A new zero-free region for Rankin–Selberg L-functions

Gergely Harcos (Alfréd Rényi Institute)
Jesse Thorner (University of Illinois, Urbana-Champaign)

Standard L-functions

Notation

Let $\mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$.

Each $\pi \in \mathfrak{F}_n$ has a standard L-function defined as an absolutely convergent Euler product

$$L(s,\pi) = \prod_p \prod_{j=1}^n \frac{1}{1-\alpha_{j,\pi}(p)p^{-s}}, \qquad \mathrm{Re}(s) > 1.$$

The original construction is due to Godement–Jacquet (1972), and it is based on Eulerian integrals of matrix coefficients.

By the work of Kondo–Yasuda (2010), one can also define the L-function in terms of "basic Hecke eigenvalues":

$$L(s,\pi) = \prod_p \left(\sum_{k=0}^n (-1)^k \lambda_{k,\pi}(p) p^{-ks} \right)^{-1}, \qquad \mathrm{Re}(s) > 1.$$

GL_1-twists

Notation

Let $\mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. Let $\mathfrak{F}_n^* \subset \mathfrak{F}_n$ be the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ whose central character is trivial on $\mathbb{R}_{>0}$.

$\chi \in \mathfrak{F}_1 \iff \chi = \chi^* |\cdot|^{it_\chi}$, with $\chi^* \in \mathfrak{F}_1^*$ a primitive Dirichlet char.

$$L(s,\chi) = L(s + it_\chi, \chi^*)$$

$\pi \in \mathfrak{F}_n \iff \pi = \pi^* \otimes |\cdot|^{it_\pi}$, with $\pi^* \in \mathfrak{F}_n^*$

$$L(s,\pi) = L(s + it_\pi, \pi^*)$$

Convexity bound

Functional equation

$$\Lambda(s,\pi) = q_\pi^{s/2} L(s,\pi) \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s+\mu_{j,\pi})$$

$$\Lambda(s,\pi) = W(\pi)\Lambda(1-s,\tilde{\pi}), \qquad \Lambda(s,\tilde{\pi}) = \overline{\Lambda(\bar{s},\pi)}$$

Analytic conductor

$$C(it,\pi) = q_\pi \prod_{j=1}^{n}(|\mu_{j,\pi}+it|+3), \quad C(\pi) = C(0,\pi).$$

Convexity bound (Molteni 2002)

Let $\pi \in \mathfrak{F}_n^*$, $j \in \mathbb{Z}_{\geq 0}$, $\sigma \in \mathbb{R}_{\geq 0}$, $t \in \mathbb{R}$. If $L(s,\pi)$ is entire, then

$$L^{(j)}(\sigma+it,\pi) \ll_{n,j,\varepsilon} C(it,\pi)^{\max(1-\sigma,0)/2+\varepsilon}.$$

If $L(s,\pi) = \zeta(s+it_0)$, then there is a correction factor for the pole.

Nonvanishing results

Theorem (Brumley 2019)

There exists a constant $c_1 = c_1(n) > 0$ *such that if* $\pi \in \mathfrak{F}_n^*$*, then* $L(\sigma + it, \pi)$ *has at most one zero* β *(real and simple) in the region*

$$\sigma \geq 1 - c_1 / \log(C(\pi)(|t| + 3)).$$

If the exceptional zero β *exists, then* $\pi = \tilde{\pi}$*.*

Theorem (Jiang–Lü–T–Wang 2021)

For every $\pi \in \mathfrak{F}_n^*$ *and* $\varepsilon > 0$*, there exists* $c_2 = c_2(\pi, \varepsilon) > 0$ *with the following property. If* $\chi \in \mathfrak{F}_1^*$ *is quadratic, then*

$$L(\sigma, \pi \otimes \chi) \neq 0, \qquad \sigma \geq 1 - c_2 C(\chi)^{-\varepsilon}.$$

Foundational results:

- de la Vallée Poussin (1899) and Siegel (1935) (classical)
- Jacquet–Shalika (1976)
- Moreno (1985)
- Hoffstein–Ramakrishnan (1995)

