# On the global Gan-Gross-Prasad conjecture for GSpin groups

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- $\mathrm{GSpin}(V_n) \subset \mathrm{GSpin}(V_{n+2\ell+1})$, and their centers share the same identity component, which is precisely $\ker(\mathrm{pr}) \cong \mathrm{GL}_1$.

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- $\psi_{\ell,a}$ a character of $N_\ell(F)\backslash N_\ell(\mathbb{A})$ associated to $\psi$ and $a \in F^\times$. Note that $L_\ell(\mathbb{A})$ acts on $\psi_{\ell,a}$ and the stabilizer is $\mathrm{GSpin}_n(\mathbb{A}) \subset L_\ell(\mathbb{A})$.

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Let $\phi$ be an automorphic form on $\mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})$ and $\phi'$ a cusp form on $\mathrm{GSpin}_n(\mathbb{A})$, such that

$$\phi(z)\phi'(z) = 1 \text{ for all } z \in \ker(\mathrm{pr})(\mathbb{A}) = \mathrm{GL}_1(\mathbb{A}). \tag{1}$$

## Definition

The **Bessel period** for the pair $(\phi, \phi')$ (with respect to the character $\psi_{\ell,a}$) is

$$\mathcal{B}^{\psi_{\ell,a}}(\phi, \phi') := \int_{\ker(\mathrm{pr})(\mathbb{A})\mathrm{GSpin}_n(F)\backslash \mathrm{GSpin}_n(\mathbb{A})} \phi^{N_\ell, \psi_{\ell,a}}(g)\phi'(g)dg$$

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$$\phi^{N_\ell, \psi_{\ell,a}}(g) := \int_{N_\ell(F)\backslash N_\ell(\mathbb{A})} \phi(ug)\psi_{\ell,a}^{-1}(u)du$$

is the Bessel-Fourier coefficient of $\phi$.

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2. Since $\phi'$ is a cusp form, the integral defining $\mathcal{B}^{\psi_{\ell,a}}$ is absolutely convergent.
3. The subgroup $H = N_\ell \rtimes \mathrm{GSpin}_n$ is called a Bessel subgroup of $\mathrm{GSpin}_{n+2\ell+1}$.

# Uniqueness of local Bessel models

- There is a local analogue of the global Bessel period: local Bessel model.

**Theorem (Y, 2025+)**

*Let $\nu$ be a place of $F$. Let $\pi_\nu, \sigma_\nu$ be irreducible admissible representations of $\mathrm{GSpin}_{n+2\ell+1}(F_\nu), \mathrm{GSpin}_n(F_\nu)$ respectively, which are of Casselman-Wallach type if $F_\nu$ is archimedean. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H(F_\nu)}(\pi_\nu \otimes \sigma_\nu, \psi_{\ell,a,\nu}) \leq 1.$$

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- The spherical case (i.e., $\ell = 0$) of the above result was proved by
  - [Emory-Takeda, 2023] when $F_\nu$ is non-archimedean,
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- Our proof is to reduce the problem to the spherical case.

# Global Gan-Gross-Prasad conjecture

*Astérisque*
**346**, 2012, p. 1–109

# SYMPLECTIC LOCAL ROOT NUMBERS, CENTRAL CRITICAL $L$-VALUES, AND RESTRICTION PROBLEMS IN THE REPRESENTATION THEORY OF CLASSICAL GROUPS

*by*

Wee Teck Gan, Benedict H. Gross & Dipendra Prasad

***Abstract.*** — In this paper, we provide a conjectural recipe for the restriction of irreducible representations of classical groups (including metaplectic groups), to certain subgroups, generalizing our earlier work on representations of orthogonal groups. Our conjectures include the cases of Bessel and Fourier-Jacobi models. In fact, it is the standard representation of the classical group, together with its orthogonal, symplectic, hermitian, or skew-hermitian form, that plays the primary role, and not the classical group alone. All of our conjectures assume the Langlands parametrization. For classical groups over local fields, the recipe involves local epsilon factors associated to the Langlands parameter and certain summands of a fixed symplectic representation of the $L$-group. For automorphic representations over global fields, it involves the central critical value of this symplectic $L$-function.

