Algebraicity of special L-values attached to Jacobi forms of higher index

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(joint with Thanasis Bouganis)

International Seminar on Automorphic Forms
20.01.2021

Goal

Prove that

$$\frac{\Lambda(\sigma/2, f, \chi)}{\pi^{e_\sigma}\langle f, f\rangle} \in \overline{\mathbb{Q}}$$

for $\sigma \in (k + 2\mathbb{Z}) \cap (2n + l + 2, k - 4n - l)$,

where

- $f \in S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$ Jacobi form, $k > 6n + 2l + 1$, $n > 1$
- $\mathcal{M}$: half-integral symmetric matrix of size $l \times l$
- χ: Dirichlet character, $\chi(-1) = (-1)^k$

Why?

$$\frac{L(\sigma, f)}{\pi^{e_\sigma} \langle f, f \rangle} \in \overline{\mathbb{Q}}$$

- Langlands conjectures: the standard L-functions of automorphic forms related to Shimura varieties may be identified with motivic L-functions.

  For them the values at certain $\sigma \in \mathbb{Z}$ are special.
- Deligne's conjecture relates them to determinants of certain period matrices, up to rational multiple.

  The values $L(\sigma, f)$ for $\sigma \in \mathbb{Z}$ are expected to be algebraic up to certain factors.
- Jacobi group may be identified with a mixed Shimura variety and Deligne's conjecture was generalized to this setting...

Outline

The talk is based on papers:

- Algebraicity of special L-values attached to Siegel–Jacobi modular forms, manuscripta math. 2020 ▸ Link
- On the analytic properties of the standard L-function attached to Siegel-Jacobi modular forms, Doc. Math. 2019 ▸ Link

Jacobi group

For $l, n \in \mathbb{N}$, we define Jacobi group

$$\boldsymbol{G}^{n,l}(\mathbb{Q}) := \{(\lambda, \mu, \kappa)g \colon \lambda, \mu \in M_{l,n}(\mathbb{Q}), \kappa \in Sym_l(\mathbb{Q}), g \in \mathrm{Sp}_n(\mathbb{Q})\},$$

whose group law can be recovered from Sp_{l+n} via an embedding

$$\boldsymbol{G}^{n,l} \ni (\lambda, \mu, \kappa)g \longmapsto \begin{pmatrix} 1_l & \lambda & \kappa - \mu\, {}^t\lambda & \mu \\ & 1_n & {}^t\mu & \\ & & 1_l & \\ & & -\,{}^t\lambda & 1_n \end{pmatrix} \begin{pmatrix} 1_l & & & \\ & a & & b \\ & & 1_l & \\ & c & & d \end{pmatrix} \in \mathrm{Sp}_{l+n}.$$

Similarly, as Sp_{l+n} acts on $\mathbb{H}_{l+n} := \{\tau \in Sym_{l+n}(\mathbb{C}) : \mathrm{Im}\,\tau > 0\}$ via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = (a\tau + b)(c\tau + d)^{-1},$$

taking $\left(\begin{smallmatrix} \tau' & w \\ {}^t w & \tau \end{smallmatrix}\right) \in \mathbb{H}_{l+n}$, we see that $\boldsymbol{G}^{n,l}$ acts on $\mathcal{H}_{n,l} := \mathbb{H}_n \times M_{l,n}(\mathbb{C})$ via

$$\boldsymbol{g} \cdot (\tau, w) = (g\tau, w(c\tau + d)^{-1} + \lambda g\tau + \mu).$$

