There are at most finitely many singular moduli that are S-units

Sebastián Herrero
(joint with Ricardo Menares and Juan Rivera-Letelier)

International Seminar on Automorphic Forms, May 2021

Notation

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

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$j : \mathbb{H} \to \mathbb{C}$ modular function with

$$j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n, \quad \text{where } q := e^{2\pi i z}.$$

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Facts:

1. j is a Hauptmodul for Γ.
2. $j(z)$ is the j-invariant of the elliptic curve $E_z \simeq \mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$.

CM points and singular moduli

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If z is CM, we call $j(z)$ a singular modulus (following Kronecker).

A question of Masser

Are there *only finitely many* singular moduli that are algebraic units?

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Motivation: In

An effective "Theorem of André" for CM-points on a plane curve (2013)

Bilu, Masser and Zannier proved that there are no pairs (j_1, j_2) of singular moduli on $X_1 \cdot X_2 = 1$.

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Natural question: Is there any such *singular unit*?

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   No singular modulus is a unit (2018)

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2. Let $\Phi_m(X, Y)$ denote the m-th modular polynomial ($m \geq 1$ integer). In

   Singular units and isogenies between CM *elliptic curves* (2019)

   Y. Li proved that $\Phi_m(j_1, j_2)$ is never an algebraic unit for j_1, j_2 singular moduli.

Differences of singular moduli

Habegger's work (2015) implies the following result: given an algebraic integer α there are at most finitely many singular moduli j such that $j - \alpha$ is an algebraic unit.

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Example: If $\alpha = 1$ then

$$j\left(\frac{1+i\sqrt{3}}{2}\right) - \alpha = 0 - 1 = -1$$

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In the case $\alpha = j_2$ is a singular modulus we have, by Y. Li's theorem with $m = 1$, that $j_1 - j_2$ is never an algebraic unit.

Fact: Differences of singular moduli are very special.

More on CM points

For z CM, define

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$$\Lambda_D = \Gamma \backslash \mathrm{CM}_D.$$

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$$\Lambda_D = \Gamma \backslash \mathrm{CM}_D.$$

Theorem (CM theory)

Λ_D is finite of cardinality $h(D)$ (class number) and $j(\Lambda_D)$ is a full Galois orbit.

Norms of differences

Given α in $\overline{\mathbb{Q}}$ define

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Gross and Zagier gave an *arithmetic formula* for $\mathrm{Nm}(j_1 - j_2)$ under certain hypotheses.

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On singular moduli (1985)

Gross and Zagier gave an *arithmetic formula* for $\mathrm{Nm}(j_1 - j_2)$ under certain hypotheses.
It is not clear how to use Gross and Zagier's formula (or extensions of it) to prove *directly* that $j_1 - j_2$ is never an algebraic unit.

Singular S-units

Fix S a finite set of prime numbers.

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Theorem (H–Menares–Rivera-Letelier, 2021)

There are at most finitely many singular moduli that are S-units.

By Bilu, Habegger and Kühne, or by Y. Li, every singular modulus is an S-unit for some finite set S.

Numerics: A. Sutherland's table[1]

D	$\prod_{z\in\Lambda_D} j(z)$	D	$\prod_{z\in\Lambda_D} j(z)$	D	$\prod_{z\in\Lambda_D} j(z)$
-3	0	-32	$2^6 5^6 23^3$	-63	$-3^6 5^{12} 17^3 41^3 47^3$
-4	$2^6 3^3$	-35	$-2^{30} 5^3$	-64	$-2^3 3^6 23^3 47^3$
-7	$-3^3 5^3$	-36	$-2^{12} 3^3 11^3 23^3$	-67	$-2^{15} 3^3 5^3 11^3$
-8	$2^6 5^3$	-39	$3^{15} 17^3 23^3 29^3$	-68	$-2^{24} 5^{12} 17^3 47^3$
-11	-2^{15}	-40	$2^{12} 3^6 5^3 29^3$	-71	$-11^9 17^6 23^3 41^3 47^3 53^3$
-12	$2^4 3^3 5^3$	-43	$-2^{18} 3^3 5^3$	-72	$2^{12} 5^6 29^3 53^3$
-15	$-3^6 5^3 11^3$	-44	$2^{12} 11^3 17^3 29^3$	-75	$2^{30} 3^6 5^1 11^3$
-16	$2^3 3^3 11^3$	-47	$-5^{15} 11^6 23^3 29^3$	-76	$2^{12} 3^9 41^3 53^3$
-19	$-2^{15} 3^3$	-48	$2^4 3^9 5^6 11^3$	-79	$-3^{15} 17^3 29^3 47^3 53^3 59^3$
-20	$-2^{12} 5^3 11^3$	-51	$2^{33} 3^6$	-80	$2^{12} 5^6 11^3 17^6 59^3$
-23	$-5^9 11^3 17^3$	-52	$-2^{12} 3^6 5^6 23^3$	-83	$-2^{48} 5^9$
-24	$2^{12} 3^6 17^3$	-55	$-3^{12} 5^6 11^3 29^3 41^3$	-84	$-2^{24} 3^{15} 47^3 59^3$
-27	$-2^{15} 3^1 5^3$	-56	$2^{24} 11^6 17^3 41^3$	-87	$3^{18} 5^{18} 23^3 53^3 59^3$
-28	$3^3 5^3 17^3$	-59	$-2^{48} 11^3$	-88	$2^{12} 3^6 5^6 17^3 41^3$
-31	$-3^9 11^3 17^3 23^3$	-60	$3^6 5^3 29^3 41^3$	-91	$-2^{30} 3^6 17^3$

