Explicit construction of Ramanujan bigraphs

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on joint work with
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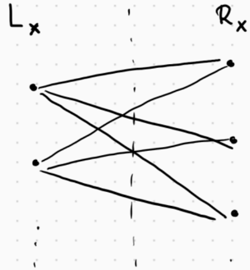
Biregular bigraphs

Graph $X = (L_X \dot{\cup} R_X \, , \, E_X)$ satisfying



bipartite : edges only between L_X- and R_X-vertices

biregular : # edges from each $\left\{ \begin{matrix} \text{left} \\ \text{right} \end{matrix} \right\}$ vertice the same

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We consider undirected graphs $X = (V_X, E_X)$ which are

- bipartite: $V_X = L_X \sqcup R_X$ such that if $(v_1, v_2) = (v_2, v_1) \in E_X$, then either ($v_1 \in L_X$ and $v_2 \in R_X$) or ($v_1 \in R_X$ and $v_2 \in L_X$).
- $(q_L + 1, q_R + 1)$-biregular: $\forall v \in L_X$: $\#\{(v, v_2) \in E_X\} = q_L + 1$ and $\forall v \in R_X$: $\#\{(v_1, v) \in E_X\} = q_R + 1$.

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- satisfying $q_L \geq q_R$, so $n_L = \#L_X \leq n_R = \#R_X$.

The adjacency matrix has got shape $\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$ with $A \in M_{n_L, n_R}(\{0, 1\})$.

Trivial eigenvalues: $\pm\sqrt{(q_L + 1)(q_R + 1)}$, 0 with multiplicity $(n_R - n_L)$.

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Non-trival eigenvalues: $\pm\lambda_2, \ldots, \pm\lambda_n$, with $n \leq n_L$.

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$$Z_X(u)^{-1} = (1-u)^{\#E_X - n_L - n_R}(1+q_R u)^{n_L - n_R}\prod_{j=1}^{n}\left(1-(\lambda_j^2 - q_L - q_R)u + q_L q_R u^2\right),$$

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Definition A (q_L+1, q_R+1)-biregular graph is a *Ramanujan bigraph* if it satisfies the Riemann hypothesis.

Alternative characterization

Proposition A finite, connected $(q_L + 1, q_R + 1)$-bigregular bipartite graph is a Ramanujan bigraph, iff for all non-trival eigenvalues λ

$$|\lambda^2 - q_L - q_R| \leq 2\sqrt{q_L q_R}.$$

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- Interlacing families of Ramanujan bigraphs exist for all $q_L, q_R \geq 2$ [Marcus, Spielman, Srivastava '14].

Bruhat-Tits trees for unitary groups

Let $E_p/\mathbb{Q}_p$ be an unramified quadratic field extension, $\Phi = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$, $U_3 = U_3(E_p/\mathbb{Q}_p, \Phi)$ unitary group defined over $\mathbb{Z}_p$.

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$$K_L = U_3(\mathbb{Z}_p) \text{ and } K_R = U_3(\mathbb{Q}_p) \cap \begin{pmatrix} \mathfrak{o}_{E_p} & \mathfrak{o}_{E_p} & p^{-1}\mathfrak{o}_{E_p} \\ \mathfrak{o}_{E_p} & \mathfrak{o}_{E_p} & p^{-1}\mathfrak{o}_{E_p} \\ p\mathfrak{o}_{E_p} & p\mathfrak{o}_{E_p} & \mathfrak{o}_{E_p} \end{pmatrix}.$$

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The graph $\mathcal{T} = \mathcal{T}_p$ is the Bruhat-Tits building (tree) of $U_3(\mathbb{Q}_p)$. It is $(p^3+1, p+1)$-biregular.

Ramanujan bigraphs from Bruhat-Tits trees

Theorem [Ballantine, Ciubotaru '11] Let Γ be a discrete, cocompact subgroup of $U_3(\mathbb{Q}_p)$ acting on $\mathcal{T}_p$ without fixed points. Then the quotient $\mathcal{T}_p/\Gamma$ is a Ramanujan bigraph **iff** $L^2(U_3(\mathbb{Q}_p)/\Gamma)^I$ satisfies the *Ramanujan hypothesis*,

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For $U_3(\mathbb{Q}_p)$, the inspection of [Rogawski 1990]'s A-packets provides a tool to prove/disprove temperedness.

