Electrical Network and Device Modelling Assignment 2

David Lynch - 758863, Daniel Landgraf, Zixiang Ren

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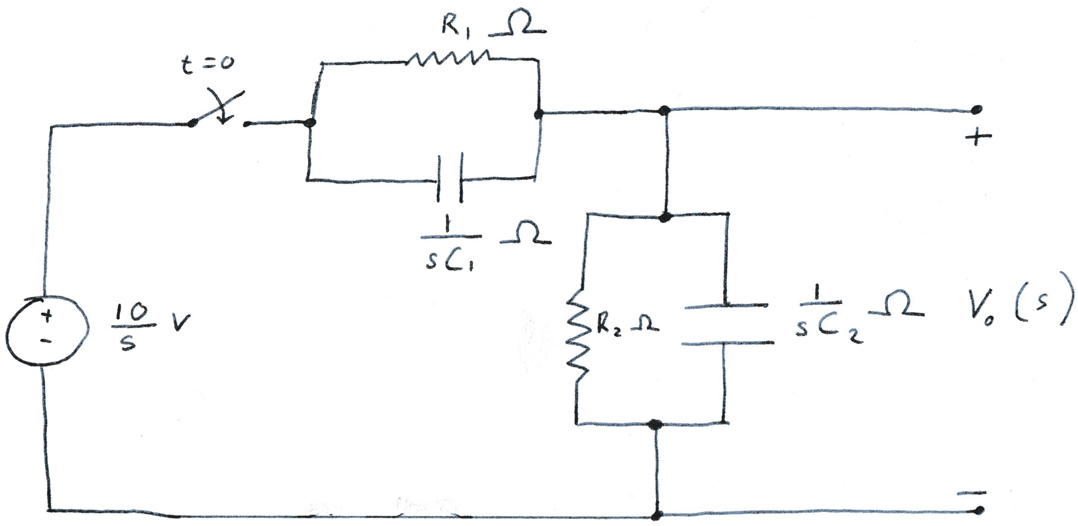
1. (a)



Where $R_1 = 1.25 \times 10^6$, $R_2 = 5 \times 10^6$

$C_1 = 4 \times 10^{-12}$, $C_2 = 16 \times 10^{-12}$

(b)

$$\begin{aligned}
Z_{eq1} &= \left(\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}}\right)^{-1} \\
&= \left(\frac{1}{R_1} + \frac{1}{\frac{1}{sC_1}}\right)^{-1} \\
&= \left(\frac{1}{1.25 \times 10^6} + s \cdot 4 \times 10^{-12}\right)^{-1} \Omega \\
&= \left(\frac{1 + 1.25 \times 10^6 \cdot (s \cdot 4 \times 10^{-12})}{1.25 \times 10^6}\right)^{-1} \Omega \\
&= \frac{1.25 \times 10^6}{1 + 1.25 \times 10^6 \cdot (s \cdot 4 \times 10^{-12})} \Omega \\
&= \frac{2.5 \times 10^{11}}{s + 2 \times 10^5} \Omega
\end{aligned}$$

(c)

$$
\begin{aligned}
Z_{eq2} &= \left(\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_2}}\right)^{-1} \\
&= \left(\frac{1}{R_2} + \frac{1}{\frac{1}{sC_2}}\right)^{-1} \\
&= \left(\frac{1}{5\times 10^6} + s\cdot 16\times 10^{-12}\right)^{-1}\ \Omega \\
&= \left(\frac{1 + 5\times 10^6\cdot(s\cdot 16\times 10^{-12})}{5\times 10^6}\right)^{-1}\ \Omega \\
&= \frac{5\times 10^6}{1+5\times 10^6\cdot(s\cdot 16\times 10^{-12})}\ \Omega \\
&= \frac{6.25\times 10^{10}}{s+1.25\times 10^4}\ A
\end{aligned}
$$

