Twisting moduli, meromorphy and zeros

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Contents

L-functions

- **Dirichlet series**: There are $a_n \in \mathbb{C}$ such that $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, converging absolutely for $\mathrm{Re}(s) > 1$.
- **Continuation**: There is an integer $m \geq 0$ such that $(s-1)^m L(s)$ continues to an entire function of finite order.
- **Functional equation**: $\Lambda(s) = \omega \overline{\Lambda(1-\bar{s})}$ for some $|\omega| = 1$ where $\Lambda(s)$ is the completed L-function:

$$\Lambda(s) = Q^s L(s) \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j), \quad Q, \lambda_j \in \mathbb{R}_{>0},\ \mu_j \in \mathbb{C},\ k \in \mathbb{Z}_{>0}.$$

- **Euler product**: $a_1 = 1$ and $\log L(s) = \sum_{n=2}^{\infty} b_n n^{-s}$ where b_n is supported on prime powers and $b_n = O\left(n^{\theta}\right)$ for some $\theta < \frac{1}{2}$.

Degree (theory)

Say $\Lambda(s)$ is the completion of $L(s)$:

$$\Lambda(s) = Q^s L(s) \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j).$$

- Roughly speaking, the degree of L is the number gamma functions appearing in Λ.
- This number is ambiguous due to the multiplication theorem for the gamma function.
- On the other hand, the sum $d = 2\sum_{j=1}^{k} \lambda_j$ is invariant and is known as the **degree**.

Degree (practice)

Completion

$\Lambda(s) = Q^s L(s) \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j)$.

Degree

$d = 2\sum_{j=1}^{k} \lambda_j$.

In all known examples:

- The λ_j can be taken to be $\frac{1}{2}$ and d is an integer.
- The Euler factor at a good prime p is the reciprocal of a degree d polynomial in p^{-s}.

$d = 1$

We will use the following normalization of the gamma function:

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

Examples

- The Riemann zeta function $\zeta(s)$ has gamma factor $\Gamma_{\mathbb{R}}(s)$.
- If χ is an Dirichlet character of parity ϵ, then $L(s, \chi)$ has gamma factor $\Gamma_{\mathbb{R}}(s + \epsilon)$.

In 1999, Kaczorowski–Perelli proved that the degree 1 elements in the Selberg class are precisely the Dirichlet L-functions.

$d > 1$

If K is a number field, then $\zeta_K(s)$ has degree $d = [K : \mathbb{Q}]$. In particular:

- If K is a real quadratic field, then the Dedekind zeta function $\zeta_K(s)$ has gamma factor $\Gamma_{\mathbb{R}}(s)^2$.
- If K is a quadratic imaginary field, then $\zeta_K(s)$ has gamma factor $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$.

Other examples:

- If ρ is a d-dimensional Artin representation then $L(s, \rho)$ has degree d.
- If π is a cuspidal automorphic representation of $\mathrm{GL}_d(\mathbb{A}_{\mathbb{Q}})$, then $L(s, \pi)$ has degree d.

Hecke

- Automorphic forms on $\mathrm{GL}(2)$ give examples of degree 2 L-functions, but a complete classification remains unknown.
- We can simplify the classification problem by specifying the invariants, such as conductor and gamma factor.
- In 1936, Hecke proved that classical newforms $f(z) = \sum_{n=1}^{\infty} a_n \exp(2\pi i n z)$ on $\Gamma_0(N)$ for $N = 1, 2, 3, 4$ are characterized by their L-functions $L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- In 1949, Maass proved the analogous result for real-analytic forms.

$N = 1$

Recall that $\Gamma_0(1)$ is generated by the matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Modularity with respect to the first generator follows from the Fourier expansion of f. By Mellin inversion, for $y > 0$ we have:

$$f(iy) = \frac{1}{2\pi i} \int_{(c)} \Lambda_f(s) y^{-s} ds$$

Applying the functional equation $\Lambda_f(s) = (-1)^{k/2} \Lambda_f(k-s)$ and shifting the contour of integration, we deduce modularity with respect to the second generator.

