Modular Forms

$\mathcal{H} = \{z = x + iy \in \mathbb{C} | y > 0\}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ acts by $z \to \frac{az+b}{cz+d}$ with factor of automorphy $j(\gamma, z) := cz + d$. Translation matrices: $T^\alpha := \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) : z \mapsto z + \alpha$.

MF: $f : \mathcal{H} \to \mathbb{C}$ (various an properties), $f(\gamma\tau) = j(\gamma, \tau)^\kappa f(\tau)$ $\forall \gamma \in \Gamma$, condition at the cusps: If $\ell \in \mathbb{P}^1(\mathbb{R})$ with Γ_ℓ parabolic and $\sigma_\ell \infty = s$ then $\sigma_\ell^{-1} \Gamma \sigma_\ell \cong \langle T^{\alpha_\ell} \rangle$, set $z_\ell := \sigma_\ell^{-1} z = x_\ell + iy_\ell$ and $q_\ell := \mathbf{e}\big(\frac{z_\ell}{\alpha_\ell}\big)$, and then $f\big|_\kappa \sigma(z) = \sum_{n \in \mathbb{Z}} a_n(y_\ell) q_\ell^n$. Γ commen with $\mathrm{SL}_2(\mathbb{Z})$: $\ell \in \mathbb{P}^1(\mathbb{Q})$, $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$, α_ℓ depends only on ℓ.

When f hol, a_n const, hol at cusp: $a_n = 0 \;\forall\, n < 0$. In many cases a_n interesting function of n ($\sigma_{\kappa-1}(n)$ for Eisenstein series, relations with number of points on elliptic curves modulo p for some cusp forms of weight 2). $\dim_{\mathbb{C}} M_\kappa(\Gamma) < \infty \Rightarrow$ relations between coeffs. $S_\kappa(\Gamma) \subseteq M_\kappa(\Gamma)$: vanishing at all cusps, decay exp there. f Mer at cusp: $a_n = 0 \;\forall\, n \ll 0$. Defines $M_\kappa^!(\Gamma)$.

For $\kappa \in \frac{1}{2}\mathbb{Z}$ need subgps of

$$\mathrm{Mp}_2(\mathbb{R}) := \big\{(\gamma, \varphi) \big|\, \gamma \in \mathrm{SL}_2(\mathbb{R}),\ \varphi : \mathcal{H} \to \mathbb{C},\ \varphi(z)^2 = j(\gamma, z)\big\}.$$

As examples, theta functions (more below), Dedekind η (with char). $\frac{1}{\eta}$ related to the partition function.

Shimura (1971) relates MF's of weight $k + \frac{1}{2}$ ($k \in \mathbb{N}$) to MF's of weight $2k$, Shintani (1975) in the other way around. More explicitly, given $f \in S_{2k}(\Gamma)$, he shows that the generating series of certain linear combinations of integrals of f are the Fourier coefficients of $g \in S_{k+1/2}(\tilde{\Gamma})$.

[Sn] Shintani, T., On the construction of holomorphic cusp forms of half-integral weight, Nagoya Math. J., vol 58, 83-126 (1975).

Lattices and Geodesics

$V := M_2(\mathbb{Q})_0$ (trace 0), quad of sgn $(2,1)$ with $Q(\lambda) := -N\det\lambda$ and $(\lambda,\mu) := N\,\mathrm{Tr}(\lambda\mu)$. $G := \mathrm{Spin}(V) \cong \mathrm{SL}_2$ over $\mathbb{Q}$ by conj. Then we have $\mathcal{H} \cong \{\text{negative lines in } V_{\mathbb{R}}\}$ by the map $z \mapsto \mathbb{R}Z^{\perp}(z)$ for $Z^{\perp}(z) := \frac{1}{\sqrt{N}y}\left(\begin{smallmatrix} x & -|z|^2 \\ 1 & -x \end{smallmatrix}\right)$ with $Q\big(Z^{\perp}(z)\big) = -1$, for which the orth comp $\mathbb{R}\Re Z(z) \oplus \mathbb{R}\Im Z(z)$ where $Z(z) := \frac{1}{\sqrt{N}}\left(\begin{smallmatrix} z & -z^2 \\ 1 & -z \end{smallmatrix}\right)$.

$L \subseteq V$ *even lattice* ($Q(L) \subseteq \mathbb{Z}$), contained in the *dual lattice* $L^* := \{\lambda \in V | (\lambda, L) \subseteq \mathbb{Z}\}$ with $D_L := L^*/L$. Stable orth grp:

$$\Gamma := \Big\{\gamma \in \mathrm{SL}_2(\mathbb{R})\Big|\ \gamma L = L,\ \gamma\big|_{D_L} = \mathrm{Id}\big|_{D_L}\Big\}, \qquad \mathrm{P}\Gamma := \Gamma/\{\pm 1\}.$$

For example, for L spanned by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\frac{1}{N}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ the dual is spanned by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\frac{1}{2N}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\frac{1}{N}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $D_L \cong \mathbb{Z}/2N\mathbb{Z}$, and $\Gamma = \Gamma_0(N)$. Related to integral binary quadratic forms. We set $\pi : \mathcal{H} \to Y := \Gamma\backslash\mathcal{H}$ and $X = Y \cup \{\text{cusps}\}$, cusps are $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$. For such ℓ, β_ℓ describes $L \cap \ell$, and then $\varepsilon_\ell := \frac{\alpha_\ell}{\beta_\ell}$ equals $\sqrt{\frac{N|D_\ell|}{8}}$.

For $\lambda \in L^*$ with $m = Q(\lambda) > 0$, set $c_\lambda := \big\{z \in \mathcal{H}\big|\ \lambda \perp Z^{\perp}(z)\big\}$—oriented geodesic, as well as $c(\lambda) := \pi(c_\lambda) \subseteq Y$.

