Electrical Network Analysis and Design
Assignment 2

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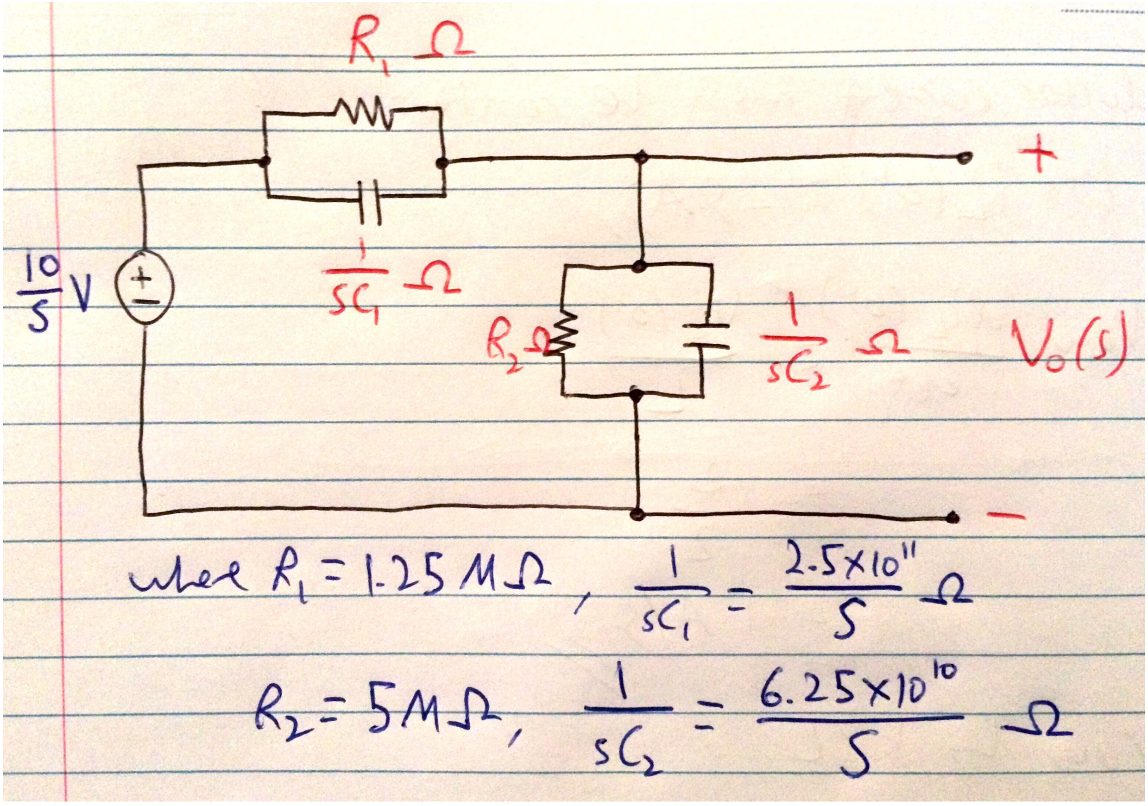
Tuesday 2:15pm - EDS 8

1. (a) The circuit is converted to the s domain:



(b)

$$
\begin{aligned}
Z_{eq1} &= \left(\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}}\right)^{-1} \\
&= \left(\frac{1}{1.25\times10^6} + s\cdot 4\times10^{-12}\right)^{-1} \\
&= \left(\frac{1 + 1.25\times10^6\cdot(s\cdot4\times10^{-12})}{1.25\times10^6}\right)^{-1} \\
&= \frac{1.25\times10^6}{1 + 1.25\times10^6\cdot(s\cdot4\times10^{-12})}
\end{aligned}
$$

$$\therefore Z_{eq1} = \frac{2.5\times10^{11}}{s + 2\times10^5}\ \Omega$$

(c)

$$\begin{aligned}
Z_{eq2} &= \left(\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_2}}\right)^{-1} \\
&= \left(\frac{1}{5 \times 10^6} + s \cdot 16 \times 10^{-12}\right)^{-1} \\
&= \left(\frac{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})}{5 \times 10^6}\right)^{-1} \\
&= \frac{5 \times 10^6}{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})}
\end{aligned}$$

$$\therefore Z_{eq2} = \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}\ \Omega$$

