Two analogues of the Rademacher symbol

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Rademacher symbol

Dedekind (1892)

$$\log \Delta(\gamma z) - \log \Delta(z) = 12\mathrm{sgn}(c)^2 \log\left(\frac{cz+d}{i\mathrm{sgn}(c)}\right) + 2\pi i \Phi(\gamma)$$

[Note] $\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24}, \ \ \gamma \in \mathrm{SL}_2(\mathbb{Z}), \ \ \gamma z := \dfrac{az+b}{cz+d}$

Rademacher (1956) defined the function $\Psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}$ by

$$\Psi(\gamma) := \Phi(\gamma) - 3\mathrm{sgn}(c(a+d))$$

For any $g \in \mathrm{SL}_2(\mathbb{Z}), \ \ \Psi(\gamma) = \Psi(-\gamma) = -\Psi(\gamma^{-1}) = \Psi(g^{-1}\gamma g)$

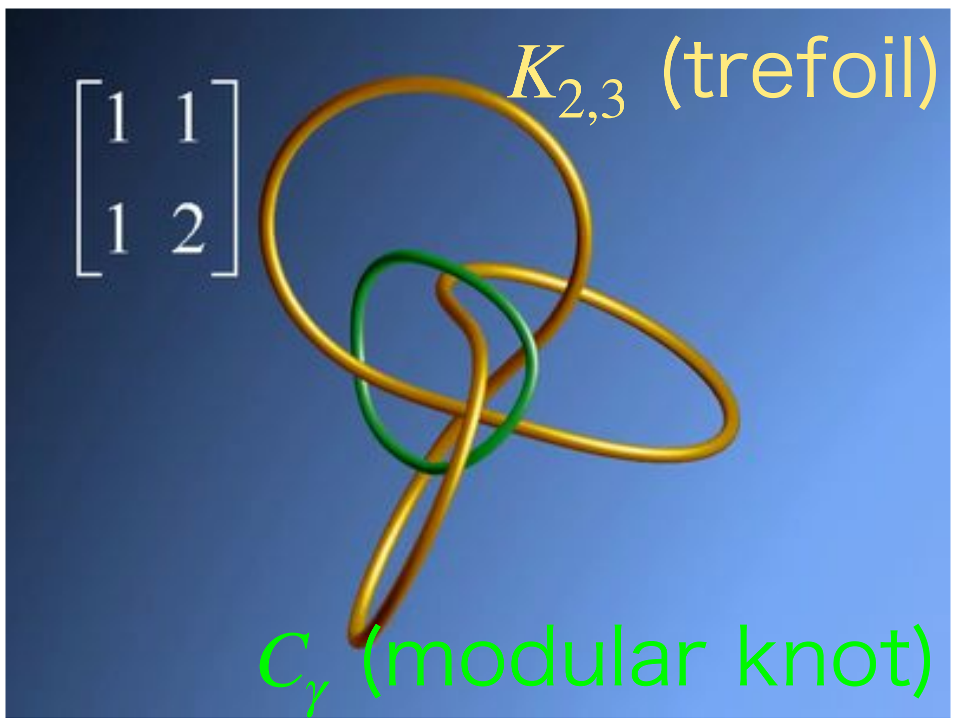
Atiyah's "omnibus theorem" : Seven definitions of $\Psi(\gamma)$!

<u>É. Ghys</u> (2007) For $\gamma \in \mathrm{SL}_2(\mathbb{Z})$: hyperbolic i.e. $|\mathrm{tr}(\gamma)| > 2$

$$\Psi(\gamma) = \mathrm{Lk}(C_\gamma, K_{2,3})$$

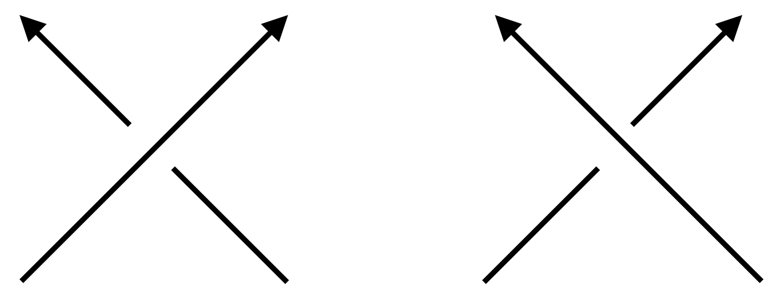
$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}) \cong S^3 - K_{2,3}$$

$\gamma =$



$$\Psi(\gamma) = \mathrm{Lk}(C_\gamma, K_{2,3}) = 0$$

Linking # $= \dfrac{\text{intersecion \#}}{2}$



$+1$ $\qquad$ -1

<u>Today</u>

Replace $\mathrm{Lk}(C_\gamma, K_{2,3})$ with $\mathrm{Lk}(\ast, \ast)$ ⇢ analogue of $\Psi(\gamma)$?

