Root systems and free algebras of modular forms

Haowu Wang (joint with Brandon Williams)

Max Planck Institute for Mathematics

International Seminar on Automorphic Forms

October 14, 2020

Motivation

- General question: determining the structure of the algebras of automorphic forms for an arithmetic group Γ on a symmetric domain $\mathcal{D}$.

Motivation

- General question: determining the structure of the algebras of automorphic forms for an arithmetic group Γ on a symmetric domain $\mathcal{D}$.
- The simplest structure: the algebra of automorphic forms is freely generated.

Motivation

- General question: determining the structure of the algebras of automorphic forms for an arithmetic group Γ on a symmetric domain $\mathcal{D}$.
- The simplest structure: the algebra of automorphic forms is freely generated.
- Difficulty: free algebra $\Longrightarrow$ The Satake-Baily-Borel compactification of the modular variety $\mathcal{D}/\Gamma$ is a weighted projective space.

Motivation

- General question: determining the structure of the algebras of automorphic forms for an arithmetic group Γ on a symmetric domain $\mathcal{D}$.
- The simplest structure: the algebra of automorphic forms is freely generated.
- Difficulty: free algebra $\Longrightarrow$ The Satake-Baily-Borel compactification of the modular variety $\mathcal{D}/\Gamma$ is a weighted projective space.
- Candidates of $\mathcal{D}$: complex balls and symmetric domains of type IV
  - Popov-Vinberg: $M_*(\Gamma)$ is free $\Longrightarrow$ Γ is generated by reflections.

Motivation

- General question: determining the structure of the algebras of automorphic forms for an arithmetic group Γ on a symmetric domain $\mathcal{D}$.
- The simplest structure: the algebra of automorphic forms is freely generated.
- Difficulty: free algebra $\Longrightarrow$ The Satake-Baily-Borel compactification of the modular variety $\mathcal{D}/\Gamma$ is a weighted projective space.
- Candidates of $\mathcal{D}$: complex balls and symmetric domains of type IV
  - Popov-Vinberg: $M_*(\Gamma)$ is free $\Longrightarrow$ Γ is generated by reflections.
- Famous example: Igusa (1962):

$$M_{2*}(\mathrm{Sp}_4(\mathbb{Z})) = \mathbb{C}[\mathcal{E}_4, \mathcal{E}_6, \Phi_{10}, \Phi_{12}].$$

Motivation

- General question: determining the structure of the algebras of automorphic forms for an arithmetic group Γ on a symmetric domain $\mathcal{D}$.
- The simplest structure: the algebra of automorphic forms is freely generated.
- Difficulty: free algebra $\Longrightarrow$ The Satake-Baily-Borel compactification of the modular variety $\mathcal{D}/\Gamma$ is a weighted projective space.
- Candidates of $\mathcal{D}$: complex balls and symmetric domains of type IV
  - Popov-Vinberg: $M_*(\Gamma)$ is free $\Longrightarrow$ Γ is generated by reflections.
- Famous example: Igusa (1962):

$$M_{2*}(\mathrm{Sp}_4(\mathbb{Z})) = \mathbb{C}[\mathcal{E}_4, \mathcal{E}_6, \Phi_{10}, \Phi_{12}].$$

- Project: to classify and construct all free algebras of modular forms

Modular forms on type IV symmetric domains

- M: even lattice of signature $(2, n)$ with bilinear form $(\cdot, \cdot)$, $n \geq 3$.
- Symmetric domain of type IV: $\mathrm{O}^+(2, n)/(\mathrm{SO}(2) \times \mathrm{O}(n))$
  $\mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}^+$
- $\mathrm{O}^+(M) \subset \mathrm{O}(M)$ preserving $\mathcal{D}(M)$
- $\widetilde{\mathrm{O}}^+(M) = \ker(\mathrm{O}^+(M) \to \mathrm{O}(M^\vee/M))$

Modular forms on type IV symmetric domains

- M: even lattice of signature $(2, n)$ with bilinear form $(\cdot, \cdot)$, $n \geq 3$.
- Symmetric domain of type IV: $\mathrm{O}^+(2, n)/(\mathrm{SO}(2) \times \mathrm{O}(n))$
  $\mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}^+$
- $\mathrm{O}^+(M) \subset \mathrm{O}(M)$ preserving $\mathcal{D}(M)$
- $\widetilde{\mathrm{O}}^+(M) = \ker(\mathrm{O}^+(M) \to \mathrm{O}(M^\vee/M))$

Definition

A modular form of weight $k \in \mathbb{N}$ for $\Gamma < \mathrm{O}^+(M)$ is a holomorphic function $F : \mathcal{D}(M)^\bullet \to \mathbb{C}$ on the affine cone $\mathcal{D}(M)^\bullet$ satisfying

Modular forms on type IV symmetric domains

- M: even lattice of signature $(2, n)$ with bilinear form $(\cdot, \cdot)$, $n \geq 3$.
- Symmetric domain of type IV: $\mathrm{O}^+(2, n)/(\mathrm{SO}(2) \times \mathrm{O}(n))$
  $\mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}^+$
- $\mathrm{O}^+(M) \subset \mathrm{O}(M)$ preserving $\mathcal{D}(M)$
- $\widetilde{\mathrm{O}}^+(M) = \ker(\mathrm{O}^+(M) \to \mathrm{O}(M^\vee/M))$

