Modular heights of unitary Shimura varieties

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Overview

Part 1

A brief introduction

- Let E/F be a CM extension of a totally real field.
- Let V be a Hermitian space over E of signature

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- Let X be the Shimura variety associated to $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(V)$ for a "suitable" compact subgroup U of $G(\mathbb{A}_f)$ of dimension n over the reflex field E. Let L be a line bundle on X equipped with an Hermitian metric.

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- Let $\mathcal{X}$ be the canonical integral model of X over $\mathcal{O}_E$. Let $\overline{\mathcal{L}}$ be the canonical arithmetic extension of L to $\mathcal{X}$.

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- Let $\mathcal{X}$ be the canonical integral model of X over $\mathcal{O}_E$. Let $\overline{\mathcal{L}}$ be the canonical arithmetic extension of L to $\mathcal{X}$.
- The goal of our work is to compute $\overline{\mathcal{L}}^{n+1}/L^n$, i.e., it is the ratio of the top arithmetic self-intersection number $\widehat{\deg}(\overline{\mathcal{L}}_U)$ of $\overline{\mathcal{L}}$ to the top geometric self-intersection number $\deg(L_U)$ of L. We call it the *modular height*.

Unitary Shimura variety

- Let $D \subset \mathbb{P}(V_{\mathbb{C}})$ be the Hermitian symmetric domain for G as follows:

$$D = \{z \in \mathbb{P}(V_{\iota,\mathbb{C}}) | q(z) < 0\}.$$

Here ι is the archimedean place of F such that V_ι has signature $(n, 1)$. It is connected, and carries a $\mathrm{U}(V_{\mathbb{C}})$-invariant complex structure. Then, for any open compact subgroup U of $G(\widehat{\mathbb{Q}})$, we have a Shimura variety with $\mathbb{C}$-points

$$X_U(\mathbb{C}) = G(\mathbb{Q}) \backslash D \times G(\widehat{\mathbb{Q}})/U.$$

In fact, such unitary Shimura variety is smooth over the reflex field E of dimension n.

A special Hermitian lattice

- Let v be a finite place of F, and ϖ_{E_v} is the uniformizing parameter of E_v. Denote by $\Lambda_v^\vee = \{x \in \mathbb{V}(E_v) : \langle x, \Lambda_v \rangle \subset \mathcal{O}_{E_v}\}$ the dual lattice of Λ_v. We introduce the following definition for an $\mathcal{O}_{E_v}$ lattice $\Lambda_v \subset \mathbb{V}(E_v)$.
  - Λ_v is *self-dual* if $\Lambda_v^\vee = \Lambda_v$;
  - Λ_v is *ϖ_{E_v}-modular* if $\Lambda_v^\vee = \varpi_{E_v}^{-1}\Lambda_v$;
  - Λ_v is *almost ϖ_{E_v}-modular* if $\Lambda_v^\vee \subset \varpi_{E_v}^{-1}\Lambda_v$ and the inclusion is of colength 1.

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- We fix a special Hermitian lattice

$$\Lambda = \prod_{v \nmid \infty} \Lambda_v \subset \mathbb{V}^\infty,$$

such that Λ_v is self-dual if v is unramified in E/F, and it is ϖ_{E_v}-modular (resp. almost ϖ_{E_v}-modular) if v is ramified in E/F with $2 \nmid n$ (resp. $2|n$).

Integral model

- Let U be an open compact subgroup of $G(\widehat{\mathbb{Q}})$, such that for any finite place v, $U_v \subset \mathrm{U}(V(E_v))$ is the stabilizer of $\Lambda(\mathcal{O}_{E_v})$. We have the following theorem.

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Theorem (Rapoport, Smithling and Zhang (2020); Qiu (2023))

Under certain restrictions on E/F, there is a regular integral model $\mathcal{X}_U$ of X_U over $\mathrm{Spec}\mathcal{O}_E$. Moreover, for any place v, $\mathcal{X}_{U,\mathcal{O}_{E_v}}$ is smooth over $\mathrm{Spec}\mathcal{O}_{E_v}$. When $F \neq \mathbb{Q}$, $\mathcal{X}_U$ is proper.

