Free algebras of modular forms on complex ball quotients

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The goal of this project was to find unitary groups whose rings of modular forms are polynomial algebras, with no relations.

Such examples are generalizations of the well-known result

$$M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$

describing modular forms for the full elliptic modular group. These examples correspond to reflection groups acting on the complex ball with finite covolume for which the (Baily–Borel compactification of the) quotient is a weighted projective space.

Motivation. Reflections and modular forms.

Let $\mathbb{H}$ be the upper half-plane.

The reflection through the imaginary axis is the (orientation-reversing) automorphism

$$\sigma : \mathbb{H} \longrightarrow \mathbb{H}, \quad x + iy \mapsto -x + iy.$$

More generally the reflection through a geodesic arc γ is

$$\varphi^{-1} \circ \sigma \circ \varphi$$

where φ is any Möbius transformation mapping γ into $i\mathbb{R}_{>0}$.

Let $\Gamma = \Gamma(2, 3, \infty)$ be the group generated by reflections through the sides of the hyperbolic triangle $\mathcal{F}$ with vertices at $\rho = e^{2\pi i/3}$, at i, and at ∞. Then $\mathcal{F}$ is a fundamental domain for Γ and its images tile $\mathbb{H}$.

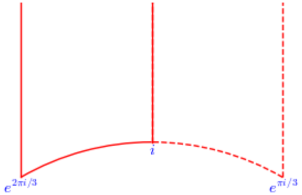

Figure: The geodesic triangle $\mathcal{F}$ and its image under the reflection σ.

Riemann mapping theorem: there is a conformal isomorphism

$$j : \mathcal{F} \xrightarrow{\sim} \mathbb{H}.$$

We can extend j to the boundary so assume it maps the sides of the triangle into $\mathbb{R} \cup \{\infty\}$. After a Möbius transformation suppose $j(\rho) = 0$, $j(i) = 1728$, $j(\infty) = \infty$.

Schwarz reflection principle: j extends across reflections $\sigma \in \Gamma$ by defining

$$j(\sigma\tau) = \overline{j(\tau)},$$

giving the analytic continuation of j to all $\mathbb{H}$.

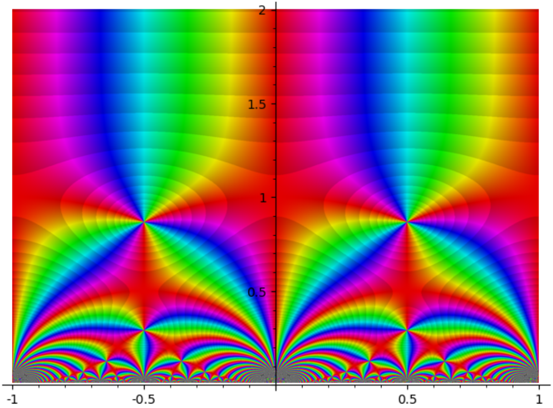

Figure: The j-invariant. The color represents $\arg j(\tau)$ and the absolute value $|j(\tau)|$ is indicated by level curves. Image made in SageMath.

Complex reflections of $\mathbb{H}$ on the other hand are holomorphic maps $\sigma : \mathbb{H} \to \mathbb{H}$ satisfying $\sigma^n = 1$ for some n with fixed point set of complex codimension one.

Complex reflections fixing a point $\tau \in \mathbb{H}$ can be constructed by taking the product of two (real) reflections through geodesic arcs intersecting in τ if they intersect with internal angle π/n.
For example composing the (real) reflection through the lines $x = 0$ and the circle $|z| = 1$ yields the complex reflection

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

through the point $\tau = i$.

Since $\mathrm{PSL}_2(\mathbb{Z})$ is the orientation-preserving subgroup of Γ the result is a *modular function*,

$$j\left(\frac{a\tau+b}{c\tau+d}\right)=j(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}).$$

We now have a map

$$j : \mathrm{PSL}_2(\mathbb{Z})\backslash\overline{\mathbb{H}} \longrightarrow \mathbb{P}^1(\mathbb{C}).$$

After computing the derivative

$$j'(\tau) = -2\pi i E_4(\tau)^2 E_6(\tau)/\Delta(\tau)$$

one can show that this is a conformal isomorphism outside of the cusps and the fixed points of complex reflections, (i.e. elliptic points), $\tau = i, e^{2\pi i/3}$ and their images under $\mathrm{PSL}_2(\mathbb{Z})$.