Definition

A modular form of weight $k \in \mathbb{N}$ for $\Gamma < \mathrm{O}^+(M)$ is a holomorphic function $F : \mathcal{D}(M)^\bullet \to \mathbb{C}$ on the affine cone $\mathcal{D}(M)^\bullet$ satisfying

$$F(t\mathcal{Z}) = t^{-k} F(\mathcal{Z}), \quad \forall t \in \mathbb{C}^*,$$
$$F(g\mathcal{Z}) = F(\mathcal{Z}), \quad \forall g \in \Gamma.$$

Modular forms on type IV symmetric domains

- M: even lattice of signature $(2, n)$ with bilinear form $(\cdot, \cdot)$, $n \geq 3$.
- Symmetric domain of type IV: $\mathrm{O}^+(2, n)/(\mathrm{SO}(2) \times \mathrm{O}(n))$
  $\mathcal{D}(M) = \{[\omega] \in \mathbb{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}^+$
- $\mathrm{O}^+(M) \subset \mathrm{O}(M)$ preserving $\mathcal{D}(M)$
- $\widetilde{\mathrm{O}}^+(M) = \ker(\mathrm{O}^+(M) \to \mathrm{O}(M^\vee/M))$

Definition

A modular form of weight $k \in \mathbb{N}$ for $\Gamma < \mathrm{O}^+(M)$ is a holomorphic function $F : \mathcal{D}(M)^\bullet \to \mathbb{C}$ on the affine cone $\mathcal{D}(M)^\bullet$ satisfying

$$F(t\mathcal{Z}) = t^{-k} F(\mathcal{Z}), \quad \forall t \in \mathbb{C}^*,$$
$$F(g\mathcal{Z}) = F(\mathcal{Z}), \quad \forall g \in \Gamma.$$

- The graded algebra $M_*(\Gamma)$ is finitely generated over $\mathbb{C}$. In particular, if $M_*(\Gamma)$ is a free algebra generated by $n+1$ forms of weights k_1, k_2, ..., k_{n+1}, then $(\mathcal{D}(M)/\Gamma)^*$ is a weighted projective space with weights $(k_1, k_2, ..., k_{n+1})$.

The first free algebra

- Igusa (1962): $M_{2*}(\mathrm{Sp}_4(\mathbb{Z}))$ is freely generated by 4 modular forms of weights 4, 6, 10, 12.

The first free algebra

- Igusa (1962): $M_{2*}(\mathrm{Sp}_4(\mathbb{Z}))$ is freely generated by 4 modular forms of weights 4, 6, 10, 12.
- $\mathrm{Sp}_4(\mathbb{Z})$ can be realized as a certain orthogonal group of signature $(2, 3)$, i.e. $\mathrm{SO}^+(2U \oplus \langle -2 \rangle)$.

The first free algebra

- Igusa (1962): $M_{2*}(\mathrm{Sp}_4(\mathbb{Z}))$ is freely generated by 4 modular forms of weights 4, 6, 10, 12.
- $\mathrm{Sp}_4(\mathbb{Z})$ can be realized as a certain orthogonal group of signature $(2, 3)$, i.e. $\mathrm{SO}^+(2U \oplus \langle -2 \rangle)$.
- $M_*(\mathrm{O}^+(2U \oplus \langle -2 \rangle))$ is freely generated by 4 modular forms of weights 4, 6, 10, 12.

The first free algebra

- Igusa (1962): $M_{2*}(\mathrm{Sp}_4(\mathbb{Z}))$ is freely generated by 4 modular forms of weights 4, 6, 10, 12.
- $\mathrm{Sp}_4(\mathbb{Z})$ can be realized as a certain orthogonal group of signature $(2,3)$, i.e. $\mathrm{SO}^+(2U \oplus \langle -2 \rangle)$.
- $M_*(\mathrm{O}^+(2U \oplus \langle -2 \rangle))$ is freely generated by 4 modular forms of weights 4, 6, 10, 12.
- $M_*(\mathrm{SO}^+(2U \oplus \langle -2 \rangle))$ is generated by 5 modular forms of weights 4, 6, 10, 12, 35 with a single relation in weight 70.

Main result

Theorem (W.-Williams 2020)

Let R be a root system of type $A_r(1 \le r \le 7)$, $B_r(2 \le r \le 4)$, $D_r(4 \le r \le 8)$, $C_r(3 \le r \le 8)$, G_2, F_4, E_6, or E_7. We define $\Gamma_R < \mathrm{O}^+(2U \oplus L_R(-1))$ as the subgroup generated by $\widetilde{\mathrm{O}}^+(2U \oplus L_R(-1))$ and $W(R)$. Then the graded algebra $M_*(\Gamma_R)$ is freely generated by $r+3$ forms of weights 4, 6, and $-k_j + 12m_j$, $1 \le j \le r+1$.

Main result

Theorem (W.-Williams 2020)

Let R be a root system of type $A_r (1 \le r \le 7)$, $B_r (2 \le r \le 4)$, $D_r (4 \le r \le 8)$, $C_r (3 \le r \le 8)$, G_2, F_4, E_6, or E_7. We define $\Gamma_R < \mathrm{O}^+(2U \oplus L_R(-1))$ as the subgroup generated by $\widetilde{\mathrm{O}}^+(2U \oplus L_R(-1))$ and $W(R)$. Then the graded algebra $M_*(\Gamma_R)$ is freely generated by $r+3$ forms of weights 4, 6, and $-k_j + 12m_j$, $1 \le j \le r+1$.