[1] https://math.mit.edu/ drew/NormsOfSingularModuli2000.pdf

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It seems like $j\left(\frac{1+\sqrt{-11}}{2}\right) = -2^{15}$ is the only singular modulus that is an S-unit for S a singleton. Is this the case?

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A. Sutherland checked this *conjecture* for discriminants D in $]-10^5, -3]$ (private communication).

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Given a singular modulus j_2, there are at most finitely many singular moduli j_1 such that $j_1 - j_2$ is an S-unit.

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Theorem (H–Menares–Rivera-Letelier, 2021)

Given a singular modulus j_2, there are at most finitely many singular moduli j_1 such that $j_1 - j_2$ is an S-unit.

We use Habegger's original strategy together with the new ingredient that for every prime number p, singular moduli are p-adically disperse.

Habegger's strategy (for singular units)

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Habegger considered the absolute logarithmic Weil height

$$h(a) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |a|_v\}$$

for a in K a number field, where

- M_K is the set of places of K,
- $|\cdot|_v$ is a representative absolute value extending $|\cdot|_p$ with p prime or ∞ (the usual field norms on $\mathbb{Q}$),
- $d_v = [K_v : \mathbb{Q}_p]$.

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First ingredient: For j a singular modulus of discriminant D we have

$$h(j) \geq A \log|D| + B,$$

with A, B absolute constants, $A > 0$.
This follows from results of Colmez (1989), and Nakkajima and Taguchi (1991).

Second ingredient: A density estimate for the number of singular moduli around 0. Given $\varepsilon > 0$ find $r > 0$ small such that

$$\frac{1}{h(D)} \left(j(\Lambda_D) \cap B(0, r) \right) \leq \varepsilon \text{ for } D \to -\infty.$$

This follows from the following equidistribution theorem for CM points.

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Theorem (Duke (1988) + Clozel and Ullmo (2004))

When $D \to -\infty$ we have

$$\frac{1}{h(D)} \sum_{z \in \Lambda_D} \delta_z \to \frac{3}{\pi} \frac{dxdy}{y^2}$$

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This step is not effective.

Third ingredient: An estimate for the Archimedean distance between a singular modulus and 0. For j a nonzero singular modulus of discriminant D we have

$$-\log|j| \leq c_\infty \log|D|,$$

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In the "$(j-\alpha)$ version" of Habegger's theorem (α algebraic integer) one needs David and Hirata-Kohno's deep lower bound for linear forms on $n = 2$ elliptic logarithms (2009).

Putting everything together

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If j is a singular unit of discriminant D, then

$$\begin{aligned} A\log|D| + B &\leq h(j) \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} d_v \log\max\{1, |a|_v\} \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K^\infty} d_v \log\max\{1, |a|_v\} \\ &= -\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K^\infty, |a|_v<1} d_v \log|a|_v, \end{aligned}$$

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Theorem (H–Menares–Rivera-Letelier, 2021)

Given $\varepsilon > 0$ *there exists* $r > 0$ *small such that*

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This follows from our identification of all limit measures of CM points in the p-adic setting.

p-adic distribution of CM points

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For simplicity, restrict to $D < 0$ fundamental discriminant.

