Chowla–Selberg phenomenon over function fields

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Outline

- Brief review of classical story.
- Gamma functions in positive characteristic and Lang–Rohrlich conjecture
- Period interpretation of special gamma values and distributions
- Thakur's recipe/conjecture on the Chowla–Selberg phenomenon over function fields

Euler's gamma function

In order to solve the interpolation problem of factorials, Euler introduced

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt, \quad \forall s > 0$$

As $\Gamma(s+1) = s \cdot \Gamma(s)$ for $\mathrm{Re}(s) > 0$, we may extend $\Gamma(s)$ to a meromorphic function on the whole complex s-plane with simple poles at $s \in \mathbb{Z}_{\leq 0}$.

The Weierstrass expression

For $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$,

$$\Gamma(s) = e^{-\gamma s} \cdot s^{-1} \prod_{n=1}^{\infty} (1 + \frac{s}{n})^{-1} \cdot e^{s/n},$$

where γ is the Euler-Mascheroni constant.

Functional equations

Reflection formula (Euler)

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Duplication formula (Legendre)

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = \sqrt{\pi}\cdot 2^{1-2s}\cdot\Gamma(2s).$$

Multiplication formula (Gauss)

For every $n \in \mathbb{N}$,

$$\Gamma(s)\Gamma(s+\frac{1}{n})\cdots\Gamma(s+\frac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}}\cdot n^{\frac{1}{2}-ns}\cdot\Gamma(ns).$$

Monomial relations

Given two complex values $\alpha, \beta \in \mathbb{C}^\times$, we write $\alpha \sim \beta$ if $\alpha/\beta \in \overline{\mathbb{Q}}^\times$.

Monomial relations

(1) (Reflection formula) For $x \in \mathbb{Q}$ with $x \notin \mathbb{Z}$, we have

$$\Gamma(x) \cdot \Gamma(1 - x) \sim \pi.$$

(2) (Multiplication formula) Let $n \in \mathbb{N}$. For $x \in \mathbb{Q}$ with $nx \notin \mathbb{Z}_{\leq 0}$, we have

$$\prod_{i=0}^{n-1} \Gamma(x + \frac{i}{n}) \sim \pi^{\frac{n-1}{2}} \cdot \Gamma(nx).$$

Diamond bracket relations

Given $x \in \mathbb{R}$, let $\langle x\rangle$ be the fractional part of x, i.e. $0 \le \langle x\rangle < 1$ and $x - \langle x\rangle \in \mathbb{Z}$. As $\langle x+n\rangle = \langle x\rangle$ for every $n \in \mathbb{Z}$, we may regard $\langle\cdot\rangle$ as a function on $\mathbb{R}/\mathbb{Z}$.

Lemma

For each $x \in \mathbb{R}/\mathbb{Z}$, one has that

$$\langle x\rangle + \langle 1-x\rangle = \begin{cases} 1, & \text{if } 0 \neq x \in \mathbb{Z} \\ 0, & \text{otherwise;} \end{cases}$$

and for each $n \in \mathbb{N}$,

$$\sum_{i=0}^{n-1} \langle x + \frac{i}{n}\rangle = \frac{n-1}{2} + \langle nx\rangle.$$

Gamma distribution

Define $\overline{\Gamma} : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ by

$$\overline{\Gamma}(x) := \frac{\Gamma(1 - \langle -x \rangle)}{\sqrt{\pi}} \cdot \overline{\mathbb{Q}}^\times \quad \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times.$$

Then $\overline{\Gamma}$ satisfies the following odd distribution property: for $0 \neq x \in \mathbb{Q}/\mathbb{Z}$,

$$\overline{\Gamma}(x) \cdot \overline{\Gamma}(-x) = 1 \cdot \overline{\mathbb{Q}}^\times \quad (\in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times),$$

and

$$\prod_{i=0}^{n-1} \overline{\Gamma}\Big(\frac{x+i}{n}\Big) = \overline{\Gamma}(x), \quad \forall n \in \mathbb{N}.$$

Lang–Rohrlich conjecture

Conjecture (Lang–Rohrlich)

Given an integer n with $n > 2$,

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}\left(\pi, \Gamma(x) \;\middle|\; x \in \frac{1}{n}\mathbb{Z} \setminus \mathbb{Z}_{\leq 0}\right) \overset{?}{=} 1 + \frac{\phi(n)}{2},$$

where ϕ is the Euler phi-function:

$$\phi(n) := n \prod_{\text{prime } p \mid n} \left(1 - \frac{1}{p}\right).$$

Chowla-Selberg formula

Let $K \subset \mathbb{C}$ be an imaginary quadratic field over $\mathbb{Q}$ with discriminant $D < -4$, let O_K be the ring of integers in K. The Kronecker limit formula implies

$$\frac{\zeta_K'(0)}{\zeta_K(0)} = -\ln\left(\frac{\sqrt{D}}{2}\right) + \frac{1}{\#\operatorname{Pic}(O_K)} \cdot \sum_{\mathfrak{A} \in \operatorname{Pic}(O_K)} \ln\left(\operatorname{Im}(z_{\mathfrak{A}}) \cdot |\Delta(z_{\mathfrak{A}})|^{\frac{1}{6}}\right),$$

where for each ideal class $\mathfrak{A} \in \operatorname{Pic}(O_K)$, we take $z_{\mathfrak{A}} \in K$ with $\operatorname{Im}(z_{\mathfrak{A}}) > 0$ so that $\mathbb{Z}z_{\mathfrak{A}} + \mathbb{Z}$ represents $\mathfrak{A}$, and Δ is the modular discriminant function:

$$\Delta(z) = (2\pi)^{12} e^{2\pi\sqrt{-1}z} \prod_{n=1}^{\infty} (1 - e^{2\pi\sqrt{-1}nz})^{24}, \quad \forall z \in \mathbb{C},\ \operatorname{Im}(z) > 0.$$

Chowla–Selberg formula

On the other hand, Lerch's formula implies

$$\frac{\zeta_K'(0)}{\zeta_K(0)} = \frac{\zeta_{\mathbb{Q}}'(0)}{\zeta_{\mathbb{Q}}(0)} + \frac{L'(0,\chi_K)}{L(0,\chi_K)}$$
$$= \ln(2\pi) + \left(-\ln D + \frac{1}{\#\operatorname{Pic}(O_K)} \sum_{i=1}^{D-1} \chi_K(r) \ln \left|\Gamma\left(\frac{r}{D}\right)\right|\right),$$

where χ_K is the quadratic character associated to $K/\mathbb{Q}$.