Jacobi forms of higher index

Holomorphic $f\colon \mathbb{H}_n \times M_{l,n}(\mathbb{C}) \to \mathbb{C}$ such that $\forall_{\boldsymbol{g}\in\Gamma}\; f|_{k,\mathcal{M}}\,\boldsymbol{g} = f$, $k \in \mathbb{Z}$, where Γ is a congruence subgroup of $\boldsymbol{G}^{n,l}(\mathbb{Q})$. We consider

$$\Gamma = \Gamma_1^n(\mathfrak{c}) := (\mathbb{Z}_{l,n}, \mathfrak{b}_{l,n}^{-1}, \mathfrak{b}_{l,l}^{-1}) \begin{pmatrix} 1_n + \mathfrak{c}\,\mathbb{Z}_{n,n} & \mathfrak{c}\mathfrak{b}^{-1} \\ \mathfrak{c}\mathfrak{b} & \mathbb{Z}_{n,n} \end{pmatrix} \cap \boldsymbol{G}^{n,l}(\mathbb{Z}),$$

where $\mathfrak{c} \in \mathbb{Z}$, $\mathfrak{b}$ fractional ideal in $\mathbb{Q}$. Then $\mathcal{M} \in \mathfrak{b}Sym_l(\mathbb{Z})$ and

$$f(\tau, w) = \sum_{\substack{t\in L \\ t\geq 0}} \sum_{r\in M} c(t,r) e(\mathrm{tr}(t\tau)) e(\mathrm{tr}({}^t r w))$$

for some lattices $L \subset Sym_n(\mathbb{Q})$, $M \subset M_{l,n}(\mathbb{Q})$, and $e(x) := e^{2\pi i x}$.

$\rightarrow$ $f \in M^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$
(cusp forms: $S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$; $c(t,r) \neq 0 \Rightarrow \begin{pmatrix} \mathcal{M} & r \\ {}^t r & t \end{pmatrix} > 0$)

Shimura: $M^n_{k,\mathcal{M}}(\mathbb{C}) = M^n_{k,\mathcal{M}}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.

Jacobi Eisenstein series (of Siegel type)

If $z \in \mathbb{H}_n \times M_{l,n}(\mathbb{C})$,

$$E_k^n(z, s; \chi) := \sum_{\gamma \in \left(\boldsymbol{P}^{n,0} \cap \Gamma\right) \backslash \Gamma} \chi(\gamma) \det(\mathrm{Im}\, z)_{\mathbb{H}_n}^{s-k/2} |_{k,\mathcal{M}}\, \gamma,$$

where $\Gamma < \boldsymbol{G}^n(\mathbb{Q})$, χ Dirichlet character,

$$\boldsymbol{P}^{n,0} := \{(0, \mu, \kappa) g \in \boldsymbol{G}^n(\mathbb{Q}) \colon g = \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \in P^n(\mathbb{Q})\}.$$

E_k^n is absolutely convergent for $k + 2\mathrm{Re}\, s > n + l + 1$.
Additionally, if $s = 0$, it is a holomorphic Jacobi form.

The (standard) L-function

Consider a (commutative) Hecke agebra generated by

$$T_r := \Gamma_1^n(\mathfrak{c}) \begin{pmatrix} {}^t r^{-1} & \\ & r \end{pmatrix} \Gamma_1^n(\mathfrak{c})$$

where

$$r \in Q(\mathfrak{c}) := \{ \underline{r} \in \mathrm{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z}) : \forall_{p \mid \mathfrak{c}} \underline{r}_p \in \mathrm{GL}_n(\mathbb{Z}_p) \}$$

and for $\mathfrak{a} \in \mathbb{Z}$ define

$$f | T(\mathfrak{a}) := \sum_{\substack{r \in Q(\mathfrak{c}) \colon \det r = \mathfrak{a} \\ \mathrm{GL}_n(\mathbb{Z}) r \mathrm{GL}_n(\mathbb{Z}) \text{ distinct}}} f|_{k,\mathcal{M}} T_r .$$

Assume $0 \neq f \in S_{k,\mathcal{M}}(\Gamma_1^n(\mathfrak{c}))$ satisfies $f | T(\mathfrak{a}) = \lambda(\mathfrak{a}) f$ for all $\mathfrak{a}$.
For a Dirichlet character χ we define

$$D(s, f, \chi) := \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a}) \lambda(\mathfrak{a})}{\mathfrak{a}^s}, \qquad \mathrm{Re}\,(s) > 2n + l + 1.$$

Theorem (Murase, Murase-Sugano, Bouganis-M.)

