Traces of CM values and geodesic cycle integrals of modular functions

Claudia Alfes-Neumann

Bielefeld University

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Outline

Starting point

- Let $n \in \mathbb{N}$ and $a(n)$ be an *interesting* function.
- Example 1: $r_2(n) = \#\{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 = n\}$
  - We have $5 = 1^2 + 2^2$, so $r_2(5) \geq 1$.
  - For which n do we have $r_2(n) \neq 0$?
- Example 2: $p(n) = \#$ ways to write n as a sum of integers $\leq n$, the partition function.
  - $p(5) = 7$, $p(100) = 190\ 569\ 292$.
  - How does $p(n)$ grow?
  - Is there a formula for $p(n)$?

Questions

- Further examples: divisor sums, special values of L-series (of elliptic curves),...
- Questions:
  - For which n is $a(n) \neq 0$?
  - Growth of $a(n)$?
  - Is there a formula for $a(n)$?
  - Can something be said on the rationality of the $a(n)$?

Generating series

- Let q be a formal variable.
- Consider:

$$f(q) = \sum_{n=1}^{\infty} a(n)q^n = a(1)q + a(2)q^2 + a(3)q^3 + \dots.$$

- Set $q = e^{2\pi i z}$, where $z \in \mathbb{H}$, $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

helpful: $\rightsquigarrow$ $f(z)$ is a modular form

Why helpful?

The operation of the modular group

- Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half plane.
- Let $\mathrm{SL}_2(\mathbb{Z}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} : ad - bc = 1\}$.
- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by *fractional linear transformations*

$$Mz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

- $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.
- Let p be prime and $\Gamma_0(p) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{p}\}$.

Now: $z \in \mathbb{H}$, $z = x + iy$, where $x, y \in \mathbb{R}$ and $q = e^{2\pi i z}$.

$\tau \in \mathbb{H}, \tau = u + iv, u, v \in \mathbb{R}$

Modular forms

Definition

Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called *modular form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$, if:

1. f is holomorphic on $\mathbb{H}$.
2. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
3. f is holomorphic in ∞.

Remark

$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ induces $z \mapsto z+1$. Therefore, f has a Fourier expansion

$$f(z) = \sum_{n \geq 0} a(n) q^n, \ q = e^{2\pi i n z}.$$

Properties of modular forms

- The space of modular forms of fixed weight (and group) is finite-dimensional.
- If two modular forms lie in the same space, there are relations between their Fourier coefficients.
- The generators are well-known (Eisenstein series and cusp forms).

Back to the generating series – representation numbers

- The function

$$\theta^2(z) = 1 + \sum_{n=1}^{\infty} r_2(n) q^n$$

  is a modular form of weight 1 (with character).
- This space is 1-dimensional and the generator is an Eisenstein series.
- The two forms agree up to a constant.
- Comparing the Fourier coefficients we obtain

$$r_2(n) = 4 \sum_{d|n, d>0 \text{ odd}} (-1)^{(d-1)/2}.$$

- $\Rightarrow$ Every prime $p \equiv 1 \pmod 4$ is a sum of two squares. (Fermat)

Back to the generating series – the partition function

- The function

$$q^{-1/24}\sum_{n=0}^{\infty} p(n)q^n \left(= \frac{1}{\eta(z)}\right)$$

  is a weakly holomorphic modular form of weight $1/2$.
- $\Rightarrow$ Hardy-Ramanujan/Rademacher: asymptotic resp. exact formula for $p(n)$ (using the circle method)
- $\Rightarrow$ Bruinier-Ono: $p(n) =$ finite sum of algebraic numbers.

More examples of generating series

- Hurwitz class numbers
- Dimensions of the irreducible representations of the Monster
- Central L-values of elliptic curves
- Cycle integrals of certain functions
- ...

Binary quadratic forms

- A binary integer quadratic form is a polynomial
$$Q(x,y) = ax^2 + bxy + cy^2$$
with $a, b, c \in \mathbb{Z}$.
- The *discriminant* of Q is defined as $D = b^2 - 4ac \equiv 0, 1 \mod 4$.
- Let $\mathcal{Q}_D$ be the set of all binary integer quadratic forms of discriminant D.
- $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_D$ with finitely many orbits, i.e. $\mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z})$ is finite.

