Eisenstein series, cotangent-zeta sums, knots, and quantum modular forms

Amanda Folsom (Amherst College)

Quantum modular forms

Let

$f : \mathbb{H} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathsf{SL}_2(\mathbb{Z}),\ \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathsf{Im}(\tau) > 0\}$

Quantum modular forms

Let

$f : \mathbb{H} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}),\ \tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$

Modular transformation:

$$f(\tau) - \epsilon^{-1}(\gamma)(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = 0$$

Quantum modular forms

Let $f : \mathbb{Q} \to \mathbb{C}, \quad \gamma := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}), \quad x \in \mathbb{Q}.$

Modular transformation:

$$f(x) - \epsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right) = ?$$

Quantum modular forms

Definition (Zagier '10)

A **quantum modular form of weight** k $(k \in \frac{1}{2}\mathbb{Z})$ is function $f : \mathbb{Q} \backslash S \to \mathbb{C}$, such that

Quantum modular forms

Definition (Zagier '10)

A **quantum modular form of weight** k ($k \in \frac{1}{2}\mathbb{Z}$) is function $f : \mathbb{Q} \backslash S \to \mathbb{C}$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the functions

Quantum modular forms

Definition (Zagier '10)

A **quantum modular form of weight** k ($k \in \frac{1}{2}\mathbb{Z}$) is function $f : \mathbb{Q}\backslash S \to \mathbb{C}$, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the functions

$$h_\gamma(x) = h_{f,\gamma}(x) := f(x) - \epsilon^{-1}(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

extend to suitably continuous or analytic functions in $\mathbb{R}$.

Quantum modular forms

Example 1.

Quantum modular forms

Example 1. For $b \in \mathbb{N}, a \in \mathbb{Z}$, $\gcd(a, b) = 1$, the **Dedekind sum**

Quantum modular forms

Example 1. For $b \in \mathbb{N}, a \in \mathbb{Z}$, $\gcd(a, b) = 1$, the **Dedekind sum**

$$s(a, b) := \sum_{n=1}^{b-1} \left(\left(\frac{n}{b}\right)\right) \left(\left(\frac{na}{b}\right)\right),$$

Quantum modular forms

Example 1. For $b \in \mathbb{N}, a \in \mathbb{Z}$, $\gcd(a, b) = 1$, the **Dedekind sum**

$$s(a, b) := \sum_{n=1}^{b-1} \left(\!\!\left(\frac{n}{b}\right)\!\!\right) \left(\!\!\left(\frac{na}{b}\right)\!\!\right),$$

where $(\!(x)\!) := x - \lfloor x \rfloor - \frac{1}{2}, \quad x \notin \mathbb{Z}.$

Quantum modular forms

Example 1. For $b \in \mathbb{N}, a \in \mathbb{Z}$, $\gcd(a, b) = 1$, the **Dedekind sum**

$$s(a, b) := \sum_{n=1}^{b-1} \left(\left(\frac{n}{b}\right)\right) \left(\left(\frac{na}{b}\right)\right),$$

where $((x)) := x - \lfloor x \rfloor - \frac{1}{2}, \quad x \notin \mathbb{Z}.$

Remark. $s(a, b)$ appears in the modular transformation for $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) \ \ (q = e^{2\pi i \tau}, \tau \in \mathbb{H}).$

Quantum modular forms

Example 1 (cont.) Define $S : \mathbb{Q} \to \mathbb{Q}$ by $S\left(\frac{a}{b}\right) := 12s(a, b)$.

Quantum modular forms

Example 1 (cont.) Define $S : \mathbb{Q} \to \mathbb{Q}$ by $S\left(\frac{a}{b}\right) := 12s(a, b)$.

Then

$$\begin{aligned} S(x) - S(x+1) &= 0, \\ S(x) + S\left(1/x\right) &= x + \frac{1}{x} \pm 3 + \frac{1}{\mathrm{Num}(x)\mathrm{Den}(x)}. \end{aligned}$$

Quantum modular forms

Example 1 (cont.) Define $S:\mathbb{Q}\to\mathbb{Q}$ by $S\left(\frac{a}{b}\right):=12s(a,b)$.

Then

$$\begin{aligned} S(x)-S(x+1)&=0,\\ S(x)+S\left(1/x\right)&=x+\frac{1}{x}\pm 3+\frac{1}{\mathrm{Num}(x)\mathrm{Den}(x)}. \end{aligned}$$

S is an **imperfect** quantum modular form

Quantum modular forms

Example 2. Kontsevich's function:

$$\begin{aligned} F(q) &:= \sum_{n=0}^{\infty} (q;q)_n \\ &= \big(1 + (1-q) + (1-q)(1-q^2) + \cdots\big) \end{aligned}$$

Quantum modular forms

Example 2. Kontsevich's function:

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n$$
$$= \big(1 + (1-q) + (1-q)(1-q^2) + \cdots\big)$$

For $n \in \mathbb{N}_0$, $(\alpha;q)_n := (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{n-1})$.

Quantum modular forms

Example 2. Kontsevich's function:

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n$$
$$= \big(1 + (1-q) + (1-q)(1-q^2) + \cdots\big)$$

For $n \in \mathbb{N}_0$, $(\alpha; q)_n := (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{n-1})$.

The function $F(q)$ converges only at roots of unity, $q = \zeta_k^h$.
$(\zeta_N := e^{2\pi i/N})$

Quantum modular forms

Example 2 (cont.) Let $\mathcal{F}(x) := e^{\pi i x/12} F(e^{2\pi i x}), \quad x \in \mathbb{Q}$.

Example 2 (cont.) Let $\mathcal{F}(x) := e^{\pi ix/12}F(e^{2\pi ix}), \quad x \in \mathbb{Q}$.

