CM146, Winter 2024
Problem Set 3: Deep learning, SVM, Kernels
Due March 3, 2024, 11:59pm PST

1 Kernels [8 pts]

(a) **(2 pts)** For any two documents $\boldsymbol{x}$ and $\boldsymbol{z}$, define $k(\boldsymbol{x}, \boldsymbol{z})$ to equal the number of unique words that occur in both $\boldsymbol{x}$ and $\boldsymbol{z}$ (i.e., the size of the intersection of the sets of words in the two documents). Is this function a kernel? Give justification for your answer.

Solution: We can show that $k(\boldsymbol{x}, \boldsymbol{z})$ is a kernel by explicitly constructing feature vectors $\phi(\boldsymbol{x})$ and $\phi(\boldsymbol{z})$ such that $k(\boldsymbol{x}, \boldsymbol{z}) = \phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{z})$. For any set of documents, we can always construct a vocabulary $\mathcal{V}$ with finite size for the words in the document set. Given the vocabulary $\mathcal{V}$, we can construct a feature mapping $\phi(\boldsymbol{x})$ for $\boldsymbol{x}$ by the following: For the ℓ^{th} word w_ℓ in $\mathcal{V}$, if w_ℓ appears in document $\boldsymbol{x}$, assign $\phi(\boldsymbol{x})_\ell$ (the ℓ^{th} element of $\phi(\boldsymbol{x})$) to be 1; otherwise set the element $\phi(\boldsymbol{x})_\ell$ to 0. Then the number of unique words common in $\boldsymbol{x}$ and $\boldsymbol{z}$ is $\phi(\boldsymbol{x}) \cdot \phi(\boldsymbol{z})$, giving us the kernel.

(b) **(3 pts)** One way to construct kernels is to build them from simpler ones. We have seen various "construction rules", including the following: Assuming $k_1(\boldsymbol{x}, \boldsymbol{z})$ and $k_2(\boldsymbol{x}, \boldsymbol{z})$ are kernels, then so are

- (scaling) $f(\boldsymbol{x})k_1(\boldsymbol{x}, \boldsymbol{z})f(\boldsymbol{z})$ for any function $f(\boldsymbol{x}) \in \mathbb{R}$
- (sum) $k(\boldsymbol{x}, \boldsymbol{z}) = k_1(\boldsymbol{x}, \boldsymbol{z}) + k_2(\boldsymbol{x}, \boldsymbol{z})$
- (product) $k(\boldsymbol{x}, \boldsymbol{z}) = k_1(\boldsymbol{x}, \boldsymbol{z})k_2(\boldsymbol{x}, \boldsymbol{z})$

Using the above rules and the fact that $k(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{z}$ is (clearly) a kernel, show that the following is also a kernel:

$$\left(1 + \left(\frac{\boldsymbol{x}}{||\boldsymbol{x}||}\right) \cdot \left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)\right)^3$$

Solution: We can construct the kernel in the following steps.

i. scaling. $k_1(\boldsymbol{x}, \boldsymbol{z}) = \frac{1}{||\boldsymbol{x}||}\frac{1}{||\boldsymbol{z}||}k(\boldsymbol{x}, \boldsymbol{z}) = \left(\frac{\boldsymbol{x}}{||\boldsymbol{x}||}\right) \cdot \left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)$

ii. sum. $k_2(\boldsymbol{x}, \boldsymbol{z}) = 1 + k_1(\boldsymbol{x}, \boldsymbol{z}) = 1 + \left(\frac{\boldsymbol{x}}{||\boldsymbol{x}||}\right) \cdot \left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)$

iii. product. $k_3(\boldsymbol{x}, \boldsymbol{z}) = k_2(\boldsymbol{x}, \boldsymbol{z})k_2(\boldsymbol{x}, \boldsymbol{z}) = \left(1 + \left(\frac{\boldsymbol{x}}{||\boldsymbol{x}||}\right) \cdot \left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)\right)^2$

iv. product. $k_4(\boldsymbol{x}, \boldsymbol{z}) = k_3(\boldsymbol{x}, \boldsymbol{z})k_2(\boldsymbol{x}, \boldsymbol{z}) = \left(1 + \left(\frac{\boldsymbol{x}}{||\boldsymbol{x}||}\right) \cdot \left(\frac{\boldsymbol{z}}{||\boldsymbol{z}||}\right)\right)^3$

For step 1, we used the scaling function $f(\boldsymbol{x}) = \frac{1}{||\boldsymbol{x}||}$. For step 2, note that $k(\boldsymbol{x}, \boldsymbol{z}) = 1$ is a valid kernel given by the constant feature mapping $\phi(\boldsymbol{x}) = 1$.