Prop: If $\frac{m}{N} \in \mathbb{Q}^2$ (split-hyper) then $\mathrm{P}\Gamma_\lambda$ triv, c_λ connects two cusps in $\mathbb{P}^1(\mathbb{Q})$, $c(\lambda) \cong c_\lambda$. Otherwise $\mathrm{P}\Gamma_\lambda$ cyclic, c_λ connects quad irrats in $\mathbb{P}^1(\mathbb{R})$, $c(\lambda) \subseteq Y$ closed geodesic.

Given $h \in D_L$ and $m \in \mathbb{Q}$, set

$$L_{m,h} := \{\lambda \in L + h | Q(\lambda) = m,\ \lambda \neq 0\}.$$

Non-empty only if $m \in \mathbb{Z} + Q(h)$. If $m \neq 0$ then $\Gamma\backslash L_{m,h}$ finite, $\mathfrak{R}_{m,h}$ set of reps.

For $f \in S_{2k}(\Gamma)$ and $m > 0$, set

$$\mathrm{Tr}_{m,h}(f) := \sum_{\lambda \in \mathfrak{R}_{m,h}} \int_{c(\lambda)} f(z)\big(\lambda, Z(z)\big)^{k-1} dz.$$

If m split-hyper, related to certain central L-values.

Shintani: For $k > 0$, $\sum_{m=1}^{\infty} \mathrm{Tr}_{m,0}(f) q^m \in S_{k+1/2}(\tilde{\Gamma})$.

To get to level 1, we obtain vector-valued modular forms.

Indefinite Theta Functions

$\mathrm{Mp}_2(\mathbb{Z}) = \big\langle T, S \big| \ S^2 = (ST)^3 = Z, \ Z^4 = (I,1)\big\rangle$, $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with $\sqrt{z} \in \mathcal{H}$ and T with $\varphi = 1$.

Weil rep: $\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}\big(\mathbb{C}[D_L]\big)$ defined by

$$\rho_L(T)\mathfrak{e}_h = \mathbf{e}\big(Q(h)\big)\mathfrak{e}_h, \quad \rho_L(S)\mathfrak{e}_h = \frac{1}{\sqrt{i|D_L|}} \sum_{g \in D_L} \mathbf{e}\big(-(g,h)\big)\mathfrak{e}_g.$$

$\tau = u + iv \in \mathcal{H}$, $z \in \mathcal{H}$, $k \in \mathbb{N}$:

$$\Theta_{k,L}(\tau, z) := \sqrt{v} \sum_{h \in D_L} \sum_{\lambda \in L+h} \big(\lambda, Z(z)\big)^k \mathbf{e}\big[Q(\lambda)\tau + \big(\lambda, Z^\perp(z)\big)^2 \tfrac{iv}{2}\big]\mathfrak{e}_h.$$

Thm: For fixed $z \in \mathcal{H}$ we have $\big(\tau \mapsto \Theta_{k,L}(\tau, z)\big) \in \mathcal{A}_{k+1/2}\big(\rho_L\big)$, and if $\tau \in \mathcal{H}$ is fixed then $\big(z \mapsto \Theta_{k,L}(\tau, z)\big) \in \mathcal{A}_{-2k}(\Gamma) \otimes_{\mathbb{C}} \mathbb{C}[D_L]$. First part by Borcherds, second is easy.

Main idea: For $f \in S_{2k}(\Gamma)$ we set

$$I_{k,L}(\tau, f) := \int_Y f(z)\Theta_{k,L}(\tau, z)d\mu(z), \qquad d\mu(z) = \frac{dxdy}{y^2},$$

which is in $\mathcal{A}_{k+1/2}\big(\rho_L\big)$. Need to show that gives Shintani. We already have the expansion, since

$$\Theta_{k,L}(\tau, z) = \sqrt{v} \sum_{h \in D_L} \sum_{m \in \mathbb{Z}+Q(h)} \Bigg[\sum_{\lambda \in L_{m,h}} \big(\lambda, Z(z)\big)^k e^{-\pi v (\lambda, Z^\perp(z))^2}\Bigg] q^m \mathfrak{e}_h$$

(plus $\sqrt{v}\mathfrak{e}_0$ when $k = 0$), can integrate for each m and h separately.

Shimura: A similar integral, in the other direction.

Proof of Shintani

Set $\mathrm{g}(\xi) := e^{-\xi^2/2}$, with decaying anti-symmetric "primitive function" $\mathrm{e}(\xi) := -\frac{\mathrm{sgn}(\xi)}{\sqrt{2}}\Gamma\big(\frac{1}{2}, \frac{\xi^2}{2}\big)$ for $\xi \neq 0$. We define

$$\psi_{k,-1}(\lambda, z) := \big(\lambda, Z(z)\big)^k \mathrm{g}\Big(\sqrt{2\pi}\big(\lambda, Z^\perp(z)\big)\Big) \qquad \text{and}$$

$$\psi_{k-1,0}(\lambda, z) := \frac{\big(\lambda, Z(z)\big)^{k-1}}{\sqrt{2\pi}} \mathrm{e}\Big(\sqrt{2\pi}\big(\lambda, Z^\perp(z)\big)\Big).$$

Summand in $\Theta_{k,L}$ is $v^{\frac{1-k}{2}}\psi_{k,-1}\big(\sqrt{v}\lambda, z\big)$, and if $L_z := -2iy^2\partial_{\overline{z}}$ is the weight lowering operator then $-L_z\psi_{k-1,0}\big(\sqrt{v}\lambda, z\big) = \psi_{k,-1}\big(\sqrt{v}\lambda, z\big)$.

