Six-dimensional sphere packing and linear programming

joint with Maria Dostert and Maryna Viazovska

Matthew de Courcy-Ireland
Stockholm University

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Plan

- Intro (sphere packing, linear programming bound)
- Main result: “a bound on the bound”

  the linear programming bound is strictly higher
  than the density of any packing in dimension 6
- Even though the lattices E_6 and E_8 are closely related,
  the method that proves what is the best packing in dimension 8
  fails in dimension 6
- We construct a “fake packing” that tricks the method,
  using modular forms

Why $1, 2, 8, 24$?

- There is a modular form (weight 3, two quadratic characters)

$$g \in M_3(\Gamma_0(48), \chi_3) \oplus M_3(\Gamma_0(48), \chi_4)$$

with Fourier expansion and transform $\widetilde{g}(z) = (-iz)^{-k} N^{-k/2} g\left(\frac{-1}{Nz}\right)$

$$g(z) = 1 + a_7 e(7z) + \sum_{n \geq 8} a_n e(nz), \quad \widetilde{g}(z) = \sum_{n=0}^{\infty} b_n e(nz)$$

all coefficients $a_n, b_n \geq 0$

$$b_0 > 0.6168$$

- Moreover

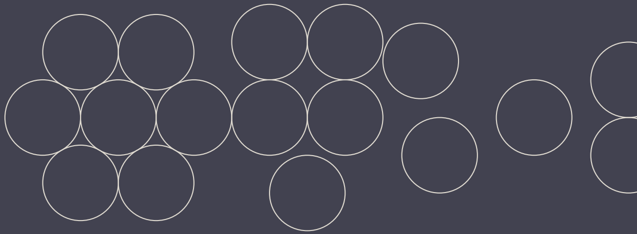
$$n \equiv 1 \bmod 4 \implies a_n = 0$$

Sphere packing

Disjoint equal-sized balls in $\mathbb{R}^D$

$$\mathcal{P} = \bigcup_{z \in S} B_r(z)$$

$$|z - z'| \geq 2r \text{ for } z \neq z' \text{ in } S$$



- Density a.k.a. packing fraction

$$\Delta(\mathcal{P}) = \limsup_{R \to \infty} \frac{\mathrm{vol}(\mathcal{P} \cap B_R)}{\mathrm{vol}(B_R)}$$

$$\Delta_D = \sup\{\Delta(\mathcal{P}) \,;\, \mathcal{P} \text{ is a packing in } \mathbb{R}^D\}$$

Δ_D known for

$$D = 1, 2, 3, 8, 24$$

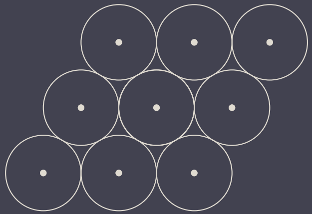
Linear programming bound

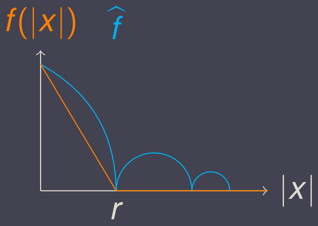
- Cohn–Elkies (2003):
  suppose a function $f : \mathbb{R}^D \to \mathbb{R}$ and $r > 0$ satisfy

$$f(x) \leq 0 \text{ for } |x| > r$$
$$\widehat{f} \geq 0$$

Then all packings in $\mathbb{R}^D$ have density at most

$$\mathrm{vol}(B_{r/2}) f(0) \div \widehat{f}(0)$$





rotation+scaling: WLOG f radial
we may fix two of $r, f(0), \widehat{f}(0)$

There are such f

- Recall

$$\widehat{f} \geq 0,\ f(x) \leq 0 \text{ for } |x| > r \implies \text{density } \leq \operatorname{vol}(B_{r/2}) f(0) \div \widehat{f}(0)$$

- There are such f in any dimension!

$$f = \mathbb{1} * \mathbb{1}, \qquad \widehat{f} = \widehat{\mathbb{1}}^2 \geq 0, \qquad \mathbb{1}(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

or

$$f(x) = P(|x|^2) \exp(-\pi |x|^2)$$

finitely many inequalities on polynomial $P \implies \checkmark$

Sharp cases

- Cohn-Elkies (2003): suppose $f(x) \leq 0$ for $|x| > r$ and $\widehat{f} \geq 0$.
  Then all sphere packings in $\mathbb{R}^D$ have density at most $\mathrm{vol}(B_{r/2}) f(0) \div \widehat{f}(0)$
$D = 1$	$f = \mathbb{1} * \mathbb{1}$ gives a sharp bound
$D = 8$	Viazovska (2017) constructs f matching the density of the E_8 lattice
$D = 24$	Cohn–Kumar–Miller–Radchenko–Viazovska (2017) similar construction for the Leech lattice
$D = 2$	the bound appears to be sharp but f remains elusive
- CKMRV (2021) same for energy when $D = 8, 24$ ("universal optimality")