$N \in \{2, 3, 4\}$

For these N, $\Gamma_0(N)$ is generated by:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}.$$

We handle the first generator as above. Modularity with respect to the second generator follows from the functional equation and the following identity:

$$\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}^{-1}.$$

Twisting

- In order to extend Hecke's argument to every N, one could try writing down generators for $\Gamma_0(N)$ and verifying modular transformation laws for each.
- In 1967, Weil discovered that this can be avoided by incorporating twisted functional equations.
- If $L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and ψ is a Dirichlet character, then we define $L_f(s,\psi) = \sum_{n=1}^{\infty} a_n \psi(n) n^{-s}$.
- If ψ is a primitive Dirichlet character mod q coprime to N, then the twisted functional equation may be written

$$\Lambda_f(s,\psi) = \omega(f,\psi)\Lambda_g(k-s,\bar{\psi}),$$

where $g(z) = f(-1/Nz)$.

Weil

Weil, 1967

Let a_n, b_n be sequences of complex numbers such that $a_n, b_n = O(n^\sigma)$ for some $\sigma > 0$, and let $\mathcal{P}$ be a suitable set of moduli. For every primitive Dirichlet character ψ of modulus $q \in \{1\} \cup \mathcal{P}$ assume that $\sum_{n=1}^\infty a_n \psi(n) n^{-s}$ and $\sum_{n=1}^\infty b_n \bar{\psi}(n) n^{-s}$ admit analytic continuation to $\mathbb{C}$ and satisfy the twisted functional equations. Then $f(z) = \sum_{n=1}^\infty a_n \exp(2\pi i n z)$ is a weight k cusp form on $\Gamma_0(N)$ and $g(z) = \sum_{n=1}^\infty b_n \exp(2\pi i n z) = f(-1/Nz)$.

The proof is a bit tricky, and we won't review it here!

Twisting moduli

- Recall that Hecke's converse theorem (1936) did not require any twisting at all for $1 \leq N \leq 4$.
- Razar (1977) found explicit finite sets of twisting moduli, depending on N.
- Assuming an Euler product, Conrey–Farmer (1995) and Conrey–Farmer–Odgers–Snaith (2007) established twistless converse theorems for $5 \leq N \leq 17$ and $N = 23$.
- Combining Conrey–Farmer and Weil, Diaconu–Perelli–Zaharescu (2002) showed that there is a single prime p such that one can take $\mathcal{P} = \{p\}$
- Bedert–Cooper–O–Zhang (2020) established twistless converse theorems for $N \in \{18, 20, 24\}$.

Modern context

- A very well-known converse theorem for automorphic representations of $\mathrm{GL}(2)$ was established by Jacquet–Langlands in 1970.
- The Jacquet–Langlands converse theorem was generalised by Cogdell–Piatetski-Shapiro to $\mathrm{GL}(n)$ in the 1990s. Their results are central to the contemporary theory, and have been applied in the proofs of certain cases of Langlands functoriality.
- For $n > 2$, it is necessary to twist by representations of $\mathrm{GL}(n-2)$.
- We are interested in twistless converse theorems for $\mathrm{GL}(2)$, and applications to critcal zeros.

$N = 5$ (I)

- The group $\Gamma_0(5)$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

- As before, modularity with respect to the first two matrices follows from the Fourier expansion and functional equation. It remains to verify modularity with respect to the final generator.
- Let Ω_f denote the annihilator ideal of f in $\mathbb{C}[\mathrm{GL}_2(\mathbb{R})^+]$.
- The Euler product implies that $T_2 f \equiv a_2 f \mod \Omega_f$, where T_2 is the Hecke operator:

$$T_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \in \mathbb{C}[\mathrm{GL}_2(\mathbb{R})^+].$$

$N = 5$ (II)

We have the Heke relation:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \equiv a_2 \bmod \Omega_f$$

Right-multiplying by $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \equiv \pm 1$, we get:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} \equiv a_2 \bmod \Omega_f.$$

Subtracting the above two equations gives:

$$\begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \bmod \Omega_f.$$

$N = 5$ (III)

We have:

$$\begin{pmatrix} 2 & 0 \\ 5 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \bmod \Omega_f.$$

Right-multiplying by $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and applying $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \equiv 1$, we get:

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \equiv 1 \bmod \Omega_f,$$

that is, f is modular with respect to the final generator of $\Gamma_0(5)$.