# Global Gan-Gross-Prasad conjecture

In 2012, Gan, Gross, and Prasad formulated a conjecture for classical groups and metaplectic groups that links the Bessel period to central $L$-value. Based on the formulation for special orthogonal groups, one can formulate the conjecture for $\mathrm{GSpin}$ groups by taking care of the center:

$$\mathcal{B}^{\psi_{\ell,a}}(\phi,\phi') := \int_{\ker(\mathrm{pr})(\mathbb{A})\mathrm{GSpin}_n(F)\backslash\mathrm{GSpin}_n(\mathbb{A})} \phi^{N_\ell,\psi_{\ell,a}}(g)\phi'(g)dg.$$

## Conjecture (Global Gan-Gross-Prasad conjecture for GSpin groups)

*Let $\pi$, $\sigma$ be irreducible tempered cuspidal representations of $\mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})$ and $\mathrm{GSpin}_n(\mathbb{A})$ respectively such that $\omega_\pi\omega_\sigma = 1$, which occur with multiplicity one in the discrete spectrum. Then the following are equivalent:*

(i) *The Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi,\phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$.*

(ii) *The central value $L(\frac{1}{2},\pi\times\sigma) \neq 0$.*

# Previous work of Jiang-Zhang

- Classical groups:
  - [Jiang-Zhang, 2020] made a breakthrough and proved the direction (i)⇒(ii) in full generality for special orthogonal and unitary groups, using **Rankin-Selberg method** together with **endoscopic classification** for these groups. (They also proved (ii)⇒(i) under some hypothesis.)

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https://doi.org/10.4007/annals.2020.191.3.2

## Arthur parameters and cuspidal automorphic modules of classical groups

By Dihua Jiang and Lei Zhang

### Abstract

The endoscopic classification via the stable trace formula comparison provides certain character relations between irreducible cuspidal automorphic representations of classical groups and their global Arthur parameters, which are certain automorphic representations of general linear groups. It is a question of J. Arthur and W. Schmid that asks *how to construct concrete modules for irreducible cuspidal automorphic representations of classical groups in term of their global Arthur parameters*? In this paper, we formulate a general construction of concrete modules, using Bessel periods, for cuspidal automorphic representations of classical groups with generic global Arthur parameters. Then we establish the theory for orthogonal and unitary groups, based on certain well expected conjectures. Among the consequences of the theory in this paper is that the global Gan-Gross-Prasad conjecture for those classical groups is proved in full generality in one direction and with a global assumption in the other direction.

# Previous work on GSpin

- GSpin groups:
  - The special case $(\mathrm{GSpin}_{n+2\ell+1}, \mathrm{GSpin}_n) = (\mathrm{GSpin}_5 \cong \mathrm{GSp}_4, \mathrm{GSpin}_2 \cong GSO_2)$ was considered by [Prasad–Takloo-Bighash 2011].

    Prasad, Dipendra (6-TIFR-SM); Takloo-Bighash, Ramin (1-ILCC-MS)
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Global GGP for $\mathrm{GSpin}$ groups is known in some very low rank special cases.

# Main result

We prove one direction ((i)⇒(ii)) of the global GGP conjecture for generic repns of $\mathrm{GSpin}$ groups.

## Theorem (Y, 2025+)

*Let $\pi$ and $\sigma$ be irreducible generic cuspidal automorphic representations of $\mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})$ and $\mathrm{GSpin}_n(\mathbb{A})$ respetcively, such that $\omega_\pi \omega_\sigma = 1$. If the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$ and $\phi_\sigma \in V_\sigma$, then $L(s, \pi \times \sigma)$ is holomorphic and non-zero at $s = \frac{1}{2}$.*

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Remarks:

- The above result works for arbitrary $n$ and $\ell$.
- The first ingredient of my proof is a new family of **Rankin-Selberg integrals** for the $L$-functions of $\mathrm{GSpin} \times \mathrm{GL}$.

  Our proof of the main result is an application of the Rankin-Selberg integral in a similar way as in [Jiang-Zhang, 2020].

# Ingredient 1: A new Rankin-Selberg integral

- $\pi$ and $\sigma$ cusp. repns of $\mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})$ and $\mathrm{GSpin}_n(\mathbb{A})$ resp. s.t. $\omega_\pi\omega_\sigma = 1$ **(Here we do not assume $\pi$, $\sigma$ are generic.)**
- $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ generic isobaric auto. repn of $\mathrm{GL}_k(\mathbb{A})$, $k > \ell$
- $\boxed{\textbf{Goal:}\text{ Construct an integral representation for } L(s, \pi \times \tau)}$ ($\sigma$ is auxiliary)

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- $$\begin{array}{cccccc} k > \ell & \Longrightarrow & \mathrm{GSpin}_n & \subset \mathrm{GSpin}_{n+2\ell+1} & \subset & \mathrm{GSpin}_{n+2k} \\ & & \text{(small)} & \text{(middle)} & & \text{(large)} \end{array}$$

  Denote $\ell' = k - \ell - 1$. Then $\mathrm{GSpin}_{n+2k} = \mathrm{GSpin}_{(n+2\ell+1)+2\ell'+1}$.

