BZSV Duality and Relative Langlands Program

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January 22, 2024

(Joint with Bryan P.J. Wang)

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- G reductive group over F; set $G = G(F)$
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An L-parameter is an equivalence class of maps

$$\phi : \Gamma_F = \mathrm{Gal}(\overline{F}/F) = \pi_1^{et}(\mathrm{Spec}(F)) \longrightarrow G^\vee$$

where $G^\vee$ is the Langlands dual group of G and equivalence is up to $G^\vee$-conjugacy. So (roughly)

$$\Phi(G) = \mathrm{Hom}(\pi_1^{et}(\mathrm{Spec}(F)), G^\vee)/G^\vee$$

Global Langlands Program

- F global field and G reductive group over F
- $G(F) \subset G(\mathbb{A}) = \prod'_v G(F_v)$; set $[G] = G(F) \backslash G(\mathbb{A})$;
- $\mathcal{A}_2(G) = \{$square-integrable automorphic forms of $G\} = \{f : [G] \to \mathbb{C}\}$.

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Main Problem: Classify the irreducible constituents of $\mathcal{A}_2(G)$.

Main Conjecture: The irreducible constituents of $\mathcal{A}_2(G)$ can be parametrized by A-parameters:

$$\Psi : \mathrm{Gal}(\overline{F}/F) \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee$$

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$$\mathrm{Irr}_{H,\chi}(G) = \{\pi : \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \chi) \neq 0\},$$

so it is the set of irreducible submodules of $C^\infty(H, \chi \backslash G)$.

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L^2-**version**: describe the spectral decomposition of $L^2(H, \chi \backslash G)$.

Expect: $\mathrm{Irr}_{H,\chi}(G)$ corresponds to L-parameters which factor through some $J^\vee \to G^\vee$. So (H, χ)-dist. reps are functorial lifts from another group J).

Relative Langlands?

Global: Have global period integral

$$\mathcal{P}_{H,\chi} : \mathcal{A}_2(G) \longrightarrow \mathbb{C}$$

defined by

$$\mathcal{P}_{H,\chi}(f) = \int_{[H]} \overline{\chi(h)} \cdot f(h)\, dh.$$

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Classify those $\pi \subset \mathcal{A}_2(G)$ such that

$$\mathcal{P}_{H,\chi,\pi} := \mathcal{P}_{H,\chi}|_{\pi} \neq 0 \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \chi).$$

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Expect:

- $\mathcal{P}_{H,\chi,\pi}$ can be factored as product of local $H(F_v)$-invariant functionals
- $\mathcal{P}_{H,\chi,\pi}$ to be related to an L-function on J.

Classical Examples

Periods	(G, H, χ)
Whittaker	$G \supset U$ (maximal unipotent) $\chi = \psi$ generic character
Symplectic	$\mathrm{GL}_{2n} \supset \mathrm{Sp}_{2n}$
Shalika	$\mathrm{GL}_{2n} \supset P_{n,n} \supset \mathrm{GL}_n^{\Delta} U$ $\chi = 1_{\mathrm{GL}_n} \otimes \psi(\mathrm{Tr}(-))$
Basic Gross-Prasad	$\mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1} \supset \mathrm{SO}_{2n}^{\Delta}$
General GP $n < m$ opp. parity	$\mathrm{SO}_n \times \mathrm{SO}_m \supset \mathrm{SO}_n^{\Delta} U$ χ generic character

Questions and Issues

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Upshot The subject of RLP is based on a number of examples but there is no systematic framework,

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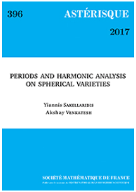
Upshot The subject of RLP is based on a number of examples but there is no systematic framework, that is, until the publication of the Asterisque volume [SV]:



Relative Langlands according to [SV]

Objects	spherical G-variety $X = H\backslash G$
Local Qn	spectral decomposition of $L^2(X)$ or $C^\infty(X)$
Global Qn	global H-periods and related L-values
Dual Data	(i) $\iota_X : X^\vee \times \mathrm{SL}_2 \to G^\vee$ (ii) (graded symplectic) representation V_X of $X^\vee$
Conjecture	(local) H-dist. π are functorial lifts from $X^\vee$ via ι_X (global) H-period of π given by $L(1/2, \sigma_\pi, V_X)$

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An example of this last point is the theta correspondence, i.e. the spectral decomposition of the Weil representation under the action of a dual pair.

