Extended modularity arising from the deformation of Riemann surfaces

Gabriele Bogo (TU Darmstadt)

Classical uniformization theory

$$X = \mathbb{P}^1 \setminus \{a_1, a_2, \ldots, a_n = \infty\} \quad \rightsquigarrow \quad P(t) := \prod_{i=0}^{n-1} (t - a_i)$$

Associated family of Fuchsian differential equations

$$L_X : \ \frac{d}{dt}\left(P(t)\frac{d}{dt}y(t)\right) + \left(\sum_{i=0}^{n-3} \rho_i t^i\right) y(t) = 0$$

The coefficients $\rho_0, \ldots, \rho_{n-4} \in \mathbb{C}$ are called accessory parameters

(the choice $\rho_{n-4} = (n/2 - 1)^2$ is fixed)

Let y_1, y_1 be linearly independent solutions $L_X y_j = 0 \quad j = 1,2$.

The multivalued function $\eta := \dfrac{y_2}{y_1} : X \to \mathbb{C}$

lifts to the universal covering $\tilde{\eta} : \widetilde{X} \to \mathbb{C}$

Accessory parameter problem

There exists a unique choice of the accessory parameters such that $\tilde{\eta} : \widetilde{X} \to \mathbb{H}$ is a biholomorphism.

In this case η can be used to construct a covering map

The accessory paramter problem is hard!

The uniformization theorem (1907) "proves" the existence of the Fuchsian parameter ρ_F. More direct approaches:

- Poincaré (around 1882) Statement and uniqueness
- Smirnov (1910) Case (0,4) with real punctures (proved existence)
- Keen, Rauch, Vasquez (1971) Case (1,1)
- Chudnovsky^2 (mostly 1980s) Case (1,1)
- Hoffmann, van Straten (2012) Case (1,e)
- Bogo (2019) Case (0,4)
- Anselmo et Al. (2019) Case (0,4)

Modular forms enter the picture

When $\widetilde{\eta}\colon \widetilde{X} \to \mathbb{H}$ is a biholomorphism, the monodromy group is a Fuchsian group $\Gamma \in \mathrm{SL}_2(\mathbb{R})$.

Let y be a holomorphic solution of the uniformizing differential equation. There exist $f \in M_2(\Gamma)$ and a Hauptmodul $t \in M_0^!(\Gamma)$ such that

$$y(t(\tau)) = \sqrt{f}(\tau)$$

locally for $\tau \in \mathbb{H}$.

We can construct the q-expansion of f and t from the uniformizing differential equation.

Construction of q-expansions

Using Frobenius method at the Fuchsian singularity t=0

$$y(\rho,t)=\sum_{n\geq 0} y_n(\rho)t^n,\quad \hat{y}(\rho,t)=\log(t)y(\rho,t)+\sum_{n\geq 0}\hat{y}_n(\rho)t^n \quad \rho=(\rho_0,\ldots,\rho_{n-4})$$

$$Q(\rho,t):=\exp\big(\hat{y}(\rho,t)/y(\rho,t)\big)=\sum_{n\geq 1} Q_n(\rho)t^n$$

$$T(\rho,Q):=Q(\rho,t)^{-1}=\sum_{n\geq 1} T_n(\rho)Q^n$$

$$F(\rho,Q):=y(\rho,T(\rho,Q))=\sum_{n\geq 0} F_n(\rho)Q^n$$

When $\rho=\rho_F$

$$Q(\rho_F,t)=ce^{2\pi i\tau}=cq,\tau\in\mathbb{H}.$$

$$t(\tau)=\sum_{n\geq 1} t_n q^n := \sum_{n\geq 1} T_n(\rho_F)c^n q^n$$

$$f(\tau)=\sum_{n\geq 0} f_n q^n := \sum_{n\geq 0} F_n(\rho_F)c^n q^n$$

Some examples

- Apéry's irrationality proof of $\zeta(2)$ (and $\zeta(3)$) (Beukers)
- Zagier's study of differential equations with integral solutions
- Chudnovskys/Thompson's study of the algebraicity of the Fuchsian parameters
- Bouw-Moeller/Moeller-Zagier's works on uniformization of Teichmueller curves (twisted modular forms)

