Counting intersection numbers on Shimura curves

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Talk outline

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2. Introduce and explain my main result
3. Explore the analogies between the two situations

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denote the classical $j-$function, where $q = e^{2\pi i\tau}$.
- If τ is an imaginary quadratic number, then $j(\tau) \in \overline{\mathbb{Q}}$.
- In fact, $j(\tau)$ generates a certain abelian extension of $\mathbb{Q}(\sqrt{D})$, and provides a solution to explicit class field theory over imaginary quadratic fields.
- In 1985, Gross and Zagier proved that $j(\tau_1) - j(\tau_2)$ had remarkable factorization properties ([GZ85]).

$J(D_1, D_2)$

Definition

Let D_1, D_2 be coprime fundamental negative discriminants, and define

$$J(D_1, D_2) = \left(\prod_{\substack{[\tau_1],[\tau_2] \\ \mathrm{disc}(\tau_i)=D_i}} j(\tau_1) - j(\tau_2)\right)^{\frac{4}{w_1 w_2}}$$

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- If $D_1, D_2 < -4$, this is the norm to $\mathbb{Q}$ of $j(\tau_1) - j(\tau_2)$, and is therefore an integer.
- In general, $J(D_1, D_2)^2$ is an integer.

ϵ function

Definition

Let p be a prime with $\left(\frac{D_1 D_2}{p}\right) \neq -1$. Define

$$\epsilon(p) := \begin{cases} \left(\dfrac{D_1}{p}\right) & \text{if } p \text{ and } D_1 \text{ are coprime;} \\ \\ \left(\dfrac{D_2}{p}\right) & \text{if } p \text{ and } D_2 \text{ are coprime.} \end{cases}$$

Extend ϵ multiplicatively, so that $\epsilon(mn) = \epsilon(m)\epsilon(n)$ when $\epsilon(m)$ and $\epsilon(n)$ are defined.

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Theorem (Gross-Zagier, 1985)

$$J(D_1,D_2)^2=\pm\prod_{\substack{x^2<D_1D_2\\ x\equiv D_1D_2 \pmod 2}} F_{GZ}\left(\frac{D_1D_2-x^2}{4}\right)$$

Gross-Zagier formula remarks

- $J(D_1, D_2)^2$ is only divisible by primes dividing a number of the form $\frac{D_1 D_2 - x^2}{4}$ for $x^2 < D_1 D_2$.

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- $J(D_1, D_2)^2$ is only divisible by primes dividing a number of the form $\frac{D_1 D_2 - x^2}{4}$ for $x^2 < D_1 D_2$.
- $F_{\mathrm{GZ}}(m)$ is either 1 or a power of a prime ℓ.
- The latter occurs if and only if ℓ is the only prime dividing m to an odd exponent for which $\epsilon(\ell) = -1$.

Proof of Gross-Zagier

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- Zagier produced an analytic proof, and Gross the algebraic proof.
- For the algebraic proof, they exploit the connection to elliptic curves, and computing $v_\ell(J(D_1, D_2)^2)$ reduces to counting isomorphisms between curves.
- This is further reduced to counting solutions to an equation in the maximal order of the quaternion algebra over $\mathbb{Q}$ ramified at ℓ and ∞.

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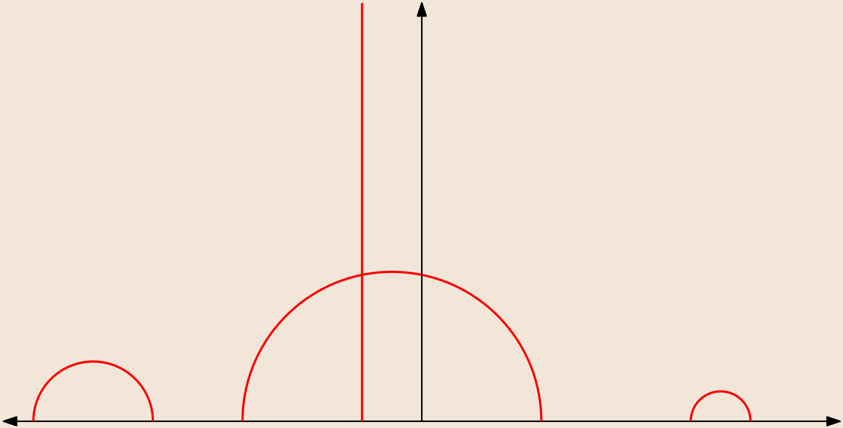
- Equip $\Gamma\backslash\mathbb{H}$ with the usual hyperbolic metric.

