

The algebraic geometry of Fourier coefficients of Poincaré series

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Summary :

- Relate Fourier coefficients of Poincaré series (of positive and negative index) with periods of algebraic varieties.
 - Algebraicity / transcendence questions.
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Basic notation :

→ upper half-plane

- $\mathbb{H} = \{ \tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0 \}$
- $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$
↳ Hecke's congruence subgroup
- acting on \mathbb{H} via $\gamma z = \frac{az+b}{cz+d}$.

Definition A weakly holomorphic modular form of weight k and level N is an holomorphic fct. $f : \mathbb{H} \rightarrow \mathbb{C}$ s.t.

$$1) \quad \forall \gamma \in \Gamma_0(N), \quad (cz+d)^{-k} f(\gamma z) = f(z)$$

$$2) \quad \forall \gamma \in SL_2(\mathbb{Z}), \quad \exists \rho \in \mathbb{R} \text{ s.t.}$$

$$(cz+d)^{-k} f(\gamma z) = O(e^{\rho \operatorname{Im} z}) \text{ as } \operatorname{Im} z \rightarrow +\infty$$

meromorphic at the cusps

\mathbb{C} -vector space: $M_k^!(\Gamma_0(N))$

Example (Poincaré series)

$$k \geq 4, \quad m \in \mathbb{Z}, \quad N \geq 1$$

$$P_{m,b,N}(z) = \sum_{\gamma \in \frac{\Gamma_0(N)}{\Gamma_\infty}} \frac{e^{2\pi i m \gamma z}}{(cz+d)^k} \quad \in M_k^!(\Gamma_0(N))$$

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

- $k = 2$ ("Hecke's trick")
- $m = 0 \rightsquigarrow$ Eisenstein series

Fourier coefficients of $f \in M_2^+(\Gamma_0(N))$:

$a_n(f) \in \mathbb{C}$ s.t.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i z}$$

Example

$m > 0, k \geq 2, N \geq 1$

$$P_{m,k,N}(z) = q^m + \sum_{n \geq 1} \left(2\pi (-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c \geq 1 \\ N \mid c}} \frac{K(m, n; c)}{c} J_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right) \right) q^n$$

$$P_{-m,k,N}(z) = q^{-m} + \sum_{n \geq 1} \left(2\pi (-1)^{\frac{k}{2}} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c \geq 1 \\ N \mid c}} \frac{K(-m, n; c)}{c} I_{k-1}\left(\frac{4\pi \sqrt{mn}}{c}\right) \right) q^n$$

where

↳ Kloosterman sum

- $K(a, b; c) = \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} e^{\frac{2\pi i}{c} (ax + bx^{-1})} \in \mathbb{R} \cap \bar{\mathbb{Q}}$

- J_{k-1}, I_{k-1} Bessel functions



What can we say about $a_n(P_{m,\xi,n})$?

- Growth in n : well understood
- Arithmetic nature: ???

Some examples:

→ doesn't look algebraic!

$$1) P_{1,12,1} = 2,84028\dots \Delta$$

$$\text{where } \Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \underbrace{c(n)}_{\in \mathbb{Z}} q^n$$

Lehmer's conjecture: $c(n) \neq 0 \quad \forall n \geq 1$

$$2) P_{-1,2,1} = -q \frac{1+j}{2q} \in \mathbb{Z}[q]$$

$$\text{where } j(z) = \frac{1}{q} + 744 + 196884q + \dots$$

Corollary (Pettersson, Rademacher): $a_n(j) \sim \frac{e^{2\pi\sqrt{n}}}{\sqrt{2} n^{3/4}}$

$$3) P_{m,4,8} \equiv 0, \quad m = 2, 4, 6, 8, \dots$$

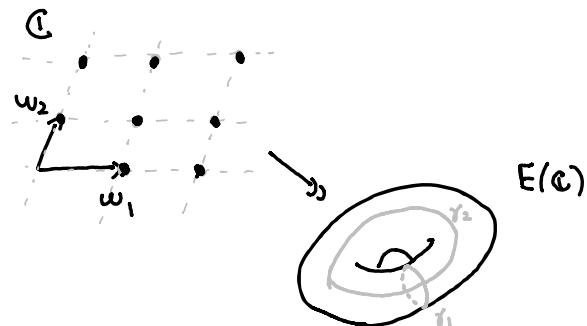
$$4) P_{-1,4,9} = \frac{1}{q} + 2q^2 - 49q^5 + 48q^8 + \dots \in \mathbb{Z}[q]$$

(Bruinier - Ono - Rhodes '08, Candelori '14)

Slogon Fourier coefficients of Poincaré series
are given by single-valued periods of
modular motives!