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- $P_k \subset \mathrm{GSpin}_{n+2k}$ a standard parabolic inside the large $\mathrm{GSpin}$ group whose Levi factor is isomorphic to $\mathrm{GL}_k \times \mathrm{GSpin}_n$.

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- $P_k \subset \mathrm{GSpin}_{n+2k}$ a standard parabolic inside the large GSpin group whose Levi factor is isomorphic to $\mathrm{GL}_k \times \mathrm{GSpin}_n$.
- Form the induced representation $\mathrm{Ind}_{P_k(\mathbb{A})}^{\mathrm{GSpin}_{n+2k}(\mathbb{A})}(\tau|\cdot|^{s-\frac{1}{2}} \otimes \sigma)$ and define an Eisenstein series associated to a section $f_{\tau,\sigma,s}$ in the induced representation:
$$E(g, f_{\tau,\sigma,s}) = \sum_{\gamma \in P_k(F)\backslash \mathrm{GSpin}_{n+2k}(F)} f_{\tau,\sigma,s}(\gamma g), \quad g \in \mathrm{GSpin}_{n+2k}(\mathbb{A})$$

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- $H' = N_{\ell'} \rtimes \mathrm{GSpin}_{n+2\ell+1}$ a Bessel subgroup of large $\mathrm{GSpin}_{n+2k}$ with
  - a unipotent subgroup

$$N_{\ell'} = \left\{ u = \begin{pmatrix} z & y & x \\ & I_{n+2\ell+2} & y' \\ & & z^* \end{pmatrix} : z \in Z_{\ell'} \right\},$$

  where $Z_{\ell'} = \{$ upper triangular unipotent matrices of size $\ell' \times \ell'\}$.
  - the middle $\mathrm{GSpin}_{n+2\ell+1}$ is the stabilizer (inside the Levi subgroup corresponding to $N_{\ell'}$) of a character $\psi_{\ell',-a}$ on $N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})$.

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$$N_{\ell'} = \left\{ u = \begin{pmatrix} z & y & x \\ & I_{n+2\ell+2} & y' \\ & & z^* \end{pmatrix} : z \in Z_{\ell'} \right\},$$

  where $Z_{\ell'} = \{$ upper triangular unipotent matrices of size $\ell' \times \ell'\}$.
  - the middle $\mathrm{GSpin}_{n+2\ell+1}$ is the stabilizer (inside the Levi subgroup corresponding to $N_{\ell'}$) of a character $\psi_{\ell',-a}$ on $N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})$.

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# Ingredient 1: A new Rankin-Selberg integral

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- $\phi_\pi \in V_\pi$ a non-zero cusp form.

- Define the global zeta integral $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s})$ as the Bessel period of $(E(\cdot, f_{\tau,\sigma,s}), \phi_\pi)$ for the pair $(\mathrm{GSpin}_{n+2k}, \mathrm{GSpin}_{n+2\ell+1})$:

$$\int_{\ker(\mathrm{pr})(\mathbb{A})\mathrm{GSpin}_{n+2\ell+1}(F)\backslash \mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})} E^{N_{\ell'},\psi_{\ell',-a}}(g, f_{\tau,\sigma,s})\phi_\pi(g)dg.$$

# Ingredient 1: A new Rankin-Selberg integral

$$\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s}) = \int_{\mathrm{ker(pr)}(\mathbb{A})\mathrm{GSpin}_{n+2\ell+1}(F)\backslash \mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})} E^{N_{\ell'},\psi_{\ell'},-a}(g, f_{\tau,\sigma,s})\phi_\pi(g)dg$$

- $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s})$ converges absolutely and uniformly in vertical strips in $\mathbb{C}$ away from the possible poles of the Eisenstein series, and hence it defines a meromorphic function on $\mathbb{C}$ with possible poles at the locations where the Eisenstein series has poles.

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- To establish an integral representation, the first step is to unfold the global integral into an Euler product of local integrals.