The accessory parameter problem is hard (reprise)

To determine the Fuchsian parameter from the surface X $\longleftrightarrow$ To describe explicitly modular forms on Fuchsian groups

Deformation of accessory parameters

The differential equation $L_X y = 0$ leads to the power series:

$$F(\rho, Q) = \sum_{n\geq 0} F_n(\rho) Q^n$$

The object of our study is the "deformation" of modular forms around the Fuchsian value.

$$\hat{f}(\rho,\tau) := \sum_{m\geq 0} \hat{f}_m(\tau)(\rho - \rho_F)^m, \quad \hat{f}_m(\tau) := \frac{\partial^m F(\rho, Q)}{\partial \rho^m}\bigg|_{\rho=\rho_F}$$

The function $\hat{f}_0(\tau) = f(\tau)$ is a modular form. What is $\hat{f}_m(\tau)$ for $m \geq 1$?

Idea: study $\hat{f}_1(\tau)$ by introducing a differential operator on (quasi)modular forms

For f and t as before and for every $i = 0,\ldots,n-4$ define

$$\partial_i f := \frac{\partial F(\rho, Q(\rho))}{\partial \rho_i}\bigg|_{\rho=\rho_F}, \quad \partial_i t := \frac{\partial T(\rho, Q(\rho))}{\partial \rho_i}\bigg|_{\rho=\rho_F}.$$

Let $g \in M_k(\Gamma)$ and write $g = f^k R(t)$ for some rational function $R(t)$

Define the **i-th deformation operator** by

$$\partial_i g := \partial_i f^k R(t).$$

Quasimodular forms

Recall that $g_0 \in \widetilde{M}_k(\Gamma)^{(\leq p)}$ if there exist holomorphic functions $g_1, \ldots, g_p : \mathbb{H} \to \mathbb{C}$ such that

$$g_0(\tau)\,|_k\gamma = \sum_{r=0}^{p} g_r(\tau)\left(\frac{c}{c\tau+d}\right)^r \qquad \text{for every } \gamma \in \Gamma.$$

Derivations on $\widetilde{M}_*(\Gamma)$: $\quad Dg_0 := (2\pi i)^{-1}\dfrac{dg_0(\tau)}{d\tau}, \quad Wg_0 := kg_0, \quad \delta g_o := g_1 .$

$\mathfrak{sl}_2(\mathbb{C})$-module structure: $\quad [W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [D, \delta] = W .$

Structure over modular forms: $\quad \widetilde{M}_k(\Gamma)^{(\leq p)} = \bigoplus_{r=0}^{p} M_{k-2r}(\Gamma)\cdot\phi^r ,$

for some $\phi \in \widetilde{M}_2(\Gamma)$ holomorphic and non modular.

Deformation on quasimodular forms

Define $\varphi := \frac{Df}{f} \in \widetilde{M}_2(\Gamma)$. It is not modular and holomorphic, so we can extend the i-th deformation operator to $\widetilde{M}_*(\Gamma)$ by

$$\partial_i \varphi := \partial_i \frac{Df}{f}$$

Eichler integrals of cusp forms

Let $h \in S_k(\Gamma)$, $h(\tau) = \sum_{m \geq 0} h_m q^m$. By $\widetilde{h}$ we denote its Eichler integral

$$\widetilde{h}(\tau) := \sum_{m \geq 1} \frac{h_m}{m^{k-1}} q^k .$$

Theorem (B., 2020)

