Algebraic proof of modular form inequalities for optimal sphere packings

Seewoo Lee

April 29, 2025. International Seminar on Automorphic Forms, Online

Goal

- We develop simple but surprisingly useful tools to study *(completely) positive quasimodular forms*.
- Using the theory, we give *algebraic* proofs of Viazovska and Cohn–Miller–Kumar–Radchenko–Viazovska's modular form inequalities for the E_8 and Leech lattice packing in dimensions 8 and 24.
- We also prove a conjecture of Kaneko and Koike for the extremal forms in the case of depth 1.

Sphere packing

Question

For given $d \geq 1$, find an optimal sphere (in fact, ball) packing of $\mathbb{R}^d$ and its density Δ_d.

Sphere packing, $d = 1$

Sphere packing, $d = 1$

Theorem

$\Delta_1 = 1.$

Sphere packing, $d = 1$

Theorem

$\Delta_1 = 1.$

Proof.

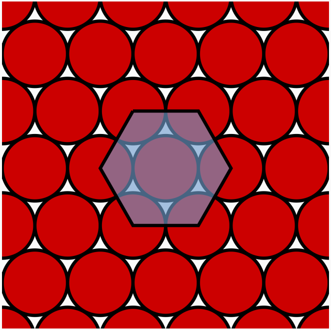
$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [2n - 1, 2n + 1] = \bigcup_{n \in \mathbb{Z}} \overline{B_1(2n)}.$ $\square$

Sphere packing, $d = 2$

Sphere packing, $d = 2$

Theorem (Thue 1890, Tóth 1942)

Hexagonal packing (A_2 lattice packing) is optimal with $\Delta_2 = \frac{\pi}{2\sqrt{3}}$.

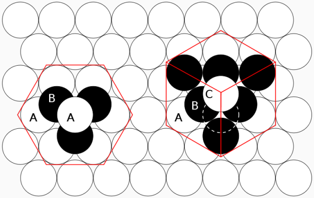


Spehere packing, $d = 3$

Spehere packing, $d = 3$

Theorem (Kepler conjecture, Hales 1998)

Cannon ball packing are optimal with $\Delta_3 = \frac{\pi}{3\sqrt{2}}$.



- Uncountably many optimal packings
- Computer-assisted, formally verified in 2014 using Isabelle + HOL light (with 20 more people)

Sphere packing, $d \geq 4$

Sphere packing, $d \geq 4$

Theorem

The following packings are optimal among **lattice** *packings.*

d	4	5	6	7	8	24
Lattice	D_4	D_5	D_6	E_7	E_8	Leech

- $d = 4, 5$ by Korkine and Zolotareff
- $d = 6, 7, 8$ by Blichfeldt
- $d = 24$ (and $d = 8$ again) by Cohn and Kumar

Conjecture

Above lattice packings are optimal among **all** *packings.*

Sphere packing

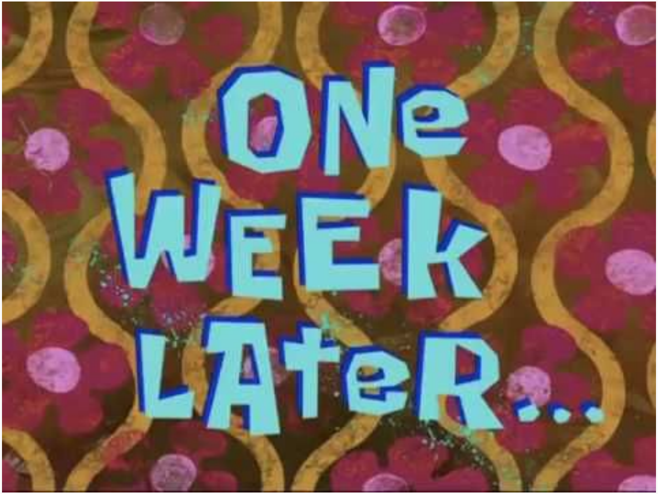
And...

Sphere packing, $d = 8$

Theorem (Viazovska, 2016 π-day on arXiv)

E_8 lattice packing is optimal with $\Delta_8 = \frac{\pi^4}{384}$.

$$E_8 = \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^8 : \sum_{i=1}^{8} x_i \equiv 0 \pmod 2 \right\} \subset \mathbb{R}^8$$


One Week Later...

Sphere packing, $d = 24$

Theorem (Cohn–Kumar–Miller–Radchenko–Viazovska, March 21st 2016 on arXiv)

Leech lattice packing is optimal with $\Delta_{24} = \frac{\pi^{12}}{12!}$.

Unique even unimodular lattice with nonzero minimial length $\lambda(\Lambda_{24}) = 2$. Can be constructed by the binary Golay code, Lorentzian lattice $II_{25,1}$, etc.

LP bound

How?

LP bound

How? We have a **Linear programming bound** for sphere packing:

Theorem (Cohn–Elkies, 2003)

Let $r > 0$. Assume that there exists a nice function $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

- $f(0) = \widehat{f}(0) > 0$,
- $f(x) \leq 0$ *for all* $\|x\| \geq r$,
- $\widehat{f}(y) \geq 0$ *for all* $y \in \mathbb{R}^d$.

Then

$$\Delta_d \leq \mathrm{vol}(B^d_{r/2}) = \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

LP bound

Sketch of the proof.

For lattice packing: let $\Lambda \subset \mathbb{R}^d$ be a lattice with minimum length r. By Poisson summation formula,

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \sum_{y \in \Lambda^*} \widehat{f}(y) \geq \frac{\widehat{f}(0)}{\mathrm{vol}(\mathbb{R}^d/\Lambda)}$$

and $f(0) = \widehat{f}(0) > 0$ gives

$$\mathrm{vol}(\mathbb{R}^d/\Lambda) \geq 1 \Leftrightarrow (\text{density}) = \frac{\mathrm{vol}(B^d_{r/2})}{\mathrm{vol}(\mathbb{R}^d/\Lambda)} \leq \mathrm{vol}(B^d_{r/2}).$$

Non-lattice packings can be approximated by a finite union of lattice packings, and the result follows similarly. $\square$

Hunt for magic function

Cohn and Elkies experimented with functions of the form (polynomial) $\times$ (gaussian), and the obtained upper bounds were surprisingly close to the conjectured bound in dimensions $d = 2, 8, 24$.

