On a twisted version of Zagier's $f_{k,D}$ function

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International Seminar on Automorphic Forms
April 20, 2021

Outline

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1. Motivation
2. The framework
3. Hyperbolic Eisenstein series at $s = 0$
   - Weight 2
   - Weight $k > 2$
4. Outlook

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- Bengoechea extended this to $D < 0$, namely $f_{k,D}$ is a meromorphic cusp form of weight $2k$, and the poles are precisely the CM points of discriminant D.

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- Kohnen, Zagier used the $f_{k,D}$ function to represent the kernel function of the Shimura and Shintani lift between integral and half integral weight cusp forms as

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- In addition, they proved that both the even periods and the weight $2k$ cycle integral of $f_{k,D}$ are rational.
- Löbrich, Schwagenscheidt as well as Alfes-Neumann, Bringmann, Schwagenscheidt extended rationality to traces of $f_{k,D}$ with $D < 0$ recently (restricting the sum to an individual equivalence class).

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where d is a fundamental discriminant dividing D. (If $D = 0$, set $d = 0$.) This function descends to $\mathcal{Q}(D)/\Gamma$.

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  - The "modular" sign function $\operatorname{sgn}(a|\tau|^2 + b\mathrm{Re}(\tau) + c)$, namely compatibility with the group actions of Γ on $\mathbb{H}$ and $\mathcal{Q}(D)$.
  - The sign function $\operatorname{sgn}([a,b,c]) := \begin{cases} \operatorname{sgn}(a) & \text{if } a \neq 0, \\ \operatorname{sgn}(c) & \text{if } a = 0. \end{cases}$

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- Todays goal is to find a condition to recover modularity.
- A future goal is to recover some rationality result.

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$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{\mathrm{Im}(\gamma\tau)^s}{j(\gamma,\tau)^k}, \qquad j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) := c\tau + d.$$

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- Matsusaka studied the case of weight 2 (twisted by $\mathrm{sgn}(Q)$ too) based on individual equivalence classes of quadratic forms in parallel.

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- We have $\left(Q \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(\tau, 1) = (c\tau + d)^2 Q(\gamma\tau, 1)$.
- We say that an integer n is represented by Q if there exist x, $y \in \mathbb{Z}$, such that $Q(x, y) = n$.

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$$\mathcal{E}_{k,D}(\tau, s) := \sum_{0 \neq Q \in \mathcal{Q}(D)/\Gamma} \chi_d(Q) \sum_{\hat{Q} \sim Q} \frac{\operatorname{sgn}\left(\hat{Q}\right)^{\frac{k}{2}} \mathrm{Im}(\tau)^s}{\hat{Q}(\tau, 1)^{\frac{k}{2}} \left|\hat{Q}(\tau, 1)\right|^s}.$$

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- The behaviour of $\mathcal{E}_{k,D}(\tau, s)$ is dictated by the sign of D.
- In the (semi-)definite case, i. e. $D \leq 0$, $Q \sim -Q$ implies $Q = 0$.

Example: Parabolic case

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- We have $Q_\pm(\tau, 1) = \pm(c\tau \pm d)^2$,

  and hence we infer

$$
\begin{aligned}
\mathcal{E}_{k,0}(\tau, s) &= 2 \sum_{\gcd(c,d)=1} \frac{\mathrm{Im}(\tau)^s}{(c\tau + d)^k \left|c\tau + d\right|^{2s}} \\
&= 2 \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{\mathrm{Im}(\gamma\tau)^s}{j(\gamma, \tau)^k}.
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  - Harmonicity with respect to the weight 2 hyperbolic Laplacian on $\mathbb{H}$, explicitly

$$0 = \Delta_2 E_2^* := \left(-v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + 2iv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\right)E_2^*.$$

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  - The function E_2^* is of at most linear exponential growth towards the cusp $i\infty$, namely $E_2^* \in O\left(\mathrm{e}^{\delta v}\right)$ for some $\delta > 0$.

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- By unfolding the sum defining $\mathcal{E}_{k,D}$, we arrive at

$$\mathcal{E}_{k,D}(\tau, s) = \frac{-iv^s}{(-1)^{\frac{k}{2}}} \sum_{m \in \mathbb{Z}} \sum_{a \geq 1} \frac{T_m(d, d', 4a)}{a^{\frac{k}{2}+s}} \int_{v-i\infty}^{v+i\infty} \frac{\mathrm{e}^{2\pi mt}\ dt}{(t^2+\lambda^2)^{\frac{k}{2}} \left|t^2+\lambda^2\right|^s}\ q^m,$$

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where $v = \mathrm{Im}(\tau)$, $\lambda = \frac{\sqrt{D}}{2a}$, and

$$T_m(d, d', c) := \sum_{\substack{b \pmod{c} \\ b^2 \equiv dd' \pmod{c}}} \chi_d\left(\left[\frac{c}{4}, b, \frac{b^2 - dd'}{c}\right]\right) \mathrm{e}^{2\pi i\left(\frac{2mb}{c}\right)}.$$

is a Salié sum.