CM points

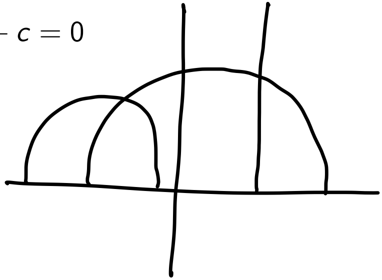
- Let $D < 0$ and $Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ be a quadratic form of discriminant $D = b^2 - 4ac$.
- The equation
$$0 = az^2 + bz + c = Q(z, 1)$$
has a unique solution $z_Q \in \mathbb{H}$.
- z_Q is called CM point associated to Q.
- z_Q CM point $\Leftrightarrow$ The elliptic curve $\mathbb{C}/(\mathbb{Z} + z_Q\mathbb{Z})$ has complex multiplication (i.e. $\mathbb{Z} \subsetneq \mathrm{End}(E)$).

Geodesics

- Let $D > 0$ and $Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ be a quadratic form of discriminant $b^2 - 4ac = D$.
- The set of solutions of

$$a|z|^2 + bx + c = 0$$

defines a geodesic c_Q in $\mathbb{H}$.



Generating series of twisted traces of CM values

- Let F be modular of weight 0.
- For $D < 0$ we define the D-th trace of F d fund. discr.

$$t_F(D) = \sum_{Q \in \mathcal{Q}_D/\mathrm{SL}_2(\mathbb{Z})} \chi_d(Q) \frac{F(z_Q)}{|\bar{\Gamma}_Q|}.$$

- Zagier: For $F(z) = J(z) = j(z) - 744$ the function

$$q^{-1} - \sum_{D<0} t_F(D) q^{-D} \qquad (q = e^{2\pi i z})$$

is a weakly holomorphic modular form of weight $3/2$.

Generating series of traces of geodesic cycle integrals

- Let $F \in S_{2k+2}$ be a cusp form.
- For $D > 0$ we define

$$\mathcal{C}(F, Q) = \int_{\Gamma_Q \backslash c_Q} F(z) Q(z,1)^k dz.$$

  and

$$t_F(D) = \sum_{Q \in \mathcal{Q}_D / \mathrm{SL}_2(\mathbb{Z})} \mathcal{C}(F, Q).$$

- Shintani:

$$\sum_{D>0} t_F(D) q^D.$$

  is a cusp form of weight $k + 3/2$.

Observation

- The generating series of the traces of CM values of a (special) modular function is again modular.
- The generating series of the traces of geodesic cycle integrals of a cusp form is again a cusp form.

Is there a general framework for such results?

Theta liftings

- For F of weight k we consider

$$I(F,\tau) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F(z)\overline{\Theta(\tau,z)}y^k \frac{dxdy}{y^2}.$$

  - $\Theta(\tau,z)$ has weight k in z.
  - $\overline{\Theta(\tau,z)}$ has weight ℓ in τ.
  - Depending on the growth of F and Θ in y we might have to *regularize* the integral. (Harvey-Moore, Borcherds)
- For *suitable* $\Theta(\tau,z)$ the coefficients of the Fourier expansion of $I(F,\tau)$ are given by the traces of CM values or geodesic cycle integrals of F.
- This is how one can (re)prove the results of Zagier and Shintani (Bruinier, Funke, Alfes, Ehlen; Niwa).

Theta liftings: changing the role of z and τ

- For F of weight ℓ we consider

$$I(F,z) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F(\tau)\overline{\Theta(\tau,z)}v^k \frac{dudv}{v^2}.$$

  - $\Theta(\tau,z)$ has weight k in z.
  - $\overline{\Theta(\tau,z)}$ has weight ℓ in τ.
  - Depending on the growth of F and Θ in v we might have to *regularize* the integral.
- This lifting is called the additive Borcherds lift.

Ex.: Shimura lift

Harmonic weak Maass forms

- Let $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ be the hyperbolic Laplace operator of weight k ($\in \frac{1}{2}\mathbb{Z}$).

