A Shimura-Shintani correspondence for rigid analytic cocycles

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International Seminar on Automorphic Forms

Overview

- Goal: to lay the foundations to develop a Shimura-Shintani style correspondence from certain modular forms of weight $k + 1/2$ to *rigid analytic cocycles* of weight $2k$ on $\mathrm{SL}_2(\mathbb{Z}[1/p])$.

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- Darmon and Vonk conjectured that the "RM values" of rigid meromorphic cocycles belong to narrow ring class fields of real quadratic fields.
- They can be thought as analogues of singular moduli for real quadratic fields.

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- So these can then be viewed as real quadratic counterparts of Borcherds' singular theta lifts.

Definition

Let $D > 0$ be a real quadratic discriminant, $k > 2$ an even integer and $z \in \mathcal{H}$. Let

$$f_k(D, z) := \sum_{disc(Q)=D} Q(z, 1)^{-k},$$

where $Q(z, 1) = az^2 + bz + c$ is a binary quadratic form with integer coefficients and discriminant D.

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Theorem (Zagier)

$f_k(D, z)$ belongs to the space $S_{2k}(SL_2(\mathbb{Z}))$ of weight $2k$ cusp forms for $SL_2(\mathbb{Z})$.

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- More precisely, we want to build a rigid analytic cocycle which should play the same role for the correspondence that we aim to build as the role played by $f_k(D,z)$ for the Shiura-Shintani correspondence.
- We will now see the connection between $f_k(D,z)$ and the Shimura-Shintani correspondence.

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- We have

$$\Omega_k(z,\tau) := (-1)^{k/2} 2^{3k-1} \sum_{D>0} D^{k-1/2} f_k(D,z) e^{2\pi i D\tau} \in S_{k+1/2} \text{ (for } \tau\text{)},$$

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$$\mathcal{S}(g)(z) = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathcal{H}} g(\tau)\overline{\Omega_k(-\overline{z},\tau)}v^{k-3/2}dudv.$$

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- RMC can be though as real quadratic analogues of meromorphic functions whose divisors are concentrated on CM points, such as those arising in the image of Borcherds' lift.
- This result of Darmon and Vonk adds evidence in favour of this analogy.

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- Let p be a prime number and let $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$ denote Drinfeld's p-adic upper-half plane.

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- Divisor(J) is a finite formal linear combination of elements of $\Gamma \setminus \mathcal{H}_p^{RM}$.

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Theorem (Darmon, Vonk)

Let ϕ be a weight $1/2$ weakly holomorphic modular form for $\Gamma_0(4p)$. Assume that ϕ is regular at all the cusps except ∞ and has integer Fourier coefficients. Let $-d$ be a negative discriminant.

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- The construction of $J_{-d,\phi}$ does not use a theta kernel.
- Our goal: to get a correspondence using a theta kernel, more precisely to define p-adic analogues of $f_k(D, z)$ and package them into a theta kernel.

The setup: $\mathcal{H}_p$ and the Bruhat-Tits Tree

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- The Ω_n^- are examples of *affinoids*.

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- Let v_0 be the vertex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We will call it *standard vertex*.

The setup: $\mathcal{H}_p$ and the Bruhat-Tits Tree

Definition

The Bruhat-Tits Tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$ is the graph $\mathcal{T}$ whose vertices are equivalence classes of lattices in $\mathbb{Q}_p^2$. Two vertices x, x' are joined by an edge if $x = [L]$, $x' = [L']$ and $pL \subsetneq L' \subsetneq L$.

- Set of vertices of $\mathcal{T}$ is denoted by $\mathcal{T}_0$, set of unordered edges $\mathcal{T}_1$, set of ordered edges $\mathcal{T}_1^*$.
- $\mathcal{T}$ is a $p+1$-regular tree. The vertices of $\mathcal{T}$ at distance n from any given vertex are in bijection with $\mathbb{P}_1(\frac{\mathbb{Z}_p}{p^n\mathbb{Z}_p})$.
- $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on $\mathcal{T}$ by $g[L] := [gL]$.
- Let v_0 be the vertex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We will call it *standard vertex*.
- Let v_1 be the vertex $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} v_0$. Let e_0 be the edge joining v_0 and v_1. We will call it *standard edge*.