Ramanujan bigraphs from Bruhat-Tits trees

Theorem [BC'11] Let $E/\mathbb{Q}$ be an imaginary quadratic extension.
Let G be the unitary group associated to a division algebra D of degree three over its center E with involution of the second kind (i.e. reducing to the Galois-automorphism on E). Assume $G(\mathbb{R})$ is compact.
Let p be a prime inert in E, and $(n, p) = 1$. Let

$$\Gamma_p(n) = ker\big(G(\mathbb{Z}[p^{-1}]) \to G(\mathbb{Z}[p^{-1}]/n\mathbb{Z}[p^{-1}])\big) \subset G(\mathbb{Q}).$$

Then, the quotient $\mathcal{T}_p/\Gamma_p(n)$ is a Ramanujan bigraph.

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Let $K = \otimes_v K_v \subset G(\mathbb{A})$ be a compact open subgroup with $K_\infty = G_\infty$, $K_p \leq K_{L,p} = G(\mathbb{Z}_p)$ of finite index and $K_p = K_{L,p}$ almost everywhere. Let $Ram(K) = \{p \mid K_p \neq K_{L,p}\}$.

Definition K is said to satisfy the *Ramanujan property* if every irreducible subrepresentations of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^K$ is tempered at all local places.

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Theorem [BEFMP'20] The Ramanujan property is satisfied by K when one of the following holds.

(1) [EP'18/20] There exists a prime $p \in Ram(E)$ such that $p \notin Ram(K)$.
(2) $K' \subset K$ for a compact open subgroup K' satisfying the Ramanujan property.
(3) $K \subset K'$ for a compact open subgroup K' satisfying the Ramanujan property, and for all p such that $K_p \neq K'_p$, the group K_p is a parahoric subgroup.

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Theorem A [BEFMP'20] K satisfies the Ramanujan property in each of the following cases.

(1) $3 \notin Ram(K)$.
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Theorem B [BEFMP'20] K **doesn't** satisfy the Ramanujan property in each of the following cases.

(1) $Ram(K) = \{3\}$, $K_3 \subset K_3(3)$ (principal congruence subgroup).
(2) $Ram(K) = \{3, q\}$, $K_3 = K_3(H)$ and $K_q = K_q(q)$.
(3) $3 \in Ram(K)$, $K_3 = K_3(H)$, for some $3 \neq q \in Ram(K)$ it holds $K_q \subset K_q(q)$.

Recall: Cayley graphs

For a group G with set of generators S such that $k = \#S < \infty$, $S = S^{-1}$, $e \notin S$, define the **Cayley graph** $X(G, S) = (V_X, E_X)$ by

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- G acts on $X(G, S)$ by multiplication from the left.
- (Regular) Ramanujan graphs arising as quotients of SL_2-trees admit an explicit description by Cayley graphs [Lubotzky, Philips, Sarnak, Montenegro; Cartwright, Steger, Mantero, Zappa]

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Definition In a group G let $S^1, \dots S^L$ be pairwise disjoint finite subsets of equal size. The pair $(G, \sqcup_{j=1}^{L} S^j)$ satisfies the **bi-Cayley axioms** if:

(1) $S = \sqcup_{j=1}^{L} S^j$ is a generating set for G, $S = S^{-1}$, and $e \notin S$.

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(1) Define the following equivalence relation on $G \times \{1, \dots L\}$:

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(2) The associated **bi-Cayley graph** $X = BCay(G, \sqcup_{j=1}^{L} S^j)$ is the $(L, \#[g, i])$-biregular bigraph with vertices $L_X = G$, $R_X = G \times \{1, \dots, L\}/\sim$ and edges $E_X = \{(g, [g, i]) \in L_X \times R_X \mid g \in G,\ i \in \{1, \dots, L\}\}$.

