Kudla's Conjecture in H^* for unitary Shimura varieties

Let k - imaginary quadratic field

$\mathcal{O}_k$ - ring of integers

Let (L, h) be a Hermitian lattice, i.e. an $\mathcal{O}_k$-module equipped with a Hermitian pairing valued in $\mathcal{O}_k$, signature $(n+1, 1)$.

Self-dual $(L^\vee = L)$, $\Gamma = \mathrm{Aut}(L)$.

The associated S.V. $X = \Gamma \backslash U(n+1,1) / U(n+1)U(1)$ dim $n+1$

is a ball quotient, usually non-compact with 0-cusps indexed by Γ-orbits of primitive isotropic $\mathcal{O}_k$-submodules $J \subseteq L$, (must be rank 1).

$$\begin{array}{ccccc} X & \subseteq & X^{BB} & = & X \cup \{0\text{-cusps}\} \\ \| & & \uparrow & & \uparrow \\ X & \subseteq & X^{tor} & = & X \cup \{\text{CM abelian vars}\}/\text{fin. groups} \end{array}$$

boundary divisors $\Delta_J = A/G$

$\mathcal{O}_k^n \cong J^\perp / J$ has signature $(n, 0)$, $A \cong J^\perp/J \otimes_{\mathcal{O}_k} \mathbb{C}/\mathcal{O}_k$

$G = \mathrm{Aut}(J^\perp/J)$ finite group

Generating series of special cycles on X, X^{tor}:

Fix $1 \le g \le n+1$ $\quad \sum_{\underline{m} \ge 0} [Z(\underline{m})]\, q^{\underline{m}}$ $\quad \underline{m}$ is semi-positive Hermitian $g \times g$ $\mathcal{O}_k$-matrix

$q^{\underline{m}} = e^{2\pi i \mathrm{Tr}(y \underline{m})}$

Thm. (KM) This series is a Hermitian modular form of weight $n+2$, $U(g,g)(\mathbb{Z})$

valued in $H^{2g}(X,\mathbb{Q})$.

$$Z(\underline{m}) = \Gamma \backslash \bigcup_{\substack{\underline{v} \in L^g \\ h(\underline{v},\underline{v}) = \underline{m}}} \operatorname{span}(\underline{v})^{\perp} \subseteq X$$

$$[Z(\underline{m})] = (\text{class of } Z(\underline{m})) \cdot c_1(\lambda^{\vee})^{g-\operatorname{rk}(\underline{m})}$$

Thm (G-Tayou) Fix $1 \le g < \frac{n+1}{2}$. In X^{tor},

- $\sum_{\underline{m} \geq 0} [\overline{Z(\underline{m})}] q^{\underline{m}}$ is Hermitian quasi-modular form of weight $n+2$ valued in $H^{2g}(X^{tor})$

- $\sum_{\underline{m} \geq 0} [\overline{Z(\underline{m})}] q^{\underline{m}} - \sum_{\{J\}/\Gamma} F_J(q)$ is Hermitian modular.

  "corrected q-series"

  quasi-modular valued in $H^{2g}(\Delta_J)$

Def. A Hermitian quasi-modular form is a holomorphic function $f: \mathcal{H}_g \to \mathbb{C}$ which admits a <u>completion</u> $f + f^*$ non-holomorphic modular

This will follow formally from a result of Freitag, Roehrig.

Previous work: CH^1 (arithmetic) Bruinier - Howard - Kudla - Rapoport - Yang

CH_0 Bruinier - Rosu - Zemel.

Lemma. If (X, Δ) is a nonsingular pair, and N_Δ the normal bundle to Δ is ample (or anti-ample) then

$$H_k(X \setminus \Delta) \oplus H_k(\Delta) \longrightarrow H_k(X) \quad \text{is } \underline{\text{surjective}}$$

$(k = 2g)$ for $k \le \dim(\Delta)$.

Pf. Use the Mayer-Vietoris sequence for $X = (X \setminus \Delta) \cup N_\Delta$, then Gysin sequence for $N_\Delta \setminus \Delta$. Use Hard-Lefschetz thm for Δ, $c_1(N_\Delta)$. ∎

Using the Lemma + Poincaré duality on X^{tor}, it's enough to study the intersection of $\sum_{\underline{m}} [\overline{Z(\underline{m})}] q^{\underline{m}}$ with test cycles $T \in H_{2g}(X^{tor}, \mathbb{Q})$

$$\underset{\substack{\| \\ T_0 + \sum_{\{J\}/\Gamma} T_J}}{T} \cdot \sum_{\underline{m}} [Z(\underline{m})] q^{\underline{m}} = \underbrace{T_0 \cdot \sum_{\underline{m}} [Z(\underline{m})] q^{\underline{m}}}_{\text{modular by Kudla-Millson}} + \underbrace{\sum_J T_J \sum_{\underline{m}} [\overline{Z(\underline{m})}] q^{\underline{m}}}_{\text{computed in each } \Delta_J}$$

