

Electrical Network and Device Modelling Assignment 2

David Lynch - 758863, Daniel Landgraf, Zixiang Ren

April 8, 2017

1. (a)
(b)

$$\begin{aligned} Z_{eq1} &= \left(\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}} \right)^{-1} \\ &= \left(\frac{1}{R_1} + \frac{1}{sC_1} \right)^{-1} \\ &= \left(\frac{1}{1.25 \times 10^6} + s \cdot 4 \times 10^{-12} \right)^{-1} \Omega \\ &= \left(\frac{1 + 1.25 \times 10^6 \cdot (s \cdot 4 \times 10^{-12})}{1.25 \times 10^6} \right)^{-1} \Omega \\ &= \frac{1.25 \times 10^6}{1 + 1.25 \times 10^6 \cdot (s \cdot 4 \times 10^{-12})} \Omega \\ &= \frac{2.5 \times 10^{11}}{s + 2 \times 10^5} \Omega \end{aligned}$$

- (c)

$$\begin{aligned} Z_{eq2} &= \left(\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_2}} \right)^{-1} \\ &= \left(\frac{1}{R_2} + \frac{1}{sC_2} \right)^{-1} \\ &= \left(\frac{1}{5 \times 10^6} + s \cdot 16 \times 10^{-12} \right)^{-1} \Omega \\ &= \left(\frac{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})}{5 \times 10^6} \right)^{-1} \Omega \\ &= \frac{5 \times 10^6}{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})} \Omega \\ &= \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} A \end{aligned}$$

(d) Using voltage division:

$$\begin{aligned}
V_o(s) &= V_{in}(s) \cdot \frac{Z_{eq2}}{Z_{eq1} + Z_{eq2}} \\
&= \frac{10}{s} \cdot \frac{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} + \frac{2.5 \times 10^{11}}{s + 2 \times 10^5}} V \\
&= \frac{10}{s} \cdot \frac{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\frac{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)}{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}} V \\
&= \frac{10}{s} \cdot \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} \cdot \frac{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)} V \\
&= \frac{6.25 \times 10^{11} \cdot (s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{3.125 \times 10^{11} \cdot s \cdot (s + 1.25 \times 10^4) \cdot (s + 5 \times 10^4)} V \\
&= \frac{2 \cdot (s + 2 \times 10^5)}{s \cdot (s + 5 \times 10^4)} V \\
&= \frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s} V
\end{aligned}$$

Using Partial fractions to continue.

$$\begin{aligned}
\frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s} &= \frac{A}{s} + \frac{B}{s + 50000} \\
A(s + 50000) + Bs &= 2(s + 200000) \\
(s = 0) : 50000A &= 400000 \\
&\implies A = 8 \\
(s = -50000) : -50000B &= 300000 \\
&B = -6 \\
\implies V_o(s) &= \frac{8}{s} - \frac{6}{s + 50000} V
\end{aligned}$$

Using inverse Laplace transform to find $v_o(t)$.

$$\begin{aligned}
v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\
&= \mathcal{L}^{-1} \left[\frac{8}{s} - \frac{6}{s + 50000} \right] V \\
&= (8 - 6e^{-50000t})u(t) V
\end{aligned}$$

(e) Using Ohm's Law in frequency domain:

$$\begin{aligned}
 V_{in}(s) &= Z_{eq} \cdot I_o(s) \\
 &= (Z_{eq1} + Z_{eq2}) \cdot I_o(s) \\
 \implies I_o(s) &= \frac{V_{in}(s)}{Z_{eq1} + Z_{eq2}} \\
 &= \frac{\frac{10}{s}}{\frac{2.5 \times 10^{11}}{s + 2 \times 10^5} + \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}} A \\
 &= \frac{\frac{10}{s}}{\frac{3.125 \times 10^{11} \cdot (s + 50000)}{(s + 12500) \cdot (s + 200000)}} A \\
 &= \frac{(s + 12500) \cdot (s + 200000)}{3.125 \times 10^{10} \cdot s \cdot (s + 50000)} A
 \end{aligned}$$

After a partial fractions expansion (add working in here)

$$I_o(s) = \frac{9}{2500000 \cdot (s + 50000)} + \frac{1}{625000 \cdot s} + \frac{1}{3.125 \times 10^{10}} A$$

Using inverse Laplace transform to find $i_o(t)$.

$$\begin{aligned}
 i_o(t) &= \mathcal{L}^{-1}[I_o(s)] \\
 &= \mathcal{L}^{-1} \left[\frac{9}{2500000 \cdot (s + 50000)} + \frac{1}{625000 \cdot s} + \frac{1}{3.125 \times 10^{10}} \right] \\
 &= 3.6 \times 10^{-6} e^{-50000t} \cdot u(t) + 1.6 \times 10^{-6} \cdot u(t) + 3.2 \times 10^{-11} \delta(t) A
 \end{aligned}$$