(d) Using voltage division:

$$
\begin{aligned}
V_o(s) &= V_{in}(s)\cdot\frac{Z_{eq2}}{Z_{eq1}+Z_{eq2}} \\
&= \frac{10}{s}\cdot\frac{\frac{6.25\times 10^{10}}{s+1.25\times 10^4}}{\frac{6.25\times 10^{10}}{s+1.25\times 10^4}+\frac{2.5\times 10^{11}}{s+2\times 10^5}}\ V \\
&= \frac{10}{s}\cdot\frac{\frac{6.25\times 10^{10}}{s+1.25\times 10^4}}{\frac{(6.25\times 10^{10})(s+2\times 10^5)+(2.5\times 10^{11})(s+1.25\times 10^4)}{(s+1.25\times 10^4)\cdot(s+2\times 10^5)}}\ V \\
&= \frac{10}{s}\cdot\frac{6.25\times 10^{10}}{s+1.25\times 10^4}\cdot\frac{(s+1.25\times 10^4)\cdot(s+2\times 10^5)}{(6.25\times 10^{10})(s+2\times 10^5)+(2.5\times 10^{11})(s+1.25\times 10^4)}\ V \\
&= \frac{6.25\times 10^{11}\cdot(s+1.25\times 10^4)\cdot(s+2\times 10^5)}{3.125\times 10^{11}\cdot s\cdot(s+1.25\times 10^4)\cdot(s+5\times 10^4)}\ V \\
&= \frac{2\cdot(s+2\times 10^5)}{s\cdot(s+5\times 10^4)}\ V \\
&= \frac{2s+4\times 10^5}{s^2+5\times 10^4\cdot s}\ V
\end{aligned}
$$

Using Partial fractions to continue.

$$
\begin{aligned}
\frac{2s+4\times 10^5}{s^2+5\times 10^4\cdot s} &= \frac{A}{s}+\frac{B}{50000} \\
A(s+50000)+Bs &= 2(s+200000) \\
(s=0):\ 50000A &= 400000 \\
\implies A &= 8 \\
(s=-50000):\ -50000B &= 300000 \\
B &= -6
\end{aligned}
$$

$$
\implies V_o(s) = \frac{8}{s} - \frac{6}{s+50000}\ V
$$

Using inverse Laplace transform to find $v_o(t)$.

$$\begin{aligned}
v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\
&= \mathcal{L}^{-1}\left[\frac{8}{s} - \frac{6}{s+50000}\right] V \\
&= (8 - 6e^{-50000t})u(t)\ V
\end{aligned}$$

(e) Using Ohm's Law in frequency domain:

$$\begin{aligned}
V_{in}(s) &= Z_{eq} \cdot I_o(s) \\
&= (Z_{eq1} + Z_{eq2}) \cdot I_o(s) \\
\implies I_o(s) &= \frac{V_{in}(s)}{Z_{eq1} + Z_{eq2}} \\
&= \frac{\frac{10}{s}}{\frac{2.5\times 10^{11}}{s+2\times 10^5} + \frac{6.25\times 10^{10}}{s+1.25\times 10^4}}\ A \\
&= \frac{\frac{10}{s}}{\frac{3.125\times 10^{11}\cdot(s+50000)}{(s+12500)\cdot(s+200000)}}\ A \\
&= \frac{(s+12500)\cdot(s+200000)}{3.125\times 10^{10}\cdot s\cdot(s+50000)}\ A \\
&= \frac{s^2 + s(12500+200000)(12500)(200000)}{3.125\times 10^{10}\cdot s^2 + s(3.125\times 10^{10})(50000)} \\
&= \frac{\frac{s^2}{3.125\times 10^{10}} + s\frac{12500+200000}{3.125\times 10^{10}} + \frac{(12500)(200000)}{3.125\times 10^{10}}}{s^2 + s(50000)} \\
&= \frac{1}{3.125\times 10^{10}} + \frac{s\left(\frac{12500+200000-50000}{3.125\times 10^{10}}\right) + \frac{(12500)(200000)}{3.125\times 10^{10}}}{s(s+50000)} \\
&= \frac{1}{3.125\times 10^{10}} + \frac{s\cdot\frac{13}{2500000} + \frac{2}{25}}{s(s+50000)}
\end{aligned}$$

Use partial fraction expansion on the s term:

$$\begin{aligned}
\frac{A}{s} + \frac{B}{s+50000} &= \frac{\frac{13}{2500000}\cdot s + \frac{2}{25}}{s(s+50000)} \\
\implies A(s+50000) + B(s) &= \frac{13}{2500000}\cdot s + \frac{2}{25} \\
(s=0) \quad \implies A\cdot 50000 &= \frac{2}{25} \\
\implies A &= \frac{1}{625000} \\
(s=-50000) \quad \implies B(-50000) &= -\left(\frac{13}{2500000}\right)(50000) + \frac{2}{25} \\
\implies B &= \left(\frac{13}{2500000}\right) - \frac{\frac{2}{25}}{50000} \\
\implies B &= \frac{9}{2500000}
\end{aligned}$$

$$\implies I_o(s) = \frac{9}{2500000 \cdot (s+50000)} + \frac{1}{625000 \cdot s} + \frac{1}{3.125 \times 10^{10}}\ A$$