Rankin–Selberg L-functions

For every pair $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$, there is a Rankin–Selberg L-function defined as an absolutely convergent Euler product

$$L(s, \pi \times \pi') = \prod_p \prod_{j=1}^{n} \prod_{j'=1}^{n'} \frac{1}{1 - \alpha_{j,j',\pi\times\pi'}(p)p^{-s}}, \qquad \mathrm{Re}(s) > 1.$$

In fact we can take

$$\alpha_{j,j',\pi\times\pi'}(p) = \alpha_{j,\pi}(p)\alpha_{j',\pi'}(p), \qquad p \nmid q_\pi q_{\pi'}.$$

The L-function is completed with suitable gamma factors

$$L_\infty(s, \pi \times \pi') = \prod_{j=1}^{n} \prod_{j=1}^{n'} \Gamma_{\mathbb{R}}(s + \mu_{\pi\times\pi'}(j, j')).$$

Analogously as before, there is a conductor $q_{\pi\times\pi'} \in \mathbb{Z}_{\geq 1}$ and a root number $W(\pi \times \pi') \in \mathbb{C}$ of modulus 1 such that

$$\Lambda(s, \pi \times \pi') = q_{\pi\times\pi'}^{s/2} L_\infty(s, \pi \times \pi') L(s, \pi \times \pi')$$

satisfies the functional equation

$$\Lambda(s, \pi\times\pi') = W(\pi\times\pi')\Lambda(1-s, \tilde{\pi}\times\tilde{\pi}') = W(\pi\times\pi')\overline{\Lambda(1-\bar{s}, \pi \times \pi')}.$$

Convexity bound

Analytic conductor

$$C(it, \pi \times \pi') = q_{\pi \times \pi'} \prod_{j=1}^{n} \prod_{j=1}^{n'} (|\mu_{j,j',\pi\times\pi'} + it| + 3).$$

This quantity is in fact $\ll_{n,n'} C(\pi)^{n'} C(\pi')^{n} (|t|+1)^{n'n}$.

Convexity bound

Let $\pi \in \mathfrak{F}_n$, $\pi' \in \mathfrak{F}_{n'}$, $j \in \mathbb{Z}_{\geq 0}$, $\sigma \in \mathbb{R}_{\geq 0}$, $t \in \mathbb{R}$. *If* $L(s, \pi \times \pi')$ *is entire, then*

$$L^{(j)}(\sigma + it, \pi \times \pi') \ll_{n,n',j,\varepsilon} C(it, \pi \times \pi')^{\max(1-\sigma,0)/2+\varepsilon}.$$

There is a correction factor when there is a pole.

$$L(s, \pi \times \pi') \text{ has pole at } s = 1 - it \iff \tilde{\pi}' = \pi \otimes |\cdot|^{it}.$$

Nonvanishing results: general pairs (π, π')

Shahidi (1981) proved that $L(s, \pi \times \pi') \neq 0$ for $\mathrm{Re}(s) \geq 1$.

Theorem (Brumley 2006 & 2013)

There exists $c_3 = c_3(n, n') > 0$ such that if $(\pi, \pi') \in \mathfrak{F}_n^* \times \mathfrak{F}_{n'}^*$, then $L(\sigma + it, \pi \times \pi') \neq 0$ when

$$\sigma \geq 1 - c_3 (C(\pi) C(\pi'))^{-n-n'} (|t| + 1)^{-nn'}.$$

Further developments:

- Moreno (1985)
- Sarnak (2004)
- Gelbart–Lapid (2006)
- Goldfeld–Li (2018)
- Humphries (2019)

Nonvanishing results: special pairs (π, π')

Theorem (Brumley 2019, Humphries–T 2022)

There exists $c_4 = c_4(n, n') > 0$ such that if $(\pi, \pi') \in \mathfrak{F}_n^* \times \mathfrak{F}_{n'}^*$ and

$$\pi = \tilde{\pi} \qquad \text{or} \qquad \pi' = \tilde{\pi}' \qquad \text{or} \qquad \pi' = \tilde{\pi},$$

then $L(\sigma + it, \pi \times \pi')$ has at most one zero β (real and simple) in

$$\sigma \geq 1 - c_4 / \log(C(\pi)C(\pi')(|t| + 3)).$$

If the exceptional zero β exists, then $(\pi, \pi') = (\tilde{\pi}, \tilde{\pi}')$ or $\pi' = \tilde{\pi}$.

