Orthogonal Eisenstein series of singular weight:

1. Orthogonal Modular Forms:

Let L be an even lattice of signature $(2,l)$, $l > 4$, $l \equiv 2 \mod 4$. Write $V = L \otimes \mathbb{Q}$, $V(\mathbb{R}) = L \otimes \mathbb{R}$.

$$L = \underbrace{U}_{\substack{\shortparallel \\ \mathbb{Z}z \oplus \mathbb{Z}z'}} \overset{\perp}{\oplus} \underbrace{U \overset{\perp}{\oplus} D}_{K}$$

U hyperbolic plane, i.e. $U = \mathbb{Z}e \oplus \mathbb{Z}f$, $(e,f) = 1$, $(e,e) = (f,f) = 0$

Consider

$$\mathbb{H}_l = \{ Z = X + iY \in K \otimes \mathbb{C} \mid q(Y) > 0 \}^+ \quad \text{pick one component}$$

$$= K \otimes \mathbb{R} \oplus i\,\mathcal{C} \qquad \mathcal{C} = \{ Y \in K \otimes \mathbb{R} \mid q(Y) > 0 \}^+$$

(",$\mathcal{H}$" written over $\mathbb{H}_l$)

$O^+(V(\mathbb{R})) \underset{\text{index } 2}{\subseteq} O(V(\mathbb{R}))$ acts $\mathbb{H}_l$.

Let L' be the dual lattice of L and

$\Gamma(L) := \ker(O^+(L) \to O(L'/L))$ „discriminant kernel"

„$SL_2(\mathbb{Z})$"

$$j(\sigma, Z) := (z, \sigma(\underbrace{z - q(Z)z + z'}_{Z_L}))$$

„factor of automorphy"

<u>Def.:</u>

A function $F : \mathbb{H}_c \to \mathbb{C}$ is called modular of weight $\kappa \in \mathbb{Z}$ w.r.t. $\Gamma(L)$, if

$$F(\sigma Z) = j(\sigma, Z)^\kappa \, F(Z), \quad \text{for all } \sigma \in \Gamma(L), Z \in \mathbb{H}_c.$$

F is a hol. modular form, if F is holomorphic and modular.

$M_\kappa(\Gamma(L))$ ~~for~~ = space of hol. mod. forms.

<u>Boundary:</u> (of $\Gamma(L)\backslash\mathbb{H}_c$)

By theory of Baily-Borel:

$$\left\{\begin{matrix}\text{0-dim cusps} \\ \text{of } \Gamma(L)\backslash\mathbb{H}_c \\ \text{(points)}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \Gamma(L)\backslash\left\{\begin{matrix}\text{isotropic} \\ \text{lines in } L\end{matrix}\right\}$$

$$\left\{\begin{array}{c}\text{1-dim cusps}\\ (\cong \Gamma\backslash \mathbb{H},\ \Gamma \subseteq SL_2(\mathbb{Q}))\end{array}\right\} \overset{1:1}{\longleftrightarrow} \Gamma(L) \Big\backslash \left\{\begin{array}{c}\text{isotropic}\\ \text{planes in } L\end{array}\right\}$$

3.7 $F : \mathcal{H}_c \to \mathbb{C} \in M_k(\Gamma(L))$ and I 1-dim cusp, one associates „boundary value" of F at I

$$F|_I : \mathbb{H} \to \mathbb{C} \in M_k(\Gamma_I).$$

For 0-dim cusps I one can define the value of F in the cusp I.

<u>Def.:</u>

$F \in M_k(\Gamma(L))$ is called cusp form if $F|_I = 0$ for all I. $\rightsquigarrow$ $S_k(\Gamma(L))$.

We write $M_k^{\partial Eis}(\Gamma(L))$ for the space of hol. mod. forms. s.t. $F|_I$ is an Eisenstein series for 1-dim. cusps I.

<u>Rem.:</u>

~~Smallest pos. weight for which M_k~~

$$M_\kappa(\Gamma(L)) = \{0\} \quad \text{for} \quad 0 < \kappa < \underbrace{\tfrac{l}{2} - 1}_{\text{singular weight}}.$$

In singular weight there cusp forms.

$\Rightarrow$ a hol. mod. form of singular weight is fully determined by its boundary values.

Eisenstein series: (for 0-dim. cusps)

Let $\lambda \in \mathrm{Iso}_0(L') = \{\lambda \in L' \text{ primitive} \mid q(\lambda) = 0\}$.

$\sigma_\lambda \in O^+(V)$, $\sigma_\lambda \lambda = z$. Define

$$\mathcal{E}_{\kappa,\lambda}(Z,s) := \sum_{\sigma \in \Gamma(L)_\lambda \backslash \Gamma(L)} q(Y)^s \big|_\kappa \sigma_\lambda \sigma, \quad \mathrm{Re}(s) > \frac{l-\kappa}{2}.$$

$$(F|_\kappa \sigma)(Z) := j(\sigma, Z)^{-\kappa} F(\sigma Z).$$

$$\mathcal{E}_{\kappa,\lambda}(Z) \text{ ``} := \text{''} \ \mathcal{E}_{\kappa,\lambda}(Z, 0).$$

From now on

$\kappa = \frac{l}{2} - 1$

Vector-valued Eisenstein series:

$\mathbb{C}[L'/L]$ vector space with basis e_γ, $\gamma \in L'/L$.