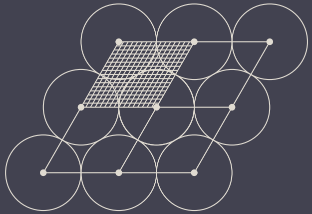
Proof sketch: density $\lesssim f(0) \div \widehat{f}(0)$

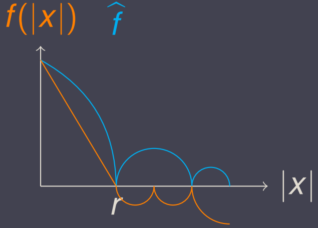
- For simplicity: assume centers of spheres form a lattice Λ $\quad (x, y \in \Lambda \implies x - y \in \Lambda)$
- Rescale: $|x| \geq r$ for $0 \neq x \in \Lambda$
  spheres have radius $r/2$
- Poisson summation

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \rho \sum_{\xi \in \Lambda^*} \widehat{f}(\xi) \geq \rho \widehat{f}(0)$$

- $1/\rho =$ volume of fundamental domain
  One ball $B_{r/2}$ per domain, so

$$\text{density} = \text{vol}(B_{r/2})\rho \leq \text{vol}(B_{r/2}) f(0) \div \widehat{f}(0)$$





Duality

Dual program: a bound on the bound

- Suppose function f and measure μ satisfy

$$
\begin{aligned}
&f(x) \leq 0 \quad \text{for} \quad |x| > r & \widehat{f} \geq 0 \\
&\mu = \delta_0 + \nu & \\
&\nu \geq 0 \text{ supported in } |x| > r & \widehat{\mu} \geq c\delta_0
\end{aligned}
$$

  Then

$$f(0) \geq \langle \mu, f \rangle = \langle \widehat{\mu}, \widehat{f} \rangle \geq c\widehat{f}(0)$$

  The LP bound is at least $c \cdot \mathrm{vol}(B_{r/2})$

- Recall

$$\widehat{f} \geq 0, \; f(x) \leq 0 \text{ for } |x| > r \implies \text{density } \leq \mathrm{vol}(B_{r/2}) f(0) \div \widehat{f}(0)$$

- Cohn (2002), Torquato–Stillinger (2006)

Zeros and spikes

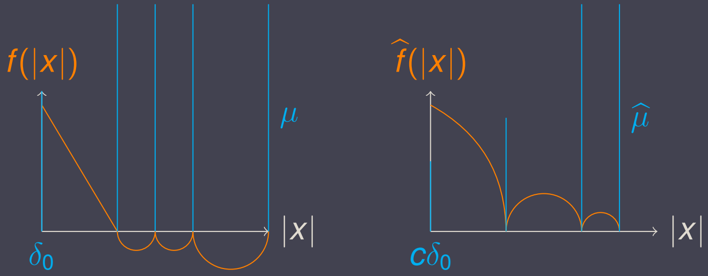
- Recall proof:

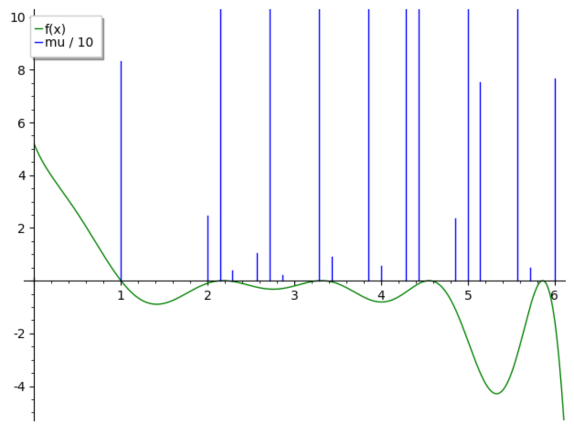
$$f(0) \geq \langle \mu, f \rangle = \langle \widehat{\mu}, \widehat{f} \rangle \geq c\widehat{f}(0)$$

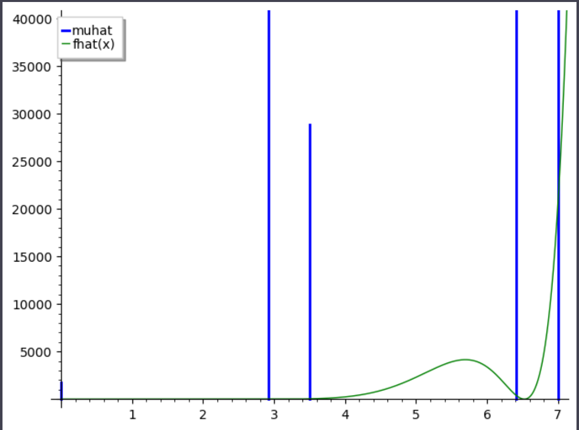
- To have equality

f should vanish on support of μ, $\widehat{f}$ on support of $\widehat{\mu}$ (except 0)

