Sato-Tate conjecture in arithmetic progressions for certain families of elliptic curves

(joint work with Kathrin Bringmann and Ben Kane)

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PLAN OF TALK

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- Part-I: Introduction to theory of equidistribution.

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- Part-II: Introduction to Sato-Tate conjecture.
- Part-III: Distribution of moments of trace of Frobenius in arithmetic progressions.

Part-I: Introduction to theory of equidistribution.

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(P.G.L. Dirichlet, 1805 – 1859)
(Source: wikipedia.org)

1842 - Dirichlet showed that there are infinitely many elements of this sequence in any neighborhood of 0.

(Leopold Kronecker, 1823 – 1891)
(Source: wikipedia.org)

1884 - Kronecker showed that this sequence is in fact dense throughout the interval [0, 1].

(P. Bohl)
(1865-1921)

(H. Weyl)
(1885-1955)

(W. Sierpinski)
(1882-1969)

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1909 - Piers Bohl.
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Question How the sequence $\{n\theta\}$ is distributed in the unit interval, when θ is irrational?

General principles of equidistribution

Let $\{x_n\}$ be a sequence of real numbers in the unit interval $[0, 1]$. For a subset I of $[0, 1]$, and for a fixed natural number N, let

$$A_I(N) := \#\{1 \leq n \leq N;\, x_n \in I\}.$$

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Definition 1

$\{x_n\}$ is said to be **equidistributed** *in the unit interval if for any $I = [a, b] \subset [0, 1]$, we have*

$$\lim_{N\to\infty} \frac{A_I(N)}{N} = b - a.$$

A sequence $\{x_n\}$ of real numbers is said to be **equidistributed** $\Leftrightarrow$ for every $I \subset [0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(x_n) = \int_0^1 \chi_I(x)dx.$$

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$\Leftrightarrow$ for all (complex valued) Riemann integrable functions $f(x)$ of period 1,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx.$$

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$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx.$$

$\Rightarrow$ For all non-zero $m \in \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} = 0.$$

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In that paper he gave a criterion for equidistribution in terms exponential sum.

Weyl's Criterion

Theorem 3

Weyl's criterion: *A sequence* $\{x_n\}$ *is e.d if and only if*

$$c_m := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(mx_n) = 0$$

for every $m \in \mathbb{Z}$, $m \neq 0$, $e(t) = e^{2\pi it}$.

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Application: If $\theta \notin \mathbb{Q}$, then $\{n\theta\}$ e.d in $[0, 1]$. If $m \neq 0$,

$$\frac{1}{N} \left| \sum_{n=1}^{N} e(mn\theta) \right|$$

$$= \frac{1}{N} \left| \frac{\sin(\pi mN\theta)}{\sin(\pi m\theta)} \right|$$

$$\to 0 \text{ as } N \to \infty.$$

Definition 4

Consider finite multi sets A_n with $\#A_n \to \infty$ as $n \to \infty$.
$\{A_n\}$ is **set-equidistributed** *with respect to a probability measure μ if for every $[a, b] \subset [0, 1]$,*

$$\lim_{n\to\infty} \frac{\#\{t \in A_n : \ t \in [a, b]\}}{\#A_n} = \int_a^b d\mu.$$

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In particular, if

$$A_n = \{x_1, x_2, ..., x_n\}$$

then the definition is the definition of an equidistributed sequence with respect to μ.

(I.J Schoenberg, 1903 - 1990) (N. Wiener, 1926 - 1964)
(Source: wikipedia.org)

Theorem 5 (Wiener-Schoenberg)

A_n is equidistributed with respect to some positive continuous measure if and only if the Weyl limits exist for every integer m and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{|m|\leq N} |c_m|^2 = 0.$$

(Van der Corput, 1890 – 1975)
(Source: wikipedia.org)

Theorem 6 (Van der Corput,1931)

If for each positive integer s, the sequence $\{x_{n+s} - x_n\}$ is equidistributed (mod 1), then the sequence $\{x_n\}$ is equidistributed (mod 1).

(Ivan Vinagradov, 1891 – 1983)
(Source: wikipedia.org)

1937 - Ivan Vinogradov; $\{p\theta\}$ is equidistributed.

Part-II: Introduction to Sato-Tate conjecture.