Theorem (Zagier)

The function $\mathcal{F}$ is a quantum modular form of weight $3/2$,

Example 2 (cont.) Let $\mathcal{F}(x) := e^{\pi i x/12} F(e^{2\pi i x}), \quad x \in \mathbb{Q}$.

Theorem (Zagier)

The function $\mathcal{F}$ is a quantum modular form of weight $3/2$, i.e.

$$\mathcal{F}(x) - \zeta_{24}^{-1}\mathcal{F}(x+1) = 0, \quad \mathcal{F}(x) \ \mp \ \zeta_8 |x|^{-\frac{3}{2}} \ \mathcal{F}(-1/x) = h(x),$$

where h is a real analytic function (except at 0).

Quantum modular forms

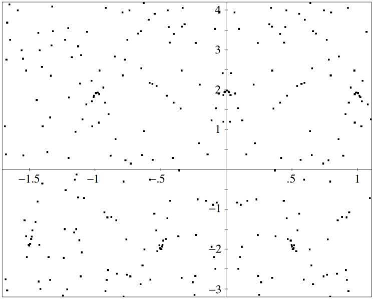

Figure 1. Graph of $\Re(f(x))$

Image credit: D. Zagier

Quantum modular forms

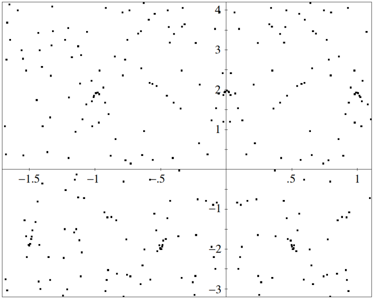

Figure 1. Graph of $\Re(f(x))$

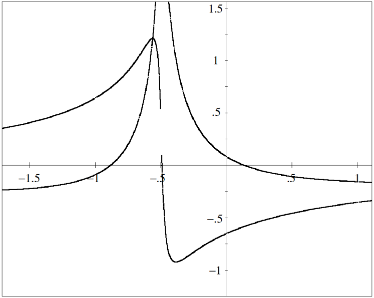

Figure 2. Graph of $\Re(h(x))$ and $\Im(h(x))$

Image credit: D. Zagier

Quantum modular forms

Objectives. Extensions of and frameworks for Examples 1 and 2.

Cotangent-zeta sums

Example 1 revisited. We have that

$$
\begin{aligned}
s(a,b) &:= \sum_{n=1}^{b-1} \left(\!\!\left(\frac{n}{b}\right)\!\!\right) \left(\!\!\left(\frac{na}{b}\right)\!\!\right) \\
&= -\frac{1}{4b} \sum_{n=1}^{b-1} \cot\left(\frac{\pi n}{b}\right) \cot\left(\frac{\pi na}{b}\right),
\end{aligned}
$$

Cotangent-zeta sums

Fix $0 < h < k$ with $\gcd(h,k) = 1$. Let $a, b \in \mathbb{N}$, $\gcd(a,b) = 1$, $a \not\equiv b \pmod{k}$.

We define the cotangent-zeta sums

$$\mathfrak{c}_s(a,b) = \mathfrak{c}_s(h,k;a,b)$$
$$:= b^{s-1} \sum_{\ell=0}^{b-1} \cot\left(\pi\left(-\frac{h}{k} + \frac{a}{b}\left(\frac{h}{k} + \ell\right)\right)\right) \zeta\left(1 - s; 1 - \frac{h + \ell k}{bk}\right),$$

where $\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$ $(\mathrm{Re}(s) > 1, \mathrm{Re}(x) > 0)$ is the Hurwitz ζ-function.

Cotangent-zeta sums

Theorem 1 (F, '20)

Fix (h,k) as above. Let $x=a/b$ as above, write $\mathfrak{c}_s(x)=\mathfrak{c}_s\left(h,k;\frac{a}{b}\right)$. The cotangent-zeta sums satisfy

$$\begin{aligned}
&\mathfrak{c}_s\left(x\right)+x^{-s}\mathfrak{c}_s\left(1/x\right)\\
&\quad=\frac{ie^{\frac{\pi is}{2}}}{(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s)}\psi_s\left(x\right)\\
&\quad-\frac{e^{\frac{\pi is}{2}}}{\sin(\pi s)}\left(1+x^{-s}e^{-\pi is}\right)\cos\left(\tfrac{\pi s}{2}\right)\left(\zeta\left(1-s;\tfrac{h}{k}\right)-\zeta\left(1-s;1-\tfrac{h}{k}\right)\right).
\end{aligned}$$

The right-hand side extends to a holomorphic function in $\mathbb{C}\backslash\mathbb{R}^{\leqslant 0}$.

Cotangent-zeta sums

Remarks.

- The cotangent-zeta sums $\mathfrak{c}_s$ are related to what Zagier calls holomorphic quantum modular forms.

Cotangent-zeta sums

Remarks.

- The cotangent-zeta sums $\mathfrak{c}_s$ are related to what Zagier calls holomorphic quantum modular forms.

  e.g., error to modularity is holomorphic on the large domain $\mathbb{C} \backslash \mathbb{R}^{\leqslant 0}$.

Cotangent-zeta sums

Remarks.

- The cotangent-zeta sums $\mathfrak{c}_s$ are related to what Zagier calls holomorphic quantum modular forms.

  e.g., error to modularity is holomorphic on the large domain $\mathbb{C}\backslash\mathbb{R}^{\leqslant 0}$.

- They transform with "weight" $s \in \mathbb{C}$.

