Global Theta Correspondence Mod p for Unitary Groups

§ Motivation

(H,G) reductive dual gp / F # field.

$\theta(h,g)$ / $(H(\mathbb{A}_F)\times G(\mathbb{A}_F))$

$$F\in \mathcal{A}(H(\mathbb{A}_F)) \rightsquigarrow \theta(F)(g)=\int_{H(F)\backslash H(\mathbb{A}_F)} \theta(h,g)\,F(h)\,dh \in \mathcal{A}(G(\mathbb{A}_F))$$

Examples: Shimura Corresp. $\begin{cases} H=SO_{2,1} \\ G=\widetilde{SL_2}\end{cases}$

Jacquet-Langlands corresp $\begin{cases} H=SU_2 \simeq B^1 \\ G=SU_{1,1}\simeq SL_2\end{cases}$

Yoshida lift $\begin{cases} H=O_4 \\ G=Sp_4\end{cases}$

Question: let π auto rep of $H(\mathbb{A}_F)$, $\theta(\pi)$ theta lifting of π to $G(\mathbb{A}_F)$. when $\theta(\pi)\neq 0$?

Answer: $\theta(\pi)\neq 0$ iff $\theta(\pi_v)\neq 0$ $\forall v$ and $L(s,\pi)$ non-zero or has a pole at $s=s_0$.

(Rallis, Kudla, Gan, Takeda, Yamana, ...)

Question: Can we establish a mod p version of theta liftings? what about $\theta(\pi)\neq 0$ in this case?

Approach 1: Weil rep mod p version $\rightsquigarrow$ theta lifting mod p

(Shin, Chinello-Turchetti, ...).

Approach 2: define integral models of theta lifting and then modulo p.

§ Automorphic forms and theta liftings

$E/\mathbb{Q}$ q.i. $(n\geq 1)$

V_H non-deg Herm / E of dim $n+1$, $H=SU(V_H)$ with $H(\mathbb{R})\simeq SU_{n+1}(\mathbb{R})$

V_G non-deg split skew-Herm / E of dim $2n$, $G=SU(V_G)$.

$$V_G = V_G^+\oplus V_G^-$$

$H(\mathbb{A}_F)\times G(\mathbb{A}_F) \underset{\text{Weil rep.}}{\curvearrowright} S(V_H\otimes V_G^+\otimes \mathbb{A}_F)$ Space of Schwartz functions

(W_λ,ρ_λ) alg rep of H with $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0)\in\mathbb{Z}^n$,

(W_τ,ρ_τ) - - - - G with $\tau=(\tau_1\geq\cdots\geq\tau_{2n-1}\geq 0)\in\mathbb{Z}^{2n-1}$.

Def: Let $K\subset H(\hat{\mathbb{Z}})$ and λ as above. A mod p auto form on $H(\mathbb{A}_f)$ of weight λ and level K is a map $F: H(\mathbb{Q})\backslash H(\mathbb{A}_f)\longrightarrow W_\lambda(\overline{\mathbb{F}}_p)$

s.t. $F(hu)=\rho_\lambda(u_p)^{-1}F(h)$ $(h\in H(\mathbb{A}_f),\ u\in K)$

$\mathcal{A}_\lambda(H,\overline{\mathbb{F}}_p)$

Rmk: We can replace $\overline{\mathbb{F}}_p$ by $\mathbb{C}_p$ and say F is p-integral if $\mathrm{Im}(F)\subset W_\lambda(\mathcal{O}_{\mathbb{C}_p})$.

F is p-primitive if $F\not\equiv 0 \pmod p$.

$$\rightsquigarrow \mathcal{A}_\lambda(H,\mathbb{C}_p)^{p\text{-int}} \xrightarrow[\text{mod } p]{} \mathcal{A}_\lambda(H,\overline{\mathbb{F}}_p).$$

Def: $K'\subset G(\hat{\mathbb{Z}})$ and let $Sh_{K'}(G)$ integral model over $\mathbb{Z}_p$ of the Shimura var asso to G and K'. A mod p auto form on G of wt τ and level K' is a global section in $H^0(Sh_{K'}(G),\widetilde{W_\tau(\overline{\mathbb{F}}_p)})$

$\mathcal{A}_\tau(G,\overline{\mathbb{F}}_p)$ the space of such auto forms.

Rmk: $F\in H^0(Sh_{K'}(G),\widetilde{W_\tau(\mathbb{C}_p)})$, $\rightsquigarrow F=\sum_S a_S q^S$ S Hermitian matrices of size $n\times n$.

$a_S\in W_\tau(\mathbb{C}_p)$.

F is p-integral if $a_S\in W_\tau(\mathcal{O}_{\mathbb{C}_p})$ $\forall S$

F is p-primitive if $F\not\equiv 0 \pmod p$.

For a Schwartz function φ on $V_H\otimes V_G^+\otimes\mathbb{A}_F$,

$$\theta_\varphi(h,g)=\sum_{x\in V_H\otimes V_G^+}\omega(h,g)\varphi(x) \quad \text{theta series asso to } \varphi.$$

$$F\rightsquigarrow \theta_\varphi(F)=\int_H \theta_\varphi(h,g)\,F(h)\,dh.$$

Thm (Kashiwara-Vergne, 78')

If F is of wt $\underline{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_n\geq 0)\in\mathbb{Z}^n$, then $\theta_\varphi(F)$ is of wt

$\underline{\tau}=\tau(\lambda)=(\lambda_1,\lambda_2,\ldots,\lambda_n,0,\ldots,0)+\underline{n+1}\in\mathbb{Z}^{2n-1}$.

