Arithmetic Quantum Chaos and L-functions

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2022.12.06

International Seminar on Automorphic Forms

Contents

An ellipse billiard table

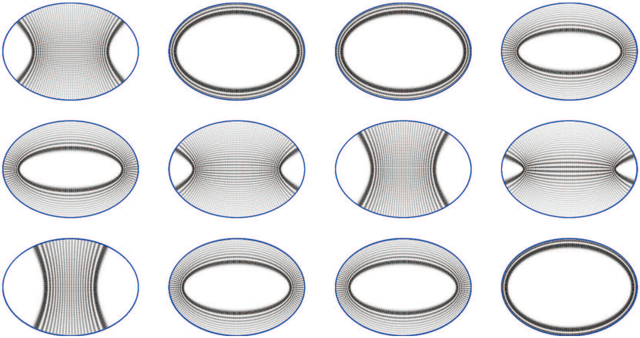


Figure 1E

A stadium billiard table

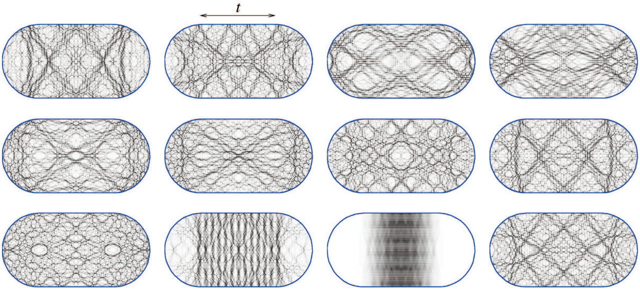


Figure 1S

A Barnett billiard table

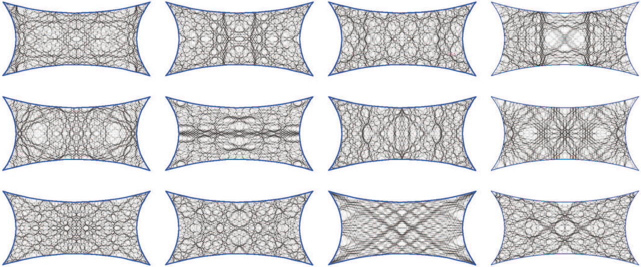


Figure 1B

In Figure 1 the domains Ω_E, Ω_S, and Ω_B are an *ellipse*, a *stadium* and a *Barnett* billiard table, respectively. Superimposed on these are the densities of a consecutive sequence of high frequency eigenfunctions (*states*, *modes*) of the Laplacian. That is, they are solutions to

$$(0) \qquad \begin{cases} \triangle\phi_j + \lambda_j\phi_j = 0 & \text{in } \Omega\,, \\ \phi\big|_{\partial\Omega} = 0 & \text{(Dirichlet boundary conditions)}, \\ \int_\Omega |\phi_j|^2 \, dxdy = 1\,. \end{cases}$$

Here $\triangle = \mathsf{divgrad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ are the eigenvalues, and the eigenfunctions are normalized to have unit L^2-norm. The sequences are of 12 consecutive modes around the 5600th eigenvalue. They are ordered from left to right and then down, and the grayscale represents the probability density $|\phi|^2$, with zero white and larger values darker.

P. Sarnak, Recent progress on the quantum unique ergodicity conjecture. *Bull. Amer. Math. Soc. (N.S.)* 48 (2011), no. 2, 211–228.

Random wave conjecture

Classical dynamics:

- M = a (compact) Riemann surface.
- T^1M = its unit tangent bundle.
- $\Phi^t : T^1M \to T^1M$ the geodesic flow.

Quantum dynamics:

- $\{\psi_j\}$ = an orthonormal basis of $L^2(M)$ consisting of the eigenfunctions of the Laplacian ($\Delta\psi_j = \lambda_j\psi_j$, with $\lambda_j = 1/4 + t_j^2$).
- Weyl's law: $\#\{t_j \leq T\} \sim c_M \cdot T^2$.

Random wave conjecture (Michael Berry 1977)

Eigenfunctions for chaotic systems are modeled by random waves in the semiclassical limit, that is, its value distribution should be Gaussian.

- The semiclassical limit $\Longleftrightarrow$ eigenvalues tend to infinity.
- Chaotic = ergodic + exponential divergence of orbits ...