Hunt for magic function

Cohn and Elkies experimented with functions of the form (polynomial) $\times$ (gaussian), and the obtained upper bounds were surprisingly close to the conjectured bound in dimensions $d = 2, 8, 24$.

One can assume that f is radial, i.e. $f(x)$ only depends on the norm $\|x\|$ of the input (by averaging over each sphere).

Hunt for magic function

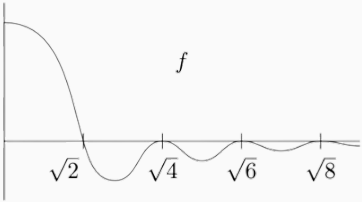
Cohn and Elkies experimented with functions of the form (polynomial) $\times$ (gaussian), and the obtained upper bounds were surprisingly close to the conjectured bound in dimensions $d = 2, 8, 24$.

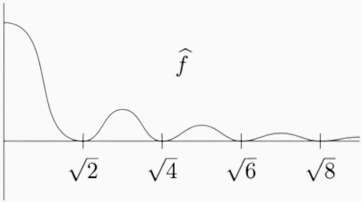
One can assume that f is radial, i.e. $f(x)$ only depends on the norm $\|x\|$ of the input (by averaging over each sphere).

If we follow the proof of LP bound that uses Poisson summation formula, both f and $\widehat{f}$ should have zeros at the nonzero lattice points, and nonpositivity (resp. nonnegativity) assumptions on f (resp. $\widehat{f}$) enforces them to be zeros of order 2 (except for the "first" zero of f).

Hunt for magic function

Hence f has a following form (for $d = 8$)





How to construct such a function? Under the philosophy of uncertainty principle, it is hard to control both f and $\widehat{f}$ at once.

Viazovska's construction

Viazovska (and colleagues) constructed the magic functions for $d = 8, 24$, using *modular forms*.

Viazovska's construction

Viazovska (and colleagues) constructed the magic functions for $d = 8, 24$, using *modular forms*.

Decompose f into Fourier eigenfunctions $f = f_+ + f_-$, where $\widehat{f_+} = f_+$ and $\widehat{f_-} = -f_-$. Viazovska write them as

$$f_\pm(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right)\int_0^\infty \varphi_\pm(t)e^{-\pi\|x\|^2 t}\mathrm{d}t,$$

where $\sin^2$ factor is included to enforce desired roots. Then $f_\pm$ being Fourier eigenfunctions correspond to $\varphi_\pm$ being "(quasi)modular forms".

Viazovska's construction

Viazovska (and colleagues) constructed the magic functions for $d = 8, 24$, using *modular forms*.

Decompose f into Fourier eigenfunctions $f = f_+ + f_-$, where $\widehat{f_+} = f_+$ and $\widehat{f_-} = -f_-$. Viazovska write them as

$$f_\pm(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right)\int_0^\infty \varphi_\pm(t)e^{-\pi\|x\|^2 t}\mathrm{d}t,$$

where $\sin^2$ factor is included to enforce desired roots. Then $f_\pm$ being Fourier eigenfunctions correspond to $\varphi_\pm$ being "(quasi)modular forms". Now the linear constraints (inequalities) on f and $\widehat{f}$ reduces to the modular inequalities

$$\varphi_+(t) + \varphi_-(t) < 0,$$
$$\varphi_+(t) - \varphi_-(t) > 0.$$

Modular forms

Definition

Let $\mathcal{H}$ be the complex upper half plane and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a **modular form of weight** k **and level** Γ if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $z \in \mathcal{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and satisfies nice growth condition at cusps.

- If $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, $f(z+1) = f(z)$ and hence f admits a Fourier expansion in $q = e^{2\pi i z}$ at ∞.

Modular forms

Examples:

- Eisenstein series

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n, \quad E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n$$

- Discriminant form (cusp form of level $\mathrm{SL}_2(\mathbb{Z})$, weight 12)

$$\Delta = (E_4^3 - E_6^2)/1728 = q \prod_{n \geq 1} (1 - q^n)^{24} = q - 24q^2 + \cdots .$$

- Jacobi thetanulle functions (level $\Gamma(2)$, weight 1/2)

$$\Theta_2 = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}\left(n + \frac{1}{2}\right)^2}, \quad \Theta_3 = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}}, \quad \Theta_4 = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}}$$

Quasimodular forms

Definition (informal)

Quasimodular forms are

- the functions act as modular forms but not exactly, or
- modular forms with E_2, or
- modular forms with differentiations.

Quasimodular forms

For example, $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$ satisfies

$$E_2\left(-\frac{1}{z}\right) = z^2 E_2(z) - \frac{6iz}{\pi}$$

and the ring of *quasi*modular forms (of level $\mathrm{SL}_2(\mathbb{Z})$) is generated by E_2, E_4, E_6, closed under the differentiation

$$f \mapsto \frac{1}{2\pi i}\frac{\mathrm{d}f}{\mathrm{d}z} = q\frac{\mathrm{d}f}{\mathrm{d}q}, \quad \sum_{n\geq 0} a_n q^n \mapsto \sum_{n\geq 0} n a_n q^n.$$

Quasimodular forms

We denote $\mathcal{QM}_w^s(\Gamma)$ for the space of quasimodular forms of weight w and *depth* $\leq s$, where depth is the degree of E_2 in the polynomial expression of the quasimodular form.

Differentiation increases weight by 2 and depth by 1, which can be computed using Ramanujan's identities

$$E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2 E_4 - E_6}{3}, \quad E_6' = \frac{E_2 E_6 - E_4^2}{2}.$$

$d = 8$

Recall that we set $f = f_+ + f_-$ where

$$f_\pm(x) = \sin^2\left(\frac{\pi\|x\|^2}{2}\right)\int_0^\infty \varphi_\pm(t)e^{-\pi\|x\|^2 t}\mathrm{d}t,$$

and find $\varphi_\pm$ such that $\widehat{f_\pm} = \pm f_\pm$. Viazovska proved that, if we put $\varphi_\pm(t) = t^2\psi_\pm(i/t)$ for some holomorphic $\psi_\pm : \mathcal{H} \to \mathbb{C}$,

$$\begin{aligned} \widehat{f_+} = f_+ &\Leftarrow \psi_+ \in \mathcal{QM}_0^{2,!}(\mathrm{SL}_2(\mathbb{Z})) \text{ such that } \dots \\ \widehat{f_-} = -f_- &\Leftarrow \psi_- \in \mathcal{QM}_{-2}^{0,!}(\Gamma(2)) \text{ such that } \dots \end{aligned}$$

Here ! stands for weakly holomorphic modular forms (i.e. allow poles at infinity). Viazovska's ansatz for $\psi_\pm$ was that $\psi_\pm\Delta$ are holomorphic modular forms.

