

# The fiber bundle method applied to triple product L-functions

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§1 Poisson Summation Conjecture

§2 Integral representation of triple product L-functions

§3 The fiber bundle method

## §1 Poisson Summation Conjecture

$F$ : number field.  $\mathbb{A}_F$ : Adele ring

Poisson Summation formula for vector spaces

$$\cdot V(\mathbb{R}) = \mathbb{R}^n, \quad V(\mathbb{Z}) = \mathbb{Z}^n$$

$$R: V(\mathbb{R}) \times GL_n(\mathbb{R}) \rightarrow V(\mathbb{R})$$

$$(v, g) \mapsto g^{-t}v$$

·  $S(V(\mathbb{R}))$ : space of rapidly decreasing functions

· Fourier transform

$$F(f)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} f(y) dy$$

$$F_v \circ R(g) = (\det g) \cdot R(g^{-t}) \cdot F_v$$

· Poisson summation formula

$$\sum_{x \in V(\mathbb{Z})} f(x) = \sum_{x \in V(\mathbb{Z})} F(f)(x)$$

Integral representation for standard L-functions for  $GL_n$

Godement-Jacquet.

· Affine  $GL_n \times GL_n$  equivariant embedding  $GL_n \hookrightarrow M_n$ .  
(generalizes  $G_m$ -equivariant embedding  $G_m \hookrightarrow G_a$  in Tate's thesis)

· Schwartz space

$$S(M_n(A_F)) = S(M_n(F_\infty)) \otimes C_c^\infty(M_n(A_F^\infty))$$

• Fourier transform

$$F : S(M_n(A_F)) \rightarrow S(M_n(A_F))$$

• P.S.F.

$$\sum_{\gamma \in M_n(F)} f(\gamma) = \sum_{\gamma \in M_n(F)} F(F)(\gamma)$$

• Zeta integral : integral representation for  $L(S, \pi)$  of  $GL_n$   
( $\pi$  : cusp. art. rep. of  $GL_n$ ).

$$Z(S, f, \psi) = \int_{GL_n(A_F)} f(g) \psi(g) |\det g|^{s + \frac{n-1}{2}} dg.$$

$f \in S(M_n(A_F))$ .  $\psi \in C(\pi)$  (matrix coefficient of  $\pi$ )

• Apply P.S.F.  $\Rightarrow$  global meromorphic continuation & functional equation of  $L(S, \pi)$

Poisson Summation Conjecture for spherical varieties.

- Braverman-Kazhdan, Ngô, L. Lafforgue, Sakellaridis.

$\forall$  affine spherical  $G$ -variety  $X$  ( $G$ -varieties w/ open dense Borel orbit).

$\Rightarrow$  Schwartz space

$$S(X(A_F)) = \bigotimes_v S_v(X(F_v))$$

w.r.t. basic functions  $\phi_x \in S_v(X(F_v))$  at each unramified places  $v$ .

Fourier transform

$$F_X : S(X(A_F)) \rightarrow S(X(A_F))$$

P.S.F.

$$\sum_{x \in X^{sm}(F)} f(x) + * = \sum_{x \in X^{sm}(F)} F_X(f)(x) + **$$

Known cases

• Braverman-Kazhdan spaces (Braverman-Kazhdan, Getz-Lin, Getz-Hu-Leslie)

$$X^\circ = \frac{G}{[P, P]} \quad X = \overline{X^\circ}^{\text{aff}}$$

$G$  : simply connected, split, simple.  $P$  : maximal parabolic.

- Triples of quadratic spaces (Getz-Lin, Getz-Ibn. Getz-Hsu-Leslie).
- Automorphic-twisted summation formula for pairs of quadratic spaces (G)
  - space of test functions on zero locus of quadratic form
  - functions are built from Whittaker coeff. of int rep of  $GL_n$ .
- Schubert varieties (Choi-Getz.)
- See also special cases by Jiang, Luo, Ngo, Shahidi, Sakurai, L. Zhang ...

Poisson summation conjecture

$\Downarrow$

F.E. & merom. cont. for fairly general Langlands L-functions.

