Cones of codimension two special cycles

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Motivation

$$X = \text{complex (quasi-)projective algebraic variety}$$

Cones in $\mathrm{Pic}(X) \otimes \mathbb{R}$ generated by (nef, ample, effective, etc) divisors have been intensely studied.

Example

- X = Fano variety $\implies \mathrm{Nef}(X)$ is rational polyhedral. (Mori)
- $X = \overline{\mathcal{M}}_{g,n} \implies \overline{\mathrm{Eff}}(X)$ is non polyhedral for $g \geq 2$ and $n \geq 2$. (Mullane)
- X = orthogonal Shimura variety of dim $\geq 3 \implies$ cone of *special divisors* is rational polyhedral (Bruinier–Möller).

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Question

What about cones of cycles in higher codimension?

Cones

$V =$ finite-dim $\mathbb{Q}$-vector space. $\mathcal{G} \subseteq V$ non-empty subset.

$\langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}} :=$ ***(convex) cone generated by $\mathcal{G}$ in V*** $=$ smallest subset of V that contains $\mathcal{G}$ and is closed under lin. comb. with non-negative coefficients in $\mathbb{Q}$.

$V_{\mathbb{R}} = V \otimes \mathbb{R}$ with Euclidean topology.

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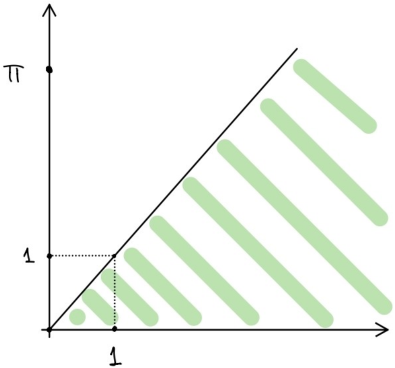
Definition

The cone $\langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}$ is:

- **pointed** if it contains no lines ($=$ subspaces of dim 1).
- **polyhedral** if $\langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}} = \langle \mathcal{G}' \rangle_{\mathbb{Q}_{\geq 0}}$ for some $\mathcal{G}' \subseteq \mathcal{G}$ finite.
- **rational** if $\overline{\langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}}$ is generated over $\mathbb{R}$ by elements of V.

Examples

$$\left\langle (1, a) \in \mathbb{Q}^2 : a \in [0, 1] \cap \mathbb{Q} \right\rangle_{\mathbb{Q}_{\geq 0}} \implies$$



pointed, polyhedral, rational

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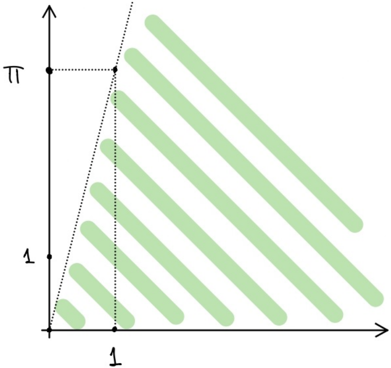
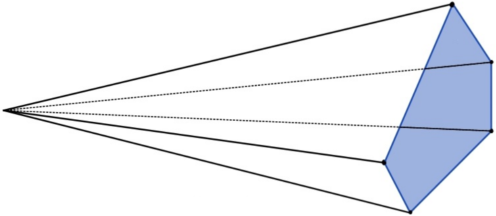
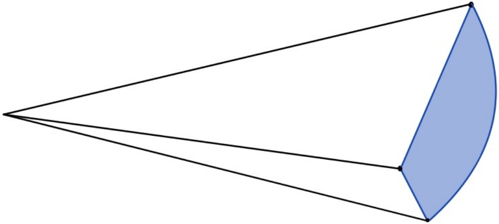


pointed, non-polyhedral, non-rational

Examples in $\mathbb{Q}^3$



polyhedral



non-polyhedral

Strategy to prove polyhedrality

Let $\mathcal{C} = \langle \mathcal{G} \rangle_{\mathbb{Q}_{\geq 0}}$ be a cone generated by $\mathcal{G}$ in V.
How to understand whether $\mathcal{C}$ is polyhedral?

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STEP 1: find all rays of $\overline{\mathcal{C}} \subseteq V_{\mathbb{R}}$ arising as "limits" of rays generated by elements of $\mathcal{G}$.
STEP 2: understand how sequences of rays generated over $\mathcal{G}$ converge towards the "limits".

