Deligne-Mostow theory and beyond

joint work with Klaus Hulek

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Reference

Today's talk is based on the preprint

K. Hulek, Y. Maeda, *Revisiting the moduli space of 8 points on* $\mathbb{P}^1$, arXiv:2211.00052.

Introduction

Introduction

I want to talk about the *intersection* between algebraic geometry and number theory.

Background (AG)

- Moduli spaces have been studied.
- GIT quotient by Mumford (1970s)
- Minimal model program (1980s)

Background (NT)

- Modular forms have been studied.
- Baily-Borel, toroidal compactifications (1960s, 1970s)
- Shimura varieties in the Langlands program (2000s)

8 points on $\mathbb{P}^1$

Let $p = \{[\Lambda_i : 1]\} \subset (\mathbb{P}^1)^8$ be a set of distinct 8 points on $\mathbb{P}^1$. Now,

$$y_0^2 \prod_{i=1}^{4} (x_0 - \lambda_i x_1) + y_1^2 \prod_{i=5}^{8} (x_0 - \lambda_i x_1) = 0$$

defines a smooth divisor C on $\mathbb{P}^1_{[x_0:x_1]} \times \mathbb{P}^1_{[y_0:y_1]}$.

Then, the minimal resolution of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along $C + (y_0 = 0) + (y_1 = 0)$ is a particular lattice-polarized K3 surface X_p, equipped with an order 4 automorphism. X_p depends only on the 8 points, thus we can relate a moduli space of K3 surfaces with 8 points on $\mathbb{P}^1$.

Period map

On the other hand, there is a period map

$$\begin{array}{ccc} \mathcal{M}_{K3} & \longrightarrow & \mathcal{D} \\ (X, \omega_X) & \longmapsto & \alpha_X(\omega_X) \end{array}$$

where $\mathcal{M}_{K3}$ is the moduli space of particular marked K3 surfaces and $\mathcal{D}$ is the type IV domain with $\dim(\mathcal{D}) = 10$. This description gives an isomorphism

$$\mathcal{M}^\circ \longrightarrow (\mathbb{B}^5 \backslash H)/\Gamma_{\mathrm{U}}$$

where $\mathcal{M}^\circ$ is the moduli space of *distinct* 8 points on $\mathbb{P}^1$, $\mathbb{B}^5$ is the type I domain with $\dim(\mathbb{B}^5) = 5$ and H is the (union of) Heegner divisor defined by roots. As we will see later, this extends to the whole of the moduli space, taking suitable compactifications.

GIT quotient: in general

Let G be an algebraic group acting on a variety X. The "quotient" X/G does *not* exist as a scheme in general even in the case of $G = \mathbb{Z}/2\mathbb{Z}$!

There are several solutions; impose some conditions on G and X, treat algebraic spaces or consider GIT quotients.

$X^{\mathrm{ss}} \subset X$: set of semistable points (w.r.t. ample line bundle L)

Then, the GIT quotient $X//_L G$ always exists and this is the unique categorical quotient of X^{ss} by G, which is the main tool to treat moduli problems.

GIT quotient: $\mathcal{M}^{\mathrm{GIT}}$

$\mathrm{SL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1$ by the linear fractional transformation.

$$\begin{aligned}
\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} &:= \mathbb{P}H^0((\mathbb{P}^1)^8, \mathscr{O}_{(\mathbb{P}^1)^8}(1, \cdots, 1))//\,\mathrm{SL}_2(\mathbb{C}) \\
&\cong (\mathbb{P}^1)^8//\,\mathrm{SL}_2(\mathbb{C}) \\
\mathcal{M}^{\mathrm{GIT}} &:= \mathbb{P}H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(8))//\,\mathrm{SL}_2(\mathbb{C}) \\
&\cong \mathbb{P}^8//\,\mathrm{SL}_2(\mathbb{C})
\end{aligned}$$

Then $\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$ (resp. $\mathcal{M}^{\mathrm{GIT}}$) is the moduli space of ordered (resp. unordered) 8 points on $\mathbb{P}^1$.

$p = (p_1, \ldots, p_8)$: semistable iff no 5 points coincide.

$p = (p_1, \ldots, p_8)$: stable iff no 4 points coincide.

H: the locus corresponding to the points such that stable but not distinct 8 points (discriminant divisor)

What's wrong with $\mathcal{M}^{\mathrm{GIT}}$?