Cotangent-zeta sums

Remark. The case $(h, k) = (0, 1)$ was originally studied by Bettin-Conrey:

$$\mathfrak{c}_0(0, 1, a, b) = -2\pi s(a, b),$$

$$\mathfrak{c}_1(0, 1, a, b) \quad \text{emerges in the Nyman-Beurling approach to RH.}$$

Cotangent-zeta sums

Remark. The case $(h,k)=(0,1)$ was originally studied by Bettin-Conrey:

$$\mathfrak{c}_0(0,1,a,b) = -2\pi s(a,b),$$
$$\mathfrak{c}_1(0,1,a,b) \ \text{ emerges in the Nyman-Beurling approach to RH.}$$

Bettin-Conrey establish quantum modularity, and and new proof of a Vasyunin sum formula appearing in the N-B RH criterion.

Cotangent-zeta sums

Example 1.

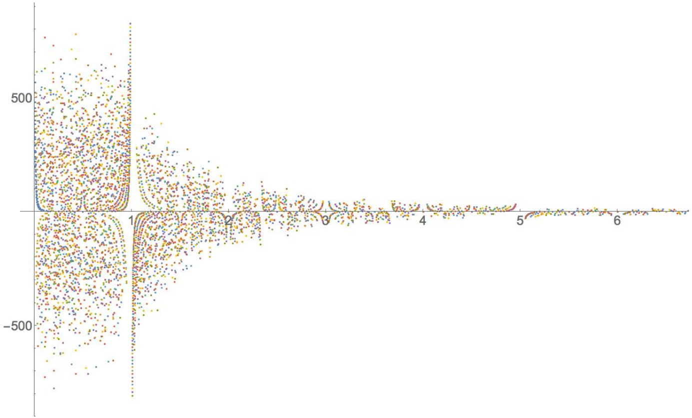

Figure: The cotangent-zeta function $\mathfrak{c}_2\left(1, 4; \frac{a}{b}\right)$.

Example 1 (cont.)

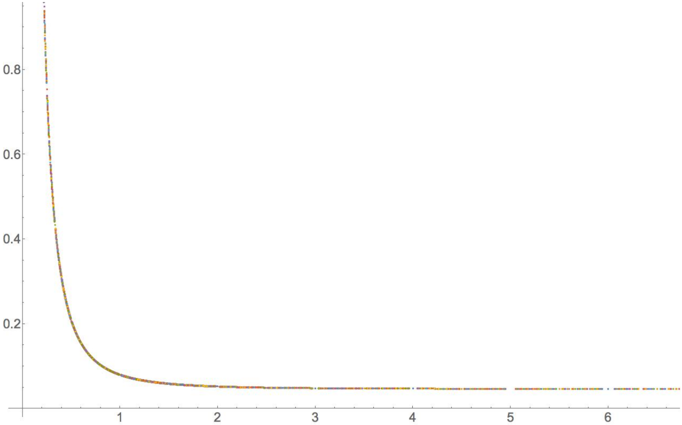

Figure: The quantum modular error $\mathfrak{c}_2\left(1, 4; \frac{a}{b}\right) + \left(\frac{b}{a}\right)^2 \mathfrak{c}_2\left(1, 4; \frac{b}{a}\right)$.

Example 2.

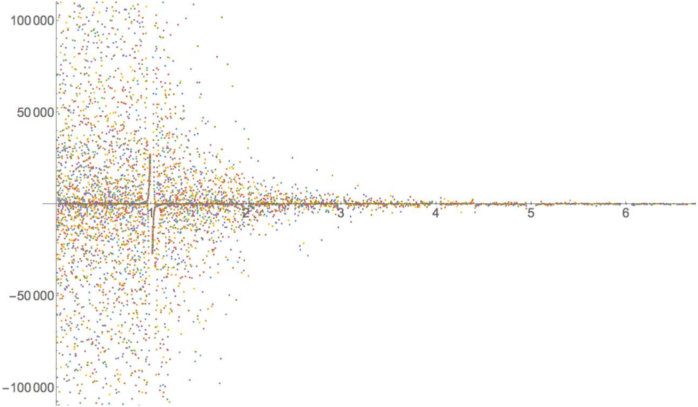

Figure: The real part of a cotangent-zeta function, $\Re\left(\mathfrak{c}_{3.3+1.2i}\left(5, 17; \frac{a}{b}\right)\right)$.

Example 2 (cont.)

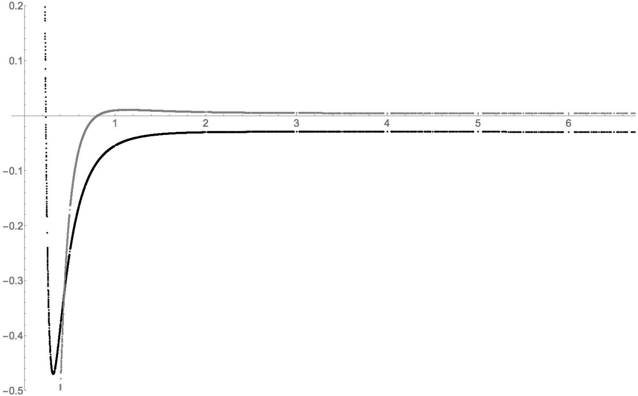

Figure: Real and imaginary quantum modular errors,
$\Re\left(\mathfrak{c}_{3.3+1.2i}\left(5,17;\frac{a}{b}\right)+\left(\frac{b}{a}\right)^{3.3+1.2i}\mathfrak{c}_{3.3+1.2i}\left(5,17;\frac{b}{a}\right)\right)$
$\Im\left(\mathfrak{c}_{3.3+1.2i}\left(5,17;\frac{a}{b}\right)+\left(\frac{b}{a}\right)^{3.3+1.2i}\mathfrak{c}_{3.3+1.2i}\left(5,17;\frac{b}{a}\right)\right)$.

