Modularity and automorphy of algebraic cycles on Shimura varieties

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Outline

- Algebraic cycles
- Automorphy for cycles on Shimura varieties
- Modularity for special cycles on Shimura varieties

Algebraic cycles

Algebraic cycles

- Let X be a smooth projective variety over a field k.
- Algebraic cycles of dim i (or i-cycles): $\mathbb{Z}\{\text{closed subvarieties of dim } i\}$.
- Chow group $\mathrm{Ch}_i(X) = \mathrm{Ch}^{\dim X - i}(X) :=$

$$\frac{\mathbb{Z}\{\text{closed subvarieties of dim } i\}}{\text{rational equivalence} = \text{deformation along } \mathbb{P}^1}$$

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- The simplest case: divisors, i.e., codimension 1.
  For a rational function f on X,

$$\mathrm{zero}(f) = \mathrm{pole}(f) \text{ in } \mathrm{Ch}^1(X).$$

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- Even simpler: let X be a curve over $\mathbb{C}$, i.e., a Riemann surface.
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- Then $Z = \partial\Gamma$ for a singular 1-chain Γ. Define

$$\{\text{Divisors of degree } 0\} \to \mathrm{Jac}_X(\mathbb{C}) = H^0(X, \Omega)^*/H^1(X, \mathbb{Z})$$
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- Abel–Jacobi theorem: kernel of this map consists of divisors $\mathrm{zero}(f) - \mathrm{pole}(f)$.
- Same is true over any field.

Algebraic cycles

- If k is a number field, $\mathrm{Jac}_X(k)$ is finitely generated (Mordell–Weil theorem, Birch–Swinnerton-Dyer conjecture).
- "Simplest" case: X is an elliptic curve E over $\mathbb{Q}$.

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- Heegner 1952:

$$\begin{tikzcd}
\text{quadratic numbers} \arrow[r] \arrow[d] & \mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\} \arrow[d] \\
\text{CM points} \arrow[r] \arrow[d] & \text{Modular curve} = \Gamma\backslash\mathbb{H},\ \Gamma \subset \mathrm{SL}_2(\mathbb{Z}) \arrow[d] \\
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- Gross–Zagier formula:

$$\text{height of a CM 0-cycle} \sim L'(E, \text{center}).$$

Algebraic cycles

Beyond divisors (k is still a number field): Beilinson–Bloch conjecture.

- For $f : X \to X$, if $f_* H^{2i-1} = \{0\}, f_* H^{2i} = \{0\}$, then $f_* \mathrm{Ch}^i(X)$ should be torsion.
- More generally, replace f by $T \in \mathrm{Corr}(X, X) = \mathrm{Ch}^{\dim X}(X \times X)$, e.g., graph of $f : X \to X$.

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- Simplest unknown case: let C be a curve, $e \in C(k)$,

$$\delta_e = \Delta - C \times e \in \mathrm{Corr}(C, C).$$

Then $\delta_e^2 \in \mathrm{Corr}(C^2, C^2)$ should annhilate 0-cycles in $\mathrm{Ch}^2(C^2)$.

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Theorem (Q. to appear)

Let $E/\mathbb{Q}$ be an elliptic curve with identity e. Then δ_e^2 annhilates CM 0-cycles in $\mathrm{Ch}^2(E^2)$.

Algebraic cycles

How about the triple product C^3?

- Diagonal 1-cycle: $\Delta_{123} = \{(x,x,x) : x \in C\} \subset C^3$.
- Modified diagonal 1-cycle (Gross–Schoen): for $e \in C(k)$, let

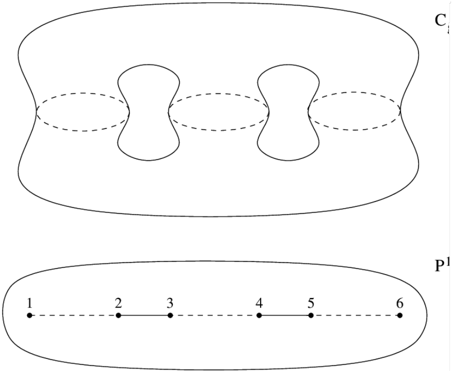
$$\begin{aligned} \Delta_{12} &= \{(x,x,e) : x \in C\}, \quad \Delta_{23} = ..., \quad \Delta_{31} = ..., \\ \Delta_{1} &= \{(x,e,e) : x \in C\}, \quad \Delta_{2} = ..., \quad \Delta_{3} = ..., \end{aligned}$$

$$\Delta_e = \Delta_{123} - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3.$$

- Cohomology class 0.