For $m \neq 0$ and h unfolding gives

$$\int_Y f(z) \sum_{\lambda \in L_{m,h}} \psi_{k,-1}\big(\sqrt{v}\lambda, z\big) d\mu(z) = \sum_{\lambda \in \mathfrak{R}_{m,h}} \int_{\Gamma_\lambda \backslash \mathcal{H}} f(z)\psi_{k,-1}\big(\sqrt{v}\lambda, z\big) d\mu(z),$$

and we multiply by $v^{\frac{1-k}{2}}$ and apply Stokes:

$$\int_{\mathcal{R}} f(z)\big(-L_z G(z)\big) d\mu(z) = \oint_{\partial\mathcal{R}} f(z)G(z)dz + \int_{\mathcal{R}} L_z f(z)G(z) d\mu(z).$$

$m < 0 \Rightarrow |\Gamma_\lambda| < \infty$, $\psi_{k,-1}\big(\sqrt{v}\lambda, z\big)$ decays at $\partial\mathcal{H}$. For $m > 0$ do the same on $\mathcal{H} \setminus c(\lambda)$, decaying except on $c(\lambda)$, different signs produce $\mathrm{Tr}_{m,h}(f)$. If $m = 0$ then for $\mathfrak{L}$ representing cusps, it is roughly

$$\sum_{\lambda \in \mathfrak{L}} \int_{\Gamma_\lambda \backslash \mathcal{H}} f(z)\psi_{k,-1}\big(\sqrt{v}\lambda, z\big) d\mu(z),$$

leaves only vanishing constant term of f. QED.

Regularized Integrals

For $f \in M^!_{2k}(\Gamma)$, the integral $I_{k,L}(\tau, f)$ diverges. For regularization, assume $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, latter with the fundamental domain

$$\mathcal{F} := \left\{z \in \mathcal{H}\middle|\ |x| \leq \tfrac{1}{2},\ |z| \geq 1\right\} \cong Y(1), \qquad \mathcal{F}_T := \{z \in \mathcal{F}|y \leq T\},$$

and then $\mathcal{F}_T(L) := \bigcup_{\ell \in \mathfrak{L}} \bigcup_{j=0}^{\alpha_\ell - 1} \sigma_\ell\{z+j|z \in \mathcal{F}_T\} \cong Y$ for $\mathfrak{L} \subseteq \mathbb{P}^1(\mathbb{Q})$ fin set of reps mod Γ. Then the *regularized Shintani lift* $I^{\mathrm{reg}}_{k,L}(\tau, f)$ is

$$\mathrm{CT}_{s=0} \lim_{T \to \infty} \int_{\mathcal{F}_T(L)} f(z)\Theta_{k,L}(\tau, z) y^{-s} d\mu(z).$$

If $k > 0$ and $f \in M^!_{2k}(\Gamma)$ has no constant terms then the result of Shintani essentially holds, except that the integral $\mathrm{Tr}_{m,h}(f)$ for $m \in N \cdot \mathbb{Q}^2$ diverges as well (will be regularized below).

For evaluating $I^{\mathrm{reg}}_{k,L}(\tau, f)$ we need $\Theta_{k,L}(\tau, z)$ near a cusp. Given ℓ, set $\Theta_{k,\ell}(\tau)$ to be

$$\sum_{h \in D_L} \sum_{0 < m \in \mathbb{Q}} \left(\iota_\ell(m,h) + \overline{\delta}_{m,0}(-1)^k \iota_\ell(m,-h)\right) \frac{\mathrm{He}_k\left(2\sqrt{2\pi m v}\right)}{(2\pi v)^{k/2}} q^m \mathfrak{e}_h.$$

Prop: $(\Theta_{k,L}\ |_{2k,z}\ \sigma_\ell)(\tau, z_\ell) = \frac{i^k y_\ell^{k+1}}{\sqrt{N}\beta_\ell}\Theta_{k,\ell}(\tau) + O(e^{-C_\ell y_\ell^2})$.

Cor: If f has no constant terms then $I^{\mathrm{reg}}_{k,L}(\tau, f)$ is just the convergent limit $\lim_{T \to \infty} \int_{\mathcal{F}_T} f(z)\Theta_{k,L}(\tau, z) d\mu(z)$. Otherwise need to subtract, for every $\ell \in \mathfrak{L}$, the function $i^k \Theta_{k,\infty}(\tau) c_\ell(0) \frac{T^k}{k}$ times $\sqrt{\frac{|D_\ell|}{8}} = \frac{\varepsilon_\ell}{\sqrt{N}}$ before taking the limit.

These ideas allow to evaluate (regularized) Shintani lifts of other modular forms, yielding interesting results. We mention the case $k = 0$, as well as lifting harmonic weak Maass forms. We do it for nearly holomorphic MF's. One reason is that we can still use Stokes with an exact differential form.

[BFI]: Bruinier, J. H., Funke, J., Imamoğlu, Ö, Regularized Theta Liftings and Periods of Modular Functions, J. reine angew. Math., vol 703, 43–93 (2015).

[ANS]: Alfes-Neumann, C., Schwagenscheidt, M., Shintani Theta Lifts of Harmonic Maass Forms, to appear in Trans. Amer. Math. Soc., https://arxiv.org/abs/1712.04491.

Nearly Holomorphic MF's

We say that f is *nearly holomorphic of depth* p if $f(z)=\sum_{l=0}^{p}\frac{f_l(z)}{y^l}$ with f_l hol, $f_p\neq 0$. This is $\mathrm{SL}_2(\mathbb{R})$-invariant. Expansion at ∞ if $T^\alpha\in\Gamma$: $f(z)=\sum_{n\in\mathbb{Z}}\sum_{l=0}^{p}\frac{c(n,l)}{y^l}\mathbf{e}(nz)$, with same conditions for nearly hol at ∞ and nearly weakly hol at ∞ (similar at other cusps).