Some cases were known:

- A_1: Igusa 1962

Main result

Theorem (W.-Williams 2020)

Let R be a root system of type $A_r (1 \le r \le 7)$, $B_r (2 \le r \le 4)$, $D_r (4 \le r \le 8)$, $C_r (3 \le r \le 8)$, G_2, F_4, E_6, or E_7. We define $\Gamma_R < \mathrm{O}^+(2U \oplus L_R(-1))$ as the subgroup generated by $\widetilde{\mathrm{O}}^+(2U \oplus L_R(-1))$ and $W(R)$. Then the graded algebra $M_*(\Gamma_R)$ is freely generated by $r+3$ forms of weights 4, 6, and $-k_j + 12m_j$, $1 \le j \le r+1$.

Some cases were known:

- A_1: Igusa 1962
- A_2, B_2: Dern-Krieg 2003, 2006
- A_3, B_3: Freitag-Hermann 2000; Klöcker 2005
- B_4, F_4: Krieg 2005, 2011

Main result

Theorem (W.-Williams 2020)

Let R be a root system of type $A_r (1 \le r \le 7)$, $B_r (2 \le r \le 4)$, $D_r (4 \le r \le 8)$, $C_r (3 \le r \le 8)$, G_2, F_4, E_6, or E_7. We define $\Gamma_R < \mathrm{O}^+(2U \oplus L_R(-1))$ as the subgroup generated by $\widetilde{\mathrm{O}}^+(2U \oplus L_R(-1))$ and $W(R)$. Then the graded algebra $M_*(\Gamma_R)$ is freely generated by $r+3$ forms of weights 4, 6, and $-k_j + 12m_j$, $1 \le j \le r+1$.

Some cases were known:

- A_1: Igusa 1962
- A_2, B_2: Dern-Krieg 2003, 2006
- A_3, B_3: Freitag-Hermann 2000; Klöcker 2005
- B_4, F_4: Krieg 2005, 2011
- C_n-tower: Vinberg 2010, 2018 (geometric method: moduli spaces of lattice polarized K3 surfaces)

Main result

Theorem (W.-Williams 2020)

Let R be a root system of type $A_r (1 \le r \le 7)$, $B_r (2 \le r \le 4)$, $D_r (4 \le r \le 8)$, $C_r (3 \le r \le 8)$, G_2, F_4, E_6, or E_7. We define $\Gamma_R < \mathrm{O}^+(2U \oplus L_R(-1))$ as the subgroup generated by $\widetilde{\mathrm{O}}^+(2U \oplus L_R(-1))$ and $W(R)$. Then the graded algebra $M_*(\Gamma_R)$ is freely generated by $r+3$ forms of weights 4, 6, and $-k_j + 12m_j$, $1 \le j \le r+1$.

Some cases were known:

- A_1: Igusa 1962
- A_2, B_2: Dern-Krieg 2003, 2006
- A_3, B_3: Freitag-Hermann 2000; Klöcker 2005
- B_4, F_4: Krieg 2005, 2011
- C_n-tower: Vinberg 2010, 2018 (geometric method: moduli spaces of lattice polarized K3 surfaces)
- The cases $A_4, A_5, A_6, A_7, E_6, E_7$ are new.

Fourier-Jacobi expansion of modular forms

Let F be a modular form of weight k for $\Gamma = \langle \widetilde{\mathrm{O}}^+(2U \oplus L(-1)), W \rangle$, $W < \mathrm{O}(L)$. We consider its Fourier and Fourier-Jacobi expansions on the tube domain

$$\mathcal{H}(L) = \{Z = (\tau, \mathfrak{z}, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\operatorname{Im} Z, \operatorname{Im} Z) > 0\}$$

at the 1-dimensional cusp determined by $2U$

Fourier-Jacobi expansion of modular forms

Let F be a modular form of weight k for $\Gamma = \langle \widetilde{\mathrm{O}}^+(2U \oplus L(-1)), W\rangle$, $W < \mathrm{O}(L)$. We consider its Fourier and Fourier-Jacobi expansions on the tube domain

$$\mathcal{H}(L) = \{Z = (\tau, \mathfrak{z}, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\operatorname{Im} Z, \operatorname{Im} Z) > 0\}$$

at the 1-dimensional cusp determined by $2U$

$$F(\tau, \mathfrak{z}, \omega) = \sum_{\substack{n,m \in \mathbb{N}, \ell \in L^\vee \\ 2nm - (\ell,\ell) \geq 0}} f(n, \ell, m) q^n \zeta^\ell \xi^m = \sum_{m=0}^{\infty} \phi_m(\tau, \mathfrak{z}) \xi^m,$$

where $q = \exp(2\pi i \tau)$, $\zeta^\ell = \exp(2\pi i (\ell, \mathfrak{z}))$, $\xi = \exp(2\pi i \omega)$.

Fourier-Jacobi expansion of modular forms

Let F be a modular form of weight k for $\Gamma = \langle \widetilde{\mathrm{O}}^+(2U \oplus L(-1)), W \rangle$, $W < \mathrm{O}(L)$. We consider its Fourier and Fourier-Jacobi expansions on the tube domain

$$\mathcal{H}(L) = \{ Z = (\tau, \mathfrak{z}, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\operatorname{Im} Z, \operatorname{Im} Z) > 0 \}$$

at the 1-dimensional cusp determined by $2U$

$$F(\tau, \mathfrak{z}, \omega) = \sum_{\substack{n, m \in \mathbb{N}, \ell \in L^\vee \\ 2nm - (\ell, \ell) \geq 0}} f(n, \ell, m) q^n \zeta^\ell \xi^m = \sum_{m=0}^{\infty} \phi_m(\tau, \mathfrak{z}) \xi^m,$$

where $q = \exp(2\pi i \tau)$, $\zeta^\ell = \exp(2\pi i (\ell, \mathfrak{z}))$, $\xi = \exp(2\pi i \omega)$. Then $\phi_m \in J^W_{k,L,m}$, i.e. ϕ_m is a W-invariant holomorphic Jacobi form of weight k and index m associated to the lattice L.