Quantum ergodicity

- $a \in C^\infty(M)$ an observable.
- $\mathrm{Op}(a) : L^2(M) \to L^2(M)$ a quantization (the multiplication operator).
- $\{\psi_j\}$ an orthonormal basis of the eigenfunctions of the Laplacian.
- Matrix elements: $\langle \mathrm{Op}(a)\psi_j, \psi_j \rangle = \langle a, |\psi_j|^2 \rangle = \int_M a|\psi_j|^2$.

Mean value of matrix elements = classical average (local Weyl law):

$$\frac{1}{N(T)} \sum_{t_j \sim T} \langle \mathrm{Op}(a)\psi_j, \psi_j \rangle \sim \int_M a, \quad \text{where } N(T) = \sum_{t_j \sim T} 1.$$

Quantum ergodicity theorem (Schnirelman 1974; Colin de Verdiere 1985; Zelditch 1987)

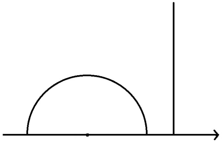
If the geodesic flow of M is ergodic, then the variance of the matrix elements vanishes, i.e.,

$$\mathrm{Var}(T) := \frac{1}{N(T)} \sum_{t_j \sim T} \left| \langle \mathrm{Op}(a)\psi_j, \psi_j \rangle - \int_M a \right|^2 \to 0, \quad T \to \infty.$$

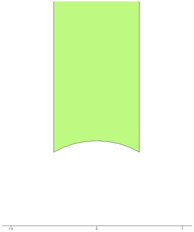
Hyperbolic surfaces

- $\mathbb{H} = \{z = x + iy : y > 0\}$ the upper half-plane with measure $\mathrm{d}\mu z = \mathrm{d}x\mathrm{d}y/y^2$.
- $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ the Laplace operator.
- Gauss curvature on $\mathbb{H}$ is negative $(= -1)$.
- Geodesics are semicircles subtended on $y = 0$ and vertical lines.

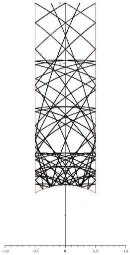
- $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ the modular group and $\Gamma \curvearrowright \mathbb{H}$ as fractional linear transforms.
- $\mathbb{X} = \Gamma \backslash \mathbb{H}$ a hyperbolic surface.
- Gauss curvature on $\mathbb{X}$ is negative $\Rightarrow$ The geodesic flow on $T^1\mathbb{X}$ is chaotic.



Geodesics in $\mathbb{H}$



$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



A projected geodesic

Maass cusp forms

A *Maass cusp form* ϕ for $\mathrm{SL}_2(\mathbb{Z})$ satisfies the automorphy condition

$$\phi(\gamma z) = \phi(z), \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \quad \forall z \in \mathbb{H},$$

the cuspidal condition

$$\int_0^1 \phi(x+iy)\mathrm{d}x = 0,$$

and is an eigenfunction of the Laplace operator Δ with eigenvalue λ_ϕ. Define the spectral parameter t_ϕ by $\lambda_\phi = 1/4 + t_\phi^2$.

Weyl's law (Selberg 1956):

$$\#\{\phi : |t_\phi| \leq T\} \sim \frac{1}{12} T^2.$$

Eigenfunctions of the modular surface

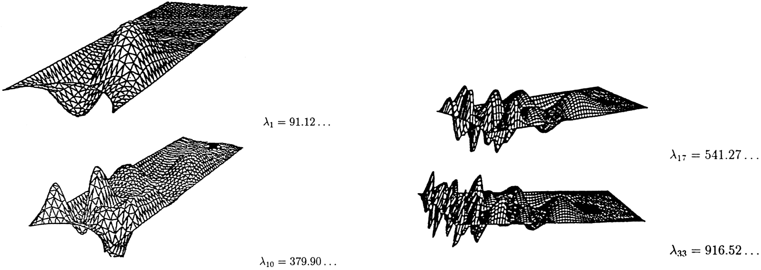
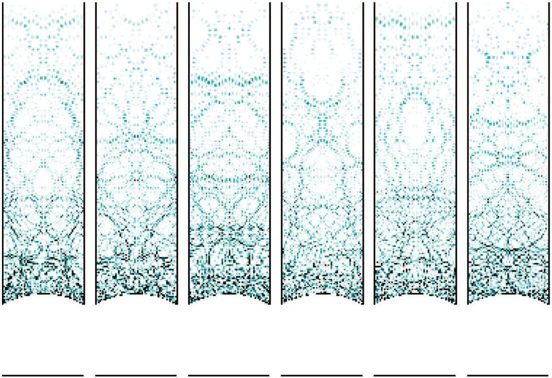


FIGURE 3. The 1^{st}, 10^{th}, 17^{th} and 33^{rd} eigenfunctions for the modular group. They are all odd with respect to the symmetry $z \longrightarrow -\bar{z}$.