Plan

1. $\mathrm{Lk}(C_\gamma, K_{2,3}) \dashrightarrow \mathrm{Lk}(C'_\gamma, C'_\sigma)$ ($\gamma, \sigma \in \mathrm{SL}_2(\mathbb{Z})$: hyperbolic)

Duke-Imamoglu-Tóth (2017) introduced $\Psi_\gamma(\sigma)$ and showed

$$\Psi_\gamma(\sigma) = \mathrm{Lk}(C'_\gamma, C'_\sigma)$$

In this talk, we give explicit formulas for $\Psi_\gamma(\sigma)$

(T. Matsusaka, A hyperbolic analogue of the Rademacher symbol, arXiv:2003.12354)

2. $\mathrm{Lk}(C_\gamma, K_{2,3}) \dashrightarrow \mathrm{Lk}(C''_\gamma, K_{p,q})$ ($\gamma \in \Gamma_{p,q}$: hyp, $K_{p,q}$: (p,q)-torus knot)

We introduce $\Psi_{p,q} : \Gamma_{p,q} \to \mathbb{Z}$ for the triangle group $\Gamma_{p,q}$

and show some arithmetic properties

(Joint work (in progress) with Jun Ueki (Tokyo Denki University))

§1. Duke-Imamoglu-Tóth's $\Psi_\gamma(\sigma)$ (summary)

1. Recall the definition of $\Psi_\gamma(\sigma)$

2. Eisenstein series $E_\gamma(z,s)$ $(\mathrm{Re}(s)>0)$, σ : hyperbolic

- $\gamma = T \dashrightarrow \lim_{s\to 0^+} E_T(z,s) \dashrightarrow E_2(z) \dashrightarrow \Psi(\sigma) = \lim_{n\to\infty} \mathrm{Re} \int_{\sigma^n z_0}^{\sigma^{n+1} z_0} E_2(z)dz$

- γ : hyp $\dashrightarrow \lim_{s\to 0^+} E_\gamma(z,s) \dashrightarrow F_\gamma(z) \dashrightarrow \Psi_\gamma(\sigma) = 4 \lim_{n\to\infty} \mathrm{Re} \int_{\sigma^n z_0}^{\sigma^{n+1} z_0} \frac{1}{2\pi i} F_\gamma(z)dz$

3. Two explicit formulas for $\Psi_\gamma(\sigma)$

Recall: Recipe for the classical Rademacher symbol $\Psi(\sigma)$

1. $F(z) := 2\pi i E_2(z) = 2\pi i \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \right)$

2. $r(\sigma, z) := (cz+d)^{-2} F(\sigma z) - F(z) = \dfrac{12c}{cz+d}$: wt 2 rational cocycle

3. $G(z) := \log \Delta(z) \overset{\frac{d}{dz}}{\to} F(z)$: primitive function of $F(z)$

4. $R(\sigma, z) := G(\sigma z) - G(z) = 12 \mathrm{sgn}(c)^2 \log \left(\dfrac{cz+d}{i \mathrm{sgn}(c)} \right) + 2\pi i \Phi(\sigma)$

5. $\Phi(\sigma) = \dfrac{1}{2\pi} \lim_{y \to \infty} \mathrm{Im}\ R(\sigma, iy)$: Dedekind symbol

6. $\Psi(\sigma) = \lim_{n \to \infty} \dfrac{\Phi(\sigma^n)}{n}$ (if σ : hyp) : Rademacher symbol

Recipe for $\Psi_\gamma(\sigma)$ ($\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, prim. $a+d>2$, $c>0$)

2. Rational cocycle (Choie-Zagier 1993)

$$r_\gamma(\sigma, z) = \sum_{\substack{g \in \Gamma_{w_\gamma} \backslash \Gamma \\ w'_{g^{-1}\gamma g} < \sigma^{-1} i\infty < w_{g^{-1}\gamma g}}} \left(\frac{1}{z - w_{g^{-1}\gamma g}} - \frac{1}{z - w'_{g^{-1}\gamma g}} \right)$$

