On the conductor of Ciani plane quartics

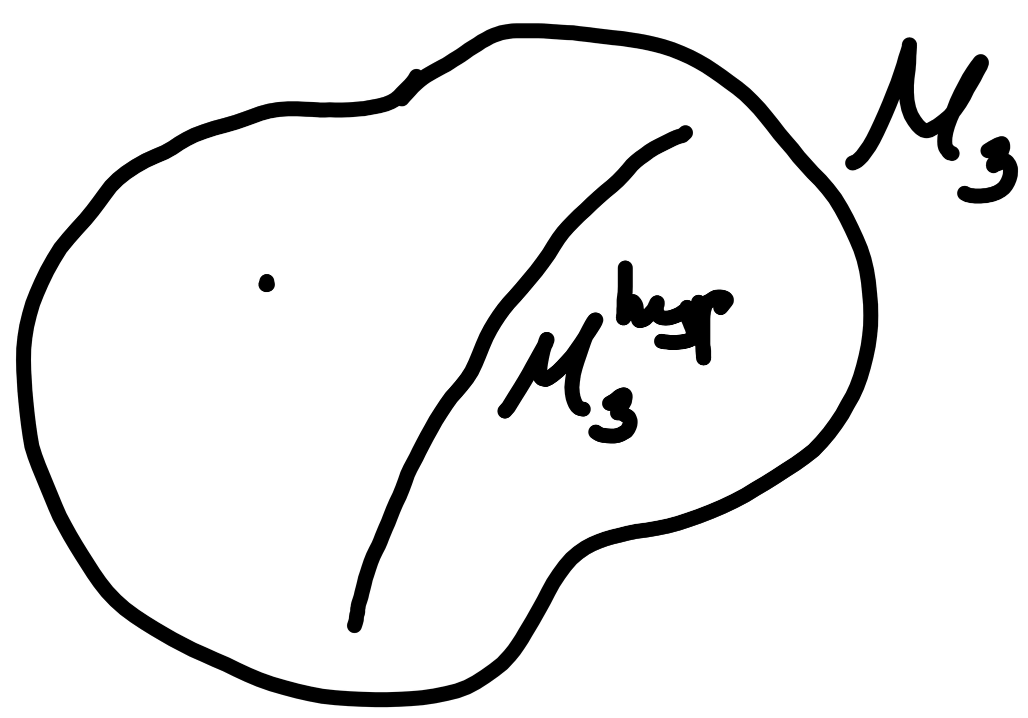
(j w w I Bouw, N Coppola and A Somoza)

genus 0 $\quad y^2 = xz$ $\quad$ dim 0

genus 1 $\quad y^2 = f(x)$, $\deg f = 3, 4$ $\quad$ dim 1

genus 2 $\quad y^2 = f(x)$, $\deg f = 5, 6$ $\quad$ dim 3

genus 3 $\quad \begin{cases} y^2 = f(x), \ \deg f = 7, 8 & \text{dim } 5 \\ F(x,y,z) = 0 & \text{dim } 6 \end{cases}$



non-hyperelliptic curves

Dixmier-Ohno invariants

7 alg independent invariants

+ 6 alg dep. ones

$\longrightarrow$ point in $\mathbb{P}^{12}_{3,6,9,9,12,12,15,15,18,18,21,21,27}$

Field of moduli = Field def. of inv $\subseteq$ Field def of curve

$\bar{k}$-isom info (geometry) $\qquad$ k-isom info (arithmetic)

[LRRS14] If $|\mathrm{Aut}_{\bar{k}}(Y)| > 2 \Rightarrow$ Field of mod = Field def curve

The largest-dimensional stratum in $\mathcal{M}_3^{\text{non-hyp}}$ with $|\mathrm{Aut}_{\bar{k}}(Y)| > 2$ is the one of Ciani quartics

