From Asai to Triple Product: Euler Systems and p-adic L-functions

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International Seminar on Automorphic Forms

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T p-adic Galois representation

$$L\text{-function for } T \xleftrightarrow{\text{Bloch–Kato}} \text{Selmer group for } T^*(1)$$

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  $\rightsquigarrow$ often need p-adic variation in families of the p-adic L-function and of the classes

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(Rankin–Selberg/Asai integral)

(B) The (twisted) triple product case

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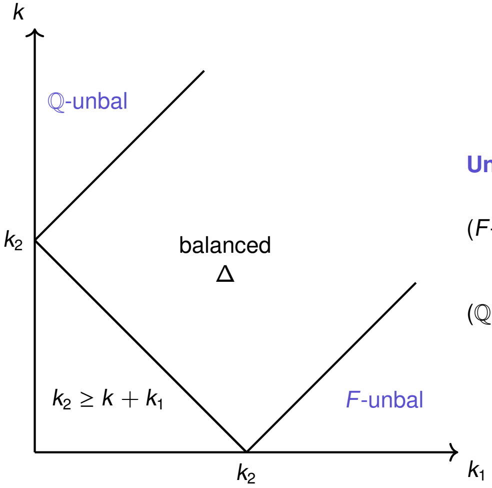

Unbalanced regions ($\epsilon = +1$)

(F-unbalanced): $k_1 \geq k + k_2$

($\mathbb{Q}$-unbalanced): $k \geq k_1 + k_2$

In both cases

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$\rightsquigarrow$ vary $\mathcal{F}$ and j (or $\mathcal{F}$ and g) in p-adic families

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Main results (A)

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Assume p splits in F. There exists $\mathcal{L}_p^{\mathrm{As}}(\underline{\Pi}) \in \mathcal{O}(\mathcal{U}_1 \times \mathcal{U}_2 \times \mathbb{Z}_p^\times)$ such that for any $(k_1, k_2, j) \in (\mathcal{U}_1 \times \mathcal{U}_2 \times \mathbb{Z}_p^\times) \cap (\mathbb{Z}^3_{\geq 0})$ such that $\boxed{k_2 \leq j \leq k_1 - 1}$

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For each $j \in \mathbb{Z}$ with $0 \leq j \leq k_2 - 2$, the Bloch–Kato logarithm of the Asai–Flach class

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Theorem (GLZ II, 4 and Bloch–Kato)

$$\mathcal{L}_p^{\mathrm{As}}(\underline{\Pi})(k_1, k_2, 1+j) \neq 0 \Rightarrow \mathrm{AF}_1^{[\underline{\Pi},\mathbf{j}]}(k_1, k_2, j) \neq 0.$$

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Under some technical assumptions on $\mathcal{F}$, for $k_2 - 1 \leq j \leq k_1 - 2$ (since $L(V^{\mathrm{As}}(\mathcal{F}), j+1) \neq 0$),

the Bloch–Kato Selmer group of $(V^{\mathrm{As}}(\mathcal{F}))^*(-j)$ is zero.

Let $\underline{\Pi}$ p-ordinary family cuspidal automorphic unitary representations of $G \times \mathrm{GL}_2$

Main results (B)

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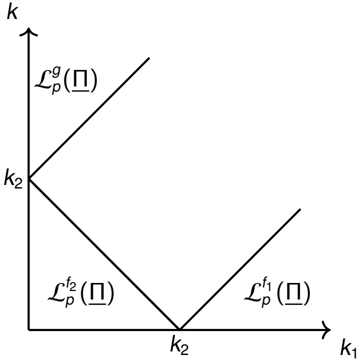


Theorem (Darmon–Rotger, [1])

There exist p-adic L-functions

$$\mathcal{L}_p^{f_2}(\underline{\Pi}),\ \mathcal{L}_p^{f_1}(\underline{\Pi}),\ \mathcal{L}_p^{g}(\underline{\Pi})$$

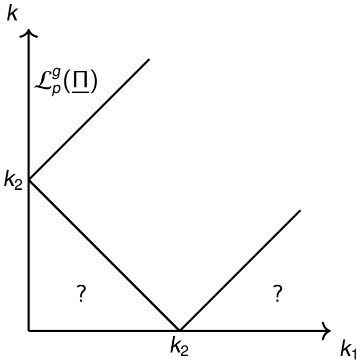
interpolating the value

$$L(\underline{\Pi}[(k_1, k_2, k)], \tfrac{1}{2})$$

for (k_1, k_2, k) in the corresponding regions.