$\Downarrow$  Converse Theorem (+ local Langlands)

Langlands functoriality in great generality,

§ 2. Integral representation for triple L-functions.

Triple product L-functions

•  $r_1, r_2, r_3 \in \mathbb{Z}_+$ ,  $\Pi = \bigotimes_{i=1}^3 \Pi_i$ ,  $\Pi_i$ : unsp. ant. rep of  $GL_{r_i}(A_F)$ .

•  $L(S, \Pi, \otimes^3) = L(S, \Pi_1 \times \Pi_2 \times \Pi_3)$

Langlands L-function defined by

$$\otimes^3 : {}^L(GL_{r_1} \times GL_{r_2} \times GL_{r_3}) \rightarrow GL_{r_1 r_2 r_3}(\mathbb{C})$$

• The case  $r_1 = r_2 = r_3 = 3$ : smallest case where their global analytic properties are unknown.

• Analytic properties + Converse theorem  $\Rightarrow$  tensor product functoriality

Ingredient of the integral representation

Groups

$R = F$ -algebra.

$H(R) := \{ (h_1, h_2, h_3) \in GL_2^{\mathbb{Z}}(R) : \det h_1 = \det h_2 = \det h_3 \}$ .

$\Sigma(R) := \{ (\varepsilon_1, \varepsilon_2, \varepsilon_3) : \varepsilon_i = \pm 1, \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1 \}$ .

$\mathbb{Z} := \mathbb{Z}_{GL_2} \times \mathbb{Z}_{GL_2} \times \mathbb{Z}_{GL_2}$

$$H^e := \mathbb{Z} \wedge^\varepsilon H = (\mathbb{Z} \times H) / \varepsilon.$$

$\mathbb{Z}$ -extension of  $H/\varepsilon$  (in the sense of Kottwitz).

$$p_1: H^e \rightarrow GL_2 \times GL_2 \times GL_2, p_2: H^e \rightarrow H/\varepsilon.$$

Assume  $r_i \geq 3$  for all  $i$ ;

$$M_i := \{(u_1, u_2) \in M_{r_i-2, 2}(K) : u_1 \wedge u_2 = 0\}$$

$$M_i^\circ = M_i - \{0\}. \quad M = \bigcap_{i=1}^3 M_i \quad M^\circ = \bigcap_{i=1}^3 M_i^\circ.$$

The space  $\mathcal{Y}$

$$- V_3 = \mathbb{G}_a^2 \otimes \mathbb{G}_a^2 \otimes \mathbb{G}_a^2$$

$$- H \curvearrowright V_3$$

$$(u_1 \otimes u_2 \otimes u_3) \cdot (h_1, h_2, h_3) = (\det h_1)^{-1} u_1 h_1 \otimes u_2 h_2 \otimes u_3 h_3.$$

This action factors through  $\varepsilon$  and pulls back along  $p_2$  to an action of  $H^e$ .

$$- GL_{r-2} \times H^e \curvearrowright M \times V_3 \quad (\text{preserving } M^\circ \times V_3).$$

$$R: ((m, v), g, h) \mapsto (g^t m p_1(h)^{-t}, v, p_2(h))$$

$$- \mathcal{Y} \subset M \times V_3 \quad (\text{preserved under } R) \quad \text{s.t.}$$

$\mathcal{Y}_{M^\circ}$  is a vector bundle of rank 4 over  $M^\circ$ .

Whittaker induction

$$- X_{r_i} = \frac{GL_{r_i}}{N_{r_i-2, 2}} \quad N_{r_i-2, 2}: \text{unipotent radical of parabolic } P_{r_i-2, 2}$$

-  $X_{r_i}$  is a spherical variety w.r.t. an action of

$$GL_{r_i-2} \times GL_2 \times GL_{r_i}$$

- We defined a  $\mathbb{G}_m, m_i^\circ$ -torsor

$$\mathcal{V}_{r_i}^\circ \rightarrow X_{r_i} \times M_i^\circ$$

equipped w/ an action  $R$  of  $GL_{r_i-2} \times GL_2 \times GL_{r_i}$

-  $\mathcal{V}_{r_i}^\circ$  is a Whittaker induction in the sense of Satake & Vogan.