Fourier-Jacobi expansion of modular forms

Let F be a modular form of weight k for $\Gamma = \langle \widetilde{\mathrm{O}}^+(2U \oplus L(-1)), W\rangle$, $W < \mathrm{O}(L)$. We consider its Fourier and Fourier-Jacobi expansions on the tube domain

$$\mathcal{H}(L) = \{Z = (\tau, \mathfrak{z}, \omega) \in \mathbb{H} \times (L \otimes \mathbb{C}) \times \mathbb{H} : (\operatorname{Im} Z, \operatorname{Im} Z) > 0\}$$

at the 1-dimensional cusp determined by $2U$

$$F(\tau, \mathfrak{z}, \omega) = \sum_{\substack{n,m\in\mathbb{N}, \ell\in L^\vee \\ 2nm-(\ell,\ell)\geq 0}} f(n,\ell,m) q^n \zeta^\ell \xi^m = \sum_{m=0}^{\infty} \phi_m(\tau,\mathfrak{z})\xi^m,$$

where $q = \exp(2\pi i\tau)$, $\zeta^\ell = \exp(2\pi i(\ell,\mathfrak{z}))$, $\xi = \exp(2\pi i\omega)$. Then $\phi_m \in J^W_{k,L,m}$, i.e. ϕ_m is a W-invariant holomorphic Jacobi form of weight k and index m associated to the lattice L. Moreover, we have the symmetric relation

$$f(n,\ell,m) = f(m,\ell,n), \quad \forall (n,\ell,m) \in \mathbb{N} \oplus L^\vee \oplus \mathbb{N}.$$

Weyl invariant Jacobi forms I

- R: irreducible root system of rank r; L_R: root lattice; $W(R)$: Weyl group;
- $\langle \cdot, \cdot \rangle$: if L_R is odd, then $\langle \cdot, \cdot \rangle := 2(\cdot, \cdot)$. L_R^*: dual lattice

Weyl invariant Jacobi forms I

- R: irreducible root system of rank r; L_R: root lattice; $W(R)$: Weyl group;
- $\langle\cdot,\cdot\rangle$: if L_R is odd, then $\langle\cdot,\cdot\rangle := 2(\cdot,\cdot)$. L_R^*: dual lattice

Let $k \in \mathbb{Z}$, $t \in \mathbb{N}$. A holomorphic function $\varphi : \mathbb{H} \times (L_R \otimes \mathbb{C}) \to \mathbb{C}$ is called a $W(R)$-invariant weak Jacobi form of weight k and index t if

$$(1) \qquad \varphi(\tau, \sigma(\mathfrak{z})) = \varphi(\tau, \mathfrak{z}), \quad \sigma \in W(R);$$

$$(2) \qquad \varphi(\tau, \mathfrak{z} + x\tau + y) = e^{-t\pi i(\langle x,x\rangle\tau + 2\langle x,\mathfrak{z}\rangle)}\varphi(\tau, \mathfrak{z}), \quad x, y \in L_R;$$

$$(3) \qquad \varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k \exp\left(t\pi i\frac{c\langle\mathfrak{z},\mathfrak{z}\rangle}{c\tau+d}\right)\varphi(\tau,\mathfrak{z});$$

$$(4) \qquad \varphi(\tau, \mathfrak{z}) = \sum_{n=0}^{\infty}\sum_{\ell\in L_R^*} f(n,\ell)e^{2\pi i(n\tau + \langle\ell,\mathfrak{z}\rangle)}.$$

Weyl invariant Jacobi forms I

- R: irreducible root system of rank r; L_R: root lattice; $W(R)$: Weyl group;
- $\langle\cdot,\cdot\rangle$: if L_R is odd, then $\langle\cdot,\cdot\rangle := 2(\cdot,\cdot)$. L_R^*: dual lattice

Let $k \in \mathbb{Z}$, $t \in \mathbb{N}$. A holomorphic function $\varphi : \mathbb{H} \times (L_R \otimes \mathbb{C}) \to \mathbb{C}$ is called a $W(R)$-invariant weak Jacobi form of weight k and index t if

$$(1) \qquad \varphi(\tau, \sigma(\mathfrak{z})) = \varphi(\tau, \mathfrak{z}), \quad \sigma \in W(R);$$

$$(2) \qquad \varphi(\tau, \mathfrak{z} + x\tau + y) = e^{-t\pi i(\langle x,x\rangle\tau + 2\langle x,\mathfrak{z}\rangle)}\varphi(\tau, \mathfrak{z}), \quad x, y \in L_R;$$

$$(3) \qquad \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(t\pi i \frac{c\langle\mathfrak{z},\mathfrak{z}\rangle}{c\tau + d}\right)\varphi(\tau, \mathfrak{z});$$

$$(4) \qquad \varphi(\tau, \mathfrak{z}) = \sum_{n=0}^{\infty} \sum_{\ell \in L_R^*} f(n, \ell) e^{2\pi i(n\tau + \langle \ell, \mathfrak{z}\rangle)}.$$

If φ further satisfies the condition

$$f(n, \ell) \neq 0 \Longrightarrow 2nt - (\ell, \ell) \geq 0$$

then φ is called a $W(R)$-invariant holomorphic Jacobi form.