Results pertaining to $s = 0$

- First, we have for any $\rho > 0$ that

$$\frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} \frac{\mathrm{e}^{2\pi m t}}{(t^2+\lambda^2)^\rho}dt = \begin{cases} \frac{\sqrt{\pi}}{\Gamma(\rho)}\left(\frac{\pi m}{\lambda}\right)^{\rho-\frac{1}{2}} J_{\rho-\frac{1}{2}}(2\pi\lambda m) & \text{if } m > 0, \\ 0 & \text{if } m \leq 0. \end{cases}$$

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  (If $m \leq 0$, the integrand is holomorphic at the cusp.)
- Secondly, the resulting expression for $\mathcal{E}_{k,D}$ was computed by Duke, Imamoḡlu, Tóth in terms of cycle integrals.

Interlude: Heegner geodesics and cycle integrals

- We associate to $Q = [a, b, c]$ the Heegner geodesic

$$S_Q := \{\tau \in \mathbb{H} : a|\tau|^2 + b\mathrm{Re}(\tau) + c = 0\},$$

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- In addition, the stabilizer Γ_Q is not trivial in this case.

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Interlude: Heegner geodesics and cycle integrals

- We associate to $Q = [a, b, c]$ the Heegner geodesic

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The integral is oriented counterclockwise if $\mathrm{sgn}(Q) > 0$, and clockwise if $\mathrm{sgn}(Q) < 0$.

A result of Duke, Imamoḡlu, Tóth

- Let

$$\phi_m(y,s) := \begin{cases} y^s & \text{if } m = 0, \\ 2\pi\sqrt{|m|\,y}\; I_{s-\frac{1}{2}}\left(2\pi\,|m|\,y\right) & \text{if } m \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

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- Averaging this seed gives rise to the weight 0 Niebur Poincaré series

$$G_m(\tau,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \phi_m\left(\mathrm{Im}(\gamma\tau),s\right)\mathrm{e}^{2\pi i m \mathrm{Re}(\gamma\tau)}, \quad \mathrm{Re}(s) > 1.$$

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- Assume $\mathrm{Re}(\rho) > 1$, and the notation and hypotheses above. Then their result states that

$$\sum_{Q\in\mathcal{Q}(D)_\sim} \chi_d(Q)\mathcal{C}_0\left(G_m(\cdot,\rho)\right)$$

$$\doteq \sum_{0<c\equiv 0 \pmod 4} \begin{cases} \frac{T_m(d,d',c)}{c^{\frac{1}{2}}} J_{\rho-\frac{1}{2}}\left(\frac{4\pi\sqrt{m^2D}}{c}\right) & \text{if } m \neq 0, \\ \frac{T_0(d,d',c)}{c^\rho} & \text{if } m = 0. \end{cases}$$

Outline

The "non-constant" terms

- Combining, we deduce that the Fourier coefficients corresponding to $m \neq 0$ are all regular at $s = 0$, and vanish for every $m < 0$.

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$$q^{-m} + O(q) =: j_m(\tau) = \lim_{s \searrow 1} \left(G_{-m}(\tau, s) - \frac{2m^{1-s}\sigma_{2s-1}(m) G_0(\tau, s)}{\pi^{-s-\frac{1}{2}} \Gamma\left(s + \frac{1}{2}\right) \zeta(2s-1)} \right).$$

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- We arrive at the Fourier coefficients

$$\frac{-2}{D^{\frac{1}{2}}} \sum_{m \geq 1} \sum_{Q \in \mathcal{Q}(D)_{\sim}} \chi_d(Q) \mathcal{C}_0\left(j_m(\cdot) + 24\sigma_1(m), Q\right) q^m.$$

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- Moreover, letting j be the modular invariant function for Γ, $j' := \frac{1}{2\pi i} \frac{\partial j}{\partial \tau}$ be the normalized derivative of j, we recall the expansion

$$\frac{j'(\tau)}{j(w) - j(\tau)} = \sum_{m \geq 0} j_m(w) q^m, \qquad \mathrm{Im}(\tau) > \mathrm{Im}(w).$$

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- One one hand, if $m = 0$, then $G_0(\tau, s)$ is the weight 0 parabolic Eisenstein series, which has a simple pole at $s = 1$.