Mass distribution



This figure depicts the densities of a sequence of Maass forms on the modular surface with shading and frequences similar to those in Figure 1. The densities are less regular than those of the Barnett stadium. From Sarnak [*BAMS* 2011].

Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$

- Let $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$. Define the Eisenstein series by

$$E(z,s) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s, \quad \operatorname{Re}(s) > 1.$$

- $E(\gamma z, s) = E(z,s)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$;
  $\Delta E(z,s) = s(1-s)E(z,s)$.

- Fourier expansion:

$$E(z,s) = y^s + \phi(s) y^{1-s} + \frac{2\sqrt{y}}{\pi^{-s}\Gamma(s)\zeta(2s)} \sum_{n \neq 0} \eta_{s-1/2}(n) K_{s-1/2}(2\pi |n| y) e(nx),$$

with $\phi(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$, $\eta_s(n) = \sum_{ab=|n|,\ a>0} (a/b)^s$,

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-(u+1/u)y/2} u^{s-1} \mathrm{d}u, \text{ and } e(x) = e^{2\pi i x}.$$

- Functional equation: $E(z,s) = \phi(s) E(z, 1-s)$.

Selberg spectral decomposition

- Denote an orthonormal basis of Hecke–Maass cusp forms by $\{\phi_j\}_{j\geq 1}$. For $\phi \in \{\phi_j\}_{j\geq 1}$, we have $\Delta\phi = (1/4 + t_\phi^2)\phi$, where $t_\phi > 1$ is the spectral parameter of ϕ.
- Denote $E_t(z) = E(z, 1/2 + it)$.

Theorem (Selberg spectral decomposition)

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$. Then we have

$$\psi(z) = \frac{3}{\pi}\langle \psi, 1\rangle + \sum_{j\geq 1}\langle \psi, \phi_j\rangle \phi_j(z) + \frac{1}{4\pi}\int_{\mathbb{R}} \langle \psi, E_t\rangle E_t(z)\mathrm{d}t.$$

Modular forms

Let $k \geq 2$ be an even integer. A holomorphic function $f : \bar{\mathbb{H}} \to \mathbb{C}$ is a **modular form** of weight k if f satisfies

$$f(\gamma z) = (cz + d)^k f(z), \quad \forall \gamma \in \mathrm{SL}_2(\mathbb{Z}),\ z \in \mathbb{H}.$$

Since $f(z+1) = f(z)$, we have the Fourier expansion

$$f(z) = a_f(0) + \sum_{n \geq 1} a_f(n) n^{\frac{k-1}{2}} e(nz).$$

If $a_f(0) = 0$, then f is called a **cusp form**.

- S_k = the space of cusp forms of weight k.
- S_k is a Hilbert space with Petersson inner product $\langle F, G \rangle = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} F(z) \overline{G(z)} \frac{\mathrm{d}x \mathrm{d}y}{y^2}$, where $F = y^{k/2} f$ and $G = y^{k/2} g$.
- $\dim S_k = k/12 + O(1)$.

Hecke eigenforms

Let $n \in \mathbb{N}$. The n-th Hecke operator $T(n)$ is defined by $(f \in S_k)$

$$(T(n)f)(z) = n^{\frac{k-1}{2}} \sum_{ad=n} d^{-k} \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right)$$

Then we have:

- $T(n) : S_k \to S_k$.
- $T(m)T(n) = \sum_{d|(m,n)} T\left(\frac{mn}{d^2}\right)$.
- $\langle T(n)f, g\rangle = \langle f, T(n)g\rangle$.
- $\exists$ an orthonormal basis H_k of S_k which consists of Hecke eigenforms.

Let $f \in H_k$. Then we have

- $T(n)f = \lambda_f(n)f$, then $a_f(n) = a_f(1)\lambda_f(n)$ and $\lambda_f(n)$ is real.
- $\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$, (in particular, $\lambda_f(n)$ is multiplicative).
- Deligne 1972 proved $|\lambda_f(n)| \leq \tau(n) = \sum_{d|n} 1 \ll n^{\varepsilon}$.

