Topographs and some infinite series

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Warm-up example

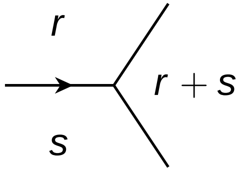
The familiar Fibonacci numbers

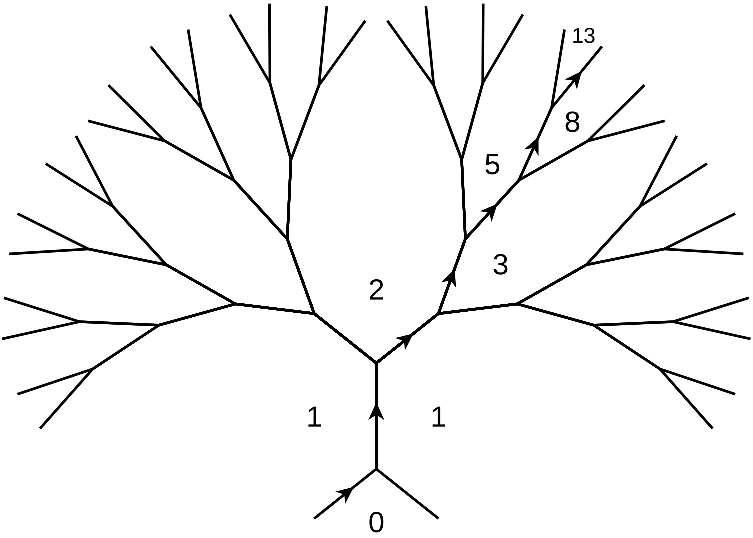
$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

have

$$\ldots, r, s, r+s, \ldots.$$

Put




13
8
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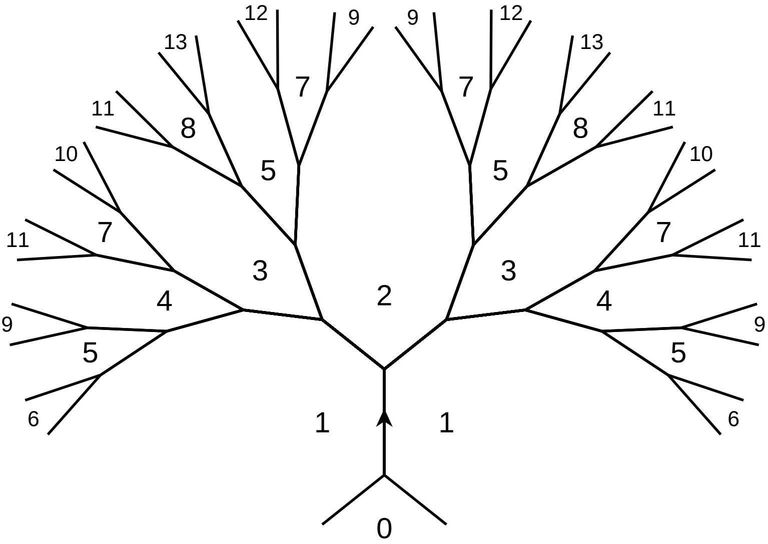
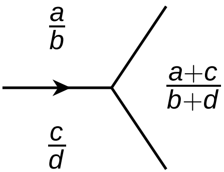


Figure: Start of the infinite Euclid tree

Recall the **mediant** of $\dfrac{a}{b}$ and $\dfrac{c}{d}$ is $\dfrac{a+c}{b+d}$