Let's compare $\mathcal{M}^{\mathrm{GIT}}$ to the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of the moduli space of genus $g \geqslant 2$ curves.

$\overline{\mathcal{M}_g}$	$\mathcal{M}^{\mathrm{GIT}}$
Smooth DM stack	Smooth Artin stack
Projective coarse moduli space	Projective coarse moduli space
Finite quotient singularities	Bad singularities
Normal crossing boundary	Not normal crossing boundary
lc model is interesting	lc model is itself

We want to find a "good" model of $\mathcal{M}^{\mathrm{GIT}}$ that behaves more like $\overline{\mathcal{M}_g}$.

(Quoted from the slide by Casalaina-Martin)

Today's goal $\sim$ seeking better compactifications $\sim$

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{K}} \arrow[r, dotted, "\Phi"] \arrow[d, "\psi"] & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \arrow[d, "\pi"] \\
\mathcal{M}^{\mathrm{GIT}} \arrow[r, "\phi"] & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}.
\end{tikzcd}$$

Today's goal is

- Describe the geometry of the spaces.
- Explain why (possibly despite appearances) $\mathcal{M}^{\mathrm{K}} \not\cong \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$.
- (See in terms of the minimal model programs.)

Definition of modular varieties

Setting

- $D = G/K$ is a Hermitian symmetric domain $(G = \mathrm{U}(1,5), D = \mathbb{B}^5)$.
- $X = D/\Gamma$ is a modular variety for an arithmetic subgroup $\Gamma \subset G$.

Example: modular curves

Let us take $G = \mathrm{SL}_2(\mathbb{R}) \supset K = \mathrm{SO}(2)$. Then,

$$\begin{array}{rcl} D = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) & \longrightarrow & \mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\} \\ g & \longmapsto & g \cdot \sqrt{-1}. \end{array}$$

In this case, $D/\mathrm{SL}_2(\mathbb{Z})$ is the (coarse) moduli space of elliptic curves. A projective variety $\overline{D/\mathrm{SL}_2(\mathbb{Z})}$ is called a modular curve.

Compactifications of modular varieties

Today's topic is compactifications of modular varieties and modular forms. There are two compactifications in this talk.

$\overline{X}^{\mathrm{BB}}$: Baily-Borel compactification

- Unique minimal compactification with worse singularities
- Higher codimensional boundaries
- Constructed by the theory of modular forms

$\overline{X}^{\mathrm{tor}}$: Toroidal compactification

- Blow-up of the BB compactification whose center is BB cusps $\overline{X}^{\mathrm{BB}} \backslash X$
- Divisorial (NC) boundaries
- Constructed by the theory of toric varieties

Moduli interpretation

$G = \mathrm{Sp}_g(\mathbb{R})$...A Siegel modular variety X is a moduli space of g- dimensional abelian varieties.

$G = \mathrm{O}^+(2,19)$...An orthogonal variety X is a moduli space of polarized K3 surfaces.

Q. What about unitary groups?

A. Deligne-Mostow theory gives a moduli interpretation to unitary modular varieties for $G = \mathrm{U}(1,n)$.

Deligne-Mostow theory

Theorem (Deligne-Mostow, 1986, Kondō, 2007)

There are isomorphisms

$$\phi_{\mathrm{ord}} : \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \cong \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$$
$$\phi : \mathcal{M}^{\mathrm{GIT}} \cong \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}},$$

which is the biggest DM variety over $\mathbb{Q}(\sqrt{-1})$. These morphisms send polystable points to Baily-Borel cusps.

Here, $\mathbb{B}^5 := \{(v_1, \dots, v_5) \in \mathbb{C}^5 \mid |v_1|^2 + \cdots + |v_5|^2 < 1\}$. The arithmetic subgroups: $\Gamma := \mathrm{U}(U \oplus U(2) \oplus D_4^{\oplus 2})$ and Γ_{ord} is the discriminant kernel.

Blow-ups (Ordered case)

On both sides, there are two blow-ups:

- the Kirwan blow-up $\varphi_1 : \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \to \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}$
- the canonical toroidal compactification $\pi_{\mathrm{ord}} : \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}$

The center of these blow-ups are equal:

polystable orbits $\cong$ Baily-Borel cusps

$$\begin{tikzcd} \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \arrow[r, dotted, "\Phi_{\mathrm{ord}}"] \arrow[d, "\varphi_1"] & \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \arrow[d, "\pi_{\mathrm{ord}}"] \\ \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \arrow[r, "\phi_{\mathrm{ord}}"] & \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}} \end{tikzcd}.$$

Lifting (Ordered case)

Gallardo-Kerr-Schaffler showed that this lifts to the unique toroidal compactification.