We ~~can~~ have unitary Weil rep $\rho_L : SL_2(\mathbb{Z}) \to GL(\mathbb{C}[L'/L])$

Def.:

Mod. form of weight k w.r.t. ρ_L is

$$f : \mathbb{H} \to \mathbb{C}[L'/L], \quad f(M\tau) = (c\tau+d)^k \rho_L(M) f(\tau), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

$$(f|_{k,L} M)(\tau) := (c\tau+d)^{-k} \rho_L(M)^{-1} f(M\tau).$$

Ex. $M_0(\rho_L) = \underbrace{\mathrm{Inv}}_{\substack{\text{inv. vectors} \\ \text{w.r.t. Weil rep.}}}(\mathbb{C}[L'/L])$.

Let $\beta \in \mathrm{Iso}(L'/L) = \{\alpha \in L'/L \mid q(\alpha) = 0 \in \mathbb{Q}/\mathbb{Z}\}$.

Define

$$E_\beta(\tau, s) := \sum_{M \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} (y^s e_\beta)|_{0,L} M, \quad \Gamma_\infty = \langle T^n \rangle.$$

⇝ . mer. cont. to $s \in \mathbb{C}$,

. functional equ. $E_\beta(\tau, s) = \frac{1}{2} \sum_{\alpha \in \mathrm{Iso}(L'/L)} \underbrace{c_\beta(\alpha, 0, s)}_{\text{certain Fourier coeff. of } E_\beta} E_\alpha(\tau, 1-s)$

. simple pole at $s=1$ with

$$\mathrm{res}_{s=1} E_\beta(\tau, s) \in \mathrm{Inv}(\mathbb{C}[L'/L]).$$

Theta-Lift:

Consider Siegel theta fctn

$$\Theta_L(\tau, Z) = \frac{v^{1/2}}{2} \sum_{\lambda \in L'} \frac{(\lambda, Z_L)^\kappa}{q(Y)^\kappa} e_\lambda\Big(iv \underbrace{q_Z(\lambda)}_{\substack{\text{a certain} \\ \text{pos. def.} \\ \text{quadratic for} \\ \text{corr. } Z}} + u q(\lambda)\Big).$$

using poisson summ. one shows Θ_L is mod. of weight 0 in τ.

$\overline{\Theta}_L$ is modular of weight κ in Z. w.r.t. $\Gamma(L)$.

Regularized theta lift

$$\Phi_\beta(Z, s) := \int_{\mathcal{F}}^{reg} \langle E_\beta(\tau, s), \Theta_L(\tau, Z) \rangle \frac{du\, dv}{v^2}$$

Consider

$$\pi_2 : \underbrace{\Gamma(L)\backslash \mathrm{Iso}_0(L')} \longrightarrow \mathrm{Iso}(L'/L)$$

this map is bijective (Eichler criterion, since we assumed that L splits 2 hyp. planes).

For $\delta \in \mathrm{Iso}(L'/L) \rightsquigarrow \mathcal{E}_{\kappa,\delta}(Z,s)$.

Thm.:

$$\Phi_\beta(Z,s) = \frac{\Gamma(s+\kappa)}{(-2\pi i)^{\kappa}\,\pi^{s}} \sum_{\substack{\delta\in \mathrm{Iso}(L'/L)\\ k_{\delta\beta}\delta=\beta}} \mathrm{ord}(\delta)^{2s+\kappa}\, \zeta_+^{k_{\delta\beta}}(2s+\kappa)\, \mathcal{E}_{\kappa,\delta}(Z,s)$$

$$\zeta_+^{k_{\delta\beta}}(2s+\kappa) = \sum_{\substack{n>0\\ n\equiv k_{\delta\beta} \bmod \mathrm{ord}(\delta)}} n^{-2s-\kappa}$$

Moreover the spaces generated by $\Phi_\beta(Z,s)$ and $\mathcal{E}_{\kappa,\delta}(Z,s)$ coincide.

$\rightsquigarrow$ meromorphic cont. + fctl. equ.