Using inverse Laplace transform to find $i_o(t)$.

$$\begin{aligned}
i_o(t) &= \mathcal{L}^{-1}\left[I_o(s)\right] \\
&= \mathcal{L}^{-1}\left[\frac{9}{2500000 \cdot (s+50000)} + \frac{1}{625000 \cdot s} + \frac{1}{3.125 \times 10^{10}}\right] \\
&= 3.6 \times 10^{-6} e^{-50000t} \cdot u(t) + 1.6 \times 10^{-6} \cdot u(t) + 3.2 \times 10^{-11}\delta(t)\ A
\end{aligned}$$

If we assume $t > 0$ only then:

$$i_o(t) = (3.6 \cdot e^{-50000t} + 1.6)\ \mu A$$

2. (a) Using KCL at the node between capacitor, resistor and inductor: $\Sigma i_{out} = 0$

$$\begin{aligned}
&\frac{V_o(s) - \frac{5}{s}}{2.5s} + \frac{V_o(s)}{2} + V_o(s) \cdot s \cdot 0.1 = 0 \\
\implies &\frac{2 \cdot V_o(s)}{5s} - \frac{2}{s^2} + \frac{V_o(s)}{2} + \frac{s \cdot V_o(s)}{10} = 0 \\
\implies &V_o(s)\left(\frac{2}{5s} + \frac{s}{10} + \frac{1}{2}\right) = \frac{2}{s^2} \\
\implies &V_o(s)\left(\frac{s^2}{10} + \frac{s}{2} + \frac{2}{5}\right) = \frac{2}{s} \\
\implies &V_o(s) \cdot (s^2 + 5s + 4) = \frac{20}{s} \\
\implies &V_o(s) = \frac{20}{s \cdot (s^2 + 5s + 4)}\ V \\
\implies &V_o(s) = \frac{20}{s \cdot (s+4) \cdot (s+1)}\ V
\end{aligned}$$

Now using partial fraction expansion:

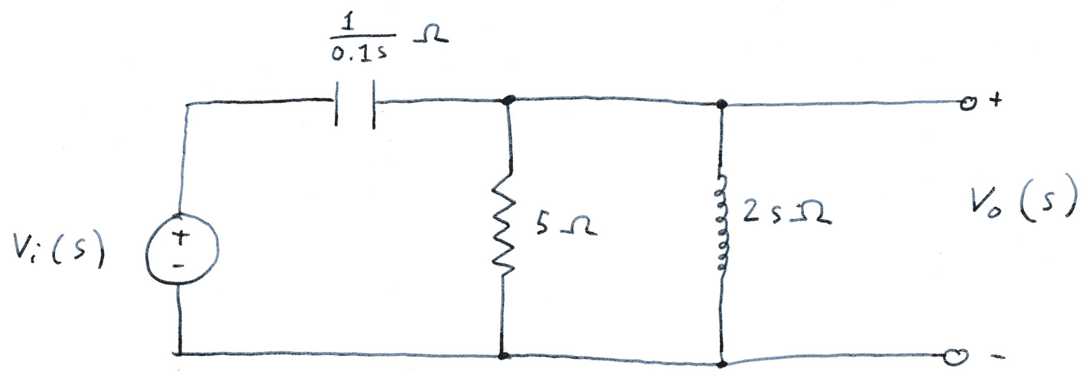
$$\begin{aligned}
&\frac{20}{s \cdot (s+4) \cdot (s+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{c}{s+1} \\
&\implies A(s+4)(s+1) + B(s)(s+1) + C(s)(s+4) = 20 \\
(s=0) &\implies A = 5 \\
(s=-1) &\implies 12B = 20 \\
&\implies B = \frac{5}{3} \\
(s=-1) &\implies -3C = 20 \\
&\implies C = \frac{-20}{3} \\
&\implies V_o(s) = \frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)}\ V
\end{aligned}$$