Hodge bundle

- Under the above complex uniformization, let Ω be the tautological bundle on D. Note that Ω carries a natural hermitian metric h_Ω defined by

$$h_\Omega(s_z) = -\langle s_z, s_z \rangle,$$

for $s_z \in \Omega_z \cong \mathbb{C}z$, $z \in D$, which is equivariant under the action of $\mathrm{U}(V_\mathbb{C})$. Let L_U be the descent of $\Omega \times 1_{G(\widehat{\mathbb{Q}})/U}$, which is an ample line bundle on X_U with $\mathbb{Q}$ coefficients.

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- The Hodge bundle L_U extends canonically to an arithmetic Hodge bundle $\overline{\mathcal{L}}_U$ on $\mathcal{X}_U$, which is also ample.
- Denote by $\hat{\xi} = (\overline{\mathcal{L}}^n)/L^n$ the degree 1 arithmetic 1-cycle on $\mathcal{X}$.
- The *modular height* of X_U with respect to $\overline{\mathcal{L}}_U$ is defined to be
$$h_{\overline{\mathcal{L}}_U}(X_U) = \frac{\widehat{\deg}(\overline{\mathcal{L}}_U)}{\deg(L_U)}.$$

Main theorem of modular height

Theorem (G. (2025))

Denote by η the quadratic character associated with E/F, and $L_f(\cdot,\eta)$ the (incomplete) Hecke L-function. When $F \neq \mathbb{Q}$, we have

$$
\begin{aligned}
h_{\overline{\mathcal{L}}}(X) = & 2\sum_{m=1}^{n} \frac{L_f'(m+1,\eta^{m+1})}{L_f(m+1,\eta^{m+1})} - (n-1)\frac{L_f'(1,\eta)}{L_f(1,\eta)} \\
& - \Big((n+1)\cdot\gamma + (n+1)\log 2\pi - \sum_{m=1}^{n}\frac{2}{m} + 1\Big)[F:\mathbb{Q}] \\
& + (n+1)\log|d_F| + \frac{n-1}{2}\log|d_{E/F}|.
\end{aligned}
$$

Here γ is the Euler constant, d_F is the discriminant of $F/\mathbb{Q}$ and $d_{E/F}$ is the norm of the relative discriminant of E/F.

Related formulas

- Modular heights of modular curves: Bost and Kuhn (2001).
- Modular heights of quaternionic Shimura curves: Kudla–Rapoport–Yang (2006) when $F = \mathbb{Q}$; Yuan (2023) when F is totally real.
- Modular heights of Hilbert modular surfaces: Bruinier–Burgos Gil–Kuhn (2007) when $F = \mathbb{Q}$.
- Modular heights of orthogonal Shimura varieties: Hörmann (2014) when $F = \mathbb{Q}$ and up to $\log \mathbb{Q}_{>0}$.
- Modular heights of Shimura varieties of unitary similitudes (GU): Bruinier–Howard (2021) when $F = \mathbb{Q}$, using a different arithmetic line bundle and a different level group.

Definition of the CM point

- Fix an orthogonal decomposition

$$V = W \oplus W^{\perp},$$

where W is a Hermitian subspace of dimension 1 of signature

$$(0,1),\ (1,0), \cdots ,\ (1,0),$$

such that the Hermitian determinants of V and W are the same.

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such that the Hermitian determinants of V and W are the same.
- Denote $U_W = U \cap \mathrm{U}(W_{\mathbb{A}_f})$. There is a natural morphism $X_{\mathbb{W},U_{\mathbb{W}}} \to X_U$. Here $X_{\mathbb{W},U_{\mathbb{W}}}$ is the zero-dimensional Shimura variety given by $\mathbb{W}$ and level group $U_{\mathbb{W}}$. Denote by $P_{\mathbb{W},U} \in \mathrm{Ch}_0(X_U)_{\mathbb{Q}}$ the normalized CM cycle of degree 1.

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- Consider the Zariski closure of $P_{\mathbb{W},U}$ in the integral model $\mathcal{X}_U$. We have a 1-cycle $\mathcal{P}_{\mathbb{W},U}$ in $\mathcal{X}_U$ of degree 1.