$d = 8$

The actual modular forms are[1]

$$\psi_+ = -\frac{(E_2E_4 - E_6)^2}{\Delta}$$
$$\psi_- = -\frac{18}{\pi^2}\frac{\Theta_2^{12}(2\Theta_2^8 + 5\Theta_2^4\Theta_4^4 + 5\Theta_4^8)}{\Delta}$$

The corresponding integrals only converge for $\|x\| > \sqrt{2}$, and one needs to analytically continue to $0 \leq \|x\| \leq \sqrt{2}$. Then the inequalities $f \leq 0$ or $\widehat{f} \geq 0$ reduces to

$$\psi_+(it) + \psi_-(it) < 0, \quad \psi_+(it) - \psi_-(it) > 0.$$

[1]Here we normalized in a slightly different way. We have $f(0) = \widehat{f}(0) = \frac{5}{4\pi}$.

$d = 8$, modular form inequalities

For simplicity, we write

$$F = (E_2 E_4 - E_6)^2$$
$$G = H_2^3(2H_2^2 + 5H_2H_4 + 5H_4^2),$$

where $H_2 = \Theta_2^4$ and $H_4 = \Theta_4^4$. Then the inequalities for f and $\widehat{f}$ reduce to

$$F(it) + \frac{18}{\pi^2} G(it) > 0,$$
$$F(it) - \frac{18}{\pi^2} G(it) < 0.$$

$d = 8$, Viazovska's proof

Viazovska's original proof uses approximations of Fourier coefficients and reduce it to finite calculations + interval arithmetic (for both inequalities).

$d = 8$, Viazovska's proof

Viazovska's original proof uses approximations of Fourier coefficients and reduce it to finite calculations + interval arithmetic (for both inequalities).

More precisely, Viazovska used a bound of Fourier coefficients of the form

$$|c(n)| \leq 2e^{4\pi\sqrt{n}}$$

that comes from the Hardy–Ramanujan formula, and write the modular forms as

$$A(t) = \psi_+(it) + \psi_-(it) = A_\bullet^{(n)}(t) + R_\bullet^{(n)}(t)$$

with $\bullet \in \{0, \infty\}$ and $A_\bullet^{(n)}(t)$ is n-th approximation of $A(t)$ as $t \to \bullet$, then prove $|R_\bullet^{(n)}(t)| \leq |A_\bullet^{(n)}(t)|$ using interval arithmetic (numerical analysis). Similar proof for $B(t) = \psi_+(it) - \psi_-(it)$.

$d = 8$, Romik's proof

Recently (2023), Romik give an alternative and much simpler proof of $d = 8$ case that does not use any of interval arithmetic.

$d = 8$, Romik's proof

Recently (2023), Romik give an alternative and much simpler proof of $d = 8$ case that does not use any of interval arithmetic.

The first inequality is "easy": we have $F(it) > 0$ and $G(it) > 0$ separately (this was not clear form Viazovska's original expression of ψ_I).

$d = 8$, Romik's proof

Recently (2023), Romik give an alternative and much simpler proof of $d = 8$ case that does not use any of interval arithmetic.

The first inequality is "easy": we have $F(it) > 0$ and $G(it) > 0$ separately (this was not clear form Viazovska's original expression of ψ_I).

But the second inequality is still "hard": we need to compare modular forms of different weights (12 and 10). Romik considered the cases $0 < t < 1$ and $t \geq 1$ separately, and used various identities and monotonicity properties.

$d = 8$, Romik's proof

For example, we have

$$\frac{\pi^2}{18}F(z) = 28800\pi^2 q^2 + 1036800\pi^2 q^3 + 14169600\pi^2 q^4 +$$
$$G(z) = 20480q^{3/2} + 2015232q^{5/2} + 41656320q^{7/2} + \cdots .$$

Both F and G have nonnegative Fourier coefficients, so $e^{3\pi t}F(it)$ and $e^{3\pi t}G(it)$ are both monotone in t.

$d = 8$, Romik's proof

For example, we have

$$\frac{\pi^2}{18} F(z) = 28800\pi^2 q^2 + 1036800\pi^2 q^3 + 14169600\pi^2 q^4 +$$
$$G(z) = 20480 q^{3/2} + 2015232 q^{5/2} + 41656320 q^{7/2} + \cdots .$$

Both F and G have nonnegative Fourier coefficients, so $e^{3\pi t} F(it)$ and $e^{3\pi t} G(it)$ are both monotone in t. Using explicit values of modular forms like

$$E_2(i) = \frac{3}{\pi}, \quad E_4(i) = \frac{3\Gamma(1/4)^8}{64\pi^6}, \quad E_6(i) = 0,$$

we get a proof for $t \geq 1$:

$$e^{3\pi t} F(it) \leq e^{3\pi} F(i) = 13130.47 \cdots < 20480 < e^{3\pi t} G(it)$$

This gives a "calculator-assisted" proof. $0 < t < 1$ is more complicated.

$d = 8$, modular form inequalities

Question

Any **algebraic** *proofs? Can we homogenize the inequality?*

$d = 8$, homogenization

Let's rewrite it as

$$\frac{F(it)}{G(it)} < \frac{18}{\pi^2}$$

which is still inhomogenous.

$d = 8$, homogenization

Let's rewrite it as

$$\frac{F(it)}{G(it)} < \frac{18}{\pi^2}$$

which is still inhomogenous. How the function on the left hand side looks like?