Theorem (Lerch 1897, Chowla-Selberg 1949)

$$\prod_{\mathfrak{A} \in \operatorname{Pic}(O_K)} \left(\operatorname{Im}(z_{\mathfrak{A}}) |\Delta(z_{\mathfrak{A}})|^{\frac{1}{6}}\right) = \left(\frac{\pi}{\sqrt{D}}\right)^{\#\operatorname{Pic}(O_K)} \cdot \prod_{r=1}^{D-1} \left|\Gamma\left(\frac{r}{D}\right)^{\chi_K(r)}\right|. \quad (1)$$

Relation with "CM periods"

Consequently, let E be an elliptic curve over $\overline{\mathbb{Q}}$ with CM by O_K. For each non-zero period ϖ of E, we have

$$\prod_{\mathfrak{A} \in \mathrm{Pic}(O_K)} \left(\mathrm{Im}(z_{\mathfrak{A}}) |\Delta(z_{\mathfrak{A}})|^{\frac{1}{6}} \right)$$

$$\sim \varpi^{2\# \mathrm{Pic}(O_K)} \sim \pi^{\# \mathrm{Pic}(O_K)} \cdot \prod_{r=1}^{D-1} \Gamma\left(\frac{r}{D}\right)^{\chi_K(r)}. \tag{2}$$

Here we denote by $x \sim y$ for $x, y \in \mathbb{C}^\times$ with $x/y \in \overline{\mathbb{Q}}^\times$.

In the function field case, we may also derive (1) via an analogue of the Kronecker limit formula established in my previous work.

Go from (1) to (2) $\Longrightarrow$ The journey begins...

Notations

- $A := \mathbb{F}_q[\theta]$, the polynomial ring with one variable θ over a finite field $\mathbb{F}_q$ with $q = p^r$ elements.
- $k := \mathbb{F}_q(\theta)$, the fraction field of A.
- $|\cdot|_\infty$: the absolute value on k normalized so that $|\theta|_\infty = q$.
- $k_\infty := \mathbb{F}_q((1/\theta))$, the completion of k with respect to $|\cdot|_\infty$.
- $\mathbb{C}_\infty$: the completion of a chosen algebraic closure $\bar{k}_\infty$ of k_∞.
- $\bar{k}$: the algebraic closure of k in $\mathbb{C}_\infty$.

$$(A, k, k_\infty, \bar{k}, \mathbb{C}_\infty) \longleftrightarrow (\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \overline{\mathbb{Q}}, \mathbb{C}).$$

Arithmetic gamma function

Given $r \in \mathbb{Z}_{\geq 0}$, let

$$D_r := \prod_{\substack{a \in A_+ \\ \deg a = r}} a.$$

Given $n \in \mathbb{Z}_{\geq 0}$, write $n = n_0 + n_1 q + \cdots + n_r q^r$ with $0 \leq n_1, \ldots, n_r < q$. The Carlitz factorial of n is defined by

$$(n)_q! := D_0^{n_0} \cdots D_r^{n_r}.$$

Goss gave the following interpolation: for $y = \sum_{i=0}^{\infty} y_i q^i \in \mathbb{Z}_p$ with $0 \leq y_i < q$,

$$\Pi_{\mathrm{ari}}(y) := \prod_{i=0}^{\infty} \overline{D}_i^{y_i}, \quad \text{where} \quad \overline{D}_i := D_i/\theta^{\deg D_i},$$

and introduced the arithmetic gamma function

$$\Gamma_{\mathrm{ari}}(y) := \Pi(y-1) \quad \in k_\infty, \quad \forall y \in \mathbb{Z}_p.$$

Relations among arithmetic gamma values

Put $\mathbb{Z}_{(p)} := \mathbb{Q} \cap \mathbb{Z}_p$. For each $y \in \mathbb{Z}_{(p)} \setminus \mathbb{Z}$, write the fractional part of $-y$ as $\sum_{i=0}^{\ell-1} y_i q^i/(q^\ell - 1)$ with $0 \leq y_i < q$. Then

$$\Gamma_{\text{ari}}(y) \sim \prod_{i=0}^{\ell-1} \Gamma_{\text{ari}}(1 - \frac{q^i}{q^\ell - 1})^{y_i},$$

and $\Gamma_{\text{ari}}(1 - a/(q-1)) \sim \tilde{\pi}^{a/(q-1)}$ for $0 < a < q-1$, where:

$$\tilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{n=1}^{\infty} (1 - \frac{\theta}{\theta^{q^n}})^{-1} \quad \text{(Carlitz fundamental period).}$$

Proposition (Goss)

Given $y \in \mathbb{Z}_{(p)}$,

$$\Gamma_{\text{ari}}(y)\Gamma_{\text{ari}}(1-y) \sim \tilde{\pi},$$

and for $n \in \mathbb{N}$ with $p \nmid n$,

$$\prod_{i=0}^{n-1} \Gamma_{\text{ari}}(y + \frac{i}{n}) \sim \tilde{\pi}^{\frac{n-1}{2}} \Gamma_{\text{ari}}(ny).$$

Geometric gamma function

On the other hand, Thakur defined

$$\Pi_{\mathrm{geo}}(x) := \prod_{a \in A_+} (1 + \frac{x}{a})^{-1}, \quad \forall x \in \mathbb{C}_\infty \setminus (-A_+),$$

and introduced the geometric gamma function: for every $x \in \mathbb{C}_\infty \setminus (-A_+ \cup \{0\})$,

$$\Gamma_{\mathrm{geo}}(x) := x^{-1} \Pi_{\mathrm{geo}}(x) = x^{-1} \prod_{a \in A_+} (1 + \frac{x}{a})^{-1} \quad \in \mathbb{C}_\infty.$$

Proposition (Thakur)

For $x \in k \setminus A$,

$$\prod_{\epsilon \in \mathbb{F}_q^\times} \Gamma_{\mathrm{geo}}(\epsilon x) \sim \tilde{\pi},$$

and for each $\mathfrak{n} \in A_+$,

$$\prod_{a \in A, \deg a < \deg \mathfrak{n}} \Gamma_{\mathrm{geo}}(\frac{x + a}{\mathfrak{n}}) \sim \tilde{\pi}^{\frac{|\mathfrak{n}|_\infty - 1}{q-1}} \cdot \Gamma_{\mathrm{geo}}(x)$$

Two-variable gamma function (by Goss)