- We also define certain stack  $\mathcal{V}_{r_i}$  over  $M_i^\circ$  that can be viewed as partial affine closure of  $\mathcal{V}_{r_i}^\circ$ .

$$\mathcal{V}_{r_i}^\circ = \mathcal{V}_{r_i, m_i^\circ}.$$

## Fiber product

$$\begin{array}{ccc} \mathcal{V} \times_M \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{V} & \longrightarrow & \mathcal{X} \\ \mathcal{X} := \bigsqcup_{i=1}^r \mathcal{X}_i & & \end{array}$$

$\mathcal{V} \times_M \mathcal{Y}$ ,  $\mathcal{X} \times_M \mathcal{Y}$  admit actions  $R$  of  $GL_{r,2} \times H^e \times GL_r$

We have equivariant map  $\mathcal{V} \times_M \mathcal{Y} \rightarrow \mathcal{X} \times_M \mathcal{Y}$

- $(\mathcal{X}_i \times_M \mathcal{Y}) / (H^e)_{\text{der}}$  is  $GL_r \times GL_r \times H^e / (H^e)_{\text{der}}$  spheres!
- $\mathcal{V} \times_M \mathcal{Y}$ : generalized Whittaker induction  
(plays similar role as Whittaker inductions)

## The space of functions

$C^\infty((\mathcal{V} \times_M \mathcal{Y})(\mathbb{A}_F), \psi)$ : space of smooth functions

$$f: GL_r(\mathbb{A}_F) \times \mathcal{Y}^e(\mathbb{A}_F) \rightarrow \mathbb{C}.$$

$$\text{s.t. } f\left(\begin{pmatrix} I_{r,2} & z \\ & I_r \end{pmatrix} g, m, v\right) = \psi(\langle m, z \rangle) f(g, m, v)$$

We have an action

$$\begin{aligned} R: C^\infty((\mathcal{V} \times_M \mathcal{Y})(\mathbb{A}_F), \psi) \times GL_{r,2} \times H^e \times GL_r(\mathbb{A}_F) \\ \longrightarrow C^\infty((\mathcal{V} \times_M \mathcal{Y})(\mathbb{A}_F), \psi) \end{aligned}$$

$$R(g, g', h) f(g_0, m, v) = f\left(\begin{pmatrix} g' & \\ & p(h) \end{pmatrix}^{-1} g_0 g, g'^t m p(h)^t, v \cdot p(h)\right)$$

$$\Theta_f(g, g', h) := \sum_{(x, m, v) \in \mathcal{X}(F) \times \mathcal{Y}^e(F)} R(g, g', h) f(x, m, v)$$

## The integral representation

$$\begin{aligned} Z(f, \psi, \psi', \Sigma, \Sigma', s_0) := \int_{[GL_{r,2}] \times [H^e] \times [GL_r]} \Theta_f(g, g', h) \psi(g) \psi'(g') \\ \times \eta_{\Sigma, \Sigma', s_0}(g, g', h) dg dg' dh \end{aligned}$$

$\varphi, \varphi' : \text{cusp forms on } GL_{r-2}, GL_r$   
 $\eta_{\underline{s}, \underline{s}', s_0} : \text{character defined using } |\det g|, |\det g'|, |\det h|.$

Thm 1 (Getz - G. - Hsu - Leslie)

For  $f$  satisfies certain local conditions,  $(\text{Re}(\underline{s}), \text{Re}(\underline{s}'), \text{Re}(s_0)) \in \mathbb{R}_{\geq 1}^2 \times \mathbb{R}_{> \frac{r}{2}}$

$Z(f, \varphi, \varphi', \underline{s}, \underline{s}', s_0)$  converges absolutely and is equal to

$$\int_{\mathbb{H}^r(\mathbb{A}_F)} \int_{GL_{r-2}(\mathbb{A}_F)} \int_{GL_r(\mathbb{A}_F)} W_{\varphi}^{\varphi}(g) W_{\varphi'}^{\varphi'}(g')$$

$$\times f((g', h)^{-1}g, g'^{-1}mop(h)^{-t}, \chi_5.P_2(h)) \eta_{\underline{s}, \underline{s}', s_0}(g, g', h) dg dg' dh$$