[Note] $w_\gamma > w'_\gamma$: fixed points of γ, $\Gamma_{w_\gamma} = \pm\langle\gamma\rangle$

1. Duke-Imamoglu-Tóth (2011) : $F_\gamma(z) = \displaystyle\sum_{n=0}^{\infty} \widetilde{\mathrm{val}}_n(\gamma) q^n$

$$\widetilde{\mathrm{val}}_n(\gamma) := \int_{z_0}^{\gamma z_0} j_n(z) \frac{-\sqrt{D}dz}{Q_\gamma(z,1)} \quad \text{(Cycle integral)}$$

[Note] $Q_\gamma(X, Y) = cX^2 + (d-a)XY - bY^2$, $D = (a+d)^2 - 4$

$$j_n(z) = q^{-n} + O(q) \in M_0^!(\mathrm{SL}_2(\mathbb{Z})), \quad r_\gamma(\sigma, z) = (cz+d)^{-2} F_\gamma(\sigma z) - F_\gamma(z)$$

3.4.5.6. $R_\gamma(\sigma, z) = G_\gamma(\sigma z) - G_\gamma(z)$, $\Phi_\gamma(\sigma) = \dfrac{2}{\pi} \lim\limits_{y\to\infty} \mathrm{Im}\, R_\gamma(\sigma, iy)$, $\Psi_\gamma(\sigma) = \lim\limits_{n\to\infty} \dfrac{\Phi_\gamma(\sigma^n)}{n}$

Question 1: Relation $E_2(z) \leftrightarrow F_\gamma(z)$?

Def (Eisenstein series)

For $\pm I \neq \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $Q_\gamma(X,Y) = cX^2 + (d-a)XY - bY^2$

$$E_\gamma(z,s) := \sum_{Q \sim Q_\gamma} \frac{\mathrm{sgn}(Q) y^s}{Q(z,1)\,|\,Q(z,1)\,|^s} \quad (\mathrm{Re}(s) > 0)$$

[Note] $Q \sim Q_\gamma \iff \exists g \in \mathrm{SL}_2(\mathbb{Z})$ s.t. $Q = Q_\gamma \circ g$

$$\mathrm{sgn}([a,b,c]) = \mathrm{sgn}(a) \text{ if } a \neq 0, \quad = \mathrm{sgn}(c) \text{ if } a = 0$$

Known: For $\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$E_T(z,s) = \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{(cz+d)^2\,|\,cz+d\,|^{2s}} \xrightarrow{s \to 0} E_2^*(z) := E_2(z) - \frac{3}{\pi y}$$

For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ (Essentially by Bringmann-Kane 2016)

$$\lim_{s \to 0^+} E_S(z,s) = \frac{\pi}{2}\left(\frac{j'(z)}{j(z)-1728} + E_2^*(z)\right), \quad \lim_{s \to 0^+} E_U(z,s) = \frac{2\pi}{3\sqrt{3}}\left(\frac{j'(z)}{j(z)} + E_2^*(z)\right)$$

Thm (M.)

For a primitive hyperbolic $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a+d > 2,\ c > 0$,

$$\lim_{s \to 0^+} E_\gamma(z,s) = -\frac{2}{\sqrt{D}}\left(F_\gamma(z) - \widetilde{\mathrm{val}}_0(\gamma) E_2^*(z)\right)$$

(idea) Fourier expansion of $E_\gamma(z,s)$ $\square$

By the definition of $E_\gamma(z,s)$ and theorem, it follows that

$$(cz+d)^{-2} F_\gamma(\sigma z) - F_\gamma(z) = r_\gamma(\sigma, z) = \sum_{\substack{g \in \Gamma_{w_\gamma} \backslash \Gamma \\ w'_{g^{-1}\gamma g} < \sigma^{-1} i\infty < w_{g^{-1}\gamma g}}} \left(\frac{1}{z - w_{g^{-1}\gamma g}} - \frac{1}{z - w'_{g^{-1}\gamma g}}\right)$$

Question 2: $\Psi_\gamma(\sigma) - \Phi_\gamma(\sigma)$?