Weyl invariant Jacobi forms II

Theorem (Wirthmüller, 1992)

If R is not of type E_8, then $J^{w,W(R)}_{*,L_R,*}$ over M_* is the polynomial algebra in $r+1$ basic $W(R)$-invariant weak Jacobi forms of weight $-k_j$ and index m_j, where $0 \le j \le r$ and

- $k_0 = 0,\ m_0 = 1;$

Weyl invariant Jacobi forms II

Theorem (Wirthmüller, 1992)

If R is not of type E_8, then $J^{w,W(R)}_{*,L_R,*}$ over M_* is the polynomial algebra in $r+1$ basic $W(R)$-invariant weak Jacobi forms of weight $-k_j$ and index m_j, where $0 \le j \le r$ and

- $k_0 = 0$, $m_0 = 1$;
- *m_j are the coefficients of the dual of the highest coroot written as a linear combination of the simple roots of R;*

Weyl invariant Jacobi forms II

Theorem (Wirthmüller, 1992)

If R is not of type E_8, then $J^{w,W(R)}_{*,L_R,*}$ over M_* is the polynomial algebra in $r+1$ basic $W(R)$-invariant weak Jacobi forms of weight $-k_j$ and index m_j, where $0 \le j \le r$ and

- $k_0 = 0$, $m_0 = 1$;
- *m_j are the coefficients of the dual of the highest coroot written as a linear combination of the simple roots of R;*
- *k_j are the degrees of the generators of the ring of $W(R)$-invariant polynomials, i.e. the exponents of $W(R)$ increased by 1.*

Weyl invariant Jacobi forms II

Theorem (Wirthmüller, 1992)

If R is not of type E_8, then $J^{w,W(R)}_{*,L_R,*}$ over M_* is the polynomial algebra in $r+1$ basic $W(R)$-invariant weak Jacobi forms of weight $-k_j$ and index m_j, where $0 \le j \le r$ and

- $k_0 = 0$, $m_0 = 1$;
- *m_j are the coefficients of the dual of the highest coroot written as a linear combination of the simple roots of R;*
- *k_j are the degrees of the generators of the ring of $W(R)$-invariant polynomials, i.e. the exponents of $W(R)$ increased by 1.*

- [W. 2018] $J^{w,W(E_8)}_{*,E_8,*}$ is not a polynomial algebra over M_*.

Weyl invariant Jacobi forms II

Theorem (Wirthmüller, 1992)

If R is not of type E_8, then $J^{w,W(R)}_{*,L_R,*}$ over M_* is the polynomial algebra in $r+1$ basic $W(R)$-invariant weak Jacobi forms of weight $-k_j$ and index m_j, where $0 \le j \le r$ and

- $k_0 = 0$, $m_0 = 1$;
- *m_j are the coefficients of the dual of the highest coroot written as a linear combination of the simple roots of R;*
- *k_j are the degrees of the generators of the ring of $W(R)$-invariant polynomials, i.e. the exponents of $W(R)$ increased by 1.*

- [W. 2018] $J^{w,W(E_8)}_{*,E_8,*}$ is not a polynomial algebra over M_*.
- [W. 2020] The Jacobian of free generators equals a theta block associated to the root system R. e.g. (A_1 case) $\phi_{0,1}\phi'_{-2,1} - \phi'_{0,1}\phi_{-2,1} = \vartheta(\tau, 2z)/\eta^3$. This observation leads to an automorphic proof of Wirthmüller's theorem (arXiv:2007.16033).

Proof of Main Theorem

1. The Fourier-Jacobi coefficients of modular forms for Γ_R are $W(R)$-invariant Jacobi forms.

Proof of Main Theorem

1. The Fourier-Jacobi coefficients of modular forms for Γ_R are $W(R)$-invariant Jacobi forms.
2. Wirthmüller 1992: For all irreducible root systems except E_8, the bigraded algebra of $W(R)$-invariant weak Jacobi forms is freely generated by $r+1$ forms over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms.

Proof of Main Theorem

1. The Fourier-Jacobi coefficients of modular forms for Γ_R are $W(R)$-invariant Jacobi forms.
2. Wirthmüller 1992: For all irreducible root systems except E_8, the bigraded algebra of $W(R)$-invariant weak Jacobi forms is freely generated by $r+1$ forms over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms.
3. We construct $r+3$ basic modular forms for Γ_R:
   (a) Two modular forms $\widetilde{E}_4$ and $\widetilde{E}_6$ of weights 4 and 6 whose first Fourier-Jacobi coefficients are respectively the Eisenstein series E_4 and E_6 on $\mathrm{SL}_2(\mathbb{Z})$;

Proof of Main Theorem

1. The Fourier-Jacobi coefficients of modular forms for Γ_R are $W(R)$-invariant Jacobi forms.
2. Wirthmüller 1992: For all irreducible root systems except E_8, the bigraded algebra of $W(R)$-invariant weak Jacobi forms is freely generated by $r+1$ forms over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms.
3. We construct $r+3$ basic modular forms for Γ_R:
   (a) Two modular forms $\widetilde{E}_4$ and $\widetilde{E}_6$ of weights 4 and 6 whose first Fourier-Jacobi coefficients are respectively the Eisenstein series E_4 and E_6 on $\mathrm{SL}_2(\mathbb{Z})$;
   (b) $r+1$ modular forms $\Phi_j \in M_{-k_j+12m_j}(\Gamma_R)$ whose first nonzero Fourier-Jacobi coefficients are $\Delta^{m_j}\phi_j$ in their m_j^{th} term, where Δ is the normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. (i.e. $\Phi_j = \Delta^{m_j}\phi_j \cdot \xi^{m_j} + O(\xi^{m_j+1})$.)