# Ingredient 1: A new Rankin-Selberg integral

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- To establish an integral representation, the first step is to unfold the global integral into an Euler product of local integrals.
- We start by unfolding the Bessel coefficient of Eisenstein series.

# Ingredient 1: A new Rankin-Selberg integral

$$E^{N_{\ell'},\psi_{\ell',-a}}(g, f_{\tau,\sigma,s}) = \int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} E(ug, f_{\tau,\sigma,s})\psi^{-1}_{\ell',-a}(u)du$$
$$= \sum_{\epsilon\in\mathcal{E}_{k,\ell}} \int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} \sum_{\delta\in P^{\epsilon}_{\ell'}(F)\backslash P_{\ell'}(F)} f_{\tau,\sigma,s}(\epsilon\delta ug)\psi^{-1}_{\ell',-a}(u)du,$$

where

- $P_{\ell'}$ is standard parabolic of large $\mathrm{GSpin}_{n+2k}$ whose Levi subgp is $\mathrm{GL}_{\ell'} \times \mathrm{GSpin}_{n+2\ell+2}$,
- $\mathcal{E}_{k,\ell}$ is a set of representatives for $P_k(F)\backslash\mathrm{GSpin}_{n+2k}(F)/P_{\ell'}(F)$,
- $P^{\epsilon}_{\ell'} = \epsilon^{-1}P_k\epsilon \cap P_{\ell'}$ is the stabilizer in $P_{\ell'}$.

# Ingredient 1: A new Rankin-Selberg integral

$$\begin{aligned} E^{N_{\ell'},\psi_{\ell'},-a}(g,f_{\tau,\sigma,s}) &= \int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} E(ug,f_{\tau,\sigma,s})\psi^{-1}_{\ell',-a}(u)du \\ &= \sum_{\epsilon\in\mathcal{E}_{k,\ell}} \int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} \sum_{\delta\in P^{\epsilon}_{\ell'}(F)\backslash P_{\ell'}(F)} f_{\tau,\sigma,s}(\epsilon\delta ug)\psi^{-1}_{\ell',-a}(u)du, \end{aligned}$$

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Let $\mathcal{N}_{\epsilon,k,\ell}$ be a set of representatives of $P^{\epsilon}_{\ell'}(F)\backslash P_{\ell'}(F)/H'(F)$, where $H'=N_{\ell'}\ltimes\mathrm{GSpin}_{n+2\ell+1}$ is the Bessel subgroup of large $\mathrm{GSpin}_{n+2k}$. Then

$$E^{N_{\ell'},\psi_{\ell'},-a}(g,f_{\tau,\sigma,s}) = \sum_{\epsilon\in\mathcal{E}_{k,\ell}}\sum_{\eta\in\mathcal{N}_{\epsilon,k,\ell}} \int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} \sum_{\delta\in H'^{\eta}(F)\backslash H'(F)} f_{\tau,\sigma,s}(\epsilon\eta\delta ug)\psi^{-1}_{\ell',-a}(u)du,$$

where $H'^{\eta}=H'\cap\eta^{-1}P^{\epsilon}_{\ell'}\eta$.

# Ingredient 1: A new Rankin-Selberg integral

Plugging in the above expression to the global zeta integral:

$$\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s}) = \sum_{\epsilon\in\mathcal{E}_{k,\ell}} \sum_{\eta\in\mathcal{N}_{\epsilon,k,\ell}} \int_{\mathrm{ker(pr)}(\mathbb{A})\mathrm{GSpin}_{n+2\ell+1}(F)\backslash\mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})} \int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} \sum_{\delta\in H'^{\eta}(F)\backslash H'(F)} f_{\tau,\sigma,s}(\epsilon\eta\delta ug)\psi^{-1}_{\ell',-a}(u)du\phi_\pi(g)dg.$$

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- If $S$ is a unipotent subgroup and $\psi_S$ is a non-trivial character on $S(F)\backslash S(\mathbb{A})$, then $\int_{S(F)\backslash S(\mathbb{A})} \psi_S(s)ds = 0$.

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$$\int_{N_{\ell'}(F)\backslash N_{\ell'}(\mathbb{A})} \sum_{\delta \in H'^{\eta}(F)\backslash H'(F)} f_{\tau,\sigma,s}(\epsilon\eta\delta ug)\psi_{\ell',-a}^{-1}(u)du\phi_\pi(g)dg.$$

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- If $U$ is the unipotent radical of a proper maximal parabolic subgroup of middle $\mathrm{GSpin}_{n+2\ell+1}$, then $\int_{U(F)\backslash U(\mathbb{A})} \phi_\pi(ug)du = 0$ due to the cuspidality of $\phi_\pi$.

