Bounds for standard L-functions

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Standard L-functions

$$L(\pi, s) = \prod_p \frac{1}{1 - \alpha_{p,1} p^{-s}} \cdots \frac{1}{1 - \alpha_{p,n} p^{-s}},$$

π: unitary cuspidal automorphic representation of GL_n over $\mathbb{Q}$.

- absolutely convergent for $\mathrm{Re}(s) > 1$
- functional equation:

$$\begin{aligned} \Lambda(\pi, s) &:= \Gamma_{\mathbb{R}}(s + \lambda_1) \cdots \Gamma_{\mathbb{R}}(s + \lambda_n) L(\pi, s) \\ &= \varepsilon_\pi C_{\mathrm{fin}}^{1/2 - s} \Lambda(\tilde{\pi}, 1 - s), \end{aligned}$$

  $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$.
- $\boxed{\lambda_1, \ldots, \lambda_n}$: "archimedean parameters" of π.

Conductors

- $C_{\mathrm{fin}} \in \mathbb{Z}_{\geqslant 1}$: “finite conductor”
- $C_\infty := \prod_{j=1}^{n}(1 + |\lambda_j|)$: “archimedean conductor”
- $C_\infty C_{\mathrm{fin}}$: “analytic conductor”

- We consider the problem of bounding $L(\pi, \frac{1}{2})$ as π varies. (This contains the problem of bounding $L(\pi, \frac{1}{2} + it)$ as t varies, because $L(\pi, \frac{1}{2} + it) = L(\pi \otimes |.|^{it}, \frac{1}{2})$.)
- Seek a bound of the shape

$$L(\pi, \tfrac{1}{2}) \ll_{n,\varepsilon} (C_\infty C_{\mathrm{fin}})^{\beta+\varepsilon}$$

  with β as small as possible.
- Convexity bound (Molteni 2002):

$$\beta \leqslant 1/4.$$

- Lindelöf hypothesis:

$$\beta = 0.$$

- Subconvexity problem: show that $\beta < 1/4$.

We focus on the "spectral aspect":

- C_∞ varies,
- C_{fin} essentially fixed (e.g., $C_{\text{fin}} = 1$).

Main result: subconvexity holds if $\lambda_1, \dots, \lambda_n$ are comparable.

Theorem 1 (N 2021)

Let $T \geqslant 1$. Assume that

$$|\lambda_1|, \dots, |\lambda_n| \in \left[\frac{T}{100}, 100T\right].$$

(Thus $C_\infty \asymp T^n$.) Then

$$L(\pi, \tfrac{1}{2}) \ll_n C_\infty^{1/4-\delta} C_{\text{fin}}^{B}$$

with

$$\delta = \frac{1}{12n^5} > 0, \quad B = \frac{1}{2} < \infty.$$

Corollary 2 (N 2021)

Let π be a unitary cuspidal automorphic representation of GL_n over $\mathbb{Q}$. Then

$$L(\pi, \tfrac{1}{2} + it) \ll_\pi (1 + |t|)^{n/4 - 1/12n^4}.$$

Earlier results

- $n = 1$: Weyl/Hardy–Littlewood 1916–1921, ...
- $n = 2$: Good 1982, ..., DFI, ..., Michel–Venkatesh 2010, ...
- $n = 3$: Li 2011, Munshi 2015, ..., Blomer–Buttcane 2020, ...
- "weak subconvexity" for GL_n (Soundararajan 2010, Soundararajan–Thorner 2019): improve by $\log^{-\delta}$
- $\mathrm{U}_{n+1} \times \mathrm{U}_n$: N 2020+, assuming U_n: anisotropic

Notation

- $G := \mathrm{GL}_{n+1}(\mathbb{R})$
- $\Gamma := \mathrm{GL}_{n+1}(\mathbb{Z})$
- $H := \mathrm{GL}_n(\mathbb{R})$
- $\Gamma_H := \mathrm{GL}_n(\mathbb{Z})$
- $\pi \hookrightarrow C^\infty(\Gamma \backslash G)$ as before.
- $G \geqslant NA$ (upper-triangular Borel), $H \geqslant N_H A_H$

Integral representations (JPSS)

Let

- $\varphi \in \pi \hookrightarrow C^\infty(\Gamma \backslash G)$,
- $\Psi \in \sigma \hookrightarrow C^\infty(\Gamma_H \backslash H)$.