Our point of view will be that the divisor of j' contains more information about the algebraic structure of modular forms than one would expect.

There are issues in the above example at cusps (e.g. j is not holomorphic at the cusps.) In some respects this project is actually easier to carry out in higher dimension since we do not have to deal with cusps.

The unitary picture: use for example the Cayley transform $\phi(\tau) = z = \frac{\tau - i}{\tau + i}$ to identify $\tau \in \mathbb{H}$ with $z \in \mathbb{D}$, the unit disc.

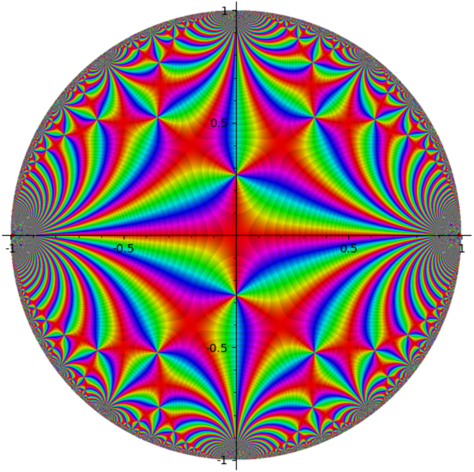

Figure: Klein j-invariant in the unit disc: $z \mapsto j(\phi^{-1} z)$.

The automorphism group $\mathrm{PSL}_2(\mathbb{R})$ of $\mathbb{H}$ becomes the automorphism group

$$\mathrm{Aut}\,\mathbb{D} = \mathrm{PU}(1,1),$$

the projective unitary group of the Hermitian form $(z_1, z_2) \mapsto |z_1|^2 - |z_2|^2$ on $\mathbb{C}^2$. Explicitly $U(1,1)$ consists of matrices

$$e^{i\theta}\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1,\ 0 \leq \theta < 2\pi$$

acting on $\mathbb{D}$ by Möbius transformations:

$$e^{i\theta}\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}.$$

For an arithmetic subgroup $\Gamma \leq \mathrm{Aut}\,\mathbb{D}$ we can consider modular forms, i.e. holomorphic functions

$$f : \mathbb{D} \longrightarrow \mathbb{C}$$

satisfying

$$f(M \cdot z) = (cz + d)^k f(z), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

together with a holomorphy condition at the cusps ("rational" points on the circle).

Typical examples for Γ come from unitary groups of Hermitian lattices L of signature $(1,1)$ defined over some order $\mathcal{O}$ in an imaginary-quadratic number field K.

Example. Conjugating $\mathrm{SL}_2(\mathbb{Z})$ by the Cayley transform gives us the *special* unitary group $\mathrm{SU}(L)$ of the unimodular Gaussian lattice

$$L = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \ z_1 + z_2 \in \mathbb{Z}[i] \text{ and } z_1 - z_2 \in \mathbb{Z}[i]\Big\} \subseteq \mathbb{C}^{1,1}.$$

The algebra of its modular forms is therefore

$$M_*(\mathrm{SU}(L)) \cong M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6].$$

The action of $\mathrm{U}(L)$ on $\mathbb{D}$ is generated by $\mathrm{SU}(L)$ and $z \mapsto iz$ so we have

$$M_*(\mathrm{U}(L)) \cong M_{4*}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, \Delta].$$

Several variables.

The automorphism group of the complex n-ball $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z| < 1\}$ is

$$\mathrm{Aut}\,\mathbb{D}^n = \mathrm{PU}(n,1).$$

For discrete subgroups $\Gamma \leq \mathrm{U}(n,1)$, the unitary group of the hermitian form

$$(z_1, ..., z_{n+1}) \mapsto |z_1|^2 + ... + |z_n|^2 - |z_{n+1}|^2,$$

we have the notion of modular forms

$$f : \mathbb{D}^n \longrightarrow \mathbb{C};$$

they should satisfy

$$f\Big((az+b)(cz+d)^{-1}\Big) = (cz+d)^k f(z)$$

for block matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Here a is an $(n \times n)$-block and d is a (1×1)-block.