Elliptic Curve

- What is an **elliptic curve**?
  For the time being, we define an elliptic curve to be any equation of the form

$$E : y^2 = x^3 + ax^2 + bx + c$$

  with $a, b, c \in \mathbb{Z}$, and such that the polynomial $x^3 + ax^2 + bx + c$ does not have repeated roots.

Theorem 7 (Siegel, 1928)

Let E be an elliptic curve given by an equation $E : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}$. Then E has only finitely many points $P = (x, y)$ with integer coordinates $x, y \in \mathbb{Z}$, i.e., $E(\mathbb{Z})$ is a finite set

The group $E(\mathbb{F}_p)$ is obviously a finite group. Indeed, it clearly has no more than $2p+1$ points.

Theorem 8 (Hasse, 1922)

Let E be an elliptic curve

$$E : y^2 = x^3 + ax + b$$

with $a, b \in \mathbb{F}_p$. Then $|\#E(\mathbb{F}_p) - p - 1| \leq 2\sqrt{p}$.

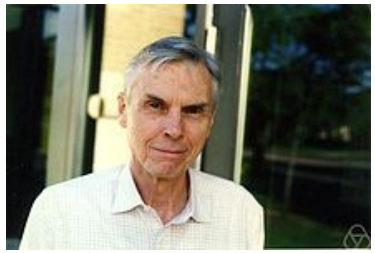
For an elliptic curve E defined over the finite field $\mathbb{F}_{p^r}$ with p^r elements (p prime, $r \in \mathbb{N}$), the *trace of Frobenius* is given by

$$\mathrm{tr}(E) = \mathrm{tr}_{p^r}(E) := p^r + 1 - \#E\left(\mathbb{F}_{p^r}\right).$$

Here $E(\mathbb{F}_{p^r})$ is the set of points on the elliptic curve over the finite field $\mathbb{F}_{p^r}$.

Sato-Tate conjecture

(Mikio Sato)



(John Tate)

(Source: wikipedia.org)

Taking $-1 \leq a \leq b \leq 1$ and a fixed elliptic curve E over $\mathbb{Q}$, it was independently conjectured by Sato and Tate that if E does not have complex multiplication, then

$$\lim_{N\to\infty} \frac{\#\{p \leq N : 2a\sqrt{p} \leq \mathrm{tr}(E_p) \leq 2b\sqrt{p}\}}{\#\{p \leq N\}} = \frac{2}{\pi}\int_a^b \sqrt{1-x^2}dx.$$

In a series of papers by Richard Taylor, Michael Harris, Nick Shepherd-Barron, David Geraghty, Laurent Clozel and Tom Barnet-Lamb, this conjecture is now a theorem.

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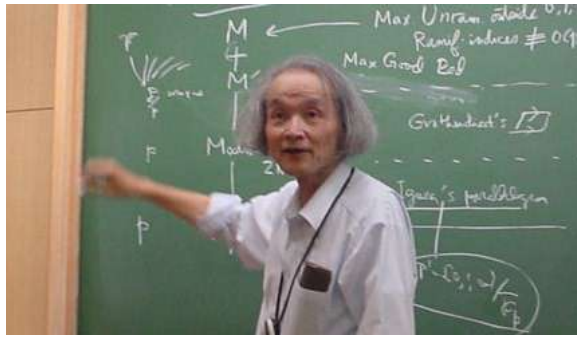
(B.J. Birch)



(Y. Ihara)

(Source: wikipedia.org)

Theorem 9 (Birch; 1968)

$\{\frac{tr_p(E)}{\sqrt{p}}\}$ *satisfy the Sato-Tate law in* $[-2, 2]$ *as* $p \to \infty$.

In other words Birch showed that

$$\sum_{E} \left(\frac{tr_p(E)}{\sqrt{p}} \right)^{2k} \sim C_k p^k \text{ as } p \to \infty,$$

where $C_k := \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

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Banks-Shparlinski; $\cdots\cdots\cdots$
For $m \in \mathbb{Z}$ and $M \in \mathbb{N}$, we restrict to the set

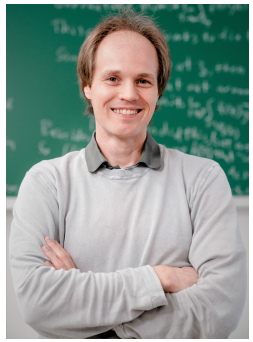
$$\mathcal{E}_{m,M,p^r} := \{E/\mathbb{F}_{p^r} : \text{tr}(E) \equiv m \pmod{M}\}.$$

