# Electrical Network and Device Modelling Assignment 2

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April 8, 2017

1. (a)

   (b)

$$\begin{aligned}
Z_{eq1} &= \left(\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}}\right)^{-1} \\
&= \left(\frac{1}{R_1} + \frac{1}{\frac{1}{sC_1}}\right)^{-1} \\
&= \left(\frac{1}{1.25 \times 10^6} + s \cdot 4 \times 10^{-12}\right)^{-1} \Omega \\
&= \left(\frac{1 + 1.25 \times 10^6 \cdot (s \cdot 4 \times 10^{-12})}{1.25 \times 10^6}\right)^{-1} \Omega \\
&= \frac{1.25 \times 10^6}{1 + 1.25 \times 10^6 \cdot (s \cdot 4 \times 10^{-12})} \Omega \\
&= \frac{2.5 \times 10^{11}}{s + 2 \times 10^5} \Omega
\end{aligned}$$

   (c)

$$\begin{aligned}
Z_{eq2} &= \left(\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_2}}\right)^{-1} \\
&= \left(\frac{1}{R_2} + \frac{1}{\frac{1}{sC_2}}\right)^{-1} \\
&= \left(\frac{1}{5 \times 10^6} + s \cdot 16 \times 10^{-12}\right)^{-1} \Omega \\
&= \left(\frac{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})}{5 \times 10^6}\right)^{-1} \Omega \\
&= \frac{5 \times 10^6}{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})} \Omega \\
&= \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} A
\end{aligned}$$

(d) Using voltage division:

$$
\begin{aligned}
V_o(s) &= V_{in}(s) \cdot \frac{Z_{eq2}}{Z_{eq1} + Z_{eq2}} \\
&= \frac{10}{s} \cdot \frac{\dfrac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\dfrac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} + \dfrac{2.5 \times 10^{11}}{s + 2 \times 10^5}}\ V \\
&= \frac{10}{s} \cdot \frac{\dfrac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\dfrac{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)}{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}}\ V \\
&= \frac{10}{s} \cdot \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} \cdot \frac{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)}\ V \\
&= \frac{6.25 \times 10^{11} \cdot (s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{3.125 \times 10^{11} \cdot s \cdot (s + 1.25 \times 10^4) \cdot (s + 5 \times 10^4)}\ V \\
&= \frac{2 \cdot (s + 2 \times 10^5)}{s \cdot (s + 5 \times 10^4)}\ V \\
&= \frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s}\ V
\end{aligned}
$$

Using Partial fractions to continue.

$$
\begin{aligned}
\frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s} &= \frac{A}{s} + \frac{B}{50000} \\
A(s + 50000) + Bs &= 2(s + 200000) \\
(s = 0):\ 50000A &= 400000 \\
\implies A &= 8 \\
(s = -50000):\ -50000B &= 300000 \\
B &= -6 \\
\implies V_o(s) = \frac{8}{s} - \frac{6}{s + 50000}\ V
\end{aligned}
$$

Using inverse Laplace transform to find $v_o(t)$.

$$
\begin{aligned}
v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\
&= \mathcal{L}^{-1}\left[\frac{8}{s} - \frac{6}{s + 50000}\right]\ V \\
&= (8 - 6e^{-50000t})u(t)\ V
\end{aligned}
$$

(e) Using Ohm's Law in frequency domain:

$$
\begin{aligned}
V_{in}(s) &= Z_{eq} \cdot I_o(s) \\
&= (Z_{eq1} + Z_{eq2}) \cdot I_o(s) \\
\implies I_o(s) &= \frac{V_{in}(s)}{Z_{eq1} + Z_{eq2}} \\
&= \frac{\frac{10}{s}}{\frac{2.5 \times 10^{11}}{s + 2 \times 10^5} + \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}} \ A \\
&= \frac{\frac{10}{s}}{\frac{3.125 \times 10^{11} \cdot (s + 50000)}{(s + 12500) \cdot (s + 200000)}} \ A \\
&= \frac{(s + 12500) \cdot (s + 200000)}{3.125 \times 10^{10} \cdot s \cdot (s + 50000)} \ A
\end{aligned}
$$

