计算方法作业 2

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李庆杨等, 数值分析, 第 5 版, 华中科大, P.162, 1,3,5,6,7,9,10,11,14,15

1. 习题 1

   以下列出编程计算的中间过程：(区间 $[x_0, x_1]$ 中点 m)

   x0 = 0.0, m = 1.0, x1 = 2.0, f(x0) = -1.0, f(m) = -1.0, f(x1) = 1.0

   x0 = 1.0, m = 1.5, x1 = 2.0, f(x0) = -1.0, f(m) = -0.25, f(x1) = 1.0

   x0 = 1.5, m = 1.75, x1 = 2.0, f(x0) = -0.25, f(m) = 0.3125, f(x1) = 1.0

   x0 = 1.5, m = 1.625, x1 = 1.75, f(x0) = -0.25, f(m) = 0.015625, f(x1) = 0.3125

   x0 = 1.5, m = 1.5625, x1 = 1.625, f(x0) = -0.25, f(m) = -0.12109375, f(x1) = 0.015625

   x0 = 1.5625, m = 1.59375, x1 = 1.625, f(x0) = -0.12109375, f(m) = -0.0537109375, f(x1) = 0.015625

   final: x0 = 1.59375, x1 = 1.625

   得到误差不超过 0.05 的近似解 $x = 1.625$

2. 习题 3

   (1) $\phi(x) = 1 + \dfrac{1}{x^2}$, $\phi'(x) = -\dfrac{2}{x^3}$, $|\phi'(x^*)| < 1$, 收敛;

   (2) $\phi(x) = (1 + x^2)^{1/3}$, $\phi'(x) = \dfrac{2}{3}x(1 + x^2)^{-2/3}$, $|\phi'(x^*)| < 1$, 收敛;

   (3) $\phi(x) = (x - 1)^{-1/2}$, $\phi'(x) = -\dfrac{1}{2}(x - 1)^{-3/2}$, $|\phi'(x^*)| > 1$, 不收敛.

   选取 (1) 来计算，粗略估根 $x^* \in [1.4, 1.5]$, $|\phi'(x)| < 0.75$. 取 $x_0 = 1.5$ 得到 $x_1 = 1.4444\ldots$, 由 $|x_k - x^*| \le \dfrac{L^k}{1-L} \times |x_1 - x_0| < \dfrac{0.75^k}{1-0.75} \times 0.1$ 知，当 k 足够大 (比如取 30) 时，能保证误差限小于 5×10^{-4}, 从而保证 4 位有效数字。计算得到 $x = 1.466$.

3. 习题 5

   迭代方程 $x = \phi(x) = x - \lambda f(x)$, $1 - \lambda M < \phi'(x) = 1 - \lambda f'(x) < 1$, 故 $|\phi'(x)| \le \max\{1, |1 - \lambda M|\}$. 由于 $0 < \lambda M < 2$，故 $|\phi'(x)| < 1$, 迭代过程收敛。

4. 习题 6

   (1) 可化为 $x = \phi^{-1}(x)$ 的形式，有 $|\phi^{-1\prime}(x)| \le 1/k < 1$, 收敛，适于迭代;

   (2) $x = \tan x \to x = \arctan x + k\pi, k \in \mathbb{Z}$. 计算 4.5 附近的根，取 $k = 1$, 迭代得: $x = 4.4934$

5. 习题 7

Newton 法：$x_{k+1}=x_k-\dfrac{x_k^3-3x_k-1}{3x_k^2-3}, x_0=2$. 编程计算得:

x0 = 2.0, x1 = phi(x0) = 1.8888888888888888

x0 = 1.8888888888888888, x1 = phi(x0) = 1.879451566951567

x0 = 1.879451566951567, x1 = phi(x0) = 1.879385244836671

x0 = 1.879385244836671, x1 = phi(x0) = 1.8793852415718169

取根为 $x=1.879$

弦截法：$x_{k+1}=x_k-\dfrac{f(x_k)}{f[x_k,x_{k-1}]}$, $f[x_k,x_{k-1}]$ 表示差商。$x_0=2, x_1=1.9$, 编程计算得:

x0 = 2.0, x1 = 1.9, f(x0) = 1.0, f(x1) = 0.1589999999999998, f[x0, x1] = 8.409999999999995, x2 = 1.8810939357907253