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Definition

A ray r of $\overline{\mathcal{C}}$ is an **accumulation ray of** $\mathcal{C}$ (w.r.t. $\mathcal{G}$) if there exists a sequence $(g_j)_{j\in\mathbb{N}}$ of *pairwise different* generators in $\mathcal{G}$, such that

$$\mathbb{R}_{\geq 0} \cdot g_j \longrightarrow r, \qquad \text{when } j \longrightarrow \infty.$$

The **accumulation cone of** $\mathcal{C}$ (w.r.t. $\mathcal{G}$) is the cone generated by the accumulation rays of $\mathcal{C}$ and by 0.

Orthogonal Shimura varieties

$L =$ even ***unimodular*** lattice of signature $(b, 2)$, $b > 2$.
$\langle\cdot,\cdot\rangle =$ sym. bilinear form, $q(\cdot) = \langle\cdot,\cdot\rangle/2$ quadratic form.

$$\mathcal{D} = \Big\{ z \in L \otimes \mathbb{C} \setminus \{0\} : \langle z, z\rangle = 0 \text{ and } \langle z, \bar{z}\rangle < 0 \Big\}/\mathbb{C}^* \quad \subset \mathbb{P}(L \otimes \mathbb{C})$$

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$\mathcal{D}$ is a b-dim complex manifold with 2 connected components.

$$\mathcal{D} = \mathcal{D}^+ \coprod \mathcal{D}^-$$

$\mathrm{O}(L) \circlearrowright \mathcal{D} \quad \rightsquigarrow \quad \mathrm{O}^+(L) =$ subgroup of $\mathrm{O}(L)$ preserving $\mathcal{D}^+$.

Definition

$\Gamma \leq \mathrm{O}^+(L)$ finite index subgroup. The **orthogonal Shimura variety** associated to Γ is

$$X_\Gamma = \Gamma \backslash \mathcal{D}^+.$$

$$\Lambda_2 = \left\{ T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n, r, m \in \mathbb{Z} \text{ and } T \geq 0 \right\} = \left\{ \begin{matrix} \text{sym. half-int. pos.} \\ \text{semi-def } 2 \times 2\text{-mat.} \end{matrix} \right\}$$

$$\Lambda_2^+ = \{ T \in \Lambda_2 : T > 0 \}.$$

The *moment matrix* of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in L^2$ is

$$q(\boldsymbol{\lambda}) = \frac{1}{2} \Big(\langle \lambda_i, \lambda_j \rangle \Big)_{i,j} = \frac{1}{2} \begin{pmatrix} \langle \lambda_1, \lambda_1 \rangle & \langle \lambda_1, \lambda_2 \rangle \\ \langle \lambda_1, \lambda_2 \rangle & \langle \lambda_2, \lambda_2 \rangle \end{pmatrix}.$$

Special cycles of codimension 2

$$\Lambda_2 = \left\{ T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n, r, m \in \mathbb{Z} \text{ and } T \geq 0 \right\} = \left\{ \begin{array}{c} \text{sym. half-int. pos.} \\ \text{semi-def } 2 \times 2\text{-mat.} \end{array} \right\}$$

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For $T \in \Lambda_2^+$, consider

$$\sum_{\substack{\boldsymbol{\lambda} \in L^2 \\ q(\boldsymbol{\lambda}) = T}} \boldsymbol{\lambda}^\perp \quad \subset \mathcal{D}^+ \quad \Longrightarrow \quad Z(T) := \Gamma \backslash \sum_{\substack{\boldsymbol{\lambda} \in L^2 \\ q(\boldsymbol{\lambda}) = T}} \boldsymbol{\lambda}^\perp \quad \subset X_\Gamma = \Gamma \backslash \mathcal{D}^+$$

Definition

$Z(T)$ is the (codimension 2) **special cycle** associated to $T \in \Lambda_2^+$. Its rational class in $\mathrm{CH}^2(X_\Gamma)$ is denoted by $\{Z(T)\}$.

Cones of special cycles

If $T \in \Lambda_2$, then $Z(T)$ is of codimension $\mathrm{rk}(T)$.

Example

$Z\left(\begin{smallmatrix} n & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is the n-th **Heegner divisor**, usually denoted by H_n.

Still possible to define a codimension 2 cycle as

$$\{Z(T)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T)},$$

where ω is the *Hodge bundle* of X_Γ.