Recall: $R_\gamma(\sigma, z) = G_\gamma(\sigma z) - G_\gamma(z)$, $\Phi_\gamma(\sigma) = \dfrac{2}{\pi} \lim_{y \to \infty} \operatorname{Im} R_\gamma(\sigma, iy)$, $\Psi_\gamma(\sigma) = \lim_{n \to \infty} \dfrac{\Phi_\gamma(\sigma^n)}{n}$

Prop: For a hyperbolic $\sigma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\Psi_\gamma(\sigma) = \frac{2}{\pi} \lim_{n \to \infty} \operatorname{Im} R_\gamma(\sigma, \sigma^n z) \quad (z \in \mathbb{H})$$

(idea) Explicit formula for $R_\gamma(\sigma, z)$ given by DIT (2017) $\square$

In other words, $$\Psi_\gamma(\sigma) = 4 \lim_{n \to \infty} \operatorname{Re} \int_{\sigma^n z_0}^{\sigma^{n+1} z_0} \frac{1}{2\pi i} F_\gamma(z) dz$$

[Note] $$\Psi(\sigma) = \lim_{n \to \infty} \operatorname{Re} \int_{\sigma^n z_0}^{\sigma^{n+1} z_0} E_2(z) dz$$

<u>Thm (M.)</u> Let σ : hyp, $w_\sigma^\infty := \lim_{n\to\infty} \sigma^n z$

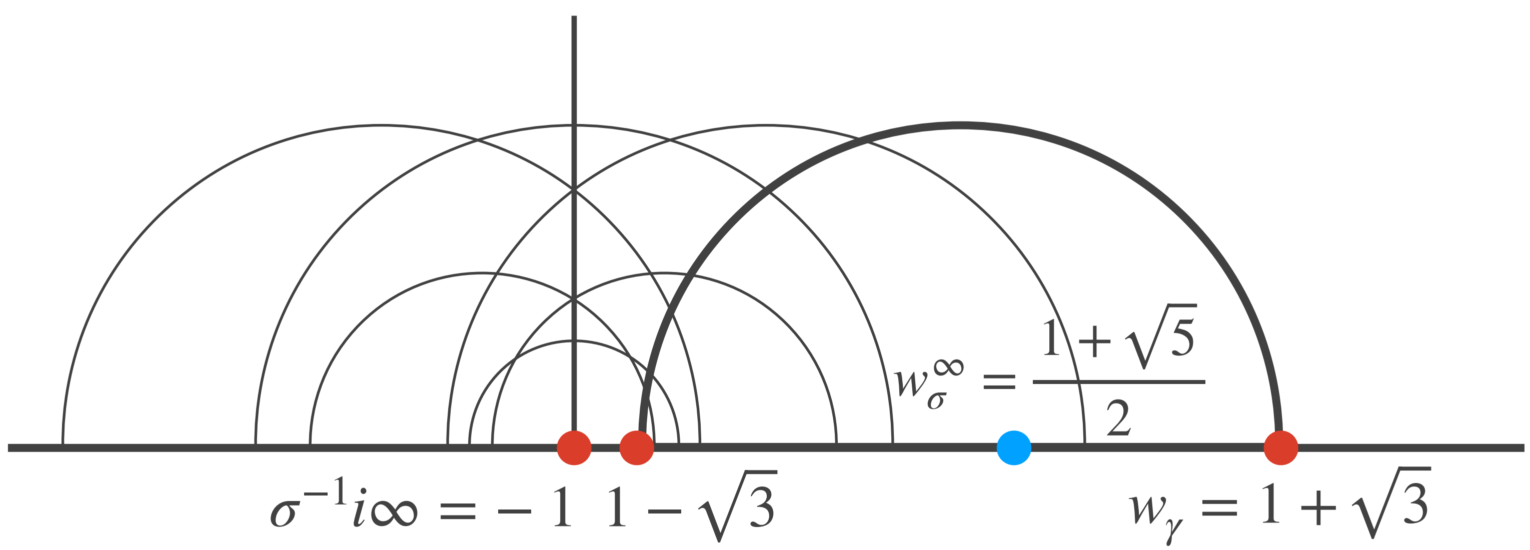
$$\Psi_\gamma(\sigma) = \Phi_\gamma(\sigma) + 2 \sum_{\substack{g\in\Gamma_{w_\gamma}\backslash\Gamma \\ w'_{g^{-1}\gamma g} < \sigma^{-1}i\infty,\, w_\sigma^\infty < w_{g^{-1}\gamma g}}} 1$$

[Note] $\Psi(\sigma) = \Phi(\sigma) - 3\mathrm{sgn}(c(a+d))$

[Note] Duke-Imamoglu-Tóth (2017)

$$\Phi_\gamma(\sigma) = -\sum_{\substack{g\in\Gamma_{w_\gamma}\backslash\Gamma \\ w'_{g^{-1}\gamma g} < \sigma^{-1}i\infty < w_{g^{-1}\gamma g}}} 1$$

Example: $\gamma = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\Phi_\gamma(\sigma) = -6$, $\Psi_\gamma(\sigma) = -4$



Question 3: Another explicit formula for $\Psi_\gamma(\sigma)$?