$\implies$ want spikes at zeros of f and $\widehat{f}$







Every fourth spike missing in our μ

$|x|^2$ on horizontal axis

- First two zeros of both f and $\widehat{f}$ align well with supports of μ, $\widehat{\mu}$
- Thanks to de Laat, Leijenhorst!
  Julia package to compute $f, \widehat{f}$ as polynomials times $e^{-\pi|x|^2}$

D	Record density	SDP bound	Dual bound	LP upper bound	
1	1			1	
2	0.906899			0.906899?	
3	0.740480	0.770271	0.770657	0.779747	$\uparrow$
4	0.616850	0.636108	0.637303	0.647705	Rupert Li
5	0.465257	0.512646	0.517236	0.524981	$\downarrow$
6	0.372947	0.410304	0.410948	0.417674	dci–
7	0.295297	0.321148	0.301191	0.327456	Dostert–
8	0.253669			0.253669	Viazovska
9	0.145774	0.191121	0.164925	0.194556	Cohn,
10	0.099615	0.143411	0.106256	0.147954	de Laat,
11	0.066238	0.106726	0.078504	0.111691	Salmon
12	0.049454	0.079712	0.083381	0.083776	$\leftarrow$
13	0.032014	0.060165	0.032522	0.062482	Cohn
14	0.021624	0.045062		0.046365	and
15	0.016857	0.033757		0.034249	Triantafillou
16	0.014708	0.023995	0.025011	0.025195	$\leftarrow$

Construction using modular forms

Cohn–Triantafillou (2021)

-

$$\mu = \sum_n a_n \delta_{\sqrt{n}} \qquad a_n = \text{coefficients of a modular form}$$

$$g(z) = \sum_n a_n \exp(2\pi i n z)$$

- Weight $k = D/2$ for dimension D. Simplest if D is a multiple of 4. Fix level $N \implies$ finite-dimensional space of g, explicit transforms $\widehat{\mu}$
- $g = \Theta_\Lambda$ theta series recovers density of lattice packing
- Eisenstein series versus cuspforms (Deligne bound)

$$a_n = a_{n,\mathrm{eis}} + a_{n,\mathrm{cusp}} \qquad a_{n,\mathrm{eis}} \approx n^{k-1}$$

$$> 0 \text{ for large } n \qquad |a_{n,\mathrm{cusp}}| < n^{(k-1)/2+o(1)}$$

$g : \mathbb{H} \to \mathbb{C}$,
$k = D/2$ integer or half-integer,
$N > 0$

$$\widetilde{g}(z) = (-iz)^{-k} N^{-k/2} g\left(\frac{-1}{Nz}\right)$$

Suppose g and $\widetilde{g}$ are both periodic

$$g(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz} \qquad \widetilde{g}(z) = \sum_{n=0}^{\infty} b_n e^{2\pi inz}$$

Then

$$\sum_{n=0}^{\infty} a_n \delta_{\sqrt{n}} \qquad \text{and} \qquad (2/\sqrt{N})^k \sum_{n=0}^{\infty} b_n \delta_{2\sqrt{n/N}}$$

are Fourier transforms of each other as distributions on $\mathbb{R}^D$,
where δ_r denotes a spherical delta at radius r.

Level N, weight k, character χ

- Recall

$$\widetilde{g}(z) = (-iz)^{-k} N^{-k/2} g\left(\frac{-1}{Nz}\right)$$

-

$$g\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k g(z)$$

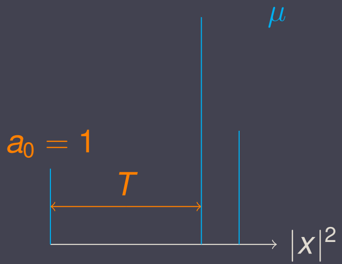
-

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \implies g, \widetilde{g} \text{ both periodic}$$

