Siegel theta series for indefinite quadratic forms

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Overview

- Vignéras' result for genus $n = 1$
- Siegel modular forms: Notation and definitions
- A generalization of Vignéras' result for genus $n \in \mathbb{N}$
- Sketch of the proof:
  - Describe suitable functions to construct theta series which transform like modular forms
  - These build a basis of a certain $n \times n$-system of pde's

Genus $n = 1$ (Vignéras '77)

- $A \in \mathbb{Z}^{m \times m}$ denotes an even symmetric non-degenerate matrix with signature (r, s).
- $p : \mathbb{R}^m \longrightarrow \mathbb{R}$, s. t. $p(u) \exp\bigl(-\pi(u^{\mathsf{t}} A u)\bigr) \in \mathcal{S}(\mathbb{R}^m)$
- Euler operator $E = u^{\mathsf{t}} \frac{\partial}{\partial u}$ and Laplace operator $\Delta_A = \left(\frac{\partial}{\partial u}\right)^{\mathsf{t}} A^{-1} \frac{\partial}{\partial u}$

Theorem (Vignéras '77)

Let $\lambda \in \mathbb{Z}$. If p fulfills

$$\left(E - \frac{\Delta_A}{4\pi}\right) p = \lambda \cdot p,$$

the series

$$\vartheta_{p,A}(z) = y^{-\lambda/2} \sum_{u \in \mathbb{Z}^m} p(u\sqrt{y}) \exp\bigl(\pi i \,(u^t A u)\, z\bigr) \quad (\mathbb{H} \ni z = x + iy)$$

transforms like a non-holomorphic modular form of weight $m/2 + \lambda$.

Applications of Vignéras' result

- S. Alexandrov, S. Banerjee, J. Manschot, B. Pioline: *Indefinite theta series and generalized error functions* (2018)
- C. Nazaroglu: *r-tuple error functions and indefinite theta series of higher-depth* (2018)
- M. Westerholt–Raum: *Indefinite theta series on cones* (2016)

Definitions and notation

Consider the Siegel upper half-plane

$$\mathbb{H}_n := \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n} \text{ symmetric}, Y > 0\}$$

and the Siegel modular group

$$\Gamma_n = \{M \in \mathbb{Z}^{2n\times 2n} \mid M^{\mathsf{t}} J M = J\}, \quad J = \begin{pmatrix} \mathrm{O} & I_n \\ -I_n & \mathrm{O} \end{pmatrix},$$

generated by $Z \mapsto Z + S$, $S = S^{\mathsf{t}} \in \mathbb{Z}^{n\times n}$ and $Z \mapsto -Z^{-1}$, which acts on $\mathbb{H}_n$ by

$$Z \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1} \quad \text{for} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Definitions and notation

Definition

We call $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ a (classical) Siegel modular form of genus n and weight k if the following conditions hold:

(a) The function f is holomorphic on $\mathbb{H}_n$.

(b) For every $M \in \Gamma_n$ we have $f(M\langle Z\rangle) = \det(CZ + D)^k f(Z)$.

(c) If $n = 1$, we additionally require f to be holomorphic for $z \to i\infty$.

The third condition is obsolete for $n > 1$ due to the Koecher principle.

Definitions and notation

- $A \in \mathbb{Z}^{m\times m}$ denotes a non-degenerate, symmetric matrix of signature (r, s).
- $H, K \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{Z}$
- $X + iY = Z \in \mathbb{H}_n$

Definition

The theta series with characteristic H and K is defined as

$$\vartheta_{H,K,p,A}(Z) = \det Y^{-\lambda/2} \cdot \sum_{U\in H+\mathbb{Z}^{m\times n}} p(UY^{1/2}) \exp\big(\pi i \operatorname{tr}(U^{\mathsf{t}}AUZ) + 2\pi i \operatorname{tr}(K^{\mathsf{t}}AU)\big).$$

Result for Siegel theta series

Theorem (R. 2020)