When $\Gamma = \mathrm{U}(L)$ for a Hermitian lattice $L/\mathcal{O}$ of signature $(n, 1)$, the quotient $\mathbb{D}^n/\mathrm{U}(L)$ can be compactified by including finitely many zero-dimensional (i.e. point) cusps corresponding to the orbits of norm zero vectors of L under $\mathrm{U}(L)$.

If $n > 1$ then Koecher's principle holds and modular forms are automatically holomorphic in the cusps.

It is more convenient to start with an abstract lattice L and view modular forms of weight k as holomorphic functions

$$F : \mathcal{A}_L \longrightarrow \mathbb{C}$$

where $\mathcal{A}_L$ is the negative cone $\{z \in L \otimes \mathbb{C} : \langle z, z \rangle < 0\}$ and modularity becomes the equations

$$F(tz) = t^{-k} F(z), \quad t \in \mathbb{C}^\times;$$

$$F(\gamma z) = F(z), \quad \gamma \in \Gamma \leq \mathrm{U}(L).$$

When $\Gamma \leq \mathrm{U}(n,1)$ the associated function on $\mathbb{D}^n$ is $f(z) := F(z, 1)$.

We want to consider the case where the algebra $M_*(\Gamma)$ is freely generated with L of signature $(n,1)$. Since $\mathbb{D}^n/\Gamma$ is n-dimensional we will need $(n+1)$ generators $f_0, ..., f_n$. Consider the map

$$F = \begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix} : \overline{\mathbb{D}^n/\Gamma} \longrightarrow \mathbb{P}(k_0, ..., k_n)$$

into weighted projective space

$$\Big(\mathbb{C}^{n+1}\backslash\{0\}\Big)/\Big((x_0, ..., x_n) \sim (\lambda^{k_0}x_0, ..., \lambda^{k_n}x_n),\ \lambda \in \mathbb{C}^\times\Big)$$

where $k_0, ..., k_n$ are the weights of $f_0, ..., f_n$.

Theorem (Wang, W.)

Let $f_0, ..., f_n \in M_*(\Gamma)$.
(i) The Jacobian determinant

$$J = J(f_0, ..., f_n) = \det \begin{pmatrix} \partial_{z_0} f_0 & ... & \partial_{z_0} f_n \\ \vdots & \vdots & \vdots \\ \partial_{z_n} f_0 & ... & \partial_{z_n} f_n \end{pmatrix}$$

(where $z_0, ..., z_n$ are any holomorphic coordinates on $\mathbb{D}^n/\Gamma$) is a cusp form of weight

$$k = k_0 + ... + k_n + n + 1$$

with the determinant character, i.e.

$$J(tz) = t^{-k} J(z), \quad J(\gamma z) = \det(\gamma) J(z),\ \gamma \in \Gamma.$$

$J \neq 0$ if and only if $f_0, ..., f_n$ are algebraically independent.

(ii) Suppose $\Gamma \leq \mathrm{U}(L)$ for a Hermitian lattice $L/\mathcal{O}_K$. Suppose Γ contains a complex reflection,

$$z \mapsto \sigma_{r,\alpha}(z) = z - (1-\alpha)\frac{\langle z, r\rangle}{\langle r, r\rangle} r$$

for some $r \in L^{\#} = \{x \in L \otimes \mathbb{C} : \ \langle x, y\rangle \in \mathcal{O}_K \text{ for all } y \in L\}$ and some unit $\alpha \in \mathcal{O}_K^{\#}$. Then J vanishes on the mirror

$$r^{\perp} = \{z \in L \otimes \mathbb{C} : \ \langle z, r\rangle = 0\}$$

and in fact

$$\operatorname{ord}(J, r^{\perp}) \equiv -1 \pmod{\operatorname{ord}\alpha}.$$

Note that $\operatorname{ord}\sigma_{r,\alpha} = \operatorname{ord}\alpha \in \{2,3,4,6\}$ i.e. $\sigma_{r,\alpha}$ can be a *biflection*, *triflection*, *tetraflection* or *hexaflection* respectively.