If we assume $t > 0$ only then:

$$i_o(t) = (3.6 \cdot e^{-50000t} + 1.6) \mu A$$

2. (a) Using KCL at the node between capacitor, resistor and inductor: $\Sigma i_{out} = 0$

$$\begin{aligned}
 &\frac{V_o(s) - \frac{5}{s}}{2.5s} + \frac{V_o(s)}{2} + V_o(s) \cdot s \cdot 0.1 = 0 \\
 \implies &\frac{2 \cdot V_o(s)}{5s} - \frac{2}{s^2} + \frac{V_o(s)}{2} + \frac{s \cdot V_o(s)}{10} = 0 \\
 \implies &V_o(s) \left(\frac{2}{5s} + \frac{s}{10} + \frac{1}{2} \right) = \frac{2}{s^2} \\
 \implies &V_o(s) \left(\frac{s^2}{10} + \frac{s}{2} + \frac{2}{5} \right) = \frac{2}{s} \\
 \implies &V_o(s) \cdot (s^2 + 5s + 4) = \frac{20}{s} \\
 \implies &V_o(s) = \frac{20}{s \cdot (s^2 + 5s + 4)} \\
 \implies &V_o(s) = \frac{20}{s \cdot (s + 4) \cdot (s + 1)}
 \end{aligned}$$

Now using partial fraction expansion:

$$\begin{aligned}
 \frac{20}{s \cdot (s+4) \cdot (s+1)} &= \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1} \\
 \Rightarrow A(s+4)(s+1) + B(s)(s+1) + C(s)(s+4) &= 20 \\
 (s=0) \Rightarrow A &= 5 \\
 (s=-1) \Rightarrow 12B &= 20 \\
 \Rightarrow B &= \frac{5}{3} \\
 (s=-1) \Rightarrow -3C &= 20 \\
 \Rightarrow C &= \frac{-20}{3} \\
 \Rightarrow V_o(s) &= \frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)}
 \end{aligned}$$

(b)

$$\begin{aligned}
 v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\
 &= \mathcal{L}^{-1} \left[\frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)} \right] \\
 &= \left(5 + \frac{5}{3}e^{-4t} - \frac{20}{3}e^{-t} \right) u(t)
 \end{aligned}$$

(c)

3. (a) NEEDS DIAGRAM!!!!

$$\begin{aligned}
 Z_{R||L} &= \left(\frac{1}{5} + \frac{1}{2s} \right)^{-1} \\
 &= \left(\frac{2s+5}{10s} \right)^{-1} \\
 &= \frac{10s}{2s+5} \\
 \Rightarrow H(s) &= \frac{V_o(s)}{V_i(s)} \\
 &= \frac{z_{R||L}}{Z_{R||L} + Z_C} \\
 &= \frac{\frac{10s}{2s+5}}{\frac{10s}{2s+5} + \frac{10}{s}} \\
 &= \frac{\frac{10s}{2s+5}}{\frac{10s^2 + 20s + 50}{2s^2 + 5s}} \\
 &= \frac{s^2}{s^2 + 2s + 5}
 \end{aligned}$$

(b) NEEDS DIAGRAM!!!!

Using phasor analysis:

$$\begin{aligned}
 V_o &= V_i \cdot \frac{\left(\frac{1}{R} + \frac{1}{j\omega L}\right)^{-1}}{\left(\frac{1}{R} + \frac{1}{j\omega L}\right)^{-1} + \frac{1}{j\omega C}} \\
 &= 10\angle 0^\circ \cdot \frac{\left(\frac{1}{5} + \frac{1}{40j}\right)^{-1}}{\left(\frac{1}{5} + \frac{1}{40j}\right)^{-1} + \frac{1}{2j}} \\
 &= 10.075\angle 5.78^\circ \text{ V}
 \end{aligned}$$

This means that the steady state response for the system when $v_i(t) = 10 \cos(20t)$ is $v_o(t) = 10.075 \cos(20t + 5.78^\circ)$ V

4. Transfer function can be simplified to:

$$\begin{aligned}
 H(s) &= \frac{2s + 10}{s^2 + 7s + 12} \\
 &= \frac{2(s + 5)}{(s + 3)(s + 4)}
 \end{aligned}$$