Weil's lemma

Definition

We say a matrix $E \in \mathrm{SL}_2(\mathbb{R})$ is elliptic if $|\mathrm{Tr}(E)| < 2$.

- Elliptic matrices have a unique fixed point in the upper half-plane $\mathcal{H}$.
- An elliptic matrix with integer coefficients necessarily has finite order.

Lemma

Say $k \in \mathbb{Z}_{\geq 0}$. If $h : \mathcal{H} \to \mathbb{C}$ is holomorphic and satisfies $h|_k E = h$ for some infinite order elliptic E, then h is constant.

Proof of Weil's lemma (I)

- Let $P \in \mathcal{H}$ denote the unique fixed point of E.
- The **Cayley transform** given by the matrix

$$K = \frac{1}{\sqrt{P - \overline{P}}} \begin{pmatrix} 1 & -P \\ 1 & -\overline{P} \end{pmatrix}$$

  takes $\mathcal{H}$ to the open unit disk $\mathcal{D}$ and $P \in \mathcal{H}$ to $0 \in \mathcal{D}$.
- The hyperbolic circles centred at P correspond under K to the Euclidean circles

$$C_r = \{re^{it} : t \in [0, 2\pi)\} \subset \mathcal{D}, \quad r < 1.$$

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Proof of Weil's lemma (II)

The transformation $R = KEK^{-1}$ on $\mathbb{P}^1(\mathbb{C})$ has the form

$$\begin{pmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{pmatrix}, \; \theta \in \mathbb{R}\backslash\mathbb{Q}.$$

Because $h = h|_k E$, the function $\tilde{h}(z) = h(K^{-1}z)$ satisfies:

$$\tilde{h}(z) = e^{ki\pi\theta}\tilde{h}(e^{2\pi i\theta}z).$$

For $k > 0$, this implies $h = 0$. For $k = 0$, we get:

$$\tilde{h}(Rz) = \tilde{h}(z).$$

Since the set $\{e^{2\pi im\theta} : m \in \mathbb{Z}\}$ is dense on the unit circle, we have:

$$\tilde{h}(z) = \tilde{h}(|z|).$$

This means that h is constant on hyperbolic circles $K^{-1}C_r$ centred at P. Because h is holomorphic, we deduce that h is constant on $\mathcal{H}$.

$N = 11$

- $\Gamma_0(11)$ is generated by the matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 11 & 6 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}.$$

- Modularity with respect to the first matrix follows from the Fourier expansion, and the second matrix follows from an argument similar to that given when $N = 5$.
- As for the third, Conrey–Farmer find the infinite order elliptic matrix $\begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix}$ which satisfies:

$$1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix} \equiv \left(1 - \begin{pmatrix} 3 & 1 \\ 11 & 4 \end{pmatrix}\right) \begin{pmatrix} 1 & -2/3 \\ 11/2 & -8/3 \end{pmatrix}.$$

- For $k > 0$, it therefore follows that $f - f|_k\gamma = 0$ as required.