Let $\underline{\Pi}$ p-ordinary family cuspidal automorphic unitary representations of $G \times \mathrm{GL}_2$

$$G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$$



Theorem (Blanco-Chacón–Fornea, [1])

There exists a p-adic L-function

$$\mathcal{L}_p^g(\underline{\Pi})$$

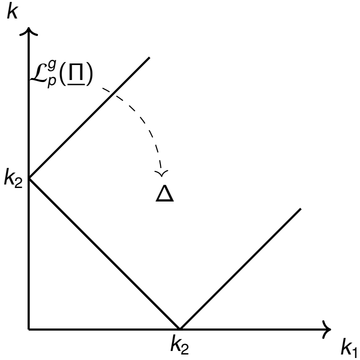
interpolating the value

$$L(\underline{\Pi}[(k_1, k_2, k)], \tfrac{1}{2})$$

for (k_1, k_2, k) in the $\mathbb{Q}$-dominant region.

Main results (B) - $\mathbb{Q}$-dominant region

In both cases:



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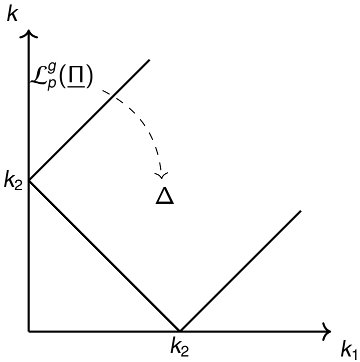

2. varying Δ in family of parallel weights $k_1 = k_2$

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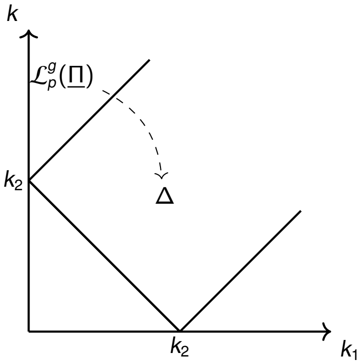

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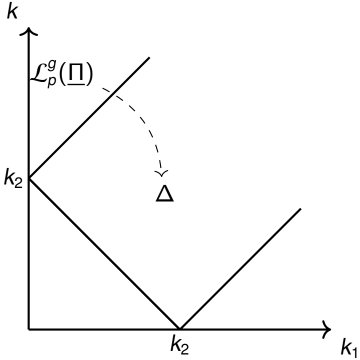

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4. specialise in weight $(1, 1, 2)$ (where $\mathcal{L}_p^g(\underline{\Pi})$ interpolates classical value)

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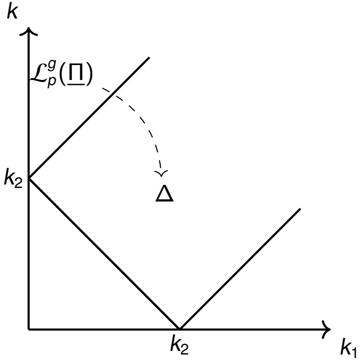

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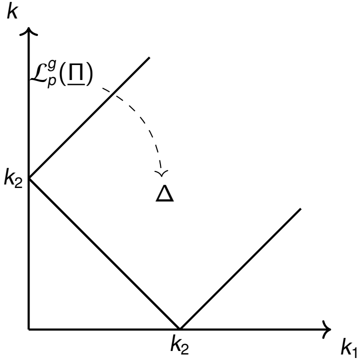

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$$(\underline{\Delta}(f_1^\alpha, f_2^\alpha), \underline{\Delta}(f_1^\alpha, f_2^\beta), \underline{\Delta}(f_1^\beta, f_2^\alpha), \underline{\Delta}(f_1^\beta, f_2^\beta))$$

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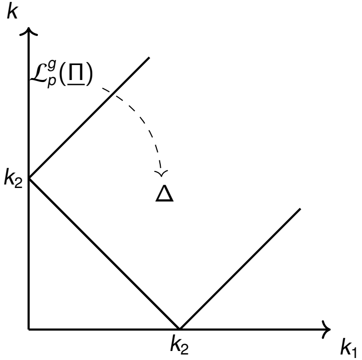