Algebraic cycles

- Gross–Schoen: Δ_e is torsion in $\mathrm{Ch}^2(C^3)$ if C is a $\mathbb{Z}/2\mathbb{Z}$-cover of $\mathbb{P}^1$, and e is a fixed point.



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Theorem (Q.– W. Zhang 2022)

(1) Suitably generalize Gross–Schoen's result to G-cover of $\mathbb{P}^1$.
(2) The unique Hurwitz curve of genus 7 has Δ_e torsion.
(3) Find explicit 1-dimensional families of non-hyperelliptic curves with Δ_e torsion.

Automorphy

Automorphy for cycles on Shimura varieties

- Let X be a Shimura variety,
  E.g., moduli of abelian varieties;
  E.g., $\Gamma\backslash\mathbb{B}_n$ where $\Gamma \subset U(n,1)$.
- Hecke algebra $\mathcal{H} \subset \mathrm{Corr}(X,X)$.

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- E.g., a Hecke correpondence on moduli of elliptic curves:

$$\begin{tikzcd}
& X_0(N) := \{E_1 \to E_2 : \ker \cong \mathbb{Z}/N\mathbb{Z}\} \arrow[dl] \arrow[dr] & \\
X = \{E\} & & X = \{E\}.
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- E.g., $X_0(N)$ itself is "the" modular curve. Analogous Hecke correpondence T_n.

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$$\{\text{rational } \mathcal{H}\text{-eigenspaces of } H^0(X, \Omega)\} \to \{\text{Elliptic curves } E \text{ over } \mathbb{Q}\},$$

relating eigenvalues of T_p with number of points of E modulo p.

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- The converse arrow: Wiles et. al. (Fermat's last theorem).
- Not really relevant in our story.

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- For a general Shimura variety X, $H^*(X,\mathbb{C})$ is a semisimple $\mathcal{H}$-module. Call a submodule automorphic.

Conjecture (Automorphy)

As an $\mathcal{H}$-module, $\mathrm{Ch}^*(X)_{\mathbb{C}}$ is semisimple and automorphic.

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As an $\mathcal{H}$-module, $\mathrm{Ch}^*(X)_{\mathbb{C}}$ is semisimple and automorphic.

- It holds for Shimura curves.

Automorphy for cycles on Shimura varieties

How about a product of Shimura curves?

Proposition (Gross–Kudla 1992)

Let C be the modular curve $X_0(N)$, $X = C^3$. Automorphy holds for $\mathcal{H}\Delta_e \subset \mathrm{Ch}^2(C^3)$ modulo Beilinson–Bloch height pairing.

- An analog of the Gross–Zagier formula (Conjecture): For a cuspidal automorphic representation π for $X = C^3$ with certain local conditions

$$\text{height of } \Delta_{e,\pi} \sim L'(\text{center}, \pi).$$

- Gross–Kudla's automorphy is necessary to formula the Conjecture.

Automorphy for cycles on Shimura varieties

Theorem (Q.– W. Zhang 2022)

(1) Automorphy holds for all cycles on an arbitrary product of Shimura curves, unconditionally.
(2) For the triple product C^3 of a Shimura curve and a cuspidal automorphic representation π with local conditions opposite to Gross–Kudla, the π-isotypic component $\Delta_{e,\pi} = \Delta_\pi = 0$

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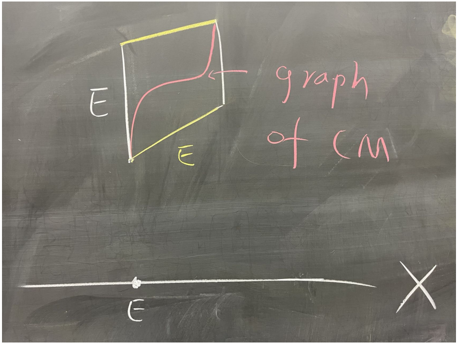
- How about more general Shimura varieties?.
- One possible approach: Modularity.

Automorphy for cycles on Shimura varieties

Interlude: Kuga-Sato varieties over modular curves



Theorem (S. Zhang 1997, for $X_0(N)$; Q. 2021, in general)

(1) Automorphy holds for CM cycles, modulo kernel of height pairing.
(2) A Gross–Zagier type formula holds for CM cycles.