(d) Using voltage division:

$$\begin{aligned}
V_o(s) &= V_{in}(s) \cdot \frac{Z_{eq2}}{Z_{eq1} + Z_{eq2}} \\
&= \frac{10}{s} \cdot \frac{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} + \frac{2.5 \times 10^{11}}{s + 2 \times 10^5}} \\
&= \frac{10}{s} \cdot \frac{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\frac{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)}{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}} \\
&= \frac{10}{s} \cdot \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} \cdot \frac{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)} \\
&= \frac{6.25 \times 10^{11} \cdot (s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{3.125 \times 10^{11} \cdot s \cdot (s + 1.25 \times 10^4) \cdot (s + 5 \times 10^4)} \\
&= \frac{2 \cdot (s + 2 \times 10^5)}{s \cdot (s + 5 \times 10^4)}
\end{aligned}$$

$$\therefore V_o(s) = \frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s}\ \text{V}$$

Now perform partial fraction expansion:

$$\begin{aligned}
\frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s} &= \frac{A}{s} + \frac{B}{s + 50000} \\
A(s + 50000) + Bs &= 2(s + 200000) \\
(s = 0):\ 50000A &= 400000 \\
\implies A &= 8 \\
(s = -50000):\ -50000B &= 300000 \\
\implies B &= -6
\end{aligned}$$

$$\therefore V_o(s) = \frac{8}{s} - \frac{6}{s+50000} \text{ V}$$

Perform inverse Laplace transform to find $v_o(t)$:

$$\begin{aligned} v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\ &= \mathcal{L}^{-1}\left[\frac{8}{s} - \frac{6}{s+50000}\right] \end{aligned}$$

$$\therefore v_o(t) = \left(8 - 6e^{-50000t}\right) u(t) \text{ V}$$

(e) Using Ohm's Law in s domain:

$$\begin{aligned} V_{in}(s) &= Z_{eq} I_o(s) \\ &= (Z_{eq1} + Z_{eq2}) I_o(s) \end{aligned}$$

$$\begin{aligned} \therefore I_o(s) &= \frac{V_{in}(s)}{Z_{eq1} + Z_{eq2}} \\ &= \frac{\frac{10}{s}}{\frac{2.5\times 10^{11}}{s+2\times 10^5} + \frac{6.25\times 10^{10}}{s+1.25\times 10^4}} \\ &= \frac{\frac{10}{s}}{\frac{3.125\times 10^{11}(s+50000)}{(s+12500)(s+200000)}} \\ &= \frac{(s+12500)(s+200000)}{3.125\times 10^{10} s(s+50000)} \text{ A} \end{aligned}$$

We note that the order of the numberator polynomial is equal to the order of the denominator polynomial, and so to perform partial fraction expansion we must first perform polynomial division:

$$\frac{(s+12500)(s+200000)}{3.125\times 10^{10} s(s+50000)} = 3.2\times 10^{-11} + \frac{5.2\times 10^{-6} s + 8\times 10^{-2}}{s(s+5\times 10^4)}$$

Now perform partial fraction expansion:

$$\begin{aligned} \frac{5.2\times 10^{-6} s + 8\times 10^{-2}}{s(s+5\times 10^4)} &= \frac{A}{s} + \frac{B}{s+5\times 10^4} \\ \therefore 5.2\times 10^{-6} s + 8\times 10^{-2} &= A(s+5\times 10^4) + Bs \end{aligned}$$

$$\begin{aligned} s = 0 &\implies 8\times 10^{-2} = A(5\times 10^4) \implies A = 1.6\times 10^{-6} \\ s = -5\times 10^4 &\implies -0.18 = B(-5\times 10^4) \implies B = 3.6\times 10^{-6} \end{aligned}$$

Now the current in the s domain can be represented in partial fraction expanded form:

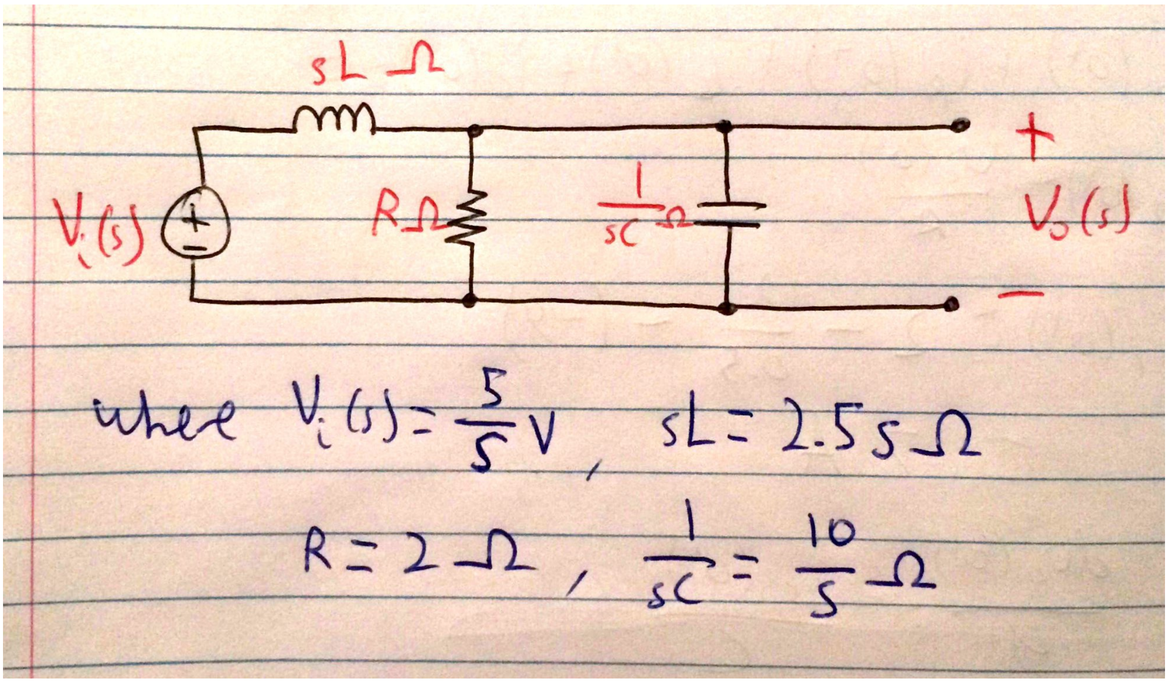
$$\begin{aligned} I_o(s) &= 3.2\times 10^{-11} + \frac{1.6\times 10^{-6}}{s} + \frac{3.6\times 10^{-6}}{s+5\times 10^4} \text{ A} \\ &= 3.2\times 10^{-5} + \frac{1.6}{s} + \frac{3.6}{s+5\times 10^4}\ \mu\text{A} \end{aligned}$$

Using inverse Laplace transform to find $i_o(t)$.

$$\begin{aligned} i_o(t) &= \mathcal{L}^{-1}\left[I_o(s)\right] \\ &= \mathcal{L}^{-1}\left[3.2\times10^{-5} + \frac{1.6}{s} + \frac{3.6}{s+5\times10^4}\right] \end{aligned}$$

$$\therefore i_o(t) = 3.2\times10^{-5}\cdot\delta(t) + \left(1.6 + 3.6\times e^{-50000t}\right)u(t)\ \mu\text{A}$$

2. (a) Convert circtuit to its s domain equivalent:



The circuit is a voltage divider:

$$\begin{aligned} V_o(s) &= V_i(s)\times\frac{Z_R||Z_C}{Z_R||Z_C+Z_L} \\ &= \frac{5}{s}\times\frac{\frac{20s}{s(2s+10)}}{\frac{20s}{s(2s+10)}+2.5s} \\ &= \frac{5}{s}\times\frac{1}{1+0.125(2s^2+10s)} \\ &= \frac{5}{s(0.25s^2+1.25s+1)} \\ &= \frac{20}{s(s+4)(s+1)} \end{aligned}$$

Perform partial fraction expansion:

$$\frac{20}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1}$$

$$\therefore 20 = A(s+4)(s+1) + Bs(s+1) + Cs(s+4)$$

Now solve for A, B and C:

$$s=0 \implies 20 = A(4)(1) \implies A = 5$$

$$s = -4 \implies 20 = B(-4)(-3) \implies B = \frac{5}{3}$$

$$s = -1 \implies 20 = C(-1)(3) \implies C = -\frac{20}{3}$$

And we arrive at $V_o(s)$ in partial fraction expanded form:

$$V_o(s) = \frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)} \text{ V}$$

(b) Perform the inverse Laplace transform:

$$\begin{aligned} v_o(t) &= \mathcal{L}^{-1}\left[V_o(s)\right] \\ &= \left(5 + \frac{5}{3}e^{-4t} - \frac{20}{3}e^{-t}\right) u(t) \text{ V} \end{aligned}$$

