Rational functions, modular forms and cotangent sums

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Goals

Our main goals are:

1. New approach to modular forms via rational functions
2. Use this mechanism for several applications (L-functions, Eichler integrals, cotangent sums, ...)

Overview:

1. Short introduction to (elliptic) modular forms
2. Weak functions and modular forms
3. Application to cotangent sums

Elliptic modular forms

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$, where $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, is called a weight $k \in \mathbb{Z}$ *modular form* for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with Nebentypus character $\chi : \Gamma \to \mathbb{C}^\times$, if the following is satisfied:

- We have $f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(M)(c\tau+d)^k f(\tau)$ for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
- The function f is holomorphic at all cusps $\mathbb{Q} \cup \{i\infty\}$.

We collect all modular forms in the space $M_k(\Gamma, \chi)$. We call a modular form vanishing in all the cusps a *cusp form*, and collect them in the subspace $S_k(\Gamma, \chi) \subset M_k(\Gamma, \chi)$.

Remark. The term congruence subgroup means, that there is some positive integer N, such that

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \subset \Gamma.$$

Classical approaches for construction:

- Eisenstein series: for positive integers M, N we define the congruence subgroup

$$\Gamma_0(M,N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| b \equiv 0 \pmod{M}, c \equiv 0 \pmod{N} \right\}$$

and two Dirichlet characters χ and ψ (mod M and N) and for $k \geq 3$

$$E_k(\chi,\psi;\tau) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \chi(m)\psi(n)(m\tau+n)^{-k}. \tag{1.1}$$

Then E_k is a weight k modular form for $\Gamma_0(M,N)$ with Nebentypus $\chi\overline{\psi}$ (non-trivial $\iff$ $(\chi\psi)(-1) = (-1)^k$). For weights $k \in \{1,2\}$ the series (1.1) does not converge absolutely, so one has to find a different approach (to sum in the "right way").

Classical approaches for construction:

- Theta functions: for an integral positive definite quadratic form $Q(x_1, x_2, ..., x_n) := {}^t x Q x$ (with even diagonal elements) one defines the corresponding theta series by

$$\Theta(Q;\tau) := \sum_{x \in \mathbb{Z}^n} q^{Q(x)/2}, \qquad q := e^{2\pi i \tau}.$$

  This is a weight $\frac{n}{2}$ modular form for $\Gamma_0(N)$, where N denotes the level of Q, i.e. NQ^{-1} is integral with even diagonal elements.
  The standard proof of this fact requires Fourier analysis, especially the Poisson summation formula.

Important properties:

- Each $f \in M_k(\Gamma)$ has a Fourier series representation $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{n/M}$, if $\Gamma(M) \subset \Gamma$.
- The spaces $M_k(\Gamma)$ have finite dimension (exact dimension formulas by Riemann-Roch theorem).
- There are no non-constant modular forms for the weights $k \leq 0$.
- We can assign to f a L-function

$$L(f, s) = \sum_{n=1}^{\infty} a(n) n^{-s}, \qquad \mathrm{Re}(s) > k.$$

It has a meromorphic continuation to $\mathbb{C}$ (if f cusp form, it is entire) with (simple) poles at most in $s = k$ and satisfies a functional equation.

Definition 2.1.

We call a meromorphic function ω on $\mathbb{C}$ weak, if the following is satisfied:

- *We have $\omega(z+1) = \omega(z)$, so ω is 1-periodic.*
- *All poles of ω are simple and at rational points.*
- *The expression $\omega(z)$ tends to zero in the strip $0 \leq \mathrm{Re}(z) < 1$ as $|z| \to \infty$.*

By Liouville's theorem there is always a decomposition

$$\omega(z) = \sum_{x \in \mathbb{Q}/\mathbb{Z}} \beta_\omega(x) h_x(z),$$

where $\beta_\omega : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}$ has discrete support, $\sum_{x \in \mathbb{Q}/\mathbb{Z}} \beta_\omega(x) = 0$ and

$$h_x(z) := \frac{e(z)}{e(x) - e(z)}, \qquad e(z) := e^{2\pi i z}.$$

The $h_x(z)$ are rational functions in $e(z)$.