Theorem (Gallardo-Kerr-Schaffler, 2021)

The birational map $\Phi_{\mathrm{ord}} : \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \to \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ is an isomorphism.

We explain that this does not hold for the *unordered* case.

The Kirwan blow-up: unordered case

On $\mathcal{M}^{\mathrm{GIT}}$, the polystable ($=$ strictly semistable with a positive dimensional stabilizer) point is the unique point $\{c_{4,4}\}$.
$R := \mathbb{C}^\times \rtimes S_2$ is the stabilizer.

$$Z_R^{\mathrm{ss}} := \{x \in (\mathbb{P}^8) \mid R \text{ fixes } x\}$$
$$\widetilde{(\mathbb{P}^8)^{\mathrm{ss}}} := Bl_{\mathrm{SL}_2(\mathbb{C})\cdot Z_R^{\mathrm{ss}}}(\mathbb{P}^8)^{\mathrm{ss}}$$

Then, the Kirwan blow-up is defined by

$$\mathcal{M}^{\mathrm{K}} := \widetilde{(\mathbb{P}^8)^{\mathrm{ss}}}//\,\mathrm{SL}_2(\mathbb{C}) \to \mathcal{M}^{\mathrm{GIT}} = \mathbb{P}^8//\,\mathrm{SL}_2(\mathbb{C}).$$

Proposition (Kirwan, 1985)

All semistable points on $\mathcal{M}^{\mathrm{K}}$ are stable.

Main results

Lifting (Unordered case)

As in the case of ordered 8 points, the setting is as follows:

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{K}} \arrow[r, dotted, "\Phi"] \arrow[d, "\psi"] & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \arrow[d, "\pi"] \\
\mathcal{M}^{\mathrm{GIT}} \arrow[r, "\phi"] & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}.
\end{tikzcd}$$

Theorem (Hulek-M, 2022)

Neither Φ nor its inverse Φ^{-1} is a morphism.

Cohomology

Theorem (Hulek-M, 2022)

All the odd degree cohomology vanishes. In even degrees, their Betti numbers are given by:

j	0	2	4	6	8	10
$\dim H^j(\mathcal{M}^{\mathrm{K}})$	1	2	3	3	2	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}})$	1	1	2	2	1	1
$\dim H^j(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}})$	1	2	3	3	2	1
$\dim H^j(\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}})$	1	43	99	99	43	1
$\dim IH^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}})$	1	8	29	29	8	1
$\dim H^j(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}})$	1	43	99	99	43	1

thus, all the Betti numbers of $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are the same.

K-equivalence

This situation remains the possibility that $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are isomorphic as abstract varieties. To discuss this problem, let us introduce the notion K-equivalence.

Recall that two projective normal $\mathbb{Q}$-Gorenstein varieties X and Y are called K-*equivalent* if there is a common resolution of singularities Z dominating X and Y birationally

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[dr, "f_Y"] & \\ X \arrow[rr, dashed, leftrightarrow] & & Y \end{tikzcd}$$

such that $f_X^* K_X \sim_{\mathbb{Q}} f_Y^* K_Y$. For K-equivalent varieties, the top intersection numbers are equal: $K_X^n = K_Y^n$, where n is the dimension of X and Y.

Why K-equivalence?

Why ask about K-equivalence?

Facts

1. Smooth K-equivalent varieties have the same Betti numbers (Batyrev).
2. Birational normal projective $\mathbb{Q}$-Gorenstein varieties with canonical singularities, such that their canonical bundles are nef, are K-equivalent (Kawamata).

We are, however, in neither of these situations: $\mathcal{M}^{\mathrm{K}}$ and $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ are singular (finite quotient singularities), Canonical bundles are not nef (birational to $\mathbb{Q}$-Fano $\mathcal{M}^{\mathrm{GIT}}$).

(Quoted from the slide by Casalaina-Martin)

Not K-equivalence

Theorem (Hulek-M, 2022)

$\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ and $\mathcal{M}^{\mathrm{K}}$ are not K-equivalent.