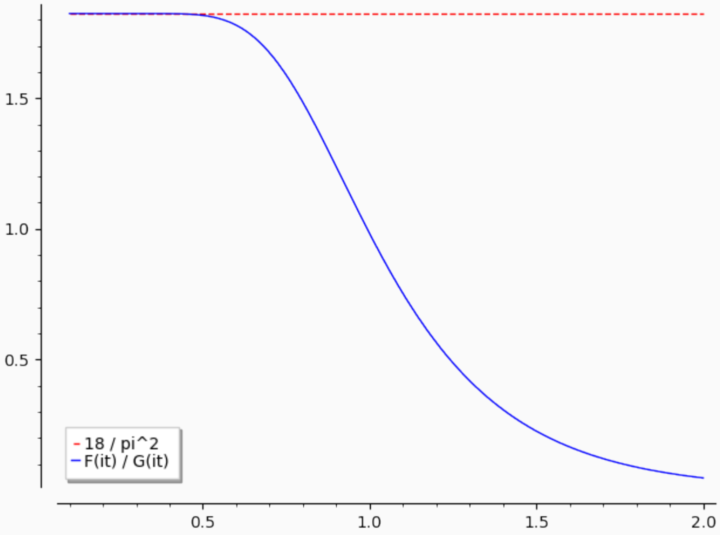
$d = 8$, homogenization

Let's rewrite it as

$$\frac{F(it)}{G(it)} < \frac{18}{\pi^2}$$

which is still inhomogenous. How the function on the left hand side looks like? Since I cannot plot it myself, let's ask SAGE...

$d = 8$, homogenization



$d = 8$, homogenization

This graph tells us what we should try:

Proposition

$$\lim_{t \to 0^+} \frac{F(it)}{G(it)} = \frac{18}{\pi^2}.$$

Proposition

The function

$$t \mapsto \frac{F(it)}{G(it)}$$

is decreasing in t.

and both turned out to be true.

$d = 8$: limit

Proof of the limit.

We have

$$\lim_{t\to 0^+} \frac{F(it)}{G(it)} = \lim_{t\to\infty} \frac{F(i/t)}{G(i/t)}$$

and F and G satisfy the following functional equations:

$d = 8$: limit

Proof of the limit.

We have

$$\lim_{t \to 0^+} \frac{F(it)}{G(it)} = \lim_{t \to \infty} \frac{F(i/t)}{G(i/t)}$$

and F and G satisfy the following functional equations:

$$F\left(\frac{i}{t}\right) = t^{12} F(it) - \frac{12t^{11}}{\pi}(E_2(it)E_4(it) - E_6(it))E_4(it) + \frac{36t^{10}}{\pi^2} E_4(it)^2,$$

$$G\left(\frac{i}{t}\right) = t^{10} H_4(it)^3 (2H_4(it)^2 + 5H_4(it)H_2(it) + 5H_2(it)^2).$$

The red terms are cusp forms, and the orange terms converges to 1. Hence the limit is $\frac{36/\pi^2}{2} = \frac{18}{\pi^2}$. □

$d = 8$: monotonicity

The monotonicity is equivalent to the *homogenous* inequality

$$F'(it)G(it) - F(it)G'(it) > 0.$$

Let's see what SAGE tells us...

$d = 8$: monotonicity

The monotonicity is equivalent to the *homogenous* inequality

$$F'(it)G(it) - F(it)G'(it) > 0.$$

Let's see what SAGE tells us... that the inequality is equivalent to

$$(H_2 + H_4)^2 H_4^2 (E_2 E_4 - E_6) \left(E_4 - \frac{1}{2} E_2 (H_2 + 2H_4) \right) > 0$$

First two terms are clearly positive, the third term is $(E_2 E_4 - E_6)(it) = 3E_4'(it) = 720 \sum_{n \geq 1} n\sigma_3(n) e^{-2\pi nt} > 0$.

$d = 8$: monotonicity

The last factor can be written as

$$E_4(it) - E_2(it)(2E_2(2it) - E_2(it)) > 0,$$

which is equivalent to

$$(E_4(it) - E_4(2it)) + (E_4(2it) - E_2(2it)^2) + (E_2(it) - E_2(2it))^2 > 0.$$

The first term is positive since

$$E_4(it) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) e^{-2\pi nt}$$

is monotone decreasing, and the second term is positive since

$$E_4(2it) - E_2(2it)^2 = -12E_2'(2it) = 288 \sum_{n \geq 1} n\sigma_1(n) e^{-4\pi nt} > 0.$$

Hence $F(it)/G(it) < \lim_{u \to 0^+} F(iu)/G(iu) = \frac{18}{\pi^2}$. $\square$

$d = 24$?

What about $d = 24$? The corresponding (quasi)modular forms are

$$\psi_+ = -\frac{F}{\Delta^2},$$
$$\psi_- = -\frac{432}{\pi^2}\frac{G}{\Delta^2},$$

where

$$F = 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2,$$
$$G = H_2^5(2H_2^2 + 7H_2H_4 + 7H_4^2).$$

$d = 24$?

$$
\begin{aligned}
F &= 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2,\\
G &= H_2^5(2H_2^2 + 7H_2H_4 + 7H_4^2).
\end{aligned}
$$

Then we need to prove the following *three* inequalities:[2]

$$
\begin{aligned}
F(it) + \frac{432}{\pi^2}G(it) &\geq 0,\\
F(it) - \frac{432}{\pi^2}G(it) &\leq 0.\\
t^{10}\left(-\frac{F(i/t)}{\Delta(i/t)^2} + \frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)^2}\right) &\geq \frac{725760}{\pi}e^{2\pi t}\left(t - \frac{10}{3\pi}\right).
\end{aligned}
$$

[2]The second inequality can only prove $\widehat{f}(r) > 0$ for $r \geq \sqrt{2}$, but not for $0 < r < \sqrt{2}$, and we need the third inequality for the remaining part.

$d = 24$?

But the "easy" inequality does not seem easy. $G(it) > 0$ is clear from the expression (and already observed by CKMRV), but for

$$F = 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2,$$

it is not clear why $F(it) > 0$.

And the second is harder, and the last inequality is much harder.

(Completely) positive quasimodular forms

To prove the 24-dimensional modular form inequalities, we develop some theory of **(completely) positive quasimodular forms**.

(Completely) positive quasimodular forms

Definition

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. We call $F \in \mathcal{QM}_w^s(\Gamma)$ a **positive quasimodular form** if it has real q-coefficients and

$$F(it) \geq 0$$

for all $t > 0$. We denote $\mathcal{QM}_w^{s,+}(\Gamma)$ for the set of positive quasimodular forms.