After a partial fractions expansion (add working in here)

$$
I_o(s) = \frac{9}{2500000 \cdot (s + 50000)} + \frac{1}{625000 \cdot s} + \frac{1}{3.125 \times 10^{10}} \ A
$$

Using inverse Laplace transform to find $i_o(t)$.

$$
\begin{aligned}
i_o(t) &= \mathcal{L}^{-1}\left[I_o(s)\right] \\
&= \mathcal{L}^{-1}\left[\frac{9}{2500000 \cdot (s + 50000)} + \frac{1}{625000 \cdot s} + \frac{1}{3.125 \times 10^{10}}\right] \\
&= 3.6 \times 10^{-6} e^{-50000t} \cdot u(t) + 1.6 \times 10^{-6} \cdot u(t) + 3.2 \times 10^{-11} \delta(t) \ A
\end{aligned}
$$

If we assume $t > 0$ only then:

$$
i_o(t) = (3.6 \cdot e^{-50000t} + 1.6) \ \mu A
$$

2. (a) Using KCL at the node between capacitor, resistor and inductor: $\Sigma i_{out} = 0$

$$
\begin{aligned}
&\frac{V_o(s) - \frac{5}{s}}{2.5s} + \frac{V_o(s)}{2} + V_o(s) \cdot s \cdot 0.1 = 0 \\
\implies &\frac{2 \cdot V_o(s)}{5s} - \frac{2}{s^2} + \frac{V_o(s)}{2} + \frac{s \cdot V_o(s)}{10} = 0 \\
\implies &V_o(s)\left(\frac{2}{5s} + \frac{s}{10} + \frac{1}{2}\right) = \frac{2}{s^2} \\
\implies &V_o(s)\left(\frac{s^2}{10} + \frac{s}{2} + \frac{2}{5}\right) = \frac{2}{s} \\
\implies &V_o(s) \cdot (s^2 + 5s + 4) = \frac{20}{s} \\
\implies &V_o(s) = \frac{20}{s \cdot (s^2 + 5s + 4)} \\
\implies &V_o(s) = \frac{20}{s \cdot (s + 4) \cdot (s + 1)}
\end{aligned}
$$

Now using partial fraction expansion:

$$
\begin{aligned}
\frac{20}{s\cdot(s+4)\cdot(s+1)} &= \frac{A}{s} + \frac{B}{s+4} + \frac{c}{s+1} \\
&\implies A(s+4)(s+1) + B(s)(s+1) + C(s)(s+4) = 20 \\
(s=0) &\implies A = 5 \\
(s=-1) &\implies 12B = 20 \\
&\implies B = \frac{5}{3} \\
(s=-1) &\implies -3C = 20 \\
&\implies C = \frac{-20}{3} \\
&\implies V_o(s) = \frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\
&= \mathcal{L}^{-1}\left[\frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)}\right] \\
&= (5 + \frac{5}{3}e^{-4t} - \frac{20}{3}e^{-t})u(t)
\end{aligned}
$$

(c)

3. (a) NEEDS DIAGRAM!!!!

$$
\begin{aligned}
Z_{R||L} &= (\frac{1}{5} + \frac{1}{2s})^{-1} \\
&= (\frac{2s+5}{10s})^{-1} \\
&= \frac{10s}{2s+5} \\
\implies H(s) &= \frac{V_o(s)}{V_i(s)} \\
&= \frac{z_{R||L}}{Z_{R||L} + Z_C} \\
&= \frac{\frac{10s}{2s+5}}{\frac{10s}{2s+5} + \frac{10}{s}} \\
&= \frac{\frac{10s}{2s+5}}{\frac{10s^2+20s+50}{2s^2+5s}} \\
&= \frac{s^2}{s^2+2s+5}
\end{aligned}
$$

(b) NEEDS DIAGRAM!!!!!