Maass forms (weight 0)

- A **weight 0 Maass form** is an eigenfunction of the hyperbolic Laplacian satisfying certain growth and automorphy properties.
- A cuspidal weight 0 Maass form with eigenvalue $\frac{1}{4} - \nu^2$ has a Fourier expansion of the form

$$f(x+iy) = \sum_{n \neq 0} a_n \sqrt{y} K_\nu(2\pi n y) \exp(2\pi i n x),$$

where K_ν is the K-Bessel function:

$$K_\nu(u) = \frac{1}{2} \int_0^\infty e^{-|u|(t+t^{-1})/2} y^{\nu-1} dt$$

Parity

- A weight 0 Maass form f with Fourier expansion $\sum_{n\neq 0} a_n \sqrt{y} K_\nu(2\pi n y)\exp(2\pi i n x)$, is **even** (resp. **odd**) if $a_n = a_{-n}$ (resp. $a_n = -a_{-n}$).
- The **parity** of f is 0 if f is even and 1 if f is odd.
- A weight 0 Maass form with Laplace eigenvalue $\frac{1}{4} - \nu^2$ and parity ϵ, has gamma factor:

$$\Gamma_{\mathbb{R}}(s + \epsilon + \nu)\Gamma_{\mathbb{R}}(s + \epsilon - \nu).$$

Maass forms (weight 1)

- Similarly, a weight k Maass form is an eigenfunction of the weight k hyperbolic Laplacian with certain growth and automorphy properties.
- A weight k Maass form has Fourier expansion similar to those in weight 0, in which K_ν is replaced by a more general Whittaker function.
- We say that a weight 1 Maass form f with eigenvalue $\frac{1}{4}-\nu^2$ has parity $\epsilon \in \{-1,1\}$ if $a_{-n} = \epsilon\nu a_n$.
- A weight 1 Maass form of parity ϵ and eigenvalue $\frac{1}{4}-\nu^2$ has gamma factor:

$$\Gamma_{\mathbb{R}}\left(s+\frac{1+\epsilon}{2}+\nu\right)\Gamma_{\mathbb{R}}\left(s+\frac{1-\epsilon}{2}-\nu\right).$$

- We will only ever consider weight 0 and weight 1 Maass forms.

Weil's lemma for real-analytic functions (I)

- The proof of Weil's Lemma shows that it remains valid for real-analytic functions in weights > 0.
- The proof of Weil's lemma in weight 0 relied crucially on the holomorphy of h.
- In fact, there are real-analytic functions invariant under the weight 0 action of infinite order elliptic matrices that are non-constant.
- The proof of Weil's Lemma shows that if a continuous function is invariant under an infinite order elliptic then it is constant on hyperbolic circles around the fixed point.
- It follows that if a continuous function is invariant under two infinite order elliptic matrices with distinct fixed points, then it is constant.

Weil's lemma for real-analytic functions (II)

In fact, one does not need to construct two infinite order elliptic matrices.

Lemma

Let $c : \mathcal{H} \to \mathbb{C}$ be a continuous function. Say there exist $E_1, E_2 \in \mathrm{SL}_2(\mathbb{R})$ such that E_1 is an elliptic matrix of infinite order with fixed point $a \in \mathcal{H}$, a is not a fixed point of E_2, and $c|_0 E_1 = c|_0 E_2 = c$, then c is constant on $\mathcal{H}$.

Proof

Indeed, since E_1 is elliptic, we know that $|\mathrm{tr}(E_1)| = |\mathrm{tr}(E_2 E_1 E_2^{-1})| < 2$. Subsequently, we deduce that $E_3 = E_2 E_1 E_2^{-1}$ is elliptic. Since E_1 has infinite order, so does E_3.

Conrey–Farmer–Maass

- The results of Conrey–Farmer apply immediately to weight 1 Maass forms, noting that the proof relies only on Weil's lemma and the axiomatic properties of group actions.
- In Bedert–Cooper–O–Zhang (2020), a second matrix is found which works for weight 0 Maass forms on $\Gamma_0(N)$ in the cases $1 \leq N \leq 12$ and $N \in \{16, 18\}$.

Critical zeros

For the remainder of this talk we will focus on the holomorphy assumptions of Weil's converse theorem. Our motivation for doing so is the non-trivial zeros of automorphic L-functions. Roughly speaking:

- The non-trivial zeros should be simple;
- Distinct L-functions should not have common zeros in the critical strip.