(c) We need to develop a more detailed s domain model of the capacitor when we assume it has a non zero voltage across it at $t = 0^-$:

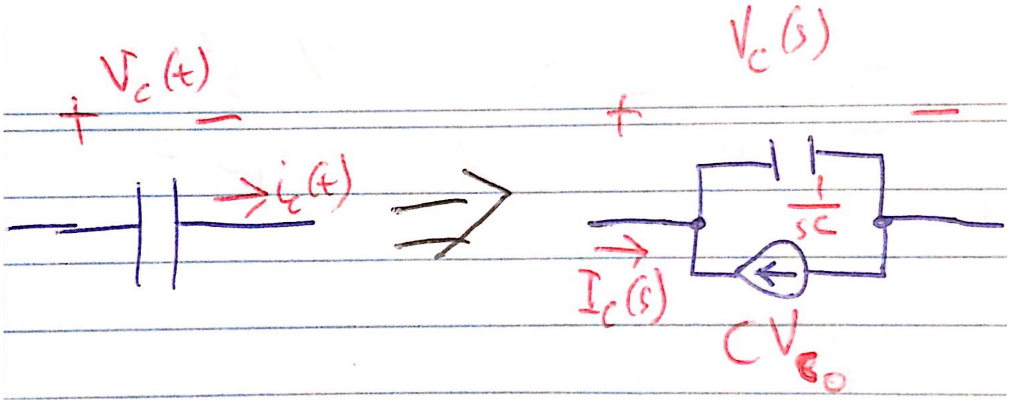
$$i_c(t) = C\frac{\mathrm{d}v_c(t)}{\mathrm{d}t}$$

Take the Laplace transform of both sides:

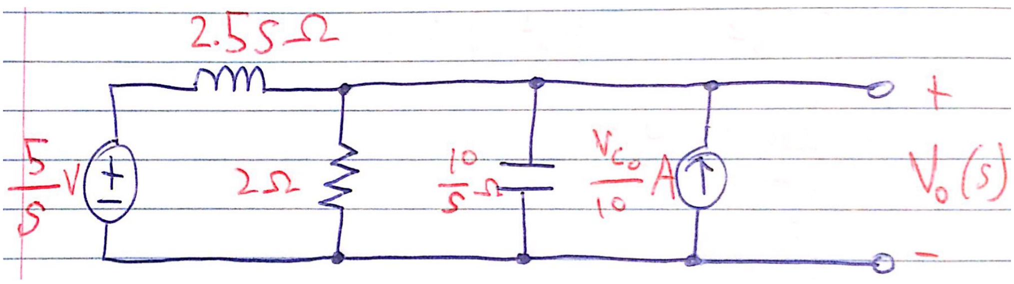
$$\begin{aligned} \mathcal{L}\left[i_c(t)\right] &= \mathcal{L}\left[C\frac{\mathrm{d}v_c(t)}{\mathrm{d}t}\right] \\ I_c(s) &= C\left(sV_c(s) - v_c(0^-)\right) \\ &= sCV_c(s) - CV_{c_o} \end{aligned}$$

Where V_{c_o} is the voltage across the capacitor at time $t = 0^-$.

The diagrammatic representation of this is a capacitor parallel to a current source whose direction is opposite to the reference current direction through the model at its terminals:



We substitute this new capacitor model into the original s domain diagram of the circuit:



We perform node voltage analysis to find $V_o(s)$:

$$\frac{V_o(s) - \frac{5}{s}}{2.5s} + \frac{V_o(s)}{2} + \frac{s \cdot V_o(s)}{10} - \frac{V_{c_o}}{10} = 0$$

Take all constants to right hand side and factor out $V_o(s)$ from what remains on left hand side:

$$V_o(s)\left(\frac{1}{2.5s} + \frac{1}{2} + \frac{s}{10}\right) = \frac{2}{s^2} + \frac{V_{c_o}}{10}$$

$$V_o(s)\left(\frac{5s^2 + 25s + 20}{50s}\right) = \frac{V_{c_o} \cdot s^2 + 20}{10s^2}$$

$$\begin{aligned} V_o(s) &= \frac{V_{c_o} \cdot s^2 + 20}{10s^2} \cdot \frac{10s}{s^2 + 5s + 4} \\ &= \frac{V_{c_o} \cdot s^2 + 20}{s(s+4)(s+1)} \end{aligned}$$