We collect all wek function ω of level d with $d|N$ (smallest positive integer, such that $\omega(z/d)$ only has poles in $\mathbb{Z}$) in the vector space W_N. The space W_N decomposes in the following way:

$$W_N = \mathfrak{P}_N \oplus \bigoplus_{d|N} \bigoplus_{\substack{\chi \bmod d \\ \chi \neq \chi_{0,d}}} \mathbb{C}\omega_\chi.$$

where $\chi_{0,d}$ is principal,

$$\omega_\chi(z) := \sum_{j=1}^{d} \chi(j) h_{j/d}(z),$$

and

$$\mathfrak{P}_N := \left\{ \omega \in W_N \middle| \omega = \sum_{d|N} c_d \sum_{j \in \mathbb{Z}/d\mathbb{Z}} \chi_{0,d}(j) h_{j/d} \right\}.$$

Define

$$\Gamma_1(M,N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M,N) \middle| a \equiv d \equiv 1 \pmod{MN} \right\}.$$

Theorem 2.2 (F., 2019, [5]).

Let $k \geq 3$ and $M, N > 1$ be integers. There is a homomorphism

$$W_M \otimes W_N \longrightarrow M_k(\Gamma_1(M,N))$$

$$\omega \otimes \eta \longmapsto \sum_{x \in \mathbb{Q}^\times} x^{k-1} \beta_\eta(x) \omega(x\tau).$$

In the case that $k = 1$ and $k = 2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1}\eta(z)\omega(z\tau)$ has a removable singularity in $z = 0$.

Idea of proof: Every pair $\omega \otimes \eta$ in $W_M \otimes W_N$ induces a holomorphic function on the union of the upper and lower half plane $\underline{\mathbb{H}} := \mathbb{H} \cup (-\mathbb{H})$ by

$$\vartheta_k : W_M \otimes W_N \longrightarrow \mathcal{O}(\underline{\mathbb{H}})$$

$$\vartheta_k(\omega \otimes \eta; \tau) := -2\pi i \sum_{x \in \mathbb{Q}^\times} \operatorname{res}_{z=x}(z^{k-1}\eta(z)\omega(z\tau)).$$

When applying contour integration to the function

$$g_\tau(z) := z^{k-1}\eta(z)\omega(z\tau)$$

one concludes with the Residue theorem the functional equation

$$\vartheta_k\left(\omega \otimes \eta; -\frac{1}{\tau}\right) = \tau^k \vartheta_k(\eta \otimes -\widehat{\omega}; \tau) + 2\pi i \operatorname{res}_{z=0}\left(z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right)\right),$$

where $\widehat{\omega}(z) := \omega(-z)$ is weak again. After showing a formula for twists $(\vartheta_k)_\psi$ (which preserves the "ϑ_k-structure") one can apply the transformation law in all (infinite) cases and use Weil's converse theorem.

Remarks:

- All modular forms constructed in this way are part of the Eisenstein space and vanish in the cusps $\tau \in \{0, i\infty\}$. For example, we conclude for non-principal primitive characters modulo M and N:

$$E_k(\chi, \psi; \tau) = \frac{\chi(-1)(-2\pi i)^k \mathcal{G}(\psi)}{N(k-1)! \mathcal{G}(\overline{\chi})} \vartheta_k(\omega_{\overline{\chi}} \otimes \omega_{\overline{\psi}}; \tau). \tag{2.1}$$

- The method is “natural“ in the sense that it does not distinguish between the cases $k = 1, 2$ and $k > 2$.
- Eisenstein series for the full modular group do not arise from weak functions, since we have $W_1 = 0$. The “reason“ is that there are no non-trivial cusp forms for the weights $k \in \{2, 4, 6, 8, 10, 14\}$ in this case.

Example 2.3.

Let $v_2(n)$ be the exponent of 2 in the prime decomposition of n. For any even $k \geq 4$ we then have that

$$f(\tau) = \sum_{n=1}^{\infty} (-1)^{n-1} (2^{v_2(n)})^{k-1} \sigma_{k-1}\left(\frac{n}{2^{v_2(n)}}\right) q^{\frac{n}{2}}$$

is an entire modular form of weight k for $\Gamma_\theta := \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$.