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3. p-adic Gross–Zagier formula
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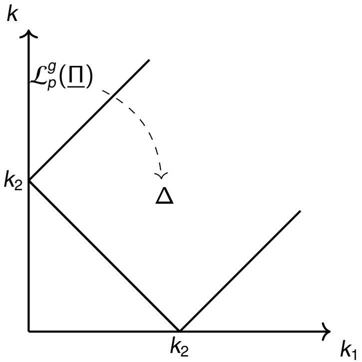

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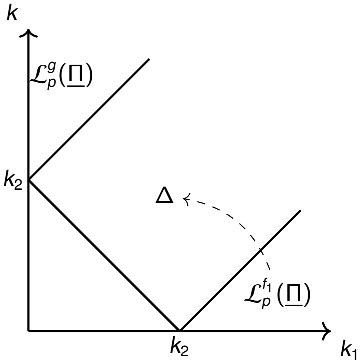

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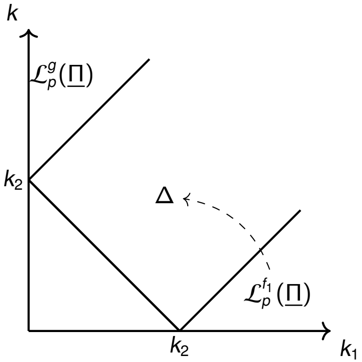

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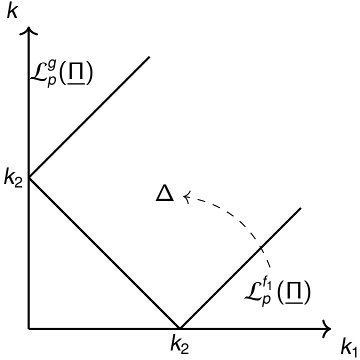

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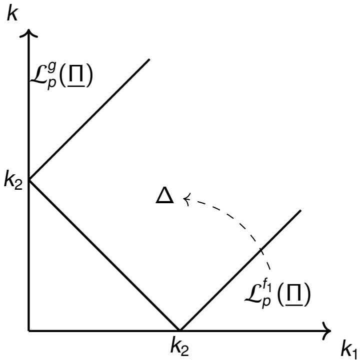

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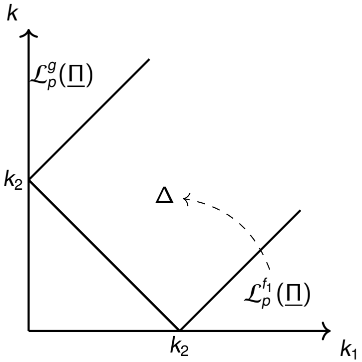

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Main results (B) - F-dominant region

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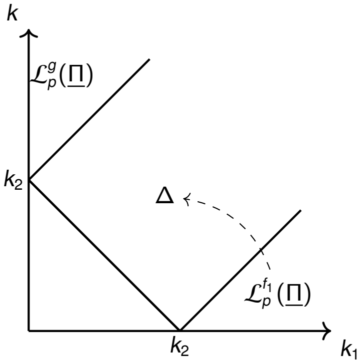

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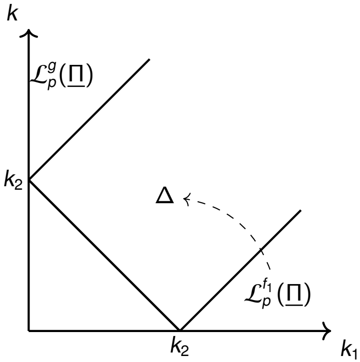

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Missing steps (B) - end goal

Rank ≥ 2 Hilbert-twisted BSD

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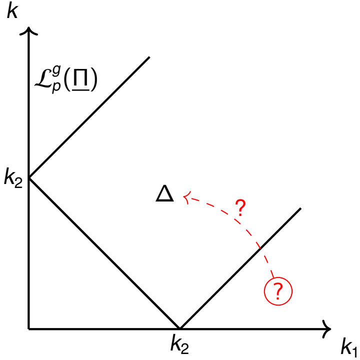

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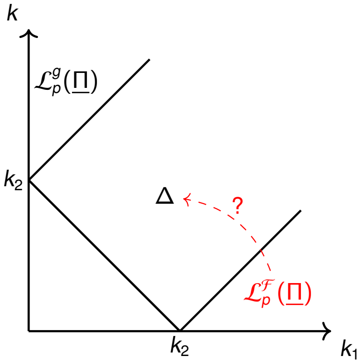

Theorem (Grossi–Graham, in progress)

There exists a p-adic L-function

$$\mathcal{L}_p^{\mathcal{F}}(\underline{\Pi})$$

interpolating the value

$$L(\underline{\Pi}[(k_1,k_2,k)], \tfrac{1}{2})$$

for (k_1,k_2,k) in the F-dominant region.