Theorem (Wang, W.)

Let $\Gamma \leq \mathrm{U}(L)$ *be a finite-index subgroup with* L *of signature* $(n, 1)$, $n \geq 2$ *and suppose* $M_*(\Gamma) = \mathbb{C}[f_0, ..., f_n]$.
(i) $J = J(f_0, ..., f_n)$ *vanishes precisely on the mirrors* $r^\perp$ *of complex reflections* $\sigma_{r,\alpha} \in \Gamma$, *to order exactly*

$$\mathrm{ord}(J; r^\perp) = -1 + \max\{\mathrm{ord}\,\alpha : \ \sigma_{r,\alpha} \in \Gamma\}.$$

(ii) Γ *is generated by complex reflections.*
(iii) Suppose $\Gamma\pi_1, ..., \Gamma\pi_s$ *are the distinct orbits of reflection mirrors under* Γ. *Then* J *satisfies a quadratic equation*

$$J^2 = \prod_{i=1}^{s} P_i(f_0, ..., f_n)$$

with irreducible polynomials P_i, *and each form* $P_i(f_0, ..., f_n)$ *vanishes only on the* Γ*-orbit* $\Gamma\pi_i$.

Theorem (Wang, W.)

Let $\Gamma \leq \mathrm{U}(L)$ be a finite-index subgroup with L of signature $(n, 1)$, $n \geq 2$. Suppose there exist modular forms $f_0, ..., f_n$ whose Jacobian J is nonzero and has divisor exactly

$$\mathrm{div}\, J = \sum_{r \in L'} \Big(-1 + \max \{ \mathrm{ord}\, \alpha : \ \sigma_{r,\alpha} \in \Gamma \} \Big) r^{\perp}.$$

Then $M_*(\Gamma) = \mathbb{C}[f_0, ..., f_n]$.

(In particular the claims of the last theorem hold.)

The mirrors of reflections of Γ weighted as above form the so-called *discriminant* divisor of the ball quotient $\mathbb{D}^n/\Gamma$.

How we use this: For some lattices one can construct a modular form J representing the discriminant divisor. The most powerful tool for this is Borcherds' theory of modular products. A $\mathrm{U}(n,1)$-version of Borcherds products was given by Hoffmann. Now if we can find *any* modular forms $f_0, ..., f_n$ with

$$\mathrm{wt}(f_0) + ... + \mathrm{wt}(f_n) + n + 1 = \mathrm{wt}(J)$$

and such that $J(f_0, ..., f_n)$ is not identically zero, (which is usually easy to check by computer) Koecher's principle implies $J(f_0, ..., f_n) = \mathrm{const} \cdot J$ and we obtain

$$M_*(\Gamma) = \mathbb{C}[f_0, ..., f_n].$$

Many easy examples come from algebras of modular forms on $\mathrm{O}(2n, 2)$.

A short review: Let L be an even $\mathbb{Z}$-lattice of signature $(n, 2)$, $n \geq 3$. The orthogonal group $\Gamma = \mathrm{O}(L)$ acts on a cone domain $\mathcal{A}_L$ consisting of vectors $z \in L \otimes \mathbb{C}$ of norm zero satisfying $\langle z, \overline{z} \rangle > 0$. (This space splits into two connected components and we fix one.)

We can define *orthogonal modular forms*, holomorphic functions

$$f : \mathcal{A}_L \longrightarrow \mathbb{C}$$

satisfying

$$f(tz) = t^{-k} f(z);$$
$$f(\gamma z) = f(z);$$

for $t \in \mathbb{C}^\times$ and $\gamma \in \Gamma$.