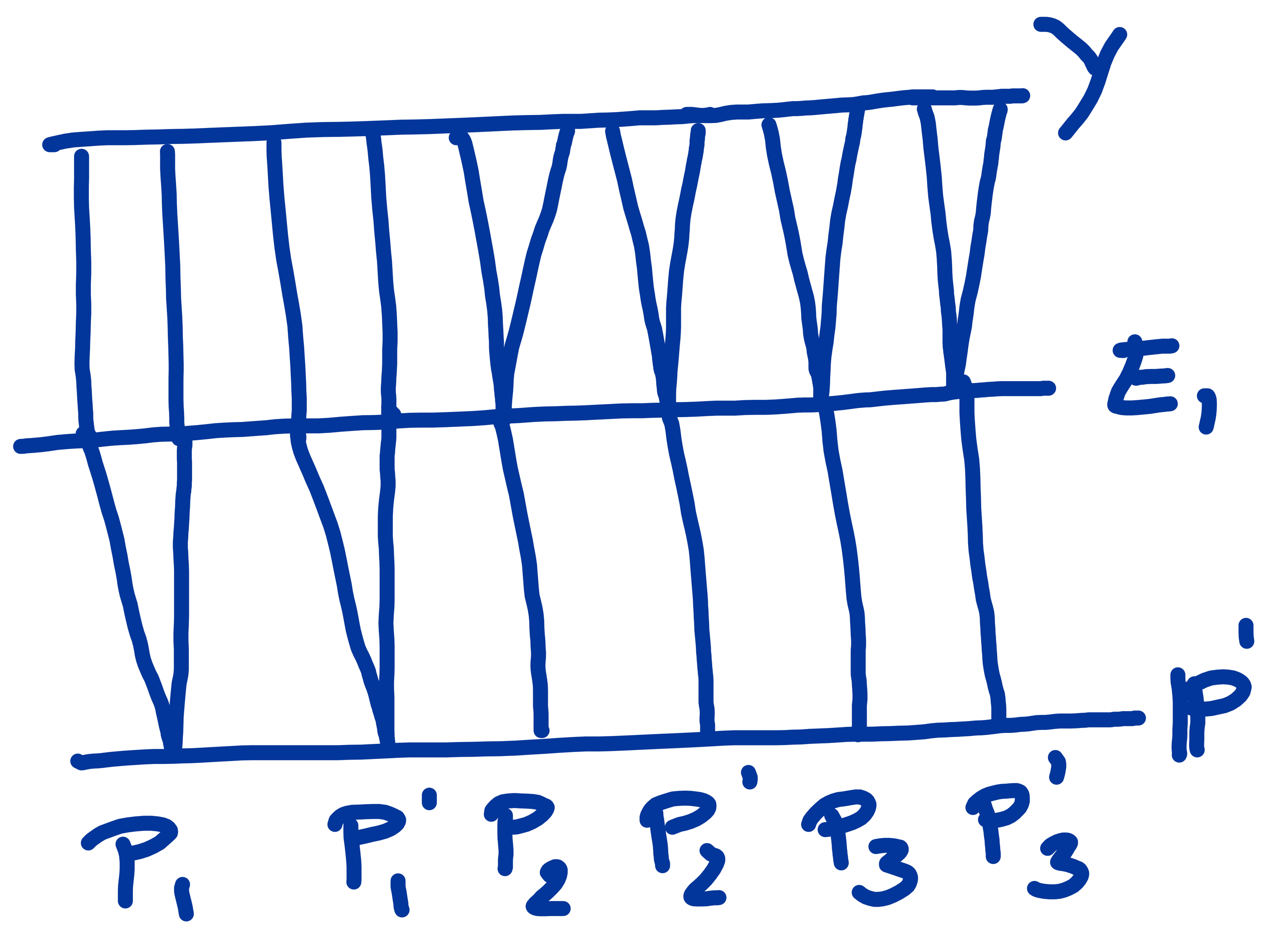
(family with no cyclic map to $\mathbb{P}^1$, i.e. non super-special, i.e. non $y^\ell = f(x)$)

Def. (Ciani quartic) is a non-hyperelliptic curve Y/k of genus 3 together with a subgroup $V \leq \mathrm{Aut}_{\bar{k}}(Y)$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and such that $g(Y/V) = 0$. V is called a Ciani subgroup

$V = \{1, \sigma_1, \sigma_2, \sigma_3\}$

$E_1 = Y/\langle \sigma_1 \rangle$

$$\begin{tikzcd} & Y \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ E_1 \arrow[dr, dash] & E_2 \arrow[d, dash] & E_3 \arrow[dl, dash] \\ & \mathbb{P}^1 & \end{tikzcd}$$