Parts of this assignment are adapted from course material by Tommi Jaakola (MIT), and Andrew Ng (Stanford), and Jenna Wiens (UMich).

(c) **(3 pts)** Given vectors $\boldsymbol{x}$ and $\boldsymbol{z}$ in $\mathbb{R}^2$, define the kernel $k_\beta(\boldsymbol{x}, \boldsymbol{z}) = (1 + \beta \boldsymbol{x} \cdot \boldsymbol{z})^3$ for any value $\beta > 0$. Find the corresponding feature map $\phi_\beta(\cdot)$[1]. What are the similarities/differences from the kernel $k(\boldsymbol{x}, \boldsymbol{z}) = (1 + \boldsymbol{x} \cdot \boldsymbol{z})^3$, and what role does the parameter β play?

Solution: To show that k_β is a kernel, simply use the same feature mapping as used by the polynomial kernel of degree 3, but first scale $\boldsymbol{x}$ by $\sqrt{\beta}$. So, in effect they are both polynomial kernels of degree 3. If you look at the resulting feature vector, the offset term 1 is unchanged, the linear terms are scaled by $\sqrt{\beta}$, the quadratic terms are scaled by β, and the cubic terms are scaled by $\beta^{1.5}$. Although the model class remains unchanged, this changes how we penalize the features during learning (from the $||\boldsymbol{\theta}||^2$ in the objective). In particular, higher-order features will become more costly to use, so this will bias more towards a lower-order polynomial.

That is,

$$\begin{aligned} k_\beta(\boldsymbol{x}, \boldsymbol{z}) &= (1 + \beta \boldsymbol{x} \cdot \boldsymbol{z})^3 \\ &= (1 + \beta (x_1 z_1 + x_2 z_2))^3 \\ &= 1 + 3\beta (x_1 z_1 + x_2 z_2) + 3\beta^2 \left(x_1^2 z_1^2 + 2 x_1 z_1 x_2 z_2 + x_2^2 z_2^2\right) \\ &\quad + \beta^3 \left(x_1^3 z_1^3 + 3 x_1^2 z_1^2 x_2 z_2 + 3 x_1 z_1 x_2^2 z_2^2 + x_2^3 z_2^3\right) \end{aligned}$$

so that

$$\phi_\beta(\boldsymbol{x}) = (1, \sqrt{3\beta} x_1, \sqrt{3\beta} x_2, \sqrt{3}\beta x_1^2, \sqrt{6}\beta x_1 x_2, \sqrt{3}\beta x_2^2, \sqrt{\beta^3} x_1^3, \sqrt{3\beta^3} x_1^2 x_2, \sqrt{3\beta^3} x_1 x_2^2, \sqrt{\beta^3} x_2^3)^T$$

The m^{th}-order terms in $\phi_\beta(\cdot)$ are scaled by $\beta^{m/2}$, so β trades off the influence of the higher-order versus lower-order terms in the polynomial. If $\beta = 1$, then $\beta^{1/2} = \beta = \beta^{3/2}$ so that $k_\beta = k$. If $0 < \beta < 1$, then $\beta^{1/2} > \beta > \beta^{3/2}$ so that lower-order terms have more weight and higher-order terms less weight; as $\beta \to 0$, k_β approaches $1 + 3\beta \boldsymbol{x} \cdot \boldsymbol{z}$ (a linear separator). If $\beta > 1$, the trade-off is reversed; as $\beta \to \infty$, only the constant and cubic terms in k_β remain.