Now suppose L has complex multiplication by some $\mathcal{O}_K$, i.e. $\mathcal{O}_K \subseteq \operatorname{End} L$. Then L has a unitary group

$$\mathrm{U}(L) = \{\gamma \in \mathrm{O}(L) :\ \gamma \text{ commutes with } \mathcal{O}_K\}$$

which acts on the cone domain

$$\mathcal{A}_L^{\mathrm{U}} = \{z \in \mathcal{A}_L :\ \mathcal{O}_K \cdot z = \operatorname{span}(z)\}$$

consisting of eigenvectors of $\mathcal{O}_K$. We have a natural restriction map

$$M_*(\mathrm{O}(L)) \longrightarrow M_*(\mathrm{U}(L)), \quad f \mapsto f\Big|_{\mathcal{A}_L^{\mathrm{U}}}.$$

Example. Suppose L is the lattice of integral (2x2)-matrices with $\det$ as its quadratic form. This is an even integral lattice of signature $(2,2)$. The orthogonal group is a semidirect product

$$\mathrm{O}(L) = (\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $(A, B) \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ acts by

$$(A, B) \cdot X = AXB^T, \quad X \in L$$

and where the $\mathbb{Z}/2\mathbb{Z}$ factor is the transpose $X \mapsto X^T$.

To a modular form $f : \mathcal{A}_L \to \mathbb{C}$ we can associate the function

$$g(z_1, z_2) := f\left(\begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix}\right), \quad z_1, z_2 \in \mathbb{H}.$$

This turns out to be an identification

$$M_*(\mathrm{O}(L)) \xrightarrow{\sim} \mathrm{Sym}^2 M_*(\mathrm{SL}_2(\mathbb{Z}))$$

to symmetric functions g which are modular forms in each variable z_1, z_2.

For any $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\omega$ we define an action on L by

$$(\alpha + \beta\omega) \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

if $(\alpha + \beta\omega)(x_{11} + \omega x_{22}) = y_{11} + \omega y_{22}$ and
$(\alpha + \beta\omega)(x_{12} + \omega x_{21}) = y_{12} + \omega y_{21}$.

Then the restriction of a form $g(z_1, z_2)$ to $\mathrm{U}(L)$ is

$$g^{\mathrm{U}}(\tau) := g(\tau, -\overline{\omega}).$$

Theorem (Wang, W.)

Suppose $\Gamma \leq \mathrm{O}(L)$ is an arithmetic subgroup for which $M_*(\Gamma) = \mathbb{C}[f_0, ..., f_{2n}]$ is freely generated, and suppose we can choose n generators $f_{n+1}, ..., f_{2n}$ whose restrictions to the unitary group vanish. Then

$$M_*(\Gamma \cap \mathrm{U}(L)) = \mathbb{C}\left[f_0\Big|_{\mathcal{A}_L^{\mathrm{U}}}, ..., f_n\Big|_{\mathcal{A}_L^{\mathrm{U}}} \right].$$

Proof sketch: $J\left(f_0\Big|_{\mathcal{A}_L^{\mathrm{U}}}, ..., f_n\Big|_{\mathcal{A}_L^{\mathrm{U}}} \right)$ occurs as a factor of $J(f_0, ..., f_{2n})$ and is therefore also nonzero.

When Γ contains $\mathrm{U}(L)$ and the field is either $\mathbb{Q}(i)$ or $\mathbb{Q}(e^{2\pi i/3})$, it is often easy to use this criterion because nonzero modular forms on $\mathrm{U}(L)$ must have weight 0 mod 4 or 0 mod 6 respectively.

Corollary

L is still of signature $(2n, 2)$.
(i) Suppose $K = \mathbb{Q}(i)$ and $M_*(\mathrm{O}(L))$ is free with exactly $(n+1)$ generators of weight 0 mod 4. Then $M_*(\mathrm{U}(L))$ is freely generated.
(ii) Suppose $K = \mathbb{Q}(e^{2\pi i/3})$ and $M_*(\mathrm{O}(L))$ is free with exactly $(n+1)$ generators of weight 0 mod 6. Then $M_*(\mathrm{U}(L))$ is freely generated.

Example. (Not really because Koecher's principle does not hold.) Let $L = \mathrm{II}_{2,2}$ be the even unimodular lattice of signature $(2,2)$. We saw earlier that $M_*(L) = \mathrm{Sym}^2 M_*(\mathrm{SL}_2(\mathbb{Z}))$. Generators are

$$g(z_1, z_2) = E_4(z_1)E_4(z_2), \quad E_6(z_1)E_6(z_2), \quad \Delta(z_1)\Delta(z_2)$$

of weights $4, 6, 12$. Therefore:

(i) If $L = \mathrm{II}_{1,1}^{\mathbb{Z}[i]}$ is the even unimodular Gaussian lattice of signature $(1,1)$ then $M_*(L)$ is freely generated in weights $4, 12$.