We call $F \in \mathcal{QM}_w^s(\Gamma)$ a **completely positive quasimodular form** if it has nonnegative real coefficients. We denote $\mathcal{QM}_w^{s,++}(\Gamma)$ for the set of completely positive quasimodular forms.

(Completely) positive quasimodular forms

We have $\mathcal{QM}_w^{s,++} \subseteq \mathcal{QM}_w^{s,+} \subseteq \mathcal{QM}_w^s$, and the two sets form a convex cone in $\mathcal{QM}_w^s$.

The inclusion is strict in general:
$\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + \cdots$ is positive but not completely positive.

Positive forms and derivatives

Proposition

1. *If F is a cusp form and $F' \in \mathcal{QM}_w^{s,+}$, then $F \in \mathcal{QM}_{w-2}^{s-1,+}$.*
2. *If $F \in \mathcal{QM}_w^{s,++}$, then $F^{(r)} \in \mathcal{QM}_{w+2r}^{s+r,++}$ for all $r \geq 0$.*

Positive forms and Serre derivatives

Definition

For $k \in \mathbb{Z}$ and $F \in \mathcal{QM}_w^s(\Gamma)$, define **Serre derivative** $\partial_k F$ of F as

$$\partial_k F = F' - \frac{k}{12} E_2 F.$$

A priori, $\partial_k F \in \mathcal{QM}_{w+2}^{s+1}(\Gamma)$. However,

Proposition

When $k = w - s$, ∂_{w-s} maps $F \in \mathcal{QM}_w^s$ to $\partial_{w-s} F \in \mathcal{QM}_{w+2}^s$.

For example, $E_2' = \frac{E_2^2 - E_4}{12}$ and $\partial_1 E_2 = -\frac{E_4}{12} \in \mathcal{QM}_4^0 = \mathcal{QM}_4^1$.

Positive forms and Serre derivatives

Proposition

Let $F = \sum_{n \geq n_0} a_n q^n \in \mathcal{QM}_w^s$ be a quasimodular form of real Fourier coefficients and $a_{n_0} > 0$. If $\partial_k F \in \mathcal{QM}_{w+2}^{s+1,+}$ for some k, then $F \in \mathcal{QM}_w^{s,+}$.

In other words, anti-Serre-derivative preserves positivity.

Positive forms and Serre derivatives

Proposition

Let $F = \sum_{n \geq n_0} a_n q^n \in \mathcal{QM}_w^s$ be a quasimodular form of real Fourier coefficients and $a_{n_0} > 0$. If $\partial_k F \in \mathcal{QM}_{w+2}^{s+1,+}$ for some k, then $F \in \mathcal{QM}_w^{s,+}$.

In other words, anti-Serre-derivative preserves positivity.

Proof.

$a_{n_0} > 0$ implies $F(it) > 0$ for large $t > 0$. From $\Delta' = E_2 \Delta$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{F(it)}{\Delta(it)^{k/12}}\right) = (-2\pi)\frac{(\partial_k F)(it)}{\Delta(it)^{k/6}} < 0.$$

$\square$

Positive forms and Serre derivatives

Proposition

Let $F = \sum_{n \geq n_0} a_n q^n \in \mathcal{QM}_w^{s,++}$. *For* $k \geq 0$ *and* $n \geq k/12$, *the* n*-th coefficient of* $\partial_k F$ *is nonnegative. Especially, if* $n_0 \geq k/12 \geq 0$, *then* $\partial_k F$ *is also completely positive.*

In other words, Serre derivative preserves complete positivity (under mild assumption on the vanishing order at cusp).

Positive forms and Serre derivatives

Proof.

From $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$, $\partial_k F$ has a q-expansion

$$\left(n_0 - \frac{k}{12}\right)a_{n_0}q^{n_0} + \left(\left(n_0 + 1 - \frac{k}{12}\right)a_{n_0+1} + 2ka_{n_0}\right)q^{n_0+1} + \cdots$$
$$+ \left(\left(n_0 + m - \frac{k}{12}\right)a_{n_0+m} + 2k\sum_{j=1}^{m}\sigma_1(m+1-j)a_{n_0+j-1}\right)q^{n_0+m} + \cdots$$

and the result follows. □

Extremal forms

Definition (Kaneko–Koike)

For a given weight w and depth s, **extremal quasimodular form of weight** w **and depth** w, $X_{w,s}$, is a quasimodular form of *largest possible vanishing order at the cusp*. More precisely, $X_{w,s}$ admits a q-expansion

$$X_{w,s} = \sum_{n \geq m} a_n q^n$$

where $m = \dim_{\mathbb{C}} \mathcal{QM}_w^s - 1$ and $a_m \neq 0$.

Examples

$$X_{6,1} = \frac{E_2E_4 - E_6}{720} = q + 18q^2 + 84q^3 + 292q^4 + 630q^5 + \cdots$$

$$X_{8,1} = \frac{-E_2E_6 + E_4^2}{1008} = q + 66q^2 + 732q^3 + 4228q^4 + 15630q^5 + \cdots$$

$$X_{4,2} = \frac{-E_2^2 + E_4}{288} = q + 6q^2 + 12q^3 + 28q^4 + 30q^5 + \cdots$$

$$X_{8,2} = \frac{-7E_2^2E_4 + 2E_2E_6 + 5E_4^2}{362880} = q^2 + 16q^3 + 102q^4 + 416q^5 + \cdots$$

$$X_{6,3} = \frac{5E_2^3 - 3E_2E_4 - 2E_6}{51840} = q^2 + 8q^3 + 30q^4 + 80q^5 + \cdots$$

Uniqueness, existence, and computation

Theorem (Pellarin)

For $1 \leq s \leq 4$*, extremal forms of weight* w *and depth* s *is unique up to constant.*

Theorem (Kaneko–Koike, Grabner)

For $1 \leq s \leq 4$*, we have recurrence relations and differential equations satisfied by the extremal forms.*

Recurrence relations, $s = 1$

For $w \equiv 0 \pmod 6$,

$$\begin{aligned} X_{w+2,1} &= \frac{12}{w+1}\partial_{w-1}X_{w,1}, \\ X_{w+4,1} &= E_4 X_{w,1}, \\ X_{w+6,1} &= \frac{w+6}{72(w+1)(w+5)}\left(E_4\partial_{w-1}X_{w,1} - \frac{w+1}{12}E_6X_{w,1}\right) \\ &= \frac{w+6}{864(w+5)}\left(E_4X_{w+2,1} - E_6X_{w,1}\right), \end{aligned}$$

and

$$X''_{w,1} - \frac{w}{6}E_2X'_{w,1} + \frac{w(w-1)}{144}(E_2^2 - E_4)X_{w,1} = 0.$$