Then

$$\int_{\Gamma_H \backslash H} \varphi \Psi = L(\pi \otimes \sigma, \tfrac{1}{2}) Z(\varphi, \Psi),$$

$$Z(\varphi, \Psi) = \int_{N_H \backslash H} W_\varphi \tilde{W}_\Psi.$$

Specialize to

$$\sigma = \{\text{Eisenstein series with parameters } (0, \dots, 0) \in \mathbb{C}^n\},$$

$$L(\pi \otimes \sigma, \tfrac{1}{2}) = L(\pi, \tfrac{1}{2})^n.$$

Better: work with "wave packet" of Eisenstein concentrated on parameters of size O(1).

Construction of vectors

Fact [NV, N]

There are “explicit,” “coadjoint microlocalized” unit vectors

$$\varphi \in \pi \hookrightarrow C^\infty(\Gamma \backslash G), \quad \Psi \in \sigma \hookrightarrow C^\infty(\Gamma_H \backslash H)$$

so that $Z(\varphi, \Psi) \approx T^{-n^2/4}$.

Explication for φ

There is a unique element $\tau \in \mathfrak{g}^*$ of the form

$$\tau = \begin{pmatrix} 0 & 0 & 0 & c_4 \\ 1 & 0 & 0 & c_3 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \end{pmatrix}$$

with eigenvalues $\{\lambda_j / T\}$. For $x \in \mathfrak{g}$,

$$\pi(x)\varphi \approx iT\langle x, \tau\rangle\varphi + \mathrm{O}(T^{1/2+\varepsilon}).$$

Construction of a reproducing kernel $\omega \in C_c^\infty(G)$

Take

$$J := G \cap \left(1 + \mathrm{O}(T^{-\varepsilon})\right) \cap \left(G_\tau + \mathrm{O}(T^{-1/2-\varepsilon})\right)$$

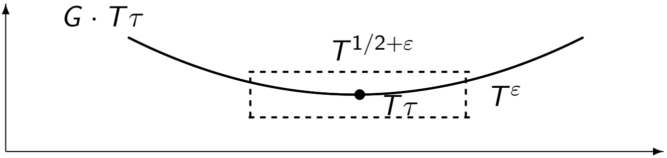
$$\chi : J \to \mathrm{U}(1),$$

$$\chi(\exp(x)) = e^{-iT\langle x,\tau\rangle}.$$

Set

$$\omega := \frac{1}{\mathrm{vol}(J)} 1_J^{\mathrm{smooth}} \chi.$$

Then $\pi(\omega)\varphi \approx \varphi$.



Pretrace inequality

Let $\eta_\pi : \mathbb{R}^\times/\mathbb{Z}^\times \to \mathrm{U}(1)$ be the central character. Set

$$\omega^\sharp(g) := \int_{z \in Z_G} \eta_\pi(z)(\omega * \omega^*)(zg)\, dz.$$

Then

$$\begin{aligned} \left|L(\pi, \tfrac{1}{2})\right|^{2n} |Z(\varphi, \Psi)|^2 &\approx \left| \int_{\Gamma_H \backslash H} \pi(\omega)\varphi \cdot \Psi \right|^2 \\ &\leqslant \int_{g, h \in \Gamma_H \backslash H} \Psi(g)\overline{\Psi(h)} \sum_{\gamma \in P\Gamma} \omega^\sharp(g^{-1}\gamma h)\, dg\, dh. \end{aligned}$$

$\sum_{\gamma \in \Gamma_H}$ contributes a "main term," addressed via amplification.
Estimating $\sum_{\gamma \in P\Gamma - \Gamma_H}$ requires the following inputs:

- A linear-algebraic fact concerning τ (consequences: "transversality," "bilinear forms estimate").
- A local L^2 growth bound for Ψ.