Let $X = \mathbb{P}^1 \simeq \mathbb{H}/\Gamma$ be a n-punctured sphere. There exist a basis $\{h_0(\tau), \ldots, h_{n-4}(\tau)\}$ of $S_4(\Gamma)$ such that for every $g_0 \in \widetilde{M}_*(\Gamma)$

$$\partial_i g_0 = 2\widetilde{h_i} D g_0 + \widetilde{h_i}' W g_0 + \widetilde{h_i}'' \delta g_0 \qquad i = 0, \ldots, n-4.$$

When g_0 is modular, the i-th deformation is given by a Rankin-Cohen bracket

$$\partial_i g = [g, \widetilde{h_i}]_1, \quad g \in M_*(\Gamma).$$

Proof

- By definition $\partial_i f = \left.\dfrac{\partial y(\rho, T(\rho, Q))}{\partial \rho_i}\right|_{\rho=\rho_F}$.

- We have $L_X\left(\dfrac{\partial y(\rho,t)}{\partial \rho_i}\right) = t^i y(\rho,t)$. It follows that $\dfrac{\partial y(\rho,t)}{\partial \rho_i}$ satisfies a Fuchsian ODE $M_i\left(\dfrac{\partial y(\rho,t)}{\partial \rho_i}\right) = 0$ of the form $M_i = L_i \circ L_X$.

- We can write $$\frac{\partial y(\rho,t)}{\partial \rho_i} = y(\rho,t)\int_0^t \frac{\int_0^{t_1} t_2^i y(\rho,t_2)\,dt_2}{y(\rho,t_1)^2 P(t_1)}\,dt_1\,,$$

where $P(t) = \prod_{j=1}^{n-1}(t-a_j)$ is determined by the punctures of X.

Proof

Using the relation $\dfrac{dQ}{Q} = \dfrac{\prod_{j=0}^{n-2}(-a_j)^n}{P(T)y^2(\rho,T)}dT$ we can write

$$\left.\frac{\partial y(\rho,t(\tau))}{\partial \rho_i}\right|_{\rho=\rho_F} = f(\tau)\int_{\tau}^{\infty}\int_{\tau_1}^{\infty} h_i(\tau)\,d\tau_2\,d\tau_1$$

where $h_i(\tau) = f^4(\tau)t^i(\tau)P(t(\tau))$.

One can prove that $h_i \in S_4(\Gamma)$ for every $i = 0,\ldots,n-4$.

This and a similar calculation for $\dfrac{\partial T(\rho,Q)}{\partial \rho_i}$ prove the statement for f and t. The generalization to modular and quasimodular forms is straigtforward. $\square$

Teichmueller theory

Teichmueller space

Let Γ be a Fuchsian group of finite type. A measurable function $\mu: \mathbb{H} \to \mathbb{C}$ is called a **Beltrami differential** if, for every $\gamma \in \Gamma$,

$$\mu(\gamma\tau)\overline{\gamma'(\tau)} = \mu(\tau)\gamma'(\tau) \qquad (\mu \in B(\Gamma)).$$

For every $\mu \in B(\Gamma)_1$ the differential equation

$$g_z = \mu(z)g_{\bar{z}}, \quad z \in \mathbb{C},$$

has a solution g^μ which restricts to a homeomorphism of $\mathbb{H}$.

Then the group $\Gamma^\mu := g^\mu\Gamma(g^\mu)^{-1}$ is Fuchsian.

The **Teichmueller space of Γ** is the space of representations

$$T(\Gamma) := \left\{p_\mu : \Gamma \to \Gamma^\mu \in \mathbb{P}\mathrm{SL}_2(\mathbb{R})\right\}/\sim \ .$$

The map $\Phi: B_1(\Gamma) \to T(\Gamma), \mu \mapsto p_\mu$, is holomorphic and defines a coordinate on $T(\Gamma)$.

The cotangent space of $T(\Gamma)$ at $\Phi(0)$ is the space $Q(\Gamma)$ of quadratic differentials on Γ (weight four cusp forms).