Kaneko–Koike conjecture

Conjecture (Kaneko–Koike)

Extremal forms of depth $1 \leq s \leq 4$ *have nonnegative q-coefficients.*

Theorem (Grabner)

Conjecture is true **for all but finitely many coefficients** *(for each form).*

Proof uses Deligne's bound: if we write $a_n = a_{n,\mathrm{Eis}} + a_{n,\mathrm{cusp}}$, $a_{n,\mathrm{Eis}} \gg a_{n,\mathrm{cusp}}$ as $n \to \infty$. Using effective version of Deligne's bound (e.g. Jenkins–Rouse), one can check nonnegativity for all n's when given w, s are small.

Kaneko–Koike conjecture for $s = 1$

Proposition (L.)

For $w \equiv 0 \pmod 6$ and $w \geq 12$, we have

Kaneko–Koike conjecture for $s = 1$

Proposition (L.)

For $w \equiv 0 \pmod 6$ and $w \geq 12$, we have

$$X'_{w,1} = \frac{5w}{72} X_{6,1} X_{w-4,1} + \frac{7w}{72} X_{8,1} X_{w-6,1}.$$

$$X'_{w+2,1} = \frac{5w}{72} X_{6,1} X_{w-2,1} + \frac{7w}{12} X_{8,1} X_{w-4,1}$$

$$X'_{w+4,1} = 240 X_{6,1} X_{w,1} + \frac{7w}{72} X_{8,1} X_{w-2,1} + \frac{5w}{72} X_{10,1} X_{w-4,1}.$$

Kaneko–Koike conjecture for $s = 1$

Proposition (L.)

For $w \equiv 0 \pmod 6$ and $w \geq 12$, we have

$$X'_{w,1} = \frac{5w}{72} X_{6,1} X_{w-4,1} + \frac{7w}{72} X_{8,1} X_{w-6,1}.$$

$$X'_{w+2,1} = \frac{5w}{72} X_{6,1} X_{w-2,1} + \frac{7w}{12} X_{8,1} X_{w-4,1}$$

$$X'_{w+4,1} = 240 X_{6,1} X_{w,1} + \frac{7w}{72} X_{8,1} X_{w-2,1} + \frac{5w}{72} X_{10,1} X_{w-4,1}.$$

Corollary

Conjecture is true for depth 1 extremal forms.

Kaneko–Koike conjecture for $s = 2$

We also have similar proof for depth 2 extremal forms of weight $w \leq 14$:

$$\begin{aligned} X'_{8,2} &= 2X_{4,2}X_{6,1} \\ X'_{10,2} &= \frac{8}{9}X_{4,2}X_{8,1} + \frac{10}{9}X_{6,1}^2 \\ X'_{12,2} &= 3X_{6,1}X_{8,2} \\ X'_{14,2} &= 3X_{4,2}X_{12,1} \end{aligned}$$

but we don't have a proof for general cases yet.

$d = 24$ inequalities

Recall that our goal is to prove the following inequalities: for

$$F = 49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2$$
$$G = H_2^5(2H_2^2 + 7H_2H_4 + 7H_4^2),$$

we have

$$F(it) + \frac{432}{\pi^2}G(it) \geq 0,$$
$$F(it) - \frac{432}{\pi^2}G(it) \leq 0,$$
$$t^{10}\left(-\frac{F(i/t)}{\Delta(i/t)^2} + \frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)^2}\right) \geq \frac{725760}{\pi}e^{2\pi t}\left(t - \frac{10}{3\pi}\right)$$

$d = 24$ inequalities: "easy"

It is clear that $G(it) > 0$ from definition. It is less clear for F, but SAGE says...

$d = 24$ inequalities: "easy"

It is clear that $G(it) > 0$ from definition. It is less clear for F, but SAGE says...

Proposition

$$\partial_{14} F =$$

$d = 24$ inequalities: "easy"

It is clear that $G(it) > 0$ from definition. It is less clear for F, but SAGE says...

Proposition

$$\partial_{14} F = 6706022400 X_{6,1} X_{12,1} \in \mathcal{QM}_{18}^{2,++}.$$

Corollary

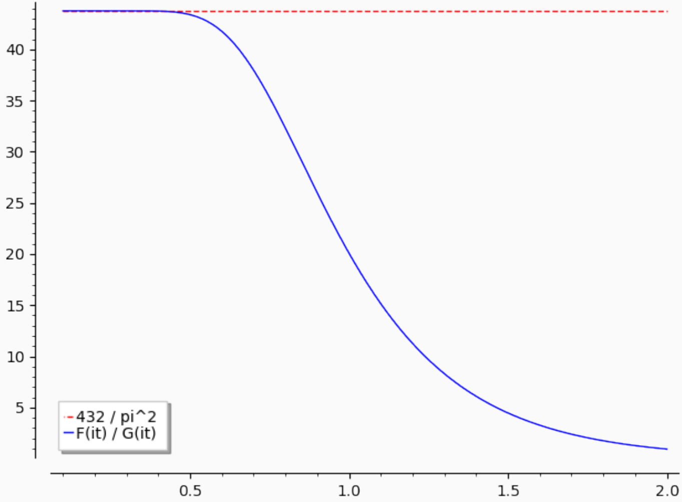
$F(it) \geq 0$ for all $t > 0$.

$d = 24$ inequalities: "hard"

For the second inequality, we have a similar plot:

$d = 24$ inequalities: "hard"

For the second inequality, we have a similar plot:



$d = 24$ inequalities: "hard"

Based on the previous observations, second (hard) inequality would follow from

Proposition

$$\lim_{t \to 0^+} \frac{F(it)}{G(it)} = \frac{432}{\pi^2}.$$

and

Proposition

The function $t \mapsto \frac{F(it)}{G(it)}$ *is strictly decreasing on* $t > 0$.

$d = 24$ inequalities: "hard"

Based on the previous observations, second (hard) inequality would follow from

Proposition

$$\lim_{t \to 0^+} \frac{F(it)}{G(it)} = \frac{432}{\pi^2}.$$

and

Proposition

The function $t \mapsto \frac{F(it)}{G(it)}$ *is strictly decreasing on* $t > 0$.