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- In other words, the point $s = 0$ is a removable singularity of the term corresponding to $m = 0$.
- Performing the computations, we obtain the term

$$\frac{-2}{D^{\frac{1}{2}}} \sum_{Q \in \mathcal{Q}(D)_{\sim}} \chi_d(Q) \mathcal{C}_0 \left(\frac{3}{\pi v}, Q \right).$$

Conclusion in weight 2

Theorem (M.)

Let $D > 0$ be a non-square discriminant, d be the positive fundamental discriminant dividing D. Then the function $\mathcal{E}_{2,D}(\tau, s)$ can be analytically continued to $s = 0$ and the continuation is given by

$$\frac{-2}{D^{\frac{1}{2}}} \sum_{m \geq 0} \sum_{Q \in \mathcal{Q}(D)/\Gamma} \chi_d(Q) \mathcal{C}_0(j_m, Q) q^m - \frac{-2}{D^{\frac{1}{2}}} E_2^*(\tau) \sum_{Q \in \mathcal{Q}(D)_\sim} \chi_d(Q) \mathcal{C}_0(1, Q).$$

for any $\tau \in \mathbb{H}$. Furthermore, if $\mathrm{Im}(\tau)$ is sufficiently large, that is τ is located above the net of geodesics $\bigcup_{Q \in \mathcal{Q}(D)} S_Q$, then we have

$$\lim_{s \to 0} \mathcal{E}_{2,D}(\tau, s) = \frac{-2}{D^{\frac{1}{2}}} \sum_{Q \in \mathcal{Q}(D)_\sim} \chi_d(Q) \mathcal{C}_0 \left(\frac{j'(\tau)}{j(\cdot) - j(\tau)} - E_2^*(\tau), Q \right).$$

Locally harmonic Maaß forms

- The function

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- Roughly speaking, such a form is a harmonic Maaß form that is permitted to have singularities on the net of geodesics $\bigcup_{Q \in \mathcal{Q}(D)} S_Q$, called "jumping singularities".

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- Roughly speaking, such a form is a harmonic Maaß form that is permitted to have singularities on the net of geodesics $\bigcup_{Q \in \mathcal{Q}(D)} S_Q$, called “jumping singularities”.
- A prominent example is the function

$$\mathcal{F}_{1-k,D}(\tau) := \frac{(-1)^k \mathcal{D}(Q)^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \sum_{Q \in \mathcal{Q}(D)} \operatorname{sgn}(Q_\tau)\, Q(\tau,1)^{k-1} \psi_k \left(\frac{\mathcal{D}(Q)\mathrm{Im}(\tau)^2}{|Q(\tau,1)|^2} \right),$$

  where $[a,b,c]_\tau := a|\tau|^2 + b\mathrm{Re}(\tau) + c$, and $\psi_k(y) := \frac{1}{2}\int_0^y t^{k-\frac{3}{2}}(1-t)^{-\frac{1}{2}}\,dt$.

Outline

Evaluating at $s = 0$

Theorem (M.)

Let $D > 0$ be a non-square discriminant, let d be the positive fundamental discriminant dividing D, and suppose that $k \geq 4$ is even. Then, we have the Fourier expansion

$$\mathcal{E}_{k,D}(\tau, 0) = \frac{(-1)^{\frac{k}{2}} 2\pi^{\frac{k}{2}}}{D^{\frac{k}{4}} \Gamma\left(\frac{k}{4}\right)^2} \sum_{m \geq 1} m^{\frac{k}{2}-1} \sum_{Q \in \mathcal{Q}(D)_{\sim}} \chi_d(Q) \mathcal{C}_0\left(G_{-m}\left(\cdot, \frac{k}{2}\right), Q\right) q^m.$$

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Modular above the net of geodesics again?

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Modular above the net of geodesics again?

Theorem (M.)

Let $2 < k \equiv 2 \pmod 4$, let D, d be as before. Suppose that $\mathrm{Im}(\tau)$ is sufficiently large, that is τ is located above the net of geodesics $\bigcup_{Q \in \mathcal{Q}(D)} S_Q$. Then $\mathcal{E}_{k,D}(\tau, 0)$ is modular of weight k for Γ.