Now put



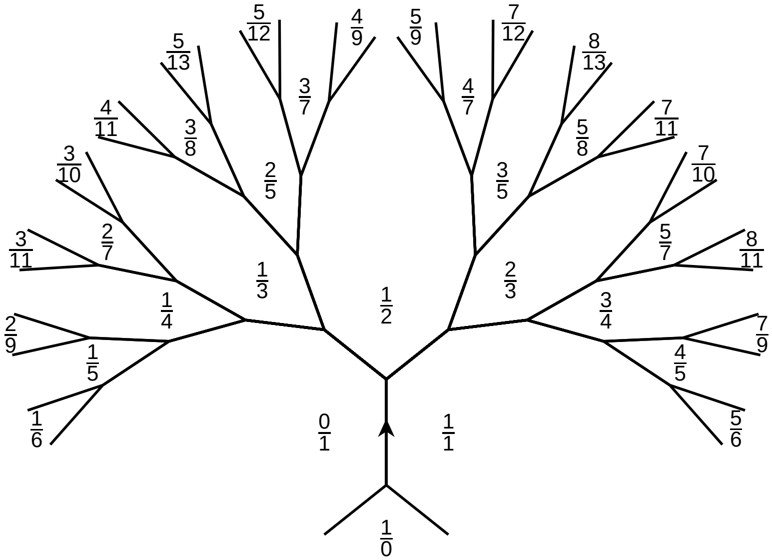


Figure: Start of the Farey tree (Stern-Brocot tree)

Each region gets a unique fraction address.

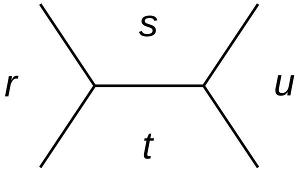
Conway's topographs

Integral binary quadratic forms

$$ax^2 + bxy + cy^2$$

have been studied for hundreds of years. They are closely related to ideal classes in the ring of integers of $\mathbb{Q}(\sqrt{D})$. J.H. Conway introduced his topographs in 1997 as a graphical way to understand these forms.

- For the same tree in the plane, start with any three adjacent integers.
- Then fill in the rest using



Conway's rule: $\quad r + u = 2(s + t)$

Examples

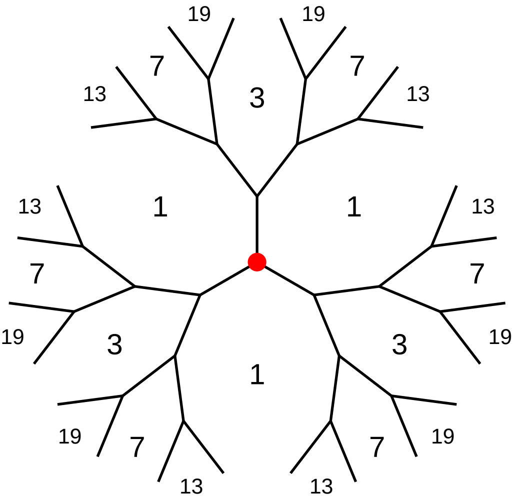


Figure: The topograph of discriminant $D = -3$

Examples

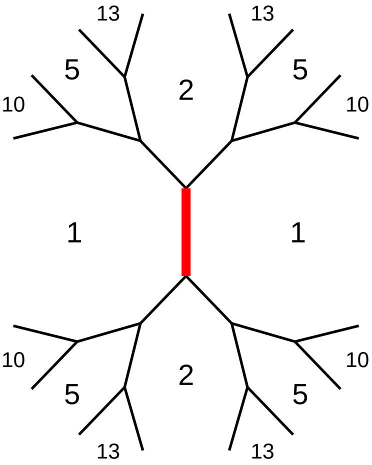
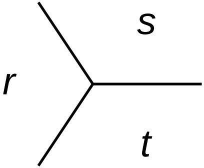


Figure: The topograph of discriminant $D = -4$

The discriminant

For three adjacent regions



set

$$D = r^2 + s^2 + t^2 - 2(rs + rt + st).$$

Then this number is the same for all adjacent regions of a particular topograph and called its **discriminant.**

Another invariant is $\gcd(r, s, t)$. We say a topograph is **primitive** if this $\gcd = 1$.

Conway's classification of topographs

Call a region with label 0 a **lake**. Call an edge between a positive region and a negative region a **river** edge. A **well** is a minimal configuration.

Four families:

- *D* < 0: then all regions positive (or all negative). Only these topographs have wells.
- *D* $= 0$. One region is a lake (or all are lakes).
- *D* > 0 a perfect square. Must be two lakes with a river connecting them.
- *D* > 0 not a perfect square. Has a single infinite periodic river. No lakes.