This is an analog to the work of Casalaina-Martin-Grushevsky-Hulek-Laza on the moduli space of cubic surfaces.

Relationship between several compactifications

$$
\begin{tikzcd}
 & \overline{\mathcal{M}}_{0,8} \arrow[dl, bend right, "\varphi_2"'] \arrow[d] \arrow[dr] & & & \\
\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \arrow[dd, "\varphi_1"'] & \overline{\mathcal{M}}_{0,8(\frac{1}{4}+\epsilon)} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \arrow[rr] \arrow[dddll, "\pi_{\mathrm{ord}}"'] & & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \arrow[dddll, "\pi"] \\
 & & & \mathcal{M}^{\mathrm{K}} \arrow[ur, dashed, "\Phi"] \arrow[dl, "\psi"'] & \\
\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \arrow[rr] \arrow[d, "\phi_{\mathrm{ord}}"', "\wr"] & & \mathcal{M}^{\mathrm{GIT}} \arrow[d, "\phi"', "\wr"] & & \\
\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}} \arrow[rr] & & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}} & &
\end{tikzcd}
$$

S_8 quotient

By taking the S_8 quotient of the diagram

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \arrow[r, "\sim"] \arrow[d, "\varphi_1"] & \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}} \arrow[d, "\pi_{\mathrm{ord}}"] \\
\mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}} \arrow[r, "\sim"] & \overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{BB}}.
\end{tikzcd}$$

we obtain

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}/S_8 \arrow[r, "\sim"] \arrow[d, "\psi"] & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \arrow[d, "\pi"] \\
\mathcal{M}^{\mathrm{GIT}} \arrow[r, "\sim"] & \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}.
\end{tikzcd}$$

Hence, our result implies $\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}}/S_8 \not\cong \mathcal{M}^{\mathrm{K}}$.

Geometry

- The toroidal boundary of $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ has 35 components and each of them isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$ (well-known).
- The toroidal boundary divisor of $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ is irreducible and isomorphic to $(\mathbb{P}(1,2,3) \times \mathbb{P}(1,2,3))/S_2$.

Key lemma

The normal bundle of the toroidal boundary in $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$ is $\mathscr{O}(-1,-1)$.

In the case of the moduli space of cubic surfaces, a similar claim was shown by the geometry of Naruki's compactification (cross ratio variety). We prove this by Hassett's program instead of using cross ratio varieties.

Proof

Outline of the proof: lifting

On the one hand,

Proposition

On $\mathcal{M}^{\mathrm{K}}$, the boundaries and discriminant divisors meet generically non-transversally.

Proof.

Local computation via the Luna slice $\square$

On the other hand,

Proposition

On $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$, the boundaries and discriminant divisors meet generically transversally.

Outline of the proof: lifting

Proof.

There is a blow-up sequence:

$$\overline{\mathcal{M}}_{0,8} \xrightarrow{\varphi_2} \mathcal{M}^{\mathrm{K}}_{\mathrm{ord}} \xrightarrow{\varphi_1} \mathcal{M}^{\mathrm{GIT}}_{\mathrm{ord}}.$$

Here, $\overline{\mathcal{M}}_{0,8}$ is a moduli space of stable maps and a normal crossing compactification of $(\mathbb{B}^5 \backslash H)/\Gamma_{\mathrm{ord}}$, where H is the discriminant divisor.

Hence, the divisors $\overline{\mathcal{M}}_{0,8} \backslash (\mathbb{B}^5 \backslash H)/\Gamma_{\mathrm{ord}}$ meet transversely, and their image by φ_2 is the union of the toroidal boundaries and the discriminant divisors. Note that the quotient by S_8 does not affect. □

Outline of the proof: lifting

Hence, we have the following reformulation:

Proposition

Let $f : X \dashrightarrow Y$ be a birational map, which is not an isomorphism, between a projective variety X and normal $\mathbb{Q}$-factorial variety Y satisfying $b_2(X) = b_2(Y)$. Then, f is not a morphism.

Outline of the proof: modular forms

Our strategy is to show that the top-self intersection numbers of the canonical bundles are not equal.

To compute them, the main tool is a specific modular form constructed by the Borherds product.

Theorem (Kondō, 2007)

There exists a modular form of weight 14 on $\mathbb{B}^5$ of level Γ_{ord} vanishing exactly on H.