Eisenstein series

Fix (h, k). For $s \in \mathbb{C}$ we define the divisor functions (on $n \in \mathbb{N}$) by

$$\sigma_s^{\pm}(h, k; n) := \sum_{\substack{dd'=n, d>0 \\ d \equiv -h \pmod{k}}} d^s \zeta_k^{\pm hd'},$$

where $\zeta_N := e^{2\pi i/N}$.

Note. $\sigma_s^{\pm}(0, 1; n) = \sigma_s(n)$.

Eisenstein series

We define

$$\mathscr{S}_s^{\pm}(h,k;\tau) := \sum_{n=1}^{\infty} \sigma_{s-1}^{\pm}(h,k;n) q^{\frac{n}{k}}$$

where $q = e^{2\pi i \tau}$, $\tau \in \mathbb{H}$, and define

$$E_s^{\pm}(h,k;\tau) := c_s \mathscr{S}_s^{\pm}(h,k;\tau) + d_s^{\pm},$$

where

$$c_s := \frac{(-2\pi i)^s}{\Gamma(s) k^{s-1}}, \qquad d_s^{\pm} := e^{-\frac{\pi i s}{2}} \cos\left(\frac{\pi s}{2}\right) Li_s(\zeta_k^{\mp h}).$$

Eisenstein series

The polylogarithm

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

Eisenstein series

The polylogarithm

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

- analytic in z, where $|z| < 1$, for any fixed $s \in \mathbb{C}$
- series also converges when $|z| = 1$ when $\Re(s) > 1$
- for fixed $s \in \mathbb{C}$, $|z| \geqslant 1$, defined by analytic continuation
- $Li_s(1) = \zeta(s)$

Eisenstein series

In the case $(h,k)=(0,1)$ we have

$$E_s^{\pm}(0,1;\tau)(d_s)^{-1} = 1 + \frac{2}{\zeta(1-s)} \sum_{n=1}^{\infty} \sigma_{s-1}(n) q^n.$$

Eisenstein series

In the case $(h, k) = (0, 1)$ we have

$$E_s^{\pm}(0,1;\tau)(d_s)^{-1} = 1 + \frac{2}{\zeta(1-s)} \sum_{n=1}^{\infty} \sigma_{s-1}(n) q^n.$$

and if $s \in 2\mathbb{N}$, $s \geqslant 4$, this is the usual modular Eisenstein series (of even positive weight $s \geqslant 4$).

Eisenstein series

Towards the proof of Theorem 1, we define the period function

$$\psi_s(h,k;\tau) := E_s^+(h,k;\tau) - \tau^{-s} E_s^-(h,k;-1/\tau).$$

- Lewis-Zagier defined spaces of period functions ψ for Maass cusp forms and real analytic Eisenstein series.
- Three-term relations: $\psi(\tau) = \psi(\tau+1) + (\tau+1)^{-2s}\psi\left(\frac{\tau}{\tau+1}\right)$.
- Analytic continuation (from $\mathbb{H}$) to $\mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$.

Eisenstein series

Theorem 2 (F '20)

The period function $\psi_s(h,k;\tau)$ satsifies

$$\psi_s(h,k;\tau) = ie^{-\frac{\pi i s}{2}} \sin\left(\frac{\pi s}{2}\right) Li_s(\zeta_k^h)\tau^{-s} + g_s(h,k;M) + \frac{(-2\pi i)^{s-1}}{2i\Gamma(s)} \int_{(-\frac{1}{2}-2M)} \frac{\Gamma(w) Z_s(h,k;w)}{\sin(\pi(w-s))(2\pi z)^w} dw,$$

where

$$g_s(h,k;M) := e^{\frac{-\pi i s}{2}} \cos\left(\frac{\pi s}{2}\right) Li_s(\zeta_k^{-h}) + \frac{i(-2\pi i)^s}{2\Gamma(s)} \zeta_{2k}^h \csc\left(\frac{\pi h}{k}\right) \zeta\left(1-s; 1-\frac{h}{k}\right) + \frac{(-2\pi i)^s}{\Gamma(s)} \sum_{n=1}^{2M} \frac{(2\pi i z)^n \zeta\left(1-n-s; 1-\frac{h}{k}\right)}{n!(1-\zeta_k^h)^{n+1}} \sum_{\nu=0}^{n-1} \left\langle {n \atop \nu} \right\rangle \zeta_k^{h(n-\nu)}$$

Eisenstein series

Theorem 2 (F '20 cont.)

and $Z_s(h,k;w) :=$

$$\sum_{\pm} e^{\pm\frac{\pi i s}{2}} Li_w(\zeta_k^{\pm h})\left(\zeta\left(1+w-s;\tfrac{h}{k}\right)e^{\pm\frac{\pi i(w-s)}{2}} - \zeta\left(1+w-s;1-\tfrac{h}{k}\right)e^{\mp\frac{\pi i(w-s)}{2}}\right).$$

Eisenstein series

Theorem 2 (F '20 cont.)

and $Z_s(h,k;w) :=$

$$\sum_{\pm} e^{\pm\frac{\pi i s}{2}} Li_w(\zeta_k^{\pm h})\Big(\zeta\big(1+w-s;\tfrac{h}{k}\big)e^{\pm\frac{\pi i(w-s)}{2}} - \zeta\left(1+w-s;1-\tfrac{h}{k}\right)e^{\mp\frac{\pi i(w-s)}{2}}\Big).$$

Further, $\psi_s(h,k;\tau)$ *extends to an analytic function on* $\mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$.