$D = 0$

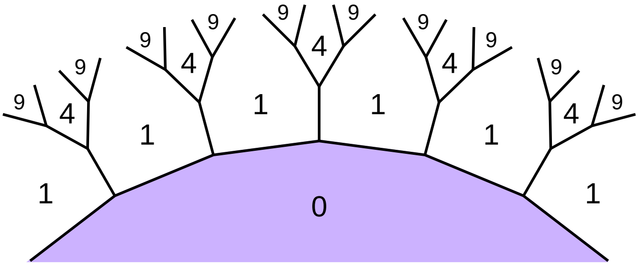


Figure: A primitive topograph with $D = 0$

Discriminant D corresponds to $\mathbb{Q}(\sqrt{D})$ so this case is 'degenerate'.

$D > 0$ a perfect square

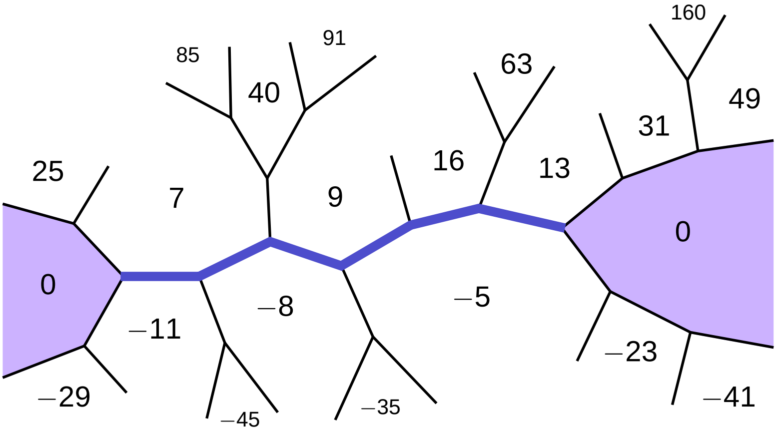


Figure: Part of a topograph of discriminant $D = 18^2$

$D > 0$ a perfect square

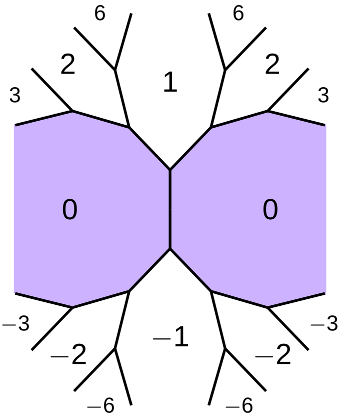


Figure: The only topograph of discriminant $D = 1$

Here the river has 0 length.

$D > 0$ a perfect square

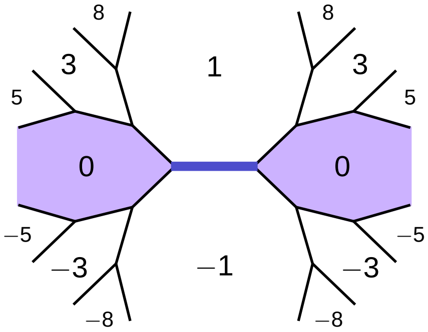


Figure: A topograph with discriminant $D = 4$

This river has length 1.

$D > 0$ not a perfect square

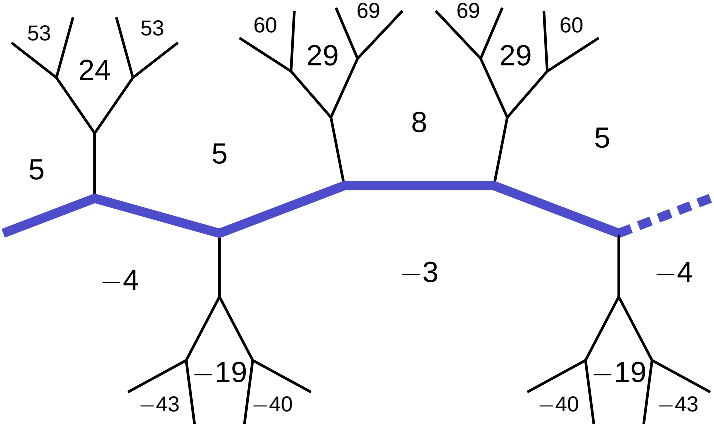
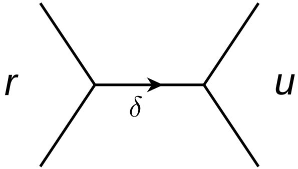