This implies that

$$14\mathscr{L} = \frac{1}{2}\overline{H/\Gamma_{\mathrm{ord}}}.$$

Here, $\mathscr{L}$ is the automorphic line bundle of weight 1 on $\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}$.

Outline of the proof: not K-equivalence

By the result of Mumford, we have

$$K_{\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}}} = 6\mathscr{L} - \frac{1}{2}\overline{H/\Gamma_{\mathrm{ord}}} - T.$$

Combining these computation, it follows

$$K_{\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}} = \pi^\star K_{\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}} + 7T$$

$$K_{\mathcal{M}^{\mathrm{K}}} = \psi^\star K_{\mathcal{M}^{\mathrm{GIT}}} + 5\mathcal{E}$$

Here, T (resp. $\mathcal{E}$) is the exceptional divisor of $\pi : \overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}} \to \overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}$ (resp. $\psi : \mathcal{M}^{\mathrm{K}} \to \mathcal{M}^{\mathrm{GIT}}$).
If these varieties are K-equivalent, it implies $(7T)^5 = (5\mathcal{E})^5$ which contradicts to

$$\mathcal{E}^5 \in \frac{1}{e}\mathbb{Z}$$

with $5 \nmid e$. This concludes the proof.

Some remarks of arithmetic and geometry

Remark

1. Freitag-Salvati-Manni and Matsumoto-Terasoma studied these Baily-Borel compactifications and computed the structure of the graded rings of modular forms.
2. More strongly, we can prove that the Kirwan blow-up $\mathcal{M}^{\mathrm{K}}$ is not a semi-toric compactification in the sense of Looijenga in terms of MMP.

Outline of the proof: cohomology

1. The cohomology of the Baily-Borel ($=$ GIT) was computed by Kirwan.
2. Combining the method of Kirwan and the representation theory of $\mathrm{SL}_2(\mathbb{C})$, $H^*(\mathcal{M}^{\mathrm{K}}_{\mathrm{ord}})$ and $H^*(\mathcal{M}^{\mathrm{K}})$ are determined from the knowledge of GIT.
3. The description of the toroidal boundaries, cohomology of the Baily-Borel and decomposition theorem due to Casalaina-Martin-Grushevsky-Hulek-Laza give $H^*(\overline{\mathbb{B}^5/\Gamma_{\mathrm{ord}}}^{\mathrm{tor}})$ and $H^*(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}})$.

Minimal model program

Theorem (Baily-Borel, 1966, Mumford, 1977)

1. $(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{BB}}, \frac{3}{4}H)$ is the log canonical model and a log minimal model.
2. $(\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}, \frac{3}{4}\widehat{H} + T)$ a log minimal model.

We can interpret our results in the context of MMP.

Theorem (Hulek-M, 2022)

1. $(\mathcal{M}^{\mathrm{K}}, \frac{3}{4}\widetilde{H} + \mathscr{E})$ is *not* a minimal model.
2. $\mathcal{M}^{\mathrm{K}}$ is not even a semi-toric compactification.

This gives another proof of some of our results.

Other cases in the Deligne-Mostow list

5, 7, 9, 10, 11 points

The Deligne-Mostow theory tells us isomorphisms between GIT quotients and the Baily-Borel compactifications for N points with $5 \leqslant N \leqslant 12$.

Theorem (Thurston, 1998, Kondō, 2007)

In the case of 5 points, both of ordered and unordered cases, the ball quotient is compact. In particular, all compactifications are isomorphic to each other.

The case of 7, 9, 10, and 11 points is outside the scope of this study, due to the *weight* of the Deligne-Mostow theory,

6, 12 points

Theorem (Hulek-M, 2022)

In the case of 6 points, all theorems in this talk hold.

For the case of 12 points, we expect but need more discussion.

Summary

Conclusion

Theorem (Hulek-M, 2022)

1. Neither Φ nor its inverse Φ^{-1} is a morphism.
2. $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ and $\mathcal{M}^{\mathrm{K}}$ are not K-equivalent, in particular not abstractly isomorphic.
3. $\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ has the normal crossing boundary, whereas $\mathcal{M}^{\mathrm{K}}$ does not.
4. ($\overline{\mathbb{B}^5/\Gamma}^{\mathrm{tor}}$ is lc (more precisely quasi-dlt), whereas $\mathcal{M}^{\mathrm{K}}$ is not.)