# Ingredient 1: A new Rankin-Selberg integral

For $\mathrm{Re}(s) \gg 0$, the global zeta integral $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s})$ unfolds to

$$\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s}) = \int_{H(\mathbb{A})\backslash \mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})} \mathcal{B}^{\psi_{\ell,a}}(\pi(g)\phi_\pi, \mathcal{J}(R(\epsilon\eta g)f_{\mathcal{W}(\tau),\sigma,s})dg,$$

where

- $H = N_\ell \rtimes \mathrm{GSpin}_n$ is the Bessel subgroup of the middle $\mathrm{GSpin}_{n+2\ell+1}$
- $\mathcal{B}^{\psi_{\ell,a}}$ is the Bessel period for $(\pi, \sigma')$, where $\sigma'$ obtained from $\sigma$ by a conjugation
- $\mathcal{J}$ denotes taking a Fourier coefficient w.r.t. adelic points of $U_0$
- $R$ denotes right translation
- $\epsilon, \eta$ are Weyl group elements coming from certain double coset decompositions
- $f_{\mathcal{W}(\tau),\sigma,s}$ belongs to $\mathrm{Ind}_{P_k(\mathbb{A})}^{\mathrm{GSpin}_{n+2k}(\mathbb{A})}(\mathcal{W}(\tau)|\cdot|^{s-\frac{1}{2}} \otimes \sigma)$, where $\mathcal{W}(\tau)$ is the Whittaker model of $\tau$.

# Ingredient 1: A new Rankin-Selberg integral

Due to the uniqueness of Bessel models for $\mathrm{GSpin}$ groups, we can decompose $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s})$ as an Euler product of local integrals.

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Local zeta integral at a place $\nu$:

$$\begin{aligned}&\mathcal{Z}_\nu(v_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma_\nu,s})\\ =&\int_{H(F_\nu)\backslash \mathrm{GSpin}_{n+2\ell+1}(F_\nu)}\int_{U_0(F_\nu)} \mathfrak{b}_\nu(\pi_\nu(g)v_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma_\nu,s}(u\epsilon\eta g))\psi_{U_0}(u)dudg\end{aligned}$$

where

- $v_{\pi_\nu} \in V_{\pi_\nu}$
- $f_{\mathcal{W}(\tau_\nu),\sigma_\nu,s} \in \mathrm{Ind}_{P_k(F_\nu)}^{\mathrm{GSpin}_{n+2k}(F_\nu)}(\mathcal{W}(\tau_\nu)|\cdot|^{s-\frac{1}{2}} \otimes \sigma_\nu)$ a section of the local induced repn
- $\mathfrak{b}_\nu \in \mathrm{Hom}_{H(F_\nu)}(\pi_\nu \otimes \sigma'_\nu, \psi_{\ell,a,\nu})$ a local Bessel functional

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The next step is to compute the local integral at a finite unramified place.

# Ingredient 1: A new Rankin-Selberg integral

We take the following data for the local unramified computation:

- $\nu$ a finite place s.t. everything is unramified
- $v^0_{\pi_\nu} \in V_{\pi_\nu}$, $v^0_{\sigma_\nu} \in V_{\sigma_\nu}$ non-zero unramified vectors
- $\mathfrak{b}_\nu$ is normalized s.t. $\mathfrak{b}_\nu(v^0_{\pi_\nu}, v^0_{\sigma_\nu}) = 1$
- $f^0_{\mathcal{W}(\tau_\nu),\sigma_\nu,s}$ the unramified section s.t. for all $a \in \mathrm{GL}_k(F_\nu)$,

$$f^0_{\mathcal{W}(\tau_\nu),\sigma_\nu,s}(e, a) = W^0_{\tau_\nu}(a) v^0_{\sigma_\nu},$$

  where $e \in \mathrm{GSpin}_{n+2\ell+1}(F_\nu)$ is the identity element, and $W^0_{\tau_\nu} \in \mathcal{W}(\tau_\nu)$ is the unramified normalized Whittaker function s.t. $W^0_{\tau_\nu}(I_k) = 1$.