Definition

The **cone of special cycles** (of codim. 2) on X_Γ is

$$\mathcal{C}_{X_\Gamma} = \langle \{Z(T)\} : T \in \Lambda_2^+ \rangle_{\mathbb{Q}_{\geq 0}} \quad \subset \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q}.$$

The **cone of rank 1 special cycles** (of codim. 2) on X_Γ is

$$\mathcal{C}'_{X_\Gamma} = \langle \{Z(T)\} \cdot \{\omega^*\} : T \in \Lambda_2, \mathrm{rk}(T) = 1 \rangle_{\mathbb{Q}_{\geq 0}} \quad \subset \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q}.$$

Main results

$L =$ even unimodular lattice of signature $(b, 2)$, $b > 2$.
$k = 1 + b/2 \implies$ even, because L is unimodular.
$M_1^k =$ space of weight k modular forms w.r.t. $\mathrm{SL}_2(\mathbb{Z})$.

Theorem (Z.)

Let X_Γ be an orthogonal Shimura variety arising from L.

1. *The cone $\mathcal{C}'_{X_\Gamma}$ is pointed, rational, polyhedral, and of dimension $\dim M_1^k$.*
2. *The accumulation cone of $\mathcal{C}_{X_\Gamma}$ is pointed, rational, polyhedral, and of the same dimension as $\mathcal{C}'_{X_\Gamma}$.*
3. *The cone $\mathcal{C}_{X_\Gamma}$ is rational, and of maximal dimension in the subspace of $\mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q}$ generated by the special cycles of codimension 2.*
4. *The cones $\mathcal{C}_{X_\Gamma}$ and $\mathcal{C}'_{X_\Gamma}$ intersect only at the origin. If the accumulation cone of $\mathcal{C}_{X_\Gamma}$ is enlarged with a non-zero element of $\mathcal{C}'_{X_\Gamma}$, the resulting cone is non-pointed.*

Open problem:

Is the cone $\mathcal{C}_{X_\Gamma}$ polyhedral?

STEP 1: Done ✔ Accumulation rays computed.

Example

Let m be a positive integer, and let $T_n = \begin{pmatrix} n & 1/2 \\ 1/2 & m \end{pmatrix} \in \Lambda_2^+$.

$$\mathbb{R}_{\geq 0} \cdot \{Z(T_n)\} \xrightarrow[n\to\infty]{} \mathbb{R}_{\geq 0} \cdot \Big(\sum_{t^2|m} \mu(t)\sigma_{k-1}(m/t^2)\{H_{m/t^2}\} \cdot \{\omega\}\Big).$$

The accumulation rays of $\mathcal{C}_{X_\Gamma}$ are *infinitely many*. This differs from the case of cones of Heegner divisors.

STEP 2: Only partial results ✗ The behaviour of sequences of rays converging to the accumulation cone is clear only in some cases.

$\rightsquigarrow$ **now:** use Siegel modular forms to find accumulation rays.

Siegel modular forms

$\mathbb{H}_2 = \{Z \in \mathbb{C}^{2\times 2} : \Im(Z) > 0\}$ = Siegel upper-half space.
$\mathrm{Sp}_4(\mathbb{Z})$ acts on $\mathbb{H}_2$ as

$$\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \mathbb{H}_2 \to \mathbb{H}_2, \qquad Z \mapsto \gamma \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Fix $k > 0$ even. A **weight *k* Siegel modular form** is a holomorphic map $F\colon \mathbb{H}_2 \to \mathbb{C}$ s.t. $F(\gamma \cdot Z) = \det(CZ + D)^k F(Z)$.

Koecher Principle: they admit a Fourier expansion

$$F(Z) = \sum_{T \in \Lambda_2} c_T(F) \exp\big(2\pi i \operatorname{tr}(TZ)\big).$$

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$M_2^k(\mathbb{Q}) = \{$ weight k Siegel mod forms with coeff in $\mathbb{Q}$ $\}$.
$M_2^k(\mathbb{Q})^* =$ dual space, finite dim., generated by the **coefficient extraction functionals**

$$c_T \colon M_2^k(\mathbb{Q}) \to \mathbb{Q}, \qquad F \mapsto c_T(F).$$

X_Γ orthogonal Shimura variety as above, of dim b.
$k = 1 + b/2 \implies$ even, because L is unimodular.

Definition

The **modular cone of weight k** is

$$\mathcal{C}_k := \langle c_T : T \in \Lambda_2^+ \rangle_{\mathbb{Q}_{\geq 0}} \qquad \subset M_2^k(\mathbb{Q})^*$$

Kudla's Modularity Conjecture, now a theorem (Bruinier–Raum), implies:

$$\begin{aligned} \psi \colon M_2^k(\mathbb{Q})^* &\xrightarrow{\text{linear}} \mathrm{CH}^2(X_\Gamma) \otimes \mathbb{Q} \\ c_T &\longmapsto \{Z(T)\} \cdot \{\omega^*\}^{2-\mathrm{rk}(T)} \end{aligned}$$

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IDEA: Study $\mathcal{C}_k$ instead of $\mathcal{C}_{X_\Gamma}$. Understand which properties of $\mathcal{C}_k$ are preserved by ψ. (Unknown if ψ is injective!)