Regularized trace: $\lambda\in L^*$ split-hyper with $m=Q(\lambda)$, $c(\lambda)$ goes from $\ell_{-\lambda}$ to ℓ_λ and we define, if f has no constant coefficients,

$$\int_{c(\lambda)}^{\mathrm{reg}} f(z)\big(\lambda,Z(z)\big)^{k-1}dz:=\int_{c(\lambda)\cap\mathcal{F}_T(L)} f(z)\big(\lambda,Z(z)\big)^{k-1}dz+$$

$$+i^k(2\sqrt{m})^{k-1}\sum_{n\neq 0}\sum_{l=0}^{p}c_{\ell_\lambda}(n,l)\left(\frac{2\pi n}{\alpha_{\ell_\lambda}}\right)^{l-k}\Gamma\left(k-l,\frac{2\pi nT}{\alpha_{\ell_\lambda}}\right)+$$

$$+(-i)^k(2\sqrt{m})^{k-1}\sum_{n\neq 0}\sum_{l=0}^{p}c_{\ell_{-\lambda}}(n,l)\left(\frac{2\pi n}{\alpha_{\ell_{-\lambda}}}\right)^{l-k}\Gamma\left(k-l,\frac{2\pi nT}{\alpha_{\ell_{-\lambda}}}\right).$$

Incomplete Γ for $n<0$: If $l<k$ well-def, for $l=k$ use PV $\int_t^\infty e^{-t}\frac{dt}{t}$, if $\mu=k-l<0$ write

$$\Gamma(\mu,t)=\frac{(-1)^\mu}{|\mu|!}\left(\Gamma(0,t)+\sum_{a=0}^{|\mu|-1}\frac{a!e^{-t}}{(-t)^{a+1}}\right).$$

This is indep of T since σ_ℓ takes $\big(\lambda,Z(z)\big)$ to $2\sqrt{m}iy_\ell$ and thus $\frac{d}{dT}=0$. With constant terms, we also have to subtract $i^k(2\sqrt{Q(\lambda)})^{k-1}$ times $\sum_{l\neq k}c_{\ell_\lambda}(0,l)\frac{T^{k-l}}{l-k}-c(0,k)\log T$ and the same with $_{\ell_{-\lambda}}$ by the same idea. $\mathrm{Tr}_{m,h}^{\mathrm{reg}}(f)$ a similar sum.

For Stokes we need additional primitive functions of higher order of $\psi_{k,-1}$. Set

$$P_\nu(\xi):=\sum_{r=0}^{\lfloor\nu/2\rfloor}\frac{\xi^{\nu-2r}}{r!(\nu-2r)!2^r},\quad Q_\nu(\xi):=\sum_{a=0}^{\nu-1}\frac{(\nu-1-a)!}{\nu!}P_{\nu-1-2a}(\xi),$$

with $P_\nu=0$ if $\nu<0$ and $Q_{-1}=1$, as well as

$$h_\nu(\xi):=P_\nu(\xi)\mathrm{e}(\xi)+Q_\nu(\xi)\mathrm{g}(\xi),\quad g_{\kappa,\nu}(\xi;\eta):=(\xi+i\eta)^\kappa h_\nu(\xi),$$

and

$$\psi_{\kappa,\nu}(\lambda,z) := \frac{\big(\lambda, Z(z)\big)^{\kappa}}{(2\pi)^{(\nu+1)/2}} h_{\nu}\Big(\sqrt{2\pi}\big(\lambda, Z^{\perp}(z)\big)\Big).$$

Lem: $h'_{\nu}(\xi) = h_{\nu-1}(\xi)$, $-L_z\psi_{\kappa,\nu}\big(\sqrt{v}\lambda, z\big) = \psi_{\kappa+1,\nu-1}\big(\sqrt{v}\lambda, z\big)$ (if none of the arguments vanish—$h'_{\nu}(\xi) = h_{\nu-1}(\xi) - \sqrt{2\pi}\cdot P_{\nu}(0)\cdot\delta_{\xi=0}$ as dist)

Cor: If f is of depth p and $k \geq 0$ then

$$\int_{\mathcal{R}} f(z)\psi_{k,-1}\big(\sqrt{v}\lambda, z\big)d\mu(z) = \sum_{\nu=0}^{p}\oint_{\partial\mathcal{R}}(L_z^{\nu}f)(z)\psi_{k-\nu-1,\nu}\big(\sqrt{v}\lambda, z\big)dz.$$

Noting that $P_{\nu}(0)$ vanishes for odd ν and equals $\frac{1}{2^b b!}$ when $\nu = 2b$ is even, we can prove:

Prop: For $h \in D_L$ and $m > 0$ not split-hyper we have

$$\lim_{T\to\infty} v^{\frac{1-k}{2}}\int_{Y_T} f(z)\sum_{\lambda\in L_{m,h}}\psi_{k,-1}\big(\sqrt{v}\lambda, z\big)d\mu(z) = \sum_{b=0}^{\lfloor p/2\rfloor}\frac{\mathrm{Tr}_{m,h}(L^{2b}f)}{(4\pi v)^b b!}.$$

Note that for $L^{2b}f$ the weight is $k-2b$.

For split-hyper m, a summand λ with $Q(\lambda) = m$ will contribute an integral along $c(\lambda)\cap\mathcal{F}_T(L)$. This yields the integral part of $\sum_{b=0}^{\lfloor p/2\rfloor}\frac{\mathrm{Tr}_{m,h}^{\mathrm{reg}}(L^{2b}f)}{(4\pi v)^b b!}$, but there are two other boundary integrals, near ℓ_{λ}. If $\eta = 2\sqrt{2\pi m v}$ then the one at ∞ gives

$$\frac{\big(2\sqrt{m}\big)^{k-1}}{-\sqrt{2\pi}\cdot\eta^k}\sum_{n\in\mathbb{Z}}e^{-2\pi nT/\alpha_{\ell_\lambda}}\sum_{l=0}^{p}\frac{l!c_{\ell_\lambda}(n,l)}{T^{l-k}}\sum_{\nu=0}^{l}\frac{(-1)^{\nu}}{(l-\nu)!}\widehat{g_{k-\nu-1,\nu}}\left(\frac{-nT}{\alpha_{\ell_\lambda}\eta};\eta\right),$$

where we define the Fourier transform

$$\widehat{g_{\kappa,\nu}}(t;\eta) := \int_{-\infty}^{\infty} g_{\kappa,\nu}(\xi;\eta)\mathbf{e}(-\xi t)d\xi.$$

Since we take the limit $T\to\infty$, we only need their behavior at the limit $t\to\infty$, as well as the value as $t = 0$ in case f has constant terms. The integral near $\ell_{-\lambda}$ yields a similar contribution, with $(-1)^k$.