L-functions: simple examples

Riemann zeta function: $\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$, $\quad \mathrm{Re}(s) > 1$.

Functional equation: $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1-s)$.

Riemann Hypothesis

The real part of every nontrivial zero of the Riemann zeta function $\zeta(s)$ is $1/2$.

If $f \in H_k$, we define L-function of f by

$$L(s,f) := \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional equation:

$$\Lambda(s,f) := \pi^{-s}\Gamma\left(\frac{s + \frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s + \frac{k+1}{2}}{2}\right)L(s,f) = i^k\Lambda(1-s,f).$$

GRH $\Rightarrow$ GLH (*i.e.* $L(1/2+it,f) \ll (k(1+|t|))^{\epsilon}$, for any $\epsilon > 0$).

QUE for Hecke–Maass cusp forms

For a test function $\psi : \mathbb{X} \to \mathbb{C}$, define

$$\mu_j(\psi) := \langle \psi, |\phi_j|^2 \rangle = \int_{\mathbb{X}} \psi(z)|\phi_j(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \mu_t(\psi) := \langle \psi, |E(*, 1/2 + it)|^2 \rangle.$$

Quantum Unique Ergodicity (Rudnick–Sarnak conjecture 1994)

For any $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$, we have

$$\mu_j(\psi) = \frac{3}{\pi} \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} + o_\psi(1), \quad \text{as } j \to \infty.$$

Theorem (Lindenstrauss 2006 & Soundararajan 2010)

QUE holds for all Hecke–Maass cusp forms for $\mathrm{SL}_2(\mathbb{Z})$.

- Luo–Sarnak 1995: $\mu_t(\psi) \sim \frac{3}{\pi} \log t^2 \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \text{as } t \to \infty.$
- Liu–Ye 2002: QUE holds for dihedral Maass forms.

QUE for Hecke eigenforms

Let $f \in H_k$. For a test function $\psi : \mathbb{X} \to \mathbb{C}$, define

$$\mu_f(\psi) := \langle \psi, y^k|f|^2 \rangle = \int_{\mathbb{X}} \psi(z) y^k |f(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

QUE Theorem (Holowinsky–Soundararajan 2010)

For any $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$, we have

$$\mu_f(\psi) = \frac{3}{\pi} \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} + o_\psi(1), \quad \text{as } k \to \infty.$$

Sarnak 2001: For CM forms f for some congruence subgroup $\Gamma_0(q)$,

$$\mu_f(\psi) \sim \frac{1}{\mathrm{vol}(\Gamma_0(q)\backslash\mathbb{H})} \int_{\Gamma_0(q)\backslash\mathbb{H}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2}, \quad \text{as } k \to \infty.$$

QUE for Hecke eigenforms

Define the imcomplete Eisenstein series by

$$E(z|w) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} w(\mathrm{Im}(\gamma z)),$$

for w being a fixed smooth function compactly supported in $(0, \infty)$.

Theorem (Holowinsky–Soundararajan 2010)

Let $f \in H_k$ and $F = y^{k/2} f$.

(i) Let ϕ be a Hecke–Maass cusp form. Then

$$\langle \phi F, F \rangle \ll_{\phi,\varepsilon} (\log k)^{-1/30+\varepsilon}.$$

(ii) Let w be a fixed smooth function compactly supported in $(0, \infty)$. Then

$$\langle E(\cdot|w) F, F \rangle - \frac{3}{\pi} \langle E(\cdot|w), 1 \rangle \ll_{w,\varepsilon} (\log k)^{-2/15+\varepsilon}.$$

Matrix elements to L-values

- The classical Rankin–Selberg theory computes the projection of $F\bar{G}$ onto the Eisenstein series and the formula is

$$|\langle E_t F, G\rangle|^2 = 2\frac{|L(1/2+it, f\times g)|^2}{L(1,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 g)|\zeta(1+2it)|^2}\mathcal{H}(k,t),$$

where

$$\mathcal{H}(k,t) = \frac{\pi^3|\Gamma(k-1/2+it)|^2}{2\Gamma(k)^2} \ll \frac{1}{k}, \quad \text{for} \quad |t| \ll k^{\varepsilon}.$$

- Watson's formula gives

$$|\langle \phi F, G\rangle|^2 = \frac{L(1/2, f\times g\times\phi)}{L(1,\mathrm{sym}^2 f)L(1,\mathrm{sym}^2 g)L(1,\mathrm{sym}^2 \phi)}\mathcal{H}(k,t_\phi).$$

- Soundararajan 2010 proved the following weak subconvexity:

$$|L(1/2+it, f\times g)|^2 \ll_{t,\varepsilon} k(\log k)^{-2+\varepsilon},$$

$$L(1/2, f\times f\times\phi) \ll_{\phi,\varepsilon} k(\log k)^{-1+\varepsilon}.$$

Effective QUE

Lester–Matomäki–Radziwiłł 2018 found a proof of effective QUE Theorem.

