Matrix Kloosterman sums

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Classical Kloosterman sums

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Basic properties

1. $K(\alpha, \beta) \in \mathbb{R}$,
2. $K(\alpha, \beta) = K(\beta, \alpha)$ and
3. if $\delta \in \mathbb{F}^*$, then $K(\alpha, \delta\beta) = K(\alpha\delta, \beta)$. Thus $K(\alpha, \delta) = K(\alpha\delta) := K(\alpha\delta, 1)$.

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$K(0) = -1$ and $K(0, 0) = q - 1$.

GL_n Kloosterman sums

Let $n \in \mathbb{N}$, $\mathrm{GL}_n(\mathbb{F})$ the group of $n \times n$ invertible matrices over $\mathbb{F}$, $\mathrm{tr} : \mathrm{GL}_n(\mathbb{F}) \to \mathbb{F}$ the matrix trace and $\psi = \mathrm{tr} \circ \varphi$.

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For $a, b \in \mathbb{F}^{n\times n}$ let $K_n(a, b) = \sum_{c \in \mathrm{GL}_n(\mathbb{F})} \psi(ac + bc^{-1})$.

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Theorem

$$K_n(a) \ll \begin{cases} q^{(3n^2-1)/4}, & \textit{if } n \textit{ is odd} \\ q^{3n^2/4}, & \textit{if } n \textit{ is even} \end{cases} \quad \textit{for any } a \in \mathbb{F}^{n \times n}.$$

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1. Explicit evaluation of the sum $K_n(a)$ (using elementary methods)
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2. Bounds (using cohomological arguments)
    - The general picture – by Weil, Grothendieck and Deligne
    - The bound on $K_n(a)$ – bounding the weights
    - Purity type results – an illustration of a theorem of Fouvry and Katz

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The sum is invariant under conjugation: for c invertible

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Let $\alpha \neq \beta \in \mathbb{F}$, $\alpha \neq 0$ and $\gamma \in \mathbb{F}_{q^2} - \mathbb{F}$.

$$\begin{aligned}
a^2 = 0 \quad & \Longrightarrow \quad K_2(a) = q \\
a \sim \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad & \Longrightarrow \quad K_2(a) = qK(\alpha)K(\beta) \\
a \sim \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \quad & \Longrightarrow \quad K_2(a) = q^3 - q^2 + qK(\alpha)^2 \\
a \sim \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \quad & \Longrightarrow \quad K_2(a) = -q^2 + qK(\alpha)^2
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a \sim_{\mathbb{F}_{q^2}} \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^q \end{pmatrix} \quad & \implies \quad K_2(a) = -qK(\gamma),
\end{aligned}$$

where in the last line the Kloosterman sum is over $\mathbb{F}_{q^2}$.

Proposition

Assume that $a = \left(\begin{array}{c|c} a_k & b \\ \hline 0 & a_l \end{array}\right)$ *with* $a_k \in \mathbb{F}^{k\times k}, a_l \in \mathbb{F}^{l\times l}, b \in \mathbb{F}^{k\times l}$ *for some* $k, l \in \mathbb{N}$ *such that* $k + l = n$ *and* a_k *and* a_l *have no common eigenvalue in* $\overline{\mathbb{F}}$. *Then* $K_n(a) = q^{kl} K_k(a_k) K_l(a_l)$.

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If a_k *and* a_l *are as above, then the linear endomorphism of* $\mathbb{F}^{k\times l}$ *given by* $v \mapsto va_l - a_k v$ *is an isomorphism.*

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Let $p_k, p_l \in \mathbb{F}_q[t]$ be the characteristic polynomials of a_k and a_l. The Cayley-Hamilton theorem implies $0 = p_k(a_k)v = vp_k(a_l)$ and $0 = vp_l(a_l)$.

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By our assumption on the eigenvalues we have that $p_k(t)$ and $p_l(t)$ are relatively prime in $\mathbb{F}[t]$, thus there are polynomials such that $p_k r_k + p_l r_l = 1$ which implies $0 = v$, hence our claim. □

Proof of the Proposition. Let $U_{[k,l]} = \left\{ \left(\begin{array}{c|c} e_k & v \\ \hline 0 & e_l \end{array} \right) \middle| v \in \mathbb{F}^{k \times l} \right\}$. Then

$$K_n(a) = \sum_x \psi(ax + x^{-1}) =$$

$$\frac{1}{q^{kl}} \sum_{u \in U_{[k,l]}} \sum_x \psi(a(u^{-1}xu) + (u^{-1}xu)^{-1}) =$$

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Now $uau^{-1} = \left(\begin{array}{c|c} a_k & b + va_l - a_k v \\ \hline 0 & a_l \end{array} \right)$ and so

$$K_n(a) = \frac{1}{q^{kl}} \sum_x \psi(ax + x^{-1}) \left(\sum_{v \in \mathbb{F}^{k \times l}} \psi_k((va_l - a_k v)x') \right),$$

where x' is the $l \times k$ matrix which we get by deleting the first k rows and last l columns of x.