For $x \in \mathbb{C}_\infty \setminus (-A_+ \cup \{0\})$ and $y = \sum_i y_i q^i \in \mathbb{Z}_p$, set

$$\Pi(x,y) := \Pi_{\text{ari}}(y)^{-1} \cdot \prod_{i=0}^{\infty} \left(\prod_{\substack{a \in A_+ \\ \deg a = i}} (1 + \frac{x}{a})^{-y_i} \right),$$

and $\Gamma(x,y) := x^{-1}\Pi(x,y-1)$. Then for $a \in A \setminus (-A_+ \cup \{0\})$,

$$\Gamma(a,y) \sim \Gamma_{\text{ari}}(y)^{-1} \quad \text{and} \quad \Gamma(x, 1 - \frac{1}{q-1}) \sim \tilde{\pi}^{\frac{-1}{q-1}} \cdot \Gamma_{\text{geo}}(x).$$

Monomial relations among two-variable gamma values

Proposition (Goss and Thakur)

Let $x \in k \setminus A$, $y \in \mathbb{Z}_{(p)}/\mathbb{Z}$, $a \in A$, and $N \in \mathbb{N}$. Write the fractional part of $-y$ as $\sum_{i=0}^{\ell-1} y_i q^i/(q^\ell - 1)$. We have:

$$\Gamma(x, N) \sim 1 \quad \text{and} \quad \Gamma(x+a, y+N) \sim \Gamma(x, y);$$

$$\Gamma(x, y) = \prod_{i=0}^{\ell-1} \Gamma(x, 1 - \frac{q^i}{q^\ell - 1})^{y_i};$$

$$\prod_{c=0}^{\ell-1} \prod_{\epsilon \in \mathbb{F}_q^\times} \Gamma(\epsilon x, q^c y) \sim 1,$$

$$\prod_{\substack{a \in A \\ \deg a < \deg \mathfrak{n}}} \Gamma(\frac{x+a}{\mathfrak{n}}, y) \sim \Gamma(x, |\mathfrak{n}|_\infty y), \quad \forall \mathfrak{n} \in A_+.$$

Algebraic independence of special gamma values

Theorem 1 (W.)

Let $\mathfrak{n} \in A_+$ and $\ell \in \mathbb{N}$. We have

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\Gamma_{\mathrm{geo}}(x), \Gamma_{\mathrm{ari}}(y), \Gamma(x,y) \,\middle|\, x \in \frac{1}{\mathfrak{n}}A \setminus (-A_+ \cup \{0\}),\ y \in \frac{1}{q^\ell - 1}\mathbb{Z}\right)$$
$$= 1 + (\ell - \frac{1}{(q-1)^{\epsilon_{\mathfrak{n}}}}) \cdot \#(A/\mathfrak{n})^\times,$$

where $\epsilon_{\mathfrak{n}} := 1$ if $\deg \mathfrak{n} > 0$ and 0 otherwise.

Remark. The algebraic independence of geometric (resp. arithmetic) gamma values was derived by Anderson-Brownawell-Papanikolas [ABP] (resp. Chang-Papanikolas-Thakur-Yu [CPTY]).

Stickelberger functions

Consider $\mathbb{F}_q(t)$ where t is another variable (transcendental over $\mathbb{C}_\infty$). Let $\mathrm{G} := \mathrm{Gal}(\mathbb{F}_q(t)^{\mathrm{sep}})/\mathbb{F}_q(t))$. Fix an $\mathbb{F}_q$-algebra embedding $\nu : \mathbb{F}_q(t)^{\mathrm{sep}} \hookrightarrow \mathbb{C}_\infty$ sending t to θ. Let $\mathrm{G}_\infty := \nu^* \, \mathrm{Gal}(k_\infty^{\mathrm{sep}}/k_\infty) \subset \mathrm{G}$.

Definition

A Stickelberger function on G is a locally constant $\mathbb{Q}$-valued function φ on G satisfying that

(1)
$$\varphi(g_1(g_2\varrho_\infty g_2^{-1}g_\infty^{-1})) = \varphi(g_1), \quad \forall g_1, g_2 \in \mathrm{G}, \ g_\infty \in \mathrm{G}_\infty.$$

(2)
$$\int_{\mathrm{G}_\infty} \varphi(g_1 g_\infty) dg_\infty = \int_{\mathrm{G}_\infty} \varphi(g_2 g_\infty) dg_\infty, \quad \forall g_1, g_2 \in \mathrm{G}.$$

The space of Stickelberger function on G is denoted by $\mathscr{S}(\mathrm{G})$.

Examples of Stickelberger functions

(1) The characteristic function $\mathbf{1}_{\mathrm{G}}$ lies in $\mathscr{S}(\mathrm{G})$.

(2) Let $\mathrm{K}(\subset \mathbb{F}_q(t)^{\mathrm{sep}})$ be a CM field over $\mathbb{F}_q(t)$ (i.e. $\mathrm{K}/\mathbb{F}_q(t)$ is separable and every place ∞^+ of the maximal totally real subfield K^+ over $\mathbb{F}_q(t)$ lying over ∞ is not split in K). Put $\mathrm{H}_{\mathrm{K}} := \mathrm{Gal}(\mathbb{F}_q(t)^{\mathrm{sep}}/\mathrm{K})$. We may identify

$$\begin{array}{ccccc} \mathrm{G}/\mathrm{H}_{\mathrm{K}} & \stackrel{\sim}{\longleftrightarrow} & \mathrm{Emb}(\mathrm{K}, \mathbb{C}_\infty) & = & J_{\mathrm{K}} \\ g\mathrm{H}_{\mathrm{K}} & \longmapsto & \nu \circ g\big|_{\mathrm{K}} & =: & \xi_\nu^g. \end{array}$$

Let I_{K} be the free abelian group generated by J_{K}, and let $\Xi = \xi_\nu^{g_1} + \cdots + \xi_\nu^{g_d} \in I_{\mathrm{K}}$ be a CM type of K (i.e. $J_{\mathrm{K}^+} = \{\xi_\nu^{g_1}\big|_{\mathrm{K}^+}, \ldots, \xi_\nu^{g_d}\big|_{\mathrm{K}^+}\}$). Suppose $[\mathrm{G}, \mathrm{G}_\infty] \subset \mathrm{H}_{\mathrm{K}}$. Then

$$\varphi_{\mathrm{K},\Xi} := \sum_{i=1}^{d} \mathbf{1}_{g_i \mathrm{H}_{\mathrm{K}}} \in \mathscr{S}(\mathrm{G}).$$