Let $0 \neq f \in S_{k,\mathcal{M}}(\Gamma_1^n(\mathfrak{c}))$ be an eigenform of all $T(\mathfrak{a})$. Under some technical assumption on the index $\mathcal{M}$ (the condition M_p^+ at $p \nmid \mathfrak{c}$),

$$\mathfrak{L}_{(\mathfrak{c})}(s,\chi) G_{(\mathfrak{c})}(s,\chi) D(s+n+l/2, f, \chi) = \prod_p L_p(\chi(p)p^{-s}, f)^{-1} =: L(s,f,\chi),$$

where

$$\mathfrak{L}_{(\mathfrak{c})}(s,\chi) := \begin{cases} \prod_{i=1}^n L_{(\mathfrak{c})}(2s+2n-2i, \chi^2), & l \in 2\mathbb{Z} \\ \prod_{i=1}^n L_{(\mathfrak{c})}(2s+2n-2i+1, \chi^2), & l \notin 2\mathbb{Z} \end{cases}$$

and

$$L_p(X,f) := \begin{cases} 1, & p \mid \mathfrak{c} \\ \prod_{i=1}^n \Big((1-\alpha_{i,p}X)(1-\alpha_{i,p}^{-1}X) \Big), & p \nmid \mathfrak{c} \end{cases}$$

where $\alpha_{i,p} \in \mathbb{C}^\times$.

The (standard) L-function

Corollary

The function $L(s,f,\chi) := \prod_p L_p(\chi(p)p^{-s},f)^{-1}$ *is absolutely convergent for* $\mathrm{Re}\,(s) > n + l/2 + 1$*, and for these* s*:*

$$L(s,f,\chi) \neq 0.$$

Doubling method

Consider a homomorphism

$$\boldsymbol{G}^{m,l} \times \boldsymbol{G}^{n,l} \to \boldsymbol{G}^{m+n,l},$$

$$((\lambda, \mu, \kappa)g), (\lambda', \mu', \kappa')g')) \longmapsto ((\lambda\, \lambda'), (\mu\, \mu'), \kappa + \kappa')(g \times g'),$$

where

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) := \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix}.$$

This map induces an embedding

$$\mathcal{H}_{m,l} \times \mathcal{H}_{n,l} \hookrightarrow \mathcal{H}_{2n,l},$$

$$(\underbrace{(\tau_1, w_1)}_{z_1}, \underbrace{(\tau_2, w_2)}_{z_2}) \longmapsto \underbrace{(\mathrm{diag}[\tau_1, \tau_2], (w_1\, w_2))}_{\mathrm{diag}[z_1, z_2]}.$$

Doubling method

Theorem (Bouganis-M.)

Let $f \in S_{k,\mathcal{M}}(\Gamma_1^n(\mathfrak{c}))$ be an eigenform of all $T(\mathfrak{a})$. Let χ be a Dirichlet character such that $\chi(-1) = (-1)^k$ and $E_k^{m+n}(z, s; \chi)$ as earlier, $m \geq n$. Then (if $\mathcal{M}$ satisfies the condition M_p^+ at $p \nmid \mathfrak{c}$)

$$\langle \underbrace{E_k^{m+n}(\mathrm{diag}[z_1, z_2], s; \chi)}_{\substack{\text{Siegel Jacobi Eisenstein series} \\ \text{on } \mathcal{H}_{m+n,l}}} f(z_2) \rangle = L(f, \chi, 2s - n - l/2) \underbrace{E_k^m(z_1, s; f^c, \chi)}_{\substack{\text{Klingen Jacobi Eisenstein series} \\ \text{on } \mathcal{H}_{m,l}}} ,$$

if $m = n$:

$$\mathrm{vol}(\Gamma_1^n(\mathfrak{c}) \backslash \mathcal{H}_{n,l}) \mathfrak{L}_{(\mathfrak{c})}(2s - n - l/2, \chi) \langle E_k^{2n}(\mathrm{diag}[z_1, z_2], s; \chi), f(z_2) \rangle =^{\overline{\mathbb{Q}}^\times}$$
$$L(2s - n - l/2, f, \chi)(G_{(\mathfrak{c})}(\chi, 2s - n - l/2))^{-1} c_{\mathcal{M},k}(s - k/2) f^c(z_1),$$

where f^c is a "conjugation" of f,
$\mathrm{Re}\, s > (m + n)/2 + l/2 + 1 \mathrm{Re}\, s > n + l/2 + 1.$