Understanding the distribution of the numbers $\mathrm{tr}(E)$ in this arithmetic progression is closely related to investigating the *weighted κ-th moment with respect to* $\mathrm{tr}(E)$ (for $\kappa \in \mathbb{N}_0$)

$$S_{\kappa,m,M}(p^r) := \sum_{\substack{E/\mathbb{F}_{p^r} \\ \mathrm{tr}(E)\equiv m \pmod{M}}} \frac{\mathrm{tr}(E)^\kappa}{\#\,\mathrm{Aut}_{\mathbb{F}_{p^r}}(E)} = \sum_{E\in\mathcal{E}_{m,M,p^r}} \frac{\mathrm{tr}(E)^\kappa}{\#\,\mathrm{Aut}_{\mathbb{F}_{p^r}}(E)}.$$

Part-III: Distribution of moments of trace of Frobenius in arithmetic progressions.

(K. Bringmann)



(B. Kane)

(Source: webpage)

(S. Pujahari)

Theorem 1

Let $m \in \mathbb{Z}$, $M \in \mathbb{N}$ and $\varepsilon > 0$ be given. Let $p > 3$ be a prime for which $p \nmid \gcd(m, M)$ and $k \in \mathbb{N}$. As $r \to \infty$, we have

$$\frac{S_{2k,m,M}\left(p^r\right)}{p^{rk} S_{m,M}\left(p^r\right)} = C_k + O_{k,p,M,\varepsilon}\left(p^{\left(-\frac{1}{2}+\varepsilon\right)r}\right),$$

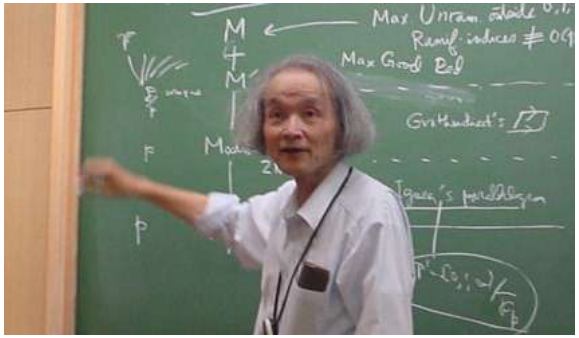
where C_k is the k-th Catlan number.

Theorem 2

Let $m \in \mathbb{Z}$, $M \in \mathbb{N}$ and $\varepsilon > 0$ be given.
For primes $p \to \infty$, we have

$$\frac{S_{2k,m,M}(p)}{p^k S_{m,M}(p)} = C_k + O_{k,M,\varepsilon}\left(p^{-\frac{1}{2}+\varepsilon}\right),$$

$$\frac{S_{2k,m,M}(p^r)}{p^{rk} S_{m,M}(p^r)} = C_k + O_{k,M,r,\varepsilon}\left(p^{-1+\varepsilon}\right) \ (r \geq 2).$$



(B.J. Birch) (Y. Ihara)

(Source: wikipedia.org)

For $M = 1$, these sums were studied by Birch and implicitly appear in the work of Ihara. They obtained a formula for these sums in terms of the trace of Hecke operators that yields the asymptotic like above.

(Wouter Castryck) (Hendrik Hubrechts)

(Source: wikipedia.org)

They studied the distribution of $\{tr_q(E) \equiv t \pmod N\}$, $N \in \mathbb{N}$ and $t \in \{1, 2, \ldots N-1\}$.

(N. Kaplan)

(I. Petrow)

(Source: webpage)

For $M = 2$, formulas for $S_{2k,m,2}$ were obtained by Kaplan and Petrow.

A special case of above theorem yields a result about elliptic curves with *M-torsion points* ($M \in \mathbb{N}$)

$$E[M] := \{P \in E : \text{ord}(P) \mid M\}.$$

Here ord(P) means the order of the point under the group law defined on elliptic curves.