Perform partial fraction expansion:

$$\frac{V_{c_o} \cdot s^2 + 20}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1}$$

$$\therefore V_{c_o} \cdot s^2 + 20 = A(s+4)(s+1) + Bs(s+1) + Cs(s+4)$$

Now solve for A, B and C:

$$s = 0 \implies 20 = A(4)(1) \implies A = 5$$

$$s = -4 \implies 16V_{c_o} + 20 = B(-4)(-3) \implies B = \frac{4V_{c_o} + 5}{3}$$

$$s = -1 \implies V_{c_o} + 20 = C(-1)(3) \implies C = -\frac{V_{c_o} + 20}{3}$$

Therefore we have $V_o(s)$ in partial fraction expanded form:

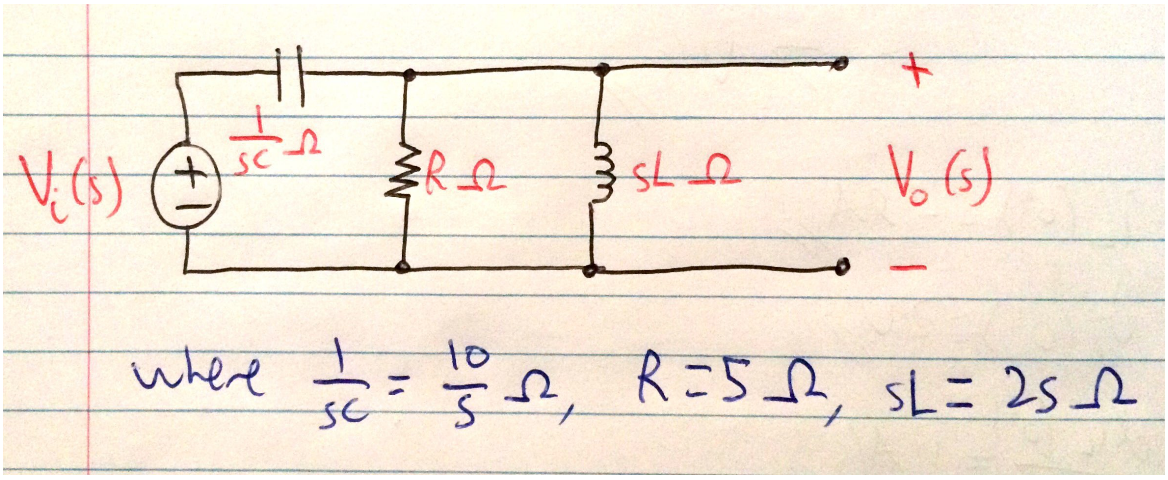
$$V_o(s) = \frac{5}{s} + \frac{4V_{c_o} + 5}{3(s+4)} - \frac{V_{c_o} + 20}{3(s+1)}$$

It is noted that only the coefficients of the $\frac{1}{s+c}$ terms are changed by a non-zero initial voltage across the capacitor.

Specifically, the $\frac{1}{s+4}$ term's coefficient changes from $\frac{5}{3}$ to $\frac{4V_{c_o}+5}{3}$ i.e. its coefficient increases for increasing values of V_{c_o}.[1]
The $\frac{1}{s+1}$ term's coefficient changes from $-\frac{20}{3}$ to $-\frac{V_{c_o}+20}{3}$ i.e. its coefficient decreases for increasing values of V_{c_o}.[1]

[1] Note that increasing here means a positive value becoming more positive or a negative value becoming less negative, and decreasing means the opposite.

3. (a) Convert circuit to its s domain equivalent, and assume no energy stored at $t = 0^-$:



Find $V_o(s)$ by recognising circuit is a voltage divider:

$$V_o(s) = V_i(s) \times \frac{\frac{5\times 2s}{5+2s}}{\frac{5\times 2s}{5+2s} + \frac{10}{s}}$$

$$\therefore H(s) = \frac{10s}{10s + \frac{50}{s} + 20}$$
$$= \frac{10s^2}{10s^2 + 20s + 50}$$

$$\therefore H(s) = \frac{s^2}{s^2 + 2s + 5}$$

(b) We note that the steady state response to the sinusoidal input will be given by the following equation:

$$v_{oSS}(t) = 10 \times |H(j20)| \cos(20t + \theta(20))$$

Where $H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}$

Therefore, find $|H(j20)|$ and $\theta(20)$:

$$H(j20) = \frac{-400}{-395 + j40}$$

$$\therefore |H(j20)| = \frac{400}{\sqrt{(-395)^2 + 40^2}}$$
$$= 1.008$$

$$\text{And } \theta(20) = \arctan\left(\frac{0}{-400}\right) - \arctan\left(\frac{40}{-395}\right)$$
$$= 180^\circ - 174.22^\circ$$
$$= 5.78^\circ$$