Let $p : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$, such that $p(U)\,\exp\big(-\pi\,\mathrm{tr}(U^{\mathsf{t}}AU)\big) \in \mathcal{S}(\mathbb{R}^{m\times n})$. If p is a solution of the $n\times n$ system of partial differential equations

$$\Big(\boldsymbol{E} - \frac{\boldsymbol{\Delta}_A}{4\pi}\Big)p = \lambda \cdot I \cdot p \quad \textit{with} \quad \boldsymbol{E} := U^{\mathsf{t}}\frac{\partial}{\partial U} \ \textit{and}\ \boldsymbol{\Delta}_A := \Big(\frac{\partial}{\partial U}\Big)^{\mathsf{t}} A^{-1}\frac{\partial}{\partial U},$$

the series

$$\vartheta_{H,K,p,A}(Z) = \det Y^{-\lambda/2} \cdot \sum_{U\in H+\mathbb{Z}^{m\times n}} p(UY^{1/2})\,\exp\big(\pi i\,\mathrm{tr}(U^{\mathsf{t}}AUZ) + 2\pi i\,\mathrm{tr}(K^{\mathsf{t}}AU)\big)$$

transforms like a vector-valued Siegel modular form of weight $m/2+\lambda$.

Result for Siegel theta series

Namely, we have for any symmetric matrix $S \in \mathbb{Z}^{n\times n}$

$$\vartheta_{H,K,p,A}(Z+S) = \exp\big(-\pi i\,\mathrm{tr}(H^{\mathsf{t}}AHS) - \pi i\,\mathrm{tr}(S_0 1_{nm} A_0 H)\big) \cdot \vartheta_{H,\widetilde{K},p,A}(Z)$$

with $\widetilde{K} := K + HS + \frac{1}{2}A^{-1}A_0 1_{mn} S_0$ and

$$\vartheta_{H,K,p,A}(-Z^{-1}) = i^{-mn/2}(-1)^{(s/2+\beta)n+\beta s}|\det A|^{-n/2}\det Z^{m/2+\lambda} \cdot \exp\big(2\pi i\,\mathrm{tr}(H^{\mathsf{t}}AK)\big) \sum_{J\in A^{-1}\mathbb{Z}^{m\times n} \bmod \mathbb{Z}^{m\times n}} \vartheta_{J+K,-H,p,A}(Z).$$

- For simplicity $H = K = \mathrm{O}$

Constructing suitable functions – A is positive definite

- P is a homogeneous polynomial of degree α, i. e. $P(UN) = \det N^\alpha P(U)$ for $N \in \mathbb{C}^{n\times n}$
- $\exp(c \operatorname{tr} \mathbf{\Delta}_A)\big(P(U)\big) := \sum_{k=0}^{\infty} \frac{c^k}{k!}(\operatorname{tr} \mathbf{\Delta}_A)^k\big(P(U)\big) \quad (c \in \mathbb{C})$

Proposition

Let A be positive definite. For $g(U) := \exp\Big(-\frac{1}{8\pi} \operatorname{tr} \mathbf{\Delta}_A\Big)\big(P(U)\big)$, the theta series

$$\vartheta_{g,A}(Z) = \det Y^{-\alpha/2} \sum_{U\in\mathbb{Z}^{m\times n}} g(UY^{1/2}) \exp\big(\pi i \operatorname{tr}(U^{\mathsf{t}} AUZ)\big)$$

transforms like a Siegel modular form of weight $m/2 + \alpha$.

- Eigenfunction with regard to the Fourier transform
- Apply Poisson summation formula

Examples

- For $m \equiv 0 \pmod 8$ we choose an even unimodular matrix A
- Let $g(U) = \exp\Bigl(-\frac{1}{8\pi}\operatorname{tr}\boldsymbol{\Delta}_A\Bigr)\bigl(P(U)\bigr)$

For

$$\vartheta_{g,A}(Z) = \det Y^{-\alpha/2} \sum_{U \in \mathbb{Z}^{m\times n}} g(UY^{1/2}) \exp\bigl(\pi i \operatorname{tr}(U^{\mathsf{t}}AUZ)\bigr),$$

we have

- $\vartheta_{g,A}(Z+S) = \vartheta_{g,A}(Z)$ for any symmetric matrix $S \in \mathbb{Z}^{n\times n}$,
- $\vartheta_{g,A}(-Z^{-1}) = \det Z^{m/2+\alpha}\,\vartheta_{g,A}(Z)$.

Thus, $\vartheta_{g,A}$ is a non-holomorphic Siegel modular form of weight $m/2+\alpha$ on the full Siegel modular group Γ_n.

