Arithmetic of Fourier coefficients of Gan-Gurevich lifts on G_2

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(joint work with Petar Bakic, Alex Horawa, and Siyan Daniel Li-Huerta)

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International Seminar on Automorphic Forms

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But not all groups admit holomorphic discrete series!

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- *: actually, only if φ has a nice level (like $\Gamma_0(N)$)

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Some examples:

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$$c_A(E_{2k}) = \zeta_A(1-2k)$$

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- For $k \geq 6$, there is a basis of level one forms with all coefficients in $\mathbb{Q}^{\mathrm{cyc}}$ (Pollack 22)

Gross's Conjecture

Let f be a cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k.

Assuming $L(1/2, f) \neq 0$, there exists a cuspidal **Gan-Gurevich lift** φ of f to G_2

Conjecture (Gross)

For all maximal totally real cubic rings A,

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- Kim-Yamauchi 24: true when $A = \mathbb{Z} \times \mathcal{O}_F$ for $F/\mathbb{Q}$ quadratic
- Today: Gross's conjecture when f is a CM form (always has level!)

Main result

- $f = f_\chi$ CM form of weight k, trivial character, and any level N, associated to $K/\mathbb{Q}$ and $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^1$.
- Assume: $L(1/2, \chi) \neq 0$.
- $\mathcal{A}_{GG}(f_\chi)$ space of "Gan-Gurevich lifts" $G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{C}$.

Theorem (Bakic–Horawa–Li-Huerta–S., in progress)

For all $\ell | N$, fix a cubic ring $A_\ell/\mathbb{Z}_\ell$, such that

$$\prod_{\ell | N} \epsilon_\ell(A_\ell, \chi_\ell) = -\epsilon(1/2, \chi^3)$$

Then $\exists$ a QMF $\varphi \in \mathcal{A}_{GG}(f_\chi)$ s.t. for A maximal outside N

$$|c_A(\varphi)|^2 = \begin{cases} L(1/2, f_\chi \otimes \rho_A)\,\mathrm{disc}(A)^{\frac{k-1}{2}} & A \otimes \mathbb{Z}_\ell = A_\ell \ \forall \ell | N \\ 0 & \textit{otherwise} \end{cases}$$

Plan of talk

1. Theory of Gan-Gurevich lifts, and role of epsilon factors
2. A construction of $\mathcal{A}_{GG}(f_\chi)$
3. Sketch of proof

1. Theory of Gan-Gurevich lifts

Arthur parameters

Langlands philosophy, G/F reductive group:

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where

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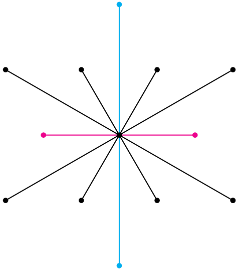
The more nontrivial the $\mathrm{SL}_2(\mathbb{C})$, the more nontempered (and degenerate) the representation. e.g. for $G = \mathrm{GL}_2$, ${}^L G = \mathrm{GL}_2(\mathbb{C})$: $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ nontrivial corresponds to the characters

Theory of Gan-Gurevich lifts: Arthur parameters

$$G = G_2,\ {}^L G = G_2(\mathbb{C})$$

$$\psi : L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_{2,\mathrm{short}}(\mathbb{C}) \times \mathrm{SL}_{2,\mathrm{long}}(\mathbb{C}) \to G_2(\mathbb{C})$$



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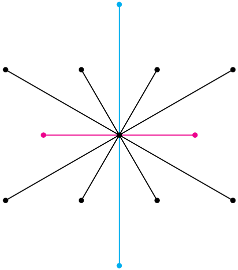

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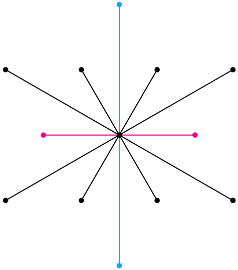