Connection with CM types

Lemma

Let I_{K}^0 be the subgroup of I_{K} generated by all CM types of K, and $\mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathrm{K}})$ be the subspace of Stickelberger functions invariant by H_{K}. The map $(\Xi \mapsto \varphi_{\mathrm{K},\Xi})$ induces an isomorphism

$$I_{\mathrm{K}}^0 \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathrm{K}}).$$

Note that for each subgroup H of G with finite index and $[\mathrm{G}, \mathrm{G}_\infty] \subset \mathrm{H}$, the fixed field K_{H} of H in $\mathbb{F}_q(t)^{\mathrm{sep}}$ is actually a CM field over $\mathbb{F}_q(t)$. As $\mathscr{S}(\mathrm{G}) = \varinjlim_{\mathrm{H}} \mathscr{S}(\mathrm{G}/\mathrm{H})$, we obtain that

Proposition

$$\left(\varinjlim_{\mathrm{K}:[\mathrm{G},\mathrm{G}_\infty]\subset \mathrm{H}_{\mathrm{K}}} I_{\mathrm{K}}^0 \right) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathscr{S}(\mathrm{G}),$$

Period distribution

For each CM field $\mathrm{K} \subset \mathbb{F}_q(t)^{\mathrm{sep}}$ with $[\mathrm{G}, \mathrm{G}_\infty] \subset \mathrm{H}_\mathrm{K}$, set

$$\widetilde{\mathscr{P}}_{\nu,\mathrm{K}} : I_\mathrm{K}^0 \longrightarrow \mathbb{C}_\infty^\times / \bar{k}^\times, \quad \Xi \longmapsto \mathcal{P}_\mathrm{K}(\xi_\nu, \Xi),$$

where $\xi_\nu := \nu\big|_\mathrm{K} \in J_\mathrm{K}$ and $\mathcal{P}_\mathrm{K}(\xi_\nu, \Xi)$ is the analogue of Shimura's "period symbol" introduced by Brownawell-Chang-Papanikolas-W. The inflation-restriction relation among period symbols induces

$$\widetilde{\mathscr{P}}_\nu : \varinjlim_{\mathrm{K}:[\mathrm{G},\mathrm{G}_\infty]\subset \mathrm{H}_\mathrm{K}} I_\mathrm{K}^0 \longrightarrow \mathbb{C}_\infty^\times / \bar{k}^\times.$$

Composing with the (inverse of the) isomorphism in the above proposition, we get an analogue of Anderson's period distribution

$$\mathscr{P}_\nu : \mathscr{S}(\mathrm{G}) \to \mathbb{C}_\infty^\times / \bar{k}^\times.$$

Shimura's conjecture

Theorem (Brownawell-Chang-Papanikolas-W.)

Let $\mathrm{K} \subset \mathbb{F}_q(t)^{\mathrm{sep}}$ be a CM field with $[\mathrm{G}, \mathrm{G}_\infty] \subset \mathrm{H}_\mathrm{K}$. Then

$$\operatorname{tr.deg}_{\bar{k}} \bar{k}\big(\mathscr{P}_\nu(\varphi) \mid \varphi \in \mathscr{S}(\mathrm{G}/\mathrm{H}_\mathrm{K})\big) = 1 + (1 - \frac{1}{[\mathrm{K} : \mathrm{K}^+]}) \cdot [\mathrm{K} : \mathbb{F}_q(t)].$$

(In particular, $\mathscr{P}_\nu$ is injective.)

The bridges between period symbols and special gamma values at fractions are built from the "Stickelberger distributions" associated to Thakur's "diamond brackets".

Thakur's diamond brackets

(Arithmetic case.) Let $\mathbb{Z}_{(p)} := \mathbb{Z}_p \cap \mathbb{Q}$. For $y \in \mathbb{Z}_{(p)}$, put $\langle y \rangle_{\mathrm{ari}}$ to be the fractional part of y.
(Geometric case.) Given $x = \sum_i \epsilon_i \theta^{-i} \in k_\infty$ and $N \in \mathbb{Z}_{\geq 0}$, put

$$\langle x \rangle_N := \begin{cases} 1, & \text{if } \epsilon_i = 0 \text{ for } 0 < i \leq N \text{ and } \epsilon_{N+1} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Define $\langle x \rangle_{\mathrm{geo}} := \sum_{N=0}^{\infty} \langle x \rangle_N$.

(Two-variable case.) Given $x \in k_\infty$ and $y \in \mathbb{Z}_{(p)}$ with $\langle y \rangle_a = \sum_{i=0}^{\ell-1} y_i \frac{q^i}{q^\ell - 1}$ and $0 \leq y_i < q$, set

$$\langle x, y \rangle := \sum_{i=0}^{\ell-1} y_i \langle x, \frac{q^i}{q^\ell - 1} \rangle, \quad \text{where} \quad \langle x, \frac{q^i}{q^\ell - 1} \rangle := \sum_{\substack{N \in \mathbb{Z}_{\geq 0} \\ N \equiv -1-i \bmod \ell}} \langle x \rangle_N.$$

Diamond bracket relations

(Arithmetic case.) for $y \in \mathbb{Z}_{(p)}$ and $N \in \mathbb{N}$ with $p \nmid N$,

$$\langle y\rangle_{\text{ari}}+\langle 1-y\rangle_{\text{ari}} = \begin{cases} 1, & \text{if } y \notin \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \sum_{i=1}^{N-1}\langle y+\frac{i}{N}\rangle_{\text{ari}} = \langle Ny\rangle_{\text{ari}}+\frac{N-1}{2}.$$

(Geometric case.) Given $x \in k$ and $\mathfrak{n} \in A_+$,

$$\sum_{\epsilon\in\mathbb{F}_q^\times}\langle \epsilon x\rangle_{\text{geo}} = \begin{cases} 1, & \text{if } x \notin A, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sum_{\substack{a\in A \\ \deg a<\deg \mathfrak{n}}} \langle x+\frac{a}{\mathfrak{n}}\rangle_{\text{geo}} = \langle \mathfrak{n}x\rangle_{\text{geo}} + \frac{|\mathfrak{n}|_\infty - 1}{q-1}.$$

Diamond bracket relations

(Two-variable case). Take $x \in k$ with $|x|_\infty < 1$ and $y \in \mathbb{Z}_{(p)}$.

$$\langle x, y\rangle + \langle x, 1-y\rangle = \begin{cases} (q-1)\langle x\rangle_{\mathrm{geo}}, & \text{if } y \notin \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\ell \in \mathbb{N}$ and $i \in \mathbb{Z}$ with $0 \leq i < \ell$,

$$\sum_{\epsilon \in \mathbb{F}_q^\times} \langle \epsilon x, \frac{q^i}{q^\ell - 1}\rangle = \begin{cases} 1, & \text{if } x \neq 0 \text{ and } \mathrm{ord}_\infty(x) \equiv -i \bmod \ell, \\ 0, & \text{otherwise.} \end{cases}$$