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$$E[M] := \{P \in E : \mathrm{ord}(P) \mid M\}.$$

Here $\mathrm{ord}(P)$ means the order of the point under the group law defined on elliptic curves. We denote the subset of torsion points of precise order M by

$$E^*[M] := \{P \in E : \mathrm{ord}(P) = M\}$$

and define

$$S^*_{\kappa,M}(p^r) := \sum_{\substack{E/\mathbb{F}_{p^r} \\ E^*[M] \neq \emptyset}} \frac{\mathrm{tr}(E)^\kappa}{\#\,\mathrm{Aut}_{\mathbb{F}_{p^r}}(E)}.$$

Corollary 3

Let M be a square free integer.

1. *As $p \to \infty$, we have*

$$\frac{S^*_{2k,M}(p)}{p^k S^*_M(p)} = C_k + O_{k,M\varepsilon}\left(p^{-\frac{1}{2}+\varepsilon}\right),$$

$$\frac{S^*_{2k,M}(p^r)}{p^{rk} S^*_M(p^r)} = C_k + O_{k,M,r,\varepsilon}\left(p^{-1+\varepsilon}\right) \quad (r \geq 2).$$

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2. *If $p > 3$ is a prime we have, as $r \to \infty$*

$$\frac{S^*_{2k,M}(p^r)}{p^{rk} S^*_M(p^r)} = C_k + O_{k,p,M,\varepsilon}\left(p^{\left(-\frac{1}{2}+\varepsilon\right)r}\right).$$

Now we consider moments of sums of Hurwitz class numbers that are of independent interest. Let $h(d)$ denote the class number.

Let $H(n) := \sum_{\substack{d^2 \mid n \\ n/d^2 \equiv 0,1 \pmod 4}} h_w(n/d^2)$ be the nth Hurwitz class number,

where

$$h_w(d) := \begin{cases} h(d)/3 & \text{if } d = -3; \\ h(d)/2 & \text{if } d = -4; \\ h(d) & \text{else.} \end{cases}$$

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Let

$$\mathcal{H}(\tau) := \sum_{n \in \mathbb{Z}} H(n) q^n$$

be the generating function for the Hurwitz class numbers.

(D. Zagier)
(Source: wikipedia.org)

Theorem 4 (Zagier; 1976)

$\mathcal{H}$ is a Mock modular forms of weight $\frac{3}{2}$.

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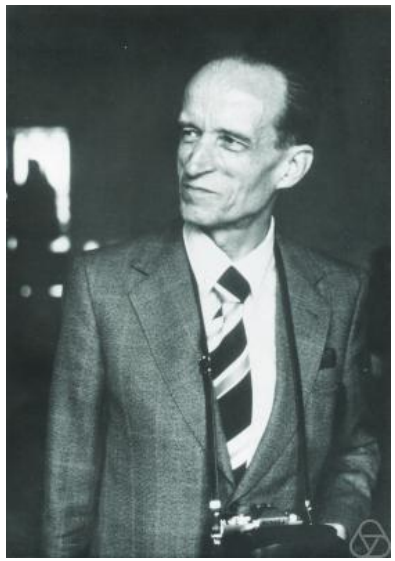
Sums of moments of these Hurwitz class numbers analogous to $S_{\kappa,m,M}$ are given by

$$H_{\kappa,m,M}(n) := \sum_{\substack{t\in\mathbb{Z} \\ t\equiv m \pmod{M}}} t^{\kappa} H\left(4n - t^2\right).$$

Sums of this type have occurred throughout the literature and satisfy many nice identities.



(M. Eichler)
(Source: wikipedia.org)

Theorem 5 (Eichler; 1956)

For $M = 1$, $\kappa = 0$, and $n = p$ prime we have the famous identity

$$H_{0,1}(p) = 2p.$$

Theorem 6

Let $m, M, k \in \mathbb{N}$ be given. As $n \to \infty$, we have

$$\frac{H_{2k,m,M}(n)}{n^k H_{m,M}(n)} = C_k + O_{k,M,\varepsilon}\left(n^{-\frac{1}{2}+\varepsilon}\right).$$

Let

$$\mathcal{E}_{p^r,t} := \{E/\mathbb{F}_{p^r} : \mathrm{tr}(E) = t\}$$

and

$$N_A(p^r; t) := \sum_{E \in \mathcal{E}_{p^r,t}} \frac{1}{\#\,\mathrm{Aut}_{\mathbb{F}_{p^r}}(E)}.$$