Finally sub these values into the equation for v_{oSS}:

$$v_{oSS}(t) = 10.075\cos(20t + 5.78°) \text{ V}$$

4. Start by converting the output response equation to its s domain equivalent:

$$\begin{aligned}
V_o(s) &= \mathcal{L}[v_o(t)] \\
&= \frac{10}{s+3} - \frac{20}{s+4} + \frac{10}{s+5} \\
&= \frac{10(s+4)(s+5) - 20(s+3)(s+5) + 10(s+3)(s+4)}{(s+3)(s+4)(s+5)} \\
&= \frac{20}{(s+3)(s+4)(s+5)}
\end{aligned}$$

Now by definition, $V_i(s) = \frac{V_o(s)}{H(s)}$, therefore:

$$\begin{aligned}
V_i(s) &= \frac{20}{(s+3)(s+4)(s+5)} \times \frac{(s+3)(s+4)}{2(s+5)} \\
&= \frac{10}{(s+5)^2}
\end{aligned}$$

Now convert the input function back to the time domain:

$$\begin{aligned}
v_i(t) &= \mathcal{L}^{-1}[V_i(s)] \\
&= 10te^{-5t}u(t) \text{ V}
\end{aligned}$$

5. (a)

$$\begin{aligned}
F(s) &= \frac{20s^2 + 141s + 315}{s\,(s^2 + 10s + 21)} \\
&= \frac{20s^2 + 141s + 315}{s(s+7)(s+3)}
\end{aligned}$$

Perform partial fraction expansion:

$$\frac{20s^2 + 141s + 315}{s(s+7)(s+3)} = \frac{A}{s} + \frac{B}{s+7} + \frac{C}{s+3}$$

$$\therefore 20s^2 + 141s + 315 = A(s+7)(s+3) + Bs(s+3) + Cs(s+7)$$

Now solve for A, B and C:

$$s = 0 \implies 315 = A(7)(3) \implies A = 15$$

$$s = -7 \implies 308 = B(-7)(-4) \implies B = 11$$

$$s = -3 \implies 72 = A(-3)(4) \implies C = -6$$

And we arrive at $F(s)$ in partial fraction expanded form:

$$F(s) = \frac{15}{s} + \frac{11}{s+7} - \frac{6}{s+3}$$

Now perform the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \left(15 + 11e^{-7t} - 6e^{-3t}\right) u(t) \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{14s^2 + 56s + 152}{(s+6)\left(s^2 + 4s + 20\right)} \\ &= \frac{14s^2 + 56s + 152}{(s+6)(s+2-j4)(s+2+j4)} \end{aligned}$$

Perform partial fraction expansion:

$$\frac{14s^2 + 56s + 152}{(s+6)(s+2-j4)(s+2+j4)} = \frac{A}{s+6} + \frac{B}{s+2-j4} + \frac{B^*}{s+2+j4}$$

$$\therefore 14s^2 + 56s + 152 = A(s+2-j4)(s+2+j4) + B(s+6)(s+2+j4) + B^*(s+6)(s+2-j4)$$

Now solve for A and B:

$$s = -6 \implies 320 = A(-4-j4)(-4+j4) \implies A = 10$$

$$s = -2 + j4 \implies -128 = B(4+j4)(j8) \implies B = 2 + j2$$

And we arrive at $F(s)$ in partial fraction expanded form:

$$\begin{aligned} F(s) &= \frac{10}{s+6} + \frac{2+j2}{s+2-j4} + \frac{2-j2}{s+2+j4} \\ &= \frac{10}{s+6} + \frac{2\sqrt{2}\angle 45^\circ}{s+2-j4} + \frac{2\sqrt{2}\angle -45^\circ}{s+2+j4} \end{aligned}$$

Now perform the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \left(10e^{-6t} + 4\sqrt{2}e^{-2t}\cos(4t + 45^\circ)\right) u(t) \end{aligned}$$