Prop: $$\Phi_\gamma(\sigma_1\sigma_2) = \Phi_\gamma(\sigma_1) + \Phi_\gamma(\sigma_2) + 2 \sum_{\substack{g \in \Gamma_{w_\gamma} \backslash \Gamma \\ w'_{g^{-1}\gamma g} < \sigma_1^{-1} i\infty,\, \sigma_2 i\infty < w_{g^{-1}\gamma g}}} 1$$

[Note] $\Phi(\sigma_1\sigma_2) = \Phi(\sigma_1) + \Phi(\sigma_2) - 3\mathrm{sgn}(c_1c_2c_{12})$ $\quad \sigma_i = \begin{pmatrix} * & * \\ c_i & * \end{pmatrix}, \ \sigma_1\sigma_2 = \begin{pmatrix} * & * \\ c_{12} & * \end{pmatrix}$

Let $\gamma = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n-1} & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m-1} & 1 \\ 1 & 0 \end{pmatrix} \quad (a_i, b_j \in \mathbb{Z}_{>0})$

$w_\gamma = [\overline{a_0, \ldots, a_{2n-1}}]$, $\quad \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} = T^a S T^{-b} S^{-1}$

We can compute $\Phi_\gamma(\sigma)$ inductively, and $\Psi_\gamma(\sigma) = \lim_{n\to\infty} \dfrac{\Phi_\gamma(\sigma^n)}{n}$

<u>Thm (M.)</u> Let $\gamma = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2n-1} & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{2m-1} & 1 \\ 1 & 0 \end{pmatrix}$,

$$\Psi_\gamma(\sigma) = -2\left(\sum_{\substack{0 \le i < 2n \\ 0 \le j < 2m}} \min(a_i, b_j) - \psi_\gamma(\sigma) \right) \in 2\mathbb{Z}_{<0}, \quad (0 \le \psi_\gamma(\sigma) \le 2mn)$$

$$\begin{aligned}
\psi_\gamma(\sigma) = \sum_{0 \le k < n} \sum_{0 \le \ell < m} \Big(& \delta(a_{2k} \ge b_{2\ell-1})\delta([\overline{b_{2\ell}, \ldots, b_{2\ell+2m-1}}] \ge [\overline{a_{2k-1}, \ldots, a_{2k-2n}}]) \\
& + \delta(a_{2k-1} \ge b_{2\ell-1})\delta([\overline{b_{2\ell}, \ldots, b_{2\ell+2m-1}}] \ge [\overline{a_{2k}, \ldots, a_{2k+2n-1}}]) \\
& + \delta(a_{2k} \ge b_{2\ell})\delta([\overline{b_{2\ell+1}, \ldots, b_{2\ell+2m}}] > [\overline{a_{2k+1}, \ldots, a_{2k+2n}}]) \\
& + \delta(a_{2k-1} \ge b_{2\ell})\delta([\overline{b_{2\ell+1}, \ldots, b_{2\ell+2m}}] > [\overline{a_{2k-2}, \ldots, a_{2k-2n-1}}]) \Big).
\end{aligned}$$

<u>Example</u>: $\gamma = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, $\sigma = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\Psi_\gamma(\sigma) = -2(4 - \psi_\gamma(\sigma)) = -4$

$$\begin{aligned}
\psi_\gamma(\sigma) &= \delta(2 \ge 1)\delta([\overline{1,1}] \ge [\overline{1,2}]) + \delta(1 \ge 1)\delta([\overline{1,1}] \ge [\overline{2,1}]) \\
&\quad + \delta(2 \ge 1)\delta([\overline{1,1}] > [\overline{1,2}]) + \delta(1 \ge 1)\delta([\overline{1,1}] > [\overline{2,1}]) \\
&= 2.
\end{aligned}$$

Plan

1. $\mathrm{Lk}(C_\gamma, K_{2,3}) \dashrightarrow \mathrm{Lk}(C'_\gamma, C'_\sigma)$ $(\gamma, \sigma \in \mathrm{SL}_2(\mathbb{Z})$: hyperbolic)