Proof of Main Theorem

1. The Fourier-Jacobi coefficients of modular forms for Γ_R are $W(R)$-invariant Jacobi forms.
2. Wirthmüller 1992: For all irreducible root systems except E_8, the bigraded algebra of $W(R)$-invariant weak Jacobi forms is freely generated by $r+1$ forms over the ring of $\mathrm{SL}_2(\mathbb{Z})$ modular forms.
3. We construct $r+3$ basic modular forms for Γ_R:
   - (a) Two modular forms $\widetilde{E}_4$ and $\widetilde{E}_6$ of weights 4 and 6 whose first Fourier-Jacobi coefficients are respectively the Eisenstein series E_4 and E_6 on $\mathrm{SL}_2(\mathbb{Z})$;
   - (b) $r+1$ modular forms $\Phi_j \in M_{-k_j+12m_j}(\Gamma_R)$ whose first nonzero Fourier-Jacobi coefficients are $\Delta^{m_j}\phi_j$ in their m_j^{th} term, where Δ is the normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. (i.e. $\Phi_j = \Delta^{m_j}\phi_j \cdot \xi^{m_j} + O(\xi^{m_j+1})$.)
4. We can kill the first Fourier-Jacobi coefficients of a given modular form by a polynomial combination of the above $r+3$ functions. If the first n Fourier-Jacobi coefficients of a modular form are zero (for a certain n which depends on the structure of the ring of weak Jacobi forms) then it is identically zero. It follows that $M_*(\Gamma_R)$ is generated by the above $r+3$ functions and hence a free algebra.

Construction of the $r+3$ basic modular forms on Γ_R I

- We use the additive lift: $J_{k,L_R,1}^{W(R)} \to M_k(\Gamma_R)$.

Construction of the $r+3$ basic modular forms on Γ_R I

- We use the additive lift: $J_{k,L_R,1}^{W(R)} \to M_k(\Gamma_R)$.
- If all generators of weak Jacobi forms have index 1, then the basic modular forms can be constructed as the additive lifts of the Jacobi-Eisenstein series of weights 4 and 6, and the holomorphic Jacobi forms $\Delta\phi_{-k_j,1}$.

Construction of the $r+3$ basic modular forms on Γ_R I

- We use the additive lift: $J^{W(R)}_{k,L_R,1} \to M_k(\Gamma_R)$.
- If all generators of weak Jacobi forms have index 1, then the basic modular forms can be constructed as the additive lifts of the Jacobi-Eisenstein series of weights 4 and 6, and the holomorphic Jacobi forms $\Delta\phi_{-k_j,1}$.
- For $D_n (4 \le n \le 8)$, E_6, E_7, there are basic weak Jacobi forms of index > 1.

Construction of the $r+3$ basic modular forms on Γ_R I

- We use the additive lift: $J^{W(R)}_{k,L_R,1} \to M_k(\Gamma_R)$.
- If all generators of weak Jacobi forms have index 1, then the basic modular forms can be constructed as the additive lifts of the Jacobi-Eisenstein series of weights 4 and 6, and the holomorphic Jacobi forms $\Delta\phi_{-k_j,1}$.
- For $D_n (4 \le n \le 8)$, E_6, E_7, there are basic weak Jacobi forms of index > 1.
- We estimate $\dim M_k(\Gamma_R)$ in terms of the dimensions of Jacobi forms:

$$M_k(\Gamma)(\xi^r) = \{F \in M_k(\Gamma) : \phi_m = 0, \text{ for all } m < r\},$$
$$J^W_{k,L,m}(q^r) = \{\phi \in J^W_{k,L,m} : \phi = O(q^r)\}.$$

Construction of the $r+3$ basic modular forms on Γ_R I

- We use the additive lift: $J^{W(R)}_{k,L_R,1} \to M_k(\Gamma_R)$.
- If all generators of weak Jacobi forms have index 1, then the basic modular forms can be constructed as the additive lifts of the Jacobi-Eisenstein series of weights 4 and 6, and the holomorphic Jacobi forms $\Delta\phi_{-k_j,1}$.
- For $D_n (4 \le n \le 8)$, E_6, E_7, there are basic weak Jacobi forms of index > 1.
- We estimate $\dim M_k(\Gamma_R)$ in terms of the dimensions of Jacobi forms:

$$M_k(\Gamma)(\xi^r) = \{F \in M_k(\Gamma) : \phi_m = 0, \text{ for all } m < r\},$$
$$J^W_{k,L,m}(q^r) = \{\phi \in J^W_{k,L,m} : \phi = O(q^r)\}.$$

We have the following exact sequence:

$$0 \longrightarrow M_k(\Gamma)(\xi^{r+1}) \longrightarrow M_k(\Gamma)(\xi^r) \xrightarrow{P_r} J^W_{k,L,r}(q^r),$$

where the map P_r sends F to its Fourier-Jacobi coefficient ϕ_r.