Geodesics in $\mathbb{H}$



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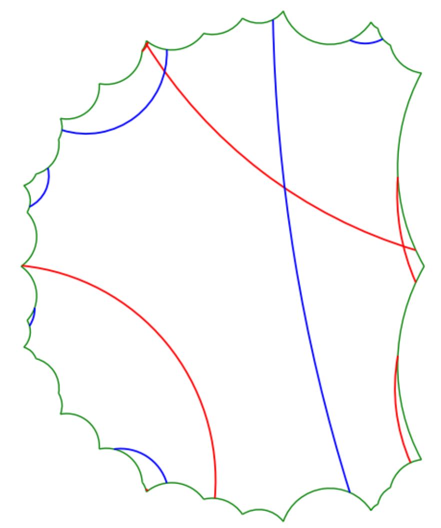
- This descends to the closed geodesic $\tilde{\ell}_\gamma$ in $\Gamma\backslash\mathbb{H}$.

Example



Intersections of closed geodesics

Definition

Let f be a function defined on transverse intersections of geodesics. Define

$$\mathrm{Int}_{\Gamma}^{f}(\gamma_1, \gamma_2) := \sum_{p \in \tilde{\ell}_{\gamma_1} \pitchfork \tilde{\ell}_{\gamma_2}} f(p)$$

to be the $f-$weighted intersection number.

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- If $\mathfrak{D} = 1$, then $\Gamma_{\mathrm{O}} = \Gamma_0(\mathfrak{M})$.
- Otherwise, the corresponding Shimura curve is compact.

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Definition

An optimal embedding of $\mathcal{O}_D$ into O is a ring homomorphism $\phi : \mathcal{O}_D \to \mathrm{O}$ that does not extend to an embedding of a larger order. Two optimal embeddings ϕ_1, ϕ_2 are equivalent if there exists an $r \in \mathrm{O}_{N=1}$ with $r\phi_1 r^{-1} = \phi_2$.

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- The set of equivalence classes optimal embeddings of $\mathcal{O}_D$ into O is denoted $\mathrm{Emb}(\mathrm{O}, D)$.

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- There is a free action of $\mathrm{Cl}^+(D)$ on $\mathrm{Emb}(\mathrm{O}, D)$, with the orbits being the orientations. In particular, $\mathrm{Emb}(\mathrm{O}, D)$ is finite.
- $\iota(\phi(\epsilon_D)) \in \Gamma_{\mathrm{O}}$ is a primitive hyperbolic element! In fact, all such elements arise in this fashion.

Recasting the question

Definition

Let ϕ_1, ϕ_2 be optimal embeddings of discriminants D_1, D_2 into $\mathcal{O}$, and let f be an intersection function. Define

$$\mathrm{Int}^f_{\mathcal{O}}(\phi_1, \phi_2) := \mathrm{Int}^f_{\Gamma_{\mathcal{O}}}(\iota(\phi_1(\epsilon_{D_1})), \iota(\phi_2(\epsilon_{D_2}))).$$

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Question

What can we say about $\mathrm{Int}^f_{\mathcal{O}}(\phi_1, \phi_2)$ *in terms of* $D_1, D_2, \mathfrak{D}, \mathfrak{M}$*?*

Reinterpreting the intersection number

- Each transverse intersection of $\tilde{\ell}_{\phi_1}, \tilde{\ell}_{\phi_2}$ corresponds to a pair of optimal embeddings ϕ_1', ϕ_2' with $\phi_i' \sim \phi_i$ and $\ell_{\phi_1'}, \ell_{\phi_2'}$ having a transverse intersection.

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- This lifting is unique up to the action of $\mathrm{O}_{N=1}$ via simultaneous conjugation.
- In other words, the set of transverse intersections bijects with

$$\{(\phi_1', \phi_2') : \phi_i \sim \phi_i', |\ell_{\phi_1'} \pitchfork \ell_{\phi_2'}| = 1\}/\sim .$$

$x-$linking

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Call ϕ_1, ϕ_2 $x-$linked if $x^2 \neq D_1 D_2$ and

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Proposition

We have $x \equiv D_1 D_2 \pmod 2$ *and*

$$\mathfrak{D}\mathfrak{M} \mid \frac{D_1 D_2 - x^2}{4}.$$

Root geodesics intersecting in the upper half plane

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- *The upper half plane root geodesics corresponding to the* $x-$*linked pair* (ϕ_1, ϕ_2) *intersect if and only if* $x^2 < D_1 D_2$.

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- *The intersection point is the upper half plane root of* $\iota(\phi_1(\sqrt{D_1})\phi_2(\sqrt{D_2}))$*, and hence corresponds to a (not necessarily optimal) embedding of the negative quadratic order* $\mathcal{O}_{x^2-D_1D_2}$.