Nice example

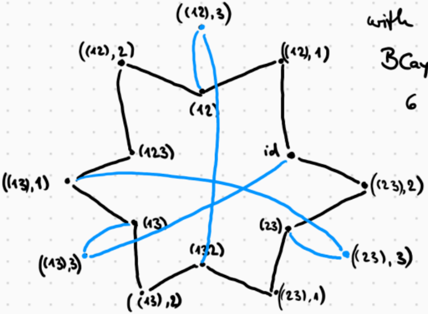
$G = S_3$

: $S = S^1 \cup S^2$

with $S^1 = \{(12)\}$, $S^2 = \{(23)\}$

$BCay(G,S) = (2,2)$-biregular, 6 left, 6 right $= 12$-Cycle

[whereas $Cay(S_3, S) = 6$-Cycle]



$\cup$:

$\tilde{S} = S^1 \cup S^2 \cup S^3$,

with $S^1 = \{(12)\}$, $S^2 = \{(23)\}$, $S^3 = \{(13)\}$

$BCay(G, \tilde{S})$: $(3,2)$-biregular, 6 left, 9 right

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- Let $H_p = \Lambda_p \pmod{3}$.
  In particular for $p = 2$, denote for $j = 0, \ldots, 5$,

$$A_j = -\frac{1}{2}\begin{pmatrix} 1 & 0 & \sqrt{-3}\zeta_6^j \\ 0 & -2 & 0 \\ \sqrt{-3}\zeta_6^{-j} & 0 & 1 \end{pmatrix}, \quad B_j = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{-3}\zeta_6^j & 0 \\ \sqrt{-3}\zeta_6^{-j} & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$C_j = -\frac{1}{2}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & \sqrt{-3}\zeta_6^j \\ 0 & \sqrt{-3}\zeta_6^{-j} & 1 \end{pmatrix}.$$

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- Then $(H_2, \sqcup S^j)$ satisfies the bi-Cayley axioms.

Instructive example

- Let $E = \mathbb{Q}(\sqrt{-3})$ and $G = U_3(E/\mathbb{Q}, \mathbf{1}_3)$.
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- Works for each inert prime p like this.

Trees as bi-Cayley graphs

Proposition [BEFMP'20] For a $(q_L + 1, q_R + 1)$-biregular tree $\mathcal{T}$, $q_L \geq q_R$, let Λ be a group acting isometrically on $\mathcal{T}$, s. th. the action is **simply transitive** on $L_{\mathcal{T}}$.

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Fix $v_0 \in L_{\mathcal{T}}$, let $v_1, \ldots, v_{q_L+1} \in R_{\mathcal{T}}$ be its neighbors, define

$$S^i := \{g \in \Lambda \mid g \neq 1, dist(gv_0, v_i) = 1\}, \quad i = 1, \ldots, q_L + 1.$$

Then $(\Lambda, \sqcup S^i)$ satisfies the bi-Cayley axioms, and the bi-Cayley graph $BCay(\Lambda, \sqcup S^i)$ is isomorphic to the tree $\mathcal{T}$.

(Simply) transitive actions

Lemma For a definite unitary group $G = U_3(E/\mathbb{Q}, \Phi)$ as before the following are equivalent.

(a) $G(\mathbb{A}) = G(\mathbb{Q}) \cdot G(\mathbb{R}) \cdot \prod_p G(\mathbb{Z}_p)$ (class number one)

(b) For each inert prime p,

$$G(\mathbb{Q}_p) = G(\mathbb{Z}[p^{-1}]) \cdot G(\mathbb{Z}_p).$$

(c) For every inert prime p the action of $G(\mathbb{Z}[p^{-1}])$ is transitive on the left vertices $L_{\mathcal{T}_p}$ of the BT-tree.

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The stabilizer of $K_{L,p} = G(\mathbb{Z}_p)$ in $G(\mathbb{Z}[p^{-1}])$ is
$$G(\mathbb{Z}) = G(\mathbb{Z}[p^{-1}]) \cap G(\mathbb{Z}_p).$$

Any subgroup Λ_p such that $G(\mathbb{Z}[p^{-1}]) = \Lambda_p \rtimes G(\mathbb{Z})$ then acts simply transitively.

Instructive example revisited

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- So Λ_p acts simply transitively on the left vertices $L_{\mathcal{T}_p}$ of the BT-tree.
- From the exact sequence $\pmod 3$

$$1 \to \Gamma_p \to \Lambda_p \to H_p \to 1$$

we find an explicit description of the quotient $\mathcal{T}_p/\Gamma_p$ by the bi-Cayley graph $BCay(H_p, \sqcup S^j)$.

Remarks

- This example isn't a Ramanujan bigraph, but it fits onto slides.
- We have more than this one example for realizing quotients of BT-trees for U_3 by bi-Cayley graphs.
- Nevertheless, these examples are rare: First, class number one for unitary groups is exeptional. Second, the definition of H_p (i.e a congruence condition) is not obvious, but rather by chance.

Thank you!