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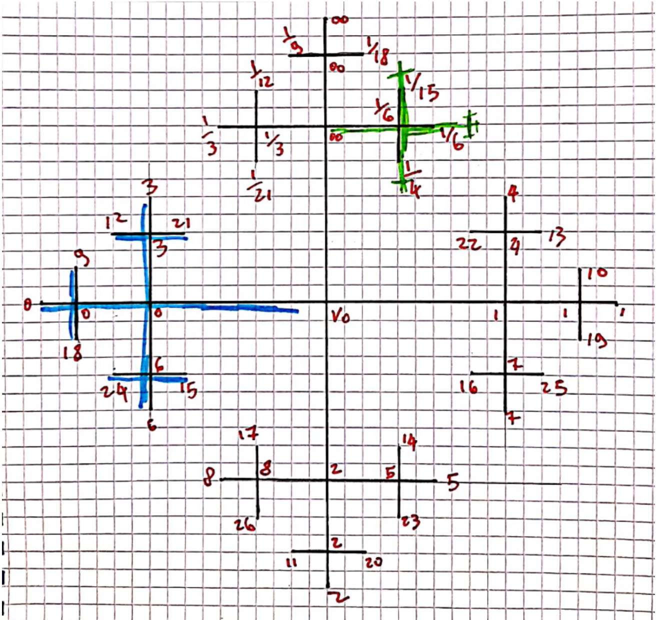

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Proposition

There is a $\mathrm{PGL}_2(\mathbb{Q}_p)$-equivariant map $r : \mathcal{H}_p \to \mathcal{T}$. We call it the *reduction map*.

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- $r^{-1}(v_0) = \{z \in \mathcal{H}_p \ s.t. \ |z - t| \geq 1, \ \text{for } t = 0, ..., p-1, \ \text{and } |z| \leq 1\}$ is the *standard affinoid*.

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- $\mathcal{H}_p$ is a *tubular neighbourhood* of $\mathcal{T}$.
- Preimages of vertices are called *affinoids*, preimages of edges are called *annuli*.

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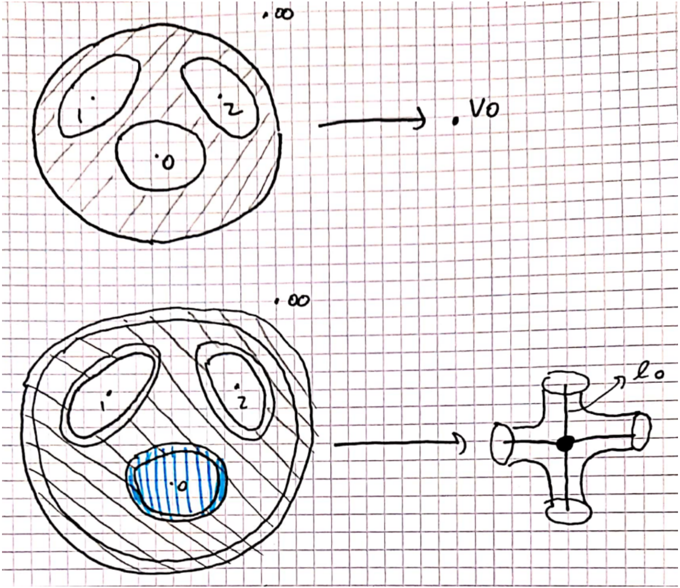