x0 = 1.9, x1 = 1.8810939357907253, f(x0) = 0.1589999999999998, f(x1) = 0.012996163275849959, f[x0, x1] = 7.722592873271078, x2 = 1.8794110601699177

x0 = 1.8810939357907253, x1 = 1.8794110601699177, f(x0) = 0.012996163275849959, f(x1) = 0.00019612871434127

f[x0, x1] = 7.606049076500403, x2 = 1.879385274283925

x0 = 1.8794110601699177, x1 = 1.879385274283925, f(x0) = 0.00019612871434127044, f(x1) = 2.4848990243242497e-07, f[x0, x1] = 7.596412413152335, x2 = 1.8793852415724437

取根为 $x=1.879$

抛物线法：$x_{k+1}=x_k-\dfrac{2f(x_k)}{\omega\pm\sqrt{\omega^2-4f(x_k)f[x_k,x_{k-1},x_{k-2}]}}$, 其中 $\omega=f[x_k,x_{k-1}]+f[x_k,x_{k-1},x_{k-2}](x_k-x_{k-1})$. 每次选取距离 x_k 最近的根。初始选取 $x_0=1, x_1=3, x_2=2$, 编程计算部分过程展示如下:

x0 = 1.0, x1 = 3.0, x2 = 2.0, f210 = 6.0 x31 = 1.8931498239234457, x32 = 0.4401835094098867, x3 = 1.8931498239234457

x0 = 3.0, x1 = 2.0, x2 = 1.8931498239234457, f210 = 6.893149823923446 x31 = 1.879135257176037, x32 = 0.7997197788090766, x3 = 1.879135257176037

x0 = 2.0, x1 = 1.8931498239234457, x2 = 1.879135257176037, f210 = 5.772285081099343 x31 = 1.8793852962191757, x32 = 0.5636774675422187, x3 = 1.8793852962191757

取根为 $x=1.879$

6. 习题 9

由均值不等式 $x_k=\dfrac{1}{2}\left(x+\dfrac{a}{x}\right)\geq\sqrt{a}$. 所以 $x_{k+1}=\dfrac{1}{2}\left(x_k+\dfrac{a}{x_k}\right)\leq\dfrac{1}{2}(x_k+x_k)=x_k$. 所以 $x_1,x_2,\cdots$ 单调递减（不增，如果有某个 $x_k=\sqrt{a}$, 那么后续都会等于该值）

7. 习题 10

$x_{k+1}-x_k=-\dfrac{f(x_k)}{f'(x_k)}$, $R_{k+1}=\dfrac{x_{k+1}-x_k}{(x_k-x_{k-1})^2}=-\dfrac{f(x_k)}{f'(x_k)}\times\dfrac{f'^2(x_{k-1})}{f^2(x_{k-1})}=-\dfrac{f'^2(x_{k-1})}{f'(x_k)}\times\dfrac{f(x_k)}{f^2(x_{k-1})}$

记 $e_k=x_k-x^*$, $\phi(x)=x-\dfrac{f(x)}{f'(x)}$, 有 $e_k=\phi'(x^*)e_{k-1}+\dfrac{\phi''}{2}(x^*)e_{k-1}^2+o(e_{k-1}^2)$, 其中 $\phi'(x^*)=0$, $\phi''(x^*)=\dfrac{f''(x^*)}{f'(x^*)}$ 故而 $e_k=\dfrac{f''(x^*)}{2f'(x^*)}e_{k-1}^2+o(e_{k-1}^2)$

所以 $\dfrac{f(x_k)}{f^2(x_{k-1})}=\dfrac{f'(x^*)e_k+o(e_k)}{f'^2(x^*)e_{k-1}^2+o(e_{k-1}^2)}=\dfrac{f'(x^*)\dfrac{f''(x^*)}{2f'(x^*)}e_{k-1}^2+o(e_{k-1}^2)}{f'^2(x^*)e_{k-1}^2+o(e_{k-1}^2)}=\dfrac{f''(x^*)+o(1)}{2f'^2(x^*)+o(1)}$