Stratification of Ciani quartics by automorphisms

$$\begin{tikzcd} & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \arrow[d, dash] & & \\ & D_4 \arrow[dl, dash] \arrow[dr, dash] & & \\ G_{16} \arrow[d, dash] \arrow[dr, dash] & & S_4 \arrow[dl, dash] \arrow[d, dash] & \\ G_{48} & G_{96} & G_{168} & \end{tikzcd}$$

dim 3

dim 2

dim 1

dim 0

Def a Ciani quartic Y is called non-special if there is a unique Ciani subgroup.

$$\left(\Longleftrightarrow \quad \mathrm{Aut}_{\bar{K}}(Y) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \right)$$

Def a standard K-model of (Y, V) is a K-model Y_1 of Y given by an equation.

$$Y_1 : \; Ax^4 + By^4 + Cz^4 + a y^2 z^2 + b x^2 z^2 + c x^2 y^2 = 0$$

such that V acts as $(x \; y \; z) \longrightarrow (\pm x \; \pm y \; z)$.

Lemma. Let $(Y_0, V)/K$ be a Ciani quartic Then there exists a Galois extension L/K with $\mathrm{Gal}(L/K) \subseteq S_3$ such that Y_0 admits a standard L-model Y_1 with $Y_0 \otimes_K L \simeq Y_1$

We introduce now the Ciani invariants.

$$\boxed{\begin{aligned} I_3 &= ABC \\ I_3' &= A(a^2 - 4BC) + B(b^2 - 4AC) + C(c^2 - 4AB) \\ I_3'' &= -4ABC + Aa^2 + Bb^2 + Cc^2 - abc \\ I_6 &= (a^2 - 4BC)(b^2 - 4AC)(c^2 - 4AB) \end{aligned}}$$

The discriminant is $\Delta(Y_1) = 2^{20} I_3 I_3''^4 I_6^2$

$$P = 8 I_3 + I_3' - I_3''$$

$$Q = -4 I_3 I_3'^3 I_6 - 27 I_3^2 I_6^2 + 18 I_3 I_3' I_6 I + I_3'^2 I - 4 I^3$$

$$I = -\frac{1}{4} \left(I_6 - I_3'^2 + 16 I_3 I_3'' + 2 I_3' I_3'' - I_3''^2 \right)$$

$$\mathcal{P}(T) = T^3 - S_1 T^2 + S_2 T - S_3$$

with $S_1 = I_3' + 12 I_3$, $S_2 = \frac{1}{4}\left(P^2 + 16 I_3 (P + I_3'') - I_6\right)$,

$S_3 = \mathcal{I}_3 P^2$ and roots $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$

The discriminant of $\mathcal{F}$ is the invariant $\mathcal{Q}$

The Ciani quartic Y is special $\Leftrightarrow \mathcal{Q}(Y) = 0$

Stable reduction.

- The reduction of a curve depends on the model
- The "best" model is the stable model. The reduction is a stable curve (at worst double points and finite automorphism group)

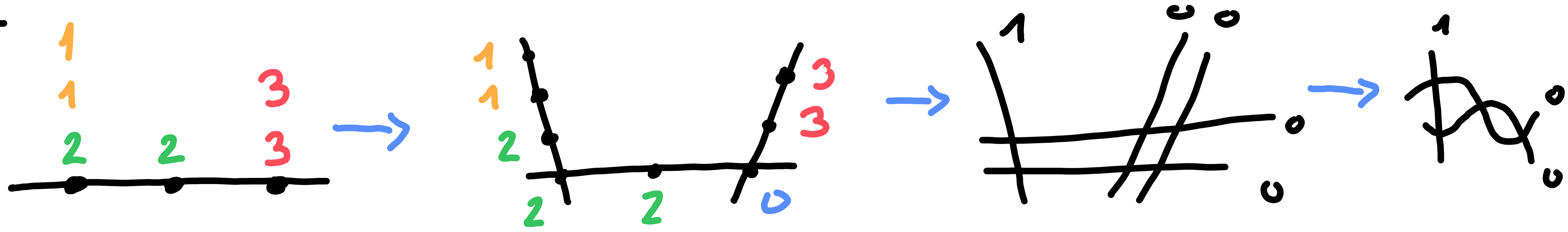
Examples.