Fourier Transforms

One advantage of making e, and with it h_ν and $g_{\kappa,\nu}$, discontinuous at 0, is that it decays strongly in both directions and Fourier transforms can be taken.

Prop: For any ν we have

$$\widehat{h_\nu}(t) = \sqrt{2\pi}\left(\frac{\mathrm{g}(2\pi t)}{(2\pi i t)^{\nu+1}} - \sum_{r=0}^{\nu} \frac{P_{\nu-r}(0)}{(2\pi i t)^{r+1}}\right).$$

This comes from the classical evaluation $\widehat{\mathrm{g}}(t) = \frac{\mathrm{g}(2\pi t)}{\sqrt{2\pi}}$ with der of dists.

The Fourier transforms $\widehat{g_{\kappa,\nu}}$ are easy to evaluate for $\kappa \geq 0$, since it $g_{\kappa,\nu}$ is h_ν times a polynomial.

Lem: Up to an error term of $o_{\varepsilon,\nu,\kappa,\eta}\left(e^{-2\pi^2(1-\varepsilon)t^2}\right)$, the value of $\widehat{g_{\kappa,\nu}}(t;\eta)$ for $\kappa \geq 0$ is

$$-\sqrt{2\pi}(i\eta)^{\kappa+1+\nu} \sum_{b=0}^{\lfloor \nu/2 \rfloor} \frac{(-1)^b}{2^b b!(\nu-2b)!\eta^{2b}} \sum_{c=\nu-2b}^{\kappa+\nu-2b} \binom{\kappa}{c+2b-\nu} \frac{c!}{(-2\pi\eta t)^{c+1}}.$$

This uses the fact that multiplying a function of ξ by $\xi + i\eta$ operates like $i\left(\eta + \frac{\partial_t}{2\pi}\right) = \frac{i}{2\pi} e^{-2\pi\eta t}\partial_t e^{2\pi\eta t}$ on the Fourier transform.

Lem: For every $\kappa \in \mathbb{Z}$ and $\eta \neq 0$ we have

$$\widehat{g_{\kappa,\nu}}(t;\eta) = -2\pi i e^{-2\pi\eta t} \int_{-\operatorname{sgn}(\eta)\infty}^{t} e^{2\pi\eta s} \widehat{g_{\kappa+1,\nu}}(s;\eta)\, ds.$$

Prop: If $\kappa \leq -1$ then the value of $\widehat{g_{\kappa,\nu}}(t;\eta)$ is $-\sqrt{2\pi}(i\eta)^{\nu+1+\kappa}$ times

$$\sum_{b=0}^{\lfloor \nu/2 \rfloor} \frac{(-1)^b}{2^b b! \eta^{2b}} e^{-2\pi\eta t} \sum_{j=0}^{|\kappa|-1} \frac{\Gamma(2b-\nu+|\kappa|-1-j, -2\pi\eta t)(2\pi\eta t)^j}{j!(|\kappa|-1-j)!}$$

plus an error term which is $o_{\varepsilon,\nu,\kappa,\eta}\left(e^{-2\pi^2(1-\varepsilon)t^2}\right)$ as $t \to -\operatorname{sgn}(\eta)\infty$, but in the other direction this error term is

$$-\frac{(2\pi)^{|\kappa|} i^{\kappa+\nu-1} \operatorname{sgn}\eta}{(|\kappa|-1)!} J_\nu(\eta) e^{-2\pi\eta t} t^{|\kappa|-1}\big(1+o(1)\big).$$

J_ν grows like a polynomial times $e^{\eta^2/2}$.

Lem: The sum $\sum_{\nu=0}^{l}\frac{(-1)^\nu}{(l-\nu)!}\widehat{g_{k-\nu-1,\nu}}(t;\eta)$ is

$$-\sqrt{2\pi}(i\eta)^k\frac{\mathrm{He}_l(\eta)}{\eta^l l!}\cdot(-2\pi\eta t)^{l-k}e^{-2\pi\eta t}\Gamma(k-l,-2\pi\eta t),$$

with error $-2\pi(i\eta)^k\frac{(-1)^k\operatorname{sgn}\eta}{(l-k)!}e^{-2\pi\eta t}(-2\pi\eta t)^{l-k}\frac{J_l(\eta)}{\eta^l}\big(1+o(1)\big)$ in case $\eta t>0$ and $l\geq k$ but decreases rapidly otherwise.

Lem: If $l\neq k$ then the sum $\sum_{\nu=0}^{l}\frac{(-1)^\nu}{(l-\nu)!}\widehat{g_{k-\nu-1,\nu}}(0;\eta)$, evaluated at $t=0$, is $-\frac{\sqrt{2\pi}(i\eta)^k}{(l-k)l!}\big(\frac{\mathrm{He}_l(\eta)}{\eta^l}-\frac{\mathrm{He}_k(\eta)}{\eta^k}\big)$. It is much nastier when $l=k$.

Prop: Take $h\in D_L$, $m>0$ split-hyper, and $T>0$ large, and assume that f has no constant terms. Then

$$v^{\frac{1-k}{2}}\int_{Y_T}f(z)\sum_{\lambda\in L_{m,h}}\psi_{k,-1}\big(\sqrt{v}\lambda,z\big)d\mu(z)$$

equals the desired sum $\sum_{b=0}^{\lfloor p/2\rfloor}\frac{\mathrm{Tr}_{m,h}^{\mathrm{reg}}(L^{2b}f)}{(4\pi v)^b b!}$ plus a linear combination of $c(n,l)J_l\big(2\sqrt{2\pi mv}\big)$ with $n<0$ and $l\geq k$, appearing only for finitely many values of m. Works also if f has constant terms $c(0,l)$ with $l\neq k$ (correcting terms in $I_{k,L}^{\mathrm{reg}}(\tau,f)$), if $c(0,k)\neq0$ an extra term of mild growth.

