Theta series and tautological cycles on orthogonal Shimura varieties

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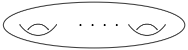
Motivation: Moduli space of curves

- Let

$$\mathbf{M}_g = \Big\{ \Sigma_g :$$



$$\Big\} / \cong$$

be the moduli space of compact Riemann surfaces of genus $g \geq 2$.

- By Teichmüller theory,

$$\mathbf{M}_g \cong \mathbf{T}/\mathrm{MCG}$$

where $\mathbf{T}$ is the **Teichmüller space** and MCG is the **mapping class group**.

- **Deligne-Mumford:** there is a projective compactification $\overline{\mathbf{M}}_g$ of $\mathbf{M}_g$, called Deligne-Mumford compactificaiton. Its boundary $\partial(\overline{\mathbf{M}}_g)$ has $[\frac{g}{2}] + 1$ irreducible components δ_i, $0 \leq i \leq [\frac{g}{2}]$, where

$$\delta_i = \Big\{ C \cup C' \mid \mathbf{gen}(C) = i,\ \mathbf{gen}(C') = g - i \Big\}.$$

Tautological cycles on $\mathbf{M}_g$

Fact: The Chow ring $\mathrm{CH}^*(\mathbf{M}_g)$ is highly complicated (e.g., potentially non-finitely generated). Mumford's tautological ring captures its most geometric/essential part.

- **Geometry from Moduli Spaces:** The moduli space $\mathbf{M}_g$ "naturally carries" a universal family of smooth curves:

$$\pi : \mathbf{M}_{g,1} \to \mathbf{M}_g, \quad \text{with fiber } \pi^{-1}([C]) = C$$

- Set

$$T_\pi = \ker\left(T_{\mathbf{M}_{g,1}} \to \pi^* T_{\mathbf{M}_g}\right).$$

  to be **relative tangent bundle**.

- **Mumford's κ-Classes:** define

$$\kappa_i = \pi_*\left(c_1(T_\pi)^{i+1}\right) \in \mathrm{CH}^i(\mathbf{M}_g).$$

  and the subring $\mathrm{R}^*(\mathbf{M}_g) \subset \mathrm{CH}^*(\mathbf{M}_g)$ generated by κ_i is called the tautological ring for $\mathbf{M}_g$.

Core Insight: The tautological ring encodes essential intersection-theoretic data on $\mathbf{M}_g$.

Structure of the Tautological Ring

Theorem (Harer, Looijenga, Faber)

Let $\mathrm{R}^*(\mathbf{M}_g) = \bigoplus_{d \geq 0} \mathrm{R}^d(\mathbf{M}_g)$. Then:

(0) **(Mumford's conjecture)** $\mathrm{R}^*(\mathbf{M}_g)$ is generated by $\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}$ and $\mathrm{R}^1(\mathbf{M}_g) = \mathrm{Pic}(\mathbf{M}_g)$.

(1) **(Vanishing)** $\mathrm{R}^d(\mathbf{M}_g) = 0$ for $d > g-2$.

(2) **(1-Dimensionality)** $\mathrm{R}^{g-2}(\mathbf{M}_g) \cong \mathbb{Q}$.

Remark.

- Properties (1)–(2) is part of **Faber's conjecture**.
  Faber's conjecture also include the **(Gorenstein Property)**, i.e. the intersection pairing
  $$\mathrm{R}^d(\mathbf{M}_g) \times \mathrm{R}^{g-2-d}(\mathbf{M}_g) \to \mathrm{R}^{g-2}(\mathbf{M}_g) \cong \mathbb{Q}$$
  is perfect, which imply a *Poincaré duality* structure on $\mathrm{R}^*(\mathbf{M}_g)$. This is still open.
- There is an analogous result on $\overline{\mathbf{M}}_g$. e.g. $\mathrm{R}^1(\overline{\mathbf{M}}_g) = \mathrm{Pic}(\overline{\mathbf{M}}_g)$ is spanned by κ_1 and irreducible components of $\partial(\overline{\mathbf{M}}_g)$.