Theorem (Humphries–T 2021)

For every $\pi \in \mathfrak{F}_n^*$ and $\varepsilon > 0$, there exists $c_5 = c_5(\pi, \varepsilon) > 0$ such that if $\chi \in \mathfrak{F}_1^*$ is quadratic, then

$$L(\sigma, \pi \otimes (\tilde{\pi} \otimes \chi)) \neq 0, \qquad \sigma \geq 1 - c_5 C(\chi)^{-\varepsilon}.$$

A new zero-free region

Goal: Extended Siegel's celebrated lower bound for Dirichlet L-functions to all GL_1-twists of $L(s, \pi \times \pi')$.

Theorem (Harcos–T)

Let $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$. For all $\varepsilon > 0$, there exists an ineffective constant $c_6 = c_6(\pi, \pi', \varepsilon) \geq 1$ such that if $\chi \in \mathfrak{F}_1$, then

$$c_6^{-1} C(\chi)^{-\varepsilon} \leq |L(\sigma, \pi \times (\pi' \otimes \chi))| \leq c_6 C(\chi)^{\varepsilon}, \qquad \sigma \geq 1 - c_6 C(\chi)^{-\varepsilon}.$$

In particular, there exists $c_7 = c_7(\pi, \pi', \varepsilon) \geq 1$ such that

$$c_7^{-1}(|t| + 1)^{-\varepsilon} \leq |L(\sigma + it, \pi \times \pi')| \leq c_7(|t| + 1)^{\varepsilon}$$

in the range

$$\sigma \geq 1 - c_7^{-1}(|t| + 1)^{-\varepsilon}.$$

Relies crucially on the group structure of $\mathfrak{F}_1$, not just $\mathfrak{F}_1^*$.

An application

The new zero-free region allows us to prove an analogue of the Siegel–Walfisz theorem for Rankin–Selberg L-functions.

Notation

For $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$, let $\Lambda_{\pi\times\pi'}(m)$ denote the m-th Dirichlet coefficient of $-L'(s, \pi \times \pi')/L(s, \pi \times \pi')$. Moreover, let

$$\mathcal{M}_{\pi\times\pi'}(x) = \begin{cases} x^{1-iu}/(1-iu), & \pi' = \tilde{\pi} \otimes |\cdot|^{iu} \\ 0, & \text{otherwise} \end{cases}$$

Theorem (Harcos–T)

Let $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$. Let $q \leq (\log x)^A$ be a positive integer coprime to the conductors of π and π', and let $a \pmod{q}$ be a reduced residue class modulo q. Then

$$\sum_{\substack{m \leq x \\ m \equiv a \ (\mathrm{mod}\ q)}} \Lambda_{\pi\times\pi'}(m) = \frac{\mathcal{M}_{\pi\times\pi'}(x)}{\varphi(q)} + O_{\pi,\pi',A}\left(\frac{x}{(\log x)^A}\right).$$

Symmetric power L-functions

Our second application is based on cases of functoriality established by Gelbart–Jacquet (1978), Kim–Shahidi (2002) and Kim (2003).

Theorem (Harcos–Thorner)

For every $\pi \in \mathfrak{F}_2$ and $\varepsilon > 0$, there exists $c_8 = c_8(\pi, \varepsilon) \geq 1$ with the following property. If $\chi \in \mathfrak{F}_1$, and $\sigma \geq 1 - c_8^{-1} C(\chi)^{-\varepsilon}$, and $n \in \{1, \ldots, 8\}$, and $r_{n,\pi,\chi} \geq 0$ is the least integer such that

$$\mathcal{L}(s, \pi, \mathrm{Sym}^n \otimes \chi) = \Big(\frac{s + it_\pi + it_\chi - 1}{s + it_\pi + it_\chi + 1}\Big)^{r_{n,\pi,\chi}} L(s, \pi, \mathrm{Sym}^n \otimes \chi)$$

holomorphically continues to $\mathrm{Re}(s) \geq 1$, then

$$c_8^{-1} C(\chi)^{-\varepsilon} \leq |\mathcal{L}(\sigma, \pi, \mathrm{Sym}^n \otimes \chi)| \leq c_8 C(\chi)^{\varepsilon}, \quad \sigma \geq 1 - c_8^{-1} C(\chi)^{-1}.$$