Finding the Laplace Transform of the output:

$$\begin{aligned}
 v_o(t) &= (10e^{-3t} - 20e^{-4t} + 10e^{-5t})u(t) \\
 V_o(s) &= \mathcal{L}[(10e^{-3t} - 20e^{-4t} + 10e^{-5t})u(t)] \\
 &= \frac{10}{s + 3} - \frac{20}{s + 4} + \frac{10}{s + 5} \\
 &= \frac{10(s + 4)(s + 5) - 20(s + 3)(s + 5) + 10(s + 3)(s + 4)}{(s + 3)(s + 4)(s + 5)} \\
 &= \frac{10s^2 + 90s + 200 - 20s^2 - 160s - 300 + 10s^2 + 70s + 120}{(s + 3)(s + 4)(s + 5)} \\
 &= \frac{20}{(s + 3)(s + 4)(s + 5)}
 \end{aligned}$$

Transfer function is defined by $H(s) = \frac{V_o(s)}{V_i(s)}$, therefore the equation can be rearranged to give $V_i(s) = \frac{V_o(s)}{H(s)}$.

$$\begin{aligned}
 V_i(s) &= \frac{V_o(s)}{H(s)} \\
 &= \frac{20}{(s + 3)(s + 4)(s + 5)} \div \frac{2(s + 5)}{(s + 3)(s + 4)} \\
 &= \frac{10}{(s + 5)^2}
 \end{aligned}$$

Taking inverse Laplace transform to find $v_i(t)$.

$$\begin{aligned}
 v_i(t) &= \mathcal{L}^{-1}[V_i(s)] \\
 &= \mathcal{L}^{-1}\left[\frac{10}{(s + 5)^2}\right] \\
 &= 10t \cdot e^{-5t}
 \end{aligned}$$

5. (a)

$$\begin{aligned} F(s) &= \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)} \\ &= \frac{20s^2 + 141s + 315}{s(s+7)(s+3)} \end{aligned}$$

Using a partial fraction expansion on $F(s)$:

$$\begin{aligned} \frac{A}{s} + \frac{B}{s+7} + \frac{C}{s+3} &= \frac{20s^2 + 141s + 315}{s(s+7)(s+3)} \\ \implies A(s+7)(s+3) + B(s)(s+3) + C(s)(s+7) &= 20s^2 + 141s + 315 \\ (s=0) \implies 21A &= 315 \\ \implies A &= 15 \\ (s=-7) \implies 28B &= 20(-7)^2 + 141(-7) + 315 \\ \implies B &= \frac{20 \cdot 49 + 141(-7) + 315}{28} \\ \implies B &= 11 \\ (s=-3) \implies -12C &= 20(-3)^2 + 141(-3) + 315 \\ \implies C &= \frac{20 \cdot 9 + 141(-3) + 315}{-12} \\ \implies C &= -6 \\ \implies F(s) &= \frac{15}{s} + \frac{11}{s+7} - \frac{6}{s+3} \end{aligned}$$

Finding inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1}\left[\frac{15}{s} + \frac{11}{s+7} - \frac{6}{s+3}\right] \\ &= (15 + 11e^{-7t} - 6e^{-3t})u(t) \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{14s^2 + 56s + 152}{(s+6)(s^2 + 4s + 20)} \\ &= \frac{14s^2 + 56s + 152}{(s+6)(s^2 + 4s + 4 - 4 + 20)} \\ &= \frac{14s^2 + 56s + 152}{(s+6)((s+2)^2 + 16)} \\ &= \frac{14s^2 + 56s + 152}{(s+6)(s+2+j4)(s+2-j4)} \end{aligned}$$

Using a partial fraction expansion and let A, B, B^* be constants where B^* is the complex conjugate of B . Then:

$$\begin{aligned}
& \frac{A}{s+6} + \frac{B}{s+2+j4} + \frac{B^*}{s+2-j4} = \frac{14s^2 + 56s + 152}{(s+6)(s+2+j4)(s+2-j4)} \\
& \Rightarrow A(s+2+j4)(s+2-j4) + B(s+6)(s+2-j4) + B^*(s+6)(s+2+j4) \\
& = 14s^2 + 56s + 152 \\
& (s = -6) \Rightarrow A(-6+2+j4)(-6+2-j4) = 14(-6)^2 + 56(-6) + 152 \\
& \Rightarrow 32A = 320 \\
& \Rightarrow A = 10 \\
& (s = -2-j4) \Rightarrow B(-2-j4+6)(-2-j4+2-j4) = 14(-2-j4)^2 + 56(-2-j4) + 152 \\
& \Rightarrow B(-32-j32) = -128 \\
& \Rightarrow B = \frac{4}{1+j} \\
& \Rightarrow B = \frac{4-j4}{2} \\
& \Rightarrow B = 2-j2 \\
& \Rightarrow B^* = 2+j2 \\
& \Rightarrow F(s) = \frac{10}{s+6} + \frac{2-j2}{s+2+j4} + \frac{2+j2}{s+2-j4}
\end{aligned}$$

Want in the form $F(s) = \frac{10}{s+6} + \frac{|K|\angle\theta}{s+\alpha-j\beta} + \frac{|K|\angle-\theta}{s+\alpha+j\beta}$ Find B and B^* in phasor form.

