G finite group. $\quad X(G) = \{(a,b) \in G \times G \mid \langle a,b \rangle = G\}$

$r: (a,b) \longmapsto (a^{-1}, b)$

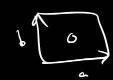
$s: (a,b) \longmapsto (b, a)$

$t: (a,b) \longmapsto (a^{-1}, ab)$

Q| What are the orbits of $\langle r,s,t \rangle \curvearrowright X(G)$

$\uparrow$ "Nielsen-equivalence classes"

Π = free gp. of rank 2 $= \langle a,b \rangle = \pi_1$(punctured torus)



$$\begin{array}{ccc} \mathrm{Aut}(\Pi) \circlearrowright & & \circlearrowleft \mathrm{Aut}(G) \\ X(G) & \xrightarrow{\sim} & \mathrm{Epi}(\Pi, G) \\ (\varphi(a), \varphi(b)) & \longleftarrow\!\shortmid & \varphi \end{array} \qquad \mathrm{Aut}(\Pi) = \langle r,s,t \rangle$$

$$\mathrm{Out}(\Pi) \circlearrowright \mathrm{Epi}^{ext}(\Pi, G) := \mathrm{Epi}(\Pi, G)/\mathrm{Inn}(G) \circlearrowleft \mathrm{Out}(G)$$

$$\mathrm{Aut}^+(\Pi) \subseteq \mathrm{Aut}(\Pi) \longrightarrow GL_2(\mathbb{Z}) \supseteq SL_2(\mathbb{Z})$$

$\searrow \mathrm{Out}(\Pi) \nearrow \cong \leftarrow$ Nielsen

$\mathrm{Out}(\Pi) \supseteq \mathrm{Out}^+(\Pi)$

- G abelian $G = \mathbb{Z}/n$

$$\begin{array}{c} \mathrm{Out}(\Pi) \curvearrowright \mathrm{Epi}^{ext}(\Pi, \mathbb{Z}/n) \\ \downarrow\cong \qquad\qquad \parallel \\ GL_2(\mathbb{Z}) \curvearrowright \mathrm{Epi}(\mathbb{Z}^2, \mathbb{Z}/n) \\ \cup \qquad\qquad\qquad \parallel \\ SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/n) \curvearrowright \mathrm{Epi}((\mathbb{Z}/n)^2, \mathbb{Z}/n) \end{array}$$

stabilizers $\sim \Gamma_1(n)$ $\quad$ trans.

$G = (\mathbb{Z}/n)^2$

stabilizers $= \Gamma(n)$ $\quad GL_2(\mathbb{Z}) \curvearrowright \mathrm{Epi}((\mathbb{Z}/n)^2, (\mathbb{Z}/n)^2)$

$\underset{2}{\underline{\phi(n)}}$ orbits, all isom.

- G metabelian (C., Deligne) all stabilizers are congruence orbits are isom.
- General G: the class of $[a,b] \in \Pi$ is preserved by the action of $\mathrm{Out}^+(\Pi)$

$\varphi([a,b]) \in \mathrm{Cl}(G)$ is the "Higman invariant" of φ.

Character Varieties

"representation variety"

$$\mathrm{Aut}(\Pi) \curvearrowright \mathrm{Hom}(\Pi, SL_2) \cong \underline{SL_2 \times SL_2} = \mathrm{Spec}\, \underline{\underline{A}}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathrm{Out}(\Pi) \curvearrowright \mathrm{Hom}(\Pi, SL_2)/\!\!/_{GL_2} = \mathrm{Spec}\, A^{GL_2}$$

$\parallel$ X_{SL_2} $\quad \varphi$

$\downarrow\cong \leftarrow$ Fricke, Brumfiel-Hilden

$$\mathrm{Out}(\Pi) \curvearrowright \mathbb{A}^3 \qquad (\mathrm{tr}\,\varphi(a), \mathrm{tr}\,\varphi(b), \mathrm{tr}\,\varphi(ab))$$