Assume  $\pi, \pi', \psi$  unramified. take  $b^{nai} \in C^\infty(\mathcal{V} \times \mathcal{S}, \psi)$  as naive basic function (analogue to the characteristic function of the integral point)

Thm 2 (Getz, - G. - Hsu - Leslie)

For  $(\text{Re}(\underline{s}), \text{Re}(\underline{s}'), \text{Re}(s_0)) \in \mathbb{R}_{\geq 1}^2 \times \mathbb{R}_{> \frac{r}{2}}$ .

$$Z(W_0, W'_0, b^{nai}, \underline{s}, \underline{s}', s_0) = \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\ell=0}^{\infty} q^{-\ell s_0} \prod_{i=1}^3 L(s'_i, \pi_i \times \pi'_i) \sum_{n=0}^{r_i-2} (-1)^{n_i} s_{0, \dots, 0, n-\ell-k_i}(\alpha_i) \frac{\text{Tr}(\Lambda^n \alpha_i)}{\text{Tr}(\Lambda^n \alpha_i')} q^{-n s'_i - k_i s_i}$$

Corollary (Getz - G. - Hsu - Leslie)

Assume  $\pi, \pi'$  tempered. Let  $\varepsilon > 0$ .  $(\text{Re}(\underline{s}), \text{Re}(\underline{s}')) \in \mathbb{R}_{> H\varepsilon}^2$ .

$$\frac{Z(W, W', b^{nai}, \underline{s}, \underline{s}', s_0)}{L(s_0, \pi^\vee, \otimes^3)} = 1 + O_\varepsilon(q^{-1-\frac{\varepsilon}{2}})$$

for  $\text{Re}(s_0) > \frac{1}{2} + \varepsilon$

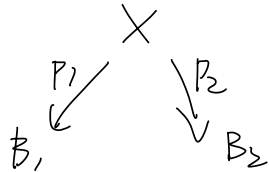
§ 3 Fiber bundle method

$F$ : number field

$G$ : reductive group  $\mathbb{F}$

$X$ : spherical variety for  $G$

Given



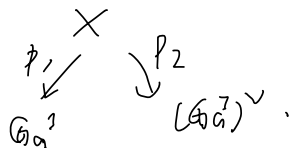
of  $G$ -schemes,

Principle: Prove P.S. conjecture for  $p_i^{-1}(b_i) \Rightarrow$  P.S. conjecture for  $X$

Example: basic affine spaces (Brouwer-Kazhdan.)

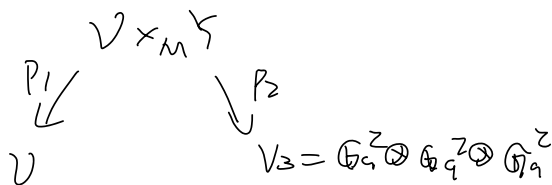
$$G = GL_3$$

$$X = \{(v, v') \in \mathbb{R}^3 \times (\mathbb{R}^3)^\vee : v'(v) = 0\}.$$



- Fibers of  $p_i$  are generically  $G_a^Z$ -torsors  $\checkmark$  skew symmetric pairing.
- Fiber bundle method  $\Rightarrow$  P.S. for  $\overline{GL_3}$  aff.
- works for  $\overline{G}^{\text{aff}}$   $\checkmark$   $G$ : split, simple, simply connected.

The case of homol.



- General fibers of  $p_2$ : Whittaker induction
- General fibers of  $p_1$ : vector spaces of rank 4

$\Leftrightarrow$

- We are proving P.S.C. for fibers of  $p_2$  known P.S. formula

Aim

Prove P.S. conjecture for  $V \times_M Y$

Apply fiber bundle method :

- construct Schwartz space

$$S(\mathbb{R}_n^d)(\mathbb{A}_F^{1,4}) \subset C^\infty(\mathbb{R}_n^d)(\mathbb{A}_F^{1,4})$$

- construct Fourier transforms and prove it has correct equivariance properties

- Prove P.S.F.