Constructing suitable functions – A has signature (r, s)

- A has signature (r, s), write $A = A^+ + A^-$ and $M = A^+ - A^-$
- $\operatorname{tr}\big((U^+)^{\mathsf{t}} A U^+\big) = \operatorname{tr}\big(U^{\mathsf{t}} A^+ U\big)$ and $\operatorname{tr}\big((U^-)^{\mathsf{t}} A U^-\big) = \operatorname{tr}\big(U^{\mathsf{t}} A^- U\big)$
- $P(U) = P_r(U^+) \cdot P_s(U^-)$ with P_r homogeneous of degree α and P_s homogeneous of degree β
- $\lambda = \alpha - \beta - s$

Proposition

For $g(U) := \exp\Big(-\frac{1}{8\pi} \operatorname{tr} \mathbf{\Delta}_M\Big)\big(P(U)\big) \exp\big(2\pi \operatorname{tr}(U^{\mathsf{t}} A^- U)\big)$, the theta series

$$\vartheta_{g,A}(Z) = \det Y^{-\lambda/2} \cdot \sum_{U \in \mathbb{Z}^{m \times n}} g(UY^{1/2}) \exp\big(\pi i \operatorname{tr}(U^{\mathsf{t}} A U Z)\big)$$

transforms like a Siegel modular form of weight $m/2 + \lambda$.

Constructing suitable functions – A has signature (r, s)

Proposition

For $g(U) := \exp\Big(-\frac{1}{8\pi}\operatorname{tr}\boldsymbol{\Delta}_M\Big)\big(P(U)\big)\exp\big(2\pi\operatorname{tr}(U^{\mathsf{t}}A^{-}U)\big)$, *the theta series*

$$\vartheta_{g,A}(Z) = \det Y^{-\lambda/2} \cdot \sum_{U\in\mathbb{Z}^{m\times n}} g(UY^{1/2})\exp\big(\pi i\operatorname{tr}(U^{\mathsf{t}}AUZ)\big)$$

transforms like a Siegel modular form of weight $m/2 + \lambda$.

- Write $A = \left(\begin{smallmatrix} I_r & \mathrm{O} \\ \mathrm{O} & -I_s \end{smallmatrix}\right)$
- Calculate the Fourier transform for the "positive definite" part with regard to $Z \in \mathbb{H}_n$ and the "negative definite" part with regard to $-\overline{Z} \in \mathbb{H}_n$
- Use the result for positive definite quadratic forms

Description of the homogeneous functions

- f is homogeneous of degree α: $f(UN) = \det N^\alpha f(U)$ for $N \in \mathbb{C}^{n\times n}$

Lemma

A function $f : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ is homogeneous of degree α if and only if it fulfills the $n \times n$- system of partial differential equations

$$\boldsymbol{E}p = \alpha \cdot I \cdot p, \quad \textit{where} \quad \boldsymbol{E} = U^{\mathsf{t}} \frac{\partial}{\partial U}.$$

- $(\mathbf{E}f)(UN) = N^{\mathsf{t}} \frac{\partial}{\partial N}\big(f(UN)\big)$
- f is homogeneous: Show $\big(\mathbf{E}f\big)(UN) = \alpha \det N^\alpha \cdot I \cdot f(U)$, then set $N = I$
- f fulfills the pde: Show $f(UN) \cdot \det N^{-\alpha} = C(U)$, again set $N = I$

Positive definite forms

- Let A denote a positive definite matrix
- Let $g(U) = \exp\left(-\frac{1}{8\pi}\operatorname{tr}\mathbf{\Delta}_A\right)\left(P(U)\right)$

Define

- the (generalized) Euler operator $\mathbf{E} = U^{\mathsf{t}}\frac{\partial}{\partial U}$
- the (generalized) Laplace operator $\mathbf{\Delta}_A = \left(\frac{\partial}{\partial U}\right)^{\mathsf{t}} A^{-1}\frac{\partial}{\partial U}$

Lemma

The function g fulfills

$$\left(\boldsymbol{E} - \frac{\mathbf{\Delta}_A}{4\pi}\right)p = \alpha \cdot I \cdot p$$

if and only if P fulfills $\boldsymbol{E}p = \alpha \cdot I \cdot p$.