Why Siegel modular forms?

Possible to compute cones explicitly, e.g. with SageMath:

$f_1, \dots, f_\ell$ basis of $S_1^k(\mathbb{Q})$ $\qquad$ $F_1, \dots, F_{\ell'}$ basis of $S_2^k(\mathbb{Q})$.

We may choose a basis of $M_2^k(\mathbb{Q})$ as

$$E_2^k, E_{2,1}^k(f_1), \dots, E_{2,1}^k(f_\ell), F_1, \dots, F_{\ell'},$$

E_2^k = Siegel–Eisenstein series,
$E_{2,1}^k(f)$ = Klingen–Eisenstein series associated to f.

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$$c_T \longleftrightarrow \begin{pmatrix} c_T(E_2^k) \\ c_T\left(E_{2,1}^k(f_1)\right) \\ \vdots \\ c_T\left(E_{2,1}^k(f_\ell)\right) \\ c_T(F_1) \\ \vdots \\ c_T(F_{\ell'}) \end{pmatrix} \in \mathbb{Q}^{1+\ell+\ell'}$$

Computation of accumulation rays

Accumulation of rays via estimate of Fourier coeff:

Let $T \in \Lambda_2^+$

If $k \equiv 2 \bmod 4$, then $c_T(E_2^k) > 0$.

$$\mathbb{R}_{\geq 0} \cdot c_T \longleftrightarrow \mathbb{R}_{\geq 0} \cdot \begin{pmatrix} c_T(E_2^k) \\ c_T\left(E_{2,1}^k(f_1)\right) \\ \vdots \\ c_T\left(E_{2,1}^k(f_\ell)\right) \\ c_T(F_1) \\ \vdots \\ c_T(F_{\ell'}) \end{pmatrix} \subset \mathbb{Q}^{1+\ell+\ell'}$$

$c_T = c_{u^t \cdot T \cdot u}$, for every $u \in \mathrm{GL}_2(\mathbb{Z})$

$\implies$ assume $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ *reduced*, i.e. $0 \leq r \leq m \leq n$.

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$$\mathbb{R}_{\geq 0} \cdot \frac{c_T}{c_T(E_2^k)} \longleftrightarrow \mathbb{R}_{\geq 0} \cdot \begin{pmatrix} 1 \\ \frac{c_T\left(E^k_{2,1}(f_1)\right)}{c_T(E_2^k)} \\ \vdots \\ \frac{c_T\left(E^k_{2,1}(f_\ell)\right)}{c_T(E_2^k)} \\ \frac{c_T(F_1)}{c_T(E_2^k)} \\ \vdots \\ \frac{c_T(F_{\ell'})}{c_T(E_2^k)} \end{pmatrix}$$

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Computation of accumulation rays

The "mixed" behaviour of the Klingen-Eisenstein series $E^k_{2,1}(f)$ has been clarified (Böcherer–Das, 2018):

$$c_T(E^k_{2,1}(f)) = \underbrace{\frac{\zeta(1-k)}{2}\sum_{t^2|m}\alpha_m(t,f)\cdot c_{\begin{pmatrix} n & r/2t \\ r/2t & m/t^2\end{pmatrix}}(E^k_2)}_{\text{Eisenstein part of } m\text{-th Fourier–Jacobi coefficient of } E^k_{2,1}(f)} + \text{cuspidal part}$$

where $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in \Lambda_2^+$ is *reduced*, i.e. $0 \le r \le m \le n$.

- $\alpha_m(t,f) = \sum_{s|t}\mu(\frac{t}{s})\frac{\sum \text{Fourier coeff of } f \text{ associated to divisors of } \frac{m}{s^2}}{\sum \text{powers of divisors of } \frac{m}{s^2}}$.
- If $\det T \to \infty$, then "cuspidal part" grows slower than $c_T(E^k_2)$.

Possible generalizations

- L non-unimodular $\implies$ *vector valued* Siegel modular forms.
- Special cycles of codim $g \geq 2$: now $T \in \Lambda_g^+$ and
$$\{Z(T)\} \in \mathrm{CH}^g(X_\Gamma) \otimes \mathbb{Q}.$$
$\implies$ consider Siegel modular forms of genus g.
- Possible to construct special cycles on orthogonal Shimura varieties defined on certain totally real extensions of $\mathbb{Q}$.
$\implies$ consider Hilbert–Siegel modular forms. (Kudla, Maeda)

Thanks for your attention!