The space $W_2^- \otimes W_2^-$ has one dimension and is generated by $\omega_2 \otimes \omega_2$, where

$$\omega_2(z) = \frac{e(z)}{e(\frac{1}{2}) - e(z)} - \frac{e(z)}{e(0) - e(z)} = -\frac{i}{\sin(2\pi z)}.$$

Hence we obtain a modular form $f \in M_k(\Gamma_\theta)$ with

$$f(\tau) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{k-1} \frac{q^{\frac{n}{2}}}{1-q^n}.$$

Rearranging the Lambert sum shows

$$\begin{aligned} f(\tau) &= \sum_{m=1}^{\infty} \sum_{\substack{n,r \\ n(2r+1)=m}} (-1)^{\frac{m}{2r+1}-1} \left(\frac{m}{2r+1}\right)^{k-1} q^{\frac{m}{2}} \\ &= \sum_{m=1}^{\infty} \sum_{\substack{u \mid m \\ u \text{ odd}}} (-1)^{\frac{m}{u}-1} \left(\frac{m}{u}\right)^{k-1} q^{\frac{m}{2}}. \end{aligned}$$

With

$$\sum_{\substack{u \mid m \\ u \text{ odd}}} (-1)^{\frac{m}{u}-1} \left(\frac{m}{u}\right)^{k-1} = (-1)^{m-1} (2^{\nu_2(m)})^{k-1} \sigma_{k-1}\left(\frac{m}{2^{\nu_2(m)}}\right)$$

the claim follows.

The case of non-positive weight

In the case $k \leq 0$ the key transformation formula

$$\vartheta_k\left(\omega \otimes \eta; -\frac{1}{\tau}\right) = \tau^k \vartheta_k(\eta \otimes -\widehat{\omega}; \tau) + 2\pi i \,\mathrm{res}_{z=0}\left(z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right)\right),$$

remains valid (since z^{k-1} is still holomorphic in $\mathbb{C}^\times$) even if we relax the conditions on $\omega \otimes \eta$. For example, if $k < 0$, we may assume that ω and η are only *bounded* in $\pm i\infty$ (*pre-weak functions*).

Questions:

- What exactly happens for non-positive k?
- What happens for "general pre-weak functions"?

- If we allow ω also to have (simple) poles at *arbitrary real numbers*, we end up with a generalized form for Eisenstein series, see [6].
- If we allow ω to have poles of arbitrary degree, we can prove transformation formulas involving a specific family of q-series, see [6].
- If we only relax the condition $\omega(\pm i\infty) = 0$ to ω is bounded at $\pm i\infty$ and consider integers $k < 0$, we obtain some insights into cotangent sums, see [6] and below.
- If $\omega \otimes \eta$ is a pair of weak functions with no poles in $z = 0$, there is a relation to Eichler integrals by a "weak function version" of Bol's identity. For example, one obtains

$$\int_{k-1} \vartheta_k(\omega_{\overline{\chi}} \otimes \omega_{\overline{\psi}}; \tau) = C\vartheta_{2-k}\left(\omega_\psi \otimes \omega_\chi; \frac{N_\chi \tau}{N_\psi}\right)$$

for some constant C, where χ, ψ are primitive characters mod $N_\chi, N_\psi > 1$ and $\int_{k-1}$ is the $k-1$-fold integral in τ.

- There are also applications to L-series of products of Eisenstein series.

Theorem 2.4 (F., 2020, [3]).

Let $\chi, \psi : \mathbb{Z}^\ell \to \mathbb{C}^\times$ be non-principal, primitive characters modulo M and N, respectively, such that $\chi_j(-1)\psi_j(-1) = (-1)^{k_j}$ for all $j = 1, \ldots, \ell$ and some weight vector $\boldsymbol{k} = (k_1, \ldots, k_\ell)$. For all $s \in \mathbb{C}$ with

$$\mathrm{Re}(s) > \max\left(|\boldsymbol{k}| - \ell - \frac{1}{2}\sum_{j=1}^{\ell}(\psi_j(-1)+1), -\frac{1}{2}\sum_{j=1}^{\ell}(\chi_j(-1)+1) \right)$$

where $|\boldsymbol{k}| = k_1 + \cdots + k_\ell$, we have for $f(\tau) := \prod_{j=1}^{\ell} E_{k_j}(\chi_j, \psi_j; \tau)$:

$$L(f,s) = \left(-\frac{2\pi i}{N}\right)^{|\boldsymbol{k}|} \prod_{j=1}^{\ell} \frac{2\mathcal{G}(\psi_j)}{(k_j-1)!} \sum_{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{N}^\ell \times \mathbb{N}^\ell} \Pi_{\boldsymbol{k}}(\boldsymbol{u}) \overline{\psi}(\boldsymbol{u}) \chi(\boldsymbol{v}) \langle \boldsymbol{u}, \boldsymbol{v} \rangle^{-s},$$

where the summation respects some specific rearrangement.

