Arithmetic of Fourier coefficients of Gan-Gurevich lifts on G_2

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Introduction: modular forms on G_2

$G =$ reductive group, $\pi = \otimes' \pi_v$ AR of $G(\mathbb{A})$.

Some π_∞ are more arithmetic than others.

Most arithmetic: π_∞ is a **holomorphic discrete series**, e.g. for SL_2, Sp_{2g}, ...

You can use the relation to Shimura varieties to construct Galois representations, study Fourier expansions, and much more

But not all groups admit holomorphic discrete series!

Introduction: modular forms on G_2

On G_2, the **quaternionic discrete series** are the most arithmetic.

π_k: QDS representation of $G_2(\mathbb{R})$ of weight $k \geq 2$.

Definition

$\varphi : G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{C}$ is a **quaternionic modular form** of weight $k \geq 2$ if φ generates the lowest K-type in π_k under $G_2(\mathbb{R})$.

Fact

The Fourier coefficients $c_A(\varphi)$ of a cuspidal QMF φ of weight k are indexed by totally real cubic rings A (and these determine φ)

- A free of rank 3 over $\mathbb{Z}$, $A \otimes \mathbb{R} = \mathbb{R}^3$ as a ring
- e.g. $A = \mathbb{Z}^3$, $A = \mathcal{O}_E$ with $E/\mathbb{Q}$ totally real cubic field

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- *: actually, only if φ has a nice level (like $\Gamma_0(N)$)

Fourier coefficients and arithmetic

The Fourier coefficients $c_A(\varphi)$ are arithmetic in nature.

Some examples:

- There exists a natural family of Eisenstein series E_{2k} of weight $2k \geq 4$, such that

$$c_A(E_{2k}) = \zeta_A(1-2k)$$

  when A is maximal. (Jiang-Rallis 97, Gan-Gross-Savin 02, Xiong 17)

- For $k \geq 6$, there is a basis of level one forms with all coefficients in $\mathbb{Q}^{\mathrm{cyc}}$ (Pollack 22)

Gross's Conjecture

Let f be a cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k.

Assuming $L(1/2, f) \neq 0$, there exists a cuspidal **Gan-Gurevich lift** φ of f to G_2

Conjecture (Gross)

For all maximal totally real cubic rings A,

$$c_A(\varphi)^2 = L(1/2, f \otimes \rho_A)\, \mathrm{disc}(A)^{\frac{k-1}{2}}$$

where ρ_A = two-dimensional Artin representation associated to A

- Kim-Yamauchi 24: true when $A = \mathbb{Z} \times \mathcal{O}_F$ for $F/\mathbb{Q}$ quadratic
- Today: Gross's conjecture when f is a CM form (always has level!)

Main result

- $f = f_\chi$ CM form of weight k, trivial character, and any level N, associated to $K/\mathbb{Q}$ and $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^1$.
- Assume: $L(1/2, \chi) \neq 0$.
- $\mathcal{A}_{GG}(f_\chi)$ space of "Gan-Gurevich lifts" $G_2(\mathbb{Q}) \backslash G_2(\mathbb{A}) \to \mathbb{C}$.

Theorem (Bakic–Horawa–Li-Huerta–S., in progress)

For all $\ell | N$, fix a cubic ring $A_\ell/\mathbb{Z}_\ell$, such that

$$\prod_{\ell | N} \epsilon_\ell(A_\ell, \chi_\ell) = -\epsilon(1/2, \chi^3)$$

Then $\exists$ a QMF $\varphi \in \mathcal{A}_{GG}(f_\chi)$ s.t. for A maximal outside N

$$|c_A(\varphi)|^2 = \begin{cases} L(1/2, f_\chi \otimes \rho_A)\, \mathrm{disc}(A)^{\frac{k-1}{2}} & A \otimes \mathbb{Z}_\ell = A_\ell \ \forall \ell | N \\ 0 & \textit{otherwise} \end{cases}$$

Plan of talk

1. Theory of Gan-Gurevich lifts, and role of epsilon factors
2. A construction of $\mathcal{A}_{GG}(f_\chi)$
3. Sketch of proof

1. Theory of Gan-Gurevich lifts

Arthur parameters

Langlands philosophy, G/F reductive group:

$$\{\text{aut. repns of } G\} \longleftrightarrow \left\{\text{``Galois representations''}\ L_{\mathbb{Q}} \to {}^L G\right\}$$

Arthur's conjecture:

$$\mathcal{A}_{\mathsf{disc}}(G) = \oplus_\psi \mathcal{A}_\psi(G)$$

where

$$\psi : L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$

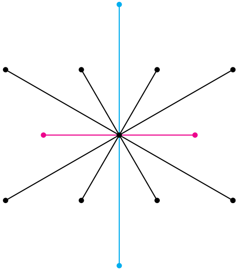
$\psi|_{L_{\mathbb{Q}}}$ is tempered, and $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic

The more nontrivial the $\mathrm{SL}_2(\mathbb{C})$, the more nontempered (and degenerate) the representation. e.g. for $G = \mathrm{GL}_2$, ${}^L G = \mathrm{GL}_2(\mathbb{C})$: $\psi|_{\mathrm{SL}_2(\mathbb{C})}$ nontrivial corresponds to the characters

Theory of Gan-Gurevich lifts: Arthur parameters

$$G = G_2,\ {}^L G = G_2(\mathbb{C})$$

$$\psi : L_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_{2,\mathrm{short}}(\mathbb{C}) \times \mathrm{SL}_{2,\mathrm{long}}(\mathbb{C}) \to G_2(\mathbb{C})$$



For any $\tau =$ cuspidal AR of PGL_2, consider ψ_τ where
$L_{\mathbb{Q}} \to \mathrm{SL}_{2,\mathrm{short}}(\mathbb{C})$ corresponds to τ.
Specialize to $\tau = \tau_\chi$ CM, $\mathcal{A}_{GG}(f_\chi) \coloneqq \mathcal{A}_{\psi_{\tau_\chi}}(G_2)$

Epsilons

Arthur has a precise conjectural description for $\mathcal{A}_\psi(G)$ in terms of **local packets** and **global multiplicities.**

Arthur's prediction for structure of global GG packet (partially known, Alonso-He-Ray-Roset 23 and BHL-HS24):

$$\mathcal{A}_{GG}(f_\chi) = \bigoplus_{(\epsilon_v)_v} m((\epsilon_v)_v) \sideset{}{'}\bigotimes_v \pi_v^{\epsilon_v}$$

where $\{\pi_v^+, \pi_v^-\}$ is a local packet depending only on χ_v, with $\pi_v^- = 0$ almost everywhere, and

$$m((\epsilon_v)_v) = \begin{cases} 1 & \prod \epsilon_v = \epsilon(1/2, \chi^3) \\ 0 & \text{else} \end{cases}$$

It is always interesting to see how arithmetic conspires to enforce the multiplicity formula. Here, it comes in via local-global obstruction for existence of Hermitian spaces.

Epsilons

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Given $\pi = \otimes'_v \pi_v^{\epsilon_v}$:

- $\varphi \in \pi$ can be a QMF only if $\epsilon_\infty = -1$.
- $\varphi \in \pi$ can have $c_A(\varphi) \neq 0$ only if $\epsilon_\ell = \epsilon_\ell(A \otimes \mathbb{Z}_\ell, \chi_\ell)$ for all $\ell | N$.

$\implies$ to see $c_A(\varphi)$, need $\prod \epsilon_\ell(A \otimes \mathbb{Z}_\ell, \chi_\ell) = -\epsilon(1/2, \chi^3)$

Main result

- $f = f_\chi$ CM form of weight k, trivial character, and any level N, associated to $K/\mathbb{Q}$ and $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^1$.
- Assume: $L(1/2, \chi) \neq 0$.
- $\mathcal{A}_{GG}(f_\chi)$ space of Gan-Gurevich lifts $G_2(\mathbb{Q}) \backslash G_2(\mathbb{A}) \to \mathbb{C}$.

Theorem (Bakic–Horawa–Li-Huerta–S., in progress)

For all $\ell | N$, fix a cubic ring $A_\ell / \mathbb{Z}_\ell$, such that

$$\prod_{\ell | N} \epsilon_\ell(A_\ell, \chi_\ell) = -\epsilon(1/2, \chi^3)$$

Then $\exists$ a QMF $\varphi \in \mathcal{A}_{GG}(f_\chi)$ s.t. for A maximal outside N:

$$c_A(\varphi) = \begin{cases} L(1/2, f_\chi \otimes \rho_A)\, \mathrm{disc}(A)^{\frac{k-1}{2}} & A \otimes \mathbb{Z}_\ell = A_\ell \ \forall \ell | N \\ 0 & \textit{otherwise} \end{cases}$$

2. construction of $\mathcal{A}_{GG}(f_\chi)$

Exceptional theta correspondences

- A theta correspondence is a construction

$$\Theta : \{\text{ARs of G}\} \rightsquigarrow \{\text{ARs of H}\}$$

  for "dual reductive pairs" H, G.
- Means can embed as mutual centralizers $H \times G \hookrightarrow \widetilde{G}$, where $\widetilde{G}$ is a group with theta functions (used as a kernel).