Modularity

Modularity for special cycles on Shimura varieties

- A modular form: f on $\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ such that $f \cdot (d\tau)^{k/2}$ is invariant by some $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.
- Let $q = e^{2\pi i \tau}$
- E.g., Jacobi–Riemann theta series

$$\theta(\tau) = \sum_{n \in \mathbb{Z}_{\geq 0}} \#\{x \in \mathbb{Z} : x^2 = n\} q^n.$$

$$\theta^2(\tau) = \sum_{n \in \mathbb{Z}_{\geq 0}} \#\{(x, y) \in \mathbb{Z} : x^2 + y^2 = n\} q^n.$$

Modularity for special cycles on Shimura varieties

- E.g., an Eisenstein series for $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \cong \{\pm 1\}$

$$E^\chi(s,\tau)|_{s=0} = * \sum_{(c,d)\in(4\mathbb{Z}\times\mathbb{Z})\backslash\{0\}} \frac{\chi(d)}{(c\tau+d)^k}$$
$$= 1 + 4\left(\sum_{n\in\mathbb{Z}_{>0}} \sum_{d|n} \chi(d)\right) q^n$$

- Siegel–Weil formula:

$$\theta^2 = E^\chi|_{s=0}.$$

So

$$\#\{(x,y)\in\mathbb{Z} : x^2+y^2=n\} = \sum_{d|n}\chi(d).$$

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- E.g., CM points, Diagonal cycles on triple curves.
- E.g., on $X = X_0(N)^2$, Hecke correspondences

$$\begin{tikzcd} & T_n \arrow[dl] \arrow[dr] & \\ X_0(N) & & X_0(N) \end{tikzcd} .$$

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- Focus on divisors. An example:

Proposition (Gross–Zagier, 1986)

For some c, $c + \sum T_n q^n$ is a modular forms valued in $\mathrm{Ch}^1(X_0(N)^2)$.

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- Modularity: geometric theta series are modular forms.

* Hirzebruch–Zagier 1976.

- Gross–Kohnen–Zagier 1987.
- Borcherds 1999.
- W. Zhang 2009; Yuan–S. Zhang–W. Zhang 2009; Liu 2011.

Modularity for special cycles on Shimura varieties

Arithmetic mixed Siegel–Weil formula.

Theorem (Gross–Zagier, 1986)

For $(n, N) = 1$, on $X_0(N)^2$,

$\langle T_n$, *CM 0-cycle of degree 0*$\rangle^* \sim$ *n-th coefficient of an explicit modular form,*

under the Heegner condition: levels are only at split places.

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- This modular form is the kernel function representing L-functions. The Gross–Zagier formula follows.

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Theorem (Q. 2022)

Replace $X_0(N)^2$ by unitary Shimura varieties over CM fields with arbitrary split levels and remove "$(n, N) = 1$" and "degree 0".

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- $$\begin{tikzcd} X \arrow[d] & & \mathcal{X} \coprod X_\infty \arrow[d] \\ \mathrm{Spec}\mathbb{Q} & & \mathrm{Spec}\mathbb{Z} \coprod \{\infty\} \end{tikzcd}.$$

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A modular form valued in $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{X})$ *for unitary Shimura varieties.*

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- Kudla–Rapoport–Yang 2006: Quaternionic Shimura curves.
- Bruinier–Burgos Gil–Kühn 2007: Hilbert modular surfaces.
- Howard–Madapusi-Pera 2020: Orthogonal Shimura varieties.
- Bruinier–Howard–Kudla–Rapoport–Yang 2020: Unitary Shimura varieties.
- All over $\mathbb{Q}$/imaginary quadratic fields with minimal level structures.

Modularity for special cycles on Shimura varieties

Geometric theta series of higher co-dimensional special cycles.

* Kudla–Millson, 1986, 1987, 1990.
- Bruinier–Westerholt-Raum 2015.
- Kudla 2022: Beilinson–Bloch conjecture implies modularity.

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- An application of modularity.

Theorem (Q. to appear)

For a "rational" automorphic representation π on a Shimura curve C, then the $\pi \boxtimes \pi$-isotypic component of $\mathrm{CM}^2(C^2)_{\mathbb{C}}$ is 0.

Modularity for special cycles on Shimura varieties

- Key input: vanishing of $\pi \boxtimes \pi$-isotypic components of constant terms of geometric theta series, including the Faber–Pandharipande cycle:

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Project (Q.)

(1) Find canonical modifications of Kudla's special cycles so that the constant terms of geometric theta series vanish.
(2) Interplay of (1) with the automorphy problem.

Arithmetic mixed Siegel–Weil formula for higher co-dimensional special cycles.

Project (Q.)

Use the archimedean part to study a conjecture of Deligne, Beilinson and Scholl:

$$L'(\pi, \textit{center}) \textit{ is a Kontsevich–Zagier period.}$$

The End

Thank you