所以 $\lim_{k\to\infty}R_k=\lim_{k\to\infty}R_{k+1}=\lim_{k\to\infty}-\dfrac{f'^2(x_{k-1})}{f'(x_k)}\times\dfrac{f''(x^*)+o(1)}{2f'^2(x^*)+o(1)}=-\dfrac{f''(x^*)}{2f'(x^*)}$

8. 习题 11

(1) 由于对称性，可只考虑大于等于 0 部分的情况，$x \geq 0, \phi(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{x^{1/2}}{\frac{1}{2}x^{-1/2}} = -x$

$\phi'(x) = -1, x \geq 0$. $x < 0$ 时也有同样的结论。所以，该迭代过程不收敛。

(2) $x \geq 0$ 时，有 $\phi(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{x^{3/2}}{\frac{3}{2}x^{1/2}} = \dfrac{1}{3}x$, $\phi'(x) = \dfrac{1}{3}$. 对 $x < 0$ 的情形，也有 $\phi'(x) = \dfrac{1}{3}$.

故该迭代过程收敛，收敛速度为线性收敛。

9. 习题 14

(1) $f(x) = x^n - a$, 迭代公式: $\phi(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{x^n - a}{nx^{n-1}}$, $x_{k+1} = \phi(x_k)$.

令 $e_k = x_k - x^* = x_k - \sqrt[n]{a}$, $e_{k+1} = x_{k+1} - x^* = \phi(x_k) - \phi(x^*) = \phi'(x^*)e_k + \dfrac{\phi''(x^*)}{2}e_k^2 + o(e_k^2)$

$\phi' = \dfrac{ff''}{f'^2}, \phi'' = \dfrac{f''f'^2 + ff'f''' - 2ff''^2}{f'^3}$, 由于 $f(x^*) = 0$, $\phi'(x^*) = 0$, $\phi''(x^*) = \dfrac{f''(x^*)}{f'(x^*)} \neq 0$

所以 $\lim_{k\to\infty} \dfrac{\sqrt[n]{a} - x_{k+1}}{(\sqrt[n]{a} - x_k)^2} = \lim_{k\to\infty} -\dfrac{e_{k+1}}{e_k^2} = -\dfrac{f''(x^*)}{2f'(x^*)} = -\dfrac{n-1}{2\sqrt[n]{a}}$

(2) $f(x) = 1 - ax^{-n}$, 迭代公式: $\phi(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{1 - ax^{-n}}{anx^{-n-1}} = \dfrac{n+1}{n}x - \dfrac{1}{an}x^{n+1}$. 同 (1), $f(x^*) = 0, \phi'(x^*) = 0, \phi''(x^*) = -\dfrac{n+1}{a}x^{*n-1} = -\dfrac{n+1}{\sqrt[n]{a}}$

所以 $\lim_{k\to\infty} \dfrac{\sqrt[n]{a} - x_{k+1}}{(\sqrt[n]{a} - x_k)^2} = -\dfrac{\phi''(x^*)}{2} = \dfrac{n+1}{2\sqrt[n]{a}}$

10. 习题 15

迭代公式 $x_{k+1} = \phi(x_k)$, $\phi(x) = \dfrac{x(x^2 + 3a)}{3x^2 + a}$, 令 $e_k = x_k - x^* = x_k - \sqrt{a}$.

计算得 $\phi' = \dfrac{3(x^2 - a)^2}{(3x^2 + a)^2}, \phi'(\sqrt{a}) = 0, \phi'' = \dfrac{48ax(x^2 - a)}{(3x^2 + a)^3}, \phi''(\sqrt{a}) = 0$.

$\phi'''(x^*) = \lim_{x\to\sqrt{a}} \dfrac{\phi''(x) - \phi''(\sqrt{a})}{x - \sqrt{a}} = \lim_{x\to\sqrt{a}} \dfrac{48ax(x + \sqrt{a})}{(3x^2 + a)^3} = \dfrac{3}{2a} \neq 0$.

这就证明了该迭代过程是三阶收敛的方法。且

$\lim_{k\to\infty} \dfrac{\sqrt{a} - x_{k+1}}{(\sqrt{a} - x_k)^3} = \lim_{k\to\infty} \dfrac{-e_{k+1}}{-e_k^3} = \dfrac{\phi'''(x^*)}{6} = \dfrac{1}{4a}$