Duke-Imamoglu-Tóth (2017) introduced $\Psi_\gamma(\sigma)$ and showed

$$\Psi_\gamma(\sigma) = \mathrm{Lk}(C'_\gamma, C'_\sigma)$$

In this talk, we give explicit formulas for $\Psi_\gamma(\sigma)$

(T. Matsusaka, A hyperbolic analogue of the Rademacher symbol, arXiv:2003.12354)

2. $\mathrm{Lk}(C_\gamma, K_{2,3}) \dashrightarrow \mathrm{Lk}(C''_\gamma, K_{p,q})$ $(\gamma \in \Gamma_{p,q}$: hyp, $K_{p,q}$: (p,q)-torus knot)

We introduce $\Psi_{p,q} : \Gamma_{p,q} \to \mathbb{Z}$ for the triangle group $\Gamma_{p,q}$

and show some arithmetic properties

(Joint work (in progress) with Jun Ueki (Tokyo Denki University))

§2. $\Psi_{p,q}(\gamma)$ for triangle group $\Gamma_{p,q}$ (summary)

1. Eisenstein series $E_2^{(p,q)}(z)$ on $\Gamma_{p,q}$

2. $\log \Delta_{p,q}(\gamma z) - \log \Delta_{p,q}(z) = 2pq \log(cz + d) + 2\pi i \psi_{p,q}(\gamma)$

3. $\widetilde{\Gamma}_{p,q} \subset \widetilde{\mathrm{SL}}_2(\mathbb{R})$: universal covering group

   $\chi_{p,q} : \widetilde{\Gamma}_{p,q} \to \mathbb{Z}$ (character), $\quad \psi_{p,q}(\gamma) = \chi_{p,q}(\widetilde{\gamma})$

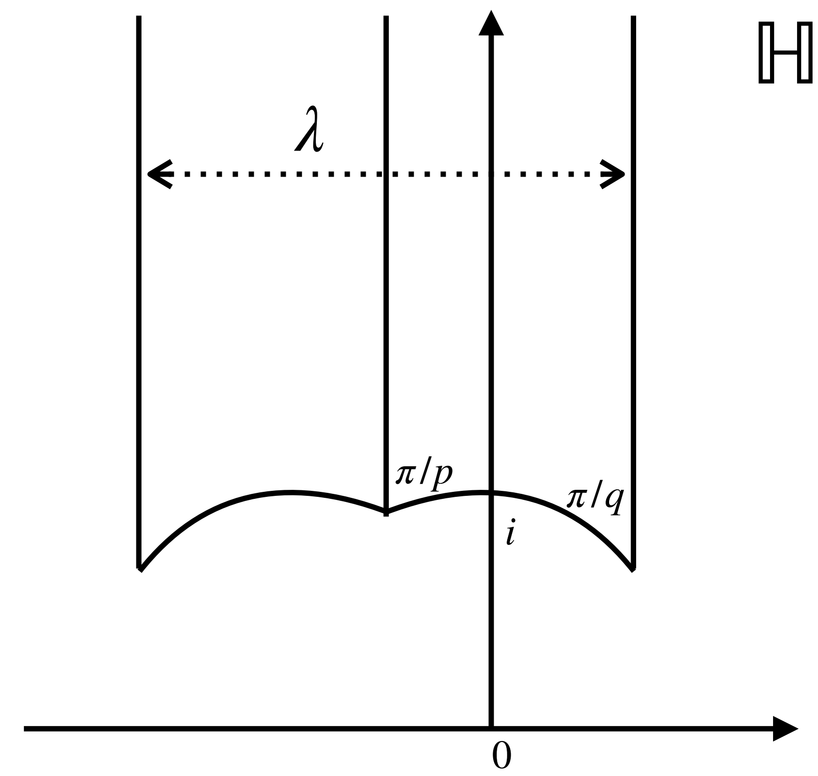
4. $\Psi_{p,q}(\gamma) = \psi_{p,q}(\gamma) + \dfrac{pq}{2}\mathrm{sgn}(\gamma)\left(1 - \mathrm{sgn}(\mathrm{tr}(\gamma))\right)$