(c)

$$F(s) = \frac{25(s+4)^2}{s^2(s+5)^2}$$

Perform partial fraction expansion:

$$\frac{25(s+4)^2}{s^2(s+5)^2} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+5)^2} + \frac{D}{s+5}$$

$$\therefore 25(s+4)^2 = A(s+5)^2 + Bs(s+5)^2 + Cs^2 + Ds^2(s+5)$$

Now solve for A, B, C and D:

$$s = -5 \implies 25 = C(-5)^2 \implies C = 1$$

$$s = 0 \implies 400 = A(5)^2 \implies A = 16$$

$$s = 1 \implies 625 = 36(16) + B(36) + 1 + D(6)$$

$$\therefore 36B + 6D = 48$$
$$6B + D = 8 \tag{1}$$

$$s = -1 \implies 225 = 16(16) - B(16) + 1 + D(4)$$

$$\therefore -16B + 4D = -32$$
$$-4B + D = -8 \tag{2}$$

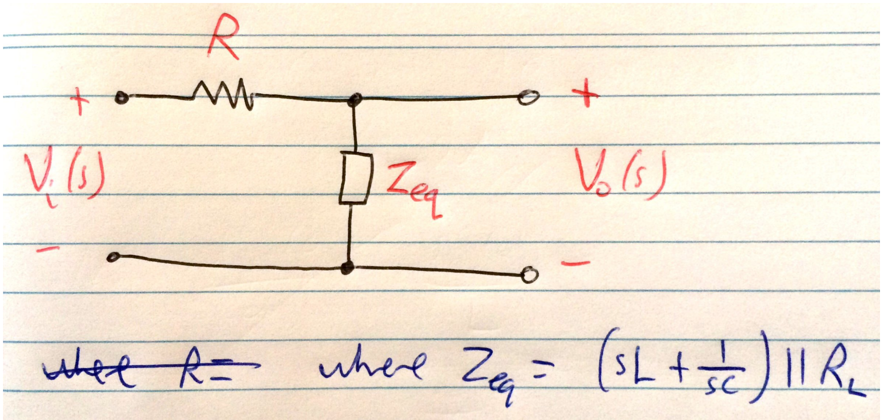
Solving equations (1) and (2) gives $B = \frac{8}{5}$ and $D = -\frac{8}{5}$. We then arrive at $F(s)$ in partial fraction expanded form:

$$F(s) = \frac{16}{s^2} + \frac{8}{5s} + \frac{1}{(s+5)^2} - \frac{8}{5(s+5)}$$

Now perform the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \left(\frac{8}{5} + 16t + e^{-5t}\left(t - \frac{8}{5}\right)\right) u(t) \end{aligned}$$

6. We convert the circuit to its s domain equivalent and combine the inductor, capacitor and load resistance into one component denoted Z_{eq}:



We note the circuit forms a voltage divider. We start by finding the impedance of the circuit across $V_o(s)$:

$$\begin{aligned} Z_{eq} &= (Z_L + Z_C) \mid\mid Z_R \\ &= \left(\frac{1}{sL + \frac{1}{sC}} + \frac{1}{R_L}\right)^{-1} \\ &= \left(\frac{sC}{s^2LC + 1} + \frac{1}{R_L}\right)^{-1} \\ &= \left(\frac{sCR_L + s^2LC + 1}{s^2LCR_L + R_L}\right)^{-1} \\ &= \frac{R_L s^2 + \frac{R_L}{LC}}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} \end{aligned}$$

$$\therefore Z_{eq} = \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}}$$

Now, $V_o(s) = V_i(s) \times \frac{Z_{eq}}{R+Z_{eq}}$, and H(s) can be found from this:

$$\begin{aligned}
H(s) &= \frac{Z_{eq}}{R + Z_{eq}} \\
&= \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} \times \left(\frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} + R \right)^{-1} \\
&= \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} \times \left(\frac{s^2(R_L + R) + \frac{R_L R}{L}s + \frac{1}{LC}(R_L + R)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} \right)^{-1} \\
&= \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2(R_L + R) + \frac{R_L R}{L}s + \frac{1}{LC}(R_L + R)} \\
&= \frac{\frac{R_L}{R_L + R}\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L R}{L(R_L + R)}s + \frac{1}{LC}}
\end{aligned}$$

$$\therefore H(s) = \frac{K\left(s^2 + \frac{1}{LC}\right)}{s^2 + K \cdot \frac{R}{L}s + \frac{1}{LC}}\ , \qquad \text{where } K = \frac{R_L}{R_L + R}$$