(ii) If $L = \mathrm{II}_{1,1}^{\mathbb{Z}[e^{2\pi i/3}]}$ is the even unimodular Eisenstein lattice of signature $(1,1)$ then $M_*(L)$ is freely generated in weights $6, 12$.

Example. Let $L = \mathrm{II}_{10,2}$ be the even unimodular lattice of signature $(10, 2)$. It was proved by Hashimoto–Ueda that

$$M_*(\mathrm{O}(L)) = \mathbb{C}[f_4, f_{10}, f_{12}, f_{16}, f_{18}, f_{22}, f_{24}, f_{28}, f_{30}, f_{36}, f_{42}].$$

Therefore:

(i) If $L = \mathrm{II}_{5,1}^{\mathbb{Z}[i]}$ is the even unimodular Gaussian lattice of signature $(5, 1)$ then $M_*(L)$ is freely generated in weights $4, 12, 16, 24, 28, 36$.

$$\color{red}{4,}\ 10,\ \color{red}{12,\ 16,}\ 18,\ 22,\ \color{red}{24,\ 28,}\ 30,\ \color{red}{36,}\ 42.$$

(ii) If $L = \mathrm{II}_{5,1}^{\mathbb{Z}[e^{2\pi i/3}]}$ is the even unimodular Eisenstein lattice of signature $(5, 1)$ then $M_*(L)$ is freely generated in weights $12, 18, 24, 30, 36, 42$.

$$4,\ 10,\ \color{red}{12,}\ 16,\ \color{red}{18,}\ 22,\ \color{red}{24,}\ 28,\ \color{red}{30,\ 36,\ 42.}$$

Gaussian lattice examples:
Here and below, U is the unimodular lattice $\mathbb{Z}^2$ with quadratic form $(x, y) \mapsto xy$, and A_n, D_n, E_n denotes an irreducible root lattice.

L	$M_*(\mathrm{O}(L))$	$M_*(\mathrm{U}(L))$
$2U \oplus 2A_1$	$4, 6, 8, 10, 12$	$4, 8, 12$
$U \oplus U(2) \oplus 2A_1$	$2, 4, 6, 8, 12$	$4, 8, 12$
$2U \oplus D_4$	$4, 6, 10, 12, 16, 18, 24$	$4, 12, 16, 24$
$2U \oplus 4A_1$	$4, 4, 6, 6, 8, 10, 12$	$4, 4, 8, 12$
$2U \oplus D_6$	$4, 6, 8, 10, 12, 12, 14, 16, 18$	$4, 8, 12, 12, 16$
$2U \oplus D_8$	$4, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18$	$4, 8, 8, 12, 12, 16$
$2U \oplus E_8$	$4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42$	$4, 12, 16, 24, 28, 36$
$U \oplus U(2) \oplus D_4$	$2, 6, 8, 10, 12, 16, 20$	$8, 12, 16, 20$

Eisenstein lattice examples. In three cases below, the ring $M_*(\mathrm{O}(L))$ is not freely generated, but $M_*(\mathrm{O}_r(L))$ is (where $\mathrm{O}_r(L)$ is the subgroup of $\mathrm{O}(L)$ generated by complex reflections) and we can argue similarly.

L	$M_*(\mathrm{O}(L))$	$M_*(\mathrm{U}(L))$
$2U \oplus A_2$	$4, 6, 10, 12, 18$	$6, 12, 18$
$U \oplus U(3) \oplus A_2$	–	$6, 12, 18$
$2U(2) \oplus A_2$	–	$6, 6, 12$
$2U \oplus D_4$	$4, 6, 10, 12, 16, 18, 24$	$6, 12, 18, 24$
$2U \oplus E_6$	–	$6, 12, 18, 24, 30$
$2U \oplus E_8$	$4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42$	$12, 18, 24, 30, 36, 42$

With a little more work we can sometimes get the structure of $M_*(\Gamma)$ for subgroups $\Gamma \leq \mathrm{U}(L)$. An important example is the discriminant kernel

$$\tilde{\mathrm{U}}(L) = \{\gamma \in \mathrm{U}(L) : \ \gamma x - x \in L \text{ for all } x \in L'\}.$$

This works because modular forms on $\tilde{\mathrm{U}}(L)$ are often modular forms with character on $\mathrm{U}(L)$.