Triangle group

(p, q) : coprime pair, $2 \leq p < q$

$$\Gamma_{p,q} = \left\langle T_{p,q} = \begin{pmatrix} 1 & 2\left(\cos\frac{\pi}{p} + \cos\frac{\pi}{q}\right) \\ 0 & 1 \end{pmatrix}, S_p = \begin{pmatrix} 0 & -1 \\ 1 & 2\cos\frac{\pi}{p} \end{pmatrix} \right\rangle$$

$(\Gamma_{2,3} = \mathrm{SL}_2(\mathbb{Z}))$



$$\lambda := 2\left(\cos\frac{\pi}{p} + \cos\frac{\pi}{q}\right), \quad U_q = \begin{pmatrix} 2\cos\frac{\pi}{q} & -1 \\ 1 & 0 \end{pmatrix} = -T_{p,q}S_p^{-1}$$

Def: $E_{2k}^{(p,q)}(z,s) = \displaystyle\sum_{\gamma \in (\Gamma_{p,q})_\infty \backslash \Gamma_{p,q}} \mathrm{Im}(z)^{s-k}\Big|_{2k} (\sigma^{-1}\gamma) \quad (\mathrm{Re}(s) > 1)$

[Note] $\sigma = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$, $\mathrm{Res}_{s=1} E_0^{(p,q)}(z,s) = \dfrac{1}{\mathrm{vol}(\Gamma_{p,q}\backslash\mathbb{H})}$

Eisenstein series of weight 2

By $\xi_{2k} E_{2k}^{(p,q)}(z,s) = (\overline{s} - k) E_{2-2k}(z,\overline{s}), \quad \xi_k := 2iy^k \overline{\frac{d}{d\overline{z}}}$

- $E_2^{(p,q),*}(z) = \lim_{s \to 1+0} E_2^{(p,q)}(z,s) \in H_2(\Gamma_{p,q})$
- $E_2^{(p,q)}(z) := E_2^{(p,q),*}(z) + \frac{1}{\mathrm{vol}(\Gamma_{p,q} \backslash \mathbb{H})} \frac{1}{y}$ is holomorphic

Lemma: For any $\gamma \in \Gamma_{p,q}$

$$(cz+d)^{-2} E_2^{(p,q)}(\gamma z) - E_2^{(p,q)}(z) = \frac{pq}{pq-p-q} \frac{c}{\pi i (cz+d)}$$

Goal: Define $\Psi_{p,q} : \Gamma_{p,q} \to \mathbb{Z}$

$$G_{p,q}(z) \xrightarrow{\frac{d}{dz}} 2\pi i r E_2^{(p,q)}(z), \quad r := pq - p - q$$

For any $\gamma \in \Gamma_{p,q}$, there exists $\psi_{p,q} : \Gamma_{p,q} \to \mathbb{C}$ s.t.

$$G_{p,q}(\gamma z) - G_{p,q}(z) = 2pq \log(cz + d) + 2\pi i \psi_{p,q}(\gamma)$$

[Note] $\operatorname{Im} \log z \in [-\pi, \pi)$

Lemma: $\psi_{p,q}(S_p) = -q, \;\; \psi_{p,q}(T_{p,q}) = r$. In particular, $\psi_{p,q}(\gamma) \in \mathbb{Z}$

Let $\Delta_{p,q}(z) := \exp G_{p,q}(z)$

$\to$ ・holomorphic ・cusp form of weight $2pq$ on $\Gamma_{p,q}$

・no zero and no pole on $\mathbb{H}$ ・$\Delta_{p,q}(z) = e^{2\pi i r z/\lambda} + \cdots$

$$\log \Delta_{p,q}(\gamma z) - \log \Delta_{p,q}(z) = 2pq \log(cz + d) + 2\pi i \psi_{p,q}(\gamma)$$

Cor (Limit formula)

$$\lim_{s\to 1+0}\left(E_0^{(p,q)}(z,s)-\frac{1}{\mathrm{vol}(\Gamma_{p,q}\backslash\mathbb{H})}\frac{1}{s-1}\right)=-\frac{1}{\mathrm{vol}(\Gamma_{p,q}\backslash\mathbb{H})}\log(y\,|\,\Delta_{p,q}(z)\,|^{1/pq})+C_{p,q}$$

For $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_{p,q}$ with $a+d>2,\ c>0$

Prop $$\int_{z_0}^{\gamma z_0}E_2^{(p,q),*}(z)dz=\frac{\psi_{p,q}(\gamma)}{pq-p-q}\quad (z_0\in S_\gamma)$$