Using phasor analysis:

$$
\begin{aligned}
V_o &= V_i \cdot \frac{\left((\frac{1}{R} + \frac{1}{j\omega L}\right)^{-1}}{\left(\frac{1}{R} + \frac{1}{j\omega L}\right)^{-1} + \frac{1}{j\omega C}} \\
&= 10\angle 0^\circ \cdot \frac{\left(\frac{1}{5} + \frac{1}{40j}\right)^{-1}}{\left(\frac{1}{5} + \frac{1}{40j}\right)^{-1} + \frac{1}{2j}} \\
&= 10.075\angle 5.78^\circ \ V
\end{aligned}
$$

This means that the steady state response for the system when $v_i(t) = 10\cos(20t)$ is $v_o(t) = 10.075\cos(20t + 5.78^\circ)\ V$

4. Transfer function can be simplified to:

$$
\begin{aligned}
H(s) &= \frac{2s+10}{s^2+7s+12} \\
&= \frac{2(s+5)}{(s+3)(s+4)}
\end{aligned}
$$

Finding the Laplace Transform of the output:

$$
\begin{aligned}
v_o(t) &= (10e^{-3t} - 20e^{-4t} + 10e^{-5t})u(t) \\
V_o(s) &= \mathcal{L}[(10e^{-3t} - 20e^{-4t} + 10e^{-5t})u(t)] \\
&= \frac{10}{s+3} - \frac{20}{s+4} + \frac{10}{s+5} \\
&= \frac{10(s+4)(s+5) - 20(s+3)(s+5) + 10(s+3)(s+4)}{(s+3)(s+4)(s+5)} \\
&= \frac{10s^2 + 90s + 200 - 20s^2 - 160s - 300 + 10s^2 + 70s + 120}{(s+3)(s+4)(s+5)} \\
&= \frac{20}{(s+3)(s+4)(s+5)}
\end{aligned}
$$

Transfer function is defined by $H(s) = \frac{V_o(s)}{V_i(s)}$, therefore the equation can be rearranged to give $V_i(s) = \frac{V_o(s)}{H(s)}$.

$$
\begin{aligned}
V_i(s) &= \frac{V_o(s)}{H(s)} \\
&= \frac{20}{(s+3)(s+4)(s+5)} \div \frac{2(s+5)}{(s+3)(s+4)} \\
&= \frac{10}{(s+5)^2}
\end{aligned}
$$

Taking inverse Laplace transform to find $v_i(t)$.

$$
\begin{aligned}
v_i(t) &= \mathcal{L}^{-1}[V_i(s)] \\
&= \mathcal{L}^{-1}[\frac{10}{(s+5)^2}] \\
&= 10t \cdot e^{-5t}
\end{aligned}
$$

5. Q5

(a)

$$
\begin{aligned}
F(s) &= \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)} \\
&= \frac{20s^2 + 141s + 315}{s(s+7)(s+3)}
\end{aligned}
$$

Using a partial fraction expansion on $F(s)$:

$$
\begin{aligned}
& \frac{A}{s} + \frac{B}{s+7} + \frac{C}{s+3} = \frac{20s^2 + 141s + 315}{s(s+7)(s+3)} \\
& \implies A(s+7)(s+3) + B(s)(s+3) + C(s)(s+7) = 20s^2 + 141s + 315 \\
(s=0) \quad & \implies 21A = 315 \\
& \implies A = 15 \\
(s=-7) \quad & \implies 28B = 20(-7)^2 + 141(-7) + 315 \\
& \implies B = \frac{20 \cdot 49 + 141(-7) + 315}{28} \\
& \implies B = 11 \\
(s=-3) \quad & \implies -12C = 20(-3)^2 + 141(-3) + 315 \\
& \implies C = \frac{20 \cdot 9 + 141(-3) + 315}{-12} \\
& \implies C = -6 \\
& \implies F(s) = \frac{15}{s} + \frac{11}{s+7} - \frac{6}{s+3}
\end{aligned}
$$

Finding inverse Laplace transform:

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}[\frac{15}{s} + \frac{11}{s+7} - \frac{6}{s+3}] \\
&= (15 + 11e^{-7t} - 6e^{-3t})u(t)
\end{aligned}
$$

(b)

$$
\begin{aligned}
F(s) &= \frac{14s^2 + 56s + 152}{(s+6)(s^2 + 4s + 20)} \\
&= \frac{14s^2 + 56s + 152}{(s+6)(s^2 + 4s + 4 - 4 + 20)} \\
&= \frac{14s^2 + 56s + 152}{(s+6)((s+2)^2 + 16)} \\
&= \frac{14s^2 + 56s + 152}{(s+6)(s+2+j4)(s+2-j4)}
\end{aligned}
$$