Theorem (Lester–Matomäki–Radziwiłł 2018)

Let $f \in H_k$. Let ψ be a smooth function, supported in the fundamental domain $\mathcal{F}$ with

$$\sup_{z \in \mathcal{F}} \left| y \frac{\partial^a}{\partial x^a} \frac{\partial^b}{\partial y^b} \psi(z) \right| \ll_{a,b} M^{a+b}, \quad z = x + iy,$$

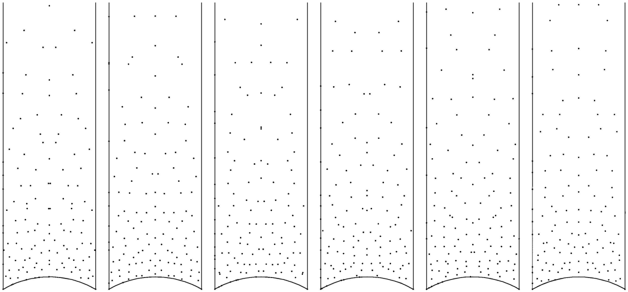
for all $a, b \leq 1$ and some $M \geq 1$. Then

$$\left| \mu_f(\psi) - \frac{3}{\pi} \int_{\mathcal{F}} \psi(z) \frac{\mathrm{d}x \mathrm{d}y}{y^2} \right| \ll_\varepsilon M^2 (\log k)^{-\eta_0 + \varepsilon},$$

for all $\varepsilon > 0$ fixed and with $\eta_0 \approx 0.0074\ldots$

Based on ideas of Holowinsky–Soundararajan and Iwaniec.

Zeros



Zeros of six holomorphic Hecke cusp forms of weight $k = 2000$ as computed by F. Stroemberg. From Sarnak [*BAMS* 2011].

Zero equidistribution

Theorem (Rudnick 2005 and Holowinsky–Soundararajan 2010)

Let $f \in H_k$ be a sequence of cuspidal Hecke eigenforms. Then as $k \to \infty$, their zeros are equidistributed in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with respect to the normalized hyperbolic measure $(3/\pi)(\mathrm{d}x\mathrm{d}y/y^2)$. That is, for any "nice" ψ,

$$\sum_{\rho \in \mathbb{X}, f(\rho)=0} \psi(\rho) \sim \frac{3}{\pi} \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Theorem (Lester–Matomäki–Radziwiłł 2018)

Let $f \in H_k$ be a sequence of cuspidal Hecke eigenforms. Let $B(z_0, r) \subset \{z \in \mathcal{F} : \mathrm{Im}(z) \leq B\}$ be the hyperbolic ball centered at z_0 and of radius r, with $B > 0$ fixed and $r \geq (\log k)^{-\delta_0/2+\varepsilon}$ where $\delta_0 = \frac{1}{4}\eta_0 \approx 0.0018\ldots$. Then as $k \to \infty$, we have

$$\frac{\#\{\rho \in B(z_0, r) : f(\rho) = 0\}}{\#\{\rho \in \mathcal{F} : f(\rho) = 0\}} = \frac{3}{\pi} \int_{B(z_0,r)} \frac{\mathrm{d}x\mathrm{d}y}{y^2} + O_B(r(\log k)^{-\delta_0/2+\varepsilon}).$$

Decorrelation

Theorem (Constantinescu 2021)

Let $f, g \in H_k$. Fix any smooth and bounded $\psi : \mathbb{X} \to \mathbb{C}$. Then we have

$$\langle \psi F, G \rangle = \delta_{f,g} \frac{3}{\pi} \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} + o_\psi(1),$$

as $k \to \infty$. Here $\delta_{f,g} = 1$ if $f = g$, 0 otherwise.

In fact, Constantinescu can deal with Hecke eigenforms with different weights.