The case where $p\geq k$ involves not only more complicated Fourier transforms, but also non-trivial terms with negative indices.

Prop: For every $h\in D_L$ and $0>m\in\mathbb{Z}+Q(h)$, the limit

$$\lim_{T\to\infty}v^{\frac{1-k}{2}}\int_{Y_T}f(z)\sum_{\lambda\in L_{m,h}}\psi_{k,-1}\big(\sqrt{v}\lambda,z\big)d\mu(z)$$

equals the sum

$$\sum_{\nu=k}^{p}\frac{4^k\sqrt{\pi}|m|^{\frac{k-1}{2}}h_\nu(2\sqrt{2\pi|m|v})\,\mathrm{Tr}_{m,h}^{(k)}(R_{2k-2\nu}^{\nu-k}L_z^\nu f)}{\sqrt{2}(4\sqrt{2\pi|m|v})^\nu(\nu-k)!}.$$

This is because in Stokes the argument of d is no longer smooth, and one must take out the point where $\lambda\perp Z(z)$. The trace is a sum over values at CM points. This decreases like a polynomial times $e^{-4\pi mv}$, and resembles in character the non-holomorphic part of harmonic weak Maass forms.

Lattice Sums and Constant Terms

For the constant terms, we shall need lattice sums of the $g_{\kappa,\nu}$'s:

$$G_{\kappa,\nu}(\omega; c, \eta) := \sum_{0\neq\xi\in\mathbb{Z}+\omega} g_{\kappa,\nu}(c\xi;\eta), \quad \omega \in \mathbb{R}/\mathbb{Z},\ c \in \mathbb{R}^\times.$$

We need only for $\eta = 0$, but evaluate as the limit $\eta \to 0$, since for $\eta \neq 0$ we can use Poisson summation.

$\frac{te^{\omega t}}{e^t-1} = \sum_{m=0}^{\infty} B_m(\omega)\frac{t^m}{m!}$ (Bernoulli pols), $B_m := B_m(0)$ (Bernoulli nums), $\mathbb{B}_m : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ Bernoulli funcs (same as B_m on $(0,1)$ except $\mathbb{B}_1(0+\mathbb{Z}) = 0$). For the latter, $\mathbb{B}_m(\omega) = -\sum_{0\neq t\in\mathbb{Z}} \frac{m!\mathbf{e}(t\omega)}{(2\pi i t)^m}$.

Prop: For $\omega \neq 0$ set $\omega_c := \operatorname{sgn} c \cdot \omega$, and then if $\kappa \geq 0$ then up to an error term of $o_{\varepsilon,\nu,\kappa,\eta}\big(e^{-2\pi^2(1-\varepsilon)/c^2}\big)$ as $c \to 0$, $G_{\kappa,\nu}(\omega; c, \eta)$ equals

$$\frac{\sqrt{2\pi}}{|c|}\Bigg[(-1)^{\nu+1}\kappa! P_{\kappa+\nu+1}(i\eta) +$$

$$+\sum_{\mu=0}^{\nu} \frac{P_{\nu-\mu}(-i\eta)}{\mu!} \sum_{m=0}^{\kappa+\mu+1} \binom{\kappa+\mu+1}{m} \frac{(i\eta)^{\mu+\kappa+1-m}|c|^m \mathbb{B}_m(\omega_c)}{\kappa+\mu+1}\Bigg].$$

For $\omega = 0$ we evaluate as $\lim_{\omega\to 0}\big(G_{\kappa,\nu}(\omega; c,\eta) - g_{\kappa,\nu}(c\omega;\eta)\big)$.

For negative κ we need the rational functions

$$F(q, -j) = \sum_{t=1}^{\infty} t^j q^t \quad \text{for} \quad |q| < 1 \quad \text{and} \quad j \in \mathbb{N},$$

the polygamma function $\psi^{(m)} = \frac{d^{m+1}}{dz^{m+1}} \log\Gamma(z)$, the function $Z_m(w)$ vanishing at $w = 0$ and having derivative

$$\big[(1-\delta_{m,0})Q_{m-1}(w) + wQ_m(w) + \big(P_{m-1}(w) + wP_m(w)\big)e^{w^2/2}\mathrm{e}(w)\big]\, w^{m-1},$$

the integral $\phi_m(\eta) := \int_0^\eta w^{2m} e^{w^2/2} dw$, the combinatorial coefficient $C(M, r) := \frac{2r}{(M-2r)!} + \frac{1-\delta_{2r,M}}{(M-1-2r)!}$ when $0 \leq 2r \leq M$, and the polynomials $\Pi_{\kappa,\nu}(\eta)$ defined by

$$\sum_{\mu=0}^{|\nu|-1} \frac{Q_\nu^{(\mu)}(-i\eta) Q_{-\kappa-\mu-1}(\eta)}{i^{\kappa+\nu+1} i^{\kappa+\mu} \mu!} - \sum_{\mu\geq 0,-\kappa} \frac{P_{\nu-\mu}(-i\eta)(\kappa+\mu)! P_{\kappa+\mu+1}(i\eta)}{i^{\kappa+\nu+1}\mu!}.$$