For $n \in \{5, 6, 7, 8\}$, the idea is to use the identity

$$L(s, \pi, \mathrm{Sym}^n \otimes \chi) = \frac{L(s, \mathrm{Sym}^4(\pi) \times (\mathrm{Sym}^{n-4}(\pi) \otimes \chi))}{L(s, \mathrm{Sym}^3(\pi) \times (\mathrm{Sym}^{n-5}(\pi) \otimes \chi\omega_\pi))}.$$

The Key Proposition

By the convexity bound for $L'(s, \pi \times (\pi' \otimes \chi))$ and the MVT, it suffices to prove the main result for $\sigma = 1$ and for $\varepsilon \in (0, \frac{1}{2})$:

$$|L(1, \pi \times (\pi' \otimes \chi))| \gg_{\pi,\pi',\varepsilon} C(\chi)^{-\varepsilon}.$$

We accomplish this by applying three times the following

Key Proposition

Let $(\pi, \pi', \chi) \in \mathfrak{F}_n \times \mathfrak{F}_{n'} \times \mathfrak{F}_1$, $\varepsilon \in (0, \frac{1}{2})$, and $\beta \in (1 - \frac{\varepsilon}{8}, 1)$. Assume that the following L-functions are entire:

$$L(s, \pi \times \pi'), \qquad L(s, \pi \times (\pi' \otimes \chi)), \qquad L(s, \pi \times (\pi' \otimes \chi^2)).$$

If $L(\beta, \pi \times \pi') = 0$, then

$$|L(1, \pi \times (\pi' \otimes \chi))| \gg_{\pi,\pi',\beta,\varepsilon} C(\chi)^{-(n+n')^2\varepsilon}.$$

A finiteness lemma for GL_1 twists

Sometimes the Key Proposition is not applicable because $L(s, \pi \times (\pi' \otimes \chi))$ or $L(s, \pi \times (\pi' \otimes \chi^2))$ has a pole.

We handle these exceptional cases with the help of the following

Finiteness Lemma

For any $(\pi, \pi') \in \mathfrak{F}_n \times \mathfrak{F}_{n'}$, there are finitely many $\chi \in \mathfrak{F}_1^*$ such that $L(s, \pi \times (\pi' \otimes \chi))$ has a pole.

Follows from the compactness of $\mathbb{A}_{\mathbb{Q}}^1/\mathbb{Q}^\times$

Step 0: Some group theory

$$\mathfrak{F}_1^{(j)} := \{\chi = \chi^* |\cdot|^{it_\chi} \in \mathfrak{F}_1 \colon {\chi^*}^j = 1\}.$$

Chain of subgroups:

$$\mathfrak{F}_1^{(1)} \leq \mathfrak{F}_1^{(2)} \leq \mathfrak{F}_1.$$

Step 1: $\chi \in \mathfrak{F}_1^{(1)}$

1. $\chi = |\cdot|^{it_\chi}$
2. Suffices to assume that $L(s, \pi \times \pi')$ is entire with a zero in $\mathrm{Re}(s) > 1 - \varepsilon'$, where $\varepsilon' = \varepsilon/[8(n+n')^2]$. (Otherwise, apply Humphries–T.)
3. "Siegel assumption": $\exists\ (\beta, \psi) \in (1-\varepsilon', 1) \times \mathfrak{F}_1^{(1)}$ depending only on (π, π', ε) such that $L(s, \pi \times (\pi' \otimes \psi))$ is entire and vanishes at $s = \beta$. Define

$$\pi'' = \pi' \otimes \psi \in \mathfrak{F}_{n'} \qquad \chi' = \overline{\psi}\chi \in \mathfrak{F}_1^{(1)}$$