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- *The angle of intersection* θ *satisfies*

$$\tan(\theta) = \frac{\sqrt{D_1 D_2 - x^2}}{x}.$$

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- Call ℓ the *level* of the intersection, and $\mathrm{sg}(\phi_1, \phi_2)\ell$ the *signed level* of the intersection.
- The level is defined for all x with $x^2 \neq D_1 D_2$, whereas the sign is only defined for $x^2 < D_1 D_2$.

Intersection number, revisited

Definition

Let $\mathrm{Emb}(\mathrm{O}, \phi_1, \phi_2, x)$ denote the set of simultaneous equivalence class pairs of optimal embeddings individually equivalent to ϕ_1, ϕ_2 that are $x-$linked.

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In particular, we see that

$$\text{Int}_{\mathcal{O}}^{f}(\phi_1, \phi_2) = \sum_{\substack{x^2 < D_1 D_2 \\ x \equiv D_1 D_2 \pmod 2}} \sum_{(\phi_1', \phi_2') \in \text{Emb}(\mathcal{O}, \phi_1, \phi_2, x)} f(\phi_1', \phi_2').$$

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Corollary

Given D_1, D_2, there are finitely many pairs $(\mathfrak{D}, \mathfrak{M})$ for which there exist optimal embeddings ϕ_1, ϕ_2 of discriminants D_1, D_2 for which $\mathrm{Int}_{\mathrm{O}}(\phi_1, \phi_2) \neq 0$.

Summing over all embeddings

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- Unfortunately, the sets $\mathrm{Emb}(\mathcal{O}, \phi_1, \phi_2, x)$ are difficult to access theoretically.
- Instead, consider

$$\mathrm{Emb}(\mathcal{O}, D_1, D_2, x) := \bigcup_{\phi_i \in \mathrm{Emb}(\mathcal{O}, D_i)} \mathrm{Emb}(\mathcal{O}, \phi_1, \phi_2, x),$$

the total $x-$linking of discriminants D_1, D_2 into $\mathcal{O}$, and

$$\mathrm{Int}^f_{\mathcal{O}}(D_1, D_2) := \sum_{\phi_i \in \mathrm{Emb}(\mathcal{O}, D_i)} \mathrm{Int}^f_{\mathcal{O}}(\phi_1, \phi_2).$$

Main result I

Theorem (Theorem 1.10 of [Ric21a])

Assume D_1, D_2 are coprime and fundamental, $\mathfrak{M} = 1$, and factorize

$$\frac{D_1 D_2 - x^2}{4} = \pm \prod_{i=1}^{r} p_i^{2e_i+1} \prod_{i=1}^{s} q_i^{2f_i} \prod_{i=1}^{t} w_i^{g_i},$$

where p_i are the primes for which $\epsilon(p_i) = -1$ that appear to an odd power, q_i are the primes for which $\epsilon(q_i) = -1$ that appear to an even power, and w_i are the primes for which $\epsilon(w_i) = 1$. Then r is even, and

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- $\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)$ *is non-empty if and only if*

$$\mathfrak{D} = p_1 p_2, \cdots p_r.$$

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- *Assume this holds. Then*

$$|\,\mathrm{Emb}(\mathrm{O}, D_1, D_2, x)| = 2^{r+1} \prod_{i=1}^{t} (g_i + 1).$$

Main result II

Theorem (Theorem 1.10 of [Ric21a])

- $\mathrm{Emb}(\mathcal{O}, D_1, D_2, x, \ell)$ *is non-empty if and only if*

$$\ell = \prod_{i=1}^{r} p_i^{e_i} \prod_{i=1}^{s} q_i^{f_i} \prod_{i=1}^{t} w_i^{g_i'},$$

where $2g_i' \leq g_i$.

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where $2g_i' \leq g_i$.

- *Assume the above holds. Let* n *be the number of indices* i *for which* $2g_i' < g_i$. *Then*

$$|\,\mathrm{Emb}(\mathcal{O}, D_1, D_2, x, \ell)| = 2^{r+n+1}.$$

Main result commentary

- In my paper, the results allow $\mathcal{O}$ to be Eichler, and D_1, D_2 to be non-fundamental and non-coprime. The only restriction is

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- We also understand how these pairs divide amongst the possible orientations of ϕ_1, ϕ_2.
- Accessing the individual terms $\text{Emb}(\mathcal{O}, \phi_1, \phi_2, x, \ell)$ does not seem to be viable with this approach.