Figure: A topograph of discriminant $D = 96$ with its periodic river

Adding edge labels to a topograph

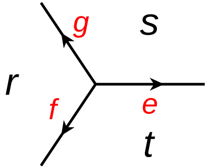
Add edge labels as follows:



$$2\delta = u - r$$

Then δ is an integer, and changing the direction of an edge switches its label's sign.

Region labels $\leftrightarrows$ edge labels



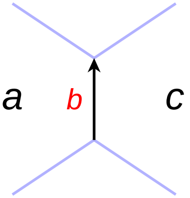
$$r = \frac{f+g}{2} \qquad e = s+t-r$$

$$s = \frac{e+g}{2} \qquad f = r+t-s$$

$$t = \frac{e+f}{2} \qquad g = r+s-t$$

Characterizing region labels

Call this configuration $[a, b, c]$



Theorem (Conway 1997)

The region labels of a topograph containing the configuration $[a, b, c]$ are

$$ax^2 + bxy + cy^2$$

for all coprime integers x and y.

Region labels example

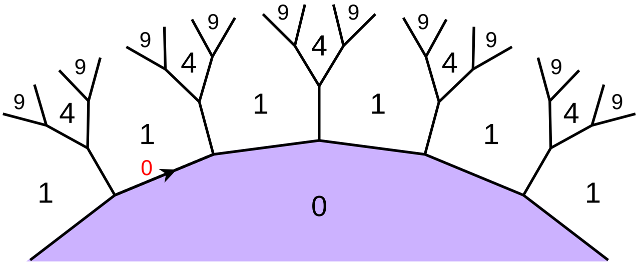


Figure: A primitive topograph with $D = 0$

This topograph contains $[1, 0, 0]$ for example. So its region labels are $1x^2 + 0xy + 0y^2 = x^2$ for $x \in \mathbb{Z}$ and $y = 1$.

Group action, equivalence

Two quadratic forms

$$q(x,y) = ax^2 + bxy + cy^2, \qquad q'(x,y) = a'x^2 + b'xy + c'y^2$$

are **equivalent** if

$$q'(x,y) = q(\alpha x + \beta y, \gamma x + \delta y)$$
$$\text{for} \quad \alpha,\beta,\gamma,\delta \in \mathbb{Z} \quad \text{with} \quad \alpha\delta - \beta\gamma = 1.$$

This is an action of $\mathrm{SL}(2,\mathbb{Z})$ on the right with

$$q|M = q(\alpha x + \beta y, \gamma x + \delta y) \quad \text{for} \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Can use $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$.

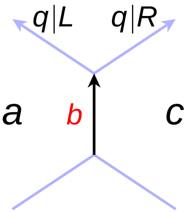
Group action, equivalence

Have

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Action on topograph:

$q = [a, b, c]$



for

$$L = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R = TST = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and S just rotates q by 180° degrees to $[c, -b, a]$.

So, as Rickards noted in 2021, each topograph is an equivalence class of forms.