Contangent sums

Definition 3.1.

(i) *For integers $m \in \mathbb{N}_0$ and pre-weak functions (that are holomorphic around $z = 0$) we define the corresponding cotangent sum*

$$C(\omega; m) := \sum_{x \in (0,1)} \beta_\omega(x) \cot^m(\pi x).$$

(ii) *Let $N \geq 2$ be an integer. To every N-periodic function ψ, for example a character mod N, and $m \geq 0$, we define*

$$C(\omega_\psi; m) := C(\psi; m) = \sum_{j=1}^{N-1} \psi(j) \cot^m\left(\frac{\pi j}{N}\right).$$

An example for a cotangent sum is

$$\sum_{j=1}^{N-1} \cot^2\left(\frac{\pi j}{N}\right) = \frac{(N-1)(N-2)}{3}, \qquad N = 2, 3, \ldots \tag{3.1}$$

This can be generalized.

Theorem 3.2 (Berndt, Yeap, 2002, [1]).

Let N and n be positive integers. Then

$$\sum_{j=1}^{N-1} \cot^{2n}\left(\frac{\pi j}{N}\right) = (-1)^n N - (-1)^n 2^{2n} \sum_{j_0=0}^{n} \left(\sum_{\substack{j_1,\ldots,j_{2n}\geq 0 \\ j_0+j_1+\cdots+j_{2n}=n}} \prod_{r=0}^{2n} \frac{B_{2j_r}}{(2j_r)!} \right) N^{2j_0}.$$

The B_n denote the *Bernoulli numbers* defined by generating series

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}.$$

We introduce two sequences $\delta_\nu(u)$ and $\delta_\nu^*(u)$ of rational numbers by

$$\delta_\nu(u) := \frac{i^{\nu+u}}{(\nu-1)!} \sum_{\ell=u-1}^{\nu-1} (-1)^{\nu+\ell-u} 2^{\nu-1-\ell} \ell! \left\{ \begin{matrix} \nu-1 \\ \ell \end{matrix} \right\} \Delta(\ell,u), \qquad (3.2)$$

where $\left\{ \begin{smallmatrix} \nu-1 \\ \ell \end{smallmatrix} \right\}$ denotes the Stirling numbers of the second kind and

$$\Delta(\ell,u) := \binom{\ell}{u} - \binom{\ell}{u-1},$$

$$\delta_{2k}^*(2\ell) := (-1)^{k+\ell} 2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k} \geq 0 \\ \ell+j_1+\cdots+j_{2k}=k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!} \qquad (3.3)$$

and

$$\delta_{2k-1}^*(2\ell-1) := (-1)^{k+\ell} 2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k-1} \geq 0 \\ 2\ell-1+2j_1+\cdots+2j_{2k-1}=2k-1}} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!}. \qquad (3.4)$$

We have also $\delta_\nu^*(u) = 0$ if $\nu + u \equiv 1 \pmod 2$.

Theorem 3.3 (F., 2020, [6]).

Let ω be a pre-weak function. Then we have for all $k \geq 1$:

$$\widetilde{L}(\omega; k) := \sum_{x \in \mathbb{R}^\times} \beta_\omega(x) x^{-k} = \pi^k \sum_{\ell=0}^{k} \delta_k(\ell) C(\omega; \ell)$$

and vice versa

$$C(\omega; k) = \sum_{\ell=1}^{k} \delta_k^*(\ell) \left(\pi^{-\ell} \widetilde{L}(\omega; \ell) - \delta_\ell(0) C(\omega; 0) \right).$$

Idea of proof.