Eisenstein series

Theorem 2 (F '20 cont.)

and $Z_s(h,k;w) :=$

$$\sum_{\pm} e^{\pm\frac{\pi i s}{2}} Li_w(\zeta_k^{\pm h})\Big(\zeta\big(1+w-s;\tfrac{h}{k}\big)e^{\pm\frac{\pi i(w-s)}{2}} - \zeta\left(1+w-s;1-\tfrac{h}{k}\right)e^{\mp\frac{\pi i(w-s)}{2}}\Big).$$

Further, $\psi_s(h,k;\tau)$ *extends to an analytic function on* $\mathbb{C}\backslash\mathbb{R}_{\leqslant 0}$.

Finally, we have the three-term period relation

$$\widetilde{\psi}_s^{\pm}(h,k;\tau) - \widetilde{\psi}_s^{\pm}(h,k;\tau+1) = (\tau+1)^{-s}\widetilde{\psi}_s^{-}\left(h,k;\tfrac{\tau}{\tau+1}\right).$$

Eisenstein series

Ingredients of proof of Thm 2, extending the case $(h, k) = (0, 1)$ by B-C:

- re-write $E_s(h, k; \tau)$ and $E_s(h, k; -1/\tau)$ as integrals

Eisenstein series

Ingredients of proof of Thm 2, extending the case $(h,k) = (0,1)$ by B-C:

- re-write $E_s(h,k;\tau)$ and $E_s(h,k;-1/\tau)$ as integrals
- shift path of integration, use residue theorem

Eisenstein series

Ingredients of proof of Thm 2, extending the case $(h,k)=(0,1)$ by B-C:

- re-write $E_s(h,k;\tau)$ and $E_s(h,k;-1/\tau)$ as integrals
- shift path of integration, use residue theorem
- Jonquière's relations:

$$\zeta\left(s;\tfrac{h}{k}\right)=\tfrac{i\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{\frac{\pi is}{2}}Li_{1-s}\left(\zeta_k^h\right)-e^{-\frac{\pi is}{2}}Li_{1-s}\left(\zeta_k^{-h}\right)\right)$$

$$Li_s\left(\zeta_k^h\right)=\tfrac{i\Gamma(1-s)}{(2\pi)^{1-s}}\left(e^{-\frac{\pi is}{2}}\zeta\left(1-s;\tfrac{h}{k}\right)-e^{\frac{\pi is}{2}}\zeta\left(1-s;1-\tfrac{h}{k}\right)\right).$$

Eisenstein series

Ingredients of proof of Thm 1 (different from B-C case (h,k)=(0,1)):

- Lipschitz summation $\Rightarrow$

$$\mathscr{S}_s^{\pm}(h,k;\tau) = k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.$$

Eisenstein series

Ingredients of proof of Thm 1 (different from B-C case (h,k)=(0,1)):

- Lipschitz summation $\Rightarrow$
$$\mathscr{S}_s^{\pm}(h,k;\tau) = k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.$$
- To compute $\lim\limits_{\tau\to a/b}\left(\mathscr{S}_s^{+}(h,k;\tau) - \tau^{-s}\mathscr{S}_s^{-}(h,k;-1/\tau)\right)$,

Eisenstein series

Ingredients of proof of Thm 1 (different from B-C case (h,k)=(0,1)):

- Lipschitz summation $\Rightarrow$
$$\mathscr{S}_s^{\pm}(h,k;\tau) = k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.$$
- To compute $\lim\limits_{\tau\to a/b}\left(\mathscr{S}_s^{+}(h,k;\tau)-\tau^{-s}\mathscr{S}_s^{-}(h,k;-1/\tau)\right)$,
- rewrite as $\sum\limits_{n\in\mathbb{N}} c_n n^{-s}$, explicitly compute the coefficients c_n.

Eisenstein series

Ingredients of proof of Thm 1 (different from B-C case (h,k)=(0,1)):

- Lipschitz summation $\Rightarrow$

$$\mathscr{S}_s^{\pm}(h,k;\tau) = k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.$$

- To compute $\lim\limits_{\tau\to a/b}\left(\mathscr{S}_s^{+}(h,k;\tau)-\tau^{-s}\mathscr{S}_s^{-}(h,k;-1/\tau)\right)$,

- rewrite as $\sum\limits_{n\in\mathbb{N}} c_n n^{-s}$, explicitly compute the coefficients c_n.

- Use Jonquière's relation for $\zeta(s;\frac{h}{k})$, properties of $\zeta(s)$,

Eisenstein series

Ingredients of proof of Thm 1 (different from B-C case (h,k)=(0,1)):

- Lipschitz summation $\Rightarrow$
$$\mathscr{S}_s^{\pm}(h,k;\tau) = k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.$$
- To compute $\lim\limits_{\tau\to a/b}\left(\mathscr{S}_s^{+}(h,k;\tau) - \tau^{-s}\mathscr{S}_s^{-}(h,k;-1/\tau)\right)$,
- rewrite as $\sum\limits_{n\in\mathbb{N}} c_n n^{-s}$, explicitly compute the coefficients c_n.
- Use Jonquière's relation for $\zeta(s;\frac{h}{k})$, properties of $\zeta(s)$,
- eventually discover the cotangent-zeta sums $\mathfrak{c}_s(h,k;a,b)$.

Eisenstein series

Ingredients of proof of Thm 1 (different from B-C case (h,k)=(0,1)):

- Lipschitz summation $\Rightarrow$

$$\mathscr{S}_s^{\pm}(h,k;\tau) = k^{s-1}\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{\substack{m\in\mathbb{N}\\ n\in\mathbb{Z}}}\frac{\zeta_k^{h(n\pm m)}}{(m\tau+n)^s}.$$

- To compute $\lim_{\tau\to a/b}\left(\mathscr{S}_s^{+}(h,k;\tau)-\tau^{-s}\mathscr{S}_s^{-}(h,k;-1/\tau)\right)$,
- rewrite as $\sum_{n\in\mathbb{N}} c_n n^{-s}$, explicitly compute the coefficients c_n.
- Use Jonquière's relation for $\zeta(s;\frac{h}{k})$, properties of $\zeta(s)$,
- eventually discover the cotangent-zeta sums $\mathfrak{c}_s(h,k;a,b)$.
- Apply Theorem 2.