- Restrict to modular forms for $\Gamma_0(N)$

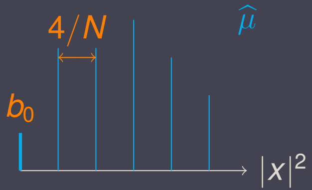
$$\mu = \sum_{n=0}^{\infty} a_n \delta_{\sqrt{n}} = \delta_0 + a_T \delta_{\sqrt{T}} + \dots$$

$$\widehat{\mu} = (2/\sqrt{N})^k \sum_{n=0}^{\infty} b_n \delta_{2\sqrt{n/N}}$$



- $k = D/2$ for dimension D
- $N \to \infty$ increases space of candidates
- If both $a_n, b_n \geq 0$ for all n, then the LP bound is at least

$$c\,\mathrm{vol}(B_{r/2}) \approx b_0 N^{-k/2} T^k$$



$N = 48, \quad T = 7, \quad b_0 \approx 0.6168$

- There is a modular form (weight 3, two quadratic characters)

$$g \in M_3(\Gamma_0(48), \chi_3) \oplus M_3(\Gamma_0(48), \chi_4)$$

  with Fourier expansion and transform

$$g(z) = 1 + a_7 e(7z) + \sum_{n \geq 8} a_n e(nz), \quad \tilde{g}(z) = \sum_{n=0}^{\infty} b_n e(nz)$$

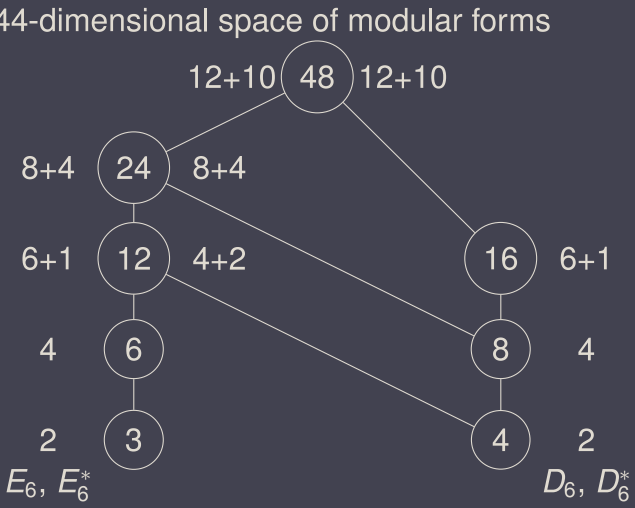
  all coefficients $a_n, b_n \geq 0$ (quadratic irrationals)

$$b_0 > 0.6168$$

- Moreover

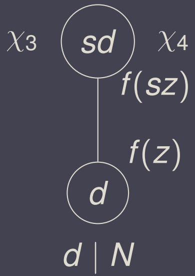
$$n \equiv 1 \bmod 4 \implies a_n = 0$$

Optimization space



$e + c$
dimensions of Eisenstein + cuspidal subspaces

$s = 2, 3$



Scaling structure of the basis

$$
\begin{array}{llll}
f_2(z) = f_1(2z), & f_7(z) = f_6(2z), & f_{24}(z) = f_{23}(2z), & f_{30}(z) = f_{29}(2z) \\
 & & f_{25}(z) = f_{23}(3z), & f_{31}(z) = f_{29}(3z) \\
f_3(z) = f_1(4z), & f_8(z) = f_6(4z), & f_{26}(z) = f_{24}(4z), & f_{32}(z) = f_{29}(4z) \\
 & & f_{27}(z) = f_{23}(6z), & f_{33}(z) = f_{29}(6z) \\
f_4(z) = f_1(8z), & f_9(z) = f_6(8z), & & \\
 & & f_{28}(z) = f_{23}(12z), & f_{34}(z) = f_{29}(12z) \\
f_5(z) = f_1(16z) & f_{10}(z) = f_6(16z) & &
\end{array}
$$

$$
\begin{array}{lll}
f_{14}(z) = f_{13}(2z) & f_{15}(z) = f_{13}(4z) & \\
f_{17}(z) = f_{16}(2z) & & \\
f_{19}(z) = f_{18}(2z) & & \\
f_{36}(z) = f_{35}(2z) & f_{37}(z) = f_{35}(4z) & \\
f_{39}(z) = f_{38}(2z) & f_{40}(z) = f_{38}(4z) & f_{42}(z) = f_{41}(3z)
\end{array}
$$

Quadratic twists

-

$$\sum_n c_n e^{2\pi i n z} \qquad \Longrightarrow \qquad \sum_n c_n \tau(n) e^{2\pi i n z}$$

Level N	$\mathrm{lcm}(N, N^* t, t^2)$
character χ mod N^*	$\chi\tau^2 \qquad (= \chi \text{ if } \tau^2 = 1)$
twist by τ mod t	

- $N = 48 = 4^2 \cdot 3$ allows twisting by $\tau = \chi_4 \qquad \Longrightarrow \quad a_n = 0$ for $n \equiv 1 \bmod 4$
- Two Eisensteins $f_{11} = f_1 \otimes \chi_4$, $f_{12} = f_6 \otimes \chi_4$
- Cuspforms

$$\begin{pmatrix} f_{20} \\ f_{21} \\ f_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} f_{13} \otimes \chi_4 \\ f_{16} \otimes \chi_4 \\ f_{18} \otimes \chi_4 \end{pmatrix}$$