$H \times G$	$\widetilde{G}$
$\mathrm{SO}_n \times \mathrm{Sp}_{2m}$	Sp_{2mn}
$U(n) \times U(m)$	Sp_{2mn}

$H \times G$	$\widetilde{G}$
$G_2 \times \mathrm{PGL}_3$	E_6
$G_2 \times \mathrm{PGSp}_6$	E_7
$G_2 \times F_4$	E_8

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$H \times G$	$\widetilde{G}$
$G_2 \times \mathrm{PU}_3$	E_6'
$G_2 \times \mathrm{PGSp}_6$	E_7
$G_2 \times F_4$	E_8

Exceptional theta correspondences and Gan-Gurevich lifts

Gan-Gurevich:

$$\begin{tikzcd}[row sep=small] \sigma & & \\ \mathrm{PGL}_2 \arrow[r, rightsquigarrow] & \mathrm{PGSp}_6 \arrow[dr, "\Theta"] & \\ & & G_2 \end{tikzcd}$$

Exceptional theta correspondences and Gan-Gurevich lifts

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Alternative approach when $\sigma =$ CM form associated to χ, cf. [BHL-HS24]

Exceptional theta correspondences and Gan-Gurevich lifts

Gan-Gurevich:

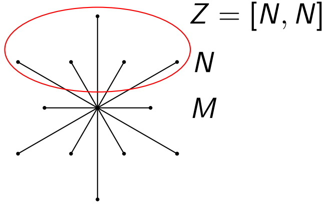
$$
\begin{tikzcd}
\sigma & & \\
\mathrm{PGL}_2 \arrow[r, rightsquigarrow] & \mathrm{PGSp}_6 \arrow[dr, "\Theta"] & \\
& & G_2 \\
U(1) \arrow[r, "\Theta_\chi"'] & \mathrm{PU}_3 \arrow[ur, "\Theta"'] & \\
\mathbb{1} & &
\end{tikzcd}
$$

Alternative approach when $\sigma =$ CM form associated to χ, cf. [BHL-HS24]

Easier to understand packet structure (comes from Howe–Piatetskii-Shapiro CAP forms on PU_3)

3. sketch of proof

What are Fourier coefficients?



$$\{\text{characters of } N(\mathbb{Q})\backslash N(\mathbb{A})\} \longleftrightarrow \{\lambda = (a, b, c, d) \in \mathbb{Q}^4\}$$
$$\psi_\lambda \longleftrightarrow \lambda$$

$$\varphi : G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{C}$$
$$c_\lambda(\varphi) :\doteq \int_{[N]} \psi_\lambda^{-1}(n)\varphi(n)dn$$

What are Fourier coefficients?

$\varphi : G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{C}$ modular form of weight $k \geq 2$

$$c_\lambda(\varphi) :\doteq \int_{[N]} \psi_\lambda^{-1}(n)\varphi(n)dn$$

- φ invariant by $P(\widehat{\mathbb{Z}}) \implies c_\lambda(\varphi)$ is supported on $\lambda \in \mathbb{Z}^4$ and depends only on the **cubic ring** A_λ
  - $A_\lambda = \mathbb{Z}^3$ with multiplication determined by $\lambda = (a, b, c, d) \in \mathbb{Z}^4$
- More generally have a cubic algebra $E_\lambda/\mathbb{Q}$
- Turns out $c_\lambda(\varphi) = 0$ unless E_λ is totally real

Proof sketch

Periods for $\Theta(\pi)$ often reduce to periods for π.

For $\varphi \in \mathcal{A}_{GG}(f_\chi)$ (any level),

$$c_\lambda(\varphi) \sim \int_{[T_E]} \rho(t) dt$$

where:

- $E = E_\lambda$ is totally real cubic étale algebra corresponding to λ.
- $T_E \hookrightarrow \mathrm{PU}_3$ is a torus embedding coming from $E \hookrightarrow \mathrm{Herm}_{3\times 3}(\mathbb{Q})$.
- $\rho =$ thing you're lifting on PU_3 (Howe-PS CAP forms).

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Periods for $\Theta(\pi)$ often reduce to periods for π.

For $\varphi \in \mathcal{A}_{GG}(f_\chi)$ (any level),

$$c_\lambda(\varphi) \sim \int_{[T_E]} \rho(t)dt$$

ρ is itself a Θ lift from $U(1)$

$$\left| \int_{[T_E]} \rho(t)dt \right|^2 \sim L(1/2, f_\chi \otimes \rho_E) L(1/2, \chi) \Delta_E^{1/2}$$

cf. Yang 97, Borade-Franzel-Girsch-Yao-Yu-Zelingher 24

Proof sketch

$$|c_\lambda(\varphi)|^2 = L(1/2, f_\chi \otimes \rho_E) L(1/2, \chi) \Delta_E^{1/2} \prod_v |I_v(\varphi_v, \lambda)|^2$$

- Additional $\Delta_E^{k/2-1}$ comes from evaluating I_∞ (using Pollack's explicit Whittaker model of minimal representation of E_6)
- $I_\ell(\varphi_\ell, \lambda)$ is hard to compute in general
  - For $\ell \nmid N$, we show $I_\ell(\varphi_\ell^{\mathrm{sph}}, \lambda) = 1$ if $A_\lambda \otimes \mathbb{Z}_\ell$ is a maximal order in a cubic étale algebra
  - For $\ell | N$, we show you can "rig" φ_ℓ so $I_\ell(\varphi_\ell, \lambda)$ is the indicator function of $A_\lambda \otimes \mathbb{Z}_\ell =$ any fixed A_ℓ

Thanks!