2 SVM [8 pts]

Suppose we are looking for a maximum-margin linear classifier *through the origin*, i.e. $b = 0$ (also hard margin, i.e., no slack variables). In other words, we minimize $\frac{1}{2}||\boldsymbol{\theta}||^2$ subject to $y_n \boldsymbol{\theta}^T \boldsymbol{x}_n \geq 1, n = 1, \ldots, N$.

(a) **(2 pts)** Given a single training vector $\boldsymbol{x} = (a, e)^T$ with label $y = -1$, what is the $\boldsymbol{\theta}^*$ that satisfies the above constrained minimization?

Solution: The SVM with one negative data point orients $\boldsymbol{\theta}$ in the opposite direction of the single data point $\boldsymbol{x}$ in order to minimize the objective function while satisfying the constraint $y_n \boldsymbol{\theta}^T \boldsymbol{x}_n = 1$. The corresponding $\boldsymbol{\theta}$ is

$$\boldsymbol{\theta}^* = -\frac{\boldsymbol{x}}{||\boldsymbol{x}||^2}.$$

[1] You may use any external program to expand the cubic.

(b) **(2 pts)** Suppose we have two training examples, $\boldsymbol{x}_1 = (1,1)^T$ and $\boldsymbol{x}_2 = (1,0)^T$ with labels $y_1 = 1$ and $y_2 = -1$. What is $\boldsymbol{\theta}^*$ in this case, and what is the margin γ?

Solution: In this case, the SVM uses both data points as support vectors such that $y_1 \boldsymbol{\theta}^T \boldsymbol{x}_1 = 1$ and $y_2 \boldsymbol{\theta}^T \boldsymbol{x}_2 = 1$. The corresponding $\boldsymbol{\theta}$ and γ are

$$\boldsymbol{\theta}^* = [-1, 2]^T, \gamma = \frac{1}{\sqrt{5}}.$$

(c) **(4 pts)** Suppose we now allow the offset parameter b to be non-zero. How would the classifier and the margin change in the previous question? What are $(\boldsymbol{\theta}^*, b^*)$ and γ? Compare your solutions with and without offset.

Solution: In this case, the SVM uses both data points as support vectors such that $y_1(\boldsymbol{\theta}^T \boldsymbol{x}_1 + b) = 1$ and $y_2(\boldsymbol{\theta}^T \boldsymbol{x}_2 + b) = 1$. The corresponding $\boldsymbol{\theta}$, b, and γ are

$$\boldsymbol{\theta}^* = [0, 2]^T, b^* = -1, \gamma = \frac{1}{2}$$

The margin for the classifier with offset is larger than the margin for the classifier without offset.

3 Implementation: Digit Recognizer [48 pts]

In this exercise, you will implement a digit recognizer in pytorch. Our data contains pairs of 28×28 images $\mathbf{x}_n$ and the corresponding digit labels $y_n \in \{0, 1, 2\}$. For simplicity, we view a 28×28 image $\mathbf{x}_n$ as a 784-dimensional vector by concatenating all the pixels together. In other words, $\mathbf{x}_n \in \mathbb{R}^{784}$. Your goal is to implement two digit recognizers (`OneLayerNetwork` and `TwoLayerNetwork`) and compare their performances.

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code and data

- code : `CS146_Winter2024_PS3.ipynb`
- data : `ps3_train.csv`, `ps3_valid.csv`, `ps3_test.csv`

Please use your *@g.ucla.edu* email id to access the code and data. Similar to *PS1*, copy the colab notebook to your drive and make the changes. Mount the drive appropriately and copy the shared data folder to your drive to access via colab. The notebook has marked blocks where you need to code.