44 equations, 44 unknowns

$x_1, \ldots, x_{44} \in \mathbb{Q}(\sqrt{3})$ are the unique solution to a system of equations:

1 equation	$a_0 = 1$	
11 equations	$a_n = 0$	for $n \equiv 1 \bmod 4$
20 equations	$a_n = 0$	for $n \in \{2, 3, 4, 6, 8, 10, 11, 12, 22, 26, 32, 38, 60, 64, 88, 90, 92, 106, 164, 1932\}$
12 equations	$b_n = 0$	for $n \in \{1, 2, 3, 4, 7, 8, 9, 10, 13, 14, 36, 82\}$

$$
\begin{aligned}
x_{10} = x_{41} = x_{43} &= 0 \\
x_1 + x_{11} &= 0 \\
x_6 + x_{12} &= 0 \\
x_{23} + x_{29} &= 0 \\
x_{25} - x_{31} &= 0 \\
x_{35} - x_{38} &= 0 \\
x_{13} + x_{20} + 3x_{21} &= 0 \\
x_{16} + x_{20} + 2x_{22} &= 0 \\
x_{18} - x_{21} - 2x_{22} &= 0
\end{aligned}
$$

Inequalities for large n

- Goal: $a_n, b_n \geq 0$ for all n
- Eisenstein series versus cuspforms
$$a_n = a_{n,\mathrm{eis}} + a_{n,\mathrm{cusp}}$$
- $a_{n,\mathrm{eis}} \geq \varepsilon n^{k-1}$
- Deligne bound
$$|a_{n,\mathrm{cusp}}| < n^{(k-1)/2+o(1)}$$
- $k = 3$ for $D = 6 \implies$ compare $\varepsilon n^2 \gg n^{1+o(1)}$
- $a_n \geq 0$ once n is large enough that $n^{1-o(1)} > 1/\varepsilon$

Vanishing on a progression

- $a_{n,\mathrm{eis}} \geq \varepsilon n^{k-1}$ fails for $n \equiv 1 \bmod 4$

$$a_n = 0 \text{ for } n \equiv 1 \bmod 4$$

- All four parts vanish (Eisenstein and cuspidal for both characters)
- e.g. Eisenstein part for χ_3
  $x_1 = -x_{11} \implies f_1$ and $f_{11} = f_1 \otimes \chi_4$ cancel
- How to show $a_{n,\mathrm{cusp}} = 0$?

Cuspidal part

- Hecke operators are skew self-adjoint on forms with a character

$$\langle T_n f, g\rangle = \chi(n)\langle f, T_n g\rangle$$

If $T_n h = \lambda(n)h$, then λ is

$$\begin{cases} \text{real} & \text{if } \chi(n) = 1 \\ \text{pure imaginary} & \text{if } \chi(n) = -1 \end{cases}$$

- Change to a Hecke basis, e.g. there is h for χ_4 so that (coefficient-wise)

$$f_{35} = 2\,\mathrm{Re}(h)$$
$$f_{38} = -f_{35} - 2\sqrt{3}\,\mathrm{Im}(h)$$

$x_{35} = x_{38} \implies$ coefficients of q^n cancel when $n \equiv 1 \bmod 4$

$n = 2^a \cdot 3^b \cdot n_0$

a, b	$\Longrightarrow$	which scalings appear? $f(2z)$, $f(3z)$, etc.
$n_0 \bmod 4$	$\Longrightarrow$	effect of twists $\otimes\chi_4$ $a_{n,\mathrm{eis}} = 0$ for $n \equiv 1 \bmod 4$
$n_0 \bmod 3, 4$	$\Longrightarrow$	real/imaginary Hecke eigenvalues $a_{n,\mathrm{cusp}} = 0$ for $n \equiv 1 \bmod 4$
Factors of n_0	$\Longrightarrow$	size of Eisenstein part

Eisenstein part

- Given character χ and weight k

$$\sigma^+(n) = \sum_{d|n} d^{k-1}\chi\left(\frac{n}{d}\right) = n^{k-1} + \dots$$

$$\begin{aligned}\sigma^-(n) &= \sum_{d|n} d^{k-1}\chi(d) = \chi(n)n^{k-1} + \dots \\ &= \chi(n)\sigma^+(n) \quad \text{(if } n, \chi \text{ share no factors)}\end{aligned}$$

- $k = 3$ for us; choose two quadratic characters χ mod 3 or 4
- Linear combination of $\sigma^{\pm}(n/s)$ for divisors s, different χ

$$a_{n,\mathrm{eis}} = X_3\sigma_3^+(n_0) + X_4\sigma_4^+(n_0) = \sigma_3^+(n_0)(X_3 + X_4\sigma_4^+ \div \sigma_3^+)$$

