Modular units for orthogonal groups of signature (2,2) & invariants for Weil representation

Modular units: $\cdot$ (meromorphic) modular forms w/ divisor concentrated on boundary.

$\cdot$ In SL_2-case studied by Kubert–Lang.

Typical example: $\eta(\tau) = q^{1/24} \cdot \prod (1-q^n)$

no zeros & poles on $\mathbb{H}$.

Here: consider modular forms for (certain) orthogonal groups of signature (2,2) of the following form

$\Gamma \subseteq SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$

$f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ meromorphic function w/ transformation property for elements of Γ.

Modular unit for Γ: modular form for Γ without zeros/poles on $\mathbb{H} \times \mathbb{H}$.

Borcherds products:

Roughly Borcherds constructed a map

$$\left\{ \begin{array}{c} \text{certain vector valued} \\ \text{modular forms} \\ \text{weight } 1 - n/2 \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{certain orthogonal} \\ \text{modular forms} \\ (2,n) \end{array} \right\}$$

$$F \longmapsto \Psi_F$$

s.t. – Ψ_F has product expansion at cusps

– divisor of Ψ_F is controlled by principal parts of F.

In particular: $n = 2$; & start with holomorphic vector valued modular forms = invariants for Weil representation.

§ Invariants of the Weil representation

L even lattice of signature (b^+, b^-), $b^+ + b^-$ is even.

L' its dual lattice

$D = L'/L$ its associated discriminant form.

Weil representation for L: unitary rep'n of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$

(we denote standard basis $e_\gamma : \gamma \in D$)

Then action of ρ_L is defined by:

$$\rho_L(T)\, e_\gamma = e(Q(\gamma)) \cdot e_\gamma$$

$$\rho_L(S)\, e_\gamma = \frac{e((b^+ - b^-)/8)}{\sqrt{|D|}} \cdot \sum_{\delta \in L'/L} e((\gamma, \delta)) \cdot e_\delta .$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad e(z) := e^{2\pi i z}$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

We want to study $\mathbb{C}[D]^{SL_2(\mathbb{Z})}$.

A subgroup $H \subseteq D$ is called

- self-orthogonal if $H \subseteq H^\perp$
- self-dual if $H = H^\perp$
- isotropic if self-orthogonal & $Q|_H \equiv 0$.

<u>Thm</u>: Assume that D contains a self-dual isotropic subgroup. Then $\mathbb{C}[D]^{SL_2(\mathbb{Z})}$ is spanned by characteristic functions of self-dual isotropic subgroups

$$v^H = \sum_{\gamma \in H} e_\gamma$$

for H self-dual isotropic subgroup.

Proof sketch:

Scheithauer: v^H these are invariants

$\{v^H\}$ form a spanning set: follow a proof of Nebe, Rains, Sloane of related result on codes, as suggested by Skoruppa.

- invariance under T: invariants are supported on isotropic vectors
- invariance under $M_u = \begin{pmatrix} * & * \\ N & u \end{pmatrix}$ where N level of D, $u \in (\mathbb{Z}/N\mathbb{Z})^*$

  then $\rho_L(M_u) e_\gamma = e_{u \cdot \gamma}$.

  $\rightsquigarrow$ space of invariants under $\langle T, M_u : u \in (\mathbb{Z}/N\mathbb{Z})^* \rangle$ generated by characteristic functions of isotropic subgroups.

- invariance under S: + linear algebra

$\rightsquigarrow$ result. $\blacksquare$

Note: Has been generalised by M. Müller.

§ The case $L = U(N) \oplus U$

U hyperbolic plane $/\mathbb{Z}$, $(\mathbb{Z}^2, Q(x,y) = xy)$

$U(N)$ scaled — " — $Q(x,y) = Nxy$.

Then
- $L = U(N) \oplus U$ has signature $(2,2)$
- $L'/L \cong (\mathbb{Z}/N\mathbb{Z})^2$
- discriminant kernel $\Gamma_L = \left\{ \left(\begin{pmatrix} a_1 & * \\ N* & * \end{pmatrix}, \begin{pmatrix} a_2 & * \\ N* & * \end{pmatrix} \right) \in Sl_2(\mathbb{Z}) \,\middle|\, a_1 a_2 \equiv 1\ (N) \right\}$

Invariants for ρ_L:

$$H_d = \left(\mathbb{Z}/d\mathbb{Z}\right) \oplus \left(\mathbb{Z}/(N/d)\mathbb{Z}\right) \subseteq L'/L \quad \text{for } d \mid N$$

$$= \left((N/d)\mathbb{Z}/N\mathbb{Z}\right) \oplus \left(d\mathbb{Z}/N\mathbb{Z}\right) \subseteq \left(\mathbb{Z}/N\mathbb{Z}\right)^2$$

These H_d are self-dual isotropic subgroups & all self-dual isotropic subgroups of L'/L are of this form.

<u>Corollary</u> (Ye, Zemel): $\{v^{H_d} : d \mid N\}$ is a basis for $\mathbb{C}[L'/L]^{SL_2(\mathbb{Z})}$.

<u>Borcherds products for T_L:</u>

<u>Thm</u>: Let $F = \sum_{d \mid N} \alpha_d v^{H_d} \in \mathbb{C}[L'/L]^{SL_2(\mathbb{Z})}$ with integers coefficients.

Then its associated Borcherds product is given by

$$\Psi_F(\tau_1, \tau_2) = \prod_{d \mid N} \eta(d\tau_1)^{\alpha_d} \cdot \eta(d\tau_2)^{\alpha_d}.$$

with some multiplier system χ, weight $\frac{1}{2}\sum_{d \mid N} \alpha_d$.

<u>Proof</u>: Just calculate product expansions. $\blacksquare$

§ Some observations for $L = U(N) \oplus U(N')$

$N' \mid N$ divisor.

In this case $L'/L \cong (\mathbb{Z}/N\mathbb{Z})^2 \overset{\perp}{\oplus} (\mathbb{Z}/N'\mathbb{Z})^2$

- list of self dual isotropic subgroups

  $\hookrightarrow$ not exhaustive in general

  $N' = p$ prime : these are all.

In this case : $\to$ generating set for $\mathbb{C}[L'/L]^{SL_2(\mathbb{Z})}$

but subject to one linear relation.

$\rightsquigarrow$ Lift this relation

<u>Corollary</u> : $\displaystyle\prod_{a=1}^{p-1} \eta(\tau + a/p) = e\left(\frac{p-1}{48}\right) \cdot \frac{\eta(p\tau)^{p+1}}{\eta(\tau) \cdot \eta(p^2\tau)}$.