We leave the first limit as an exercise for audiences.

$d = 24$ inequalities: "hard"

Monotonicity is equivalent to

$$\mathcal{L}_{1,0} := F'G - FG' > 0.$$

We use Serre derivative trick again. Note that

$$\mathcal{L}_{1,0} = 1342429609328640 0q^{\frac{11}{2}} + 494781198866841600q^{\frac{13}{2}} + O(q^{\frac{15}{2}})$$

and has weight 32 and depth 2. If we apply $\partial_{30} = \partial_{32-2}$, we get

$$\mathcal{L}_{2,0} := (\partial_{14}^2 F)G - F(\partial_{14}^2 G) = \partial_{30}\mathcal{L}_{1,0}$$

(where $\partial_{14}^2 = \partial_{16}\partial_{14}$) and it is enough to show that $\mathcal{L}_{2,0}$ is positive.

$d = 24$ inequalities: “hard” (New proof)

Now, surprisingly, F and G satisfy the following differential equations:

$d = 24$ inequalities: "hard" (New proof)

Now, surprisingly, F and G satisfy the following differential equations:

$$\partial_{14}^2 F = \frac{14}{9} E_4 F + c\Delta X_{8,2},$$
$$\partial_{14}^2 G = \frac{14}{9} E_4 G$$

for $c = 548674560$. This gives

$$\mathcal{L}_{2,0} = c\Delta X_{8,2} G > 0$$

and we get $\mathcal{L}_{1,0} > 0$. $\square$

$d = 24$ inequalities: "hard" (New proof)

- Kaneko and Zagier introduced a modular differential operator[3]

$$L_{2,k} := \partial_k^2 - \frac{k(k+2)}{144} E_4 : \mathcal{M}_k(\Gamma) \to \mathcal{M}_{k+4}(\Gamma)$$

and the above identities show $L_{2,14} F > 0$ and $L_{2,14} G = 0$.

- Similar proof also works for $d = 8$ case: we have

$$L_{2,10} F = \partial_{10}^2 F - \frac{5}{6} E_4 F = 172800 \Delta X_{4,2} > 0,$$
$$L_{2,10} G = \partial_{10}^2 G - \frac{5}{6} E_4 G = -640 \Delta H_2 < 0$$

and this gives $\partial_{22} \mathcal{L}_{1,0} = \mathcal{L}_{2,0} > 0$.

[3] Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials, 1998

$d = 24$ inequalities: "harder"

We have one more inequality left:

$$t^{10}\left(-\frac{F(i/t)}{\Delta(i/t)^2} + \frac{432}{\pi^2}\frac{G(i/t)}{\Delta(i/t)^2}\right) \geq \frac{725760}{\pi}e^{2\pi t}\left(t - \frac{10}{3\pi}\right)$$

for $t \geq 1$. Note that $0 \leq t < 1$ case follows from "hard" inequality.

$d = 24$ inequalities: “harder”

LHS is positive (for all $t > 0$) due to “hard” inequality, and RHS is nonpositive for $t \leq \frac{10}{3\pi}$. Hence it is enough to prove the inequality for $t > \frac{10}{3\pi}$.

Now, the follwoing simple inequality removes exponential term:

Proposition

For all $t > 0$, $\Delta(it) < e^{-2\pi t}$.

$d = 24$ inequalities: "harder"

LHS is positive (for all $t > 0$) due to "hard" inequality, and RHS is nonpositive for $t \leq \frac{10}{3\pi}$. Hence it is enough to prove the inequality for $t > \frac{10}{3\pi}$.

Now, the follwoing simple inequality removes exponential term:

Proposition

For all $t > 0$, $\Delta(it) < e^{-2\pi t}$.

Proof.

$$\Delta(it) = e^{-2\pi t} \prod_{n \geq 1} (1 - e^{-2\pi n t})^{24} < e^{-2\pi t}.$$

$\square$

$d = 24$ inequalities: "harder"

Using the above inequality & substitute t with $1/t$, the inequality reduces to

$$\frac{432}{\pi^2} - \frac{F(it)}{G(it)} \geq \frac{725760\Delta(it)}{G(it)} \left(\frac{1}{\pi t^3} - \frac{10}{\pi^2 t^2}\right)$$

for $0 < t < \frac{3\pi}{10}$.

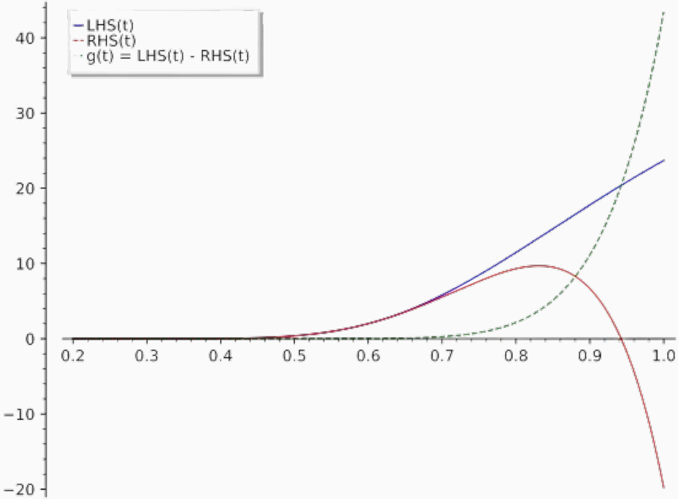
$d = 24$ inequalities: "harder"

Using the above inequality & substitute t with $1/t$, the inequality reduces to

$$\frac{432}{\pi^2} - \frac{F(it)}{G(it)} \geq \frac{725760\Delta(it)}{G(it)} \left(\frac{1}{\pi t^3} - \frac{10}{\pi^2 t^2} \right)$$

for $0 < t < \frac{3\pi}{10}$. SAGE, please tell me something again...

$d = 24$ inequalities: "harder"



$d = 24$ inequalities: "harder"

From this, we can try to prove:

Proposition

The function

$$g(t) := \frac{432}{\pi^2} - \frac{F(it)}{G(it)} - \frac{725760\Delta(it)}{G(it)}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right)$$

is monotone increasing in t *for* $0 < t < \frac{3\pi}{10}$ *and* $\lim_{t\to 0^+} g(t) = 0$. *Especially, we have* $g(t) > 0$ *for all* $0 < t < \frac{3\pi}{10}$.