$r: (x,y,z) \longmapsto (x, y, xy - z)$

$s: \quad " \quad \longmapsto (y, x, z)$

$t: \quad " \quad \longmapsto (x, z, y)$

$$X_{SL_2} \cong \mathbb{A}^3$$

$$\begin{array}{ccccc} \varphi & & \downarrow T & & (x,y,z) \\ \downarrow & & & & \downarrow \\ \mathrm{tr}\,\varphi([a,b]) & & \mathbb{A}^1 & & x^2+y^2+z^2-xyz-2 \end{array}$$

"Markoff surface" $\quad$ "trace invariant" (preserved by $\mathrm{Out}(\Pi)$)

$\mathbb{A}^3 \supseteq \mathbb{X} := T^{-1}(-2): \quad x^2+y^2+z^2-xyz = 0$

Prop $\forall p \geq 3 \quad \mathrm{Epi}(\Pi, SL_2(p))_{T=-2}/GL_2(p) \xrightarrow{\sim} \mathbb{X}^*(p) := \mathbb{X}(\mathbb{F}_p) - \{(0,0,0)\}$

Thm (Markoff, 1879) $\mathrm{Out}(\Pi) \curvearrowright$ trans. $\mathbb{X}(\mathbb{Z}_{\geq 0}) - \{(0,0,0)\}$

Thm (Bourgain, Gamburd, Sarnak, 2016)

Let $\mathbb{E}_{bgs} := \{p \text{ prime} \mid \mathrm{Out}(\Pi) \text{ is not trans. on } \mathbb{X}^*(p)\}$

(a) $\forall \varepsilon > 0, \ \#\{p \leq x \mid p \in \mathbb{E}_{bgs}\} = O(x^\varepsilon)$

(b) $\forall \varepsilon > 0, \ \exists$ large orbit $\mathcal{C}(p) \subseteq \mathbb{X}^*(p)$ s.t.

$$|\mathbb{X}^*(p) - \mathcal{C}(p)| \leq p^\varepsilon \qquad (|\mathbb{X}^*(p)| \sim p^2)$$

Conj (BGS, Baragar 1991) $\mathbb{E}_{bgs} = \emptyset$

Thm 1 (C. 2021) Every $\mathrm{Out}^+(\Pi)$-orbit on $\mathbb{X}^*(p)$ has size $\equiv 0 \mod p$.

$\overset{(b)}{\Longrightarrow} \mathbb{E}_{bgs}$ is finite

Moduli of elliptic curves w/ G-structures

work $/\mathbb{C}$, $Z(G) = 1$

$\mathcal{M}(G)$ = "moduli stack of smooth admissible G-covers of elliptic curves"

only branched above $O \in E$

$$\begin{tikzcd} \mathrm{Out}(\Pi) \curvearrowright \mathrm{Epi}^{ext}(\Pi, G) \arrow[r] \arrow[d] & \mathcal{M}(G) \arrow[d, "\text{finite étale}"] & C \arrow[r] \arrow[d, mapsto] & E \\ \mathrm{Spec}\,\mathbb{C} \arrow[r, "E"] & \mathcal{M}(1) = \text{"}[\mathcal{H}/SL_2(\mathbb{Z})]\text{"} & E & \end{tikzcd}$$

"moduli stack of ell. curves

$\pi_1(\mathcal{M}(1)) = SL_2(\mathbb{Z}) = \mathrm{Out}^+(\Pi)$

$$\{\text{conn. comps of } \mathcal{M}(G)\} \xrightarrow{\sim} \{\mathrm{Out}^+(\Pi) \text{ orbits on } \mathrm{Epi}^{ext}(\Pi, G)\}$$

$$[\mathcal{H}/\Gamma_\varphi] \longleftarrow \mathrm{Out}^+(\Pi)\cdot\varphi$$

$$\Gamma_\varphi = \mathrm{Stab}_{SL_2(\mathbb{Z})}(\varphi)$$

BGS-conj $\iff \underbrace{\mathcal{M}(SL_2(p))_{T=-2}/GL_2(p)}_{\mathcal{M}_p}$ is connected

Thm 1 $\iff$ every $\mathcal{M} \in \pi_0(\mathcal{M}_p)$ has $\deg(\mathcal{M}/\mathcal{M}(1)) \equiv 0 \mod p$

Applications

"all but finitely many"