Rmk: $(K_{H,\infty})_{\mathbb{C}}\simeq SL_{n+1}(\mathbb{C})$, $\underline{(K_{G,\infty})_{\mathbb{C}}}\simeq \underline{S}(GL_n(\mathbb{C})\times GL_n(\mathbb{C}))$

$(K_{H,\infty})_{\mathbb{C}}\times(K_{G,\infty})_{\mathbb{C}}\curvearrowright \mathbb{C}[M_{n+1,2n}]$ $\quad M_{n+1,2n}=V_H\otimes V_G^+$

(W_τ,ρ_τ) rep of G can be described as the space of pluri-harmonic polynomials on $M_{n+1,2n}$ with values in W_λ.

(P is pluri-harmonic if $x\mapsto P((g_1,g_2)x)$ is harmonic in x. $\left(\forall (g_1,g_2)\in (K_{G,\infty})_{\mathbb{C}}\right)$)

There is a unique element $\underline{I_\lambda(x)}\in \mathrm{Hom}(W_\tau,W_\lambda)[M_{n+1,2n}]$

$$I_\lambda(x)P=P(x)\ (\forall P\in W_\tau)$$

§ Main Result:

Fix an integer $N\geq 1$ and $K_N=\mathrm{Ker}(H(\hat{\mathbb{Z}})\to H(\mathbb{Z}/N\mathbb{Z}))$.

$$\varphi=\otimes_v'\varphi_v:\ \begin{cases}\varphi_\infty(x)=I_\lambda(x)\\ \varphi_\ell(x)=\mathbb{1}_{v_0+NV^+(\mathbb{Z}_\ell)}(x) \quad \left(v_0=\begin{pmatrix}1_n\\0\end{pmatrix}\in V^+(\mathbb{Z}_\ell)\right)\end{cases}$$

Thm (Z, 23').

Assume (W_λ,ρ_λ) is an irre rep of $H(\mathbb{F}_p)$. Then for any p-primitive $F\in\mathcal{A}_\lambda(H,K_N,\mathbb{C}_p)$, its theta lifting $\theta_\varphi(F)$ is also p-primitive.

Cor: For $F\in\mathcal{A}_\lambda(H,K_N,\mathbb{C})$, $F\neq 0$, $\theta_\varphi(F)$ is non-zero.

(Choose prime $\ell\neq p$ split in $E/\mathbb{Q}$ $\rightsquigarrow H(\mathbb{Q}_\ell)\simeq SL_{n+1}(\mathbb{Q}_\ell)$)

Sketch of proof of thm: We chose some particular Hermitian matrices $S_k\in Her_{n,n}(E)$ s.t. the Fourier coeff a_S of $\theta_\varphi(F)$ looks like

$a_{S_k}\sim\langle I_\lambda(S_k),F(g_k)\rangle$ for some $g_k\in H(\mathbb{Q}_\ell)$ of the form $\begin{pmatrix}1_n & *\\ 0 & 1\end{pmatrix}$

Thm (Ratner, 95')

Let G be a semisimple ℓ-adic alg gp and $\Gamma\subset G$ discrete and cocompact subgp. For any unipotent subgp U, $\overline{\Gamma U}$ is of the form ΓH for a closed subgp H of G.

$\rightsquigarrow \left[G=SL_2(\mathbb{Q}_\ell),\ U=\begin{pmatrix}1&*\\0&1\end{pmatrix}.\rightsquigarrow \overline{\Gamma U}=G\right]$

$G=SL_{n+1}(\mathbb{Q}_\ell)$, $U=\left\{\begin{pmatrix}1_n&*\\0&1\end{pmatrix}\right\}\rightsquigarrow\overline{\Gamma U}=G$. $\boxtimes$

§ Applications

Mod p J-L corresp à la Serre

① $S_k(N,\overline{\mathbb{F}}_p)$ (defined on supersingular e.c. / $\overline{\mathbb{F}}_p$)

$\quad = S_k = S_{k+p^2-1}$ ($\Leftarrow$ all s.s. e.c. / $\overline{\mathbb{F}}_p$ are defined over $\mathbb{F}_{p^2}$)

② B quaternion alg ram at $p\infty$. $\rightsquigarrow D_k(N,\overline{\mathbb{F}}_p) = D_k$ mod p auto form on $B^\times\backslash B^\times_{\mathbb{A}}$ of wt k.

Thm (Serre, 96')

$S_k\simeq D_k$ compatible with Hecke operators T_ℓ $(\ell\nmid Np)$.

Our result $\begin{cases}H=SU_2=B^1 & (n=1)\\ G=SU_{1,1}=SL_2\end{cases}$

Bloch-Kato conjecture

$n=2$, π auto rep of $H(\mathbb{A})$, $F\in\pi$

Rallis inner product formula $\quad \underline{\dfrac{L(1,F,St)}{\Omega}}=\dfrac{\langle\theta_\varphi(F),\theta_\varphi(F)\rangle}{\langle F,F\rangle\,\Omega}$

Modularity lifting thms for H and G

$$\langle F,F\rangle\rightsquigarrow \underline{Sel(Ad(\rho_F))}$$

$$\frac{\langle\theta_\varphi(F),\theta_\varphi(F)\rangle}{\Omega}\rightsquigarrow\underline{Sel(Ad(\rho_{\theta_\varphi(F)}))}$$

$$\rightsquigarrow \frac{L(1,F,St)}{\Omega}\doteq \chi(Sel(\rho_F,St)) \quad \text{p-part of Bloch-Kato for } \pi.$$