Higher dimensional case: K3 surfaces

Definition

- A compact complex surface S is a K3 surface if
  1. $\pi_1(\mathrm{S}) = 1$,
  2. $\mathrm{H}^0(\mathrm{S}, \Omega^2_{\mathrm{S}})$ is spanned by a nowhere degenerated 2-form.
- A **quasi-polarized K3 surface of genus g** is a pair (S, L) where S is a K3 surface, $\mathrm{L} \in \mathrm{Pic}(\mathrm{S})$ is a nef line bundle with $\mathrm{L}^2 = 2g - 2$.
- Let
$$\mathbf{F}_g = \Big\{(\mathrm{S}, \mathrm{L}) : \text{quasi-polarized K3 surfaces of genus } g\Big\}/\cong$$
be the coarse moduli space of quasi-polarized K3 surfaces.

Global Torelli Theorem (Pjateckiĭ-Šapiro, Šafarevič)

$\mathbf{F}_g$ is isomorphic to a 19-dimensional quasi-projective Shimura variety via the period map, equipped with a canonical compactification $\overline{\mathbf{F}}_g$, called the **Baily-Borel compactification**.

Tautological ring of F_g

- Motivated from Mumford's work, Marian-Oprea-Pandaripande defined the κ-classes on family of quasi-polarized K3 surfaces $\pi : \mathcal{X} \to B$ as below

$$\kappa_{a_1,\ldots,a_k,b} := \pi_*(\mathcal{L}_1^{a_i} \cdots \mathcal{L}_k^{a_k} c_2(T_\pi)^b)$$

  where $\mathcal{L}_i \in \mathrm{Pic}(\mathcal{X})$.

- Let $\mathrm{R}^*(\mathbf{F}_g) \subseteq \mathrm{CH}^*(\mathbf{F}_g)$ be the subring generated by κ-classes on families of K3 surfaces over the **higher Noether-Lefschetz loci** of $\mathbf{F}_g$.

- **MOP Conjecture**:
  - (0) Generators: $\mathbf{R}^*(\mathbf{F}_g)$ is generated by components in the higher Noether-Lefschetz loci.
  - (1) Vanishing: $\mathbf{R}^d(\mathbf{F}_g) = 0$ for $d > 19 - 2$
  - (2) 1-Dim: $\mathbf{R}^{17}(\mathbf{F}_g) \cong \mathbb{Q}$

- (0) has been confirmed by Yin-Pandaripande, and also Bergero-L. (in cohomology) via different methods.

Current Progress

A Comparative Overview

Moduli space of curves

- $\mathrm{Pic}(\mathbf{M}_g) \cong \mathbb{Z}$
- $\mathrm{Pic}(\overline{\mathbf{M}}_g) \cong \mathbb{Z}^{[\frac{g}{2}]+2}$
- $\mathrm{R}^*(\mathbf{M}_g)$ is finitely generated
- $\mathrm{R}^{>g-2}(\mathbf{M}_g) = 0$
- $\mathrm{R}^{g-2}(\mathbf{M}_g) \cong \mathbb{Q}$

Moduli space of quasi-polarized K3 surfaces

- $\mathrm{Pic}(\mathbf{F}_g) \cong \mathbb{Z}^{d(g)},\ d(g) \approx [\frac{19g}{24}] + 1$
- $\mathrm{Pic}(\overline{\mathbf{F}}_g) \cong \mathbb{Z}$
- It remains open whether $\mathrm{R}^*(\mathbf{F}_g)$ is finitely generated
- **Petersen '19**: $\mathrm{R}^{>\dim \mathbf{F}_g - 2}(\mathbf{F}_g) = 0$ modulo homologous equivalence
- **Canning-Oprea-Pandharipande '24:** $\mathrm{R}^{\dim \mathbf{F}_g - 2}(\mathbf{M}_g) \cong \mathbb{Q}$ when $g = 2$.