Thm: If $\kappa \le -1$, $\omega \ne 0$, and $\eta \ne 0$, then $G_{\kappa,\nu}(\omega; c, \eta)$ equals

$$\sqrt{\pi}\sum_{\mu=0}^{|\kappa|-1} \frac{P_{\nu-\mu}(-i\eta)\big[\psi^{(|\kappa|-\mu-1)}\big(1-\omega_c-\frac{i\eta}{|c|}\big)-(-1)^{\kappa+\mu}\psi^{(|\kappa|-\mu-1)}\big(\omega_c+\frac{i\eta}{|c|}\big)\big]}{\sqrt{2}|c|^{|\kappa|-\mu}\mu!(|\kappa|-\mu-1)!}+$$

$$+\sqrt{\pi}\frac{P_{\nu+\kappa+1}(-i\eta)(2\log|c|+\gamma+\log 2)}{\sqrt{2}|c|\cdot(|\kappa|-1)!}+\frac{\sqrt{2\pi}}{|c|}\sum_{\mu=0}^{|\kappa|-1}\frac{P_{\nu-\mu}(-i\eta)Z_{|\kappa|-\mu-1}(\eta)}{\mu!(-i\eta)^{|\kappa|-\mu-1}}+$$

$$+\frac{\sqrt{2\pi}}{|c|}\sum_{\mu=|\kappa|}^{|\nu|}\frac{P_{\nu-\mu}(-i\eta)}{\mu!}\sum_{m=0}^{\kappa+\mu+1}\binom{\kappa+\mu+1}{m}\frac{(i\eta)^{\kappa+\mu+1-m}|c|^m\mathbb{B}_m(\omega_c)}{\kappa+\mu+1}+$$

$$+\frac{\sqrt{2\pi}}{|c|}i^{\kappa+\nu+1}\Pi_{\kappa,\nu}(\eta)+\frac{\sqrt{2\pi}}{|c|}e^{\eta^2/2}\mathrm{e}(\eta)\left[\frac{iQ_{\nu+\kappa+1}(-i\eta)}{(|\kappa|-1)!}-\sum_{\mu=0}^{|\kappa|-2}\frac{i^{|\kappa|-\mu-1}P_{|\kappa|-\mu-2}(\eta)P_{\nu-\mu}(-i\eta)}{\mu!(|\kappa|-1-\mu)}\right]+$$

$$+\sum_{j=0}^{|\kappa|-1}\left(\frac{-2\pi}{|c|\operatorname{sgn}\eta}\right)^{j+1}\frac{F\big(e^{2\pi(i\delta\omega-|\eta/c|)},-j\big)}{j!}\left[i^{j+1}e^{\eta^2/2}\frac{Q_{\nu+\kappa+j+1}(-i\eta)}{(|\kappa|-j-1)!}+\right.$$

$$\left.+\sum_{\mu=0}^{|\kappa|-1-j}\frac{P_{\nu-\mu}(-i\eta)}{i^{\kappa+\mu+1}\mu!}\left[\sum_{r=0}^{\lfloor(|\kappa|-\mu-j)/2\rfloor}\frac{C(|\kappa|-\mu-j,r)\phi_{|\kappa|-\mu-j-r-1}(\eta)}{2^r r!\eta^{|\kappa|-\mu-j-1}}-\frac{\overline{\delta}_{\mu,|\kappa|-j-1}e^{\eta^2/2}P_{|\kappa|-\mu-j-2}(\eta)}{|\kappa|-1-j-\mu}\right]\right]$$

plus an error of $o_{\varepsilon,\kappa,\nu,\eta}(e^{-2\pi^2(1-\varepsilon)/c^2})$.

In the combination $\sum_{\nu=0}^{l}\frac{(-1)^\nu}{(l-\nu)!}G_{k-1-\nu,\nu}(\omega;c,\eta)$ there are cancelations, in particular we have:

Lem: The sum $\sum_{\nu=0}^{l}\frac{(-1)^\nu\Pi_{k-1-\nu,\nu}(\eta)}{(l-\nu)!}$ equals $-\frac{\mathrm{He}_k(\eta)}{l!(k-l)}$ in case $l<k$ and equals $\sum_{b=1}^{l}\frac{(-1)^{k+b-1}(l-k+b-1)!}{b!(l-b)!(l-k)!}Q_{2b-1-k}(\eta)$ for $l \ge k$.

There are additional simplifications in the limit $\eta \to 0$:

Prop: If $\omega \ne 0$ then the value of $\sum_{\nu=0}^{l}\frac{(-1)^\nu}{(l-\nu)!}G_{k-1-\nu,\nu}(\omega;c,0)$ is

$$\frac{\sqrt{2\pi}}{|c|(k-l)}\left[P_l(0)|c|^{k-l}\mathbb{B}_{k-l}(\omega_c)-\frac{i^k\,\mathrm{He}_k(0)}{l!}\right]+o_{\varepsilon,k,l}(e^{-2\pi^2(1-\varepsilon)/c^2})$$

for $l<k$, and in case $l \ge k$ it equals

$$\frac{\sqrt{2\pi}}{|c|}P_l(0)\left[\frac{\psi^{(l-k)}(1-\omega_c)+(-1)^{l-k}\psi^{(l-k)}(\omega_c)+\delta_{l,k}(2\log|c|+\gamma+\log 2)}{2|c|^{l-k}(l-k)!}+\right.$$

$$\left.+\overline{\delta}_{l,k}\frac{i^{l-k}Q_{l-k-1}(0)}{l-k}\right]+\frac{2\pi}{|c|}Q_l(0)i^{l-k+1}\left[\frac{(-2\pi)^{l-k}F\big(\mathbf{e}(\omega_c),k-l\big)}{|c|^{l-k}(l-k)!}+\frac{\delta_{l,k}}{2}\right]+$$

$$+\frac{\sqrt{2\pi}i^k}{|c|}\sum_{b=1}^{l}\frac{(-1)^{k+b-1}(l-k+b-1)!}{b!(l-b)!(l-k)!}Q_{2b-1-k}(0)+o_{\varepsilon,k,l}(e^{-2\pi^2(1-\varepsilon)/c^2}).$$