**Local unramified computation:**

$$\mathcal{Z}_\nu(v^0_{\pi_\nu}, f^0_{\mathcal{W}(\tau_\nu),\sigma_\nu,s}) = \frac{L(s, \pi_\nu \times \tau_\nu)}{L(s + \frac{1}{2}, \sigma_\nu \times \tau_\nu \otimes \omega_{\pi_\nu}) L(2s, \tau_\nu, \rho \otimes \omega_{\pi_\nu})}$$

where

$$\rho = \begin{cases} \wedge^2 & \text{if } n \text{ is even} \\ \mathrm{Sym}^2 & \text{if } n \text{ is odd.} \end{cases}$$

# Ingredient 1: A new Rankin-Selberg integral

## Theorem (Y, 2025+)

*Assume $(\pi, \sigma')$ has a non-zero Bessel period. There is a choice of data so that*

$$\mathcal{Z}(\phi_\pi, f_{\tau,\sigma,s}) = \frac{L^S(s, \pi \times \tau)}{L^S(s+\frac{1}{2}, \sigma \times \tau \otimes \omega_\pi) L^S(2s, \tau, \rho \otimes \omega_\pi)} \cdot \mathcal{Z}_S(\phi_\pi, f_{\tau,\sigma,s})$$

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- When $\pi$ is globally generic, $\tau$ is cuspidal, and $P_k$ is the Siegel parabolic subgroup of $\mathrm{GSpin}_{n+2k}$, such construction appears in a recent work of [Asgari-Cogdell-Shahidi, 2024+] using Eisenstein series associated to $\mathrm{Ind}_{P_k(\mathbb{A})}^{\mathrm{GSpin}_{n+2k}(\mathbb{A})}(\tau|\cdot|^{s-\frac{1}{2}} \otimes \omega_\pi^{-1})$.

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- In the special case $n = k = 2$, $\ell = 1$, the above result recovers an integral representation of [Furusawa 1993] for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

# Ingredient 2: Image of functorial transfer for generic reps

## Theorem (Asgari-Shahidi, 2006, 2014)

*Let $G_n$ be a quasisplit $\mathrm{GSpin}_{2n+1}$ or $\mathrm{GSpin}_{2n}$, and let $\pi$ be an irreducible generic cuspidal representation of $G_n(\mathbb{A})$. Then $\pi$ has a unique functorial transfer to an automorphic representation $\Pi$ of $\mathrm{GL}_{2n}(\mathbb{A})$. The representation $\Pi$ satisfies $\Pi \cong \widetilde{\Pi} \otimes \omega_\pi$. The representation $\Pi$ is an isobaric sum of the form*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$$

*where each $\Pi_i$ is an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A})$ such that for each $1 \leq i \leq r$, the complete L-function $L(s, \Pi_i, \wedge^2 \otimes \omega_\pi^{-1})$ has a pole at $s = 1$ in the odd case and $L(s, \Pi_i, \mathrm{Sym}^2 \otimes \omega_\pi^{-1})$ has a pole at $s = 1$ in the even case. Moreover, $\Pi_i \cong \widetilde{\Pi}_i \otimes \omega_\pi$ for each $i$, $\Pi_i \not\cong \Pi_j$ if $i \neq j$, and $n_1 + \cdots + n_r = 2n$.*

*Conversely, any automorphic representation $\Pi$ of $\mathrm{GL}_{2n}(\mathbb{A})$ satisfying the above conditions is a functorial transfer of some irreducible generic cuspidal representation $\pi$ of $G_n(\mathbb{A})$.*

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

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- **Step 1.** We have a Rankin-Selberg integral

$$\mathcal{Z}(\phi_\pi, f_{\tau,\sigma',s}) = \prod_\nu \mathcal{Z}_\nu(\phi_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma'_\nu,s})$$
$$= \frac{L^S(s, \pi \times \tau)}{L^S(s+\frac{1}{2}, \sigma' \times \tau \otimes \omega_\pi) L^S(2s, \tau, \rho \otimes \omega_\pi)} \cdot \mathcal{Z}_S(\phi_\pi, f_{\tau,\sigma',s})$$

  Note that $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma',s})$ unfolds to the Bessel period of $(\pi, \sigma)$.