Linear algebraic aside

$\mathfrak{g} = M_{n+1}(\mathbb{R}) \geqslant \mathfrak{h} = M_n(\mathbb{R})$. $\mathfrak{g}^* \cong \mathfrak{g}$. $\mathfrak{h}^* \cong \mathfrak{h}$.
Let $\tau \in \mathfrak{g}^*$. Write

$$\tau = \begin{pmatrix} \tau_0 & b \\ c & d \end{pmatrix}$$

with $\tau_0 \in \mathfrak{h}^*$.

Theorem (N 2020)

Assume that

$$\mathrm{ev}(\tau) \cap \mathrm{ev}(\tau_0) = \emptyset.$$

Then for all

- $x \in \mathfrak{g}_\tau - \mathfrak{z}_G$,
- $z \in \mathfrak{z}_H - \{0\}$,

we have

$$[x, [z, \tau]] \notin [\mathfrak{h}, \tau].$$

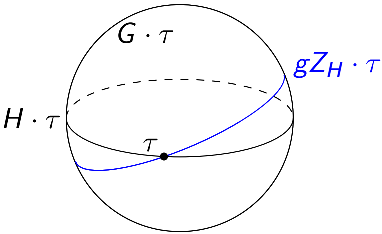
Transversality

Corollary (N 2020)

For small noncentral $g \in G_\tau$, the varieties

$$gZ_H \cdot \tau, \qquad H \cdot \tau$$

meet transversally.



Bilinear forms estimate

Corollary (N 2020)

Let $u_1, u_2 \in L^2(H)$. Suppose

- $u_2(zy) \approx u_2(y)$ for small $z \in Z_H$, and
- g is not too close to HZ_G.

Then

$$\int_{x,y\in\Omega} \left| u_1(x) u_2(y) \omega^\sharp(x^{-1}gy) \right| \, dx \, dy \ll \|u_1\|_2 \|u_2\|_2 \|\omega^\sharp\|_\infty T^{-n^2/2-1/4+\varepsilon}.$$

Growth bound for the Eisenstein series Ψ

Each $h \in \Gamma_H \backslash H$ may be written

$$h = \Gamma_H a x, \quad (a, x) \in A_H^{\mathrm{dom}} \times \Omega,$$

where

- $A_H^{\mathrm{dom}} := \{\mathrm{diag}(a_1, \ldots, a_n) : a_1 \geqslant \cdots \geqslant a_n\}$,
- $\Omega \subseteq H$: compact.

$$\text{Haar measure on } \Gamma_H \backslash H \approx \frac{da}{\delta_H(a)}\, dx.$$

Normalize so that $\|\Psi\|_{L^2(\Gamma_H \backslash H)} = 1$.

Theorem (N 2021)

$$\frac{\|\Psi\|^2_{L^2(a\Omega)}}{\delta_H(a)} \ll \min(a_1^{-1}, a_n)^n T^{\varepsilon}.$$

Proof involves $\int_{\Gamma_H \backslash H} |\Psi(h)|^2 \sum_{v \in \mathbb{Z}^n} \phi(vh)\, dh$, $h \mapsto h^{-t}$.