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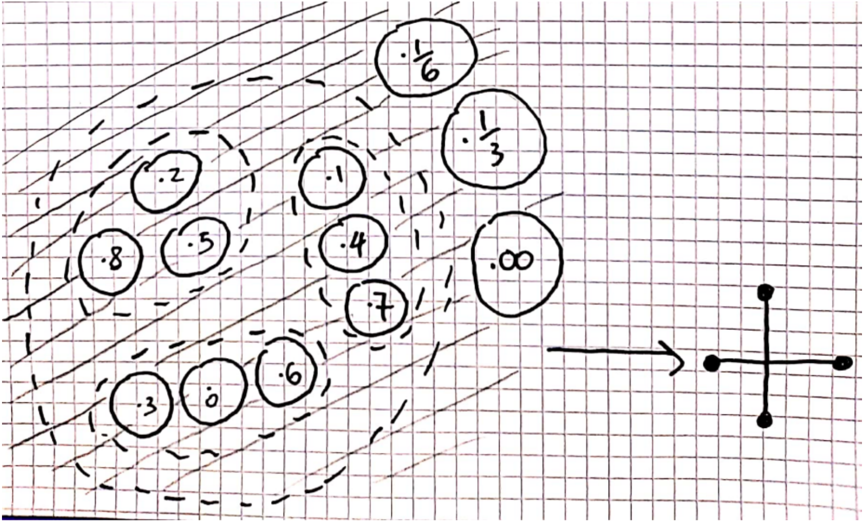

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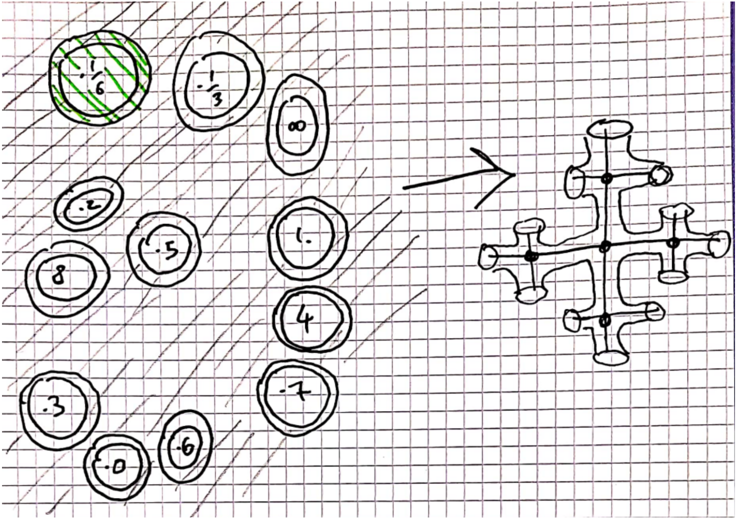

The setup: rigid analytic and meromorphic functions

Definition

A rigid analytic function is a $\mathbb{C}_p$-valued function f on $\mathcal{H}_p$, such that its restriction to any affinoid is a uniform limit, with respect to the sup norm, of rational functions on $\mathbb{P}^1(\mathbb{C}_p)$ having poles outside the affinoid.

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Definition

A rigid meromorphic function is the quotient of two rigid analytic functions.

Rigid meromorphic cocycles

- For all $k \geq 0$, the weight k action of Γ on rigid analytic and meromorphic functions is defined as

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right), \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

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A rigid meromorphic (resp. analytic) cocycle of weight $k > 0$ is a class in $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_k)$ (resp. in $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}_k)$).

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- Goal: to realise a Shimura-Shintani style correspondence from certain modular forms of weight $k + 1/2$ to $\mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}_{2k})$.

Modular symbols

Definition

A Γ-invariant $\mathcal{A}_k$-valued modular symbol is a function $m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \mathcal{A}_k$ that, for all $r, s, t \in \mathbb{P}_1(\mathbb{Q})$, satisfies

$$m\{r,s\} = -m\{s,r\} \quad \text{and} \quad m\{r,s\} + m\{s,t\} = m\{r,t\},$$

as well as

$$m\{\gamma r, \gamma s\}|\gamma = m\{r,s\} \quad \text{for all } \gamma \in \Gamma.$$

We denote the space of such functions as $\mathrm{MS}^\Gamma(\mathcal{A}_k)$. The space of Γ-invariant $\mathcal{M}_k$-valued modular symbols $\mathrm{MS}^\Gamma(\mathcal{M}_k)$ is defined simililarly.