Consider the elliptic curve

$$E : y^2 + y = x^3 - x^2 - 10x - 20$$



Periods

$$\omega_1 = \int_{\delta_1} \frac{dx}{2y+1}$$

$$= 1.269\dots$$

Quasi-periods

$$\eta_1 = \int_{\delta_1} x \frac{dx}{2y+1}$$

$$= -2.214\dots$$

$$\omega_2 = \int_{\delta_2} \frac{dx}{2y+1}$$

$$= 0.634\dots + i 1.458\dots$$

$$\eta_2 = \int_{\delta_2} x \frac{dx}{2y+1}$$

$$= -1.107\dots + i 2.405\dots$$

One can numerically verify that:

$$\alpha_1(P_{1,2,11}) = 1.696\dots = -\frac{2\pi i}{w_1 \bar{w}_2 - \bar{w}_1 w_2}$$

$$\alpha_1(P_{-1,2,11}) = -0.952\dots = \frac{\bar{w}_1 \eta_2 - \bar{w}_2 \eta_1}{w_1 \bar{w}_2 - \bar{w}_1 w_2} - 1$$

Reason: $E = X_0(11)$ (over \mathbb{C} , compact. of $\Gamma_0(11) \backslash \mathbb{H}$)

$$\text{and } S = \begin{pmatrix} w_1 & \eta_1 \\ w_2 & \eta_2 \end{pmatrix}^{-1} \begin{pmatrix} \bar{w}_1 & \bar{\eta}_1 \\ \bar{w}_2 & \bar{\eta}_2 \end{pmatrix} = \frac{1}{2\pi i} \begin{pmatrix} \bar{w}_1 \eta_2 - \bar{w}_2 \eta_1 & \bar{\eta}_1 \eta_2 - \eta_1 \bar{\eta}_2 \\ w_1 \bar{w}_2 - \bar{w}_1 w_2 & w_1 \bar{\eta}_2 - w_2 \bar{\eta}_1 \end{pmatrix}$$

is the single-valued period matrix of $H^1(X_0(11))$.

Periods (Kontsevich-Zagier)

X/\mathbb{Q} smooth affine \rightsquigarrow motives $H^n(X)$

$$\text{comp: } H_{\text{de}}^n(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_B^n(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

$$[\omega] \mapsto ([\tau] \mapsto \int_{\sigma} \omega)$$

Period matrix:

$$P = \begin{pmatrix} \int_{\sigma_1} w_1 & \dots & \int_{\sigma_r} w_1 \\ \vdots & & \vdots \\ \int_{\sigma_r} w_1 & \dots & \int_{\sigma_r} w_r \end{pmatrix} \in GL_r(\mathbb{C})$$

Single-valued periods (Brown-Dupont '20)

(combinations of integrals of the form $\int_{\sigma} w_1 \bar{y}_1$,
archimedean analogs of p -adic periods)

Complex conj. $X(\mathbb{C}) \rightarrow X(\mathbb{C}) \rightsquigarrow F_\infty : H_B''(X) \rightarrow H_B''(X)$

induces via comp

$$s : H_{\mathbb{R}}''(X) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H_{\mathbb{R}}''(X) \otimes_{\mathbb{Q}} \mathbb{R}$$

Single-valued period matrix:

$$S = P^{-1} \bar{P} = P^{-1} F_\infty P \in GL_r(\mathbb{R})$$

Modular motives: $k \geq 2$ min, $N \geq 1$

weakly holomorphic
wpforms ($a_0=0$)
with an $\in \mathbb{Q}$ &
with $\in \mathbb{Z}$

$$\frac{S_k^!(\Gamma_0(N))_{\mathbb{Q}}}{D^{k-1} M_k^!(\Gamma_0(N))_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow[\sim]{\text{comp}} H_{\text{par}}^!(\Gamma_0(N), V_{k-2}) \otimes_{\mathbb{Q}} \mathbb{C}$$

"Eichler-Shimura"

subquotient of $H^{k-1} \left(\underbrace{E \times_{Y_0(N)} \cdots \times_{Y_0(N)} E}_{k-2} \right)$

- $M(2, N) = H^1(X_0(N))$
 - Periods are periods and "quasi-periods" of modular forms (cf. Brown-Hain '18)
 - Single-valued periods ?
-