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$\mathbb{A}^1_{\mathrm{Berk}}$ can be defined as the set of multiplicative semi-norms on $\mathbb{C}_p[X]$ extending $|\cdot|_p$ on $\mathbb{C}_p$, with a suitable topology (locally compact, arc-connected),

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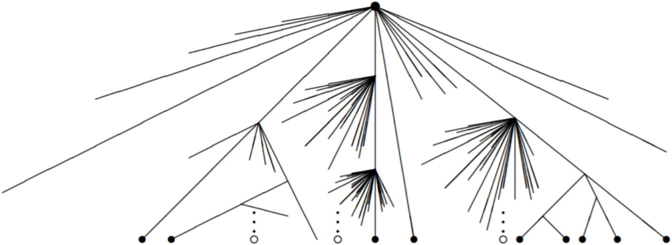
$$\mathbb{C}_p \hookrightarrow \mathbb{A}^1_{\mathrm{Berk}}, \qquad z \mapsto \iota(z)$$

is defined by $\iota(z)(f) = |f(z)|_p$ for f in $\mathbb{C}_p[X]$.
$\mathbb{C}_p$ is dense in $\mathbb{A}^1_{\mathrm{Berk}}$.

Above the unit disc in $\mathbb{C}_p$ we have the following picture[2]



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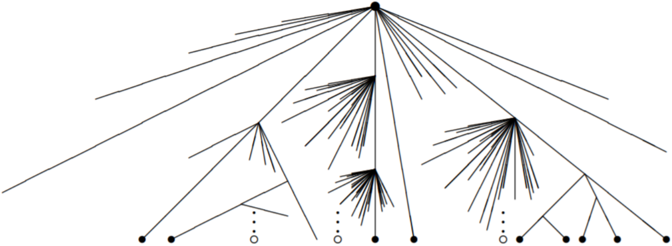

At the top we have the Gauss point ζ defined by

$$\zeta(a_0 + a_1 X + \ldots + a_n X^n) = \max\{|a_0|_p, |a_1|_p, \ldots, |a_n|_p\}.$$

[2] Illustration of Joe Silverman

Convergence towards the Gauss point

Convergence towards the Gauss point

Theorem (H–Menares–Rivera-Letelier, 2020)

1. *For fundamental discriminants $D < 0$ with $\left(\frac{D}{p}\right) = 1$ we have*

$$\frac{1}{h(D)} \sum_{z \in \Lambda_D} \delta_{j(z)} \to \delta_\zeta$$

weakly on $\mathbb{A}^1_{\mathrm{Berk}}$.

2. *This is not the case for fundamental discriminants $D < 0$ with $\left(\frac{D}{p}\right) \neq 1$.*

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$$\Lambda_D \subset \Lambda_{\mathfrak{D}} = \{E \text{ ell. curve over } \overline{\mathbb{Q}}_p \text{ with } \operatorname{End}(\widehat{E}) \simeq \mathcal{O}_{\mathfrak{D}}\} \subset Y(\overline{\mathbb{Q}}_p)$$

where $\widehat{E}$ is the *formal group* of E.

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For a p-adic discriminant $\mathfrak{D}$ the set $\Lambda_{\mathfrak{D}}$ is compact and there exists a (unique) Borel probability measure $\nu_{\mathfrak{D}}$ with support $\Lambda_{\mathfrak{D}}$ such that for fundamental discriminants $D < 0$ with $D \in \mathfrak{D}$ we have

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There are 3 (for $p > 2$) or 7 (for $p = 2$) p-adic fundamental discriminants.

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The strategy is essentially the same for differences of singular moduli that are S-units.

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F. Campagna shows that $S_0 = \{p \text{ prime }, p \equiv 1 \bmod 3\}$ every singular S-unit is a singular unit, hence there are none.

We can use Campagna's result to extend ours to certain classes of infinite sets S of prime numbers (larger than S_0).

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General question: What about more general Hauptmoduln?

The method seems to extend without major difficulties to the case of differences of singular moduli that are S-units for any Hauptmodul of a genus zero subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$ that is algebraically related to the j-function.

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1. The λ-invariants: there are six of them, they satisfy

$$2^8(1-\lambda+\lambda^2)^3 - j\lambda^2(1-\lambda)^2 = 0.$$

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Weber's computations show that infinitely many singular moduli for these functions are algebraic units.

Note that 0 is not a singular modulus for any of these functions.

Fin

¡Muchas gracias!