矩阵理论 作业 10

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1. 习题 1

$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{bmatrix}$ 首先求其 Jordan 标准型：容易计算 $\det(xE - A) = (x-2)^4$. 考虑

$B = A - 2E = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & 3 & -3 \\ 4 & -1 & 3 & -3 \end{bmatrix}$, $B^2 = O$. 求得 $Bx = 0$ 的解空间上的一组基 $\alpha_1 = [-1,-1,0,-1]^\top, \alpha_2 = [0,0,3,3]^\top$. 容易构造 $\beta_1 = [0,1,0,0]^\top, \beta_2 = [0,0,1,0]^\top$ 使 $B\beta_1 = \alpha_1, B\beta_2 = \alpha_2$. 容易验证 $\beta_1, \beta_2, \alpha_1, \alpha_2$ 线性无关，故构造得到 $S = [\alpha_1, \beta_1, \alpha_2, \beta_2] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ -1 & 0 & 3 & 0 \end{bmatrix}$. 求得 $S^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1/3 & 0 & 0 & 1/3 \\ 1 & 0 & 1 & -1 \end{bmatrix}$. 从而 $S^{-1}AS = J = \begin{bmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{bmatrix} = 2E + B_1$. 其中 B_1 是幂零的 Jordan 阵, $B_1^2 = O$.

(1) $\sin(2+x) = \sin(2) + \cos(2) \cdot x - \dfrac{\sin(2)}{2} x^2 + \cdots$, $\sin(J) = \sin(2E + B_1) = \sin(2)E + \cos(2)B_1$.

$$\sin(A) = S \cdot \sin(J) \cdot S^{-1} = S \cdot \begin{bmatrix} \sin(2) & \cos(2) & & \\ & \sin(2) & & \\ & & \sin(2) & \cos(2) \\ & & & \sin(2) \end{bmatrix} \cdot S^{-1}$$

由于 $\sin A = (\sin 2)SES^{-1} + (\cos 2)SB_1S^{-1} = (\sin 2)E + (\cos 2)(A - 2E)$,

具体计算可得 $\sin(A) = \begin{bmatrix} \sin 2 + \cos 2 & -\cos 2 & 0 & 0 \\ \cos 2 & -\cos 2 + \sin 2 & 0 & 0 \\ 3\cos 2 & 0 & 3\cos 2 + \sin 2 & -3\cos 2 \\ 4\cos 2 & -\cos 2 & 3\cos 2 & \sin 2 - 3\cos 2 \end{bmatrix}$.

(2) $\mathrm{e}^J = \mathrm{e}^2(E+B_1) = \mathrm{e}^2 \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$, $\mathrm{e}^A = S \cdot \mathrm{e}^J \cdot S^{-1}$.

由于 $\mathrm{e}^A = \mathrm{e}^2 \cdot S(E+B_1)S^{-1} = \mathrm{e}^2(E+A-2E) = \mathrm{e}^2(A-E)$,

具体计算可得 $\mathrm{e}^A = \mathrm{e}^2 \cdot \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 4 & -3 \\ 4 & -1 & 3 & -2 \end{bmatrix}$.