# Growth of Bianchi modular forms

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# Backgrounds

Let $S_k(\Gamma_1(N))$ be the space of cuspidal modular forms of weight $k$ and principal level $N \geq 5$. If $k \geq 2$, one has a dimension formula for $\dim_{\mathbb{C}} S_k(\Gamma_1(N))$ via the Riemann-Roch theorem. When $k \geq 3$ and $N$ fixed, $\dim_{\mathbb{C}} S_k(\Gamma_1(N))$ is a linear function in terms of $k$.
Next consider the case $F$ is quadratic over the rational number field. In this case the notion of weight is indexed by two positive even integers $k_1, k_2$. If $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$ is a congruence subgroup, we use $S_{k_1,k_2}(\Gamma)$ to denote the space of cuspidal modular forms of weight $k_1, k_2$ and level $\Gamma$. If F is real quadratic, Shimizu (1963) has proven that

$$\dim_{\mathbb{C}} S_{k_1,k_2}(\Gamma) \sim C \cdot k_1 k_2$$

for a constant C depending only on $\Gamma$. But if F is imaginary quadratic, things become much more complicated. Unlike the real situation, $S_{k,k}(\Gamma)$ does not grow quadratically!

# Cohomological reinterpretation

Let $K_f \subset \mathrm{SL}_2(\mathbb{A}_F^\infty)$ be compact open level subgroup, and $\Gamma = K_f \cap \mathrm{SL}_2(F)$. We consider the following locally symmetric space as a double quotient

$$Y(K_f) := \mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A}_F^) \times \mathrm{SL}_2(\mathbb{C}) / K_f \times \mathrm{SU}_2(\mathbb{C}).$$

It is an arithmetic hyperbolic 3-manifold with universal cover $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2(\mathbb{C}) \simeq H^3$. Consider the representation $W_k := \mathrm{Sym}^{k/2-1} \otimes \overline{\mathrm{Sym}}^{k/2-1}$ of $\mathrm{SL}_2(\mathbb{C})$, it descends to a local system on $Y(K_f)$. As a consequence of the Eichler-Shimura isomorphism, we have

$$\dim_{\mathbb{C}} H_c^1(Y(K_f), W_k) = \dim_{\mathbb{C}} S_{k,k}(\Gamma).$$

# Conjecture for Bianchi modular forms

**Conjecture**

If $F$ is imaginary quadratic, there exists a constant $c$ depending only on $K_f$ such that

$$\dim_{\mathbb{C}} S_k(K_f) \leq c \cdot k.$$

This conjecture is supported by experimental data of Finis-Grunewald-Tirao and the work of Calegari-Mazur (for Hida families). And such an upper bound of linear growth rate is sharp from base change of classical elliptic modular forms. A trivial upper bound is $\dim_{\mathbb{C}} S_{\mathbf{k}}(K_f) \leq O(k^2)$.

# Previous progresses

Finis-Grunewald-Tirao (2010) established a $\log k$ saving bound

$$\dim_{\mathbb{C}} S_k(K_f) \ll_{K_f} \frac{k^2}{\log k},$$

using trace formula.
Simon Marshall (2012) proved a power saving bound

$$\dim_{\mathbb{C}} S_k(K_f) \ll_{\epsilon,K_f} k^{5/3+\epsilon},$$

using Emerton's completed homology.
Yongquan Hu (2021) proved a better power saving bound

$$\dim_{\mathbb{C}} S_k(K_f) \ll_{\epsilon,K_f} k^{3/2+\epsilon},$$

making full uses of the $\mathrm{SL}_2(\mathbb{Q}_p)$-action.

# Main global result

From now on we assume $F$ is imaginary quadratic.

**Theorem (Fu)**

*The conjecture holds true and we have*

$$\dim_{\mathbb{C}} S_k(K_f) \ll_{K_f} k.$$

# Completed homology

We pick a prime such that $p$ splits in $F$. Let $G$ be a compact open subgroup of $\mathrm{SL}_2(F_p)$, $K^p \subset \mathrm{SL}_2(\mathbb{A}_F^{p,\infty})$ be a tame level.
Let's first recall Emerton's completed homology (2006) and list two important properties for us (Calegari-Emerton, 2009):

$$\widetilde{H}_\bullet(K^p) := \varprojlim_s \varprojlim_{K_p \subset G} H_\bullet(Y(K_pK^p), \mathbb{Z}/p^s\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

- Because $\mathrm{SL}_2(\mathbb{C})$ does not admit discrete series, $\widetilde{H}_*(K^p)$ is a finitely generated torsion $\mathbb{Q}_p[[G]]$-module.
- There is a spectral sequence

$$E_2^{i,j} = H_i(G, \widetilde{H}_j(K^p) \otimes W_{\mathbf{k}}) \Longrightarrow H_{i+j}(Y(GK^p), W_{\mathbf{k}}), \tag{1}$$

  for $K_f = GK^p$.