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$L_{\mathbb{Q}} \to \mathrm{SL}_{2,\mathrm{short}}(\mathbb{C})$ corresponds to τ.
Specialize to $\tau = \tau_\chi$ CM, $\mathcal{A}_{GG}(f_\chi) \coloneqq \mathcal{A}_{\psi_{\tau_\chi}}(G_2)$

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Arthur's prediction for structure of global GG packet (partially known, Alonso-He-Ray-Roset 23 and BHL-HS24):

$$\mathcal{A}_{GG}(f_\chi) = \bigoplus_{(\epsilon_v)_v} m((\epsilon_v)_v) \bigotimes_v{}' \pi_v^{\epsilon_v}$$

where $\{\pi_v^+, \pi_v^-\}$ is a local packet depending only on χ_v, with $\pi_v^- = 0$ almost everywhere, and

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$\implies$ to see $c_A(\varphi)$, need $\prod \epsilon_\ell(A \otimes \mathbb{Z}_\ell, \chi_\ell) = -\epsilon(1/2, \chi^3)$

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2. construction of $\mathcal{A}_{GG}(f_\chi)$

Exceptional theta correspondences

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$$\Theta : \{\text{ARs of G}\} \rightsquigarrow \{\text{ARs of H}\}$$

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$G_2 \times \mathrm{PGSp}_6$	E_7
$G_2 \times F_4$	E_8

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Exceptional theta correspondences and Gan-Gurevich lifts

Gan-Gurevich:

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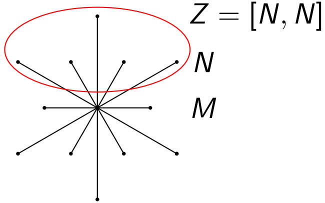
Easier to understand packet structure (comes from Howe–Piatetskii-Shapiro CAP forms on PU_3)

3. sketch of proof

What are Fourier coefficients?



$$\{\text{characters of } N(\mathbb{Q})\backslash N(\mathbb{A})\} \longleftrightarrow \{\lambda = (a, b, c, d) \in \mathbb{Q}^4\}$$
$$\psi_\lambda \longleftrightarrow \lambda$$

$$\varphi : G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{C}$$
$$c_\lambda(\varphi) :\doteq \int_{[N]} \psi_\lambda^{-1}(n)\varphi(n)dn$$

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- Turns out $c_\lambda(\varphi) = 0$ unless E_λ is totally real

Proof sketch

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For $\varphi \in \mathcal{A}_{GG}(f_\chi)$ (any level),

$$c_\lambda(\varphi) \sim \int_{[T_E]} \rho(t)dt$$

where:

- $E = E_\lambda$ is totally real cubic étale algebra corresponding to λ.
- $T_E \hookrightarrow \mathrm{PU}_3$ is a torus embedding coming from $E \hookrightarrow \mathrm{Herm}_{3\times 3}(\mathbb{Q})$.
- $\rho =$ thing you're lifting on PU_3 (Howe-PS CAP forms).

Proof sketch

Periods for $\Theta(\pi)$ often reduce to periods for π.

For $\varphi \in \mathcal{A}_{GG}(f_\chi)$ (any level),

$$c_\lambda(\varphi) \sim \int_{[T_E]} \rho(t)dt$$

ρ is itself a Θ lift from $U(1)$

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$$\left| \int_{[T_E]} \rho(t)dt \right|^2 \sim L(1/2, f_\chi \otimes \rho_E) L(1/2, \chi) \Delta_E^{1/2}$$

cf. Yang 97, Borade-Franzel-Girsch-Yao-Yu-Zelingher 24

Proof sketch

$$|c_\lambda(\varphi)|^2 = L(1/2, f_\chi \otimes \rho_E) L(1/2, \chi) \Delta_E^{1/2} \prod_v |I_v(\varphi_v, \lambda)|^2$$

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  - For $\ell | N$, we show you can "rig" φ_ℓ so $I_\ell(\varphi_\ell, \lambda)$ is the indicator function of $A_\lambda \otimes \mathbb{Z}_\ell =$ any fixed A_ℓ

Thanks!