Positive definite forms

- We additionally assume that $g(U)\,\exp\bigl(-\pi\,\mathrm{tr}(U^tAU)\bigr) \in \mathcal{S}(\mathbb{R}^{m\times n})$
- From $\Bigl(\mathbf{E} - \frac{\mathbf{\Delta}_A}{4\pi}\Bigr)g = \alpha \cdot I \cdot g$ we get $\mathrm{tr}\Bigl(\mathbf{E} - \frac{\mathbf{\Delta}_A}{4\pi}\Bigr)g = \alpha \cdot n \cdot g$
- Apply Vignéras: g is a polynomial and so is P
- This gives us a finite basis of all solutions (for homogeneous polynomials of degree α we can explicitly determine a basis)

Indefinite forms

- A has signature (r, s), write $A = A^+ + A^-$ and $M = A^+ - A^-$
- $g(U) = \exp\left(-\frac{1}{8\pi}\operatorname{tr}\boldsymbol{\Delta}_M\right)\big(P(U)\big)\,\exp\big(2\pi\operatorname{tr}(U^{\mathsf{t}}A^-U)\big)$, where $P(U) = P_r(U^+)\cdot P_s(U^-)$, where P_r is only defined on the subspace U^+, where A is positive definite, and homogeneous of degree α (and P_s analogously for the negative definite part)

Lemma

The function g fulfills

$$\left(\boldsymbol{E} - \frac{\boldsymbol{\Delta}_A}{4\pi}\right)p = \lambda\cdot I\cdot p \quad (\lambda = \alpha - \beta - s).$$

- Diagonalize A and split up the equation

Indefinite forms

- We set $A = \left(\begin{smallmatrix} I_r & \mathrm{O} \\ \mathrm{O} & -I_s \end{smallmatrix}\right)$ and write $\mathcal{D} := \left(\mathbf{E} - \frac{\mathbf{\Delta}_A}{4\pi}\right)$ as $\mathcal{D} = \mathcal{D}_{U_r} + \mathcal{D}_{U_s}$
- Any solution f must be a product of a function f_r defined on U_r and a function f_s defined on U_s
- We have $\mathcal{D}_{U_r} f_r = C_r \cdot f_r$ and $\mathcal{D}_{U_s} f_s = C_s \cdot f_s$ for some matrices C_r, C_s with $C_r + C_s = \lambda \cdot I$
- For genus $n = 1$ we can immediately deduce the claim, for higher genus n we have to show that $C_r = \alpha \cdot I$ holds.

Indefinite forms

Lemma

Let $p : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}$ denote a polynomial, which solves the system of partial differential equations

$$\boldsymbol{E}p = C \cdot p \quad (C \in \mathbb{C}^{n\times n}).$$

If p is not the zero function, the matrix C has the form $C = \alpha \cdot I$ for some $\alpha \in \mathbb{N}_0$.

- Show the claim for $m = n = 2$
- Reduce the general case to a 2×2-system

Indefinite forms

- A has signature (r, s), write $A = A^+ + A^-$ and $M = A^+ - A^-$
- $g(U) = \exp\bigl(-\frac{1}{8\pi} \operatorname{tr} \boldsymbol{\Delta}_M\bigr)\bigl(P(U)\bigr) \exp\bigl(2\pi \operatorname{tr}(U^{\mathsf{t}} A^- U)\bigr)$, where $P(U) = P_r(U^+) \cdot P_s(U^-)$, where P_r is only defined on the subspace U^+, where A is positive definite, and homogeneous of degree α (and P_s analogously for the negative definite part)

Proposition

The functions g build an (infinite) basis for all functions p such that $p(U) \exp\bigl(-\pi \operatorname{tr}(U^t A U)\bigr) \in \mathcal{S}(\mathbb{R}^{m \times n})$ and

$$\Bigl(\boldsymbol{E} - \frac{\boldsymbol{\Delta}_A}{4\pi}\Bigr)p = \lambda \cdot I \cdot p \quad (\lambda = \alpha - \beta - s).$$

Examples in the hyperbolic case

For signature $(m-1,1)$ one can generalize Zwegers' construction ('02):

- Build ϑ_f, where f consists of a harmonic polynomial of degree 1 defined on the subspace where the quadratic form is positive definite and an exponential factor defined on the subspace where it is negative definite.

Thank you for your attention!

- Preprint *Siegel theta series for indefinite quadratic forms* on arXiv: 2009.08230

References:

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- E. Freitag: Siegelsche Modulfunktionen, *Grundlehren der mathematischen Wissenschaften, Bd. 254, Berlin, 1983.*
- M.-F. Vignéras: Séries thêta des formes quadratiques indéfinies, *Serre JP., Zagier D.B. (eds) Modular Functions of One Variable VI. Lecture Notes in Mathematics* ***627****, 1977.*