International Seminar on Automorphic Forms

Slope of Siegel modular forms: Geometric applications

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24 January 2023

In this talk we will consider the slope of Siegel modular forms and I will discuss some applications to the moduli space of complex principally polarized abelian varieties

$$X = \mathcal{A}_g = \{(A, \Theta), / A \,\text{abelian variety},\ \Theta \,\text{principal polarization}\}$$

and possibly to the generalized Kuga's varieties.

The orbifold $\mathcal{A}_g$ can be written as an arithmetic quotient

Definitions

$\mathcal{H}_g := \{\tau \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : \tau^t = \tau; \mathrm{Im}\,\tau > 0\}$ the Siegel space

$\mathrm{Sp}(2g, \mathbb{R})$ the symplectic group

$\mathrm{Sp}(2g, \mathbb{R}) \times \mathcal{H}_g \to \mathcal{H}_g$ the action as

$\tau \to \sigma\tau := (A\tau + B)(C\tau + D)^{-1},\ \sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$

$\mathcal{A}_g := \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$.

Modular Forms

Scalar modular forms, i.e $f : \mathcal{H}_g \to, \mathbb{C}$ holomorphic such that

$$f(\sigma \cdot \tau) = det(C\tau + D)^k f(\tau), \ \ \sigma \in \mathrm{Sp}(2g, \mathbb{Z})$$

plus a condition at ∞ when $g = 1$.

Modular forms have Fourier and Fourier-Jacobi expansions. I will use the second

$$\tau = \begin{pmatrix} \tau' & z \\ z' & w \end{pmatrix}$$

$\tau' \in \mathcal{H}_{g-1}$, $z \in \ \mathbb{C}^{g-1}$, $w \in \mathcal{H}_1$, $\ q = exp(2\pi i w)$

$$f(\tau) = \sum_{r \geq 0} f_r(\tau', z) q^r$$

They are cusp forms $\iff$ $r \gneq 0$.

The vanishing order of f at the boundary is given by

$$b = (1/2)min_{r \in \mathbb{Z}}\{f_r(\tau', z) \neq 0\}$$

The value b indicates the order of the vanishing of the modular form along the boundary

We define the slopes

$$sl(f) = k/b$$

$$sl(\mathcal{A}_g) = \ inf\{sl(f)/f \ modular \ form\}$$

Geometric interpretation

We consider the partial compactification of $\mathcal{A}_g$ given by t he blow up of the partial Satake compactification $\mathcal{A}_g \cup \mathcal{A}_{g-1}$ along the boundary component .
Set-theoretically it is $\mathcal{A}_g^p = \mathcal{A}_g \cup \mathcal{X}_{g-1}/\pm 1$

When $g \geq 3$, $Pic(\mathcal{A}_g^p)$ is equal to

$$\mathbb{Z}\lambda \oplus \mathbb{Z}\delta$$

with λ the class of cocycle $det(CZ+D)$ and δ the class of the boundary divisor. Thus the class of the strictly transform of the divisor of a modular form f is

$$(f) = k\lambda - b\delta$$

Around 50 years ago Mumford raised the question about the Kodaira dimension of $\mathcal{A}_g$.

$\Omega^N(\tilde{\mathcal{A}}_g)$ is the space of holomorphic differential forms of top degree, i.e. $N = g(g+1)/2$.

$\tilde{\mathcal{A}}_g$ is a smooth compactification of $\mathcal{A}_g$

The general question about the Kodaira dimension of $\mathcal{A}_g$ is:

How does

$$\mathrm{dim}\Omega^N(\tilde{\mathcal{A}}_g)^{\otimes k} = ck^t,\ t \leq N$$

grow?

Tai (1982) proved that, when $g \geq 5$,

$$\omega \in \Omega^N(\mathcal{A}_g^0)^{\otimes k}$$

a section of the $k-$canonical bundle on the smooth part of $\mathcal{A}_g$

$$\omega = f(\tau) d\tau^{\otimes k}, \quad d\tau = \wedge_{1 \leq i \leq j \leq g} d\tau_{ij}$$

with $f(\tau)$ a modular form of weight $k(g+1)$
extends to a smooth compactification $\iff$ $sl(f) \leq g+1$.

$\mathcal{A}_g$ is of general type if $t = N$, i.e $\kappa(\mathcal{A}_g) = N \iff$ there exists f with $sl(f) < g+1$.

$\kappa(\mathcal{A}_g) = -\infty \iff sl(\mathcal{A}_g) > g+1$.

If $sl(\mathcal{A}_g) = g+1$, then $\kappa(\mathcal{A}_g) \geq 0$

At the begin of 80's Tai, Freitag, Mumford proved that $\mathcal{A}_g$ is of general type if $g \geq 7$.

Meanwhile Clemens, Donagi, Verra, etc.. proved that if $g \leq 5$, then $\kappa(\mathcal{A}_g) = -\infty$.
Using estimate of the dimensions of space of Jacobi's forms appearing in the Fourier- Jacobi's expansion

Tai proved that there are sufficiently many pluri-canonical forms which extend when $g \geq 9$,

so that in these cases $\mathcal{A}_g$ is of general type.