This spectral sequence provides us an upper bound

$$\dim_{\mathbb{Q}_p} H_q(Y(K_f), W_k) \leq \sum_{i+j=q} \dim_{\mathbb{Q}_p} H_i(G, \widetilde{H}_j(K^p) \otimes W_k).$$

We may assume $G$ is the first principal subgroup (example) so that $G$ is *uniform pro-$p$*. It suffices for us to prove

**Theorem (Fu)**

$$\dim_{\mathbb{Q}_p} H_*(G, \widetilde{M} \otimes W_k) \ll_{\widetilde{M}} k$$

*for any finitely generated torsion $\mathbb{Q}_p[[G]]$-module $\widetilde{M}$.*

We can associate to (such a uniform pro-p group) $G$ a $\mathbb{Z}_p$-Lie algebra $\mathfrak{g}_0$ by the fundamental work of Lazard (1965) such that

# Microlocalisation

$\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{sl}_{2,\mathbb{Q}_p} \bigoplus \mathfrak{sl}_{2,\mathbb{Q}_p}$, along with the *completed universal enveloping algebra*

$$\widehat{U(\mathfrak{g}_0)} := \varprojlim_a \left( \frac{U(\mathfrak{g}_0)}{p^a U(\mathfrak{g}_0)} \right), \ \widehat{U(\mathfrak{g})} := \widehat{U(\mathfrak{g}_0)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

We pass our problem to Lie algebra level by a *microlocalisation* of Ardakov-Wadsley (2013) $\mathbb{Q}_p[[G]] \to \widehat{U(\mathfrak{g})}$ which is a flat ring extension, and $\widehat{M} := \widehat{U(\mathfrak{g})} \otimes_{\mathbb{Q}_p[[G]]} \widetilde{M}$.

# Comparison of algebraic quotients

Let $\widetilde{M}_k, \widehat{M}_k$ be respectively the maximal $W_k$-quotients of $\widetilde{M}, \widehat{M}$. For example, $(\mathbb{Q}_p[[G]])_k \simeq \mathrm{End}_{\mathbb{Q}p}(W_k) \simeq (\widehat{U(\mathfrak{g})})_k$ as a Peter-Weyl theorem in our setting. We prove a comparison of algebraic quotients using results from Ardakov-Wadsley.

## Theorem (Fu)

*If $\widetilde{M}$ is a finitely generated Iwasawa module over $\mathbb{Q}_p[[G]]$, then*

$$\widetilde{M}_k \xrightarrow{\sim} \widehat{M}_k.$$

# Genericity of Iwasawa modules

As $\mathfrak{g} \simeq \mathfrak{sl}_{2,\mathbb{Q}_p} \bigoplus \mathfrak{sl}_{2,\mathbb{Q}_p}$, $\widehat{U(\mathfrak{g})}$ contains central Casimir elements $\Delta_1, \Delta_2$ ($\Delta = \frac{1}{2}h^2 - h + 2ef$, e $= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ). For $(\lambda_1, \lambda_2) \in \mathbb{Z}_p^2$, we consider the quotient algebra $\widehat{U(\mathfrak{g})}_\lambda := \widehat{U(\mathfrak{g})}/(\Delta_1 - \lambda_1, \Delta_2 - \lambda_2)$.

**Theorem (Ardakov-Wadsley, Fu)**

*For any pair $(\lambda_1, \lambda_2) \in \mathbb{Z}_p^2$, the following composition of maps is injective*

$$\mathbb{Q}_p[[G]] \to \widehat{U(\mathfrak{g})} \to \widehat{U(\mathfrak{g})}_\lambda.$$

For any $\delta \in \mathbb{Q}_p[[G]]$, we may assume $\delta \in \widehat{U(\mathfrak{g}_0)}$ up to multiplication of a p-power. Since $\mathbb{Z}_p^2$ is compact, we can find a positive integer $n_\delta$ such that $\widehat{U(\mathfrak{g}_0)}_\lambda/p^{n_\delta}$ for all $\lambda \in \mathbb{Z}_p^2$.

# Sketch of proof

By a homological degree-shifting argument, we reduce to prove

$$\dim_{\mathbb{Q}_p} H_0(G, \widetilde{M} \otimes W_k) = \dim_{\mathbb{Q}_p} \mathrm{Hom}_{\mathbb{Q}_p[[G]]}(\widetilde{M}, W_k) \ll_{\widetilde{M}} k$$

for a cyclic torsion module $\widetilde{M} \simeq \mathbb{Q}_p[[G]]/\mathbb{Q}_p[[G]] \cdot \delta$.
By our comparison, it suffices to prove

$$\dim_{\mathbb{Q}_p}(\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})} \cdot \delta)_k \ll_\delta k^3.$$

It can be proven that taking maximal $W_k$-quotient is right exact, we apply it to the short exact sequence of left $\widehat{U(\mathfrak{g})}$-modules

$$0 \to \widehat{U(\mathfrak{g})} \to \widehat{U(\mathfrak{g})} \to \widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})} \cdot \delta \to 0.$$

We prove this theorem by relating this dimension to the dimension of a Poincaré–Birkhoff–Witt filtration on $\widehat{U(\mathfrak{g})}_\lambda$, induced from the filtration on $\widehat{U(\mathfrak{g})}$.

# Summary of ingredients

There exists a positive integer $\alpha$ such that

$$\delta \in \mathrm{Fil}_\alpha(\widehat{U(\mathfrak{g}_0)}/p^{n_\delta}).$$

Eventually we prove the image of $(\widehat{U(\mathfrak{g}_0)})_k \to (\widehat{U(\mathfrak{g}_0)})_k$ is of dimension at least $(\frac{k}{2}-\alpha)^4$, therefore

$$\dim_{\mathbb{Q}_p}(\widehat{U(\mathfrak{g})}/\widehat{U(\mathfrak{g})}\cdot\delta)_k \leq (\frac{k}{2})^4 - (\frac{k}{2}-\alpha)^4.$$

In summary, we crucially use 1. the PBW-filtration on $\widehat{U(\mathfrak{g})}$ 2. genericity of $\delta$ as an element in image of the Iwasawa algebra (in some sense, this means Iwasawa modules are generic) 3. We also need to utilize the integral structure of $\widehat{U(\mathfrak{g})}$ and its image in the endomorphism ring of $W_k$.