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Lemma

$\mathrm{MS}^{\Gamma}(\mathcal{M}_k) \cong \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{M}_k)$ and $\mathrm{MS}^{\Gamma}(\mathcal{A}_k) \cong \mathrm{H}^1_{\mathrm{par}}(\Gamma, \mathcal{A}_k)$.

The main object

- Let $\mathcal{F}_D(\mathbb{Z}[1/p])$ denote the set of binary quadratic forms of discriminant D with coefficients in $\mathbb{Z}[1/p]$, equipped with the natural action of Γ.

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Theorem (N.)

Let $k \geq 1$ be an odd integer and let D be non-square. For all $(r, s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, the infinite sum

$$J_{k,D}\{r, s\}(z) := \sum_{Q \in \mathcal{F}_D(\mathbb{Z}[1/p])} (\gamma_Q \cdot (r, s)) \cdot Q(z, 1)^{-k}$$

converges to a rigid meromorphic function of $z \in \mathcal{H}_p$, which is rigid analytic when $\left(\frac{D}{p}\right) = 1$. The function $J_{k,D} : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \mathcal{M}_{2k}$ is a rigid meromorphic cocycle of weight $2k$.

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$$f = \sum \alpha_i g_i,$$
- to f we can then associate the rigid analytic cocycle $\sum \alpha_i m_i$.

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But if D is a square then $J_{k,D}$ is not defined, so we prove instead

Theorem (N.)

If D is not a square, then $J_{k,D}$ is the D-th coefficient of a weight $k + 1/2$ modular form $\widehat{\Omega}(q)$ with coefficients in $\mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$.

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- Apply a linear map $S_{2k}(\Gamma_0(p)) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$ sending $c_D \mapsto J_{k,D}$.
- This will give a modular form with coefficients in $\mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$.

The coefficients of $\bar{\Omega}$

- Let $\mathcal{F}_D^{(p)}(\mathbb{Z}) := \{Q(x,y) = ax^2 + bxy + cy^2 \text{ where } p|a,\ b^2 - 4ac = D \text{ and } a,b,c \in \mathbb{Z}\}$.

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Proposition (N.)

Let

$$f_{k,D}^{(p),s}(z) := \sum_{Q \in \mathcal{F}_D^{(p),s}(\mathbb{Z})} Q(z,1)^{-k}.$$

Then $f_{k,D}^{(p),s}(z)$ converges to a form in $S_{2k}(\Gamma_0(p))$.

The periods of $f_{k,D}^{(p),s}(z)$

Theorem (N.)

$\bar{\Omega}(q) := \sum_{D\geq 1} D^{k-1/2} f_{k,D}^{(p),s}(z) \cdot q^D$ is a modular form of weight $k+1/2$ and coefficients in $S_{2k}(\Gamma_0(p))$.

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The periods of $f_{k,D}^{(p),s}(z)$

Definition

Let $f \in S_{2k}(SL_2(\mathbb{Z}))$ and let n be an integer with $0 \leq n \leq 2k-2$. We define

$$r_n(f) := \int_0^\infty f(it) t^n dt,$$

$$r^+(f)(x) := \sum_{\substack{0 \leq n \leq 2k-2 \\ n \text{ even}}} (-1)^{n/2} \binom{2k-2}{n} r_n(f) x^{2k-2-n}.$$

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We can do something similar for $f_{k,D}^{(p),s}(z)$, which is the analogue of level p of $f_k(D,z)$.

Proposition (Kohnen, Zagier)

If D is not a square and k is even, then

$$r^{+}(f_k(D,z))(x) = \sum_{\substack{a,b,c \in \mathbb{Z} \\ a<0<c \\ b^2-4ac=D}} (ax^2+bx+c)^{k-1}.$$

The periods of $f_{k,D}^{(p),s}(z)$

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Proposition (N.)