Short description of the methods used by Freitag and Mumford, since I will use later

In $\mathcal{A}_g$ there is a distinguish divisor N_0 parametrizing principally polarized abelian varieties (X, Θ)

such that the divisor Θ is singular.

We have that this divisor has two irreducible components

$$N_0 = \theta_{null} \cup N_0'.$$

θ_{null} parametrizes p.p.a.v. (A, Θ) with a singular point of order two;

N_0' parametrizes the other cases.

In the general case of θ_{null} the theta divisor has only a singular point that is an ordinary singularity, i.e,. the hessian matrix has maximal rank.

2) In the general case has of N_0' the theta divisor has two singular points $\pm x$ that are ordinary

We have two modular forms F_θ and F' vanishing exactly on these divisors

Freitag proved that $sl(F_{theta}) = 8 + \frac{1}{2^{g-3}}$
Mumford proved that $sl(F') \asymp 6 + \frac{12}{g+1}$

Meanwhile Tai proved also that

$$lim_{g \to \infty} sl(\mathcal{A}_g) = 0$$

Thus it remained the case of Kodaira's dimension of $\mathcal{A}_g$ when $g = 6$

Recently, we proved the following

Theorem (Dittmann, ..., Scheithauer)
The Kodaira dimension of $\mathcal{A}_6$ is non-negative.

Idea of the proof

We gave an explicit section of the bicanonical bundle on $\mathcal{A}_6$, i.e a modular form f of degree 6, weight 14 and vanishing order 2.

We used theta series with pluriharmonic coefficients

V is an euclidean vector space of dimension m, $<,>$ is the scalar product and $L \subset V$ is an even unimodular lattice, $h = (h_1, \ldots, h_g)$ a sequence of g vectors in $V_{\mathbb{C}}$ such that

$$Q(h) := Q(h,h) =< h_i, h_j >= 0, \quad Q(h,\overline{h}) =< h_i, \overline{h_j} >> 0$$

$$\Theta_{L,h,k} = \sum_{x \in L^g} det((Q(x,h))^k exp(\pi i Tr(Q(x)\tau)$$

is a Siegel cusp form of weight $m/2 + k$ with Fourier coefficients

$$a(T) = \sum_{x \in L^g / Q(x) = T} det(Q(x,h)^k$$

$\Theta_{L,h,k} \neq 0$ implies vanishing order $\geq (1/2) min_{0 \neq x \in L} Q(x)$.

Candidate when $g = 6$

$\Lambda =$ Leech's lattice, hence $m = 24$, $k = 2$, thus $\Theta_{\Lambda,h,2}$ has weight 14 and vanishing at least 2 since Leech's lattice is even and has no roots,i.e $Q(x) \geq 4$.

We proved that $\Theta_{\Lambda,h,2} \neq 0$.

What is known about the slope of $\mathcal{A}_g$. Few cases are known

g	1	2	3	4	5	6
sl	12	10	9	8	$54/7$	$5.3 \leq \cdot \leq 7$
form	Δ	F_{null}	F_{null}	F'	F'	?

For $g \leq 4$, these divisors are rigid, i.e. up to a power or a multiplicative constant these are unique with this slope.
When $g = 5$ it is unknown.

Moving Slope

E an effective divisor on $\mathcal{A}_g^p$;
$[E] = a\lambda - b\delta$

We say that it is moving if the base locus of the linear system $|nE|$ has codimension at least 2 for $n >> 0$.

This is equivalent to require the existence of a rational map $\mathcal{A}_g^p \to \mathbb{P}^1$

Obviously, up to a scalar multiple, to E corresponds a modular form f of weight na and vanishing nb.

Hence we can define the moving slope of A_g as

$$sl_{mov}(\mathcal{A}_g) = inf\{sl(E)/E \; is \; moving\}$$

$$sl_{mov}(\mathcal{A}_g) \geq sl(\mathcal{A}_g)$$

We have the equality if and only if $sl(\mathcal{A}_g)$ is not realized by a rigid divisor .

What is known about the moving slope

g	1	2	3	4	5	6
sl	12	10	9	8	$54/7$	$5.3 \leq \cdot \leq 7$
mov	∞	12	$28/3$	$34/4$	$\leq 542/70$	?

Candidate for the moving slope

Theorem (Grushevsky, Ibukiyama, Mondello, ———-)
For every $g \geq 2$ and every integer $a \geq \frac{g}{2}$ there exists a differential operator $\mathcal{D}_{g,a}$
acting on the space of genus g Siegel modular forms of weight a that satisfies the following properties:

(i) if F is a genus g Siegel modular form of weight a and vanishing order b along the boundary, then $\},\dashv(\mathcal{F})$ is a Siegel modular form of weight $ga+2$ and of vanishing order $\beta \geq gb$ along the boundary;

(ii) the restriction of $\mathcal{D}_{g,a}(F)$ to the zero locus of F is equal to the restriction of $g!\det(\partial F)$.