========= TODO : START =========

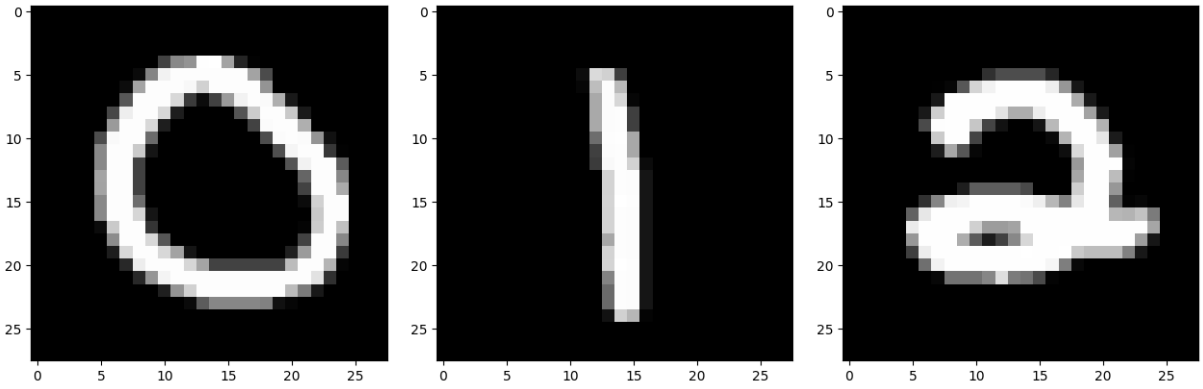
========= TODO : END =========

Note: For parts (b)-(h), you are expected to copy-paste your code as a part of the solution in the submission pdf. Tip: If you are using LaTeX, check out the Minted package (example) for code highlighting.

Data Visualization and Preparation [10 pts]

(a) **(2 pts)** Select three training examples with *different labels* and print out the images by using `plot_img` function. Include those images in your report.

Solution: Any three training examples with different labels (i.e. the images are of digits 0, 1, 2) works. One set of three training examples is shown below:



(b) **(3 pts)** The loaded examples are numpy arrays. Convert the numpy arrays to PyTorch tensors.

Solution:

```
X_train = torch.from_numpy(X_train)
y_train = torch.from_numpy(y_train)
X_valid = torch.from_numpy(X_valid)
y_valid = torch.from_numpy(y_valid)
X_test = torch.from_numpy(X_test)
y_test = torch.from_numpy(y_test)
```

(c) **(5 pts)** Prepare `train_loader`, `valid_loader`, and `test_loader` by using `TensorDataset` and `DataLoader`. We expect to get a batch of pairs $(\mathbf{x}_n, y_n)$ from the dataloader. Please set the batch size to 10 for all dataloaders and shuffle to True for `train_loader`. We need to set shuffle to True for `train_loader` so that we are training on randomly selected minibatches of data.

You can refer https://pytorch.org/docs/stable/data.html for more information about `TensorDataset` and `DataLoader`.

Solution:

```
train_loader = DataLoader(TensorDataset(X_train, y_train), batch_size=10, shuffle=True)
valid_loader = DataLoader(TensorDataset(X_valid, y_valid), batch_size=10, shuffle=False)
test_loader = DataLoader(TensorDataset(X_test, y_test), batch_size=10, shuffle=False)
```

One-Layer Network [15 pts]

For one-layer network, we consider a *784–3* network. In other words, we learn a 784×3 weight matrix $\mathbf{W}$. Given a $\mathbf{x}_n$, we can compute the probability vector $\mathbf{p}_n = \text{Softmax}(\mathbf{W}^\top \mathbf{x}_n)$, where $\mathbf{p}_{n,c}$

denotes the probability of class c. Then, we focus on the *cross entropy loss*

$$-\sum_{n=1}^{N}\sum_{c=1}^{C}\mathbf{1}(c=y_n)\log(\mathbf{p}_{n,c})$$

where N is the number of examples, C is the number of classes, and $\mathbf{1}$ is the indicator function.

(d) **(5 pts)** Implement the constructor of `OneLayerNetwork` with `torch.nn.Linear` and implement the `forward` function to compute the outputs of the single fully connected layer i.e. $\mathbf{W}^\top\mathbf{x}_n$. Notice that we do not compute the softmax function here since we will use `torch.nn.CrossEntropyLoss` later. The bias term is included in `torch.nn.Linear` by default, do *not* disable that option.

You can refer to https://pytorch.org/docs/stable/generated/torch.nn.Linear.html for more information about `torch.nn.Linear` and refer to https://pytorch.org/docs/stable/generated/torch.nn.CrossEntropyLoss.html for more information about using `torch.nn.CrossEntropyLoss`.