$$\begin{aligned}
|B| &= |B^*| = \sqrt{(2)^2 + (-2)^2} \\
&= 2\sqrt{2} \\
\angle B &= \arctan\left(\frac{2}{-j2}\right) \\
&= -45^\circ \\
\Rightarrow \angle B^* &= 45^\circ \\
\Rightarrow F(s) &= \frac{10}{s+6} + \frac{2\sqrt{2}\angle-45^\circ}{s+2+j4} + \frac{2\sqrt{2}\angle45^\circ}{s+2-j4}
\end{aligned}$$

Finding inverse Laplace Transform:

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}\left[\frac{10}{s+6} + \frac{2\sqrt{2}\angle-45^\circ}{s+2+j4} + \frac{2\sqrt{2}\angle45^\circ}{s+2-j4}\right] \\
&= (10e^{-6t} + 4\sqrt{2}e^{-2t} \cos(4t + 45^\circ))u(t)
\end{aligned}$$

(c)

$$F(s) = \frac{25(s+4)^2}{s^2(s+5)^2}$$

Using a partial fraction expansion:

Let $F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5} + \frac{D}{(s+5)^2}$ where A, B, C , and D , are constants.

$$\begin{aligned}
& \Rightarrow A(s)(s+5)^2 + B(s+5)^2 + C(s)^2(s+5) + D(s)^2 = 25(s+4)^2 \\
& \Rightarrow A(s^3 + 10s^2 + 25s) + B(s^2 + 10s + 25) + C(s^3 + 5s^2) + D(s^2) = 25s^2 + 200s + 400
\end{aligned}$$

Equating co-efficients gives:

$$\begin{aligned}A + C &= 0 \\10A + B + 5C + D &= 25 \\25A + 10B &= 200 \\25B &= 400 \\\implies B &= 16 \\\implies A &= \frac{200 - 160}{25} \\&= \frac{8}{5} \\\implies C &= -\frac{8}{5} \\\implies D &= 1 \\\implies F(s) &= \frac{8}{5s} + \frac{16}{s^2} + \frac{-8}{5(s+5)} + \frac{1}{(s+5)^2}\end{aligned}$$

Taking the inverse Laplace transform:

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}[F(s)] \\f(t) &= \mathcal{L}^{-1}\left[\frac{8}{5s} + \frac{16}{s^2} + \frac{-8}{5(s+5)} + \frac{1}{(s+5)^2}\right] \\&= \left(\frac{8}{5} + 16t - \frac{8}{5}e^{-5t} + te^{-5t}\right)u(t)\end{aligned}$$

6. Transforming circuit to the frequency (s) domain:

DIAGRAM NEEDED!!!!!!!!!!!!

$$\begin{aligned}
z_{eq} &= \left(\frac{1}{R_L} + \frac{1}{sL + \frac{1}{sC}} \right)^{-1} \\
&= \left(\frac{1}{R_L} + \frac{1}{\frac{s^2LC + 1}{sC}} \right)^{-1} \\
&= \left(\frac{1}{R_L} + \frac{sC}{s^2LC + 1} \right)^{-1} \\
&= \left(\frac{s^2LC + 1 + sR_LC}{R_L(s^2LC + 1)} \right)^{-1} \\
&= \frac{R_L(s^2LC + 1)}{s^2LC + 1 + sR_LC} \\
Z_{eq} + R &= \frac{R_L(s^2LC + 1)}{s^2LC + 1 + sR_LC} + R \\
&= \frac{R_L(s^2LC + 1) + R(s^2LC + 1 + sR_LC)}{s^2LC + 1 + sR_LC} \\
H(s) &= \frac{V_o(s)}{V_i(s)} \\
&= \frac{Z_{eq}}{Z_{eq} + R} \\
&= \frac{R_L(s^2LC + 1)}{s^2LC + 1 + sR_LC} \div \frac{R_L(s^2LC + 1) + R(s^2LC + 1 + sR_LC)}{s^2LC + 1 + sR_LC} \\
&= \frac{R_L(s^2LC + 1)}{s^2LC + 1 + sR_LC} \times \frac{s^2LC + 1 + sR_LC}{R_L(s^2LC + 1) + R(s^2LC + 1 + sR_LC)} \\
&= \frac{R_L(s^2LC + 1)}{R_L(s^2LC + 1) + R(s^2LC + 1 + sR_LC)} \\
&= \frac{R_L(s^2LC + 1)}{s^2LC(R + R_L) + sRR_LC + R + R_L} \\
&= \frac{\frac{R_L}{R + R_L} \left(s^2 + \frac{1}{LC} \right)}{s^2 + s \frac{RR_L}{L(R + R_L)} + \frac{1}{LC}}
\end{aligned}$$