For $\mathfrak{n} \in A_+$,

$$\sum_{\substack{a \in A \\ \deg a < \deg \mathfrak{n}}} \langle x + \frac{a}{\mathfrak{n}}, y\rangle = \langle \mathfrak{n}x, |\mathfrak{n}|_\infty y\rangle - \langle |\mathfrak{n}|_\infty y\rangle_{\mathrm{ari}} + |\mathfrak{n}|_\infty \langle y\rangle_{\mathrm{ari}}.$$

Stickelberger distributions

$\mathbb{Z}_{(p)}/\mathbb{Z} \simeq \mu_\infty \subset \bar{k}$

$k/A \simeq$ "Carlitz torsions"

Given $\mathfrak{n} \in \mathbb{F}_q[t]_+$, let $C_{\mathfrak{n}}^*(t,z) \in \mathbb{F}_q[t,z]$ be the $\mathfrak{n}$-th Carlitz cyclotomic polynomial. For $\ell \in \mathbb{N}$, let $O_{\mathfrak{n},\ell} := \mathbb{F}_{q^\ell}[t,z]/(C_{\mathfrak{n}}^*(t,z))$ and $\mathrm{K}_{\mathfrak{n},\ell}$ be the fraction field of $O_{\mathfrak{n},\ell}$, called the $(\mathfrak{n},\ell)$-th cyclotomic function field over $\mathbb{F}_q(t)$. We may assume that $\mathrm{K}_{\mathfrak{n},\ell} \subset \mathbb{F}_q(t)^{\mathrm{sep}}$ for every $\mathfrak{n},\ell$, and take an embedding $\nu_1 : \mathbb{F}_q(t)^{\mathrm{sep}} \hookrightarrow \mathbb{C}_\infty$ satisfying that $t \mapsto \theta$ and

$$z \longmapsto \exp_{\mathcal{C}}\left(\frac{\tilde{\pi}}{\mathfrak{n}(\theta)}\right) \quad \in \bar{k}.$$

When identifying k/A (resp. $\mathbb{Z}_{(p)}/\mathbb{Z}$) with the Carlitz torsions in $\bar{k}$ (resp. $\overline{\mathbb{F}}_q^\times$), we may define an action $\star$ of G on k/A (resp. $\mathbb{Z}_{(p)}/\mathbb{Z}$) via ν_1.

Stickelberger distributions

Definition

Given $x \in k/A$ and $y \in \mathbb{Z}_{(p)}/\mathbb{Z}$, set

$$\begin{aligned}
\mathrm{St}_{\mathrm{ari}}(y)(\varrho) &:= \langle -\varrho \star y \rangle_{\mathrm{ari}}, \\
\mathrm{St}_{\mathrm{geo}}(x)(\varrho) &:= \langle \varrho \star x \rangle_{\mathrm{geo}} - \frac{1}{q-1}, \\
\mathrm{St}(x,y)(\varrho) &:= \langle \varrho \star x, -\varrho \star y \rangle - \langle -\varrho \star y \rangle_{\mathrm{ari}}.
\end{aligned}$$

Proposition

For every $x \in k/A$ and $y \in \mathbb{Z}_{(p)}/\mathbb{Z}$, we have that

$$\mathrm{St}_{\mathrm{ari}}(y),\ \mathrm{St}_{\mathrm{geo}}(x),\ \mathrm{St}(x,y)\ \in \mathscr{S}(\mathrm{G}).$$

Moreover, the diamond bracket relations assure the "distribution properties" of $\mathrm{St}_{\mathrm{ari}}$, $\mathrm{St}_{\mathrm{geo}}$, and St.

$$\begin{aligned}
\mathrm{St}_{ari} &: \mathbb{Z}_{(p)}/\mathbb{Z} \\
\mathrm{St}_{geo} &: k/A \to \mathscr{S}(G) \\
\mathrm{St} &: k/A \times \mathbb{Z}_{(p)}/\mathbb{Z}
\end{aligned}$$

$$\mathscr{S}(G) \xrightarrow{\mathcal{P}_{v_1}} \mathbb{C}_\infty^\times/\bar{k}^\times$$

Gamma distribution

Composing with the period distribution $\mathscr{P}_{\nu_1}$ introduced before, we have the following:

Theorem 2 (W.)

$$\mathscr{P}_{\nu_1} \circ \mathrm{St}_{\mathrm{ari}} = \hat{\Gamma}_{\mathrm{ari}} : \mathbb{Z}_{(p)}/\mathbb{Z} \longrightarrow \mathbb{C}_\infty^\times/\bar{k}^\times,$$
$$\mathscr{P}_{\nu_1} \circ \mathrm{St}_{\mathrm{geo}} = \hat{\Gamma}_{\mathrm{geo}} : k/A \longrightarrow \mathbb{C}_\infty^\times/\bar{k}^\times,$$
$$\mathscr{P}_{\nu_1} \circ \mathrm{St} = \hat{\Gamma} : k/A \times \mathbb{Z}_{(p)}/\mathbb{Z} \longrightarrow \mathbb{C}_\infty^\times/\bar{k}^\times,$$

where $\hat{\Gamma}_{\mathrm{ari}}$, $\hat{\Gamma}_{\mathrm{geo}}$, and $\hat{\Gamma}$ are the "arithmetic, geometric, and two-variable gamma distributions", respectively.

Gamma distribution

(Arithmetic case.) Define $\tilde{\Gamma}_{\mathrm{ari}} : \mathbb{Z}_{(p)}/\mathbb{Z} \longrightarrow \mathbb{C}_\infty^\times$ by

$$\tilde{\Gamma}_{\mathrm{ari}}(y) := \Gamma_{\mathrm{ari}}(1 - \langle -y \rangle_{\mathrm{ari}}).$$

(Geometric case.) Define $\tilde{\Gamma}_{\mathrm{geo}} : k/A \longrightarrow \mathbb{C}_\infty^\times$ by

$$\tilde{\Gamma}_{\mathrm{geo}}(x) := \tilde{\pi}^{\frac{-1}{q-1}} \cdot \begin{cases} \Gamma_{\mathrm{geo}}(\{x\}), & \text{if } \{x\} \neq 0, \\ 1, & \text{if } \{x\} = 0. \end{cases}$$

Here $\{x\} \in k_\infty$ with $|\{x\}|_\infty < 1$ and $x - \{x\} \in A$.