Construction of the $r+3$ basic modular forms on Γ_R II

- By the symmetric relation $f(n,\ell,m)=f(m,\ell,n)$, we get

$$\dim M_k(\Gamma_R) \leq \sum_{r=0}^{\infty} \dim J_{k,L_R,r}^{W(R)}(q^r) \leq \sum_{r=0}^{\infty} \dim J_{k-12r,L_R,r}^{w,W(R)}.$$

Construction of the $r+3$ basic modular forms on Γ_R II

- By the symmetric relation $f(n,\ell,m) = f(m,\ell,n)$, we get

$$\dim M_k(\Gamma_R) \leq \sum_{r=0}^{\infty} \dim J_{k,L_R,r}^{W(R)}(q^r) \leq \sum_{r=0}^{\infty} \dim J_{k-12r,L_R,r}^{w,W(R)}.$$

- (The root system C_8):
  - The "orthogonal Eisenstein" series

$$\mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{8,0}, \mathcal{E}_{8,1}, \mathcal{E}_{10,0}, \mathcal{E}_{10,1}, \mathcal{E}_{12,0}, \mathcal{E}_{12,1}, \mathcal{E}_{14,0}, \mathcal{E}_{16,0}, \mathcal{E}_{18,0}$$

satisfy no algebraic relations in weights less than 20.

Construction of the $r+3$ basic modular forms on Γ_R II

- By the symmetric relation $f(n,\ell,m) = f(m,\ell,n)$, we get

$$\dim M_k(\Gamma_R) \leq \sum_{r=0}^{\infty} \dim J_{k,L_R,r}^{W(R)}(q^r) \leq \sum_{r=0}^{\infty} \dim J_{k-12r,L_R,r}^{w,W(R)}.$$

- (The root system C_8):
  - The "orthogonal Eisenstein" series

$$\mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{8,0}, \mathcal{E}_{8,1}, \mathcal{E}_{10,0}, \mathcal{E}_{10,1}, \mathcal{E}_{12,0}, \mathcal{E}_{12,1}, \mathcal{E}_{14,0}, \mathcal{E}_{16,0}, \mathcal{E}_{18,0}$$

    satisfy no algebraic relations in weights less than 20.
  - **Proof**: We choose $v \in D_8$ with $v^2 = 2m$. The pull-backs of the above series (by taking $\mathfrak{z} = z \cdot v$) are additive lifts of the pull-backs of the Jacobi Eisenstein series, and they are Siegel paramodular forms of level m. It suffices to prove the same claim for paramodular forms.

Construction of the $r+3$ basic modular forms on Γ_R II

- By the symmetric relation $f(n,\ell,m)=f(m,\ell,n)$, we get

$$\dim M_k(\Gamma_R) \leq \sum_{r=0}^{\infty} \dim J_{k,L_R,r}^{W(R)}(q^r) \leq \sum_{r=0}^{\infty} \dim J_{k-12r,L_R,r}^{w,W(R)}.$$

- (The root system C_8):
  - The "orthogonal Eisenstein" series

$$\mathcal{E}_4, \mathcal{E}_6, \mathcal{E}_{8,0}, \mathcal{E}_{8,1}, \mathcal{E}_{10,0}, \mathcal{E}_{10,1}, \mathcal{E}_{12,0}, \mathcal{E}_{12,1}, \mathcal{E}_{14,0}, \mathcal{E}_{16,0}, \mathcal{E}_{18,0}$$

    satisfy no algebraic relations in weights less than 20.
  - **Proof**: We choose $v \in D_8$ with $v^2 = 2m$. The pull-backs of the above series (by taking $\mathfrak{z} = z \cdot v$) are additive lifts of the pull-backs of the Jacobi Eisenstein series, and they are Siegel paramodular forms of level m. It suffices to prove the same claim for paramodular forms.
  - We then derive

$$\dim M_k(\Gamma_{C_8}) = \sum_{r=0}^{\infty} \dim J_{k-12r,D_8,r}^{w,W(C_8)}, \quad \text{for } k \leq 20.$$

    This yields the existence of the basic modular forms on Γ_{C_8}.

Some examples

- $R = A_1$: $\Gamma_R = \mathrm{O}^+(2U \oplus A_1(-1))$. The $W(A_1)$-invariant weak Jacobi forms has generators of weights and indices $(0, 1)$ and $(-2, 1)$. Thus the generators of orthogonal modular forms have weights 4, 6, 10, 12.

Some examples

- $R = A_1$: $\Gamma_R = \mathrm{O}^+(2U \oplus A_1(-1))$. The $W(A_1)$-invariant weak Jacobi forms has generators of weights and indices $(0,1)$ and $(-2,1)$. Thus the generators of orthogonal modular forms have weights 4, 6, 10, 12.
- $R = B_4$: $\Gamma_R = \mathrm{O}^+(2U \oplus 4A_1(-1))$, weights: $4, 4, 6, 6, 8, 10, 12$.
- $R = A_7$: $\Gamma_R = \widetilde{\mathrm{O}}^+(2U \oplus A_7(-1))$, weights: $4, 4, 5, 6, 6, 7, 8, 9, 10, 12$.
- $R = C_8$: $\Gamma_R = \mathrm{O}^+(2U \oplus D_8(-1))$, weights: $4, 6, 8, 8, 10, 10, 12, 12, 14, 16, 18$.
- $R = E_7$: $\Gamma_R = \mathrm{O}^+(2U \oplus E_7(-1))$, weights: $4, 6, 10, 12, 14, 16, 18, 22, 24, 30$.
- $R = E_6$: $\Gamma_R = \widetilde{\mathrm{O}}^+(2U \oplus E_6(-1))$, weights: $4, 6, 7, 10, 12, 15, 16, 18, 24$