Each Δ_J is a finite gp quotient of E^n where $E = \mathbb{C}/\mathcal{O}_k$

$$\overline{Z}(\underline{m}) \cap \Delta_J \quad \text{pulls back to} \quad \sum_{\substack{\underline{v} \in (J^\perp/J)^g \\ h(\underline{v},\underline{v}) = \underline{m}}} \operatorname{span}(\underline{v})^\perp \subseteq E^n$$

$$T_J \cdot \sum_{\substack{\underline{v} \in (J^\perp/J)^g \\ h(\underline{v},\underline{v}) = \underline{m}}} \overline{Z(\underline{m})} q^{m} = \sum_{\underline{v} \in (J^\perp/J)^g} P(\underline{v})\, q^{h(\underline{v},\underline{v})} \quad \text{weighted theta series}$$

$P : (J^{\perp}/J)^g \longrightarrow \mathbb{Q}$ is a polynomial with the transformation law

$\forall A \in GL_g(\mathbb{C})$ $$P(\underline{v}\, A) = |\det(A)|^2 P(\underline{v}).$$

Freitag polynomials. $\mathcal{F}_{n,g}$

They have a Laplace operator $\Delta : \mathcal{F}_{n,g} \to \mathcal{F}_{n,g-1}$

Thm (Roehrig, Freitag)

If P is harmonic then $\sum_{\underline{v}} P(\underline{v})\, q^{h(\underline{v},\underline{v})}$ is modular wt $n+2$

If P is not harmonic, then we have the completion:

$X+iY \in \mathcal{H}_g$ $$\det(Y)^{-1} \sum_{\underline{v}} \exp\left(-\frac{\Delta}{4\pi}\right)(P)\left(\underline{v}\, Y^{1/2}\right) q^{h(\underline{v},\underline{v})}$$

non-holomorphic of weight $n+2$ $\Rightarrow$ quasi-modularity

For the corrected q-series, we need to know how far P is from being harmonic. This depends only on where T_J sits in the Lefschetz decomposition of $H_{2g}(\Delta_J, \mathbb{Q}) \cong H_{2g}(E^n, \mathbb{Q})$

$c_1(N^{\vee}_{\Delta_J})$ is ample on Δ, so it pulls back to polarization Θ

$$\Theta \text{ on } E^n = (\text{pos. const.})\ \Theta_h$$

width of the cusp

Thm (G-Tayou) $$\bigoplus_{g=0}^{n} H_{g,g}(E^n) \xrightarrow{\cong} \bigoplus_{g=0}^{n} \mathcal{F}_{n,g}$$

intertwining the $\mathfrak{sl}_2$-representations on both sides lowering operator $\cap \Theta$ lowering operator = Laplace op.

raising operator adjoint raising operator = Euler operator

Ham $= g - 2n$

We have a linear map $\vartheta : \mathcal{F}_{n,g} \longrightarrow \mathrm{QHerm}(n+2)$

$$P \longmapsto \sum_{\underline{v} \in (J^{\perp}/J)^{g}} P(\underline{v}) q^{h(\underline{v},\underline{v})}$$

Lefschetz filt. $\longmapsto$ depth filtration on QHerm.

$\sum_{m \geq 0} [\overline{Z(\underline{m})}] q^{\underline{m}}$ pulls back to a natural element of

$\alpha \in H^{g,g}(E^n) \otimes \mathcal{F}_{n,g}$ annihilated by $\mathfrak{sl}_2$.

we can express α as a sum of Lefschetz components.

$\exists$ universal projectors in $\mathcal{U}(\mathfrak{sl}_2)$ onto each component

$$\Pi_{\mathrm{prim}} = \sum_{j=0}^{g} (-1)^j \frac{(m+1)!}{j!\,(m+j+1)!} E^j F^j \quad \text{where } n - 2g = m$$

A few words about the orthogonal case: $O(2, n+2)$

$(\Gamma \backslash \mathbb{D})^{\mathrm{II}}$, Δ_{II} satisfies the splitting lemma so the same technique can be applied in that setting

$\Delta_J =$ fin. quotient of $\mathcal{E}^n$

modular family of elliptic curves.

$$H_{g,g}(\mathcal{E}^n, \mathbb{Q}) \xrightarrow[]{\text{not isom } g \geq 2} \mathcal{F}^{\text{ortho}}_{n,g} \cong H^0(G(g,n), \mathcal{O}(2))$$

$$\mathcal{F}^{\text{ortho}}_{n,g} = \{ P : \mathbb{Q}^{1\times g} \to \mathbb{Q} \mid P(\underline{v}A) = \det(A)^2 P(\underline{v}) \}$$

<u>Corollary</u>. There exist cycles $T_J \in \Delta_J$ orthogonal to all $\overline{Z(\underline{m})}$.