Then, for a prime $p > 3$ and $r \in \mathbb{N}$ we have

$$2N_A(p^r; t) = \begin{cases} H(4p^r - t^2) & \text{if } t^2 < 4p^r,\ p \nmid t, \\ H(4p) & \text{if } t = 0 \text{ and } r \text{ is odd}, \\ \frac{1}{2}\left(1 - \left(\frac{-1}{p}\right)\right) & \text{if } t = 0 \text{ and } r \text{ is even}, \\ \frac{1}{3}\left(1 - \left(\frac{-3}{p}\right)\right) & \text{if } t^2 = p^r, \\ \frac{1}{12}(p-1) & \text{if } t^2 = 4p^r, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch of proof

Let $G_{k,m,M}(n) := \sum_{t \equiv \pm m \pmod{M}} p_{2k}(t,n) H\left(4n - t^2\right)$, where $p_{2k}(t,n)$ denotes the $(2k)$-th coefficients in the Taylor expansion of $(1 - tX + nX^2)^{-1}$.

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- Since $C_0 = 1$, the claim holds trivially for $k = 0$.

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- Since $C_0 = 1$, the claim holds trivially for $k = 0$.
- For $k \geq 1$,

$$G_{k,m,M}(n) + \frac{1}{2^{2k} \cdot k!} \lambda_{2k+1,m,M}(4n)$$

is the n-th coefficient of a weight $2k+2$ cusp form., where

$$\lambda_{\ell,m,M}(n) := 2 \sum_{\pm} \sum_{\substack{t > s \geq 0 \\ t^2 - s^2 = n \\ t \equiv \pm m \pmod{M}}} (t-s)^{\ell}.$$

continued...

- $$\lambda_{\ell,m,M}(n) \leq n^{\frac{\ell}{2}} \lambda_{m,M}(n) \ll_{\varepsilon} n^{\frac{\ell}{2}+\varepsilon}.$$

continued...

- $$\lambda_{\ell,m,M}(n) \leq n^{\frac{\ell}{2}} \lambda_{m,M}(n) \ll_{\varepsilon} n^{\frac{\ell}{2}+\varepsilon}.$$

- By Deligne's bound it thus may be bound against $O_{k,M,\varepsilon}(n^{k+\frac{1}{2}+\varepsilon})$. The implied constant in the error term a priori depends on m as well, but by taking the maximum over all of the choices of $m \pmod M$.

continued...

- $$\lambda_{\ell,m,M}(n) \leq n^{\frac{\ell}{2}} \lambda_{m,M}(n) \ll_{\varepsilon} n^{\frac{\ell}{2}+\varepsilon}.$$

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- $$G_{k,m,M}(n) \ll_{k,M,\varepsilon} n^{k+\frac{1}{2}+\varepsilon}.$$

Theorem 10

Let $k \in \mathbb{N}$, $m \in \mathbb{Z}$, and $M \in \mathbb{N}$ be given.

(1) *For a fixed prime $p > 3$, as $r \to \infty$ we have*

$$S_{2k+1,m,M}(p^r) = O_{k,M,\varepsilon}\left(p^{(k+1+\varepsilon)r}\right).$$

(2) *For $r \in \mathbb{N}$ fixed, as $p \to \infty$ we have*

$$S_{2k+1,m,M}(p^r) = O_{k,M,\varepsilon}\left(p^{(k+1+\varepsilon)r}\right).$$

Theorem 11

Let $k \in \mathbb{N}_0$ be given. Then

$$\frac{H_{2k+1,m,M}(n)}{n^{k+\frac{1}{2}} H_{m,M}(n)} = O_{k,M,\varepsilon}\left(n^{-\frac{1}{2}+\varepsilon}\right), \quad H_{2k+1,m,M}(n) = O_{k,M,\varepsilon}\left(n^{k+1+\varepsilon}\right).$$

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Theorem 12

Let $m \in \mathbb{Z}$ and $M \in \mathbb{N}$ be given. The $x_E := \frac{tr_q(E)}{\sqrt{q}}$ for $E \in \mathcal{E}_{m,M}(p^r)$ are equidistributed with respect to the Sato–Tate measure. Specifically, we have

$$\lim_{p\to\infty} \Pr_{\text{Aut}}\left(a \leq \frac{tr_q(E)}{\sqrt{q}} \leq b : E \in \mathcal{E}_{m,M}(p^r)\right) = \int_a^b \mu(x)dx.$$


THANK YOU