- The identity $\widetilde{L}(\omega;k) := \sum_{x\in\mathbb{R}^\times} \beta_\omega(x)x^{-k} = \pi^k \sum_{\ell=0}^{k} \delta_k(\ell)C(\omega;\ell)$ is proved by looking at the local Taylor expansions of pre-weak functions $\omega(z)$ in $z=0$ and using the limit formula

$$\widetilde{L}(\omega;k) = 2\pi i \ \mathrm{res}_{z=0}\left(z^{-k}\omega(z)\right). \tag{3.5}$$

- Identity $C(\omega;k) = \sum_{\ell=1}^{k} \delta_k^*(\ell)\left(\pi^{-\ell}\widetilde{L}(\omega;\ell) - \delta_\ell(0)C(\omega;0)\right)$ is more involved. It makes use of classical formulas for cotangent sums (in order to find the independent δ^* by comparing coefficients between polynomials in N). We skip the details.

Theorem 3.4 (F., 2020, [6]).

Let ω be a pre-weak function. Let $K|\mathbb{Q}$ be a field extension (not necessarily finite) and $m \in \mathbb{N}$ be any positive integer. Assume that $C(\omega; 0) \in K$. Then we have

$$\frac{\widetilde{L}(\omega; 1)}{\pi}, \frac{\widetilde{L}(\omega; 2)}{\pi^2}, \cdots, \frac{\widetilde{L}(\omega; m)}{\pi^m} \in K \iff C(\omega; 1), \cdots, C(\omega; m) \in K.$$

Proof. We can express the terms $\widetilde{L}(\omega; k)\pi^{-k} - C(\omega; 0)\delta_k(0)$ as rational combinations of $C(\omega; m)$, $1 \leq m \leq k$ and vice versa the terms $C(\omega; k)$ as rational combinations of $\widetilde{L}(\omega; m)\pi^{-m} - C(\omega; 0)\delta_m(0)$. Since $\delta_m(0) \in \mathbb{Q}$ for all $m \geq 0$, the claim follows with $C(\omega; 0) \in K$. $\square$

For example, $\left(1 - \frac{1}{N^{2k}}\right)\zeta(2k) \in \mathbb{Q}\pi^{2k}$ for all integers $k > 0$ implies $\sum_{j=1}^{N-1} \cot^m\left(\frac{\pi j}{N}\right) \in \mathbb{Q}$ (Berndt, Yeap) for all integers $m > 0$, and vice versa.

We can use this theorem to investigate cotangent sums and L-functions in more detail. We motivate this by the following well-known result.

Theorem 3.5 (Berndt, Zaharescu, 2004, [2]).

Let $k > 0$. For odd, real and primitive characters mod k we have the formula

$$C(\chi; 1) = 2\sqrt{k}h(-k),$$

where $h(-k)$ is the class number of $\mathbb{Q}(\sqrt{-k})$.

The value $h(-k)$ is closely linked to $L(\chi, 1)$ by the class number formula!

Theorem 3.6 (F., 2020, [6]).

Let χ^+ be an even and χ^- be an odd primitive character modulo $N > 1$ and $m \geq 1$ be an integer. We have the explicit formulas

$$C(\chi^+; 2m) = \mathcal{G}(\chi^+) \sum_{\ell=1}^{m} (-1)^{\ell-1} 2^{2\ell} \delta^*_{2m}(2\ell) \frac{B_{2\ell,\overline{\chi^+}}}{(2\ell)!}. \tag{3.6}$$

and

$$C(\chi^-; 2m-1) = i\mathcal{G}(\chi^-) \sum_{\ell=1}^{m} (-1)^{\ell-1} 2^{2\ell-1} \delta^*_{2m-1}(2\ell-1) \frac{B_{2\ell-1,\overline{\chi^-}}}{(2\ell-1)!}. \tag{3.7}$$

Here, $\mathcal{G}(\chi)$ is the usual Gauss sum and $B_{n,\chi}$ are the generalized Bernoulli numbers.

Corollary 3.7 (F., 2020, [6]).

Let p be a prime and χ be the Legendre symbol modulo p. Then we have for all $m \in \mathbb{N}$

$$\sqrt{p}C(\chi; m) \in \mathbb{Q}.$$

Proof. For the Legendre symbol χ we have the identity

$$\mathcal{G}(\chi) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod 4 \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

Since χ is real, we have $B_{n,\overline{\chi}} = B_{n,\chi} \in \mathbb{Q}$ for all n and the claim follows with Theorem 3.6. □

We can apply this formalism also to Dirichlet series with trigonometric coefficients.