→ This relates to $\mathrm{Lk}(C_\gamma,K_{p,q})$

[Note] $G_r\backslash\widetilde{\mathrm{SL}}_2(\mathbb{R})\cong S^3-K_{p,q}$ ((p,q)-torus knot)

$\Gamma_{p,q}\backslash\mathrm{SL}_2(\mathbb{R})\cong L(pq-p-q,p-1)-\overline{K}_{p,q}$ (in Lens space)

The universal covering group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$

2-cocycle $W : \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathbb{Z}$

$$W(\gamma_1, \gamma_2) = \frac{1}{2\pi i}\left(\log j(\gamma_1, \gamma_2 z) + \log j(\gamma_2, z) - \log j(\gamma_1\gamma_2, z)\right)$$

[Note] $j(\gamma, z) = cz + d, \quad \mathrm{Im}\log z \in [-\pi, \pi)$

Def:

$$\mathrm{sgn}(\gamma) = \begin{cases} \mathrm{sgn}(c) & \text{if } c \neq 0 \\ \mathrm{sgn}(a) = \mathrm{sgn}(d) & \text{if } c = 0 \end{cases}$$

$\mathrm{sgn}(\gamma_1)$	$\mathrm{sgn}(\gamma_2)$	$\mathrm{sgn}(\gamma_1\gamma_2)$	$W(\gamma_1, \gamma_2)$
1	1	-1	1
-1	-1	1	-1
	otherwise		0

Prop: $$\psi_{p,q}(\gamma_1\gamma_2) = \psi_{p,q}(\gamma_1) + \psi_{p,q}(\gamma_2) + 2pqW(\gamma_1, \gamma_2)$$

If $f : \Gamma_{p,q} \to \mathbb{C}$ satisfies the above, then $f = \psi_{p,q}$

The universal covering group $\widetilde{\mathrm{SL}}_2(\mathbb{R}) = \{(\gamma, n) \mid \gamma \in \mathrm{SL}_2(\mathbb{R}), n \in \mathbb{Z}\}$,

[Note] $(\gamma_1, n_1) \cdot (\gamma_2, n_2) = (\gamma_1\gamma_2, n_1 + n_2 + W(\gamma_1, \gamma_2))$

Let $\widetilde{\Gamma}_{p,q} = \{(\gamma, n) \mid \gamma \in \Gamma_{p,q}, n \in \mathbb{Z}\} \subset \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and $\tilde{\gamma} = (\gamma, 0)$

Consider an additive character $\chi_{p,q} : \widetilde{\Gamma}_{p,q} \to \mathbb{Z}$

Generators satisfy $(\widetilde{S}_p)^p = (\widetilde{U}_q)^q \quad \to \quad p \cdot \chi_{p,q}(\widetilde{S}_p) = q \cdot \chi_{p,q}(\widetilde{U}_q)$

Thm Let $\chi_{p,q}(\widetilde{S}_p) = -q$ and $\chi_{p,q}(\widetilde{U}_q) = -p$. For any $\gamma \in \Gamma_{p,q}$

$$\psi_{p,q}(\gamma) = \chi_{p,q}(\tilde{\gamma})$$

Def $$\Psi_{p,q}(\gamma) = \psi_{p,q}(\gamma) + \frac{pq}{2}\mathrm{sgn}(\gamma)\left(1 - \mathrm{sgn}(\mathrm{tr}(\gamma))\right)$$

Thm (M.-Ueki) For any $\gamma, g \in \Gamma_{p,q}$

$$\Psi_{p,q}(\gamma) = \Psi_{p,q}(-\gamma) = -\Psi_{p,q}(\gamma^{-1}) = \Psi_{p,q}(g^{-1}\gamma g)$$

If $\gamma \in \Gamma_{p,q}$ is not elliptic, $\Psi_{p,q}(\gamma^n) = n\Psi_{p,q}(\gamma)$

Thm (M.-Ueki) For a prim. hyp. $\gamma \in \Gamma_{p,q}$ with $a + d > 2$, $c > 0$

Let $1 \leq n \leq r$ s.t. $n = (2pq)^{-1}\Psi_{p,q}(\gamma) \in \mathbb{Z}/r\mathbb{Z}$

- We define n modular knots $C_\gamma^{(j)}$ in $S^3 - K_{p,q}$ $(0 \leq j \leq n)$
- $\mathrm{Lk}(C_\gamma^{(j)}, K_{p,q}) = \dfrac{1}{\gcd(n, r)}\Psi_{p,q}(\gamma) \in \mathbb{Z}$