Effective Decorrelation

Theorem (H. 2022)

Let $f, g \in H_k$ and $f \neq g$. Let M be a positive constant such that $1 \leq M \leq \log k$. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$ satisfying

$$y^{a+b} \frac{\partial^a}{\partial x^a} \frac{\partial^b}{\partial y^b} \psi(z) \ll_{a,b} M^{a+b}, \quad \text{for any } a, b \in \mathbb{Z}_{\geq 0}.$$

Assume that $\psi|_{\mathcal{F}}$ has support contained in the interior of $\mathcal{F}$ and $\psi(x+iy) = 0$ if $y \geq BM$ for some absolute constant $B > 1$. Then we have

$$\langle \psi F, G \rangle \ll_\varepsilon M^{5/3} (\log k)^{-\delta+\varepsilon},$$

for any $\delta \leq 1.19 \times 10^{-41}$.

Tools: Waston's formula + Soundararajan–Thorner's weak subconvexity + Holowinsky's method + Iwaniec's optimization.

EQQUE and Equidistribution for "near" Hecke eigenforms

Definition

Let $J \geq 1$ be a positive integer. Define the set of *J-near Hecke eigenforms* by

$$H_k^{(J)} = \Big\{ f \in S_k : \text{there are distinct } f_j \in H_k,\ (j = 1, 2, \ldots, J), \\ \text{and } (c_1, c_2, \ldots, c_J) \in \mathbb{C}^J \text{ such that } f = \sum\nolimits_{1 \leq j \leq J} c_j f_j \ \Big\}.$$

Theorem (H. 2022)

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$ be as in Main Theorem. Let $f \in H_k^{(J)}$ and $\|f\|_2 = 1$. Then we have

$$\mu_f(\psi) = \frac{3}{\pi}\langle \psi, 1 \rangle + O_\varepsilon(JM^{5/3}(\log k)^{-\delta+\varepsilon}), \quad \delta \leq 1.19 \times 10^{-41}.$$

Corollary (H. 2022)

Let $f \in H_k^{(J)}$ with $J \leq (\log k)^{\delta - \varepsilon}$. Then the zeros of $f(z)$ (counting with multiplicity) are equidistributed in $\mathbb{X}$ as $k \to \infty$.

Three Hecke eigenforms

Let $f \in H_k$ and $h \in H_{2k}$. Denote $F = y^{k/2} f$ and $H = y^k h$. Then we have $f^2 = \sum_{h \in H_{2k}} \langle F^2, H \rangle h$. One can conjecture that

$$\langle F^2, H \rangle \ll k^{-1/2+\varepsilon}.$$

This is a consequence of the Grand Riemann Hypothesis for the triple product L-functions $L(s, f \times f \times h) = L(s, h) L(s, \mathrm{sym}^2 f \times h)$. Unconditionally we can prove

$$\langle F^2, H \rangle \ll k^{1/6+\varepsilon}. \tag{1}$$

Blomer–Khan–Young 2013 proved an upper bound for L^4-norm of f:

$$\|F\|_4^4 = \langle F^2, F^2 \rangle = \sum_{h \in H_{2k}} |\langle F^2, H \rangle|^2 \ll k^{1/3+\varepsilon}.$$

This also proves (1) by positivity of each term in the above summation.

Moreover, we have $\langle F^2, H \rangle \ll k^{-1/3+\varepsilon}$ for all but $O(k^{1-\varepsilon})$ forms $h \in H_{2k}$.

The cubic moment problem

Let ϕ be a Hecke–Maass cusp form and be real valued. Assume

$$\|\phi\|_2^2 = \int_{\mathbb{X}} \phi(z)^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} = 1.$$

QUE due to Lindenstrauss and Soundararajan: For any $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$,

$$\int_{\mathbb{X}} \psi(z)\phi(z)^{2} \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_{\mathbb{X}} \psi(z)\frac{3}{\pi}\frac{\mathrm{d}x\mathrm{d}y}{y^2} + o_\psi(1), \quad \text{as } \lambda_\phi \to \infty.$$

The cubic moment problem: For any $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$,

$$\int_{\mathbb{X}} \psi(z)\phi(z)^{3} \frac{\mathrm{d}x\mathrm{d}y}{y^2} = o_\psi(1), \quad \text{as } \lambda_\phi \to \infty?$$

Watson 2002 proved

$$\int_{\mathbb{X}} \phi(z)^3 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \ll \lambda_\phi^{-1/12+\varepsilon}, \quad \forall \varepsilon > 0.$$

The cubic moment of Hecke–Maass cusp forms

Theorem (H. 2022)

Fix any $\psi \in \mathcal{C}_c^\infty(\mathbb{X})$. Then we have

$$\int_{\mathbb{X}} \psi(z)\phi(z)^3 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \ll \lambda_\phi^{-\delta}.$$

for any $\delta < 1/24$, as $\lambda_\phi \to \infty$.