We will discuss the generalization of this picture and the motivations.

General Setting: Shimura variety of orthogonal type

- M: an integral lattice of signature $(2, n)$.
- $\mathbf{D} = \{x \in \mathbb{P}(M \otimes \mathbb{C}) \mid x^2 = 0,\ (x, \bar{x}) > 0\}$
- Γ: a congruence subgroup of the stable orthogonal group $\widetilde{\mathrm{O}}(M)$.
- **Baily-Borel compactification**: The Shimura variety $\mathrm{Sh}_\Gamma(M) = \Gamma\backslash\mathbf{D}$ is a quasi-projective variety, and it admits a canonical compactification:

$$\overline{\mathrm{Sh}}_\Gamma(M) = \mathrm{Proj} \bigoplus_k \mathrm{H}^0(\mathrm{Sh}_\Gamma(M), \lambda^{\otimes k}),$$

  where λ is the **Hodge line bundle** on $\mathrm{Sh}_\Gamma(M)$.
- **Example.** When $M = \langle 2-2g\rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}(-1)$ and $\Gamma = \widetilde{\mathrm{O}}(M)$, $\mathrm{Sh}_\Gamma(M) \cong \mathbf{F}_g$.
- **Geometric properties**: $\mathrm{Sh}_\Gamma(M)$ has only quotient singularities and is therefore **$\mathbb{Q}$-factorial**, while $\overline{\mathrm{Sh}}_\Gamma(M)$ can have **canonical singularities**.

Tautological cycles on $\mathrm{Sh}(M)$

- On $\mathrm{Sh}_\Gamma(M)$, there are natural cycles of the form
$$\Gamma_v \backslash v^\perp \to \Gamma \backslash D$$
where $v \subseteq M$ is a negative definite subspace.
- When $\mathrm{Sh}_\Gamma(M) = \mathbf{F}_g$, these are irreducible components of higher Noether-Lefschetz loci.
- Let $\mathrm{R}^*(\mathrm{Sh}_\Gamma(M))$ be the subring of $\mathrm{CH}^*(\mathrm{Sh}_\Gamma(M))$ generated by all natural cycles.

A conjectural description is

Conjecture and questions

Assume $\dim \mathrm{Sh}_\Gamma(M) > 3$. Then

1. The cycle class map $\mathrm{R}^*(\mathrm{Sh}_\Gamma(M)) \to \mathrm{H}^*(\mathrm{Sh}_\Gamma(M), \mathbb{Q})$ is injective
2. $\mathrm{R}^*(\mathrm{Sh}_\Gamma(M))$ is finitely generated (A stronger version of Kudla's modularity).
3. $\mathrm{R}^{>\dim -2}(\mathrm{Sh}_\Gamma(M)) = 0$.

4? $\mathrm{R}^{\dim -2}(\mathrm{Sh}_\Gamma(M)) = \mathbb{Q}$ for $\Gamma = \widetilde{\mathrm{O}}(M)$?

Special Divisors on Orthogonal Shimura Varieties

The problem (4?) for $\Gamma = \widetilde{\mathrm{O}}(M)$ is related to the study of divisors on $\mathrm{Sh}_\Gamma(M)$.

- **Heegner divisors:** For $m \in \mathbb{Q}^{\geq 0}$ and $\gamma \in M^\vee/M$, define

$$\mathrm{H}_{m,\gamma} = \begin{cases} \Gamma\backslash \sum\limits_{\substack{v \in M+\gamma \\ \frac{v^2}{2} = -m}} v^\perp, & \text{if } (m,\gamma) \neq (0,0), \\ -\lambda, & \text{if } (m,\gamma) = (0,0). \end{cases}$$

  which can be regarded as an element in $\mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}_\Gamma(M))$.

- **Bergeron-L-Millson-Moeglin:** If $\mathrm{rank}(M) \geq 5$, then $\mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}_\Gamma(M))$ is spanned by the irreducible components of $\mathrm{H}_{m,\gamma}$.