BZSV

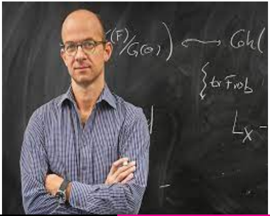
The above issues are (to some extent) resolved in the 400-page preprint [BZSV] of Ben-Zvi, Sakellaridis and Venkatesh:

RELATIVE LANGLANDS DUALITY

DAVID BEN-ZVI, YIANNIS SAKELLARIDIS AND AKSHAY VENKATESH

Abstract. We propose a duality in the relative Langlands program. This duality pairs a Hamiltonian space for a group G with a Hamiltonian space under its dual group $\check{G}$, and recovers at a numerical level the relationship between a period on G and an L-function attached to $\check{G}$; it is an arithmetic analog of the electric-magnetic duality of boundary conditions in four-dimensional supersymmetric Yang–Mills theory.

This is a draft. We anticipate making another round of changes before submitting it for publication. All comments are very welcome! In particular, if we have failed to attribute or properly reference a work it is most likely due to either ignorance or forgetfulness - please tell us.



Relative Langlands according to [BZSV]

Objects	Hyperspherical Hamiltonian G-variety M
Local Qn	spectral decomp. of quantization Π_M of M
Global Qn	spectral decomp. of theta function of Π_M
Dual Data	Hyperspherical Hamiltonian $G^\vee$-variety $M^\vee$
Conjecture	(local) Galois action has fixed point on $M^\vee$ (global) L-function arises fro Galois rep. on tangent spaces of fixed points

Comparing [BZSV] with [SV]

	[SV]	[BZSV]
Objects	spherical X	hyperspherical M
Spectral Qn	$L^2(X)$	Quantization of M
Dual Data	$(X^\vee, \iota_X, V_X)$	hyperspherical $M^\vee$
Conj.	Factor though ι_X	Galois-fixed points on $M^\vee$

Hyperspherical Hamiltonian G-varieties M

(a) symplectic G-variety: M is a a symplectic variety with G acting as symplectomorphisms. Then $\mathcal{O}(M)$ is a G-equivairant Poisson algebra.

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If X is a spherical G-variety (affine smooth), then $M = T^*X$ is hyperspherical.

Structure Theory of Hyperspherical Varieties

Suppose one is given:

- $\iota : H \times \mathrm{SL}_2 \longrightarrow G$, with $H \subset Z_G(\iota(\mathrm{SL}_2))$ a spherical subgroup;
- S a (graded) symplectic (finite dim) representation of H.

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Let $\{h, e, f\}$ be $\mathfrak{sl}_2$-triple associated to $\iota|_{\mathrm{SL}_2}$. Then define a G-variety by:

$$M = (S \times_{\mathfrak{h}^*} (f + \mathfrak{g}^e)) \times^H G.$$

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Observe that the above data is of the same type as the dual data from [SV]:

$$\iota_X : X^\vee \times \mathrm{SL}_2 \longrightarrow G^\vee$$

and V_X a (graded symplectic) rep. of $X^\vee$.

3 Basic Examples

M	Data for construction
$T^*(H\backslash G)$	$\iota : H \to G,\ S = 0$
$T^*_e(U\backslash G)$	$\iota : \mathrm{SL}_2 \to G$ (regular SL_2), $S = 0$
$(W,\ G \subset \mathrm{Sp}(W))$	$\iota : G \to G,\ S = W$

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A mixed example: $S = 0$ and

$$\iota : \mathrm{GL}_n \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}_{2n} \quad \text{(tensor product)}.$$

Then

$$M = \{\begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix} : B \in M_n\} \times^{\mathrm{GL}_n^{\Delta}} \mathrm{GL}_{2n} = T_e(\mathrm{GL}_n^{\Delta} U \backslash \mathrm{GL}_{2n})$$

Call this the Shalika variety since it gives rise to the Shalika period.

Quantization

Quantization refers to the following philosophy:

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One does not have this, but many standard constructions in symplectic geometry can be quantized, i.e. have natural representation theoretic counterparts, as realized by Kirillov, Guillemin-Sternberg, Kazhdan etc.

Classical vs. Quantization

Classical	Quantum
M	(ρ_M, V_M)
$C(M,\mathbb{R}) \subset C(M,\mathbb{C})$	$\mathrm{Herm}(V_M) \subset \mathrm{End}(V_M)$
$\mu : M \to \mathfrak{g}^*$	?
$\mu^* : C(\mathfrak{g}^*) \to C(M,\mathbb{C})$ (pullback)	$\rho_M : C^*(G) \to \mathrm{End}(V_M)$ (C^*-alg. module)
Coisotropic	Multiplicity-free
$\left(M \times^G_{\mathfrak{g}^*} \{0\}\right)$ (Symplectic reduction)	$(V_M)_G$ (G-coinvariants)

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The last line is often expressed as:

Quantization commutes with Reduction.