Solution:

```
class OneLayerNetwork(torch.nn.Module):
    def __init__(self):
        super(OneLayerNetwork, self).__init__()

        ### ========== TODO : START ========== ###
        ### part d: implement OneLayerNetwork with torch.nn.Linear
        self.layer_1 = torch.nn.Linear(784, 3)
        ### ========== TODO : END ========== ###

    def forward(self, x):
        # x.shape = (n_batch, n_features)

        ### ========== TODO : START ========== ###
        ### part d: implement the foward function
        outputs = self.layer_1(x)
        ### ========== TODO : END ========== ###
        return outputs
```

(e) **(2 pts)** Create an instance of `OneLayerNetwork`, set up a criterion with `torch.nn.CrossEntropyLoss`, and set up a SGD optimizer with learning rate 0.0005 by using `torch.optim.SGD`.

You can refer to https://pytorch.org/docs/stable/optim.html for more information about `torch.optim.SGD`.

Solution:

```
model_one = OneLayerNetwork()
criterion = torch.nn.CrossEntropyLoss()
optimizer = torch.optim.SGD(model_one.parameters(), lr=0.0005)
```

(f) **(8 pts)** Implement the training process. This includes forward pass, initializing gradients to zeros, computing loss, loss.backward, and updating model parameters. If you implement

everything correctly, after running the `train` function in main, you should get results similar to the following.

```
Start training OneLayerNetwork...
| epoch  1 | train loss 1.075380 | train acc 0.453333 | valid loss ...
| epoch  2 | train loss 1.021195 | train acc 0.553333 | valid loss ...
| epoch  3 | train loss 0.972527 | train acc 0.626667 | valid loss ...
| epoch  4 | train loss 0.928296 | train acc 0.710000 | valid loss ...
...
```

Solution:

```python
def train(model, criterion, optimizer, train_loader, valid_loader, epochs=31):
    train_loss_list = []
    valid_loss_list = []
    train_acc_list = []
    valid_acc_list = []
    for epoch in range(1, epochs):
        model.train()
        for batch_X, batch_y in train_loader:
            ### ========== TODO : START ========== ###
            ### part f: implement the training process
            model.zero_grad()
            output = model(batch_X)
            loss = criterion(output, batch_y)
            loss.backward()
            optimizer.step()
            ### ========== TODO : END ========== ###

        train_loss = evaluate_loss(model, criterion, train_loader)
        valid_loss = evaluate_loss(model, criterion, valid_loader)
        train_acc = evaluate_acc(model, train_loader)
        valid_acc = evaluate_acc(model, valid_loader)
        train_loss_list.append(train_loss)
        valid_loss_list.append(valid_loss)
        train_acc_list.append(train_acc)
        valid_acc_list.append(valid_acc)

        print(f"| epoch {epoch:2d} | train loss {train_loss:.6f} | train acc {train_acc:.6

    return train_loss_list, valid_loss_list, train_acc_list, valid_acc_list
```

Two-Layer Network [7 pts]

For two-layer network, we consider a *784–400–3* network. In other words, the first layer will consist of a fully connected layer with 784×400 weight matrix $\mathbf{W}_1$ and a second layer consisting of 400×3 weight matrix $\mathbf{W}_2$. Given a $\mathbf{x}_n$, we can compute the probability vector $\mathbf{p}_n = \text{Softmax}(\mathbf{W}_2^\top \sigma(\mathbf{W}_1^\top \mathbf{x}_n))$, where $\sigma(.)$ is the element-wise sigmoid function. Again, we focus on the

cross entropy loss, hence the network will implement $\mathbf{W}_2^\top \sigma(\mathbf{W}_1^\top \mathbf{x}_n)$ (note the softmax will be taken care of implicitly in our loss). The bias term is included in `torch.nn.Linear` by default, do *not* disable that option.

(g) **(5 pts)** Implement the constructor of `TwoLayerNetwork` with `torch.nn.Linear` and implement the `forward` function to compute $\mathbf{W}_2^\top \sigma(\mathbf{W}_1^\top \mathbf{x}_n)$.