Corollary A

Corollary (W.-Williams 20)

Let R be a root system in Main Theorem. For any weak Jacobi form $\phi \in J^{w,W(R)}_{k,L_R,m}$, there exists a modular form of weight $k+12m$ for Γ_R whose first non-zero Fourier-Jacobi coefficient is $(\Delta^m \phi)\cdot \xi^m$. Moreover, we have the equality

$$\dim M_k(\Gamma_R) = \sum_{r=0}^{\infty} \dim J^{w,W(R)}_{k-12r,L_R,r}.$$

Corollary B

Let k be a positive integer. A formal series of Jacobi forms is an element

$$\Psi(Z) = \sum_{m=0}^{\infty} \psi_m \xi^m \in \prod_{m=0}^{\infty} J^W_{k,L,m}.$$

We call Ψ a formal Fourier-Jacobi expansion of weight k if it satisfies

$$f_m(n,\ell) = f_n(m,\ell), \quad m,n \in \mathbb{N}, \ell \in L^\vee,$$

where $f_m(n,\ell)$ are Fourier coefficients of ψ_m. We denote the space of such expansions by $FM_k(\Gamma)$.

Corollary B

Modularity of formal Fourier-Jacobi expansions

For all Γ_R in Main Theorem, we have that $FM_k(\Gamma_R) = M_k(\Gamma_R)$ for any $k \in \mathbb{N}$. In other word, every formal Fourier-Jacobi expansion is convergent on the tube domain $\mathcal{H}(L)$ and defines an orthogonal modular form.

Corollary B

Modularity of formal Fourier-Jacobi expansions

For all Γ_R in Main Theorem, we have that $FM_k(\Gamma_R) = M_k(\Gamma_R)$ for any $k \in \mathbb{N}$. In other word, every formal Fourier-Jacobi expansion is convergent on the tube domain $\mathcal{H}(L)$ and defines an orthogonal modular form.

Proof.

The Fourier-Jacobi expansion of modular forms defines the injective map

$$M_k(\Gamma) \to FM_k(\Gamma), \quad F \mapsto \text{Fourier-Jacobi expansion of } F.$$

Using a similar argument, we get $\dim FM_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim J^{w,W}_{k-12r,L,r}$. We then prove the surjectivity of the above map by Corollary A. $\square$

Corollary B

Modularity of formal Fourier-Jacobi expansions

For all Γ_R in Main Theorem, we have that $FM_k(\Gamma_R) = M_k(\Gamma_R)$ for any $k \in \mathbb{N}$. In other word, every formal Fourier-Jacobi expansion is convergent on the tube domain $\mathcal{H}(L)$ and defines an orthogonal modular form.

Proof.

The Fourier-Jacobi expansion of modular forms defines the injective map

$$M_k(\Gamma) \to FM_k(\Gamma), \quad F \mapsto \text{Fourier-Jacobi expansion of } F.$$

Using a similar argument, we get $\dim FM_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim J^{w,W}_{k-12r,L,r}$. We then prove the surjectivity of the above map by Corollary A. □

Remark: This nice property is only known to hold in the A_1 case (Aoki 2000) and for Siegel modular forms (Bruinier-Braum 2015). The modularity for Siegel modular forms + Zhang Wei's thesis ⇒ Kudla's conjecture on the modularity of generating series of special cycles for orthogonal Shimura varieties.

Remarks I

- The bigraded ring of $W(E_8)$-invariant weak Jacobi forms is not a free algebra so our method does not apply to the E_8 root system.

Remarks I

- The bigraded ring of $W(E_8)$-invariant weak Jacobi forms is not a free algebra so our method does not apply to the E_8 root system.
- Hashimoto–Ueda 2014 The graded ring of modular forms on $\mathrm{O}^+(2U \oplus E_8(-1))$ is freely generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42.

Remarks I

- The bigraded ring of $W(E_8)$-invariant weak Jacobi forms is not a free algebra so our method does not apply to the E_8 root system.
- Hashimoto–Ueda 2014 The graded ring of modular forms on $\mathrm{O}^+(2U \oplus E_8(-1))$ is freely generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42.
- We do not need to consider modular forms associated to root systems of rank > 8 due to the following result:

Remarks I

- The bigraded ring of $W(E_8)$-invariant weak Jacobi forms is not a free algebra so our method does not apply to the E_8 root system.
- Hashimoto–Ueda 2014 The graded ring of modular forms on $\mathrm{O}^+(2U \oplus E_8(-1))$ is freely generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42.
- We do not need to consider modular forms associated to root systems of rank > 8 due to the following result:
  - Shvartsman-Vinberg 2017 Let Γ be an arithmetic subgroup of $\mathrm{O}^+_{2,n}$. When $n > 10$, the graded algebra $M_*(\Gamma)$ is never free.

Remarks II

Theorem (W. 2020)

Let $M = 2U \oplus L(-1)$ be an even lattice of signature $(2, n)$. Let $\Gamma < \mathrm{O}^+(M)$ be a subgroup containing $\widetilde{\mathrm{O}}^+(M)$. If $M_*(\Gamma)$ is a free algebra, then Γ must be

- $\mathrm{O}^+(2U \oplus E_8(-1))$;
- *one of the 25 groups in Main Theorem.*

Thank you very much!