Main theorem of CM point

- Under the complex uniformization, this CM cycle is represented by

$$P_{\mathbb{C}} = [(e), 1] \in G(\mathbb{Q})\backslash D \times G(\widehat{\mathbb{Q}})/U.$$

Here (e) represents the negative line given by $e = \prod_v e_v$, and e is the generator of $\mathbb{W}$.

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Denote by η the quadratic character associated with E/F, and $L_f(\cdot, \eta)$ the (incomplete) Hecke L-function. We have

$$\overline{\mathcal{L}} \cdot \mathcal{P} = -\frac{L_f'(0,\eta)}{L_f(0,\eta)} - \frac{1}{2}\log d_{E/F}.$$

Here $d_{E/F}$ is the norm of the relative discriminant of E/F.

Related results

- Heights of CM points on quaternionic Shimura curves: Kudla–Rapoport–Yang (2006) when $F = \mathbb{Q}$; Yuan–Zhang (2015) when F is totally real, and they proved the average Colmez conjecture using the height formula.
- Heights of CM points on orthogonal Shimura varieties: Andreatta-Goren-Howard-Madapusi-Pera (2015) when $F = \mathbb{Q}$, and they proved the average Colmez conjecture using the height formula.

Part 2

Gross–Zagier formula

- Let N be a positive integer and $f \in S_2(\Gamma_0(N))$ a newform of weight 2. Let $K \subset \mathbb{C}$ be an imaginary quadratic field and χ a character of $\mathrm{Pic}(\mathcal{O}_K)$. Form the L-series $L(f, \chi, s)$ as the Rankin–Selberg convolution of the L-series $L(f, s)$ and the L-series $L(\chi, s)$.
- Assume that K has an odd fundamental discriminant D, and satisfies the following *Heegner condition*: every prime factor of N is split in K. Then the sign of the functional equation of the L-series $L(f, \chi, s)$ is -1 and hence $L(f, \chi, 1) = 0$.

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- Let $X_0(N)$ be the modular curve over $\mathbb{Q}$. For every nonzero ideal $\mathcal{I}$ of $\mathcal{O}_K$, let $P_{\mathcal{I}}$ denote the point on $X_0(N)(\mathbb{C})$ representing the isogeny $\mathbb{C}/\mathcal{I} \to \mathbb{C}/\mathcal{I}\mathcal{N}^{-1}$.

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- Form a point in the Jacobian $J_0(N)$ of $X_0(N)$ using the cusp ∞ by

$$P_\chi = \sum_{[\mathcal{I}] \in \mathrm{Pic}(\mathcal{O}_K)} [P_{\mathcal{I}} - \infty] \otimes \chi([\mathcal{I}]) \in J_0(N)(H) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Gross–Zagier formula

Theorem (Gross–Zagier formula (1986))

Denote by $P_\chi(f)$ *the f-isotypical component of* P_χ *in* $J_0(N)(H) \otimes_{\mathbb{Z}} \mathbb{C}$ *under the action of the Hecke operators. Denote by h the class number of K, and u half of the number of units of* $\mathcal{O}_K$*. Then*

$$\langle P_\chi(f), P_\chi(f)\rangle_{\mathrm{NT}}^{H} = \frac{hu^2|D|^{\frac{1}{2}}}{8\pi^2(f,f)} \cdot L'(f,\chi,1).$$

Here (f,f) *is the Peterson inner product of f, and* $\langle P_\chi(f), P_\chi(f)\rangle_{\mathrm{NT}}^{H}$ *denotes the Neron–Tate height over H.*

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- In 2012, Yuan–Zhang–Zhang proved a generalized Gross–Zagier formula over quaternionic Shimura curve.

A big picture–Kudla program

- In 1997, Stephen S. Kudla started the Kudla program which studies the relation among (arithmetic) special cycles and their relations with values/derivatives of Eisenstein series and L-functions. Starting with low dimensional orthogonal Shimura varieties, a lot of progress has been made in last few years in both orthogonal and unitary cases.

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- The core objects of the Kudla program are generating series formed by special cycles on Shimura varieties, as well as their arithmetic versions on integral models. Roughly speaking, the generating series are actually analogies of classical theta series in both geometric and arithmetic senses. They can be considered as a type of theta series taking values in the Chow groups or in cohomology groups.