(Two-variable case.) Define $\tilde{\Gamma} : k/A \times \mathbb{Z}_{(p)}/\mathbb{Z} \longrightarrow \mathbb{C}_\infty^\times$ by

$$\tilde{\Gamma}(x, y) := \begin{cases} \Gamma(\{x\}, 1 - \langle -y \rangle_{\mathrm{ari}}), & \text{if } \{x\} \neq 0, \\ \Gamma_{\mathrm{ari}}(1 - \langle -y \rangle_{\mathrm{ari}})^{-1}, & \text{if } \{x\} = 0. \end{cases}$$

Then $\hat{\Gamma}_{\mathrm{ari}}, \hat{\Gamma}_{\mathrm{geo}}$, and $\hat{\Gamma}$ are the corresponding induced maps to $\mathbb{C}_\infty^\times / \bar{k}^\times$.

Proof of the Lang–Rohrlich conjecture (Sketch)

We first point out that the functions $\mathrm{St}(x, y)$ are in fact "evaluators" of certain Artin L-values (via inner product with characters). This implies the "universality" of the Stickelberger distributions $\mathrm{St}_{\mathrm{ari}}$, $\mathrm{St}_{\mathrm{geo}}$, and St. Consequently, for each $\mathfrak{n} \in \mathbb{F}_q[t]_+$ and $\ell \in \mathbb{N}$, put

$$\mathrm{H}_{\mathfrak{n},\ell} := \mathrm{Gal}(\mathbb{F}_q(t)^{\mathrm{sep}}/\mathrm{K}_{\mathfrak{n},\ell}) \subset \mathrm{G}.$$

Then we get

$$\mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathfrak{n},\ell}) = \left\langle \mathrm{St}(x, y) \;\middle|\; x \in \frac{1}{\mathfrak{n}(\theta)}A/A,\ y \in \frac{1}{q^{\ell}-1}\mathbb{Z}/\mathbb{Z} \right\rangle_{\mathbb{Q}}.$$

Proof of the Lang–Rohrlich conjecture (Sketch)

Therefore

$$
\begin{aligned}
& \operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\Gamma_{\mathrm{geo}}(x), \Gamma_{\mathrm{ari}}(y), \Gamma(x,y) \;\middle|\; x \in \frac{1}{\mathfrak{n}(\theta)}A \setminus (A_+ \cup \{0\}), y \in \frac{1}{q^\ell - 1}\mathbb{Z}\right) \\
= \; & \operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\tilde{\Gamma}(x,y) \;\middle|\; x \in \frac{1}{\mathfrak{n}(\theta)}A/A, \; y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}\right) \\
= \; & \operatorname{tr.deg}_{\bar{k}} \bar{k}\left(\mathscr{P}_{\nu_1}(\varphi) \;\middle|\; \varphi \in \mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathfrak{n},\ell})\right) \\
= \; & 1 + \left(1 - \frac{1}{[\mathrm{K}_{\mathfrak{n},\ell} : \mathrm{K}^+_{\mathfrak{n},\ell}]}\right) \cdot [\mathrm{K}_{\mathfrak{n},\ell} : \mathbb{F}_q(t)] \\
= \; & 1 + \left(1 - \frac{1}{\ell(q-1)^{\epsilon_{\mathfrak{n}}}}\right) \cdot \left(\ell \cdot \#(A/\mathfrak{n}(\theta))^\times\right) \\
= \; & 1 + \left(\ell - \frac{1}{(q-1)^{\epsilon_{\mathfrak{n}}}}\right) \cdot \#(A/\mathfrak{n}(\theta))^\times .
\end{aligned}
$$

$\mathcal{P}_{\nu_1} \cdot St_* = \hat{\Gamma}_*$

$\square$

Chowla–Selberg phenomenon

Given $\mathfrak{n} \in \mathbb{F}_q[t]_+$ and $\ell \in \mathbb{N}$, recall that

$$\mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathfrak{n},\ell}) = \left\langle \mathrm{St}(x,y) \;\middle|\; x \in \frac{1}{\mathfrak{n}(\theta)}A/A,\ y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z} \right\rangle_{\mathbb{Q}}.$$

Let K be a CM field over $\mathbb{F}_q(t)$ with $\mathrm{K} \subset \mathrm{K}_{\mathfrak{n},\ell}$. Then

$$\mathrm{H}_{\mathrm{K}} \supset \mathrm{H}_{\mathfrak{n},\ell}, \quad \text{and so} \quad \mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathrm{K}}) \subset \mathscr{S}(\mathrm{G}/\mathrm{H}_{\mathfrak{n},\ell}).$$

Hence for each generalized CM type Ξ of K (i.e. Ξ is a sum of CM types of K), we may express

$$\varphi_{\mathrm{K},\Xi} = \sum_{x \in \frac{1}{\mathfrak{n}(\theta)}A/A} \sum_{y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}} m_{x,y}\, \mathrm{St}(x,y) \quad \text{with } m_{x,y} \in \mathbb{Q}.$$

Therfore

$$\left(\mathcal{P}_{\mathrm{K}}(\xi_{\nu_1}, \Xi) =\right) \quad \mathscr{P}_{\nu_1}(\varphi_{\mathrm{K},\Xi}) = \prod_{x \in \frac{1}{\mathfrak{n}(\theta)}A/A} \prod_{y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}} \hat{\Gamma}(x,y)^{m_{x,y}}.$$

Chowla–Selberg phenomenon

Theorem 3 (W.)

Let E_ρ be a CM abelian t-module over $\bar{k}$ with generalized CM type (K, Ξ) where $\mathrm{K} \subset \mathrm{K}_{\mathfrak{n},\ell}$ for some $\mathfrak{n} \in \mathbb{F}_q[t]_+$ and $\ell \in \mathbb{N}$. The space of quasi-periods of E_ρ is spanned by

$$\prod_{x \in \frac{1}{\mathfrak{n}(\theta)} A/A} \prod_{y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}} \tilde{\Gamma}(\varrho \star x, \varrho \star y)^{m_{x,y}}, \qquad \text{for } \varrho \in \mathrm{G}/\mathrm{H}_{\mathrm{K}}.$$