Here setting $F_{ij} = \frac{\partial F}{\partial \tau_{ij}}$
we have

$$(\partial F) = \begin{pmatrix} F_{11} & F_{21}/2 & \dots & F_{1g}/2 \\ F_{12}/2 & F_{22} & \dots & F_{2g}/2 \\ \dots & \dots & \dots & \dots \\ F_{g1}/2 & F_{g2}/2 & \dots & F_{gg} \end{pmatrix}$$

Consequence : Assume that the slope is realized by a form F such that $det(\partial F) \neq 0$ when $F = 0$, then

$$sl_{mov}(\mathcal{A}_g) \leq sl(\mathcal{A}_g) + \frac{2}{gb}$$

This happens in all known cases

When $g = 2, 3$ F_{null} is the form

When $g = 4, 5$ F' is the form

In all these cases the condition about $det(\partial F)$ is satisfied.

This is consequence of Andreotti- Mayer's results on the singularities of the divisor Θ, in fact

$x \in \Theta$ a singular point, we set

$$\theta_{ij}(\tau, x) = \frac{\partial^2 \theta(\tau, x)}{\partial z_i \partial z_j}$$

and

$$\partial^2 \theta(\tau, x) = \begin{pmatrix} \theta_{11}(\tau, x) & \theta_{12}(\tau, x) & \dots & \theta_{1g}(\tau, x) \\ \theta_{12}(\tau, x) & \theta_{22}(\tau, x) & \dots & \theta_{2g}(\tau, x) \\ \dots & \dots & \dots & \dots \\ \theta_{1g}(\tau, x) & \theta_{2g}(\tau, x) & \dots & \theta_{gg}(\tau, x) \end{pmatrix}$$

Thus at the the generic x we get

$$\partial^2 \theta(\tau, x) = c(\partial F)$$

and it has rank g.

Comment

Recently in Slopes of Siegel cusp forms and geometry of compactified Kuga varieties,

we (Poon, ... Sankaran) considered Kuga's varieties

$$X_g = (\mathbb{C}^g \times \mathbb{H}_g)/(\mathbb{Z}^{2g} \ltimes \mathrm{Sp}(2g, \mathbb{Z})$$

and the n-folds X_g^n.
We have been able to prove a theorem similar to Tai's theorem, namely
A holomorphic differential forms of top degree

$$\omega = f(\tau) d\tau^{\otimes k}, \quad d\tau \wedge dz^{\otimes n}$$

with

$$d\tau = \wedge_{1 \le i \le j \le g} d\tau_{ij}, \ \ dz = \wedge_{1 \le l \le g} dz_l$$

extends to a smooth compactification of X_g^n when $g + n \ge 6$ and $sl(f) \le g + n + 1$.

Thus we have

Theorem: Suppose that $g \geq 3$ and $n \geq 1$.

Then the Kodaira dimension $\kappa(X_g^n)$ of X_g^n satisfies

- $\kappa(X_g^n) = g(g+1)/2$ if $g+n \geq 7$ and $(g,n) \neq (3,5)$, $(4,3)$ or $(3,4)$,
- $\kappa(X_3^5) = \kappa(X_4^3) = 0$, and
- $\kappa(X_g^n) = -\infty$ otherwise, in particular $\kappa(X_3^4) = -\infty$.

.

We state

Problems

I assume $g \geq 3$. F a modular form relative to the (full) symplectic group

F factorizes in irreducible factors

$$F = f_1^{r_1} \dots f_k^{r_k}$$

with f_j modular forms wrt any subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ of finite index.
We proved in Universally irreducible subvarieties of Siegel moduli spaces
Theorem: (Mondello , ...) The loci in $\mathbb{H}_g$ defined by $f_j = 0$ are irreducible

Hence it make sense to raise the following
Question Let us assume $g \geq 3$ and F a modular form whose zero set is irreducible in $\mathbb{H}_g$, then is $det(\partial F) \not\equiv 0$ when $F = 0$?

$g = 4$ the form F' is the Schottky's form that has weight 8. It is the equation of $\mathcal{M}_4$ (the moduli space of smooth compact Riemann surfaces of genus 4) in $\mathcal{A}_4$
The form $\mathcal{D}_{4,8}(F')$ has weight 34 and vanishing 4.

It describes, in $\mathcal{M}_4$, the locus θ_{null}.
This locus is described in $\mathcal{A}_4$ by a form of weight 68 and vanishing 8.
Hence the restriction of $\mathcal{D}_{4,8}(F')$ to $\mathcal{M}_4$ (as Teichmuller modular form)
can be considered at the square root of the classical form giving θ_{null}.

Question Does this result lead to a different investigation about Teichmüller modular forms?

In fact the standard example of a Teichmüller modular form that is not the restriction of a Siegel modular form
is obtained considering the square root of theta constants.