Remarks

- $\underline{\Pi}$ is a p-ordinary family of unitary cuspidal automorphic representations of $G \times \mathrm{GL}_2$ over open affinoid of (three variable) weight space

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  $\leftsquigarrow$ higher Hida theory (Boxer–Pilloni)

- Rankin–Selberg/triple product and $\mathbb{Q}$-dominant twisted triple product:

$$\langle \iota^\star \omega_\Pi, \eta_\phi \rangle, \quad \omega_\Pi \in H^0(X_G) \text{ and } \eta_\phi \in H^1(X_{\mathrm{GL}_2}) \simeq H^0(X_{\mathrm{GL}_2})^\vee$$

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- Previous works with this strategy:

  "Eis classes ES": $\mathrm{GSp}_4 \supset \mathrm{GL}_2 \times \mathrm{GL}_2$ [Loeffler–Pilloni–Skinner–Zerbes]

  "cycles ES": $U(1, 2n-1) \supset U(1, n-1) \times U(0, n)$ [Graham]

- higher Hida theory complex: $\mathrm{R}\Gamma^1_c\left(\omega_G^{(2-\kappa_1,\kappa_2)}\right)$

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Theorem

There exists a free rank-1 $\mathcal{O}(U)$-submodule $H_{\underline{\Pi}}$ of

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$$H_{\underline{\Pi}}[(k_1,k_2)] \simeq H^1\left(X_G, \omega_G^{(2-k_1,k_2)}\right)^{\mathrm{ord}}[\underline{\Pi}[(k_1,k_2)]].$$

Higher Hida theory (after Boxer–Pilloni) and p-adic L-functions

- higher Hida theory complex: $\mathrm{R}\Gamma^1_c\left(\omega_G^{(2-\kappa_1,\kappa_2)}\right)$
- choice of suitable Hecke operators at $\mathfrak{p}_1, \mathfrak{p}_2$ to cut out ordinary part

  $\rightsquigarrow$ classicality

Theorem

There exists a free rank-1 $\mathcal{O}(U)$-submodule $H_{\underline{\Pi}}$ of

$$H^1\left(\mathrm{R}\Gamma^1_c\left(\omega_G^{(2-\kappa_1,\kappa_2)}\right)^{ord}\right)\otimes\mathcal{O}(U)$$

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- $\eta_{\underline{\Pi}}$ basis of $H_{\underline{\Pi}}$

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- (A) Katz' p-adic Eisenstein measure

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- (A) Katz' p-adic Eisenstein measure

  (B) $\Delta^{\kappa'}\phi_{\underline{\sigma}}$, $\underline{\sigma}$ family of car of GL_2, $\phi_{\underline{\sigma}}$ basis of $H^0(\omega^{\kappa}_{\mathrm{GL}_2})^{\mathrm{ord}}[\underline{\sigma}]$

Higher Hida theory (after Boxer–Pilloni) and p-adic L-functions

- higher Hida theory complex: $\mathrm{R}\Gamma^1_c\left(\omega_G^{(2-\kappa_1,\kappa_2)}\right)$
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- $\eta_{\underline{\Pi}}$ basis of $H_{\underline{\Pi}}$
- (A) Katz' p-adic Eisenstein measure

  (B) $\Delta^{\kappa'}\phi_{\underline{\sigma}}$, $\underline{\sigma}$ family of car of GL_2, $\phi_{\underline{\sigma}}$ basis of $H^0(\omega^{\kappa}_{\mathrm{GL}_2})^{\mathrm{ord}}[\underline{\sigma}]$
- $\iota_p^{\star}\eta_{\underline{\Pi}} \in H^1(\omega_{\mathrm{GL}_2}^{2-(\kappa_1-\kappa_2)})$