Thakur's recipe/conjecture

Corollary

Let E_ρ be a Drinfeld $\mathbb{F}_q[t]$-module over $\bar{k}$ with full-CM by an imaginary field $\mathrm{K} \subset \mathrm{K}_{\mathfrak{n},\ell}$. Write

$$\mathbf{1}_{\mathrm{H}_\mathrm{K}} = \sum_{x \in \frac{1}{\mathfrak{n}(\theta)} A/A} \sum_{y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}} m_{x,y}\, \mathrm{St}(x,y) \quad \text{with } m_{x,y} \in \mathbb{Q}.$$

For each non-zero period ϖ of E_ρ, we have that

$$\left(\mathscr{P}_{\nu_1}(\mathbf{1}_{\mathrm{H}_\mathrm{K}}) =\right) \quad \varpi \sim \prod_{x \in \frac{1}{\mathfrak{n}(\theta)} A/A} \prod_{y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}} \tilde{\Gamma}(x,y)^{m_{x,y}}.$$

Chowla–Selberg formula over function fields

Let $\mathfrak{n} \in \mathbb{F}_q[t]_+$ and $\ell \in \mathbb{N}$. Given a CM field $\mathsf{K} \subset \mathsf{K}_{\mathfrak{n},\ell}$, put $\mathsf{G}_\mathsf{K} := \mathrm{Gal}(\mathsf{K}/\mathbb{F}_q(t)) = \mathsf{G}/\mathsf{H}_\mathsf{K}$. For each $\varphi \in \mathscr{S}(\mathsf{G}_\mathsf{K})$, we have

$$\varphi = \frac{1}{\#\mathsf{G}_\mathsf{K}}\left(\sum_{g\in\mathsf{G}}\varphi(g)\right)\mathbf{1}_{\mathsf{G}_\mathsf{K}} + \frac{1}{\#\mathsf{G}_\mathsf{K}}\sum_{\chi\in\widehat{\mathsf{G}}_\mathsf{K}\setminus\widehat{\mathsf{G}}_{\mathsf{K}^+}}\left(\sum_{g\in\mathsf{G}}\varphi(g)\overline{\chi}(g)\right)\chi.$$

On the other hand, note that

$$(\mathbb{F}_q[t]/\mathfrak{n})^\times \times \mathbb{Z}/\ell\mathbb{Z} \cong \mathsf{G}/\mathsf{H}_{\mathfrak{n},\ell} \twoheadrightarrow \mathsf{G}_\mathsf{K}.$$

For each $\chi \in \widehat{\mathsf{G}}_\mathsf{K} \setminus \widehat{\mathsf{G}}_{\mathsf{K}^+}$, viewing χ as a character on $(\mathbb{F}_q[t]/\mathfrak{n})^\times \times \mathbb{Z}/\ell\mathbb{Z}$, the "evaluator" property of the Stickelberger functions implies that

$$\chi = \sum_{a\in(\mathbb{F}_q[t]/\mathfrak{c}_\chi)^\times}\sum_{i\in\mathbb{Z}/\ell\mathbb{Z}}\frac{\overline{\chi}(a, i+\deg\mathfrak{c}_\chi)}{L_A(0,\overline{\chi})}\cdot \mathrm{St}\Big(\frac{a(\theta)}{\mathfrak{c}_\chi(\theta)}, \frac{q^i}{1-q^\ell}\Big).$$

Chowla–Selberg formula over function fields

For $\mathfrak{c} \mid \mathfrak{n}$, $a \in (\mathbb{F}_q[t]/\mathfrak{c})^\times$, $i \in \mathbb{Z}/\ell\mathbb{Z}$, and $g \in \mathrm{G}_{\mathrm{K}}$, define

$$n_{\mathfrak{c}}(g, a, i) := \sum_{\substack{\chi \in \widehat{\mathrm{G}}_{\mathrm{K}} \setminus \widehat{\mathrm{G}}_{\mathrm{K}^+} \\ \mathfrak{c}_\chi = \mathfrak{c}}} \frac{\chi(g)\chi(a, i + \deg \mathfrak{c})}{L_A(0, \chi)} \quad \in \mathbb{Q}.$$

Then

$$\varphi = \frac{1}{[\mathrm{K} : \mathbb{F}_q(t)]} \left(\sum_{g \in \mathrm{G}_{\mathrm{K}}} \varphi(g) \right) \cdot \mathbf{1}_{\mathrm{G}_{\mathrm{K}}}$$

$$+ \frac{1}{[\mathrm{K} : \mathbb{F}_q(t)]} \sum_{\mathfrak{c} \mid \mathfrak{n}} \sum_{a \in (\mathbb{F}_q[t]/\mathfrak{c})^\times} \sum_{i \in \mathbb{Z}/\ell\mathbb{Z}} \left(\sum_{g \in \mathrm{G}_{\mathrm{K}}} \varphi(g) n_{\mathfrak{c}}(g, a, i) \right) \cdot \mathrm{St}\left(\frac{a(\theta)}{\mathfrak{c}(\theta)}, \frac{q^i}{1 - q^\ell}\right).$$

Chowla–Selberg formula over function fields

Theorem 4 (W.)

Let E_ρ be a CM abelian t-module over $\bar{k}$ with generalized CM type (K, Ξ) where $\mathrm{K} \subset \mathrm{K}_{\mathfrak{n},\ell}$ for some $\mathfrak{n} \in \mathbb{F}_q[t]_+$ and $\ell \in \mathbb{N}$. Write $\Xi = \sum_{g \in \mathrm{G}_\mathrm{K}} m_g \xi_{\nu_1}^g$. The space of quasi-periods of E_ρ is spanned by

$$\tilde{\pi}^{\frac{\mathrm{wt}(\Xi)}{[\mathrm{K}:\mathrm{K}^+]}} \cdot \prod_{g \in \mathrm{G}_\mathrm{K}} \prod_{\mathfrak{c} \mid \mathfrak{n}} \prod_{a \in (\mathbb{F}_q[t]/\mathfrak{c})^\times} \prod_{i \in \mathbb{Z}/\ell\mathbb{Z}} \tilde{\Gamma}\Big(\frac{a(\theta)}{\mathfrak{c}(\theta)}, \frac{q^i}{1-q^\ell}\Big)^{\frac{n_\mathfrak{c}(gg_0, a, i) m_g}{[\mathrm{K}:\mathbb{F}_q(t)]}}, \quad \text{for } g_0 \in \mathrm{G}_\mathrm{K}.$$

In particular, when E_ρ is a CM Drinfeld $\mathbb{F}_q[t]$-module, we get that for every non-zero period ϖ of E_ρ,

$$\varpi^{[\mathrm{K}:\mathbb{F}_q(t)]} \sim \tilde{\pi} \cdot \prod_{\mathfrak{c} \mid \mathfrak{n}} \prod_{a \in (\mathbb{F}_q[t]/\mathfrak{c})^\times} \prod_{i \in \mathbb{Z}/\ell\mathbb{Z}} \tilde{\Gamma}\Big(\frac{a(\theta)}{\mathfrak{c}(\theta)}, \frac{q^i}{1-q^\ell}\Big)^{n_\mathfrak{c}(\mathrm{id}_\mathrm{K}, a, i)}.$$

Chowla–Selberg formula over function fields

(1) The above result agrees with Thakur's formula when $\mathrm{K} = \mathbb{F}_{q^\ell}(t)$ (constant field extension) or $K = \mathbb{F}_q(\sqrt[q-1]{-t})$ (Carlitz t-torsions).