If D is not a square and k is odd, then the *odd* period polynomial r^- of $f_{k,D}^{(p),s}$ is

$$r^-(f_{k,D}^{(p),s})(x) = \sum_{[a,b,c]\in\mathcal{F}_D^{(p),s}(\mathbb{Z})} (ax^2+bx+c)^{k-1},$$

where $[a,b,c]$ is a simple form.

The modular symbol attached to periods of $f_{k,D}^{(p),s}(z)$

To any $f \in S_{2k}(\Gamma_0(p))$ we can associate $\bar{\kappa}_f \in \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$ defined by:

$$\bar{\kappa}_f\{r, s\}(x) := \int_r^s f(z)(x-z)^{k-1} dz,$$

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For any $f \in S_{2k}(\Gamma_0(p))$, we let $\kappa_f\{r,s\} := \bar{\kappa}_f\{r,s\} - \bar{\kappa}_f\{-r,-s\}|\widetilde{I}$. Then $\kappa_f \in \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$.

The modular symbol attached to periods of $f_{k,D}^{(p),s}(z)$

To any $f \in S_{2k}(\Gamma_0(p))$ we can associate $\bar{\kappa}_f \in \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$ defined by:

$$\bar{\kappa}_f\{r,s\}(x) := \int_r^s f(z)(x-z)^{k-1}dz,$$

where the integral is over the geodesic in the upper half plane joining r, s.

Definition

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Proposition (N.)

Up to a constant we have:

$$\kappa_{f_{k,D}^{(p),s}(z)}\{r,s\}(x) = D^{1/2-k} \sum_{Q \in \mathcal{F}_D^{(p),s}(\mathbb{Z})} (\gamma_Q \cdot (r,s))Q(x,1)^{k-1}.$$

The map $\mathfrak{p}$ is defined as $f \mapsto \kappa_f$.

A Schneider-Teitelbaum lift for rigid analytic cocycles

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- Γ acts on $\mathcal{P}_{2k-2}$ by

$$(h|\gamma)(z) := (cz+d)^{2k-2} h\left(\frac{a\tau+b}{c\tau+d}\right), \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

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- Moreover, we will use the isomorphism

$$\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}^{\vee}) \cong \ \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee})).$$

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$$c(\bar{e}) = -c(e), \quad \text{and} \sum_{s(e)=v} c(e) = 0, \quad \text{for all } v \in \mathcal{T}_0 \text{ and } e \in \mathcal{T}_1^*.$$

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- So we want a map $\mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee})) \xrightarrow{ST} \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$.

A Schneider-Teitelbaum lift for rigid analytic cocycles

Note that $c \in \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee}))$ is a collection of harmonic cocycles $c\{r,s\}$ indexed on $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$.

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$$\int_{U(e)} P(t) d\mu_{c\{r,s\}}(t) = (c\{r, s\}(e))(P),$$

where $P \in \mathcal{P}_{2k-2}$ and $U(e)$ is the p-adic ball associated to the edge e.

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- The map $\mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee)) \xrightarrow{ST} \mathrm{MS}^\Gamma(\mathcal{A}_{2k})$ will be given by $c \mapsto f$ where

$$f\{r,s\}(z) := \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_{c\{r,s\}}(t).$$

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- that the expression makes sense,
- that $f\{r,s\}(z)$ is in $\mathcal{A}_{2k}$ and that $f\{r,s\}$ is a Γ-invariant modular symbol.

Note that $c\{r,s\} \mapsto c\{0,\infty\}$ identifies $\mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee}))$ with a subset of $C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee})$ of cocycles c satisfying the relations

$$c|(1+S)=0, \quad c|(1+U+U^2)=0, \quad c|D=c,$$

where

$$S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \quad U=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix}, \quad D=\begin{pmatrix}p & 0\\ 0 & 1/p\end{pmatrix}.$$

A Schneider-Teitelbaum lift for rigid analytic cocycles

So it is enough to show that for any $c \in C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee})$ satisfying the conditions above the integral

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_c(t)$$

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satisfies certain bounds, hence the distribution can be extended uniquely to the space of $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic except for a pole at ∞ of order at most $2k-2$.

Thank you!