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- **Step 2.** Let $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ be an irreducible unitary generic isobaric automorphic representation of $\mathrm{GL}_k(\mathbb{A})$ associated to distinct $\tau_1, \cdots, \tau_r$, such that $\widetilde{\tau}_i \cong \tau_i \otimes \omega_\sigma^{-1}$ for each $1 \leq i \leq r$. We can prove:

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- **Step 2.** Let $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$ be an irreducible unitary generic isobaric automorphic representation of $\mathrm{GL}_k(\mathbb{A})$ associated to distinct $\tau_1, \cdots, \tau_r$, such that $\widetilde{\tau}_i \cong \tau_i \otimes \omega_\sigma^{-1}$ for each $1 \leq i \leq r$. We can prove:
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  - (ii) The Eisenstein series $E(\cdot, f_{\tau,\sigma',s})$ can possibly have a pole at $s = 1$ of order at most $r$.

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

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  (ii) The Eisenstein series $E(\cdot, f_{\tau,\sigma',s})$ can possibly have a pole at $s = 1$ of order at most $r$.
  (iii) The following are equivalent:
    - The Eisenstein series $E(\cdot, f_{\tau,\sigma',s})$ has a pole at $s = 1$ of order $r$.
    - $L(s, \tau_i, \rho \otimes \omega_\sigma^{-1})$ has a pole at $s = 1$ for $i = 1, \cdots, r$, and $L(s, \sigma' \times \tau \otimes \omega_\sigma^{-1})$ is non-zero at $s = \frac{1}{2}$.

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

- **Step 3.** Normalize the local zeta integral by

$$\mathcal{Z}_\nu^*(\phi_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma'_\nu,s}) = \frac{\mathcal{Z}_\nu(\phi_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma'_\nu,s})}{\mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho)},$$

  where

$$\mathcal{L}(s, \tau_\nu, \pi_\nu, \sigma_\nu; \rho) := \frac{L(s, \pi_\nu \times \tau_\nu)}{L(s + \frac{1}{2}, \sigma_\nu \times \tau_\nu \otimes \omega_{\pi_\nu}) L(2s, \tau_\nu, \rho \otimes \omega_{\pi_\nu})}.$$

# Sketch of proof of one direction of GGP for generic repns

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Let $\Pi$ be the functorial transfer of $\pi$ (existence due to Asgari-Shahidi) and let $\tau = \Pi \otimes \omega_\pi^{-1}$. We can show that

$$\mathcal{L}(s,\tau,\pi,\sigma;\rho) = \prod_\nu \mathcal{L}(s,\tau_\nu,\pi_\nu,\sigma_\nu;\rho)$$

has a pole at $s = 1$ of order $r$ (coming from numerator), and

$$\mathcal{Z}_S^*(\phi_\pi, f_{\tau,\sigma',s}) = \prod_{\nu\in S} \mathcal{Z}_\nu^*(\phi_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma'_\nu,s})$$

is holomorphic at $s = 1$ for any choice of the smooth sections $f_{\tau,\sigma',s}$ (by using the identity $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma',s}) = \mathcal{L}(s,\tau,\pi,\sigma;\rho)\cdot\mathcal{Z}_S^*(\phi_\pi, f_{\tau,\sigma',s})$ and Step 2 (ii)).

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

- **Step 4.** We can prove that for fixed $s = s_0 \in \mathbb{C}$, if the local Bessel functional is non-zero at places $\nu \in S$, then there is a choice of data so that the finite product of local zeta integrals over $S$, $\mathcal{Z}_S(\phi_\pi, f_{\tau,\sigma',s})$, is non-zero at $s = s_0$. This step is very technical.

  The finite product $\mathcal{Z}_S(\phi_\pi, f_{\tau,\sigma',s})$ is defined as $\mathcal{Z}_S(\phi_\pi, f_{\tau,\sigma',s}) = \prod_{\nu \in S} \mathcal{Z}(v_{\pi_\nu}, f_{\mathcal{W}(\tau_\nu),\sigma'_\nu,s})$.

  The main technical challenge is to construct a section $f_{\mathcal{W}(\tau_\nu),\sigma'_\nu,s}$ satisfying the property.

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

- **Step 5.** Recall that we assumed the global Bessel period $\mathcal{B}^{\psi_{\ell,a}}$ is non-zero. We can prove that there exists a choice of data so that both
  - $\mathcal{Z}_S^*(\phi_\pi, f_{\tau,\sigma',s})$ at $s = 1$
  - the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma)$ for $(\phi_\pi, \phi_\sigma)$

  are simultaneously non-zero.

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

- **Step 6.** With the above choice of data, we can prove that the global integral $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma',s})$ has a pole at $s = 1$ of order $r$. Deduce that $E(\cdot, f_{\tau,\sigma',s})$ has a pole at $s = 1$ of order $r$.