Example 2 revisited. Quantum modularity of Kontsevich's

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n.$$

q-hypergeometric sums

Let
$u(m,n) := \#\{\text{size } n \text{ strongly unimodal sequences with rank } m\}$.

q-hypergeometric sums

Let
$u(m,n) := \#\{\text{size } n \text{ strongly unimodal sequences with rank } m\}$.

Generating function:

$$U(w;q) = \sum_{n\geqslant 0}\sum_{m\in\mathbb{Z}} u(m,n)w^m q^n = \sum_{n=0}^{\infty}(-wq;q)_n(-w^{-1}q;q)_n q^{n+1}.$$

q-hypergeometric sums

Let $\mathcal{U}(x) := e^{-\pi i x/12} U(-1; e^{2\pi i x})$.

Let $\mathcal{U}(x) := e^{-\pi i x/12} U(-1; e^{2\pi i x})$.

Theorem (Bryson-Ono-Pittman-Rhoades)

(1) For any root of unity ζ, we have that

$$F(\zeta^{-1}) = U(-1; \zeta).$$

q-hypergeometric sums

Let $\mathcal{U}(x) := e^{-\pi i x/12} U(-1; e^{2\pi i x})$.

Theorem (Bryson-Ono-Pittman-Rhoades)

(1) For any root of unity ζ, we have that

$$F(\zeta^{-1}) = U(-1; \zeta).$$

(2) For $x \in \mathbb{Q} \cup \mathbb{H} \backslash \{0\}$, we have that

$$\mathcal{U}(x) + (-ix)^{-\frac{3}{2}} \mathcal{U}(-1/x) = h(x),$$

where h is a real analytic function (except at 0).

Questions:

- What to make of the "duality" $F(\zeta^{-1}) = U(-1;\zeta)$?
- The results hold for $U(w;q)$ when viewed as a one-variable function in x (with $q = e(x)$), with $w = -1$ fixed.

  Is there more to say, when considering the second variable w?

Quantum Jacobi forms

Eichler-Zagier's Jacobi forms (1980s)

$+$ Zagier's quantum modular forms (2010) led us to define...

Quantum Jacobi forms

Eichler-Zagier's Jacobi forms (1980s)

$+$ Zagier's quantum modular forms (2010) led us to define...

Definition (Bringmann-F)

A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **and index** $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form**
$\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ such that $\forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$,

Quantum Jacobi forms

Eichler-Zagier's Jacobi forms (1980s)

+ Zagier's quantum modular forms (2010) led us to define...

Definition (Bringmann-F)

A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **and index** $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ such that $\forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$,

$$h_\gamma(z;\tau) := \phi(z;\tau) - \epsilon_1^{-1}(\gamma)(c\tau+d)^{-k} e\left(\frac{-mcz^2}{c\tau+d}\right) \phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right),$$

and

Quantum Jacobi forms

Eichler-Zagier's Jacobi forms (1980s)

$+$ Zagier's quantum modular forms (2010) led us to define...

Definition (Bringmann-F)

A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **and index** $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ such that $\forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$,

$$h_\gamma(z;\tau) := \phi(z;\tau) - \epsilon_1^{-1}(\gamma)(c\tau+d)^{-k} e\left(\frac{-mcz^2}{c\tau+d}\right)\phi\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right),$$

and

$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) - \epsilon_2^{-1}(\lambda,\mu)e(m(\lambda^2\tau+2\lambda z))\phi(z+\lambda\tau+\mu;\tau),$$

Quantum Jacobi forms

Eichler-Zagier's Jacobi forms (1980s)

$+$ Zagier's quantum modular forms (2010) led us to define...

Definition (Bringmann-F)

A **weight** $k \in \frac{1}{2}\mathbb{Z}$ **and index** $m \in \frac{1}{2}\mathbb{Z}$ **quantum Jacobi form** $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{C}$ such that $\forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z} \times \mathbb{Z}$,

$$h_\gamma(z;\tau) := \phi(z;\tau) \\ - \epsilon_1^{-1}(\gamma)(c\tau + d)^{-k} e\left(\frac{-mcz^2}{c\tau + d}\right) \phi\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right),$$

and

$$g_{(\lambda,\mu)}(z;\tau) := \phi(z;\tau) \\ - \epsilon_2^{-1}(\lambda,\mu) e(m(\lambda^2\tau + 2\lambda z))\phi(z + \lambda\tau + \mu;\tau),$$

satisfy a suitable property of continuity or analyticity in $\mathbb{R} \times \mathbb{R}$.

q-hypergeometric multisums

Let $t \in \mathbb{N}$.

q-hypergeometric multisums

Let $t \in \mathbb{N}$.
Hikami-Lovejoy:

$$U_t(w;q)$$

$$:= q^{-t} \sum_{k_t \geqslant \cdots k_1 \geqslant 1}^{\infty} (-wq;q)_{k_t-1}(-w^{-1}q;q)_{k_t-1} q^{k_t} \prod_{j=1}^{t-1} q^{k_j^2} \begin{bmatrix} k_{j+1}+k_j-j+2\sum_{\ell=1}^{j-1} k_\ell \\ k_{j+1}-k_j \end{bmatrix},$$

q-hypergeometric multisums

Let $t \in \mathbb{N}$.
Hikami-Lovejoy:

$$U_t(w;q)$$
$$:= q^{-t} \sum_{k_t \geqslant \cdots k_1 \geqslant 1}^{\infty} (-wq;q)_{k_t-1}(-w^{-1}q;q)_{k_t-1} q^{k_t} \prod_{j=1}^{t-1} q^{k_j^2} \left[\begin{array}{c} k_{j+1}+k_j-j+2\sum_{\ell=1}^{j-1} k_\ell \\ k_{j+1}-k_j \end{array} \right]_q$$