Definition

A function $f : \mathbb{H} \to \mathbb{C}$ is called *harmonic weak Maass form of weight k for* $\mathrm{SL}_2(\mathbb{Z})$ if the following hold:

- f is smooth and $\Delta_k f = 0$.
- $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.
- There is a Fourier polynomial $P_f(z) = \sum_{n\le 0} c^+(n) q^n \in \mathbb{C}[q^{-1}]$, such that $f(z) - P_f(z) = \mathcal{O}(e^{-Cy})$ for $y \to \infty$ for a $C > 0$.

Fourier expansion

Lemma (Bruinier–Funke)

A harmonic weak Maass form of weight k ($k \neq 1$) has a Fourier expansion of the form

$$f(z) = \underbrace{\sum_{n \gg -\infty} c_f^+(n) q^n}_{\text{holomorphic part } f^+} + \underbrace{\sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|v) q^n}_{\text{nonholomorphic part } f^-},$$

at the cusp ∞. Here, $\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$ is the incomplete Γ-function.

Notation

Let:

- $M_k(p) :=$ the space of modular forms of weight k for $\Gamma_0(p)$
- $S_k(p) :=$ the space of cusp forms of weight k for $\Gamma_0(p)$
- $M_k^!(p) :=$the space of weakly holomorphic modular forms of weight k for $\Gamma_0(p)$
- $H_k(p) :=$ the space of harmonic weak Maass forms of weight k for $\Gamma_0(p)$

Relation to classical modular forms

Lemma (Bruinier–Funke)

Define $\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}}$. *This gives a map*

$$\xi_k : H_k \to S_{2-k}.$$

We have

- $\xi_k(f) = \xi_k(f^-)$.
- ξ_k *is surjective.*

Theta liftings of harmonic weak Maass forms

(jt w/ M. Schwagenscheidt)

$F(z) \mapsto I(F,\tau)$

$F \in H_{2k}$

- $k>0$, Shintani $\longrightarrow$ $H_{1/2+k}$: • holomorphic part = generating series of the (regularized) traces of cycle integrals of F
- $k=0$, Kudla-Millson $\longrightarrow$ Bruinier-Funke (generalization of Don Zagier's result)
- $k\leq 0$, Millson $\longrightarrow$ $H_{1/2-k}$: • holomorphic part = generating series of CM values of $R^k_{-2k} F$

$F \in H_0 \overset{\text{Millson}}{\longmapsto} f \in H_{1/2}$

many more: $\rightarrow$ Li, Zemel, Bringmann, ... BIF

Theta liftings of harmonic weak Maass forms: symplectic to orthogonal

$F(\tau) \mapsto I(F, z)$

$F \in H_{3/2-k}$

$k=1$, Siegel: Bruinier-Ono: weight 0, smooth outside a certain divisor

$k=1$, Millson: Hoevel: weight 0, "locally" harmonic

$k>1$, Siegel / Millson: AN-Bruinier-Schwagenscheidt: weight $2k$, smooth outside a certain divisor

Elliptic curves

- Let E be an elliptic curve over $\mathbb{Q}$ of conductor p

$$E : y^2 = x^3 + ax + b \quad (a, b \in \mathbb{Q}).$$

- Let $L(E, s)$ be the Hasse-Weil-Zeta function of E.
- We consider twists of E with a fundamental discriminant Δ

$$E(\Delta) : \Delta y^2 = x^3 + ax + b.$$

- Mordell-Weil: $E(\mathbb{Q}) \simeq E(\mathbb{Q})^{\text{tors}} \oplus \mathbb{Z}^r$, r is the rank of E.

The Modularity Theorem and the BSD Conjecture

- Modularity Theorem (Wiles/...): For every E there is a ~~cusp~~ new form $G_E(z) = \sum_{n>0} a_E(n)q^n \in S_2^{\text{new}}(p)$ such that

$$L(E(\Delta), s) = L(G_E, \chi_\Delta, s) \left(= \sum_{n=1}^{\infty} \chi_\Delta(n) a_E(n) n^{-s} \right).$$

- BSD Conjecture:

$$L(E, s) = c \cdot (s-1)^r + \text{higher order terms},$$

where $c \neq 0$ und $r = \mathrm{rank}(E)$.