$s=1$:

Thm.:

Let $k = \frac{l}{2} - 1$. Then $\Phi_\beta(Z,s)$ has a simple pole at $s=1$

with residue

$$\operatorname{res}_{s=1} \Phi_\beta(Z,s) = \Phi(Z, \operatorname{res}_{s=1} E_\beta(\tau,s))$$

$$= \frac{\Gamma(k)\,\zeta(k)}{(-2\pi i)^k} \langle \operatorname{res}_{s=1} E_\beta(\tau,s), e_0 \rangle \quad \longleftarrow \text{const. term}$$

$$+ \sum_{\substack{\lambda \in K' \setminus \{0\} \\ q(\lambda)=0 \\ \lambda \in \overline{E}}} \underbrace{\sum_{n \mid \lambda} n^{k-1} \langle \operatorname{res}_{s=1} E_\beta(\tau,s), e_{\frac{\lambda}{n}} \rangle}\, e(\lambda, Z). \qquad (1)$$

$$\in M_k^{\partial \mathrm{Eis}}(\Gamma(L))$$

$$\begin{tikzcd}
\langle E_\beta(\tau,s) \rangle \arrow[r, "\operatorname{res}_{s=1}"] \arrow[d, "\Phi"', "\text{bij.}"] & \operatorname{Inv}(\mathbb{C}[L'/L]) \arrow[d, "\Phi"', "\text{bij.?}"] \\
\langle \mathcal{E}_{k,\beta}(Z,s) \rangle \arrow[r, "\operatorname{res}_{s=1}", "\text{surj?}"'] & M_k^{\partial \mathrm{Eis}}(\Gamma(L))
\end{tikzcd}$$

<u>Thm.:</u> $\operatorname{Inv}(\mathbb{C}[L'/L]) \xrightarrow{\Phi} M_k^{\partial \mathrm{Eis}}(\Gamma(L))$ bijective. $\Big]$ much easier for higher weight

weight

Proof: (Sketch).

For $F \in M_\kappa(\Gamma(L))$ we def

$$\Phi^*(\tau, F) := \int_{\Gamma(L)\backslash \mathbb{H}_\ell} F(Z)\, \Theta_L(\tau, Z)\, q(Y)^\kappa \frac{dXdY}{q(Y)^\ell}$$

$[\kappa = \frac{\ell}{2} - 1]$ ~~and~~ Then we have

$$\langle \Phi^*(\tau, F), v \rangle_{Pet} = \langle F, \Phi(Z, v) \rangle_{Pet} \qquad \begin{array}{l} F \in M_\kappa(\Gamma(L)) \\ v \in \mathrm{Inv}(\mathbb{C}[L'/L]) \end{array}$$

so Φ^*, Φ are adjoint to eachother.

Now consider the weight κ Laplace operator Ω_κ on $\mathbb{H}_\ell$. It satisfies

$$\underset{\text{in } Z}{\Omega_\kappa} \Theta_L = \underset{\text{in } \tau}{\Delta_0} \Theta_L \qquad \Delta_0 \text{ hyp. Laplace on } \mathbb{H}.$$

so that if $F \in M_\kappa(\Gamma(L))$.

$$\Delta_0 \Phi^*(\tau, F) = \int_{\Gamma(L)\backslash \mathbb{H}_\ell} F(Z)\, \underbrace{\Delta_0 \Theta_L}_{= \Omega_\kappa \Theta_L}\, q(Y)^\kappa \frac{dXdY}{q(Y)^\ell}$$

$\kappa\; dXdY$

$$\overset{\Omega_\kappa \text{ self-adjoint}}{=} \int_{\Gamma(2)\backslash \mathbb{H}_c} \big(\underbrace{\Omega_\kappa F(Z)}_{=0}\big) \overline{\Theta_L(\tau, Z)}\, q(Y) \frac{\cdots}{q(X)^c}$$

$$= 0$$

$\rightsquigarrow$ One can show that $\Phi^*(\tau, F)$ has only polynomial growth. (harmonic of weight 0)

$\rightsquigarrow$ growth comes from the const. Fourier coeff.

This coeff. can be calculated and is given by

$$\frac{\Gamma(1/2)}{2(2\pi)^{1/2}} \sum_{\gamma \in \mathrm{Iso}(L'/L)} \sum_{\substack{\delta \in \mathrm{Iso}(L'/L) \\ \gamma = k_{\delta\gamma}\delta}} \zeta_+^{k_{\delta\gamma}} (1-\kappa)\, F(\delta)\, e_\gamma. \quad (2)$$

$\rightsquigarrow$ $\Phi^*(\tau, F)$ bounded, harmonic of weight 0

$\dashv$ invariant vector given by (2).

$$M_\kappa^{\partial \mathrm{Eis}} \underset{\Phi}{\overset{\Phi^*}{\rightleftarrows}} \mathrm{Inv}(\mathbb{C}[L'/L]).$$

Have to prove that Φ^*, Φ are inj.

$\rightsquigarrow$ can be done easily by looking at (1), (2)

$\square$

If L does not split 2 hyp. planes over $\mathbb{Z}$.

usually no surjectivity / injectivity.

We can still describe the image explicitly.

injective $\Leftrightarrow$ L splits a hyp plane over $\mathbb{Z}$

$\rightsquigarrow$ L max but not unimodular

$\Rightarrow \operatorname{Inv}(\mathbb{C}[L'/L]) = \{0\} \simeq M_k^{\partial \mathrm{Eis}}(\Gamma(L))$

Moreover $\underbrace{\text{if}}_{\text{additionally}}$ $l = 6, 10, 14, 18, 22, 30,$ then

$k = 2, 4, 6, 8, 10, 14$

and $M_k(\Gamma(L)) = M_k^{\partial \mathrm{Eis}}(\Gamma(L))$.