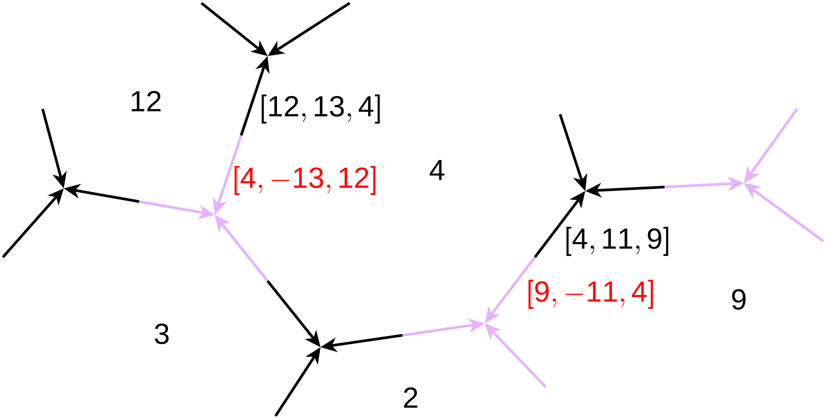


Figure: Visualizing all forms in an equivalence class

(Our earlier example with the Farey tree comes from linear forms $ax + by$.)

Class numbers

The **class number** $h(D)$ is the number of equivalence classes of primitive forms of discriminant D. Here, $q = [a, b, c]$ is primitive means $\gcd(a, b, c) = 1$.

Gauss made famous conjectures about them in 1801. Two are:

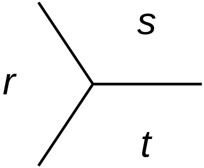
(1) For $D < 0$ (fundamental) have $h(D) = 1$ only for $D = -1, -2, -7, -11, -19, -43, -67, -163$.

(2) Have $h(D) = 1$ for infinitely many (fundamental) $D > 0$.

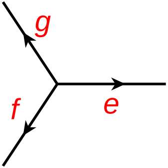
(1) was proved in the 1950s by Heegner, though not believed at first. (2) is still open.

Dirichlet provided well-known formulas for $h(D)$ involving values of L-functions and Kronecker symbols. We obtain an elementary formula when $D < 0$ by counting topographs using their well configurations as follows.

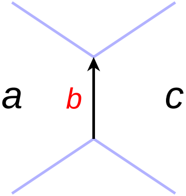
The discriminant also equals:



$$D = r^2 + s^2 + t^2 - 2(rs + rt + st)$$



$$D = -ef - fg - ge$$



$$D = b^2 - 4ac$$

Counting topographs when $D < 0$

Theorem (O'S.)

Suppose $D < -4$. Put $m := |D|$ if D is odd and $m := |D|/4$ otherwise. Then

$$h(D) = 2 \sum_{\substack{e>f>g>0 \\ ef+eg+fg=m}} 1 + \sum_{\substack{e,f>0 \\ e^2+2ef=m}} 1 + \sum_{\substack{e>f>0 \\ ef=m}} 1,$$

where the sums are over pairs or triples of integers with $\gcd = 1$. In the first two sums, the pairs or triples should be all odd if D is odd, and not all odd if D is even. The last sum is only included when D is even.

Seems to be new. Related to work of Mordell in 1923.
For example $h(-31) = 3$.

Sums of three squares

Let $r_3(n)$ be the number of ways to write n as a sum of squares of 3 integers. Let $r_3'(n)$ be the number of ways with 3 integers with $\gcd = 1$.

Our theorem combined with Krammer's 1993 identity

$$(-1)^{n+1} r_3(n) = 4 \sum_{\substack{e,f,g>0 \\ ef+eg+fg=n}} (-1)^{e+f+g} + 6 \sum_{\substack{e,f>0 \\ ef=n}} (-1)^{e+f}$$

gives a quick proof of a result of Gauss that is often quoted in papers (but never proved): for $n > 3$,

$$r_3'(n) = \begin{cases} 12h(-4n), & \text{if } n \equiv 1, 2 \bmod 4 \\ 24h(-n), & \text{if } n \equiv 3 \bmod 8 \\ 0, & \text{if } n \equiv 0, 4, 7 \bmod 8. \end{cases}$$

Some infinite series

Theorem (Hurwitz 1905)

Let $\mathcal{T}$ be any topograph of discriminant $D < 0$ then

$$|D|^{3/2} \sum_{r \succ {s \atop t} \in \mathcal{T}} \frac{1}{|rst|} = 4\pi$$

where we sum over all vertices of $\mathcal{T}$, (each vertex contributing one term).

