Arithmetic volumes of unitary Shimura varieties

(joint work with B. Howard)

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Modular curves

Consider the moduli stack of elliptic curves

$$Y \to \operatorname{Spec} \mathbb{Z}.$$

Have $Y(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

- Let $\pi : E \to Y$ be the universal elliptic curve.
- Determines line bundle of modular forms of weight 1:

$$\omega_{E/Y} = \pi_* \Omega^1_{E/Y} \in \operatorname{Pic}(Y).$$

Petersson (also known as Hodge or L^2) metric:

- Let $\tau \in Y(\mathbb{C})$, let E_τ be the corresponding elliptic curve, and $s_\tau \in \omega_{E/Y,\tau} = H^0(E_\tau(\mathbb{C}), \Omega^1_{E_\tau(\mathbb{C})})$. Set

$$\|s_\tau\|^2 = \left| \frac{1}{2\pi i} \int_{E_\tau(\mathbb{C})} s_\tau \wedge \overline{s_\tau} \right|.$$

- Get metrized line bundle

$$\hat{\omega}_{E/Y} \in \widehat{\operatorname{Pic}}(Y).$$

Extension to compactification

The bundle $\omega_{E/Y}$ extends to the Deligne-Rapoport compactification $\bar{Y}$ of Y.

- Metric does not extend smoothly across the boundary.
- It has logarithmic singularity.
- Get a class in the generalized arithmetic Picard group

$$\hat{\omega}_{E/Y} \in \widehat{\mathrm{Pic}}(\bar{Y})$$

  in the sense of Burgos-Kramer-Kühn.

Complex volume

The Chern form

$$\mathrm{ch}(\hat{\omega}_{E/Y}) = -dd^c \log \|s\|^2 = -dd^c \log \left|s(\tau)^2 v\right| = \frac{1}{4\pi} \frac{du\, dv}{v^2}$$

is integrable. Here $\tau = u + iv \in \mathbb{H}$.

Proposition

The complex volume of Y is

$$\mathrm{vol}_{\mathbb{C}}(\hat{\omega}_{E/Y}) = \int_{Y(\mathbb{C})} \mathrm{ch}(\hat{\omega}_{E/Y}) = \frac{1}{12} = -\zeta(-1).$$

- Given by special value of zeta function.

Arithmetic volume

Arithmetic intersection theory (Arakelov, Gillet-Soulé, Burgos-Kramer-Kühn) gives

- arithmetic Chern class

$$\widehat{\mathrm{Pic}}(\bar{Y}) \to \widehat{\mathrm{CH}}^1(\bar{Y}), \quad \hat{\mathcal{L}} \mapsto \widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log \|s\|^2),$$

- intersection pairing

$$\widehat{\mathrm{CH}}^1(\bar{Y}) \times \widehat{\mathrm{CH}}^1(\bar{Y}) \to \widehat{\mathrm{CH}}^2(\bar{Y}),$$

- arithmetic degree $\widehat{\deg} : \widehat{\mathrm{CH}}^2(\bar{Y}) \to \mathbb{R}$.

Define the *arithmetic volume* by

$$\widehat{\mathrm{vol}}(\hat{\omega}_{E/Y}) = \widehat{\deg}(\hat{\omega}_{E/Y} \cdot \hat{\omega}_{E/Y}).$$

Arithmetic volume of the modular curve

Theorem (Bost, Kühn, $\sim$1999)

$$\widehat{\mathrm{vol}}(\hat{\omega}_{E/Y}) = -\frac{1}{2}\,\mathrm{vol}_{\mathbb{C}}(\hat{\omega}_{E/Y})\left(1 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)$$

- Given by logarithmic derivative of zeta function.

Idea of the proof

Take two sections s, t of $\hat{\omega}_{E/Y}^{\otimes k}$ whose divisors intersect properly on $\bar{Y}(\mathbb{C})$. Compute

$$\widehat{\mathrm{vol}}(\hat{\omega}_{E/Y}) = \frac{1}{k^2}\left((\mathrm{div}(s), \mathrm{div}(t))_{fin} + (\widehat{\mathrm{div}}(s), \widehat{\mathrm{div}}(t))_\infty\right),$$

$$(\widehat{\mathrm{div}}(s), \widehat{\mathrm{div}}(t))_\infty = -\int_{Y(\mathbb{C})} \log\|s\| \,\mathrm{ch}(\hat{\omega}_{E/Y}^{\otimes k}) - (\log\|t\|)[\mathrm{div}(s)].$$

- Take $k = 12$, and $s = E_4^3$, $t = \Delta$.
- Then $(\mathrm{div}(s), \mathrm{div}(t))_{fin} = 0$, and $\|E_4^3\| = |E_4^3(\tau)|v^6$, and

$$(\widehat{\mathrm{div}}(s), \widehat{\mathrm{div}}(t))_\infty = -12 \int_{Y(\mathbb{C})} \log\|E_4^3\| \frac{du\,dv}{4\pi v^2} - (\log\|\Delta\|)(\tfrac{1+i\sqrt{3}}{2}).$$

- Evaluating the right hand side gives the result.