There exists a linear map $Q(\Gamma) \to B(\Gamma)$ given by

$$h(\tau) \mapsto \overline{h(\tau)}\Im(\tau)^2$$

Weight four cusp forms give Beltrami differentials, but in general not "small enough"

Deformations from the uniformizing ODE

Let $\{h_0, \ldots, h_{n-4}\}$ be the basis of $S_4(\Gamma) = Q(\Gamma)$ considered in Theorem 1. Define

$$v_i(\tau) \; := \; \overline{h_i(\tau)}\Im(\tau)^2 \in B(\Gamma) \qquad i = 0, \ldots, n-4.$$

Let $\epsilon > 0$ be such that $\epsilon v_i \in B_1(\Gamma)$ and let $g^{\epsilon v_i}$ be the homeomorphic solutions of the associated differential equation.

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "g^{\varepsilon v_i}"] \arrow[d, "t"'] & \mathbb{H} \arrow[d, "t^{\varepsilon v_i}"] \\
\mathbb{H}/\Gamma \arrow[r, "G^{\varepsilon v_i}"] & \mathbb{H}/\Gamma^{\varepsilon v_i}
\end{tikzcd}$$

t and $t^{\epsilon v_i}$ are Hauptmodules,
$G^{\epsilon v_i}$ is holomorphic in ϵ,
$g^{\epsilon v_i}$ and $t^{\epsilon v_i}$ are real-analytic in ϵ.

Theorem 2 (B., 2020)

Let $t \colon \mathbb{H} \to X$ and $t^{\epsilon v_i} \colon \mathbb{H} \to X^{\epsilon v_i}$ be modular functions. Then

$$\partial_i t = \left.\frac{\partial t^{\epsilon v_i}}{\partial \bar{\epsilon}}\right|_{\bar{\epsilon}=0} \qquad i = 0,\ldots,n-4.$$

Corollary (Ahlfors)

Let h_i and $g^{\epsilon v_i}$ be as above and denote $g_\tau^{\epsilon v_i} := \frac{dg^{\epsilon v_i}}{d\tau}$. Then

$$\left.\frac{\partial f_{\tau\tau\tau}^{\epsilon v_i}}{\partial \bar{\epsilon}}\right|_{\bar{\epsilon}=0} = -\frac{h_i}{2}.$$

Proof

- Let $m_0, \ldots, m_{n-4}$ and $m_0^{\epsilon v_i}, \ldots, m_{n-4}^{\epsilon v_i}$ be the accessory parameters related to t and $t^{\epsilon v_i}$ respectively. We have $m_i = m_i(\rho)$ for every $i = 0, \ldots, n-4$.
- The theorem reduces to proving $\left.\dfrac{\partial m_j(\rho)}{\partial \rho_i}\right|_{\rho=\rho_F} = \left.\dfrac{\partial m_j^{\epsilon v_i}}{\partial \bar{\epsilon}}\right|_{\bar{\epsilon}=0}$.
- There is a linear isomorphism $J : Q(\Gamma) \to D_2(X)$ between quadratic differentials and a space of rational functions with poles at the punctures of X.
- One finds $\left.\dfrac{\partial m_j^{\epsilon v_i}}{\partial \bar{\epsilon}}\right|_{\bar{\epsilon}=0} = \mathrm{Res}_{t=a_j} J(h_i) = \mathrm{Res}_{t=a_j}\left(\dfrac{t^i}{P(t)}\right) = \left.\dfrac{\partial m_j(\rho)}{\partial \rho_i}\right|_{\rho=\rho_F}$.

$\square$

Extended modular forms (work in progress)

Vector-valued modular forms

Recall that $\partial_i f = [f, \widetilde{h_i}]$ for every $f \in M_k(\Gamma)$.