Duke, Imamoğlu and Tóth in [DIT 2021] reconsidered and extended Hurwitz's work. They used the Poincaré series $P(\tau; s_1, s_2, s_3)$ for $\tau \in \mathbf{H}$.

It is defined as

$$P(\tau; s_1, s_2, s_3) := \sum_{\gamma \in \Gamma} \mathcal{H}(\gamma\tau; s_1, s_2, s_3)$$

for $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ and the usual action $\gamma\tau$. Here

$$\mathcal{H}(\tau; s_1, s_2, s_3) := \frac{\mathrm{Im}(\tau)^{s_1+s_2+s_3}}{|\tau|^{2s_2}|\tau - 1|^{2s_3}}.$$

Proof requires:

- $P(\tau; 1, 1, 1) = 3\pi/2$
- zero $z_q = \frac{-b+\sqrt{D}}{2a}$ of $ax^2 + bx + c$ for $q = [a, b, c]$
- $\gamma z_q = z_{q|\gamma^{-1}}$
- $\mathcal{H}(z_q; 1, 1, 1) = \frac{|D|^{3/2}}{8} \frac{1}{ac(a+b+c)}$.

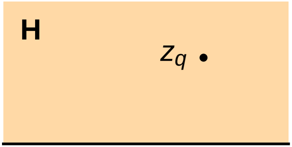
Theorem (DIT 2021, topograph version)

Let $\mathcal{T}$ be any topograph of non-square discriminant $D > 0$. Define $\mathcal{T}_\star$ to equal $\mathcal{T}$ except that all the river edges are relabeled with $\sqrt{D}$ when directed rightwards. Then

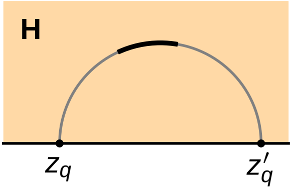
$$D^{3/2} \sum_{\substack{f \nwarrow \, \xrightarrow{g} \\ \swarrow e} \;\in\; \mathcal{T}_\star} \frac{1}{|efg|} = 2 \log \varepsilon_D ,$$

where we sum over all vertices of $\mathcal{T}_\star$ modulo the river period.

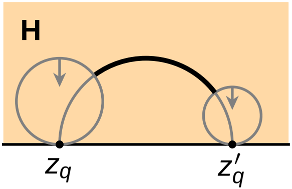
Here $2\varepsilon_D = u + \sqrt{D}v$ from the minimal positive solution to $u^2 - Dv^2 = 4$.



$D < 0$



$D > 0$ non-square



$D > 0$ square

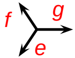
Define the period 1 function

$$W_1(x) := 2\mathrm{Re} \int_0^{\infty} \frac{y}{y^2+1} \cdot \frac{1}{e^{\pi(y+2ix)}-1}\, dy.$$

Theorem (O'S.)

Let $\mathcal{T}$ be any topograph of square discriminant $D = m^2 > 1$. Define $\mathcal{T}_\star$ as before. Denote by r and s the congruence classes mod m of the lake adjacent region labels. Then

$$W_1\left(\frac{r}{m}\right) + W_1\left(\frac{s}{m}\right) + m^3 \sum \frac{1}{|efg|} = 2\log\left(\frac{m}{2\gcd(m,r)}\right)$$

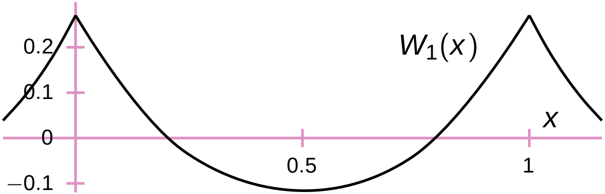


$\in \mathcal{T}_\star$

where we sum over all vertices of $\mathcal{T}_\star$ that are not on a lake.

Mystery function?

$$W_1(x) = 2\mathrm{Re} \int_0^\infty \frac{y}{y^2+1} \cdot \frac{1}{e^{\pi(y+2ix)}-1}\, dy$$



See my arXiv paper for more:

Topographs for binary quadratic forms and class numbers, 2024