Generalizations

For any PEL Shimura variety M with universal abelian scheme $\pi : A \to M$ we can try to compute the arithmetic volume of

$$\omega_{A/M} = \pi_* \Omega_{A/M}^{\dim(A)} \in \mathrm{Pic}(M)$$

equipped with the Petersson metric as above.

- Kudla-Rapoport-Yang: Quaternionic Shimura curves
- B.-Burgos-Kühn: Hilbert modular surfaces
- Jung-Pippich: Siegel threefold
- Hörmann: $\mathrm{O}(n, 2)$ Shimura varieties (up to few 'bad' primes).

GU$(n-1,1)$ Shimura varieties

Fix $K \subset \mathbb{C}$ imaginary quadratic, assume $D = -\operatorname{disc}(K)$ is odd.
For $n \geq 1$, let $M_{(n-1,1)}(\mathbb{C})$ be the moduli space of (A, ι, ψ), where

- A abelian variety over $\mathbb{C}$ of dimension n,
- $\iota : \mathcal{O}_K \to \operatorname{End}(A)$ an $\mathcal{O}_K$-action such that the induced action on $\operatorname{Lie}(A) \cong \mathbb{C}^n$ has signature $(n-1,1)$,
- $\psi : A \to A^\vee$ a principal polarization compatible with ι.

Pappas, Krämer: There exists a flat regular integral model

$$M_{(n-1,1)} \to \operatorname{Spec} \mathcal{O}_K$$

of dimension n. Has a canonical toroidal compactification $\bar{M}_{(n-1,1)}$.

$\mathrm{GU}(n-1,1)$ Shimura varieties

There is a decomposition

$$M_{(n-1,1)} = \bigsqcup_V M_V,$$

where V runs over all similarity classes of K-hermitian spaces of signature $(n-1,1)$ that admit a self-dual $\mathcal{O}_K$-lattice.

- *Similarity* means isometric after possibly rescaling the hermitian form by a positive rational factor.
- For n even: similar=isometric. For n odd: all V are similar.
- M_V is integral model of a $\mathrm{GU}(V)$ Shimura variety.

The universal abelian scheme $\pi : A \to M_V$ gives Hodge bundle

$$\hat{\omega}_{A/M_V} \in \widehat{\mathrm{Pic}}(\bar{M}_V).$$

Some notation

Let $\varepsilon(\cdot) = \left(\frac{\cdot}{D}\right)$ be the quadratic Dirichlet character of $K/\mathbb{Q}$.
For $k \in \mathbb{Z}_{>0}$ put

$$\beta_k(s) = \frac{D^{k/2}\Gamma(s+k)L(2s+k,\varepsilon^k)}{2^k\pi^{s+k}} \times \prod_{\ell \mid D} \begin{cases} 1, & k=1 \\ \left[1+\left(\frac{-1}{\ell}\right)^{\frac{k}{2}}\mathrm{inv}_\ell(V)\ell^{-s-\frac{k}{2}}\right], & k \geq 2 \text{ even} \\ \left[1+\left(\frac{-1}{\ell}\right)^{\frac{k-1}{2}}\mathrm{inv}_\ell(V)\ell^{-s-\frac{k-1}{2}}\right]^{-1}, & k \geq 2 \text{ odd,} \end{cases}$$

where

- we understand $L(s,\varepsilon^k) = \zeta(s)$ for k even, and
- $\mathrm{inv}_\ell(V) = (\det(V), -D)_\ell \in \{\pm 1\}$ is the local invariant of V.