An example from sum of two squares

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Theorem (Jacobi (1829))

Let

$$r(n) := \#\{(x, y) \in \mathbb{Z}^2 : n = x^2 + y^2\}$$

be the representation number of n. Then

$$r(n) = 4\Big(\sum_{d|n,\ d\equiv 1 \mod 4} 1 - \sum_{d|n,\ d\equiv 3 \mod 4} 1\Big).$$

An example from sum of two squares

Proof.

Consider the following *Jacobi's theta series*:

$$\theta := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots , \ \theta^2 = \sum_{n\geq 0} r(n)q^n.$$

where $q = e^{2\pi i\tau}$. It can be checked that

$$\theta(\tau) \in M_{\frac{1}{2}}(\Gamma_1(4)), \ \theta^2(\tau) \in M_1(\Gamma_1(4)).$$

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Note that $M_1(\Gamma_1(4))$ is generated by the *Eisenstein series*

$$E_1^{\chi} = 1 + c_1^{\chi} \cdot \sum_{n\geq 1} \Big(\sum_{d|n} \chi(d) \Big) q^n.$$

Then by checking the constant coefficient, we conclude the theorem. $\square$

Weil representation

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- Before the further discussion, we give a brief introduction of the Weil representation associated to $\mathrm{U}(1,1)(F_v)$. Assume V is a Hermitian space over E, $(G,H) = (\mathrm{U}(1,1), \mathrm{U}(V))$ is a reductive dual pair, and $\phi \in \mathcal{S}(V(\mathbb{A}_E))$ is a Schwartz function. The Weil representation is an action r of the group $\mathrm{U}(1,1)(F_v) \times \mathrm{U}(V(E_v))$ on V_v given as follows:
  - $r(h)\Phi(x) = \Phi(h^{-1}x), \quad h \in \mathrm{U}(V(E_v))$;
  - $r(m(a))\Phi(x) = \chi_{V,v}(a)|a|_{E_v}^{\dim V/2}\Phi(ax), \quad a \in E_v^\times$;
  - $r(n(b))\Phi(x) = \psi(bq(x))\Phi(x), \quad b \in F_v$;
  - $r(w)\Phi = \gamma(V_v, q)\widehat{\Phi}, \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

  Here

$$m(a) = \begin{pmatrix} a & \\ & \bar{a}^{-1} \end{pmatrix}, \; n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \; w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Theta series and Eisenstein series

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- Define the *Siegel Eisenstein series*

$$E(s,g,\phi) = \sum_{\gamma\in P(F)\backslash G(F)} \delta(\gamma g)^s r(\gamma g)\phi(0), \quad g\in G(\mathbb{A}). \tag{2}$$

It has a meromorphic continuation to $s\in\mathbb{C}$.

Siegel–Weil formula

Theorem (Siegel–Weil formula in unitary case, Ichino (2004))

There exists a constant κ such that

$$\kappa \cdot \int_{H(F)\backslash H(\mathbb{A})} \theta(g, h, \phi)dh = E(0, g, \phi).$$

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- The Siegel–Weil formula gives a precise identity of the form

$$\text{theta integral} \longleftrightarrow \text{special value of Siegel Eisenstein series.}$$

Local Siegel–Weil formula

- For any $a \in F$, the a-th Whittaker function is defined by

$$W_a(s,g,\Phi) = \int_{\mathbb{A}} \delta(wn(b)g)^s r(wn(b)g)\Phi(0)\psi(-ab)db;$$

$$W_{a,v}(s,g,\Phi_v) = \int_{F_v} \delta_v(wn(b)g)^s r(wn(b)g)\Phi_v(0)\psi(-ab)db.$$

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- By the standard theory,

$$E(s, g, \phi_2) = \delta(g)^s r(g)\phi_2(0) + W_0(s, g, \phi_2) + \sum_{a \in F^\times} W_a(s, g, \phi_2).$$

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Theorem (local Siegel–Weil formula)

Suppose $q(x_a) = a$, there exists a constant κ_v such that

$$\kappa_v \cdot \int_{E_v^1} r(g, h)\Phi_v(x_a)dh = W_{a,v}(0, g, \Phi_v).$$

An example from Hurwitz class number formula

- For any positive integer D, the Hurwitz class number $H(D)$ is the weighted size of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of positive definite binary quadratic forms

$$ax^2 + bxy + cy^2, \quad b^2 - 4ac = -D, \; a, b, c \in \mathbb{Z}.$$

The forms equivalent to $a(x^2 + y^2)$ and $a(x^2 + xy + y^2)$ are counted with multiplicities $\frac{1}{2}$ and $\frac{1}{3}$ respectively.