Solution:

```
class TwoLayerNetwork(torch.nn.Module):
    def __init__(self):
        super(TwoLayerNetwork, self).__init__()
        ### ========== TODO : START ========== ###
        ### part g: implement TwoLayerNetwork with torch.nn.Linear
        self.layer_1 = torch.nn.Linear(784, 400)
        self.sigmoid = torch.nn.Sigmoid()
        self.layer_2 = torch.nn.Linear(400, 3)

        ### ========== TODO : END ========== ###

    def forward(self, x):
        # x.shape = (n_batch, n_features)

        ### ========== TODO : START ========== ###
        ### part g: implement the foward function
        outputs = self.layer_2(self.sigmoid(self.layer_1(x)))

        ### ========== TODO : END ========== ###
        return outputs
```

(h) **(2 pts)** Create an instance of `TwoLayerNetwork`, set up a criterion with `torch.nn.CrossEntropyLoss`, and set up a SGD optimizer with learning rate 0.0005 by using `torch.optim.SGD`. Then train `TwoLayerNetwork`.

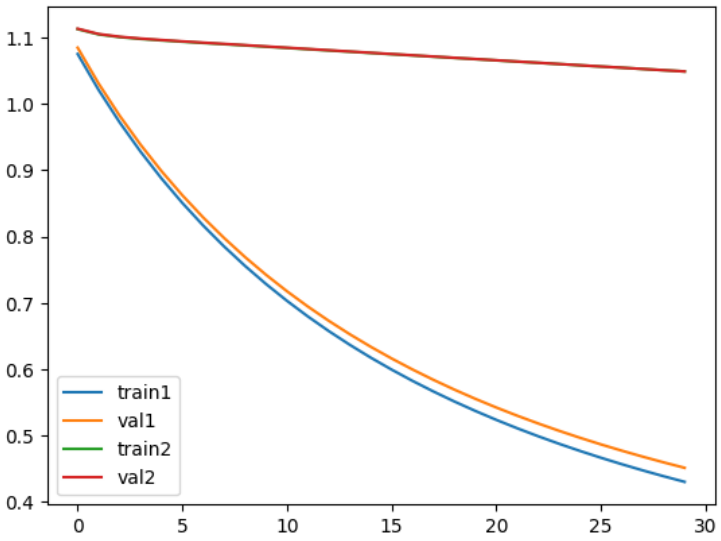
Solution:

```
model_two = TwoLayerNetwork()
criterion = torch.nn.CrossEntropyLoss()
optimizer = torch.optim.SGD(model_two.parameters(), lr=0.0005)
```

Performance Comparison [16 pts]

(i) **(3 pts)** Generate a plot depicting how `one_train_loss`, `one_valid_loss`, `two_train_loss`, `two_valid_loss` varies with epochs. Include the plot in the report and describe your findings.

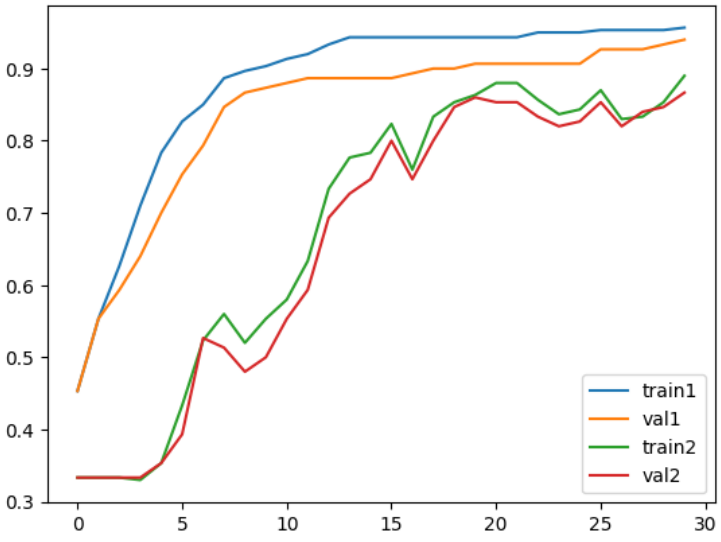
Solution:



From the above plot, we see that the training and validation losses for the one layer network decrease rapidly while the training and validation losses for the two layer network do not decrease much.

(j) **(3 pts)** Generate a plot depicting how `one_train_acc`, `one_valid_acc`, `two_train_acc`, `two_valid_acc` varies with epochs. Include the plot in the report and describe your findings.