As before, limit part is easy and left as an exercise for you.

$d = 24$ inequalities: "harder"

Direct computation shows that $\mathrm{d}g/\mathrm{d}t > 0$ is equivalent to

$$\mathcal{L}_{1,0}(it) - 725760\Delta(it)\left[(\partial_{12}G)(it)\left(\tfrac{1}{\pi t^3} - \tfrac{10}{3\pi^2 t^2}\right) - G(it)\left(\tfrac{3}{2\pi^2 t^4} - \tfrac{10}{3\pi^3 t^3}\right)\right] > 0.$$

If we denote above as $\widetilde{\mathcal{L}}_{1,0}$, then $\widetilde{\mathcal{L}}(\frac{3\pi i}{10}) > 0$ and it is enough to prove $\partial_{30}\widetilde{\mathcal{L}}_{1,0}(it) > 0$ for $0 < t < \frac{3\pi}{10}$. Surprisingly, ΔG factors out and it reduces to the positivity of

$$7560 X_{8,2}(it) - \tfrac{37E_4(it) - E_2(it)^2}{24}\left(\tfrac{1}{\pi t^3} - \tfrac{10}{3\pi^2 t^2}\right) - E_2(it)\left(\tfrac{3}{4\pi^2 t^4} - \tfrac{5}{3\pi^3 t^3}\right) + \left(\tfrac{3}{\pi^3 t^5} - \tfrac{5}{\pi^4 t^4}\right).$$

$d = 24$ inequalities: "harder"

$$7560X_{8,2}(it) - \frac{37E_4(it)-E_2(it)^2}{24}\left(\frac{1}{\pi t^3} - \frac{10}{3\pi^2 t^2}\right) - E_2(it)\left(\frac{3}{4\pi^2 t^4} - \frac{5}{3\pi^3 t^3}\right) + \left(\frac{3}{\pi^3 t^5} - \frac{5}{\pi^4 t^4}\right).$$

If we denote this as $h(t)$, then $t^{-8}h(1/t)$ can be written as

$$\frac{1}{t^8}h\left(\frac{1}{t}\right) = 7560X_{8,2}(it) + \frac{1}{\pi t}\left[\left(\frac{3}{10} - \frac{1}{\pi t}\right)J_1(it) + \frac{3}{40}J_2(it) + \frac{7}{4}J_3(it)\right]$$

where

$$J_1 = \frac{5}{3}E_2' - \frac{1}{4}E_2 + \frac{1}{4}E_4$$
$$J_2 = E_2 - E_6$$
$$J_3 = 3E_4' + \frac{9}{10}E_6 - \frac{9}{10}E_4$$

so it is enough to prove $J_k(it) > 0$ for $\frac{1}{t} < \frac{3\pi}{10} \Leftrightarrow t > \frac{10}{3\pi}$.

$d = 24$ inequalities: "harder"

We can compute Fourier coefficients of these forms explicitly, and prove that J_1 and J_2 are completely positive. For J_3, we have $J_3 = \sum_{n \geq 1} a_n q^n$ with $a_1 > 0$ and $a_n < 0$. Hence

$$t \mapsto e^{2\pi t} J_3(it) = a_1 + \sum_{n \geq 2} a_n e^{-2\pi n t}$$

is increasing, and

$$e^{2\pi t} J_1(it) > e^{2\pi} J_1(i) = e^{2\pi} \left(\frac{3}{\pi} - \frac{9}{10} \right) E_4(i) > 0 \Rightarrow J_3(it) > 0$$

for $t \geq 1$, hence for $t > \frac{10}{3\pi}$. $\square$

Future works

What's next?

Characterize (completely) positive quasimodular forms

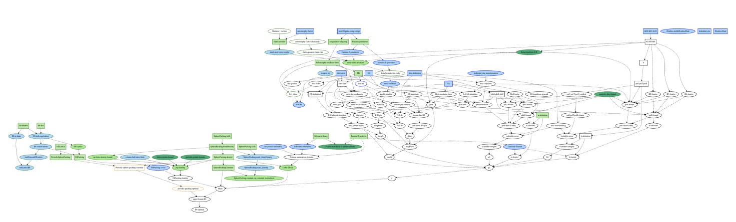
- Generating functions of geometric objects
  - (Kaneko–Zagier) $X_{6,3}$ counts certain coverings of elliptic curves.
- "Generators" of $\mathcal{QM}^{+}_{w,s}$ and $\mathcal{QM}^{++}_{w,s}$?
- (Sakai–Tsutsumi) Higher levels?
  - **Theorem** (L.) Let $\mathcal{D}_w$ be the unique level $\Gamma_0(2)$ extremal form of weight w and depth 1. Then $\mathcal{D}_w$ and $\mathcal{D}'_w$ are positive.
  - **(almost) Theorem** (L.) Extremal quasimodular forms of level $\Gamma_0(2)$, depth 2 and weight w are completely positive if and only if $w \in \{4, 6, 8, 10, 12, 14, 18\}$.
- Better (algebraic) proof for the inequalities associated to universality of E_8 and Leech lattices?
  - (L.) There is an *algebraic* proof for the inequality (3b).

Other LP problems

- (Cohn–Triantafillou) Dual LP
- (Brougain–Clozel–Kahane, Cohn–Gonçalves) Uncertainty principle
- Any results that are "uniform" in dimensions?
  - (Feigenbaum–Grabner–Hardin) (F, G) pairs for other dimensions $\equiv 0 \pmod 4$, although they do *not* give a magic function (except for $d = 8, 24$)
  - **Theorem** (L.) Kaneko–Koike conjecture for depth 2 implies positivity of FGH's "$(-1)^{d/4}$" family of modular forms.
- We need new summation formulae and new (magic) functions

Make formalization easier

- thefundamentaltheor3m / **Sphere-Packing-Lean** (WIP)
- Lead by Sidharth Hariharan (for their master's thesis!) and with Chris Birkbeck, Gareth Ma, Maryna Viazovska, ...
- Blueprint:



- Will be open to public on June and announced on *Big Proof* conference.

Paper: `arxiv.org/abs/2406.14659`

Code: `github.com/seewoo5/posqmf`

Thank you!