Prop: For $\omega=0$, the sum $\sum_{\nu=0}^{l}\frac{(-1)^{\nu}}{(l-\nu)!}G_{k-1-\nu,\nu}(0;c,0)$ takes the value

$$\frac{\sqrt{2\pi}}{|c|(k-l)}\left[\overline{\delta}_{l,k-1}P_l(0)|c|^{k-l}B_{k-l}-\frac{i^k\operatorname{He}_k(0)}{l!}\right]+\delta_{l,k-1}Q_{k-1}(0)$$

if $l<k$, and when $l\geq k$ its value is

$$\frac{\sqrt{2\pi}}{|c|}P_l(0)\left[\frac{\psi^{(l-k)}(1)\big[1+(-1)^{l-k}\big]+\delta_{l,k}(2\log|c|+\gamma+\log 2)}{2|c|^{l-k}(l-k)!}+\overline{\delta}_{l,k}\frac{i^{l-k}Q_{l-k-1}(0)}{l-k}\right]+$$

$$-\overline{\delta}_{l,k}\frac{2\pi}{|c|}Q_l(0)i^{l-k+1}\frac{(-2\pi)^{l-k}B_{l-k+1}}{|c|^{l-k}(l-k+1)!}+\frac{\sqrt{2\pi}i^k}{|c|}\sum_{b=1}^{l}\frac{(-1)^{k+b-1}(l-k+b-1)!}{b!(l-b)!(l-k)!}Q_{2b-1-k}(0),$$

both up to the error term $o_{\varepsilon,k,l}\big(e^{-2\pi^2(1-\varepsilon)/c^2}\big)$.

Set $\Phi_w(\omega)$ to be $-\frac{\mathbb{B}_w(\omega)}{w}$ if $w>0$ and $-\frac{\psi^{(|w|)}(1-\omega)+(-1)^w\psi^{(|w|)}(\omega)}{2\cdot|w|!}$ in case $w\leq 0$, completed with argument 1 if $\omega=0$, and the contribution of ℓ to $\operatorname{Tr}_{0,h}^{\mathrm{reg}}(f)$ is $\sqrt{\frac{|D_\ell|}{8}}=\frac{\varepsilon_\ell}{\sqrt{N}}$ times $\iota_\ell(0,h)$ times $c_\ell(0,0)(\sqrt{N}\beta_\ell)^k\Phi_k\big(\frac{k_{\ell,h}}{\beta_\infty}\big)$.

Prop: The contribution of the cusp ℓ to the integral

$$v^{\frac{1-k}{2}}\int_{Y_T}f(z)\sum_{\lambda\in L_{0,h}}\psi_{k,-1}\big(\sqrt{v}\lambda,z\big)d\mu(z)$$

is $\sqrt{\frac{|D_\ell|}{8}}=\frac{\varepsilon_\ell}{\sqrt{N}}$ times $\iota_\ell(0,h)$ times

$$-\sum_{l=0}^{p}\frac{l!c_\ell(0,l)\sqrt{N}\beta_\infty T^{k-1-l}}{(2\pi)^{k/2}v^{\frac{k-1}{2}}}\sum_{\nu=0}^{l}\frac{(-1)^{\nu}}{(l-\nu)!}G_{k-1-\nu,\nu}\big(\tfrac{k_h}{\beta_\infty};\tfrac{\sqrt{2\pi Nv}\beta_\infty}{T},0\big).$$

This equals $\sum_{b=0}^{\lfloor p/2\rfloor}\frac{\operatorname{Tr}_{0,h}^{\mathrm{reg}}(L^{2b}f)}{(4\pi v)^b b!}$ (with weight $k-2b$), plus the correction terms from the regularization, plus additional terms with $l\geq k$. The additional terms involve half-integral powers of $\frac{1}{v}$ if $l\neq k$, and more complicated expressions in case $l=k$. All vanish if f has no constant terms.

The Regularized Shintani Lift

We define

$$I^{\mathrm{nh}}_{k,L}(\tau,f) := \sum_{b=0}^{\lfloor p/2\rfloor}\sum_{h\in D_L}\sum_{0\leq m\in\mathbb{Z}+Q(h)} \frac{\mathrm{Tr}^{(\mathrm{reg})}_{0,h}(L^{2b}f)}{(4\pi v)^b b!} q^m \mathfrak{e}_h$$

(which is nearly holomorphic of depth $\left\lfloor \frac{p}{2}\right\rfloor$). We set $I^{\mathrm{neg}}_{k,L}(\tau,f)$ to be

$$\sum_{h\in D_L}\sum_{0>m\in\mathbb{Z}+Q(h)}\sum_{\nu=k}^{p}\frac{4^k\sqrt{\pi}|m|^{\frac{k-1}{2}}h_\nu(2\sqrt{2\pi|m|v})\,\mathrm{Tr}^{(k)}_{m,h}(R^{\nu-k}_{2k-2\nu}L^{\nu}_{z}f)}{\sqrt{2}(4\sqrt{2\pi|m|v})^\nu(\nu-k)!} q^m\mathfrak{e}_h$$

(plus some contribution from the constant terms), and get

Thm (Li-Z): The Shintani lift $\tau\mapsto I_{k,L}(\tau,f)$ of $f\in\widetilde{M}^{!}_{2k}(\Gamma)$ is given by

$$I_{k,L}(\tau,f) = I^{\mathrm{nh}}_{k,L}(\tau,f)+I^{\mathrm{neg}}_{k,L}(\tau,f)+I^{\mathrm{prin}}_{k,L}(\tau,f)+I^{\mathrm{const}}_{k,L}(\tau,f)+I^{\mathrm{cor}}_{k,L}(\tau,f),$$

where $I^{\mathrm{prin}}_{k,L}(\tau,f)$ is a finite sum of exponentially increasing functions based on $m\in N\mathbb{Q}^2$ with coefficients $c(n,l)$ with $n<0$ and $l\geq k$, $I^{\mathrm{const}}_{k,L}(\tau,f)$ depends only on the coefficient $c(0,k)$, and $I^{\mathrm{cor}}_{k,L}(\tau,f)$ is a small correction term appearing only with $c(0,k-1)$. For $k=0$ we have to add $\sqrt{v}\int_Y^{\mathrm{reg}} f(z)d\mu(z)\mathfrak{e}_0$.

Cor: If $p<k$ then $\tau\mapsto I_{k,L}(\tau,f)$ is nearly holomorphic (no poles) of weight $k+\frac{1}{2}$ and depth $\left\lfloor\frac{p}{2}\right\rfloor$.

In particular, this is always the case for anisotropic lattices, where Γ has no cusps at all and no regularization is necessary (then it is cuspidal).