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

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  This follows from
  - $\mathcal{Z}(\phi_\pi, f_{\tau,\sigma',s}) = \mathcal{L}(s, \tau, \pi, \sigma; \rho) \cdot \mathcal{Z}_S^*(\phi_\pi, f_{\tau,\sigma',s})$
  - $\mathcal{L}(s, \tau, \pi, \sigma; \rho)$ has a pole at $s = 1$ of order $r$ (Step 3)
  - $\mathcal{Z}_S^*(\phi_\pi, f_{\tau,\sigma',s})$ at $s = 1$ is non-zero (Step 5)

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

- **Step 7.** Combining Step 2 and Step 6 we conclude that

$$L(s, \sigma \times \pi) = L(s, \sigma' \times \pi) = L(s, \sigma' \times \Pi) = L(s, \sigma' \times \tau \otimes \omega_\sigma^{-1})$$

is holomorphic and non-zero at $s = \frac{1}{2}$.

# Sketch of proof of one direction of GGP for generic repns

Assume $\pi$, $\sigma$ are generic and the Bessel period $\mathcal{B}^{\psi_{\ell,a}}(\phi_\pi, \phi_\sigma) \neq 0$ for some $\phi_\pi \in V_\pi$, $\phi_\sigma \in V_\sigma$. We want to show $L(\frac{1}{2}, \sigma \times \pi) \neq 0$.

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is holomorphic and non-zero at $s = \frac{1}{2}$.

This completes the proof of one direction of global GGP for generic repns.

# Application to reciprocal non-vanishing of Bessel periods

As another application of the Rankin-Selberg integral, we obtain a reciprocal non-vanishing result of Bessel periods, relating the non-vanishing of Bessel periods for two pairs of $\mathrm{GSpin}$ groups.

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- Recall

$$\mathrm{GSpin}_n \subset \mathrm{GSpin}_{n+2\ell+1} \subset \mathrm{GSpin}_{n+2k}$$

$$\text{(small)} \xrightarrow{2\ell+1} \text{(middle)} \xrightarrow{2\ell'+1} \text{(large)}$$

  Here we have two pairs of groups:

$$(\mathrm{GSpin}_{n+2k}, \mathrm{GSpin}_{n+2\ell+1}) \quad \text{and} \quad (\mathrm{GSpin}_{n+2\ell+1}, \mathrm{GSpin}_n).$$

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- When the Eisenstein series $E(\cdot, f_{\tau,\sigma',s})$ has a pole at $s = 1$ of order $r$, we denote by $\mathcal{E}_{\tau\otimes\sigma'}$ the $r$-th iterated residue at $s = 1$ of $E(\cdot, f_{\tau,\sigma',s})$.

# Application to reciprocial non-vanishing of Bessel periods

## Theorem (Y, 2025+)

*Let $\pi$ and $\sigma$ be irreducible generic cuspidal automorphic representations of $\mathrm{GSpin}_{n+2\ell+1}(\mathbb{A})$ and $\mathrm{GSpin}_n(\mathbb{A})$ respectively, such that $\omega_\pi\omega_\sigma = 1$. Let $\Pi$ be the functorial transfer of $\pi$ and let $\tau = \Pi \otimes \omega_\pi^{-1}$. Assume that the residue $\mathcal{E}_{\tau\otimes\sigma'}$ is non-zero. The following are equivalent:*

(1) *The Bessel period $\mathcal{B}^{\psi_{\ell'},-a}$ for $(\mathcal{E}_{\tau\otimes\sigma'}, \pi)$ is non-zero for some choice of data.*

(2) *The Bessel period $\mathcal{B}^{\psi_{\ell},a}$ for $(\pi, \sigma)$ is non-zero for some choice of data.*

# Application to reciprocial non-vanishing of Bessel periods

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1. *The Bessel period $\mathcal{B}^{\psi_{\ell'},-a}$ for $(\mathcal{E}_{\tau\otimes\sigma'}, \pi)$ is non-zero for some choice of data.*
2. *The Bessel period $\mathcal{B}^{\psi_{\ell},a}$ for $(\pi, \sigma)$ is non-zero for some choice of data.*

(1) $\Rightarrow$ (2): is an immediate consequence of the Rankin-Selberg integral.

(2) $\Rightarrow$ (1): is a byproduct of the above proof of one direction of global GGP (using Step 1 - Step 6).

Thank you!