2, \pi_{\mathbb{C}}^{\sigma})^{\omega} \right)$

- for $\chi = \mathbb{1}$, similar results by Grobner, Harris, Lin.

<u>Gan-Gross-Prasad</u> : • $\mathcal{L}(1/2, \Pi, \mathbb{1}) \overset{\checkmark}{\rightsquigarrow} |\mathcal{P}(\phi)|^2$

BP-Lin-Zhang-Zhu

$P: \pi \to \mathbb{C}$

π rep. of $U(n)\times U(n+1)(\mathbb{A})$

BC $\downarrow$ Π

$n=1$ Waldspurger

• $\mathcal{L}'(1/2, \Pi) \overset{?}{=} ht.\left(\begin{array}{l}\text{algebraic cycle} \\ \text{on } Sh(U(n)\times U(n+1))\end{array}\right)$

$n=1$ Gross-Zagier

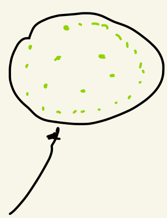
3. p-adic interpolation

Let $\mathcal{Y} = \mathrm{Spec}\left(\mathbb{Z}_p[[\overbrace{F^\times \backslash \mathbb{A}^\times / F_\infty^\times \hat{O}_F^{p,\times}}^{(\text{finite})\times \mathbb{Z}_p^d}]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)$

$= \{$continuous p-adic characters unramified outside p$\}$

$\cup$

$Y(p^\infty)_{\mathbb{Q}_p} = \bigcup_N Y(p^N)_{\mathbb{Q}_p} = \{$finite characters " $\}$

$\mathcal{Y} = \sqcup$ polydiscs  ...

$\cup$

$Y(p^\infty) =$ infinite set of pts

$Y(p^\infty) \subset \hat{\mathbb{Z}}_p$

$\chi \quad\quad \cup \quad r$

$\downarrow$

$\zeta_{p^n} - 1 = \chi(\gamma) - 1 \in D$

<u>In general</u>: - $\mathbb{C}$-L-functions are defined by analytic continuation $L(\pi, s)$ from $\mathrm{Re}(s) >> 0$ to all $\mathbb{C}$.

- p-adic L-functions are defined by interpolation from $\{L(\pi, \chi) \mid \chi \in Y(p^\infty)\}$ to $L_p(\pi) \in \mathcal{O}(Y)$

<u>In our case</u>: Let π nice rep. $/ L \supseteq \mathbb{Q}_p$.

<u>Def</u> - We say that π_m is <u>ordinary</u> at v / p

if $\pi_m^{N_m(\mathcal{O}_{F,v})} \neq 0$

$$N(\mathcal{O}_{F,v}) = \left(\begin{pmatrix} 1 & & \mathcal{O}_{F_v} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right)$$

$(\Rightarrow \pi_m^{I_m} \neq 0)$

& $\exists \underset{\neq 0}{w} \in \pi_m^{I_m}$ s.t. $\forall t = \varpi_v^\lambda \quad \lambda \in \mathbb{Z}_{\geq 0}$,

$$U_t w = \underbrace{\alpha(t)}_{} w$$

$$\alpha(t) \in \mathcal{O}_L^\times$$

<u>Ex</u> $\pi_m = \mathrm{Sym}^{m-1} \pi_{A E}$ is ordinary at v $\quad A/F = \mathbb{Q}$ ell. curve

if A has ord. reduction at v

$A[p](\bar{\mathbb{F}}_p) \neq 0$.

<u>Thm</u> – Suppose that for $m = n, n+1$, & all $v | p$,

Π_m is v-ordinary, + for $v|p$, either $\Pi_{n,v}$ is unramified or v splits in E.

then there is $\mathcal{L}_p(\Pi) \in \mathcal{O}(\mathcal{Y})$ s.t.

$$\mathcal{L}_p(\Pi)\big|_{Y(p^\infty)} = e_p(\Pi)\, \mathcal{L}(\Pi)$$

where $e_p(\Pi, \chi)$ = p-local interpolation factor as conjectured by Coates – Perrin – Riou.

<u>Conj</u> – $\mathcal{L}_p'(\Pi, 0) = ht\,(\,GGP \text{ cycle}\,)$

---

4. <u>Jacquet - Rallis relative - Trace formula</u>

We consider: $H_1 = GL_{n/E} \hookrightarrow G = GL_{n/E} \times GL_{n+1/E}$

$$g \longmapsto \left(g, \begin{pmatrix} g & \\ & 1 \end{pmatrix}\right)$$

$$H_2 = GL_{n/F} \times GL_{n/F} \hookrightarrow G$$

$$\rightsquigarrow \quad P_{1,\chi}(\phi) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \phi(h_1)\,\chi(\det h_1)\,dh_1 \quad \text{R-S period} \qquad P_i : L^2([G]) \to \mathbb{C}$$

$\updownarrow$ $\phi \in \pi$

$L(s, \pi_n \times \pi_{n+1})$

$$P_2(\phi) = \int_{H_2(F)\backslash H_2(\mathbb{A})} \phi(h_2)\ \text{char.}(h_2)\,dh_2 \qquad \text{char.} \leftarrow \eta_{E/F}$$