(2) Let E_ρ be a Drinfeld $\mathbb{F}_q[t]$-module of rank 2 over $\bar{k}$ with CM by K, where ∞ is tamely ramified in K. Let $\mathfrak{d}$ be the discriminant of $O_{\mathrm{K}}/\mathbb{F}_q[t]$. Then $K \subset \mathrm{K}_{\mathfrak{d},2}$. Let χ_{K} be the quadratic character of $\mathrm{K}/\mathbb{F}_q(t)$. Then for $a \in (\mathbb{F}_q[t]/\mathfrak{d})^\times$ and $i \in \mathbb{Z}/2\mathbb{Z}$, we get

$$n_{\mathfrak{d}}(\mathrm{id}_{\mathrm{K}}, a, i) = \frac{w_{\mathrm{K}}\chi_{\mathrm{K}}(a, i + \deg \mathfrak{d})}{\#\operatorname{Pic}(O_{\mathrm{K}})}, \qquad \text{where } w_{\mathrm{K}} := \frac{\#\mathbb{F}_{\mathrm{K}}^\times}{\#\mathbb{F}_q^\times}.$$

Hence for every non-zero period ϖ of E_ρ, we have that

$$\varpi \sim \sqrt{\tilde{\pi}} \cdot \prod_{a \in (\mathbb{F}_q[t]/\mathfrak{d})^\times} \prod_{i \in \mathbb{Z}/2\mathbb{Z}} \tilde{\Gamma}\Big(\frac{a(\theta)}{\mathfrak{d}(\theta)}, \frac{q^i}{1-q^2}\Big)^{\frac{w_{\mathrm{K}}\chi_{\mathrm{K}}(a, i+\deg \mathfrak{d})}{2\#\operatorname{Pic}(O_{\mathrm{K}})}}.$$

Deligne-Gross-type period conjecture

Let M be a pure uniformizable dual t-motive over $\bar{k}$, and $\mathrm{H}(\mathrm{M})$ be its "Hodge-Pink structure" of M. Given a Hodge-Pink substructure $\mathrm{H} = (H, W_\bullet H, \mathfrak{q})$ of $\mathrm{H}(\mathrm{M})$, suppose it has *full-CM* by a field $\mathrm{K} \subset \mathrm{K}_{\mathfrak{n},\ell}$. Then H is regarded as a one-dimensional vector space over K.

Note that the underlying space $\mathbb{C}_\infty \otimes_{\mathrm{k}} H$ is identified with a subspace of the de Rham space $H_{\mathrm{dR}}(\mathrm{M}, \mathbb{C}_\infty)$. Take $\omega \in \mathbb{C}_\infty \otimes_{\mathrm{k}} H \subset H_{\mathrm{dR}}(\mathrm{M}, \mathbb{C}_\infty)$ to be an "eigen-differential" under the K-multiplication, i.e. there exists an embedding $\iota_\omega : \mathrm{K} \hookrightarrow \mathbb{C}_\infty$ such that

$$\alpha^*\omega = \iota_\omega(\alpha) \cdot \omega, \quad \forall \alpha \in \mathrm{K}.$$

Deligne-Gross-type period conjecture

We derive that:

Theorem 5 (W.)

Suppose ω is algebraic, i.e. $\omega \in H_{\mathrm{dR}}(\mathrm{M}, \bar{k})$. Then for every cycle $\gamma \in H_{\mathrm{Betti}}(\mathrm{M})$ such that the period integral $\int_\gamma \omega \neq 0$, we have that

$$\int_\gamma \omega \sim \prod_{x \in \frac{1}{\mathfrak{n}}A/A} \prod_{y \in \frac{1}{q^\ell - 1}\mathbb{Z}/\mathbb{Z}} \tilde{\Gamma}(x, y)^{\varepsilon(x,y)},$$

where the exponents $\varepsilon(x, y)$ are described via the embedding ι_ω and the decomposition of the "Hodge-Pink-type" of H'.

Working in progress: v-adic counterpart

Following Morita's construction in the p-adic case, Goss also introduced the v-adic arithmetic gamma function: for $y = \sum_{i=0}^{\infty} y_i q^i \in \mathbb{Z}_p$ with $0 \leq y_i < q$, set

$$\Gamma_{\mathrm{ari},v}(y+1) = \prod_{i=0}^{\infty} D_{i,v}^{y_i}, \quad \text{where} \quad D_{i,v} := \prod_{\substack{a \in A_+ \\ \deg a = i,\, v \nmid a}} a.$$

Theorem 6. (Chang-W.-Yu)

Given a positive integer ℓ, we have that

$$\operatorname{tr.deg}_k k\big(\Gamma_{\mathrm{ari},v}(z) \bigm| z \in \mathbb{Q} \text{ with } (q^\ell - 1)z \in \mathbb{Z}\big) = \ell - \gcd(\ell, \deg v).$$

Consequently, the algebraic relations among v-adic arithmetic gamma values at rational p-adic integers are generated by the monomial relations coming from the functional equations and Thakur's analogue of Gross–Koblitz formula.

Working in progress: v-adic counterpart

Remark:

- One-side inequality "$\leq$" in Theorem 6 comes from the the functional equations and Thakur's analogue of Gross–Koblitz formula. To show the opposite innequality, we express the v-adic arithmetic gamma values in terms of the v-adic periods of the "crystalline–de Rham comparison isomorphism" for Carlitz t-motives with complex multiplication by constant field extensions (a v-adic Chowla–Selberg-type formula).

- The v-adic geometric gamma functions was introduced by Thakur in the 90's, and Ting-Wei Chang is currently extending this definition to the "two-variable" one and demonstrating a "geometric version" of the Gross–Koblitz–Thakur formula. Our ultimate goal is to establish the whole Chowla–Selberg phenomenon for the v-adic CM periods in the function field context.

The end. Thank you very much for your attention.