Comparison to Gross-Zagier

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- $$J(D_1, D_2)^2 = \pm \prod_{\substack{x^2 < D_1 D_2 \\ x \equiv D_1 D_2 \pmod 2}} F_{\mathrm{GZ}}\left(\frac{D_1 D_2 - x^2}{4}\right)$$

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- $$\mathrm{Int}_{\mathcal{O}}(D_1, D_2) = \sum_{\substack{x^2 < D_1 D_2 \\ x \equiv D_1 D_2 \pmod 2}} F\left(\frac{D_1 D_2 - x^2}{4}\right)$$

Comparison to Gross-Zagier

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- $v_\ell\left(F_{\mathrm{GZ}}\left(\frac{D_1D_2-x^2}{4}\right)\right)\neq 0$ if and only if $\ell=\prod_{i=1}^r p_i$
- If this holds, then

$$v_\ell\left(F_{\mathrm{GZ}}\left(\frac{D_1D_2-x^2}{4}\right)\right)=(e_1+1)\prod_{i=1}^{t}(g_i+1)=(e_1+1)\sum_{d\mid\frac{D_1D_2-x^2}{4\ell}}\epsilon(d).$$

- $F\left(\frac{D_1D_2-x^2}{4}\right)\neq 0$ if and only if $\mathfrak{D}=\prod_{i=1}^r p_i$
- If this holds, then

$$F\left(\frac{D_1D_2-x^2}{4}\right)=2^{r+1}\prod_{i=1}^{t}(g_i+1)=2^{r+1}\sum_{d\mid\frac{D_1D_2-x^2}{4\mathfrak{D}}}\epsilon(d).$$

Comparison to Gross-Zagier

- The total unweighted intersection number of positive discriminants, $\mathrm{Int}_{\mathcal{O}}(D_1, D_2)$, behaves like the exponents of primes in the factorization of $J(D_1, D_2)^2$ for negative discriminants.

Comparison to Gross-Zagier

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- The total unweighted intersection number of positive discriminants, $\mathrm{Int}_{\mathcal{O}}(D_1, D_2)$, behaves like the exponents of primes in the factorization of $J(D_1, D_2)^2$ for negative discriminants.
- The components $\mathrm{Int}_{\mathcal{O}}^{f}(\phi_1, \phi_2)$ should behave like exponents of primes in the factorization of $j(\tau_1) - j(\tau_2)$ for an appropriate f.
- This indicates that there should exist some function J defined on real quadratic irrationalities for which the exponents of primes dividing $J(\tau_1) - J(\tau_2)$ are precisely $\mathrm{Int}_{\mathcal{O}}^{f}(\phi_1, \phi_2)$.

Darmon-Vonk

Figure: Henri Darmon

Figure: Jan Vonk

Darmon-Vonk

- If (ϕ_1', ϕ_2') is $x-$linked of level ℓ and q is a prime, we define their $q-$intersection by
$$\mathsf{sg}(\phi_1', \phi_2')(1 + v_q(\ell)).$$
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Darmon-Vonk

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$$\mathrm{sg}(\phi_1', \phi_2')(1 + v_q(\ell)).$$
Denote the $q-$weighted intersection number by $\mathrm{Int}_{\mathrm{O}}^{q}$.
- In [DV20], given τ_1, τ_2 real quadratic points corresponding to coprime fundamental discriminants D_1, D_2 and a prime $p \leq 13$, Darmon and Vonk $p-$adically construct a $J_p(D_1, D_2)$, which is conjecturally algebraic and belonging to the compositum of ring class fields associated to D_1, D_2.

Darmon-Vonk

Conjecture (Conjecture 4.26 of [DV20])

Let $\mathfrak{q}$ lie above the integer prime $q \neq p$. If q is split in $\mathbb{Q}(\sqrt{D_1})$ or $\mathbb{Q}(\sqrt{D_2})$, then $v_{\mathfrak{q}}(J_p(\tau_1, \tau_2)) = 0$. Otherwise, let $\mathcal{O}$ be a maximal order in the quaternion algebra ramified at p, q. Then there exist optimal embeddings ϕ_1, ϕ_2 of discriminants D_1, D_2 into $\mathcal{O}$ for which

$$v_{\mathfrak{q}}(J_p(\tau_1, \tau_2)) = \mathrm{Int}_{\mathcal{O}}^{q}(\phi_1, \phi_2).$$

Computational evidence

- I have written methods to compute (among other related things) optimal embeddings and intersection numbers in PARI ([The21]), and the package is publicly hosted on GitHub ([Ric21b]).

Computational evidence

- I have written methods to compute (among other related things) optimal embeddings and intersection numbers in PARI ([The21]), and the package is publicly hosted on GitHub ([Ric21b]).
- I computed the intersection numbers $\mathrm{Int}_{\mathcal{O}}^{q}(\phi_1, \phi_2)$ for all pairs with $D_1 = 5, 13$ and $D_2 \leq 1000$, and compiled it into a 600 page document. On the other side, Jan Vonk computed the $q-$adic valuations of $J_p(\tau_1, \tau_2)$ for many of these examples, and the data matched perfectly.

Acknowledgments and References

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