Flicker-Rallis period

detects $\pi^\vee \cong \pi^c$

related to $L(As)$

For $f \in \mathcal{H} = C_c^\infty(G(\mathbb{A}))$, consider

$$I(f,\chi) = \mathrm{Tr}^{P_1 \otimes P_2}_{\langle,\rangle} R(f) = \sum_{\phi \in ON \text{ of } L^2([G])} P_{1,\chi}(R(f)\phi)\,P_2(\phi)$$

$$\mathrm{Tr}^{B}_{\langle,\rangle}(A) = \sum_{\phi \text{ o.n. basis of } V,\ \text{wrt } \langle,\rangle.} B(A\phi,\phi)$$

spectral exp^n

$$= \sum_{\pi \text{ cuspidal}} \underbrace{I_\pi(f,\chi)}$$

$\parallel$ JPSS, FR

$$\frac{1}{4}\,\mathcal{L}(\tfrac{1}{2}, \pi \otimes \chi) \cdot \prod_v I_{\pi,v}(f_v,\chi_v)$$

f "quasi-cuspidal":

$R(f)\phi = 0$

if ϕ not cuspidal

$$\boxed{R(f) \rightsquigarrow K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)}$$

geometric expn

$$= \sum_{\gamma \in H_1(F) \backslash C(F) / H_2(F)} \underbrace{O_\gamma(f, \chi)}_{\text{orbital integrals}} = \prod_v O_{\gamma,v}(f, \chi_v)$$

$$= \int_{H_1 \times H_2(F_v)} f(x^{-1} \gamma y)\, \chi(x)\, dx\, dy$$

$\underline{\underline{f_\infty}}$: "Gaussian function" $\to I_{\pi_\infty} = 1$

$O_\gamma(f_\infty, \chi) = \begin{cases} 1 \\ 0 \end{cases}$ depending on γ.

Rationality of $\mathcal{L}(1/2, \pi)$ (avg. over all π) $\Leftarrow$ Rationality of $I(\pi, \chi)$ for nice f

$\Updownarrow$

Rationality of $O_\gamma(f, \chi)$

- if γ "nice" (reg. semisimple) $\to O_{\gamma,v}$ finite sum.
- in general, γ "kind of nice" (regular)

W. Lu

$$O_{\gamma,v} = \underbrace{L_{\gamma,v}(\chi)}_{\uparrow} \underbrace{O^{\#}_{\gamma,v}}_{\leftarrow \text{ finite sum}}$$

abelian function $L_\gamma(1 - j \cdot (\eta \circ \chi)^{t} \circ N_{F/F})$

$\in \mathbb{Q}$

Lemme, for nice Π, there exists 'nice f':

Techniques of ("Isolation of cuspidal spectrum")

- in span of $f^{\infty} f_{\infty}$, $\underbrace{f_{\infty}}_{\text{Gaussian}}$
- quasi-cuspidal
  $R(f)\phi = 0 \quad \phi \in \Pi' \not\cong \Pi$.
- $I_{\Pi_v}(f_v, \chi_v) \neq 0$

$\Rightarrow$ Proof of Rationality theorem.

5. p-adic Relative Trace formula

Thm - there exists

- a distribution $I : \mathcal{H}^p \to \mathcal{O}(\mathcal{Y})$
  $I(f^p, -)$ χ : cts char. of $F^\times \backslash \mathbb{A}^\times / \hat{\mathcal{O}}_F^{\times, p}$
- $\mathcal{O}_\gamma(f_1^p, -) \in \mathcal{O}(\mathcal{Y})$
- a generalized Radon measure valued in $\mathcal{O}(\mathcal{Y})$
  on $\underbrace{H_1 \backslash G / H_2}_{}(\mathbb{A}^p)$ $\quad d\mu(\gamma)$

s.t.

$$\sum_{\substack{\pi \text{ cuspidal} \\ \text{ordinary} \\ \text{nice}}} \mathcal{L}_p(\pi) \prod_{v \nmid p} I_{\pi_v}(f_v, \cdot) = \tilde{I}(f^p) = \int_{B(\mathbb{A}^p)}^{B} \underline{O_\gamma(f^p, -)}\ \boxed{d\mu(\gamma)}$$

Proof:

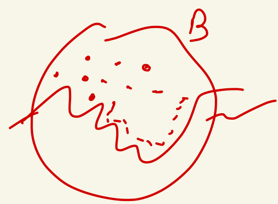
- $O^H_{\gamma, v}$ = poly. in $x(\varpi_v)$

  $v \nmid p\infty$

- $x \longmapsto L^{(p)}_\gamma(x)$

  interpolated by Deligne Ribet '80

- $O_{\gamma,p}(\underbrace{f_{p,N}}_{\text{reg.}}, \chi) = \begin{cases} 1 & \in B_N(F_p) \\ 0 & \notin B_N(F_p) \end{cases}$



$$d\mu(\gamma) = \lim_{N \to \infty} \delta_{B(F)^\circ \cap B_N(F_p)} \quad \text{p-adically bounded.}$$