Eisenstein part, even $n = 2^a \cdot 3^b \cdot n_0$

$$
\begin{aligned}
a_{n,\mathrm{eis}} = \\
& x_1\sigma_3^+(n) &+&\ x_6\sigma_3^-(n) &+&\ x_{23}\sigma_4^+(n) &+&\ x_{29}\sigma_4^-(n) \\
& x_2\sigma_3^+(n/2) &+&\ x_7\sigma_3^-(n/2) &+&\ x_{24}\sigma_4^+(n/2) &+&\ x_{30}\sigma_4^-(n/2) \\
& x_3\sigma_3^+(n/4) &+&\ x_8\sigma_3^-(n/4) &+&\ x_{26}\sigma_4^+(n/4) &+&\ x_{32}\sigma_4^-(n/4) \\
& x_4\sigma_3^+(n/8) &+&\ x_9\sigma_3^-(n/8) \\
& x_5\sigma_3^+(n/16) &+&\ x_{10}\sigma_3^-(n/16) \\
& & & &+&\ x_{25}\sigma_4^+(n/3) &+&\ x_{31}\sigma_4^-(n/3) \\
& & & &+&\ x_{27}\sigma_4^+(n/6) &+&\ x_{33}\sigma_4^-(n/6) \\
& & & &+&\ x_{28}\sigma_4^+(n/12) &+&\ x_{34}\sigma_4^-(n/12)
\end{aligned}
$$

- $\sigma(2^a 3^b n_0) = \sigma(2^a)\sigma(3^b)\sigma(n_0)$, $\qquad \sigma^-(n_0) = \chi(n_0)\sigma^+(n_0)$
- Collect terms: $a_{n,\mathrm{eis}} = X_3\sigma_3^+(n_0) + X_4\sigma_4^+(n_0) = \sigma_3^+(n_0)(X_3 + X_4\sigma_4^+ \div \sigma_3^+)$

$$a_{n,\mathrm{eis}} = \sigma_3^+(n_0)(X_3 + X_4\sigma_4^+(n_0) \div \sigma_3^+(n_0)) \geq \varepsilon n^2 \leftarrow \text{to show}$$

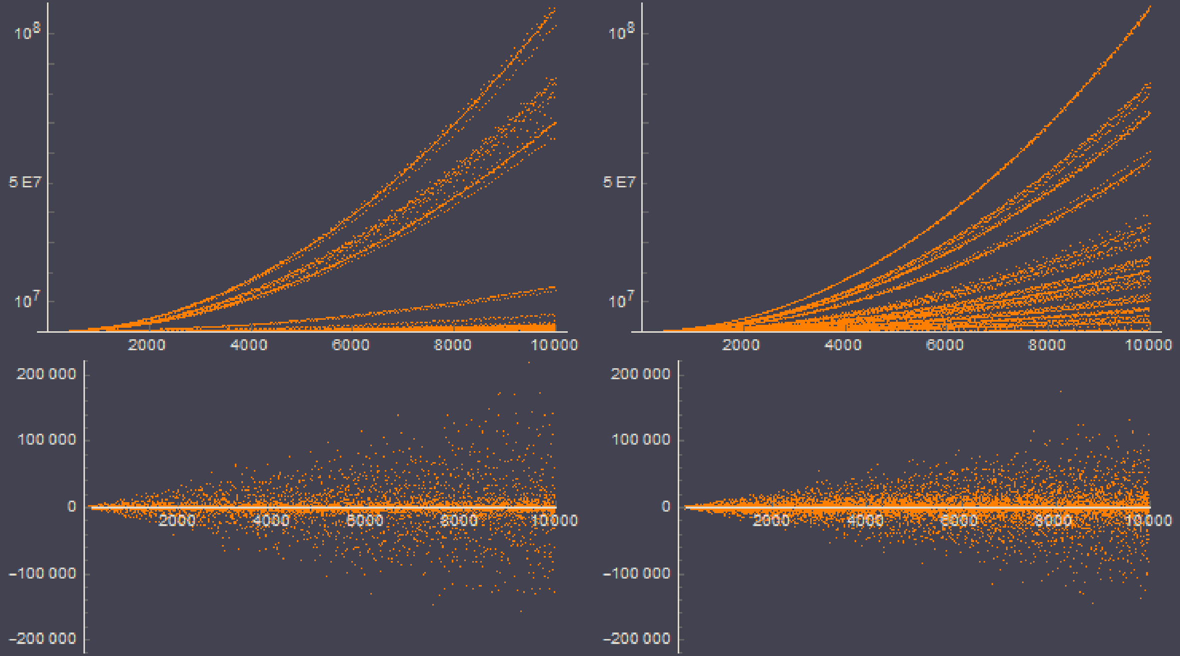
$$\sigma^+(n) = \sum_{d|n} d^2\chi(n/d) \implies \sigma^+(p) = p^2 \pm 1$$