4. Apply Key Proposition with $(\pi'', \chi', \varepsilon')$ in place of $(\pi', \chi, \varepsilon)$
5. Claimed lower bound follows since $L(\beta, \pi \times \pi'') = 0$ and the following L-functions are entire:

$$L(s, \pi \times \pi''), \qquad L(s, \pi \times (\pi'' \otimes \chi')), \qquad L(s, \pi \times (\pi'' \otimes \chi'^2)).$$

6. Formal consequence: Claimed lower bound also holds for all χ in any fixed coset of $\mathfrak{F}_1^{(1)}$ within $\mathfrak{F}_1$

Step 2: $\chi \in \mathfrak{F}_1^{(2)}$

1. $\chi = \chi^* |\cdot|^{it_\chi}$ and ${\chi^*}^2 = 1$
2. Suffices to assume that $L(s, \pi \times (\pi' \otimes \chi))$ is entire with a zero in $\mathrm{Re}(s) > 1 - \varepsilon'$, where $\varepsilon' = \varepsilon/[8(n+n')^2]$. (Otherwise, χ is in a fixed coset of $\mathfrak{F}_1^{(1)}$.)
3. "Siegel assumption": $\exists\ (\beta, \psi) \in (1-\varepsilon', 1) \times \mathfrak{F}_1^{(2)}$ depending only on (π, π', ε) such that $L(s, \pi \times (\pi' \otimes \psi))$ is entire and vanishes at $s = \beta$. Define

$$\pi'' = \pi' \otimes \psi \in \mathfrak{F}_{n'} \qquad \chi' = \overline{\psi}\chi \in \mathfrak{F}_1^{(2)}$$

4. Apply Key Proposition with $(\pi'', \chi', \varepsilon')$ in place of $(\pi', \chi, \varepsilon)$.
5. Claimed lower bound follows since $L(\beta, \pi \times \pi'') = 0$ and the following L-functions are entire:

$$L(s, \pi \times \pi''), \qquad L(s, \pi \times (\pi'' \otimes \chi')), \qquad L(s, \pi \times (\pi'' \otimes \chi'^2)).$$

6. Formal consequence: Claimed lower bound also holds for all χ in any fixed coset of $\mathfrak{F}_1^{(2)}$ within $\mathfrak{F}_1$

Step 3: the general case

1. $\chi = \chi^* |\cdot|^{it_\chi}$ and χ^* arbitrary
2. Suffices to assume that $L(s, \pi \times (\pi' \otimes \chi))$ is entire with a zero in $\mathrm{Re}(s) > 1 - \varepsilon'$, where $\varepsilon' = \varepsilon/[8(n+n')^2]$.
   (Otherwise, χ is in a fixed coset of $\mathfrak{F}_1^{(1)}$.)
3. "Siegel assumption": $\exists\ (\beta, \psi) \in (1-\varepsilon', 1) \times \mathfrak{F}_1$ depending only on (π, π', ε) such that $L(s, \pi \times (\pi' \otimes \psi))$ is entire and vanishes at $s = \beta$. Define

$$\pi'' = \pi' \otimes \psi \in \mathfrak{F}_{n'}, \qquad \chi' = \overline{\psi}\chi \in \mathfrak{F}_1$$

4. Suffices to have $L(s, \pi \times (\pi'' \otimes \chi'^2))$ entire.
   (Otherwise, χ is in a fixed coset of $\mathfrak{F}_1^{(2)}$.)
5. Apply Key Proposition with $(\pi'', \chi', \varepsilon')$ in place of $(\pi', \chi, \varepsilon)$.
6. Claimed lower bound follows since $L(\beta, \pi \times \pi'') = 0$ and the following L-functions are entire:

$$L(s, \pi \times \pi''), \qquad L(s, \pi \times (\pi'' \otimes \chi')), \qquad L(s, \pi \times (\pi'' \otimes \chi'^2)).$$

Proof of the Key Proposition (1 of 3)

If $\pi \otimes \chi^* = \pi$ or $\pi' \otimes \chi^* = \pi'$, then we may assume that $|t_\chi| > 1$, because the left-hand side of the bound equals $|L(1 + it_\chi, \pi \times \pi')|$.