1. $\forall' p \qquad \mathbb{X}(\mathbb{Z}) \twoheadrightarrow \mathbb{X}(\mathbb{F}_p)$ "strong approximation" (at p)

2. $\forall' p \quad \mathrm{Out}(\pi) \curvearrowright \mathbb{X}(\mathbb{Z}_p)$ is minimal (ie orbits are dense)

3. $\forall' p$ can count flat geodesics on $\mathcal{H}/_{\Gamma(p) \cap SL_2(\mathbb{Z})'}$

4. $\mathrm{genus}(\mathcal{M}_p) = \frac{1}{12} p^2 + O(p^{3/2}) \qquad p(p^2-1) = |SL_2(p)|$

5. $\mathcal{M}_p$ has bad reduction at all $\ell \mid p^2-1$ (and $\ell = p$ if $p \equiv 1\ (4)$)

"Proof" of Thm 1

$\mathcal{M} \subseteq \mathcal{M}(G)_{\text{ram index } e}$

admissible G-cover of ram. index e

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi"] & \mathcal{E} \arrow[r, "\tilde{f}"] \arrow[d] & \mathcal{E}(1) \arrow[d] \\ & \mathcal{M} \arrow[r, "f"] \arrow[u, bend right, "\sigma"'] \arrow[ul, bend left, "\tau"] & \mathcal{M}(1) \arrow[u, bend right, "\sigma_1"'] \end{tikzcd}$$

$\pi \circ \tau = \sigma$

λ

"Hodge bundle"

$$\underline{\tau^* \Omega_{\mathcal{C}/\mathcal{M}}^{\otimes e}} \cong \tau^* \pi^* \Omega_{\mathcal{E}/\mathcal{M}} \cong \sigma^* \Omega_{\mathcal{E}/\mathcal{M}} = \sigma^* \tilde{f}^* \Omega_{\mathcal{E}(1)/\mathcal{M}(1)} = \underline{f^* \overbrace{\sigma^* \Omega_{\mathcal{E}(1)/\mathcal{M}(1)}}}$$

$$e \cdot \deg \tau^* \Omega_{\mathcal{C}/\mathcal{M}} = \deg(f) \cdot \underbrace{\deg \lambda}_{\frac{1}{24}}$$

$$\Longrightarrow \deg(f) = 24 e \cdot \deg \tau^* \Omega_{\mathcal{C}/\mathcal{M}}$$

Problems

1. $\mathcal{M}(G)$ is not compact! Use "$\mathrm{Adm}(G)$" instead
   can't talk about degree!
   compactification of $\mathcal{M}(G)$
2. $\deg \tau^* \Omega_{\mathcal{C}/\mathcal{M}}$ may not $\in \mathbb{Z}$!
   $m :=$ largest possible denominator
3. τ may not exist!
   Let R = comp. of ramification divisor $\quad (R \subseteq \mathcal{C})$
   $d := \deg(R/\mathcal{M})$

<u>Thm</u> (C., 2021) $\quad \mathcal{M} \subseteq \mathcal{M}(G)_{\text{ram. index } e}$

$$\deg(\mathcal{M} \to \mathcal{M}(1)) \equiv 0 \bmod \frac{12e}{\gcd(12e, md)}$$

Need to control m, d!

<u>Thm</u> (C. 2021) $e \in \mathbb{Z}_{\geq 0}$, ℓ prime

$r := \mathrm{ord}_\ell(e)$

$s := \mathrm{ord}_\ell(|G|) - r$

$j := \inf\{ j \geq 0 \mid G \text{ does not contain a proper normal subgp of order div. by } \ell^{j+1} \}$

If $\mathcal{M} \subseteq \mathcal{M}(G)_{\text{Higman}}$, then

$$\deg(\mathcal{M} \to \mathcal{M}(1)) \equiv 0 \bmod \ell^{\left\lceil \frac{r-3s-j}{2} \right\rceil}$$

Ie, if $\varphi : \pi \twoheadrightarrow G$ w/ $|\varphi([a,b])| = e$, then

$$|\mathrm{Out}^+(\pi) \cdot \varphi| \equiv 0 \bmod \ell^{\lceil \cdots \rceil}$$