Main result

Theorem (B.-Howard)

i) The complex volume $\mathrm{vol}_{\mathbb{C}}(\hat{\omega}_{A/M_V})$ *of* M_V *is*

$$\int_{M_V(\mathbb{C})} \mathrm{ch}(\hat{\omega}_{A/M_V})^{n-1} = \beta_1(0)\cdots\beta_n(0)\cdot\begin{cases} 2^{n-1} & n \text{ even,} \\ 2^{n-o(D)} & n \text{ odd.}\end{cases}$$

ii) The arithmetic volume is

$$\widehat{\mathrm{vol}}(\hat{\omega}_{A/M_V}) = \left[\frac{2\beta_1'(0)}{\beta_1(0)} + \cdots + \frac{2\beta_n'(0)}{\beta_n(0)} + \log(D) - nC_0(n)\right] \times \mathrm{vol}_{\mathbb{C}}(\hat{\omega}_{A/M_V}),$$

where

$$C_0(n) = 2\log\left(\frac{4\pi e^{\gamma}}{\sqrt{D}}\right) + (n-4)\left(\frac{L'(0,\varepsilon)}{L(0,\varepsilon)} + \frac{\log(D)}{2}\right).$$

Example

If $n = 3$ then the generic fiber of M_V is a *Picard modular surface*. We obtain in this case:

$$\mathrm{vol}_{\mathbb{C}}(\hat{\omega}_{A/M_V}) = \frac{D^3}{2^{2+o(D)}\pi^6} \cdot L(1,\varepsilon)\zeta(2)L(3,\varepsilon),$$

$$\widehat{\mathrm{vol}}(\hat{\omega}_{A/M_V}) = \Bigg[4\frac{L'(1,\varepsilon)}{L(1,\varepsilon)} + 4\frac{\zeta'(2)}{\zeta(2)} + 4\frac{L'(3,\varepsilon)}{L(3,\varepsilon)} + \log(D) + 5 - 6\gamma - 6\log(\pi) - 3C_0(3)\Bigg] \cdot \mathrm{vol}_{\mathbb{C}}(\hat{\omega}_{A/M_V}).$$

- If n is odd, the volumes only depend on n but not on V.

Idea of the proof

- Use induction on n.
- If $n = 2$ we have $\mathrm{sig}(V) = (1, 1)$.
  Use $\mathrm{SU}(1, 1) \cong \mathrm{SL}_2(\mathbb{R})$ to relate M_V to a modular curve (or a quaternionic Shimura curve).
  Reduce to the result of Bost, Kühn (or Kudla-Rapoport-Yang).
- If $n > 2$, use Borcherds products on M_V to relate the volume of M_V to volumes of $M_{V'}$ for hermitian spaces V' of smaller dimension.
- Vary the Borcherds product relations to simplify the computations.

Kudla-Rapoport divisors

Convenient to replace M_V by its finite etale cover $M_{(1,0)} \times M_V$, where $M_{(1,0)}$ is the moduli space of elliptic curves with CM by $\mathcal{O}_K$.

- Any point $(E, A) \in M_{(1,0)} \times M_V$ has associated $\mathcal{O}_K$-module $\mathrm{Hom}_{\mathcal{O}_K}(E, A)$ with positive definite hermitian form:

$$\begin{aligned} \langle x, y \rangle &= \psi_E^{-1} \circ y^\vee \circ \psi_A \circ x \\ &\in \mathrm{End}_{\mathcal{O}_K}(E) \cong \mathcal{O}_K \end{aligned}$$

$$\begin{tikzcd} E \arrow[r, "x"] \arrow[d, "\psi_E"'] & A \arrow[d, "\psi_A"] \\ E^\vee & A^\vee. \arrow[l, "y^\vee"] \end{tikzcd}$$

$Z(m)$: moduli stack of triples (E, A, x), where

- $(E, A) \in M_{(1,0)} \times M_V$.
- $x \in \mathrm{Hom}_{\mathcal{O}_K}(E, A)$ and $\langle x, x \rangle = m$.

A nice special case

Let $p \equiv 1 \pmod{D}$ be a prime. Let V' be the K-hermitian space of signature $(n-2,1)$ whose local invariants satisfy

$$\mathrm{inv}_\ell(V') = (p, -D)_\ell \cdot \mathrm{inv}_\ell(V)$$

for all places $\ell \leq \infty$.

- Equivalently, $V' = x^\perp \subset V$ of any $x \in V$ with $\langle x, x \rangle = p$.
- V' admits a self-dual $\mathcal{O}_K$-lattice.

Proposition

$$Z(p) \text{ “}{=}\text{” } (p^{n-1} + 1) \cdot M_{V'}.$$

- Pull-back of $\hat{\omega}_V$ via each $M_{V'} \to Z(p) \to M_V$ is $\hat{\omega}_{V'}$ (up to some explicit vertical divisors).