Using a partial fraction expansion and let $A, B, B^*$ be constants where $B^*$ is the complex conjugate of $B$. Then:

$$
\begin{aligned}
&\frac{A}{s+6} + \frac{B}{s+2+j4} + \frac{B^*}{s+2-j4} = \frac{14s^2+56s+152}{(s+6)(s+2+j4)(s+2-j4)} \\
&\implies A(s+2+j4)(s+2-j4) + B(s+6)(s+2-j4) + B^*(s+6)(s+2+j4) \\
&\quad =14s^2+56s+152 \\
(s=-6) &\implies A(-6+2+j4)(-6+2-j4) = 14(-6)^2+56(-6)+152 \\
&\implies 32A = 320 \\
&\implies A = 10 \\
(s=-2-j4) &\implies B(-2-j4+6)(-2-j4+2-j4) = 14(-2-j4)^2+56(-2-j4)+152 \\
&\implies B(-32-j32) = -128 \\
&\implies B = \frac{4}{1+j} \\
&\implies B = \frac{4-j4}{2} \\
&\implies B = 2-j2 \\
&\implies B^* = 2+j2 \\
&\implies F(s) = \frac{10}{s+6} + \frac{2-j2}{s+2+j4} + \frac{2+j2}{s+2-j4}
\end{aligned}
$$

Want in the form $F(s) = \frac{10}{s+6} + \frac{|K|\angle\theta}{s+\alpha-j\beta} + \frac{|K|\angle-\theta}{s+\alpha+j\beta}$ Find $B$ and $B^*$ in phasor form.

$$
\begin{aligned}
|B| = |B^*| &= \sqrt{(2^2)+(-2)^2} \\
&= 2\sqrt{2} \\
\angle B &= \arctan\left(\frac{2}{-j2}\right) \\
&= -45^\circ \\
\implies \angle B^* &= 45^\circ \\
\implies F(s) &= \frac{10}{s+6} + \frac{2\sqrt{2}\angle-45^\circ}{s+2+j4} + \frac{2\sqrt{2}\angle 45^\circ}{s+2-j4}
\end{aligned}
$$

Finding inverse Laplace Transform:

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}\left[\frac{10}{s+6} + \frac{2\sqrt{2}\angle-45^\circ}{s+2+j4} + \frac{2\sqrt{2}\angle 45^\circ}{s+2-j4}\right] \\
&= (10e^{-6t} + 4\sqrt{2}e^{-2t}\cos(4t+45^\circ))u(t)
\end{aligned}
$$

(c)

$$F(s) = \frac{25(s+4)^2}{s^2(s+5)^2}$$

Using a partial fraction expansion:
Let $F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5} + \frac{D}{(s+5)^2}$ where $A$, $B$, $C$, and $D$, are constants.

$$
\begin{aligned}
&\implies A(s)(s+5)^2 + B(s+5)^2 + C(s)^2(s+5) + D(s)^2 = 25(s+4)^2 \\
&\implies A(s^3+10s^2+25s) + B(s^2+10s+25) + C(s^3+5s^2) + D(s^2) = 25s^2+200s+400
\end{aligned}
$$

Equating co-efficients gives:

$$\begin{aligned}
A + C &= 0 \\
10A + B + 5C + D &= 25 \\
25A + 10B &= 200 \\
25B &= 400 \\
\implies B &= 16 \\
\implies A &= \frac{200 - 160}{25} \\
&= \frac{8}{5} \\
\implies C &= -\frac{8}{5} \\
\implies D &= 1 \\
\implies F(s) &= \frac{8}{5s} + \frac{16}{s^2} + \frac{-8}{5(s+5)} + \frac{1}{(s+5)^2}
\end{aligned}$$

Taking the inverse Laplace transform:

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
f(t) &= \mathcal{L}^{-1}\left[\frac{8}{5s} + \frac{16}{s^2} + \frac{-8}{5(s+5)} + \frac{1}{(s+5)^2}\right] \\
&= \left(\frac{8}{5} + 16t - \frac{8}{5}e^{-5t} + te^{-5t}\right)u(t)
\end{aligned}$$

6. Q6