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Theorem (Kronecker (1860), Gierster (1883), Hurwitz (1885))

If m is not a perfect square, then

$$\sum_{dd'=m}\max\{d,d'\}=\sum_{t\in\mathbb{Z},4m-t^2>0}H(4m-t^2).$$

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Proof.

Denote by

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On the one hand, using the cohomological equivalence of $Z(1)$,

$$\langle Z(m), Z(1)\rangle_X = \sum_{dd'=m} \max\{d, d'\}.$$

An example from Hurwitz class number formula

Proof.

Denote by

$$Y(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H},\ X = Y \times Y.$$

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On the other hand, using the moduli interpretation,

$$Z(m) \cap Z(1) = \{(E, E), E \xrightarrow{\deg m} E\} = \sum_{t\in\mathbb{Z}, 4m-t^2>0} H(4m - t^2).$$

Idea of generating series

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- In one word, generating series is the geometric analogue of theta series, first proposed by Kudla, and is also called "geometric theta series" in literature. Unlike the classical theta series, it coefficients are special cycles on the Shimura variety. It is modular (or automorphic) in the sense that its composition with any linear functional on Chow group (with some convergence condition) is a Siegel modular form.

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- Many mathematicians have studied the modularity of generating series in different cases, including Kudla, Borcherds, Bruinier-Raum, Yuan–Zhang–Zhang and Liu.

Special divisors

- Let $x \in U\backslash V(E)$, such that the one-dimensional subspace $W_0 \subset V$ generated by x is a positive definite subspace of dimension 1. There is an orthogonal decomposition $V = W_0 \oplus W_0^{\perp}$. For any $g \in G(\widehat{\mathbb{Q}})$, define the *Kudla special divisor* on X_U

$$Z(x,g)_U := \{(z,hg) \in D \times G(\widehat{\mathbb{Q}}) \big| z \in D_{W_0}, h \in G_{W_0}(\widehat{\mathbb{Q}})\}.$$

  Here $D_{W_0} = \{v \in D \big| \langle v, W_0 \rangle = 0\}$ is a subspace of D, and $G_{W_0}(\widehat{\mathbb{Q}})$ is a subgroup of $G(\widehat{\mathbb{Q}})$ fixing every point in W_0.
- When W_0 is not positive definite, we define

$$Z(x,g)_U = 0.$$

Generating series (of special divisors)

- For any Schwartz function $\Phi \in \mathcal{S}(V(\mathbb{A}_{E,f}))^U$ and $t \in F_+$, define the weighted special divisors

$$Z_t = Z_t(\Phi)_U := \sum_{x \in U \backslash V_f, \langle x,x \rangle = t} \Phi(x) Z(x)_U.$$

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- Define the Kudla's generating function of divisors in the Chow group $\mathrm{Ch}^1(X_U, \mathbb{C})$ with complex coefficients as follows:

$$Z_\Phi(\tau) := [L^\vee]\Phi(0) + \sum_{t \in F_+} Z_t q^t. \tag{3}$$

Here $q = \prod_{k=1}^d e^{2\pi i \tau_k}$ with $\tau = (\tau_k)_{k=1}^d \in \mathcal{H}^d$, and $d = [F : \mathbb{Q}]$.

- Note that this generating series can also be written as $Z(g, \Phi)$, which is a function of g and Φ.