For n_0 not divisible by 2 or 3,

$$0.94999 < \prod_{p\equiv 7 \bmod 12} \frac{p^2-1}{p^2+1} \leq \frac{\sigma_4^+(n_0)}{\sigma_3^+(n_0)} \leq \prod_{p\equiv 5 \bmod 12} \frac{p^2+1}{p^2-1} < 1.09696$$

$$0.9429 < \prod_{\substack{p\equiv 2 \bmod 3 \\ p\neq 2}} (1-p^{-2}) \leq \frac{\sigma_3^+(n_0)}{n_0^2} \leq \prod_{p\equiv 1 \bmod 3} (1+p^{-2}) < 1.0336$$

$$0.9631 < \prod_{\substack{p\equiv 3 \bmod 4 \\ p\neq 3}} (1-p^{-2}) \leq \frac{\sigma_4^+(n_0)}{n_0^2} \leq \prod_{p\equiv 1 \bmod 4} (1+p^{-2}) < 1.0545$$


10^8
5 E7
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5 E7
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2000
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200 000
100 000
0
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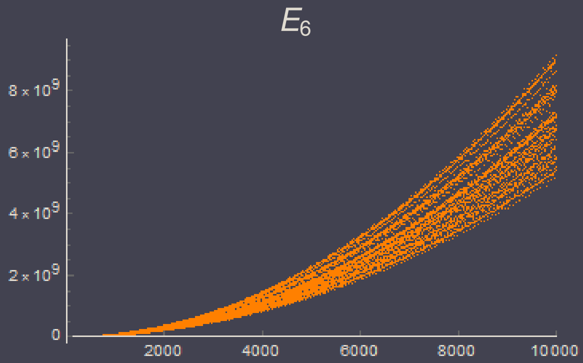
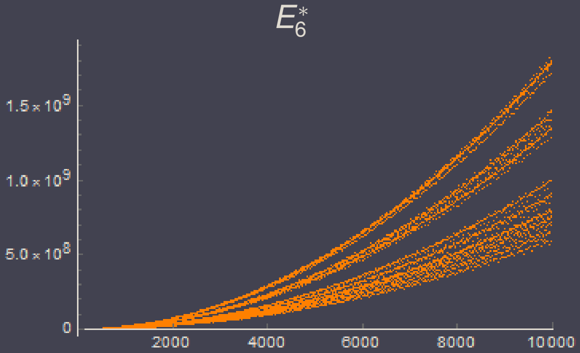

E_6
8 x 10^9
6 x 10^9
4 x 10^9
2 x 10^9
0
2000
4000
6000
8000
10000


E_6^*
1.5 x 10^9
1.0 x 10^9
5.0 x 10^8
0
2000
4000
6000
8000
10000

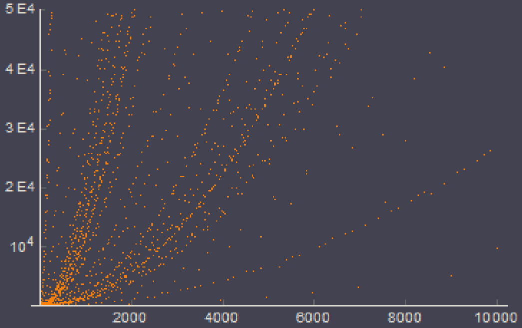
Computer assistance (GP/Pari)

- $a_{n,\mathrm{eis}} \geq \varepsilon n^2$
  $\varepsilon = 8.7536 \times 10^{-6}$
- $|a_{n,\mathrm{cusp}}| \leq C \cdot n\sigma_0(n)$
  $C = 21.6161$
  $\sigma_0(n)$ number of divisors
- Nicolas–Robin (1983), Wigert

$$\sigma_0(n) \leq \exp(1.07 \log n \div \log\log n)$$

- The bounds show $a_n \geq 0$ for

$$n \geq 5347177639 \approx 5 \times 10^9$$



- Check remaining cases!
- No need unless $a_{n,\mathrm{eis}} - Cn\sigma_0(n) < 0$
  Most such n have many factors, making it easier to compute $a_{n,\mathrm{cusp}}$

Thanks! Summary

- Why does a method that works perfectly in dimension 8 not work in other dimensions? (fake packings)

- How to interpolate mixed data on f and $\widehat{f}$? (modular forms can be useful)

Open problems:

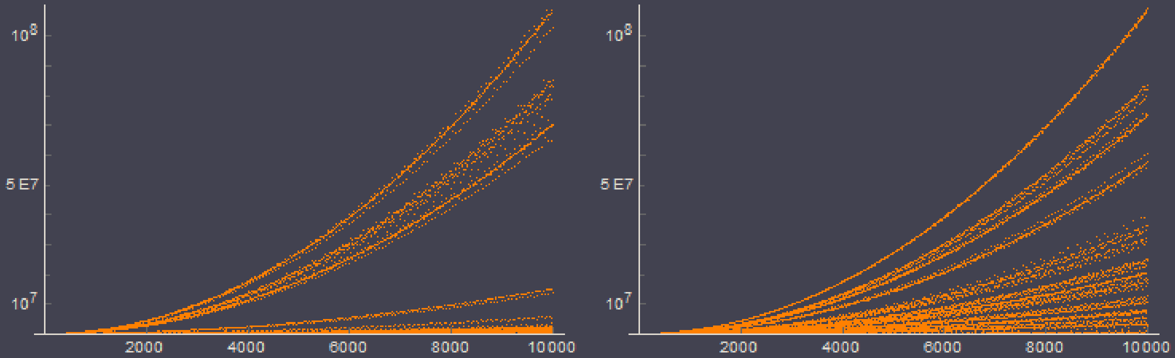
- Prove LP bound is sharp for $D = 2$

- Estimate the asymptotics as $D \to \infty$

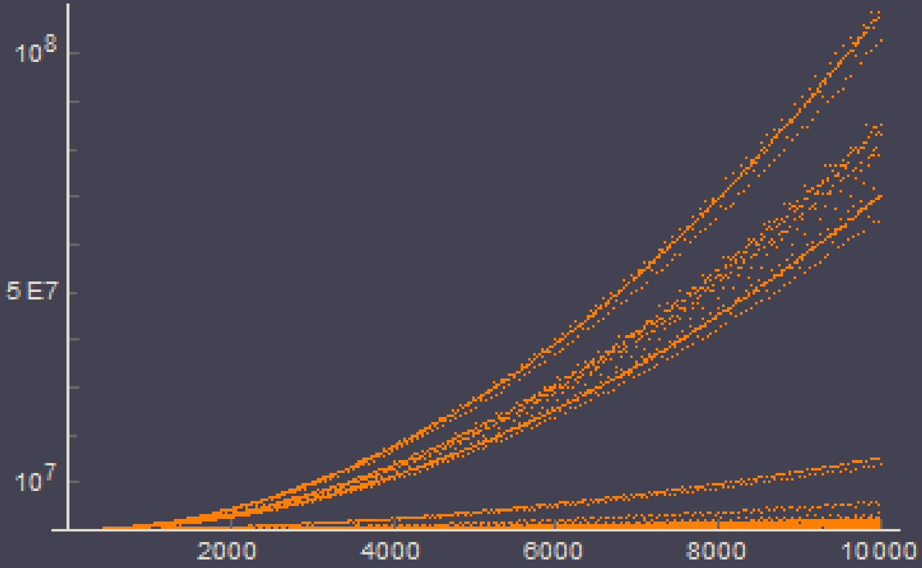
- Prove there are only finitely many sharp cases ($D = 1, 2, 8, 24$)

bonus slides

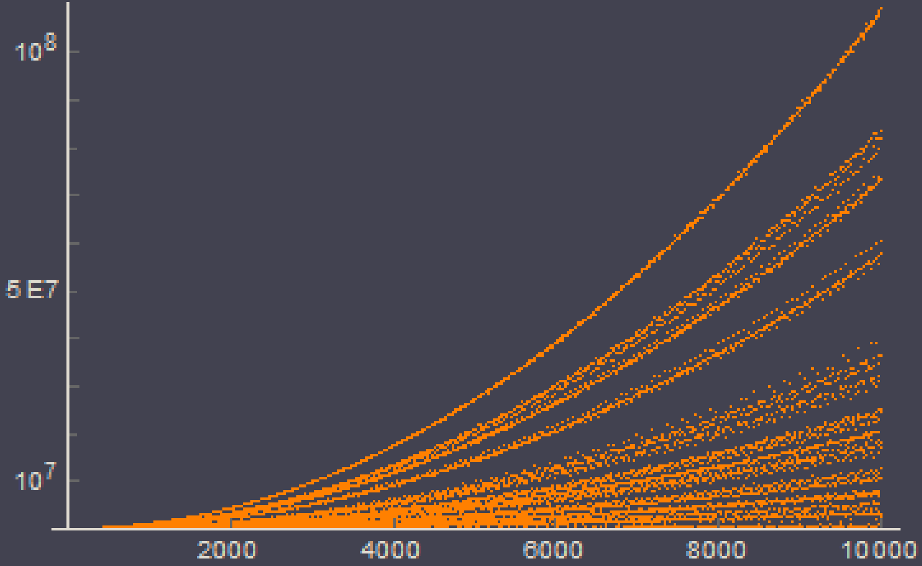
Eisenstein part



$a_{n,\mathrm{eis}}$



$b_{n,\mathrm{eis}}$



Cuspidal part