Algebraicity - main theorem

$$\Lambda(s, \boldsymbol{f}, \chi) = L(2s - n - l/2, \boldsymbol{f}, \chi) \begin{cases} L_{\mathfrak{c}}(2s - l/2, \chi), & \text{if } l \in 2\mathbb{Z}, \\ 1, & \text{if } l \notin 2\mathbb{Z}. \end{cases}$$

Theorem (Bouganis-M.)

Let $n > 1$ and assume that $\mathcal{M}$ satisfies the condition M_p^+ at $p \nmid \mathfrak{c}$. Let $k > 6n + 2l + 1$, $0 \neq \boldsymbol{f} \in S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$ an eigenfunction of all $T(\mathfrak{a})$ and χ a Dirichlet character such that $\chi(-1) = (-1)^k$. Then:

$$\frac{\Lambda(\sigma/2, \boldsymbol{f}, \chi)}{\pi^{e_\sigma} \langle \boldsymbol{f}, \boldsymbol{f} \rangle} \in \overline{\mathbb{Q}},$$

where e_σ explicit and $\sigma \in (2\mathbb{Z} + k) \cap (2n + l + 1, k - 4n - l)$ (so that the normalised Eisenstein series $E_k^{2n}(z, \sigma/2; \chi)$ is nearly holomorphic and of bounded growth).

Main theorem - sketch of the proof

First we show that for any $\boldsymbol{g} \in M^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$,

$$\langle \boldsymbol{g}, \boldsymbol{f} \rangle \in \pi^{\cdots} \Lambda(\sigma/2, \boldsymbol{f}, \chi)\overline{\mathbb{Q}}.$$

Enough to consider $\boldsymbol{g} \in S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$: apply to $\boldsymbol{g}$ projection to cusp forms. Define the space

$$V(\boldsymbol{f}) := \{\widetilde{\boldsymbol{f}} \in S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}}) : \widetilde{\boldsymbol{f}}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\widetilde{\boldsymbol{f}} \text{ for all } \mathfrak{a}\}.$$

Hecke operators $T(\mathfrak{a})$ are normal and preserve $S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$, so

$$S^n_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}}) = V(\boldsymbol{f}) \oplus U$$

for some $\overline{\mathbb{Q}}$-vector space U orthogonal to $V(\boldsymbol{f})$.

$\rightarrow$ Enough to consider $\boldsymbol{g} \in V(\boldsymbol{f})$.

Main theorem - sketch of the proof

Doubling identity for $\tilde{\boldsymbol{f}} \in V(\boldsymbol{f})$ at $s = \sigma/2$:

$$\mathrm{vol}(\Gamma_1^n(\mathfrak{c})\backslash\mathcal{H}_{n,l})\mathfrak{L}(\sigma - n - l/2, \chi)\langle E_k^{2n}(\mathrm{diag}[z_1, z_2], \sigma/2; \chi), \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} L(\sigma - n - l/2, \boldsymbol{f}, \chi)(G_{(\mathfrak{c})}(\chi, \sigma - n - l/2))^{-1} c_{\mathcal{M},k}(\sigma/2 - k/2)\tilde{\boldsymbol{f}}^c(z_1),$$

$$\mathrm{vol}(\Gamma_1^n(\mathfrak{c})\backslash\mathcal{H}_{n,l})\mathfrak{L}(\sigma - n - l/2, \chi)\langle E_k^{2n}(\mathrm{diag}[z_1, z_2], \sigma/2; \chi), \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} L(\sigma - n - l/2, \boldsymbol{f}, \chi)\underbrace{(G_{(\mathfrak{c})}(\chi, \sigma - n - l/2))^{-1}}_{\in \overline{\mathbb{Q}}^\times} c_{\mathcal{M},k}(\sigma/2 - k/2)\tilde{\boldsymbol{f}}^c(z_1),$$

and suitable χ.