Borchers products

Theorem (Borcherds, BHKRY)

Let

$$f = \sum_m c(m)q^m \in M^{!,\infty}_{2-n}(\Gamma_0(D), \varepsilon^n)$$

be a weakly holomorphic modular form with integral coefficients $c(m)$*. There is a rational section* $\Psi(f)$ *of*

$$\omega_V^{\otimes c(0)} \in \mathrm{Pic}(\bar{M}_V)$$

such that:

- $\mathrm{div}(\Psi(f)) = \sum_{m>0} c(-m)Z(m)$ *(up to explicit vertical and boundary components),*
- *induced section on* $\bar{M}_V(\mathbb{C})$ *is given by regularized theta lift.*

The inductive argument: Complex volume

For simplicity, assume that we can chose $f \in M^{!,\infty}_{2-n}(\Gamma_0(D), \varepsilon^n)$ with

$$f = q^{-p} + c(0) + c(1)q + \dots$$

for a prime $p \equiv 1 \pmod{D}$.

- Proposition: $c(0) = \frac{p^{n-1}+1}{\beta_n(0)}$.
- The Borcherds product $\Psi(f)$ gives in $H^2(\bar{M}_V(\mathbb{C}), \mathbb{C})$:

$$[Z(p)] = c(0) \cdot \mathrm{ch}(\hat{\omega}_V).$$

$$\begin{aligned} \Rightarrow \quad \int_{M_V(\mathbb{C})} \mathrm{ch}(\hat{\omega}_V)^{n-1} &= \frac{1}{c(0)} \int_{Z(p)} \mathrm{ch}(\hat{\omega}_V)^{n-2} \\ &= \frac{p^{n-1}+1}{c(0)} \int_{M_{V'}(\mathbb{C})} \mathrm{ch}(\hat{\omega}_{V'})^{n-2} \\ &= \beta_n(0) \int_{M_{V'}(\mathbb{C})} \mathrm{ch}(\hat{\omega}_{V'})^{n-2}. \end{aligned}$$

The inductive argument: Arithmetic volume

The Borchers product $\Psi(f)$ gives in $\widehat{\mathrm{CH}}^1(\bar{M}_V)$ the relation

$$\hat{\omega}_V^{\otimes c(0)} = \widehat{\mathrm{div}}\,(\Psi(f)) = (Z(p), -\log\|\Psi(f)\|^2).$$

Hence,

$$\begin{aligned}
\widehat{\mathrm{vol}}(\hat{\omega}_V) &= \widehat{\deg}(\hat{\omega}_V \cdots \hat{\omega}_V) \\
&= \frac{1}{c(0)} \widehat{\deg}\left(\hat{\omega}_V \cdots \hat{\omega}_V \cdot \widehat{\mathrm{div}}(\Psi(f))\right) \\
&= \frac{p^{n-1}+1}{c(0)} \cdot \widehat{\mathrm{vol}}(\hat{\omega}_{V'}) - \int_{M_V(\mathbb{C})} \log\|\Psi(f)\|^2 \cdot \mathrm{ch}(\hat{\omega}_V)^{n-1} \\
&= \beta_n(0) \cdot \widehat{\mathrm{vol}}(\hat{\omega}_{V'}) + B'(p, s_0)\,\mathrm{vol}_{\mathbb{C}}(\hat{\omega}_V).
\end{aligned}$$

Here $B(p,s) =$ p-th coefficient of an Eisenstein series of weight n.

- Proposition: $B(p,s) = \frac{1}{\beta_n(s-s_0)} \cdot \prod_{\ell | pD}(\text{local factor in } \ell^{s-s_0})$.

Integrals of automorphic Green functions

Here we have used the following result (due to Kudla and B.-Kühn in the orthogonal case).

Theorem

Let $f = \sum_m c(m) q^m \in M^{!,\infty}_{2-n}(\Gamma_0(D), \varepsilon^n)$*. Then*

$$-\int_{M_V(\mathbb{C})} \log \|\Psi(f)\|^2 \cdot \mathrm{ch}(\hat{\omega}_V)^{n-1} = \mathrm{vol}_{\mathbb{C}}(\hat{\omega}_V) \sum_{m>0} c(-m) B'(m, s_0),$$

where $G(\tau, s) = \sum_{m \geq 0} B(m, s) q^m$ *is a weight* n *Eisenstein series for* $\Gamma_0(D)$ *and* $s_0 = \frac{n-1}{2}$*.*

Upshot: All contributions to $\widehat{\mathrm{vol}}(\hat{\omega}_V)$ are expressed in terms of $\beta_1(s), \dots, \beta_n(s)$ and various corrections factors from primes $\ell \mid pD$, which one must keep track of.

Thank you for your attention!