- When M contains two hyperbolic lattices and $\Gamma = \widetilde{\mathrm{O}}(M)$, by using Eichler's criteria, $\mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}(M))$ is spanned by Heegner divisors.

- If $\dim \mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}_\Gamma(M)) > 1$, Problem (4?) likely has a negative answer.

Picard group of Baily-Borel compactification

Theorem 1 (Huang-L-Müller-Ye)

Suppose

1. M contains two hyperbolic lattice;
2. $\mathrm{rank}(M) > 10$;
3. $M \otimes \mathbb{Z}_p$ contains three hyperbolic for all p.

If $\Gamma = \widetilde{\mathrm{O}}(M)$, then

$$\mathrm{Pic}(\overline{\mathrm{Sh}_\Gamma}(M)) \cong \mathbb{Z}$$

is spanned by some multiple of the extended Hodge line bundle $\bar{\lambda}$.

Remark:

- It will be interesting to known if $\dim \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{Sh}_\Gamma}(M)) = 1$ for more general Γ and M with $\mathrm{rank}(M) \geq 5$ or not, (In fact, we find that this also holds for 2-elementary lattice containing two hyperbolics)
- The proof heavily use the closed relation between Heegner divisors and vector-valued modular forms.

Modular Forms

- **Vector-Valued version:** Given $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ (or its double cover $\mathrm{Mp}_2(\mathbb{Z})$), a vector-valued function $f : \mathbb{H} \to V$ is called a **modular form of weight k of type ρ** if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \cdot \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}(f(z)), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

- If the constant terms in its Fourier expansion are all 0, it is called a **cusp form**. Set $\mathbf{Cusp}_k(\rho)$: space of cusp forms of weight k and type ρ.
- e.g. the **Eisenstein series**

$$\mathbf{E}_{2k}(z) = \frac{1}{2\zeta(2k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^{2k}}.$$

  is a modular form of weight $2k$.
- If ρ is a Weil representation, a main source to construct vector-valued modular forms is from Borcherds' theta series.

Theta Series

- **Borcherds' Theta Series**
  - for any even lattice L, there is a Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ attached to L
  $$\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \to \mathbb{C}[L^\vee/L]$$
  - if L is **definite** and h is a harmonic polynomial of degree $d > 0$, the **vector-valued Theta series**
  $$\Theta_{L,h}(z) = \sum_{v \in L^\vee} h(v) q^{\frac{v^2}{2}} \mathbf{e}_v,$$
  is an element in $\mathbf{Cusp}_{d+\frac{\mathrm{rank}(L)}{2}}(\rho_L)$, where $\mathbf{e}_v$ is the standard basis of $\mathbb{C}[L^\vee/L]$.
  - Set $\mathbf{Cusp}_k^\theta(\rho_L) \subseteq \mathbf{Cusp}_k(\rho_L)$ to be the subspace generated by $\Theta_{\Lambda,h}$ for all definite lattice $\Lambda \in \mathbf{Gen}(L)$.

A stronger version via Borcherds' theta lifting

Set $\mathrm{Sh}(M) = \mathrm{Sh}_{\widetilde{\mathrm{O}}(M)}(M)$. Bocherds introduces the **singular theta lifting**, which enable us to give an explicit description of $\mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}(M))$ via modular forms.

Set

- $\mathbf{ACusp}_k(\rho_M)$: space generated by $\mathbf{Cusp}_k(\rho_M)$ and an Eisenstein series of weight k.
- M : even lattice of signature $(2, n)$ containing two hyperbolic lattice

There is an isomorphism

$$\Upsilon : \mathrm{Pic}(\mathrm{Sh}(M))_{\mathbb{Q}} \cong \mathbf{ACusp}(\rho_M)^{\vee},$$

sending the Heegner divisor $\mathrm{H}_{m,\gamma}$ to the **coefficients functional**

$$\mathbf{ACusp}_{\frac{n+2}{2}}(\rho_M) \longrightarrow \mathbb{Q}$$

$$\sum_{m' \in \mathbb{Q}, \gamma' \in M^{\vee}/M} a_{m',\gamma'} q^{m'} \mathbf{e}_{\gamma'} \mapsto a_{m,\gamma}$$

Local obstruction for extending Heegner divisors

Question: when an element in $\mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}(M))$ can be extended to the boundary?