$$\mathrm{vol}(\Gamma_1^n(\mathfrak{c})\backslash\mathcal{H}_{n,l})\mathfrak{L}(\sigma - n - l/2, \chi)\langle E_k^{2n}(\mathrm{diag}[z_1, z_2], \sigma/2; \chi), \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} L(\sigma - n - l/2, \boldsymbol{f}, \chi)\underbrace{(G_{(\mathfrak{c})}(\chi, \sigma - n - l/2))^{-1}}_{\in \overline{\mathbb{Q}}^\times} c_{\mathcal{M},k}(\sigma/2 - k/2)\tilde{\boldsymbol{f}}^c(z_1),$$

and suitable χ.

Main theorem - sketch of the proof

$$\langle G(z_1, z_2; \sigma/2), \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} \pi^{\cdots} \Lambda(\sigma/2, \boldsymbol{f}, \chi)\, \tilde{\boldsymbol{f}}^c(z_1)$$

$$\langle G(z_1, z_2; \sigma/2), \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} \pi^{\cdots} \Lambda(\sigma/2, \boldsymbol{f}, \chi) \tilde{\boldsymbol{f}}^c(z_1) \qquad /e(\mathrm{tr}(\ldots))_{z_1}$$

... and evaluate at $z_1 = \boldsymbol{\omega} = (\omega, v\, {}^t(\omega\ 1_n))$, where ω a CM point of $\mathbb{H}_n$, $v \in M_{l,2n}(\mathbb{Q})$...

$$\langle \underbrace{G_*(\boldsymbol{\omega}, z_2; \sigma/2)}_{\text{Siegel mod. form in } \omega}\ \mathfrak{p}_k(\omega)^{-1}, \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} \pi^{\cdots} \Lambda(\sigma/2, \boldsymbol{f}, \chi)\ \underbrace{\tilde{\boldsymbol{f}}^c_*(\boldsymbol{\omega})}_{\text{Siegel mod. form in } \omega}\ \mathfrak{p}_k(\omega)^{-1}$$

$$\langle \underbrace{G_*(\boldsymbol{\omega}, z_2; \sigma/2)\mathfrak{p}_k(\omega)^{-1}}_{\in N^{n,D}_{k,\mathcal{M}}(\overline{\mathbb{Q}}) \cap \mathcal{B}^n_{k,\mathcal{M}}}, \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} \pi^{\cdots} \Lambda(\sigma/2, \boldsymbol{f}, \chi)\, \tilde{\boldsymbol{f}}^c_*(\boldsymbol{\omega})\, \mathfrak{p}_k(\omega)^{-1}$$

$$\Big\| \qquad \text{projection to cusp forms (in } V(\boldsymbol{f})\text{)}$$

$$\langle g_{\boldsymbol{\omega}}(z_2), \tilde{\boldsymbol{f}}(z_2)\rangle$$

Main theorem - sketch of the proof

$$\langle g_{\boldsymbol{\omega}}(z_2), \tilde{\boldsymbol{f}}(z_2)\rangle =^{\overline{\mathbb{Q}}^\times} \pi^{\cdots}\, \Lambda(\sigma/2, \boldsymbol{f}, \chi)\, \underbrace{\tilde{\boldsymbol{f}}^c_*(\boldsymbol{\omega})\mathfrak{p}_k(\omega)^{-1}}_{\in \overline{\mathbb{Q}}}$$

Note:

$$V(\boldsymbol{f}) = \mathrm{span}_{\overline{\mathbb{Q}}}\{g_{\boldsymbol{\omega}} : \boldsymbol{\omega}\}$$

because $\Lambda(\sigma/2, \boldsymbol{f}, \chi) \neq 0$ and the CM points are dense in $\mathbb{H}_n$.