Let $p_{h_i}(\gamma;\tau) = r_{i,2}(\gamma)\tau^2 + r_{i,1}(\gamma)\tau + r_{i,0}(\gamma)$ be the period polynomial of h attached to $\gamma \in \Gamma$. Then

$$\begin{pmatrix} \partial_\rho f \\ \tau^2 f' + 2\tau f \\ \tau f' + f \\ f' \end{pmatrix}(\gamma\tau) = \begin{pmatrix} 1 & r_{i,2}(\gamma) & r_{i,1}(\gamma) & r_{i,0}(\gamma) \\ 0 & a^2 & 2ab & b^2 \\ 0 & ac & ad+bc & bd \\ 0 & c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} \partial_\rho f \\ \tau^2 f' + 2\tau f \\ \tau f' + f \\ f' \end{pmatrix} (c\tau + d)^k, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This can be proved by a direct computation or (better) via monodromy considerations.

Representations and quasimodular forms

$V_s := \mathrm{Sym}^s(\mathbb{C})$, $\mathrm{sym}^s\colon \Gamma \to V_s$, the symmetric tensor representation

is the restriction of the irreducible representation $\mathrm{SL}_2(\mathbb{R}) \to V_s$.

$$\mathrm{sym}^0(\gamma) = 1 \qquad \mathrm{sym}^1(\gamma) = \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathrm{sym}^2(\gamma) = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

$M_k(\Gamma, V_s), S_k(\Gamma, V_s)$ spaces of weight k vvmf with respect to sym^s.

Theorem (Kuga-Shimura, Choie-Lee)

There is a bijection $\quad M_k(\Gamma, V_s) \xrightarrow{\sim} \widetilde{M}_{k+s}(\Gamma)^{(\leq s)}$

Theorem (B., 2020)

The following short sequence is exact

$$0 \longrightarrow M_{r+2}(\Gamma, V_s) \longrightarrow \mathrm{Ext}^1_\Gamma(V_s, V_r) \longrightarrow S_{r+2}(\Gamma, V_s) \longrightarrow 0\,.$$

By the Choie-Lee theorem, this means that equivalence classes of extensions of symmetric tensor representations are induced by quasimodular forms.

Proof (not the best one)

- Identify $\mathrm{Ext}^1_\Gamma(V_s, V_r) \simeq H^1(\Gamma, V_r \otimes V_s)$
- $H^1(\Gamma, V_r \otimes V_s) \simeq \bigoplus H^1(\Gamma, V_i)$
- Use Eichler-Shimura for $H^1(\Gamma, V_i)$ and identify the spaces with vvmf

Idea: use the previous result to construct a space of homolorphic functions on $\mathbb{H}$ associated to (the periods of) a given quasimodular form.

Let $g_0 \in \widetilde{M}_k(\Gamma)^{(\leq p)}$. Several extensions are induced by g_0, namely:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{r+2}(\Gamma, V_s) & \longrightarrow & \mathrm{Ext}^1_\Gamma(V_s, V_r) & \longrightarrow & S_{r+2}(\Gamma, V_s) & \longrightarrow & 0\,. \\ & & \wr\uparrow & & & & & & \\ & & \widetilde{M}_{r+s+2}(\Gamma)^{(\leq s)} & & & & & & \end{array}$$

The quasimodular form g_0 induces representations classes $[V_{s,r}(g_0)] \in \mathrm{Ext}^1_\Gamma(\mathrm{V}_s, \mathrm{V}_r)$ where r, s are positive integers such that

$r + s + 2 = k$ and $s \geq p$.

Extended modular forms

Let $g_0 \in \widetilde{M}_k(\Gamma)^{(\leq p)}$ and let $V_{r,s}(g_0)$ be a representation induced by g_0.

$$\begin{pmatrix} h_s \\ \vdots \\ h_0 \\ f_r \\ \vdots \\ f_0 \end{pmatrix}(\gamma(\tau)) = \left(\begin{array}{c|c} \mathrm{sym}^s(\gamma) & B(\gamma) \\ \hline 0 & \mathrm{sym}^r(\gamma) \end{array}\right) \begin{pmatrix} h_s \\ \vdots \\ h_0 \\ f_r \\ \vdots \\ f_0 \end{pmatrix}(c\tau + d)^l$$

Call h_0 extended modular form of weight $s + l$ associated to $V_{s,r}(g_0)$.