The **Grand Simplicity Hypothesis** asserts more precisely that non-real zeros in the critical strip should be linearly independent over $\mathbb{Q}$.

Multiple real zeros

L-functions may have multiple zeros on the real line, corresponding to interesting arithmetic.

Examples

1. Let $K/\mathbb{Q}$ be a number field such that $\mathcal{O}_K$ has rank $r > 1$. The Dedekind zeta function $\zeta_K(s)$ has a zero of order r at $s = 0$.
2. Let $E/\mathbb{Q}$ be an elliptic curve of rank $r > 1$. The conjecture of Birch–Swinnerton-Dyer implies that the Hasse–Weil L-function $L(E, s)$ has a zero of order r at $s = 1$.

Multiple imaginary zeros

- Artin L-functions may have multiple zeros in the critical strip corresponding to multiplicity in the decomposition of a Galois representation.
- Indeed, let ρ_1 and ρ_2 be continuous complex representations of $\mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right)$. We have

$$L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2).$$

- In the Selberg class, we say an L-function is primitive if it is not a product of lower degree L-functions.

Common zeros

Given two distinct L-functions L_1 (resp. L_2) in the Selberg class, let d_1 (resp. d_2) denote the degree and let S_1 (resp. S_2) denote the set of zeros in the critical strip.

- A lower bound for the number of elements in the symmetric difference $(S_1 \backslash S_2) \bigcup (S_2 \backslash S_1)$ up to a finite height was given by Murty–Murty (1994).
- Under certain orthogonality hypotheses, a lower bound for the asymmetric difference $S_1 \backslash S_2$ when $d_1 = d_2$ was established by Bombieri–Perelli (1998).
- When $d_2 - d_1 \leq 0$, a lower bound for $S_1 \backslash S_2$ was proved by Srinivas (2003).
- Much less is known in the cases where $d_2 - d_1 > 0$.

$d_2 - d_1 = 1$

Booker, 2013

Let π_1 (resp. π_2) be a unitary cuspidal automorphic representation of $\mathrm{GL}_{d_1}(\mathbb{A}_{\mathbb{Q}})$ (resp. $\mathrm{GL}_{d_2}(\mathbb{A}_{\mathbb{Q}})$) such that $d_2 - d_1 = 1$, then $\Lambda(s, \pi_2)/\Lambda(s, \pi_1)$ has infinitely many poles.

Proof idea

If the quotient had finitely many poles, then it would be a completed Dirichlet L-function. This violates the cuspidality of π_2, which gives rise to a primitive element in the Selberg class.

We can use degree 2 converse theorems to extend this argument to more general quotients, however we will encounter non-standard Euler factors and poles in the critical strip.

Weil with poles

Booker–Krishnumurthy, 2013

Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset \mathbb{C}$ satisfy $|a_n|, |b_n| = O(n^{\sigma})$, for some $\sigma \in \mathbb{R}_{>0}$. Let $\mathcal{P}$ be a set of primes such that $\{p \in \mathcal{P} : p \equiv u \pmod{v}\}$ is infinite for every $u, v \in \mathbb{Z}_{>0}$ with $(u, v) = 1$ and $p \nmid N$ for any $p \in \mathcal{P}$. If, for all $q \in \mathcal{P} \cup \{1\}$, $\Lambda_f(s, \psi)$ and $\Lambda_g(s, \overline{\psi})$ admit **meromorphic** continuation to $\mathbb{C}$ and satisfy the functional equation, and there is a polynomial $P(s) \in \mathbb{C}[s]$ such that $P(s)\Lambda_f(s, \mathbf{1})$ is entire, then $f(z) := \sum_{n=1}^{\infty} a_n \exp(2\pi i n z)$ is a modular form on $\Gamma_0(N)$ of weight k and $g(z) = f(-1/Nz)$.