Gaussian lattices; discriminant kernel.

L	$M_*(\widetilde{\mathrm{O}}(L))$	$M_*(\widetilde{\mathrm{U}}(L))$
$2U \oplus D_4$	$4, 6, 8, 8, 10, 12, 18$	$4, 8, 8, 12$
$U \oplus U(2) \oplus D_4$	$2, 4, 4, 4, 4, 6, 10$	$4, 4, 4, 4$
$2U(2) \oplus D_4$	$2, 2, 2, 2, 2, 2, 6$	$2, 2, 2, 2$
$2U \oplus D_6$	$4, 6, 6, 8, 10, 12, 14, 16, 18$	$4, 6, 8, 12, 16$
$2U \oplus D_8$	$4, 4, 6, 8, 10, 10, 12, 12, 14, 16, 18$	$4, 4, 8, 12, 12, 16$

Eisenstein lattices; discriminant kernel.

L	$M_*(\widetilde{\mathrm{O}}(L))$	$M_*(\widetilde{\mathrm{U}}(L))$
$2U \oplus A_2$	$4, 6, 9, 10, 12$	$6, 9, 12$
$U \oplus U(3) \oplus A_2$	$1, 3, 3, 3, 4$	$3, 3, 3$
$2U(2) \oplus A_2$	$2, 2, 2, 2, 3$	$2, 2, 3$
$2U(3) \oplus A_2$	$1, 1, 1, 1, 1$	$1, 1, 1$
$2U \oplus D_4$	$4, 6, 8, 8, 10, 12, 18$	$6, 8, 12, 18$
$2U(2) \oplus D_4$	$2, 2, 2, 2, 2, 2, 6$	$2, 2, 2, 6$
$2U \oplus E_6$	$4, 6, 7, 10, 12, 15, 16, 18, 24$	$6, 12, 15, 18, 24$

Other cases. We found several examples of free algebras of modular forms for $\mathrm{U}(n,1)$ that do not come from a free algebra of modular forms for $\mathrm{O}(2n,2)$. I will work out one example of the method.

Consider the group

$$\Gamma := \mathrm{SU}(2,1) \cap \mathrm{SL}_3(\mathbb{Z}[i])$$

of determinant one (3×3)-matrices with entries in $\mathbb{Z}[i]$ that preserve the Hermitian form $|z_1|^2 + |z_2|^2 - |z_3|^2$ on $\mathbb{C}^3$. It was shown by Resnikoff and Tai (1978) that $M_*(\Gamma)$ is generated in weights $4, 8, 10, 12, 17$.

To interpret this, we identify Γ as

$$\ker \det : \mathrm{U}(L) \longrightarrow \mathbb{Z}[i]^{\times},$$

where $L \subseteq \mathbb{C}^{2,1}$ is the lattice of vectors (z_1, z_2, z_3) with $z_1, z_2 \pm z_3 \in \mathbb{Z}[i]$.

The underlying lattice over $\mathbb{Z}$ is $2U \oplus 2A_1$. So we view $L = 2U \oplus 2A_1$ with an action of $\mathbb{Z}[i]$.

We showed earlier that $M_*(\mathrm{O}(L)) = \mathbb{C}[X_4, X_6, X_8, X_{10}, X_{12}]$ and therefore $M_*(\mathrm{U}(L)) = \mathbb{C}[X_4, X_8, X_{12}]$.