Completion of the proof

Taking a Siegel domain for $\Gamma_H \backslash H$, we need to bound

$$\int_{a,b \in A_H^{\mathrm{dom}}} \sum_{\gamma \in P\Gamma - \Gamma_H} \int_{x,y \in \Omega} \left| \Psi(ax) \overline{\Psi(by)} \omega^\sharp(x^{-1}a^{-1}\gamma by) \right| \, d(\cdots).$$

We apply the "bilinear forms estimate" to $\int_{x,y \in \Omega}$. We then count the number of γ that contribute. This eventually yields

$$\frac{|L(\pi, \frac{1}{2})|^{2n}}{T^{n(n+1)/2}} \ll T^{-\delta} + T^{-1/4 + C\delta} \int_{\substack{a \in A_H^{\mathrm{dom}} \\ \det(a) \asymp 1}} \|\Psi\|_{L^2(a\Omega)}^2 \mathcal{N}(a) \, \frac{da}{\delta_H(a)},$$

where

$$\mathcal{N}(a) := \prod_{j=1}^{n} \max(a_j, a_j^{-1}).$$

Finally, we apply the "growth estimate" for Ψ, and use that $\min(a_1^{-1}, a_n)^n \ll \mathcal{N}(a)^{-1}$.

Summary of proof ingredients

1. Temperedness of unitary Eisenstein series
2. Convexity bound! (Bounds for $L(\pi, s)$ when $s \approx 1$)
3. Standard global arguments (matrix counting, ...)
4. Construction of test vectors φ and Ψ, following [NV].
5. Symmetries of φ ("asymptotics of Kirillov model, Bessel functions, ..."; uses microlocal calculus for Lie group representations from earlier papers)
6. Linear algebra concerning τ
7. Growth bounds for Ψ (estimates for pseudo local Rankin–Selberg integrals, plus degenerate variants)
8. In practice, the construction of the Eisenstein series Ψ is much more complicated than we have indicated. Convenient to arrange that the underlying test function be exactly invariant under all normalized intertwining operators. Construct such a function by convolving a distribution on $U_H \backslash H$, related to Jacquet integrals, by a suitable function on $A_H \times H$.

Matrix counting

Lemma

For $a, b \in A_H^{\mathrm{dom}}$, the set

$$\left\{\gamma \in \Gamma : a^{-1}\gamma b \ll 1\right\}$$

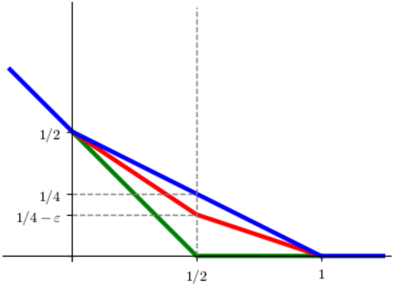
is empty unless $a \approx b$, in which case it has size

$$\ll \delta_H(a)\mathcal{N}(a), \quad \mathcal{N}(a) := \prod_{j=1}^{n} \max(a_j, a_j^{-1}).$$

Convexity principle

$$L(\pi,\sigma) \ll (C_\infty C_{\mathrm{fin}})^{\beta(\sigma)+\varepsilon}$$

Convexity: $\beta(\frac{1}{2}) \leqslant 1/4$.
Lindelöf: $\beta(\frac{1}{2}) = 0$.
Subconvexity: $\beta(\frac{1}{2}) < 1/4$.



Moment method

short family

Take π on GL_{n+1}, say $C_{\mathrm{fin}} = 1$.
Choose a family $\mathcal{F} \ni \pi$.

$$\left|L(\pi, \tfrac{1}{2})\right|^{2n} \leqslant \textstyle\sum_{\pi' \in \mathcal{F}} \left|L(\pi', \tfrac{1}{2})\right|^{2n} \ll |\mathcal{F}| \ll T^{n(n+1)/2}$$

assuming Lindelöf

$$\implies |L(\pi, \tfrac{1}{2})| \ll |\mathcal{F}|^{1/2n} \ll T^{(n+1)/4} \asymp C_\infty^{1/4}.$$

Amplification method: shrink $\mathcal{F}$ by $T^{-\delta}$ by incorporating Hecke eigenvalues at primes $p \leqslant T^\kappa$. Leads to $L(\pi, \frac{1}{2}) \ll C_\infty^{1/4-\delta'}$.