Harmonic Maass forms as *generating* series of central L-derivatives – Approach 1

differential whose coeff. are given by c_E

constructed as an odd. backwards lift of G_E

modularity thm

$$\begin{tikzcd} & G_E \in S_2^{\mathrm{new}}(p) \arrow[r, leftrightsquigarrow] \arrow[d, "\text{Shintani}"] & E \\ f_E \in H_{\frac{1}{2}}(4p) \arrow[r, "\xi_{1/2}"] & g_E \in S_{\frac{3}{2}}^{\mathrm{new}}(4p). & \end{tikzcd}$$

$\xi_{1/2}(f) = \xi_{1/2}(f^-)$

Theorem (Bruinier-Ono)

For a fundamental discriminant $\Delta > 0$ with $\left(\frac{\Delta}{p}\right) = 1$ we have

$$L'(G_E, \chi_\Delta, 1) = 0 \iff c_E^+(\Delta) \in \mathbb{Q}.$$

$$\underset{L'(E, \chi_\Delta, 1)}{\overset{\shortparallel}{}}$$

Harmonic Maass forms as *generating* series of central L-derivatives – Approach 2

construct a "canonical" F_E using Weierstrass' $\wp$-fct

$$\begin{tikzcd}
F_E \in H_0(p) \arrow[r, "\xi_0"] \arrow[d, "\text{Millson}"] & G_E \in S_2^{\text{new}}(p) \arrow[r, leftrightsquigarrow] \arrow[d, "\text{Shintani}"] & E \arrow[ll, bend right] \\
f_E \in H_{\frac{1}{2}}(4p) \arrow[r, "\xi_{1/2}"] & g_E \in S_{\frac{3}{2}}^{\text{new}}(4p). &
\end{tikzcd}$$

$f_E = \sum c_E^+(n) q^n$

CM-traces of F_E

Theorem (Alfes)

For a fundamental discriminant $\Delta > 0$ with $\left(\frac{\Delta}{p}\right) = 1$ we have

$$L'(G_E, \chi_\Delta, 1) = 0 \;\Leftrightarrow\; c_E^+(\Delta) \in \mathbb{Q}.$$

The coefficient $c_E^+(\Delta)$ as a quotient of periods

$$\sum_{Q \in \mathcal{Q}_\Delta \backslash SL_2(\mathbb{Z})} Z_Q \qquad f_E \longrightarrow g_E \quad \text{with } G_E \downarrow g_E$$

Theorem (Bruinier)

There is a unique differential $\zeta_\Delta(f_E)$ of the third kind with residue divisor $\sum_{n<0} c_E^+(n) Z_\Delta(n)$ that satisfies:

- the first Fourier coefficient of $\zeta_\Delta(f_E)$ vanishes,
- for all Hecke operators T we have:
  $T\zeta_\Delta(f_E) - \lambda_{G_E}(T)\zeta_\Delta(f_E) = \frac{dF}{F}$, $F \in \mathbb{C}(X)^\times$,

and

$$c_E^+(\Delta) = \frac{\Re\left(\int_{c_{G_E}} \zeta_\Delta(f)\right)}{\sqrt{\Delta}\int_{c_{G_E}} \omega_G}.$$

Generalization to higher weight

$$\begin{tikzcd} F \in H_{-2k}(p) \arrow[r, "\xi_0"] \arrow[d, "\text{Millson}"] & G \in S_{2k+2}^{\mathrm{new}}(p) \arrow[d, "\text{Shintani}"] \\ f \in H_{\frac{1}{2}-k}(4p) \arrow[r, "\xi_{1/2}"] & g \in S_{\frac{3}{2}+k}^{\mathrm{new}}(4p). \end{tikzcd}$$

Kuga-Sato variety

- The coefficient c_f^+ is now given as the CM trace of $R_{2k}^k(F)$.
- AN-Bruinier-Schwagenscheidt:

$$c_f^+(\Delta) = \frac{\Re\left(\int_{c_G} \zeta_\Delta(f,z) Q_G(z,1)^k dz\right)}{(G,G)},$$

where:

- c_G and Q_G are chosen such that $\int_{c_G} G(z) Q_G(z,1)^k dz = (G,G)$.
- $\zeta_\Delta(f,z)$ is given as a certain linear combination of the functions $f_{k,d}(z) = \sum_{Q\in\mathcal{Q}_{d\Delta}} Q(z,1)^{-k}$.

Thank you!