Solution:



From the above plot, we see that the training and validation accuracies for the one layer network increase steadily. The training and validation accuracies for the two layer network also increase but with occasional dips. The two layer network's accuracies are lower than the one layer network's accuracies.

(k) **(3 pts)** Calculate and report the test accuracy of both the one-layer network and the two-layer network. How can you improve the performance of the two-layer network ?

Solution:

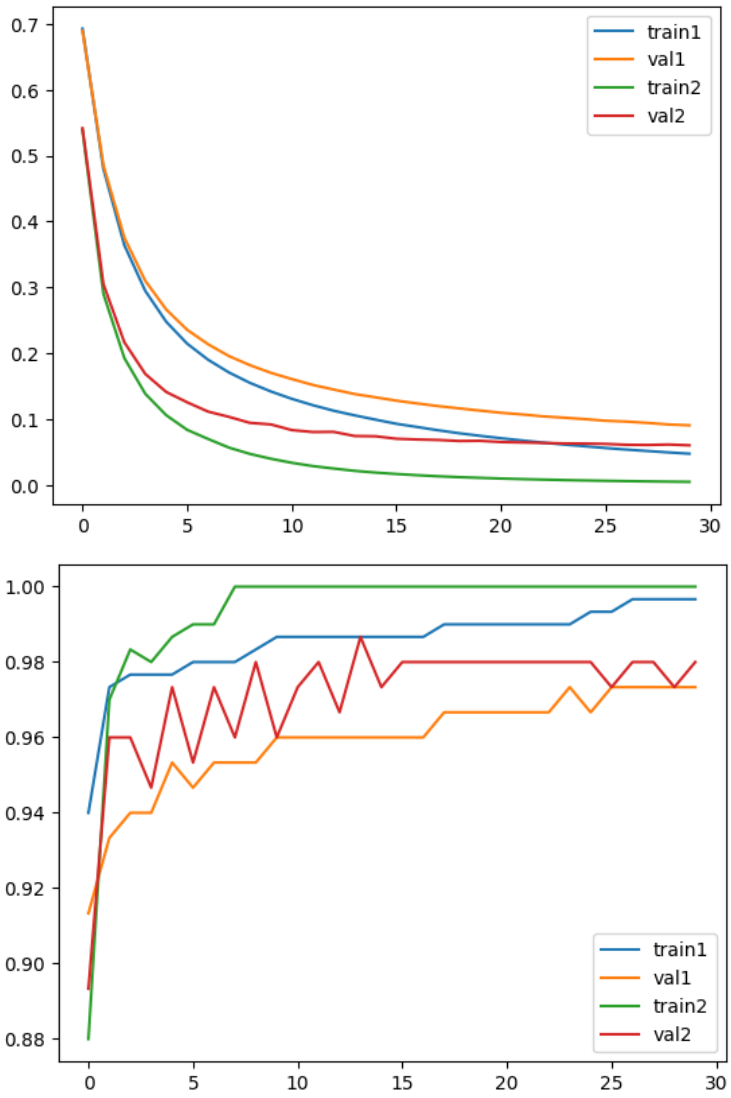
One-layer network test accuracy (rounded): 0.960
Two-layer network test accuracy (rounded): 0.893

There are a variety of potential ways to improve the performance of the two-layer network such as using a more complex optimizer such as Adam, training for more epochs, and training with a different learning rate schedule.

(l) **(7 pts)** Replace the SGD optimizer with the Adam optimizer and do the experiments again. Show the loss figure, the accuracy figure, and the test accuracy. Include the figures in the report and describe your findings.

You can refer to https://pytorch.org/docs/stable/optim.html for more information about `torch.optim.Adam`.

Solution: Loss plot (top) and accuracy plot (bottom) are below.



One-layer network test accuracy (rounded): 0.967
Two-layer network test accuracy (rounded): 0.973

Briefly, the two layer network's performance (whether we use loss or accuracy as the metric) improves significantly when training with Adam as opposed to SGD. While the one layer network's performance also improves when training with Adam, the two layer network's performance improves much more so that the two layer network outperforms the one layer network, opposite of what was observed when training with SGD.