F, Hikami:

$$F_t(w;q)$$
$$:= q^t(-w)^t \sum_{k_t \geqslant \cdots \geqslant k_1 \geqslant 0}^{\infty} (-w)^{k_t}(-wq;q)_{k_t} \prod_{j=1}^{t-1} q^{k_j(k_j+1)}(-w)^{2k_j} \left[\begin{array}{c} k_{j+1} \\ k_j \end{array} \right]_q$$

Notation.

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q := \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}$$

is the q-binomial coefficient.

q–hypergeometric multisums

When $t = 1$, we have

$$\begin{aligned} U_1(w;q) &= q^{-1}U(w;q), \\ F_1(-1;q) &= qF(q). \end{aligned}$$

q–hypergeometric multisums

Let $w = e(z)$ and $q = e(\tau)$. Define

$$\mathcal{F}_t(z;\tau) := (1-w)q^{\frac{(2t-1)^2}{16t+8}-t}w^{-\frac{1}{2}}F_t(-w;q),$$

$$\mathcal{U}_t(z;\tau) := (1-w)q^{\frac{(2t-1)^2}{16t+8}-t}w^{-\frac{1}{2}}U_t(-w;q^{-1}).$$

Quantum Jacobi forms

Theorem (F, '19)

1. *$\mathcal{F}_t(z;\tau) = \mathcal{U}_t(z;\tau)$ is a quantum Jacobi form of weight $1/2$, index $-t - \frac{1}{2}$.*

Quantum Jacobi forms

Theorem (F, '19)

1. *$\mathcal{F}_t(z;\tau) = \mathcal{U}_t(z;\tau)$ is a quantum Jacobi form of weight $1/2$, index $-t-\frac{1}{2}$. In particular,*

$$\mathcal{F}_t(z;\tau) - (2\beta_t^2\tau+1)^{-\frac{1}{2}}\chi_{2\beta_t,1}^{-1} e\left(\frac{2\beta_t^3 z^2}{8(2\beta_t^2\tau+1)}\right)\mathcal{F}_t\left(\frac{z}{2\beta_t^2\tau+1};\frac{\tau}{2\beta_t^2\tau+1}\right)$$
$$= -\frac{1}{2}\int_0^\infty \frac{\sum_{j=1}^{4}\chi(\alpha_t^{(j)})\sum_{\pm} g_{-\frac{\alpha_t^{(j)}}{2\beta_t}+\frac{3\mp1}{4},-\beta_t z}\left(\frac{2}{\beta_t}+it\right)}{\sqrt{-i\left(\frac{2}{\beta_t}+it-4\beta_t\tau\right)}}\,dt,$$

Quantum Jacobi forms

Theorem (F, '19)

1. *$\mathcal{F}_t(z;\tau) = \mathcal{U}_t(z;\tau)$ is a quantum Jacobi form of weight $1/2$, index $-t-\frac{1}{2}$. In particular,*

$$\mathcal{F}_t(z;\tau) - (2\beta_t^2\tau+1)^{-\frac{1}{2}}\chi_{2\beta_t,1}^{-1} e\left(\frac{2\beta_t^3 z^2}{8(2\beta_t^2\tau+1)}\right)\mathcal{F}_t\left(\frac{z}{2\beta_t^2\tau+1};\frac{\tau}{2\beta_t^2\tau+1}\right)$$

$$= -\frac{1}{2}\int_0^\infty \frac{\sum_{j=1}^{4}\chi(\alpha_t^{(j)})\sum_{\pm} g_{-\frac{\alpha_t^{(j)}}{2\beta_t}+\frac{3\mp 1}{4},-\beta_t z}\left(\frac{2}{\beta_t}+it\right)}{\sqrt{-i(\frac{2}{\beta_t}+it-4\beta_t\tau)}}dt,$$

and the error to Jacobi transformation in $\mathbb{Q}\times\mathbb{Q}$ extends to a C^∞ function in $\mathbb{R}\times\mathbb{R}$.

Quantum Jacobi forms

Theorem (F, '19)

1. *$\mathcal{F}_t(z;\tau) = \mathcal{U}_t(z;\tau)$ is a quantum Jacobi form of weight $1/2$, index $-t-\frac{1}{2}$. In particular,*

$$\mathcal{F}_t(z;\tau) - (2\beta_t^2\tau+1)^{-\frac{1}{2}}\chi_{2\beta_t,1}^{-1} e\left(\frac{2\beta_t^3 z^2}{8(2\beta_t^2\tau+1)}\right)\mathcal{F}_t\left(\frac{z}{2\beta_t^2\tau+1};\frac{\tau}{2\beta_t^2\tau+1}\right)$$

$$= -\frac{1}{2}\int_0^\infty \frac{\sum_{j=1}^{4}\chi(\alpha_t^{(j)})\sum_{\pm} g_{-\frac{\alpha_t^{(j)}}{2\beta_t}+\frac{3\mp 1}{4},-\beta_t z}\left(\frac{2}{\beta_t}+it\right)}{\sqrt{-i(\frac{2}{\beta_t}+it-4\beta_t\tau)}}\,dt,$$

and the error to Jacobi transformation in $\mathbb{Q}\times\mathbb{Q}$ extends to a C^∞ function in $\mathbb{R}\times\mathbb{R}$.