(b)

$$\begin{aligned}
v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\
&= \mathcal{L}^{-1}\left[\frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)}\right] \\
&= (5 + \frac{5}{3}e^{-4t} - \frac{20}{3}e^{-t})u(t)\ V
\end{aligned}$$

(c)

3. (a)



$$
\begin{aligned}
Z_{R||L} &= (\frac{1}{5} + \frac{1}{2s})^{-1} \\
&= (\frac{2s+5}{10s})^{-1} \\
&= \frac{10s}{2s+5} \\
\implies H(s) &= \frac{V_o(s)}{V_i(s)} \\
&= \frac{z_{R||L}}{Z_{R||L} + Z_C} \\
&= \frac{\frac{10s}{2s+5}}{\frac{10s}{2s+5} + \frac{10}{s}} \\
&= \frac{\frac{10s}{2s+5}}{\frac{10s^2+20s+50}{2s^2+5s}} \\
&= \frac{s^2}{s^2+2s+5}
\end{aligned}
$$

(b) Using phasor analysis:

$$
\begin{aligned}
V_o &= V_i \cdot \frac{\left((\frac{1}{R} + \frac{1}{j\omega L} \right)^{-1}}{\left(\frac{1}{R} + \frac{1}{j\omega L} \right)^{-1} + \frac{1}{j\omega C}} \\
&= 10\angle 0^\circ \cdot \frac{\left(\frac{1}{5} + \frac{1}{40j} \right)^{-1}}{\left(\frac{1}{5} + \frac{1}{40j} \right)^{-1} + \frac{1}{2j}} \\
&= 10.075\angle 5.78^\circ \ V
\end{aligned}
$$

This means that the steady state response for the system when $v_i(t) = 10\cos(20t)$ is $v_o(t) = 10.075\cos(20t + 5.78°)\ V$

4. Transfer function can be simplified to:

$$\begin{aligned} H(s) &= \frac{2s+10}{s^2+7s+12} \\ &= \frac{2(s+5)}{(s+3)(s+4)} \end{aligned}$$

Finding the Laplace Transform of the output:

$$\begin{aligned} v_o(t) &= (10e^{-3t} - 20e^{-4t} + 10e^{-5t})u(t)\ V \\ V_o(s) &= \mathcal{L}[(10e^{-3t} - 20e^{-4t} + 10e^{-5t})u(t)]\ V \\ &= \frac{10}{s+3} - \frac{20}{s+4} + \frac{10}{s+5}\ V \\ &= \frac{10(s+4)(s+5) - 20(s+3)(s+5) + 10(s+3)(s+4)}{(s+3)(s+4)(s+5)}\ V \\ &= \frac{10s^2 + 90s + 200 - 20s^2 - 160s - 300 + 10s^2 + 70s + 120}{(s+3)(s+4)(s+5)}\ V \\ &= \frac{20}{(s+3)(s+4)(s+5)}\ V \end{aligned}$$

Transfer function is defined by $H(s) = \frac{V_o(s)}{V_i(s)}$, therefore the equation can be rearranged to give $V_i(s) = \frac{V_o(s)}{H(s)}$.

$$\begin{aligned} V_i(s) &= \frac{V_o(s)}{H(s)} \\ &= \frac{20}{(s+3)(s+4)(s+5)} \div \frac{2(s+5)}{(s+3)(s+4)}\ V \\ &= \frac{10}{(s+5)^2}\ V \end{aligned}$$

Taking inverse Laplace transform to find $v_i(t)$.

$$\begin{aligned} v_i(t) &= \mathcal{L}^{-1}[V_i(s)] \\ &= \mathcal{L}^{-1}\left[\frac{10}{(s+5)^2}\right]\ V \\ &= 10t \cdot e^{-5t}\ V \end{aligned}$$

5. (a)

$$\begin{aligned} F(s) &= \frac{20s^2 + 141s + 315}{s(s^2 + 10s + 21)} \\ &= \frac{20s^2 + 141s + 315}{s(s+7)(s+3)} \end{aligned}$$