- For $x \in \overline{\mathrm{Sh}}_{\Gamma}(M)$, the **local Picard group** at x is defined as:

$$\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{Sh}}_{\Gamma}(M), x) := \lim_{x \in U} \mathrm{Pic}_{\mathbb{Q}}(U \cap \mathrm{Sh}(M)).$$

- There is a restriction map:

$$\mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}_{\Gamma}(M)) \to \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{Sh}}_{\Gamma}(M), x).$$

  Some multiple of $H \in \mathrm{Pic}(\mathrm{Sh}_{\Gamma}(M))$ can be extended to x if its image in $\mathrm{Pic}(\overline{\mathrm{Sh}}_{\Gamma}(M), x)$ is trivial.

- **Example:** $\lambda^{\otimes n}$ can be extended to the boundary for sufficiently large n.

- **Extension of λ (by Lan)**: If Γ is neat, then λ can be extended to $\overline{\mathrm{Sh}}_{\Gamma}(M)$.

Obstruction from theta series

Fact: the boundary components of $\overline{\mathrm{Sh}}(M)$ correspond to isotropic planes $J \subseteq M$.

Theorem (Bruinier-Freitag)

Suppose

- M : lattice of signature $(2, n)$ and $\mathrm{U}^{\oplus 2} \subseteq M$;
- $J \subseteq M$: an isotropic plane;
- $\partial_J \subseteq \overline{\mathrm{Sh}}(M) - \mathrm{Sh}(M)$: the boundary component corresponds to J;
- $\mathrm{H} = \sum_{\gamma \in M^\vee/M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} \mathrm{H}_{m,\gamma}$

Then H is trivial in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{Sh}}(M), x)$ for general $x \in \partial_J$ if and only if

$$\sum_{\gamma \in M^\vee/M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} c_{m,\gamma} = 0$$

for all theta series $\Theta_{J^\perp/J,h} = \sum_{m,\gamma} c_{m,\gamma} q^m \mathbf{e}_\gamma$, where h runs over all harmonic polynomials of degree 2.

Observations and question

- **1st observation:** as M contains two hyperbolic, then

$$\left\{ J^{\perp}/J \mid J \subseteq M \text{ is an isotropic plane } \right\}$$

contains the genus of negative definite lattices of rank $n-2$ with discriminant group $\cong M^{\vee}/M$.

- **A necessary condition**: if $\mathrm{H} \in \mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}(M))$ can be extended to $\overline{\mathrm{Sh}}(M)$, then

$$\Upsilon(H)(f) = \sum_{\gamma \in M^{\vee}/M} \sum_{m \in \mathbb{Q}} a_{m,\gamma} c_{m,\gamma} = 0$$

for all $f \in \mathbf{Cusp}^{\theta}_{\frac{n+2}{2}}(\rho_M)$.

- **2nd observation:** $\mathrm{H} \in \mathrm{Pic}_{\mathbb{Q}}(\mathrm{Sh}(M))$ is propotional to λ if and only if

$$\Upsilon(H)(f) = 0$$

for all $f \in \mathbf{Cusp}_{\frac{n+2}{2}}(\rho_M)$.

- **Question:** Is $\mathbf{Cusp}^{\theta}_{\frac{n+2}{2}}(\rho_M) = \mathbf{Cusp}_{\frac{n+2}{2}}(\rho_M)$? If this holds, then $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathrm{Sh}}(M))$ is spanned by λ.

Cusp forms as theta series

The proof of Theorem 1 relies on the following theta lifting result

Theorem 2 (Theta lifting)

With the assumption as before.