Examples of Quantization

M	Π_M
$T^*(X)$, X affine smooth spherical	$L^2(X)$
$T_e^*(U\backslash G) = (e+\mathfrak{g}^f)\times G$ (e regular nilpotent)	$L^2(U,\psi\backslash G)$ (Whittaker/Gelfand-Graev module)
$T_e^*(\mathrm{GL}_n^{\Delta} U\backslash \mathrm{GL}_{2n})$ (Shalika variety)	$L^2(\mathrm{GL}_n U,\psi\backslash \mathrm{GL}_{2n})$ (Shalika module)
symplectic vector space $W = X + X^*$	Weil representation $L^2(X)$
$(V\otimes W, \mathrm{O}(V)\times \mathrm{Sp}(W))$	Theta correspondence for $\mathrm{O}(V)\times\mathrm{Sp}(W)$

Summary

	[SV]	[BZSV]
Objects	spherical X	hyperspherical M
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What is gained from [SV] to [BZSV]:

- Scope of RLP expanded (e.g. to include theta correspondence)
- there is now a clear symmetry between the basic object M and the dual data $M^\vee$

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- Period invariant: M gives rise to a theta function $\Theta_M \in \mathcal{A}(G))$. Each $\pi \subset \mathcal{A}_2(G)$ then gives:

$$\mathcal{P}_M(\pi) := \sum_{\{\phi_i\} \subset \pi} |\langle \Theta_M, \phi_i \rangle_{\mathrm{Pet}}|^2.$$

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- *L*-function-invariant: through Galois action on tangent spaces of Galois-fixed points, get an invariant by considering special L-value

$$\mathcal{L}_M : \{\text{A-parameters valued in } G\} \longrightarrow \mathbb{C}$$

The above discussion led [BZSV] to the following

Conjecture

There is an involutive duality

$$\{\textit{hyperspherical } G\textit{-var.}\} \longleftrightarrow \{\textit{hyperspherical } G^{\vee}\textit{-var.}\}$$

such that if $M \longleftrightarrow M^{\vee}$, *then*

$$\mathcal{P}_M = \mathcal{L}_{M^{\vee}}$$

and

$$\mathcal{P}_{M^{\vee}} = \mathcal{L}_M.$$

Duality Conjecture

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and

$$\mathcal{P}_{M^\vee} = \mathcal{L}_M.$$

Note that in both equations, the domains of the LHS and RHS are identified by the Langlands correspondence.

Examples of Duality [BZSV]

M	$M^\vee$
point	$T_e(U\backslash G)$
$T^*(X)$	$V_X \times^{X^\vee} G^\vee$
$T^*(\mathrm{Sp}_{2n}\backslash\mathrm{GL}_{2n})$ (symplectic period)	$T_e(\mathrm{GL}_n^\Delta U\backslash\mathrm{GL}_{2n})$ (Shalika period)
$T^*(SO_{2n}^\Delta\backslash(\mathrm{SO}_{2n}\times\mathrm{SO}_{2n+1}))$ (Basic Gross-Prasad)	$V_{2n}\otimes W_{2n}$ (Equal Rank Theta Corr.)
$(V\otimes W, \mathrm{O}(V)\times\mathrm{Sp}(W))$	General GP-varieties

Special Case (Joint with Bryan Wang)

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The associated G-variety M just depends on a unipotent conjugacy class $e \in G$. The quantization of M_e is a generalized Whittaker/Gelfand-Graev G-module:

$$\Pi_e = \mathrm{Ind}_{H \cdot U}^{G} 1_H \otimes \psi.$$

Results I

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Proposition

Assume $G = \mathrm{O}_{2n}$ for simplicity. Then M_e is hyperspherical if and only if e belongs to the following list:

- $e = [2n - r, 1^r]$, *r odd (hook type)*
- $e = (2^n)$ *(Shalika type)*
- $e = (3,3)$, $(4,4)$ *or* $(6,6)$ *(sporadic type)*

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Theorem

For $e \in G = \mathrm{O}_{2n}$ of hook type.

$$M_e^\vee = M_{e^\vee},$$

where $e^\vee \in G^\vee = \mathrm{O}_{2n}$ is also a nilpotent element of hook type. More precisely, the relation $e \longleftrightarrow e^\vee$ is depicted by the following diagram.

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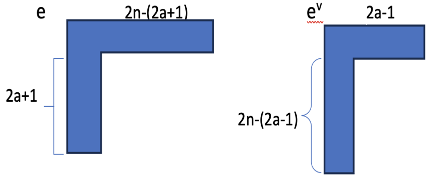

Remarks on the proof

- The proof of the theorem involves resolving two spectral decomposition problems (for M and $M^\vee$ resp.) , and showing that the answer can be described in terms of the dual variety ($M^\vee$ and M resp.) as dictated by the BZSV conjecture.

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- In particular, we use the results of Gomez and Zhu on the transfer of generalized Whittaker models under theta correspondence.
- Bryan has extended these local results to the global setting and the L^2-setting, allowing us to resolve the L^2 and global version of the BZSV conjecture.

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In other words,

Hyperspherical Duality "commutes" with Reduction.

Thank You for Your Attention!