Proof of the Proposition - continued.

By the lemma we have

$$\sum_{v\in\mathbb{F}^{k\times l}} \psi_k((va_l - a_k v)x') = \sum_{v\in\mathbb{F}^{k\times l}} \psi_k(vx') = \begin{cases} 0, & \text{if } x' \neq 0 \\ q^{kl}, & \text{if } x' = 0. \end{cases}$$

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Denote $a \in JN_n(\alpha)$ if $a \in \mathbb{F}^{n\times n}$ is Jordan normal form with unique eigenvalue $\alpha \in \mathbb{F}$.

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   With the insight of Will Sawin:
   Let $K(\alpha) = -\lambda - \bar{\lambda}$ with $|\lambda| = \sqrt{q}$ and a as above, then

$$K_n(a) = q^{n(n-1)/2} \sum_{d=0}^{n} \#\left(V \leq \mathbb{F}^n \,|\, \dim(V) = d, aV \subseteq V\right) \lambda^d \bar{\lambda}^{n-d}.$$

The general picture – by Weil, Grothendieck and Deligne

Let $\mathbb{F}_m \subset \bar{\mathbb{F}}$ the degree m extension of $\mathbb{F}$ – the field with q^m elements. Let $\mathrm{Tr}_m = \mathrm{Tr}_{\mathbb{F}_m|\mathbb{F}}$ the field trace map and $\varphi_m = \mathrm{Tr}_m \circ \varphi$.

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Let $\mathbb{A}^1$ be the affine line and X a quasiprojective variety and $f : X \to \mathbb{A}^1$ be a regular morphism. Then by the Grothendieck trace formula

$$\sum_{x \in X(\mathbb{F}_m)} \varphi_m(f(x)) = \sum_{i=0}^{2\dim(X)} (-1)^i \sum_{j=1}^{d_i} (\lambda_j^i)^m,$$

where $X(\mathbb{F}_m)$ is the set of $\mathbb{F}_m$-points of X and the $\lambda_j^i \in \mathbb{C}$ are the eigenvalues of the geometric Frobenius acting on certain cohomology complex.

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By Deligne's work on the Weil conjectures it is known that the eigenvalues are Weil numbers, pure of weight $w \leq i$, i.e. they are algebraic and $|\iota(\lambda_j^i)| = q^{w/2}$ for all embedding $\iota : \mathbb{Q}(\lambda_j^i) \to \mathbb{C}$.

Classical Kloosterman sums revisited

Let $X = \mathbb{A}^1 - \{0\}$ and f be the morphism which is given by $\gamma \mapsto \alpha\gamma + \beta\gamma^{-1}$.Then $\sum_{x \in X(\mathbb{F}_m)} \varphi_m(f(x)) = K(\alpha, \beta, \mathbb{F}_m)$.

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Then in the corresponding cohomology complex the terms with index more than $n^2 + 2n(w)$ vanish, where

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Thus $d_i = 0$ if $i > n^2 + 2n(w)$ and all the weights are at most $n^2 + 2n(w)$.

The general bound

To finish the proof we need the following steps:

- If $a \in JN_n(\alpha)$ then $K_n(a) \ll \begin{cases} q^{(3n^2-1)/4}, & \textit{if } n \textit{ is odd} \\ q^{3n^2/4}, & \textit{if } n \textit{ is even} \end{cases}.$

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Remarks:

- $K_n(\alpha e) = \begin{cases} 2K(\alpha)q^{(3n^2-3)/4} + O(q^{(3n^2-3)/4-1}), & \text{if } n \text{ is odd} \\ q^{3n^2/4} + O(q^{3n^2/4-1}), & \text{if } n \text{ is even} \end{cases}$

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- $|K_n(a)| \leq |K_n(\alpha e)|$ for any $a \in JN_n(\alpha)$.

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Let $X \subset \mathbb{A}^n$ be a affine variety and $f \in \mathbb{Z}[x_1, x_2, \ldots, x_N]$ there exist subvarieties $\mathbb{A}^N \supset V_1 \supset V_2 \supset \cdots \supset V_N$, $\dim(V_j) \leq N - j$ such that for all $a \in \mathbb{A}^N - V_j$ we have

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- Let $N = n^2 + 1$. Then $\mathrm{GL}_n \subset \mathbb{A}^N = V(\mathbb{Z}[x, y])$ is a closed subvariety $V(\det(x)y = 1)$. Let $f : x \mapsto \mathrm{tr}(x^{-1})$ – this is indeed a polynomial in $y\mathbb{Z}[x]$.

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  These conditions can also be formulated as polynomial equations in the variables, which makes possible to get certain V_j-s explicitly.

Thank you for your attention!