- $\mathbf{E}^{(2)}_{k,M}$: the vector-valued Siegel Eisenstein series of weight k and type $\rho^{(2)}_M$;
- Set $\vartheta(z, z') = \partial_d \mathbf{E}^{(2)}_{k,M}(z, z')$, where ∂_d is the Eichler-Zagier's differential operator.

For $d > 0$ and $k = d + \mathrm{rank}(M)$, there is an injective map

$$\Psi : \mathbf{Cusp}_k(\rho_M) \to \mathbf{Mod}_k(\rho_M) \tag{1}$$

given by

$$f \mapsto \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle f(z), \vartheta(z, -\overline{z'}) \rangle \frac{\mathrm{d}x\mathrm{d}y}{y^{2-k}}$$

whose image is exactly $\mathbf{Cusp}^{\theta}_k(\rho_M)$. In other words, $\mathbf{Cusp}_k(\rho_M) = \mathbf{Cusp}^{\theta}_k(\rho_M)$.

Proof of Theorem 2

Image of Ψ: $\Psi(f) \in \mathbf{Cusp}_k^\theta(\rho_M)$ follows from a dedicated computation by using Siegel-Weil formula.

Injectivity of Ψ:

- The idea is to use **vector-valued Hecke operators:** for each $\alpha \in \mathbb{Z}$, there is an operator

$$\mathbf{T}_{\alpha^2} : \mathbf{Cusp}_k(\rho_M) \to \mathbf{Cusp}_k(\rho_M)$$

defined by

$$\mathbf{T}_{\alpha^2}(f) = \alpha^{k-2} \sum_i \sum_{\gamma \in G} \left(f_\gamma \mid_k [\tilde{\delta}_i]\right) \otimes \left(\mathbf{e}_\gamma \mid [\tilde{\delta}_i]\right),$$

where $f = \sum_{\gamma \in G} f_\gamma \mathbf{e}_\gamma$ and $\tilde{\delta}_i$ are coset representatives of $\mathrm{Mp}_2(\mathbb{Z})$.

Proof of Theorem 2

- Properties of Hecke operators **(Bruinier-Stein)**
  - (i) They are self-adjoint with respect to the Petersson inner product.
  - (ii) Set $N := \min\left\{m \in \mathbb{Z}_{>0} \mid \; mx^2 = 0, \; \forall x \in M^\vee/M\right\}$. The Hecke operators
  $$\{\mathbf{T}_{\alpha^2} : \gcd(\alpha, N) = 1\}$$
  generate a commutative subalgebra.
  - (iii) $\mathbf{T}_{\alpha^2} \circ \mathbf{T}_{\beta^2} = \mathbf{T}_{(\alpha\beta)^2}$ if $\gcd(\alpha, \beta) = 1$.
- For $k > d + 3$, we have
  $$\Psi = C \sum_{\alpha=1}^{\infty} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-d}}.$$
  for some non-zero constant C.

Theta lifting as Heck operators

- **Injectivity of** $\sum_{\alpha=1}^{\infty} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-d}}$.
  - $\mathbf{Cusp}_k(\rho_M)$ is spanned by simultaneous eigenforms of $\mathbf{T}_{\alpha^2}$ with $\gcd(\alpha, N) = 1$.
  - let $f \in \mathbf{Cusp}_k(\rho_M)$ be a non-zero simultaneous eigenform with eigenvalue ψ_{α^2}, then

$$\left(\sum_{\substack{\alpha \geq 1 \\ \gcd(\alpha,N)=1}} \frac{\mathbf{T}_{\alpha^2}}{\alpha^{2k-2-d}} \right)(f) = \left(\sum_{\substack{\alpha \geq 1 \\ \gcd(\alpha,N)=1}} \frac{\psi_{\alpha^2}}{\alpha^{2k-2-d}} \right) f = L(f, 2k-2-d) \cdot f \neq 0.$$

  - The case $p \mid N$ needs addition attention. It makes use of the isotropic lift of modular forms.

Thanks!