Real-analytic version

Neururer–O (2018), Hochfilzer–O (2019)

Let $\epsilon \in \{0,1\}$, $\nu \in \mathbb{C}$, and a_n, b_n be sequences of complex numbers such that $|a_n|, |b_n| = O(n^\sigma)$ for some $\sigma \in \mathbb{R}_{>0}$. Let $\mathcal{P}$ be a set of primes such that $\{p \in \mathcal{P} : p \equiv u \pmod{v}\}$ is infinite for every $u, v \in \mathbb{Z}_{>0}$ with $(u,v) = 1$ and $p \nmid N$ for any $p \in \mathcal{P}$. If, for all $q \in \mathcal{P} \cup \{1\}$, $\Lambda_f(s,\psi)$ and $\Lambda_g(s,\overline{\psi})$ admit **meromorphic** continuation to $\mathbb{C}$ and satisfy the functional equation, and there is a polynomial $P(s) \in \mathbb{C}[s]$ such that $P(s)\Lambda_f(s,\mathbf{1})$ is entire, then $f(z) := \sum_{n\neq 0} a_n \sqrt{y} K_\nu(2\pi ny)\exp(2\pi inx)$ is a Maass form on $\Gamma_0(N)$ of weight 0 and $g(z) = f(-1/Nz)$.

(NO) applies when $\nu \neq 0$;

(HO) applies when $\nu = 0$.

Sym^2

Given a Maass form f, its symmetric square L-function is given by:

$$L(\mathrm{Sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a_{n^2}}{n^s}.$$

It's gamma factor is given by:

$$\pi^{-3s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right),$$

and so $\mathrm{Sym}^2 f$ has degree 3. Gelbart–Jacquet showed that $\mathrm{Sym}^2 f$ defines an isobaric automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$.

$d_2 - d_1 = 2$ (I)

Neururer–O, 2018

Let $\xi(s)$ denote the completed Riemann zeta function. If f is a (non-CM) Maass form on $\Gamma_0(N)$, then $\Lambda(\mathrm{Sym}^2 f, s)/\xi(s)$ has infinitely many poles.

Proof idea

The quotient $\Lambda_2(s)/\Lambda_1(s)$ has the functional equation of a Maass form with eigenvalue $\frac{1}{4} - \nu^2$. If it has finitely many poles, then the converse theorem of Neururer–O implies that it is the L-function of a Maass form. This gives an impossible linear dependence of Euler products.

Using different methods, Raghunathan (1999) obtained this result for modular forms f.

$d_2 - d_1 = 2$ (II)

Hochfilzer–O, 2019

Let $\Lambda_1(s)$ (resp. $\Lambda_2(s)$) be a completed primitive Artin L-function of degree d_1 (resp. d_2) such that $d_2 - d_1 \leq 2$. If the gamma factors cancel as required, then $\Lambda_2(s)/\Lambda_1(s)$ has infinitely many poles.

Proof idea

The quotient $\Lambda_2(s)/\Lambda_1(s)$ has the functional equation of a Maass form with eigenvalue $\frac{1}{4}$ or a weight 1 modular form. If the quotient has finitely many poles, then by the converse theorems of Hochfilzer–O and Booker–Krishnumurthy, it is the L-function of a Maass or modular form. We argue as before.

Artin

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_d(\mathbb{C})$ be a continuous irreducible representation. The following are standard conjectures about the Artin L-function $L(s, \rho)$:

- **Weak**. $L(s, \rho)$ continues to an entire function on $\mathbb{C}$.
- **Strong**. There is a cuspidal automorphic representation π on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ such that $L(s, \pi) = L(s, \rho)$.

The strong Artin conjecture implies the weak.

Twistless

- Say $d = 2$. If enough twists are holomorphic, then it is known that $L(s, \rho)$ is automorphic.
- A general Artin L-function is known to be meromorphic.
- Combining this knowledge with the converse theorem of Hochfilzer–O, we deduce that if a single twist is holomorphic then $L(s, \rho)$ is automorphic.
- If fact, if a single twist has only finitely many poles, then $L(s, \rho)$ is automorphic.
- This reproves a result of Booker (2003).

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