- $x^4 + y^4 + \pi z^4 = 0$ has bad reduction modulo π but the curve has potentially good (quartic) reduction
- $(x^2 + y^2 + z^2)^2 + \pi(x^2y^2 + y^2z^2 + z^2x^2) = 0$ has bad reduction modulo π but the curve has potentially good (hyperelliptic!) reduction
- $x^4 + y^4 + z^4 - (2+\pi)x^2y^2 + z^2(x^2 + y^2) = 0$ has (geometrically) bad reduction

[LLLR21] Characterisation of primes dividing the discriminant

[BCKKLS21] Characterisation of reduction type in terms of Ciani invariants valuations ($p \neq 2$)

Ex



Admissible covers theory

Liu ($g=2$), Bouw-Wewers ($y^\ell = f(x)$), Dokchitser-Dokchitser-Maistret-Morgan ($y^2 = f(x)$)

The conductor exponent is a number that gives information on the reduction type of the jacobian of Y, and on the Galois representation associated to it.

Remark for example elliptic curves appear by conductor in the Cremona tables or the newest version, the LMFDB. Ogg '67 $f_p(E) \leq v_p(\Delta(E))$

Let Y/K with potentially good reduction. Let M be the minimal (Galois) extension over which Y has a model with good reduction $\mathcal{Y}$. Then $G = \mathrm{Gal}(M/K)$ acts on $\bar{\mathcal{Y}}$. Let $Z = \bar{\mathcal{Y}}/G$. Assume M/K is tamely ramified. Then

$$\boxed{f_p(Y) = 2g(Y) - 2g(Z)} \quad [BW17]$$

Co $f_p(Y) \in \{0, 2, 4, 6\}$

Cor $p \neq 2,3$, Y/K Ciani quartic

Y good reduction $\Longleftrightarrow f_p = 0$

Reconstruction of Ciani quartics from its invariants

Thm Let $(I_3, I_3', I_3'', I_6) \in K^4$. Assume $\Delta = 2^{20} I_3 I_3''^4 I_6^2 \neq 0$. Let L/K be the splitting field of $\mathcal{P}$ $(L = K(\mathcal{A}, \mathcal{B}, \mathcal{C}))$

Assume $P \neq 0$. Then

$$Y_1 \quad \mathcal{A}x^4 + \mathcal{B}y^4 + \mathcal{C}z^4 + P(x^2y^2 + y^2z^2 + z^2x^2) = 0 \;/L$$

has invariants $(P^2 I_3, P^2 I_3', P^2 I_3', P^4 I_6)$ and discriminant $\Delta(Y_1) = \Delta\, P^{18}$

Prop If $3 \mid [L\ K]$ then

$$\phi = \begin{pmatrix} 1 & \mathcal{A} & \mathcal{A}^2 \\ 1 & \mathcal{B} & \mathcal{B}^2 \\ 1 & \mathcal{C} & \mathcal{C}^2 \end{pmatrix} \cdot \; Y_0 \longrightarrow Y_1$$

defines Y_0/K with $\boxed{\Delta(Y_0) = \Delta\, P^{18} Q^{36}}$

We get equivalent results when $P = 0$

OUR RESULTS

$p > 3$, $\underline{I} = (I_3\ I_3'\ I_3''\ I_6)$ pot. good red normalised

& non-special ($\alpha \neq 0$)

* Potentially good quartic reduction

$v(I_3) = v(I_3'') = v(I_6) = 0, \quad v(I_3') \geq 0$

$v(\alpha) = 0$	$v(\alpha) > 0$
Then there exists a standard Ciani model Y/K with good reduction. $f_p(Y) = 0$. The other twists have $f_p(Y') = 4$	Then there exist a standard Ciani model Y/K and for all twists $f_p(Y') = 4$

* Potentially good hyper reduction

$v(I_3) = 0, \quad v(I_3') \geq e, \quad v(I_3'') = 2e, \quad v(I_6) = 3e$

There exists a model Y/K and for all twists the conductor exponent is given by:

		$[L:K]$		
		1	2	3
e	odd	6	2	6
	even	0	4	4

<u>Ex</u> $Y/\mathbb{Q}_p^{nr}$ Ciani quartic and non-special $p > 3$ of pot good red. Then $f_p(Y) \leq v_p(\Delta(Y))$

<u>Example</u>. $I_3 = I_3'' = I_6 = 1$

$I_3' = -6 \implies P = 1, \Delta = 2^{20}$

$\mathcal{P} = T^3 - 6T^2 + 8T - 1$

$Q = 229$

$\Delta(Y_0) = 2^{20} \; 229^{36}, \quad f_{229}(Y_0) = 4$