Now for $\tilde{\boldsymbol{f}} = \boldsymbol{f}$ and any $\boldsymbol{g} \in V(\boldsymbol{f})$:

$$\langle \boldsymbol{g}, \boldsymbol{f}\rangle \in \pi^{\cdots}\, \Lambda(\sigma/2, \boldsymbol{f}, \chi)\overline{\mathbb{Q}}.$$

Taking $\boldsymbol{g} = \boldsymbol{f}$ and suitable $\boldsymbol{\omega}$:

$$\frac{\pi^{\cdots}\, \Lambda(\sigma/2, \boldsymbol{f}, \chi)}{\langle \boldsymbol{f}, \boldsymbol{f}\rangle} =^{\overline{\mathbb{Q}}^\times} \frac{\langle g_{\boldsymbol{\omega}}, \boldsymbol{f}\rangle}{\langle \boldsymbol{f}, \boldsymbol{f}\rangle} \in \overline{\mathbb{Q}}.$$

Nearly holomorphic Jacobi forms

A C^∞ function $f(\tau, w) : \mathcal{H}_{n,l} \to \mathbb{C}$ such that

1. $\forall_{\gamma\in\Gamma}\, f|_{k,\mathcal{M}}\gamma = f$, for some $k, \mathcal{M}, \Gamma$,
2. f is holomorphic with respect to w
   and nearly holomorphic with respect to τ;

then

$$f(\tau, w) = \sum_{\substack{t\in\frac{1}{2}Sym_n(\mathbb{Z})\\ t\geq 0}} \sum_{r\in M_{l,n}(\mathbb{Z})} p_{t,r}(\mathrm{Im}\,(\tau)^{-1})e(\mathrm{tr}(t\tau))e(\mathrm{tr}({}^t rw)),$$

where $p_{t,r}$ are polynomial functions on $Sym_n(\mathbb{R})$ of total degree at most D;

we write $N^{n,D}_{k,\mathcal{M}}(\Gamma)$ or $N^{n,D}_{k,\mathcal{M}}(\Gamma, \overline{\mathbb{Q}})$.

Functions of bounded growth

$f \in N^{n,D}_{k,\mathcal{M}}(\Gamma)$ is of bounded growth if

$$\int\!\!\int\!\!\int\!\!\int_{\bigcup_{g \in \Gamma_\infty \backslash \Gamma} g\mathcal{F}_\Gamma} |f(z)|\, e^{-2\pi \mathrm{tr}(ty + {}^t r v)} \Delta_{\mathcal{M},k}(z) dz < \infty$$

for all (t, r) such that $t > 0$ and $4t - {}^t r \mathcal{M}^{-1} r > 0$, where

$$\Delta_{\mathcal{M},k}(z) dz = (\det y)^{k-l-n-1} e(2i \mathrm{tr}({}^t v \mathcal{M} v y^{-1})) dx dy du dv$$

and $z = (\tau, w)$, $\tau = x + iy$, $w = u + iv$.

Example

- $S^n_{k,\mathcal{M}}(\Gamma)$ if $k > 2n + l$,
- (normalized) $E^{2n}_k(z, \sigma/2; \chi)$ and its pullback $E^{2n}_k(\mathrm{diag}[z_1, z_2], \sigma/2; \chi)$, if $\sigma \in (2n + l/2 + 1, k - 4n - l)$, $\sigma \in k + 2\mathbb{Z}$.

Holomorphic & cuspidal projection

Theorem (B.-M.; Jacobi Poincaré series and reproducing kernel)

For (t, r) such that $t > 0$ and $4t - {}^t r\mathcal{M}^{-1}r > 0$, if $k > 2n + l$, then

$$P_{t,r}(\tau_1, w_1) := \sum_{\boldsymbol{g} \in Z_l \Gamma_\infty \backslash \Gamma} \overline{\chi(\boldsymbol{g})} e(\mathrm{tr}(t\tau_1 + {}^t r w_1))|_{k,\mathcal{M}}\, \boldsymbol{g} \in S^n_{k,\mathcal{M}}(\Gamma, \chi).$$