Geometric Siegel–Weil formula

- Denote by

$$\deg_L(Z(g,\Phi)) := Z(g,\Phi)\cdot c_1(L)^{n-1},$$

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- To summarize, geometric Siegel–Weil formula is a precise identity of the form

$$\text{generating series} \longleftrightarrow \text{special value of Siegel Eisenstein series.}$$

Gross–Keating formula

- Let $\mathcal{Y}, \mathcal{X}$ be the canonical integral models of Y and X over $\mathbb{Z}$.
- The divisor $Z(m)$ naturally extends to a divisor $\mathcal{Z}(m)$ on $\mathcal{X}$.
- $E(\tau, s)$ is th Siegel Eisenstein series on $Sp(6)$ of weight 2. Note that $E(\tau, 0) = 0$.

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Theorem (Gross–Keating (1993); Gross–Kudla–Zagier (1997))

Assume there is no positive definite binary quadratic form representing m_1, m_2, m_3 simultaneously. Then (up to an explicit constant)

$$\langle \mathcal{Z}(m_1), \mathcal{Z}(m_2), \mathcal{Z}(m_3) \rangle_{\mathcal{X}} = \sum_{T = \begin{pmatrix} m_1 & * & * \\ * & m_2 & * \\ * & * & m_3 \end{pmatrix}} E'_T(\tau, 0).$$

Idea of arithmetic generating series

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- Similarly, one can ask at least two questions:
  - Does there exist an arithmetic generating series, such that the coefficients are arithmetic special cycles on the Shimura varieties?
  - Does there exist Siegel-Weil formulas in arithmetic sense? In other words, does there exist some correspondence between arithmetic intersection numbers of arithmetic generating series and special derivative of Eisenstein series?

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  - Does there exist Siegel-Weil formulas in arithmetic sense? In other words, does there exist some correspondence between arithmetic intersection numbers of arithmetic generating series and special derivative of Eisenstein series?
- In general, both questions remain open. But there are still a lot of progress.

Modularity of arithmetic generating series

Problem (Kudla's modularity problem, divisor case)

Can we find arithmetic divisors $\widehat{\mathcal{Z}}(x)$ on $\mathcal{X}$ extending $Z(x)$, explicitly and canonically, such that

$$\widehat{\mathcal{Z}}_{\Phi}(\tau) := [\overline{\mathcal{L}}^{\vee}] + \sum_{t \in F_+} \widehat{\mathcal{Z}}_t q^t \tag{4}$$

is modular. We call this the arithmetic generating series.

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- Here are some known cases: Kudla–Rapoport–Yang for quaternionic Shimura curves over $\mathbb{Q}$ in 2006, Bruinier–Burgos Gil–Kuhn for Hilbert modular surfaces over $\mathbb{Q}$ in 2007, Howard–Madapusi Pera for orthogonal Shimura varieties over $\mathbb{Q}$ in 2020, and Bruinier–Howard–Kudla–Rapoport–Yang for unitary Shimura varieties over imaginary quadratic fields with self-dual lattice level structures in 2020.

Admissible arithmetic extensions

- Apart from the above-mentioned ones, an idea by S. Zhang (2020) to treat Kudla's modularity problem is to apply his notion of *admissible arithmetic extensions*. Roughly speaking, fixing the arithmetic line bundle $\overline{\mathcal{L}}$, for each Z_t, there is a unique $\overline{\mathcal{L}}$-admissible extension $\widehat{\mathcal{Z}_t}$ of Z_t to $\mathcal{X}$.

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- Inspired by this idea, as well as the proof of modularity for generating series on the unitary Shimura variety by Liu in 2011, Qiu solved the problem for generating series of divisors over unitary Shimura varieties (in a situation consistent with our setup) in 2023.
- A key idea is the following observation, which is called almost modularity in literature:

$$\widehat{\mathcal{Z}}_\Phi(\tau) \cdot \mathcal{P}_0$$

is modular for any arithmetic 1-cycle $\mathcal{P}_0$ such that $\deg \mathcal{P}_{0,E} = 0$.