Here, $g_{a,b}(u) := \sum_{n\in a+\mathbb{Z}} ne(n^2u/2+nb)$.

Quantum Jacobi forms

Theorem (cont.)

2. *The function* $\mathcal{F}_t(z; -\tau)$ *is a mock Jacobi form of weight* $1/2$ *and index* $-t - \frac{1}{2}$ *(e.g., transforms appropriately in* $\mathbb{C} \times \mathbb{H}$*).*

Quantum Jacobi forms

Theorem (F, '19)

Let $t \in \mathbb{N}$. For any $N \in \mathbb{N}$, we have the duality

$$F_t(-q^N; q^{-1}) = U_t(-q^N; q) \in \mathbb{Z}[q].$$

Quantum Jacobi forms

Theorem (F, '19)

Let $t \in \mathbb{N}$. For any $N \in \mathbb{N}$, we have the duality

$$F_t(-q^N; q^{-1}) = U_t(-q^N; q) \in \mathbb{Z}[q].$$

In particular, for any $b \mid k$, we have that

$$F_t(-\zeta_b^a; \zeta_k^{-h}) = U_t(-\zeta_b^a; \zeta_k^h).$$

Modularity & Duality Results

- Zagier:
  - Quantum modular, F_1, $w = -1$.
- Bryson-Ono-Pittman-Rhoades :
  - Quantum/Mock modular, duality, F_1 and U_1, $w = -1$.
- Bringmann-F:
  - Quantum/Mock Jacobi, U_1.
- F-Ki-Truong Vu-Yang:
  - Quantum modular, duality, F_1 and U_1, $w = \zeta_b^a$.
- Hikami-Lovejoy:
  - Quantum modular, duality, F_t and U_t, $w = -1$.
- F:
  - Quantum/Mock Jacobi, duality, F_t and U_t

Quantum modular forms and knots

Let $J_N(K;q) :=$ **N-colored Jones polynomial for a knot K.**

Quantum modular forms and knots

Let $J_N(K;q) :=$ **N-colored Jones polynomial for a knot K.**

We consider the torus knots $T_{(2,2t+1)}$.

Quantum modular forms and knots

Let $J_N(K;q) :=$ **N-colored Jones polynomial for a knot K.**

We consider the torus knots $T_{(2,2t+1)}$.

Proof of duality theorem. We show, using work of Hikami-Lovejoy, for $t \in \mathbb{N}$

Quantum modular forms and knots

Let $J_N(K;q) :=$ **N-colored Jones polynomial for a knot K.**

We consider the torus knots $T_{(2,2t+1)}$.

Proof of duality theorem. We show, using work of Hikami-Lovejoy, for $t \in \mathbb{N}$

$$F_t(-q^{-N};q) = J_N(T_{(2,2t+1)};q),$$
$$U_t(-q^{-N};q) = J_N(T^*_{(2,2t+1)};q).$$

Quantum modular forms and knots

Let $J_N(K;q) :=$ **N-colored Jones polynomial for a knot K.**

We consider the torus knots $T_{(2,2t+1)}$.

Proof of duality theorem. We show, using work of Hikami-Lovejoy, for $t \in \mathbb{N}$

$$F_t(-q^{-N};q) = J_N(T_{(2,2t+1)};q),$$
$$U_t(-q^{-N};q) = J_N(T^*_{(2,2t+1)};q).$$

The claimed duality result now follows from the fact that

$$J_N(K;q) = J_N(K^*;q^{-1}),$$

where $K^* =$ mirror image of K.

Quantum modular forms and knots

Remark. The above proof + quantum theorem reveal quantum properties of colored Jones polynomials

$$J_N(T_{(2,2t+1)};\zeta), \quad J_N(T^*_{(2,2t+1)};\zeta).$$

Example:

Quantum modular forms and knots

Example: For $K = T_{(2,3)}$, the right-handed trefoil ($t = 1$ case),

$$J_N(T_{(2,3)}; q) = q^{1-N} \sum_{n=0}^{N} q^{-nN} (q^{1-N}; q)_n,$$

Quantum modular forms and knots

Example: For $K = T_{(2,3)}$, the right-handed trefoil ($t = 1$ case),

$$J_N(T_{(2,3)}; q) = q^{1-N} \sum_{n=0}^{N} q^{-nN} (q^{1-N}; q)_n,$$

$$J_N(T^*_{(2,3)}; q) = \sum_{n=0}^{N} q^{n} (q^{1-N}; q)_n (q^{1+N}; q)_n.$$

Example: For $K = T_{(2,3)}$, the right-handed trefoil ($t = 1$ case),

$$J_N(T_{(2,3)}; q) = q^{1-N} \sum_{n=0}^{N} q^{-nN} (q^{1-N}; q)_n,$$

$$J_N(T^*_{(2,3)}; q) = \sum_{n=0}^{N} q^n (q^{1-N}; q)_n (q^{1+N}; q)_n.$$

Thus, for $N \equiv -ab'h' (\mathrm{mod}\ k)$, where $bb' = k$, $hh' \equiv -1 (\mathrm{mod}\ k)$, we have

$$J_N(T_{(2,3)}; \zeta_k^{-h}) = F_1(-\zeta_b^a; \zeta_k^{-h}),$$

and

$$J_N(T^*_{(2,3)}; \zeta_k^{h}) = U_1(-\zeta_b^a; \zeta_k^{h}).$$

Quantum modular forms and knots

Related work: Garoufalidis, Hikami, Lê, Lovejoy, Osburn, Zagier...

Quantum modular forms

Thank you