Theorem 3.8 (F., 2020, [4]).

Let $k > 0, N > 0$ *and* $m \geq 0$ *be integers, such that* $k + m \equiv 0 \pmod 2$. *Then we have the formula*

$$\sum_{\substack{n>0 \\ n \not\equiv 0 \pmod N}} \frac{\cot^m\left(\frac{n\pi}{N}\right)}{n^k} = \left(\frac{\pi}{N}\right)^k \sum_{j=1}^{\frac{m+k}{2}} a_{k,m}(2j)\zeta(2j)\pi^{-2j}\left(N^{2j}-1\right), \quad (3.8)$$

where the rational numbers $a_{k,m}(j)$ *are given by*

$$a_{k,m}(j) = \sum_{\ell=0}^{k} \delta_k(\ell)\delta^*_{m+\ell}(j). \quad (3.9)$$

Idea of proof. Use the connection between L-series and cotagent sums *twice*.

- Use $C(\omega; k) = \sum_{\ell=1}^{k} \delta_k^*(\ell) \left(\pi^{-\ell}\widetilde{L}(\omega; \ell) - \delta_\ell(0) C(\omega; 0)\right)$ to obtain

$$\sum_{\ell=1}^{N-1} \cot^{2n}\left(\frac{\pi\ell}{N}\right) = (N-1)(-1)^n + 2\sum_{\ell=1}^{n} \delta_{2n}^*(2\ell)\zeta(2\ell)\pi^{-2\ell}\left(N^{2\ell} - 1\right).$$

- Using this time $\widetilde{L}(\omega; k) = \pi^k \sum_{\ell=0}^{k} \delta_k(\ell) C(\omega; \ell)$ we obtain

$$\sum_{\substack{n>0 \\ n\not\equiv 0 \pmod N}} \frac{\cot^m\left(\frac{n\pi}{N}\right)}{n^k} = \frac{1}{2}\left(\frac{\pi}{N}\right)^k \sum_{\ell=0}^{k} \delta_k(\ell) \sum_{n=1}^{N-1} \cot^{\ell+m}\left(\frac{\pi n}{N}\right).$$

On the right hand side we can substitute the above expressions for the cotangent sums.

Some explicit examples: We have for all positive integers N, k

$$\sum_{\substack{n>0 \\ n\not\equiv 0 \pmod N}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^2} = \frac{(N^4-5N^2+4)\pi^2}{90N^2},$$

$$\sum_{\substack{n>0 \\ n\not\equiv 0 \pmod N}} \frac{\cot\left(\frac{n\pi}{N}\right)}{n^3} = \frac{(N^4-5N^2+4)\pi^3}{90N^3},$$

$$\sum_{\substack{n>0 \\ n\not\equiv 0 \pmod N}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^4} = \frac{(N^6-7N^4+14N^2-8)\pi^4}{945N^4},$$

$$\sum_{\substack{n>0 \\ n\not\equiv 0 \pmod N}} \frac{\cot^2\left(\frac{n\pi}{N}\right)}{n^{2k}} = \frac{\zeta(2k+2)}{\pi^2}N^2 - \frac{2}{3}\zeta(2k) + O\left(\frac{1}{N^2}\right).$$

Similar results exist for twisted trigonometric series.

Theorem 3.9 (F., 2020, [4]).

Let $N > 1$, $k > 0$ and $m \geq 0$ be integers, and χ be a primitive Dirichlet character modulo N. Assume that $k + m + \frac{1-\chi(-1)}{2} \equiv 0 \pmod{2}$. Then we have the formula

$$\sum_{\substack{n>0 \\ n \not\equiv 0 \pmod{N}}} \frac{\chi(n)\cot^m\left(\frac{n\pi}{N}\right)}{n^k} = \sum_{j=1}^{m+k} a_{k,m}(j) L(\chi; j) \pi^{k-j} N^{j-k} \tag{3.10}$$

where the rational numbers $a_{k,m}(j)$ are given as above.

Thank you for your attention!

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