This is because for $k > 2n + l$ and every $z_2 \in \mathcal{H}_{n,l}$,

$$P_{k,\mathcal{M}}(z_1, z_2) = \sum_{\boldsymbol{g} \in Z_l \backslash \Gamma} \overline{\chi(\boldsymbol{g})} \det(\tau_1 + \tau_2)^{-k} e(-\mathrm{tr}(\mathcal{M}[w_1 - w_2](\tau_1 + \tau_2)^{-1}))|^{(1)}_{k,\mathcal{M}}\, \boldsymbol{g}$$

is a cusp form in z_1 in $S^n_{k,\mathcal{M}}(\Gamma, \chi)$. Moreover, then

$$\begin{aligned} K(z_1, z_2) &= C_1 P_{k,\mathcal{M}}((\tau_1, w_1), (-\bar{\tau}_2, \bar{w}_2)) \\ &= C_2 \sum_{t,r} \det(4t - {}^t r\mathcal{M}^{-1}r)^{k-(n+l+1)/2} P_{t,r}(\tau_1, w_1) e(-\mathrm{tr}(t\bar{\tau}_2 + {}^t r\bar{w}_2)) \end{aligned}$$

satisfies

$$Hol(f)(z_2) := \langle f(z_1), K(z_1, z_2)\rangle \in S^n_{k,\mathcal{M}}(\Gamma, \chi), \quad f \in N^{n,D}_{k,\mathcal{M}}(\Gamma) \cap \mathcal{B}^n_{k,\mathcal{M}}(\Gamma)$$

and $\langle f, g\rangle = \langle Hol(f), g\rangle$ *for all $g \in S^n_{k,\mathcal{M}}(\Gamma, \chi)$.*

Thank you for your attention!

The (standard) L-function

Consider a (commutative) Hecke agebra generated by

$$T_r := \Gamma_1^n(\mathfrak{c}) \begin{pmatrix} {}^t r^{-1} & \\ & r \end{pmatrix} \Gamma_1^n(\mathfrak{c})$$

where

$$r \in Q(\mathfrak{c}) := \{\underline{r} \in \mathrm{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z}) : \forall_{p|\mathfrak{c}} \underline{r}_p \in \mathrm{GL}_n(\mathbb{Z}_p)\}$$

and for $\mathfrak{a} \in \mathbb{Z}$ define

$$f|T(\mathfrak{a}) := \sum_{\substack{r \in Q(\mathfrak{c}) :\, \det r = \mathfrak{a} \\ \mathrm{GL}_n(\mathbb{Z}) r \mathrm{GL}_n(\mathbb{Z}) \text{ distinct}}} f|_{k,\mathcal{M}} T_r.$$

Assume $0 \neq f \in S_{k,\mathcal{M}}(\Gamma_1^n(\mathfrak{c}))$ satisfies $f|T(\mathfrak{a}) = \lambda(\mathfrak{a}) f$ for all $\mathfrak{a}$.
For a Dirichlet character χ we define

$$D(s, f, \chi) := \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})\lambda(\mathfrak{a})}{\mathfrak{a}^s}, \qquad \mathrm{Re}\,(s) > 2n + l + 1.$$

The (standard) L-function

In fact, if $\det r \nmid \mathfrak{c}$,

$$r \in \{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in M_n(\mathbb{Z}) : 0 < a_1 \mid a_2 \mid \ldots \mid a_n,\ \gcd(a_n, \mathfrak{c}) = 1\}.$$

This is because we assume $\mathcal{M}$ satisfies the condition M_p^+ at $p \nmid \mathfrak{c}$:

- $$\forall_{\mathbb{Z}_p\text{-lattice } M\subset\mathbb{Q}_p^l} \left(\mathbb{Z}_p^l \subset M, \quad \forall_{x\in M}\, {}^t x\mathcal{M}x \in \mathbb{Z}_p \quad \Rightarrow \quad M = \mathbb{Z}_p^l\right),$$
- $$\{x \in (2\mathcal{M})^{-1}\mathbb{Z}_p^l :\ p\, {}^t x\mathcal{M}x \in \mathbb{Z}_p\} = \mathbb{Z}_p^l.$$