A conceptual proof

- Suppose that $k > 1$, $D > 0$ is a non-square discriminant, and $\tau \in \mathbb{H} \setminus \bigcup_{Q \in \mathcal{Q}(D)} S_Q$. Then, the function $\mathcal{F}_{1-k,D}$ satisfies

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\right)^{2k-1}\mathcal{F}_{1-k,D}(\tau) = -\frac{(2k-2)!}{(4\pi)^{2k-1}}D^{\frac{1}{2}-k}f_{k,D}(\tau).$$

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- Consequently, we define a twisted version

$$\widetilde{\mathcal{F}}_{1-k,D}(\tau) \coloneqq \frac{(-1)^k \mathcal{D}(Q)^{\frac{1}{2}-k}}{\binom{2k-2}{k-1}\pi} \sum_{Q \in \mathcal{Q}(D)_\sim} \chi_d(Q)$$
$$\times \sum_{\hat{Q} \sim Q} \mathrm{sgn}(\hat{Q})\,\mathrm{sgn}\left(\hat{Q}_\tau\right) \hat{Q}(\tau,1)^{k-1} \psi_k\left(\frac{\mathcal{D}(Q)\mathrm{Im}(\tau)^2}{\left|\hat{Q}(\tau,1)\right|^2}\right)$$

of the locally harmonic Maaß form $\mathcal{F}_{1-k,D}$.

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- By Bol's identity, we obtain

$$\left(R_{2-k}^{k-1}\widetilde{\mathcal{F}}_{1-\frac{k}{2},D}\right)(\tau) = (-1)^k\Gamma(k-1)D^{\frac{1-k}{2}}\mathcal{E}_{k,D}(\tau,0),$$

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- The iterated raising yields weight k modularity.

A (short) technical proof

1. Let $L_\kappa^n := L_{\kappa-2n+2} \circ \ldots \circ L_{\kappa-2} \circ L_\kappa$ be the iterated Maaß lowering operator. We use a result by Alfes-Neumann, Schwagenscheidt:

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   Let $h\colon \mathbb{H} \to \mathbb{C}$ be a weak Maaß form of weight $2-2\kappa$. Then we have

$$\mathcal{C}\left(L_{2-2\kappa}^{-\kappa-\ell+2} h, Q\right) \doteq \mathcal{C}\left(L_{2-2\kappa}^{-\kappa-\ell} h, Q\right), \text{ if } \ell \leq -\kappa.$$

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2. Let $\Phi_{2-k,-m}$ be the weight $2-k$ Maaß-Poincaré series of index $-m$. Then we have

   $$\left(L_0^{\frac{k}{2}-1} G_{-m}\right)\left(w, \frac{k}{2}\right) \doteq \frac{1}{|m|^{\frac{k}{2}-1}} \Phi_{2-k,-m}(w)$$

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   This can be seen by rewriting the seed of G_m in terms of the M-Whittaker function, and then differentiate it iteratively.

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3. Recall one of Petersson's Poincaré series, namely

$$\mathbb{P}_k(z_1, z_2) := \mathrm{Im}(z_2)^{k-1} \sum_{\gamma \in \Gamma} \left(\frac{1}{(z_1 - z_2)(z_1 - \overline{z_2})^{k-1}} \right) \Big|_{k,z_1} \gamma.$$

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We have the following identity due to Bringmann, Kane:

$$\sum_{m \geq 1} \Phi_{2-k,-m}(w) q^m \doteq \mathbb{P}_k(\tau, w), \quad \mathrm{Im}(\tau) > \max\left(\mathrm{Im}(w), \frac{1}{\mathrm{Im}(w)}\right).$$

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4. By $L_\kappa^0 := \mathrm{Id}$, we arrive at the identity

$$\mathcal{E}_{k,D}(\tau, 0) \doteq \sum_{Q \in \mathcal{Q}(D)_\sim} \chi_d(Q) \mathcal{C}_{2-k}\left(\mathbb{P}_k(\tau, \cdot), Q\right),$$

   whenever $2 < k \equiv 2 \pmod 4$ and $\mathrm{Im}(\tau)$ is sufficiently large.

Outline

Understand the local behaviour

- We found two locally harmonic Maaß forms, which are twisted traces of the functions

$$\mathcal{C}_0\left(\frac{j'(\tau)}{j(\cdot)-j(\tau)} - E_2^*(\tau), Q\right), \qquad \mathcal{C}_{2-k}\left(\mathbb{P}_k(\tau,\cdot), Q\right).$$

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- "Refine" $\mathcal{E}_{k,D}(\tau,0)$ by letting

$$\widetilde{\mathcal{E}_{k,D}}(\tau)\coloneqq\sum_{0\neq Q\in\mathcal{Q}(D)/\Gamma}\chi_d\left(Q\right)\sum_{\hat{Q}\sim Q}\frac{\mathbb{1}_{\hat{Q}}(\tau)^{\frac{k}{2}}}{\hat{Q}(\tau,1)^{\frac{k}{2}}},\quad \mathbb{1}_Q(\tau)\coloneqq\begin{cases}1 & \text{if } \tau\notin A_Q,\\ -1 & \text{if } \tau\in A_Q,\end{cases}$$

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- This idea appears in Matsusaka's first version as well.

Recover rationality

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3. Rationality can be obtained via this local polynomial.

Thank you very much!