Taking an nth root of a factor in the Jacobian $J(X_4, X_8, X_{12})$ we obtain modular forms f_1, f_2, f_3 (with character), each vanishing with simple zeros only on the mirrors of a $\mathrm{U}(L)$-conjugacy class of reflections:

$$\operatorname{div} f_1 = \Delta_{1/4} = \bigcup_{\substack{r \in L' \\ \langle r,r \rangle = 1/4}} r^\perp;$$

$$\operatorname{div} f_2 = \Delta_{1/2} = \bigcup_{\substack{r \in L' \\ \text{primitive} \\ \langle r,r \rangle = 1/2}} r^\perp;$$

$$\operatorname{div} f_3 = \Delta_1 = \bigcup_{\substack{r \in L' \\ \text{primitive} \\ \langle r,r \rangle = 1}} r^\perp;$$

These forms can also be constructed as Borcherds products. From this we find the weights $\mathrm{wt}(f_1) = 2$, $\mathrm{wt}(f_2) = 3$, $\mathrm{wt}(f_3) = 12$.

One can check that the divisors $\Delta_{1/4}$ and $\Delta_{1/2}$ are tetraflective, i.e. if $\langle r, r \rangle = 1/4$ or $\langle r, r \rangle = 1/2$ then the tetraflection

$$\sigma_{r,i} : z \mapsto z - (1-i)\frac{\langle z, r \rangle}{\langle r, r \rangle} r$$

preserves our lattice (although it acts nontrivially on L'/L). On the other hand Δ_1 only admits biflections.

Then f_1 and f_2 have (distinct) characters χ_1 and χ_2 of order four and f_3 transforms with a character χ_3 of order two; and χ_1, χ_2, χ_3 generate the group of characters of $\mathrm{U}(L)$.

Let $\tilde{\mathrm{U}}_r(L)$ be the subgroup of the discriminant kernel $\tilde{\mathrm{U}}(L)$ generated by reflections.

Theorem

$M_*(\tilde{\mathrm{U}}_r(L)) = \mathbb{C}[E_4, f_1^2, f_2^2]$ is generated in weights $4, 4, 6$.

Here E_4 is the Eisenstein series.

Proof.

The reflections that act trivially on L'/L are precisely the biflections through the mirrors that make up $\Delta_{1/4}$, $\Delta_{1/2}$, Δ_1, so the discriminant divisor is

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = \mathrm{div}(f_1 f_2 f_3).$$

E_4, f_1, f_2 are algebraically independent because the restrictions of E_4 and f_2 to any $r^\perp$ with $\langle r, r\rangle = 1/4$ are algebraically independent (E_4 becomes an Eisenstein series; f_2 becomes a cusp form) and f_1 vanishes on $r^\perp$ by definition. So $J(E_4, f_1^2, f_2^2)$ is nonzero and has weight 17 so it equals the discriminant form $f_1 f_2 f_3$ up to constant, and we are done.

Remark. The discriminant kernel $\tilde{\mathrm{U}}(L)$ is *not* generated by reflections. Its ring of modular forms is

$$M_*(\tilde{\mathrm{U}}(L)) = \mathbb{C}[E_4, f_1^4, f_1^2 f_2^2, f_2^4]$$

modulo an obvious relation.

We also recover

$$M_*(\mathrm{U}(L)) = \mathbb{C}[E_4, f_1^4, f_2^4]$$

in weights $4, 8, 12$.

Now the group considered by Resnikoff–Tai was

$$\Gamma = \ker(\det) = \ker(\chi_1\chi_2\chi_3).$$

The characters of $\mathrm{U}(L)$ that are trivial on Γ are exactly the four powers of $\chi_1\chi_2\chi_3$, corresponding to the modular forms $f_1 f_2 f_3$, $f_1^2 f_2^2$, $f_1^3 f_2^3 f_3$, 1. Adding these to the generators of $M_*(\mathrm{U}(L))$ we get

$$\begin{aligned} M_*(\Gamma) &= \mathbb{C}[E_4, f_1^4, f_2^4, f_1 f_2 f_3, f_1^2 f_2^2, f_1^3 f_2^3 f_3] \\ &= \mathbb{C}[E_4, f_1^4, f_1^2 f_2^2, f_2^4, f_1 f_2 f_3] \end{aligned}$$

generated in weights $4, 8, 10, 12, 17$ as expected.

Thank you for listening!