We introduce the auxiliary L-function $D(s) = L(s, \Pi \times \tilde{\Pi})$, where

$$\Pi = \pi \boxplus \pi \otimes \chi \boxplus \tilde{\pi}' \boxplus \tilde{\pi}' \otimes \bar{\chi}.$$

Hoffstein–Ramakrishnan: $D(s)$ has has nonneg. Dirichlet coeff's. Also, $D(s)$ factors as

$$L(s, \pi \times \tilde{\pi})^2 L(s, \pi' \times \tilde{\pi}')^2 L(s, \pi \times (\pi' \otimes \chi))^2 L(s, \tilde{\pi} \times (\tilde{\pi}' \otimes \bar{\chi}))^2$$
$$L(s, \pi \times (\tilde{\pi} \otimes \chi)) L(s, \pi' \times (\tilde{\pi}' \otimes \chi)) L(s, \tilde{\pi} \times \tilde{\pi}') L(s, \pi \times (\pi' \otimes \chi^2))$$
$$L(s, \pi \times (\tilde{\pi} \otimes \bar{\chi})) L(s, \pi' \times (\tilde{\pi}' \otimes \bar{\chi})) L(s, \pi \times \pi') L(s, \tilde{\pi} \times (\tilde{\pi}' \otimes \bar{\chi}^2)).$$

Hypotheses $\implies$ poles of $D(s)$ is a set $\mathcal{S} \subset \{1, 1 - it_\chi, 1 + it_\chi\}$. Moreover, $D(\beta) = 0$, hence $\mathcal{S}$ is also the set of poles of $D(s)x^{s-\beta}\Gamma(s - \beta)$ in the half-plane $\mathrm{Re}(s) > 0$.

Proof of the Key Proposition (2 of 3)

Let us choose $x \geq 1$ later. The Residue Theorem gives that

$$1 \ll \sum_{s_0 \in \mathcal{S}} \operatorname*{Res}_{s=s_0} D(s) x^{s-\beta} \Gamma(s-\beta) + \frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} D(s) x^{s-\beta} \Gamma(s-\beta)\, ds.$$

The order of the pole $s_0 = 1$ is always $m = 4$, while the order of the pole $s_0 = 1 \pm it_\chi$ is $m \in \{1, 2\}$ depending on how many of the equations $\pi \otimes \chi^* = \pi$ and $\pi' \otimes \chi^* = \pi'$ hold true.

$$\sum_{s_0 \in \mathcal{S}} \operatorname*{Res}_{s=s_0} D(s) x^{s-\beta} \Gamma(s-\beta) \ll_{\pi,\pi',\beta,\varepsilon} |L(1, \pi \times (\pi' \otimes \chi))| C(\chi)^{(n+n')^2 \varepsilon/4}$$

(Long residue theorem computation...)

Proof of the Key Proposition (3 of 3)

Let
$$Q = C(\chi)^{(n+n')^2}.$$

By convexity,
$$1 \ll_{\pi,\pi',\beta,\varepsilon} \left(|L(1,\pi \times (\pi' \otimes \chi))| + Qx^{-1/2}\right)(Qx)^{\varepsilon/4}.$$

Choose
$$x = \max\left(1, Q^2|L(1,\pi \times (\pi' \otimes \chi))|^{-2}\right).$$

If $x = 1$, then the desired lower bound on $L(1)$ is immediate. Otherwise,
$$x = Q^2|L(1,\pi \times (\pi' \otimes \chi))|^{-2} > 1.$$

Solve for $|L(1,\pi \times (\pi' \otimes \chi))|$:
$$|L(1,\pi \times (\pi' \otimes \chi))| \gg_{\pi,\pi',\beta,\varepsilon} Q^{-3\varepsilon/(4-2\varepsilon)} > Q^{-\varepsilon}.$$