Remark:

- It is a surprise that a power saving upper bound can be proved for the smooth cubic moment $\int_{\mathbb{X}} \psi(z)\phi(z)^3 \frac{\mathrm{d}x\mathrm{d}y}{y^2}$, since for the second moment (QUE) one can not obtain any rate of decay unconditionally.
- This confirms the smooth cubic case of the random wave conjecture in this setting, which captures the sign changes locally.

Cubic moment to L-functions

- Fix any $\psi(z) \in \mathcal{C}_c^\infty(\mathbb{X})$ with $\psi \geq 0$ and $\int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = 1$. By Hölder's inequality,

$$\int_{\mathbb{X}} \psi(z)\phi(z)^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \leq \Big(\int_{\mathbb{X}} \psi(z)|\phi(z)|^3 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \Big)^{2/3} \Big(\int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2} \Big)^{1/3}.$$

  By the QUE, i.e. $\int_{\mathbb{X}} \psi(z)\phi(z)^2 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \sim \frac{3}{\pi} \int_{\mathbb{X}} \psi(z) \frac{\mathrm{d}x\mathrm{d}y}{y^2}$, we get

$$\int_{\mathbb{X}} \psi(z)|\phi(z)|^3 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \geq \left(\frac{3}{\pi}\right)^{3/2} + o(1),$$

  which is bounded below by a constant.
- By Parseval's formula and Watson's formula

$$\int_{\mathbb{X}} \phi_k(z)\phi(z)^3 \frac{\mathrm{d}x\mathrm{d}y}{y^2} \rightsquigarrow \sum_{j\geq 1} \langle \phi^2, \phi_j\rangle\langle\phi_j, \phi_k\phi\rangle$$

$$\rightsquigarrow t_\phi^{-3/2} \sum_{t_j - t_\phi \ll 1} L(1/2, \phi_j)^{1/2} L(1/2, \phi_j \times \mathrm{Sym}^2\,\phi)^{1/2} L(1/2, \phi_j \times \phi \times \phi_k)^{1/2}.$$

Moments of L-functions

We may prove $\sum_{t_j - t_\phi \ll 1} L(1/2, \phi_j \times \mathrm{Sym}^2 \phi) \ll t_\phi^{3/2+\varepsilon}$ and $\sum_{t_j - t_\phi \ll 1} L(1/2, \phi_j) L(1/2, \phi_j \times \phi \times \phi_k) \ll t_\phi^{3/2+\varepsilon}$ when $t_k \ll 1$.

Theorem (H. 2022)

Let ϕ be a Hecke–Maass cusp form with the spectral parameter $t_\phi = T > 0$. Let $T^{1/3+\varepsilon} \leq M \leq T^{1/2-\varepsilon}$. Then we have

$$\sum_{T-M \leq t_j \leq T+M} L(1/2, \phi_j \times \mathrm{Sym}^2 \phi) \ll T^{1+\varepsilon} M.$$

In particular, we have $\sum_{t_j - t_\phi \ll 1} L(1/2, \phi_j \times \mathrm{Sym}^2 \phi) \ll t_\phi^{4/3+\varepsilon}$, and the subconvexity bound $L(1/2, \phi_j \times \mathrm{Sym}^2 \phi) \ll t_\phi^{4/3+\varepsilon}$ when $|t_j - t_\phi| \leq t_\phi^{1/3}$.

Such first moment was considered in Xiaoqing Li (*Ann. Math.* 2011) for a fixed self dual GL(3) form $\mathrm{Sym}^2 \phi$, i.e. $t_\phi \ll 1$. Li can prove Lindelöf on average bounds as T goes to infinity when $T^{3/8+\varepsilon} \leq M \leq T^{1/2-\varepsilon}$. See McKee–Sun–Ye and Lin–Nunes–Qi for improvements of Li's result.

Thank you very much!

非常感谢!