Theorem (F. '20) Let $S = (s_{ij})_{1 \leq i, j \leq r} \in GL_r(\mathbb{R})$ be a single-valued period matrix of $M(k, N)$. Then:

$$\mathbb{Q}(s_{ij} : 1 \leq i, j \leq r) = \mathbb{Q}(\alpha_n(p_m, q, N) : m, n \in \mathbb{Z})$$

Easy corollaries:

- $\mathbb{Q}(\alpha_n(p_m, q, N) : m, n \in \mathbb{Z})$ is finitely generated.
- If $M(k, N) = 0$, then $\alpha_n(p_m, q, N) \in \mathbb{Q} \ \forall m, n$

e.g. $M(2, 1) = H^1(X_0(1)) = H^1(\mathbb{P}^1) = 0$

$$p_{-1, 2, 1} = \frac{1}{q} - 196884q + 42987520q^2 + \dots$$

$\Leftrightarrow S_2(\Gamma_0(N)) = 0$

Assume $M(k, N)$ has rank 2 ($\Leftrightarrow \dim S_k(\Gamma_0(N)) = 1$)

Let $f \in S_k(\Gamma_0(N))_{\mathbb{Q}}$ and $g \in S_k^!(\Gamma_0(N))_{\mathbb{Q}}$ induce a basis of $M(k, N)_{\mathbb{Q}} = S_k^!(\Gamma_0(N))_{\mathbb{Q}} / D^{k-1} M_{k-k}^!(\Gamma_0(N))_{\mathbb{Q}}$ and $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ be the corresponding single-valued period matrix.

Theorem (F. '20) For every $m \geq 1$, there is $h_m \in M_{2-k}^!(\Gamma_0(N))_{\mathbb{Q}}$ s.t., for every $n \geq 1$,

$$a_n(P_{m,k,n}) = -\frac{(k-2)!}{m^{k-1}} \alpha_m(f) \alpha_n(f) \frac{1}{s_{21}}$$

$$a_n(P_{-m,k,n}) = \frac{(k-2)!}{m^{k-1}} \alpha_m(f) \alpha_n(f) \frac{s_{11}}{s_{21}} + r_{m,n}$$

$$\text{where } r_{m,n} = \frac{(k-2)!}{m^{k-1}} \alpha_m(f) \alpha_n(g) + n^{k-1} a_n(h_m).$$

$$\text{Note: } s_{21} \doteq (f, f)_{P_{0,k}}, \quad s_{11}/s_{21} \doteq C_p^+(1)$$

Corollary If f has CM, then $a_n(P_{-m,k,n}) \in \mathbb{Q}$.

$$\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} \bar{2} & \bar{5} \\ 0 & \bar{1} \end{pmatrix} \Rightarrow \frac{s_{11}}{s_{21}} = \frac{5}{2-q}$$

$$\text{e.g. } (k, N) = (4, 9) \text{ CM by } \mathbb{Q}(\sqrt{-3}) \rightsquigarrow a_n(P_{-m,4,9}) \in \mathbb{Q}.$$

How to prove?

- Explicit description of

$$s : M(k, N)_{\mathbb{R}} \otimes \mathbb{R} \longrightarrow M(k, N)_{\mathbb{R}} \otimes \mathbb{R}$$

via harmonic Meass forms

$$\begin{array}{ccc}
 \xi_{z-k}(F) = \frac{1}{(Im z)^{k-1}} \frac{\partial F}{\partial z} & & D = g \frac{d}{dx} \\
 \curvearrowright \frac{1}{(4\pi)^{k-1}} \xi_{z-k} & \curvearrowright H_{2-k}^!(\Gamma_0(N)) & \curvearrowright \frac{1}{(k-2)!} D^{k-1} \\
 M_k^!(\Gamma_0(N)) & \curvearrowright & M_k^!(\Gamma_0(N)) \\
 \downarrow & & \downarrow \\
 \frac{M_k^!(\Gamma_0(N))}{D^{k-1} M_{k-1}^!(\Gamma_0(N))} & \xrightarrow{s \otimes c_{tr}} & \frac{M_k^!(\Gamma_0(N))}{D^{k-1} M_{k-1}^!(\Gamma_0(N))} \\
 \curvearrowleft (\mathbb{Q} \oplus M(k, N)_{\mathbb{R}}) \otimes \mathbb{C} & \curvearrowright "flipping operator"
 \end{array}$$

- Bringmann - Ono '07

$$\Rightarrow s([P_{m,k,N}]) = -[P_{-m,k,N}]$$

- "Linear algebra" ...