An arithmetic Siegel–Weil formula for divisors

Theorem (G. (2025))

There exists a decomposition

$$\sum_{t \in F_+} \frac{\mathcal{Z}_t \cdot \overline{\mathcal{L}}^n}{\deg_L(X)} = -E_*(0, g, \Phi) h_{\overline{\mathcal{L}}}(Z(y_0)) + \sum_{y \in U \setminus V_f} r(g)\Phi(y) D(y),$$

where

$$h_{\overline{\mathcal{L}}}(Z(y_0)) = \frac{\mathcal{Z}(y_0) \cdot (\overline{\mathcal{L}}^n)}{\deg_L(Z(y_0))}$$

is the modular height of $Z(y_0)$, a fixed unitary Shimura variety of dimension $n-1$. Here $E_*(0, g, \Phi)$ is the non-constant term of an Eisenstein series. Each $D(y)$ is a real number associated to vector y, which essentially comes from the derivative of Eisenstein series and can be computed explicitly.

Arithmetic Siegel–Weil formula

- In general, constructing and proving the arithmetic Siegel–Weil formula is a big project.
- Locally at infinity (the Green function part), it was proved by Yifeng Liu in unitary case in 2011 and by Bruinier–Yang in orthogonal case in 2020. Garcia and Sankaran proved both unitary and orthogonal cases in more general setting in 2019.

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- In the global case, Li and W. Zhang prove an identity between the arithmetic degree of Kudla–Rapoport cycles of full rank and the derivative of nonsingular Fourier coefficients of the incoherent Eisenstein series in 2019, and Ryan Chen extends the identity to corank 1 in 2024.

Part 3

Main idea

- In one word, we try to match the arithmetic intersection number of arithmetic generating series and special value (or derivative) of L-function, then compute the modular height formula via induction on the dimension.

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- In one word, we try to match the arithmetic intersection number of arithmetic generating series and special value (or derivative) of L-function, then compute the modular height formula via induction on the dimension.
- Roughly speaking, consider the arithmetic intersection

$$\widehat{\mathcal{Z}}(g, \Phi) \cdot (\overline{\mathcal{L}}^n / L^n).$$

Then the constant term is the modular height of unitary Shimura variety, while the non-constant terms are mainly the arithmetic degree of special divisors. By definition, special divisors can be viewed as unitary Shimura varieties of lower dimension.

- We also need to find a suitable "analytic" side to match this arithmetic intersection number, in other words, we need to construct an arithmetic Siegel–Weil formula.

Difference series

- We consider the following difference series

$$\mathcal{D}(g,\Phi) := \mathrm{Pr}\, I'(0,g,\Phi) + \Big(\widehat{\mathcal{Z}}_*(g,\Phi) - \frac{\deg_L(Z_*(g,\Phi))}{\deg_L(X)}\overline{\mathcal{L}}\Big)\cdot(\mathcal{P}-\hat{\xi}),$$

  which is a cusp form for $g \in \mathrm{U}(1,1)(\mathbb{A})$.
- $\mathrm{Pr}\, I'(0,g,\Phi)$ is the *holomorphic projection* of some mixed theta-Eisenstein series, which is called the *derivative series*.
- $\mathcal{P}$ is an arithmetic 1-cycle of degree 1 given by a CM point, and $\widehat{\xi}$ is $\overline{\mathcal{L}}^n/L^n$. $\Big(\widehat{\mathcal{Z}}_*(g,\Phi) - \frac{\deg_L(Z_*(g,\Phi))}{\deg_L(X)}\overline{\mathcal{L}}\Big)\cdot(\mathcal{P}-\hat{\xi})$ is the arithmetic intersection number of the so-called *arithmetic generating series* and a degree 0 arithmetic cycle, which is called the *height series*.
- By considering the constant term in $\mathcal{D}(g,\Phi)$, we obtain an induction formula of the modular height.

Relation with previous work

- The idea of our proof is based on the subsequent works by Yuan–Zhang–Zhang on the Gross–Zagier formula, Yuan–Zhang on the average Colmez conjecture, and Yuan on the modular height of quaternionic Shimura curve.
- Below is an intuitive explanation of the connections between these works on quaternionic Shimura curve:

$$\mathcal{P} \cdot \mathcal{P} \longrightarrow \mathcal{P} \cdot \overline{\mathcal{L}} \longrightarrow \overline{\mathcal{L}} \cdot \overline{\mathcal{L}}.$$

  From left to right, the Schwartz function becomes more general, and the parts requiring specific calculations gradually increase.

The End