Denote by $\mathrm{Ext}_{s+1}\big(\Gamma, \mathrm{V}_{s,r}(\mathrm{g}_0)\big)$ the vector space of such functions

Theorem (B., 202?)

Let $g_0 \in \widetilde{M}_k(\Gamma)^{(\leq p)}$ and consider all the representations $V_{s,r}(g_0)$ induced by g_0. Define $\mathrm{Ext}_l(\Gamma, g_0) := \oplus_{(r,s)} \mathrm{Ext}_l(\Gamma, V_{s,r}(g_0))$.
The space $\mathrm{Ext}_*(\Gamma, g_0) := \oplus_{l \geq l_0} \mathrm{Ext}(\Gamma, g_0)$ is closed under differentiation, and has a $\mathfrak{sl}_2(\mathbb{C})$-module structure.

Examples

- Quasimodular forms (trivial extensions)
- Eichler integrals and their derivatives
- Deformations of accessory parameters (previous example)
- Depth one elliptic multiple zeta values ($\Gamma = \mathrm{SL}_2(\mathbb{Z})$)

Elliptic multiple zeta values

Consider the Jacobi theta function $\theta_\tau(u) := \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+1/2)^2} e^{(n+1/2)u}$

and the Kronecker function

$$F(u,\alpha,\tau) \;:=\; \frac{\theta'_\tau(0)\,\theta_\tau(u+\alpha)}{\theta_\tau(u)\,\theta_\tau(\alpha)} \;=\; \sum_{n\geq 0} f_n(u,\tau)(2\pi i\alpha)^{n-1}$$

A depth one elliptic multiple zeta value is a linear combination of the following functions (suitably normalized)

$$A_{n,r}(\tau) = \int_0^1 \frac{(2\pi i)^{r-1}}{(r-1)!} f_n(u,\tau)\,du$$

Extended modular forms for $\mathbb{E}_4'$

Let $\mathbb{E}_4$ denote the Eisenstein series of weight 4 on $\mathrm{SL}_2(\mathbb{Z})$

The quasimodular form $\mathbb{E}_4'$ induces four extensions $V_{4,0}, V_{3,1}, V_{2,2}, V_{1,3}$.

For instance, $V_{1,3}$ is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & -4 & -6 & -4 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 & 1 & 0 & -5 & 0 \\ 1 & 0 & 0 & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Proposition (Zerbini, 2017)

The elliptic multiple zeta values $\hat{A}_{1,4}, A_{2,3}, A_{3,2}, A_{4,1} \in \mathrm{Ext}_*(\mathrm{SL}_2(Z), \mathbb{E}_4')$

Extended modular forms for $\mathbb{E}_4'$ II

We can use this knowledge to reinterpret some known relations for elliptic MZV in terms of extensions

$$\mathscr{A}(X, Y; \tau) := \sum_{n\geq 0, r\geq 1} \frac{A_{n,r(\tau)}}{(2\pi i)^{r-1}} X^{n-1} Y^{r-1} \qquad \mathscr{G}(X; \tau) := \sum_{n\geq -1} n c_n \mathbb{E}_n(\tau) X^{n-1}$$

The formula (Zerbini)

$$\frac{\partial}{\partial \tau} \mathscr{A}(X, Y; \tau) = (1 - e^Y) \mathscr{G}(X; \tau) - Y \frac{\partial}{\partial X} \mathscr{A}(X, Y; \tau)$$

can be explained using the fact that the space of extended modular forms attached to $\mathbb{E}_4'$ is closed under differentiation and that to a quasimodular form correspond finitely many extensions.

THANK YOU!