Using a partial fraction expansion on $F(s)$:

$$\begin{aligned}
& \frac{A}{s}+\frac{B}{s+7}+\frac{C}{s+3}=\frac{20s^2+141s+315}{s(s+7)(s+3)} \\
& \implies A(s+7)(s+3)+B(s)(s+3)+C(s)(s+7)=20s^2+141s+315 \\
(s=0) \quad & \implies 21A=315 \\
& \implies A=15 \\
(s=-7) \quad & \implies 28B=20(-7)^2+141(-7)+315 \\
& \implies B=\frac{20\cdot 49+141(-7)+315}{28} \\
& \implies B=11 \\
(s=-3) \quad & \implies -12C=20(-3)^2+141(-3)+315 \\
& \implies C=\frac{20\cdot 9+141(-3)+315}{-12} \\
& \implies C=-6 \\
& \implies F(s)=\frac{15}{s}+\frac{11}{s+7}-\frac{6}{s+3}
\end{aligned}$$

Finding inverse Laplace transform:

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}[\frac{15}{s}+\frac{11}{s+7}-\frac{6}{s+3}] \\
&= (15+11e^{-7t}-6e^{-3t})u(t)
\end{aligned}$$

(b)

$$\begin{aligned}
F(s) &= \frac{14s^2+56s+152}{(s+6)(s^2+4s+20)} \\
&= \frac{14s^2+56s+152}{(s+6)(s^2+4s+4-4+20)} \\
&= \frac{14s^2+56s+152}{(s+6)((s+2)^2+16)} \\
&= \frac{14s^2+56s+152}{(s+6)(s+2+j4)(s+2-j4)}
\end{aligned}$$

Using a partial fraction expansion and let A, B, B^* be constants where B^* is the complex conjugate of B. Then:

$$\begin{aligned}
&\frac{A}{s+6}+\frac{B}{s+2+j4}+\frac{B^*}{s+2-j4}=\frac{14s^2+56s+152}{(s+6)(s+2+j4)(s+2-j4)}\\
&\implies A(s+2+j4)(s+2-j4)+B(s+6)(s+2-j4)+B^*(s+6)(s+2+j4)\\
&=14s^2+56s+152\\
(s=-6)\ &\implies A(-6+2+j4)(-6+2-j4)=14(-6)^2+56(-6)+152\\
&\implies 32A=320\\
&\implies A=10\\
(s=-2-j4)\ &\implies B(-2-j4+6)(-2-j4+2-j4)=14(-2-j4)^2+56(-2-j4)+152\\
&\implies B(-32-j32)=-128\\
&\implies B=\frac{4}{1+j}\\
&\implies B=\frac{4-j4}{2}\\
&\implies B=2-j2\\
&\implies B^*=2+j2\\
&\implies F(s)=\frac{10}{s+6}+\frac{2-j2}{s+2+j4}+\frac{2+j2}{s+2-j4}
\end{aligned}$$

Want in the form $F(s)=\frac{10}{s+6}+\frac{|K|\angle\theta}{s+\alpha-j\beta}+\frac{|K|\angle-\theta}{s+\alpha+j\beta}$ Find B and B^* in phasor form.

$$\begin{aligned}
|B|=|B^*|&=\sqrt{(2^2)+(-2)^2}\\
&=2\sqrt{2}\\
\angle B&=\arctan\left(\frac{2}{-j2}\right)\\
&=-45^\circ\\
\implies\ \angle B^*&=45^\circ\\
\implies\ F(s)&=\frac{10}{s+6}+\frac{2\sqrt{2}\angle-45^\circ}{s+2+j4}+\frac{2\sqrt{2}\angle45^\circ}{s+2-j4}
\end{aligned}$$

Finding inverse Laplace Transform:

$$\begin{aligned}
f(t)&=\mathcal{L}^{-1}[F(s)]\\
&=\mathcal{L}^{-1}\left[\frac{10}{s+6}+\frac{2\sqrt{2}\angle-45^\circ}{s+2+j4}+\frac{2\sqrt{2}\angle45^\circ}{s+2-j4}\right]\\
&=(10e^{-6t}+4\sqrt{2}e^{-2t}\cos(4t+45^\circ))u(t)
\end{aligned}$$

(c)

$$F(s)=\frac{25(s+4)^2}{s^2(s+5)^2}$$

Using a partial fraction expansion:
Let $F(s)=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+5}+\frac{D}{(s+5)^2}$ where A, B, C, and D, are constants.

$$\begin{aligned}
&\implies A(s)(s+5)^2+B(s+5)^2+C(s)^2(s+5)+D(s)^2=25(s+4)^2\\
&\implies A(s^3+10s^2+25s)+B(s^2+10s+25)+C(s^3+5s^2)+D(s^2)=25s^2+200s+400
\end{aligned}$$

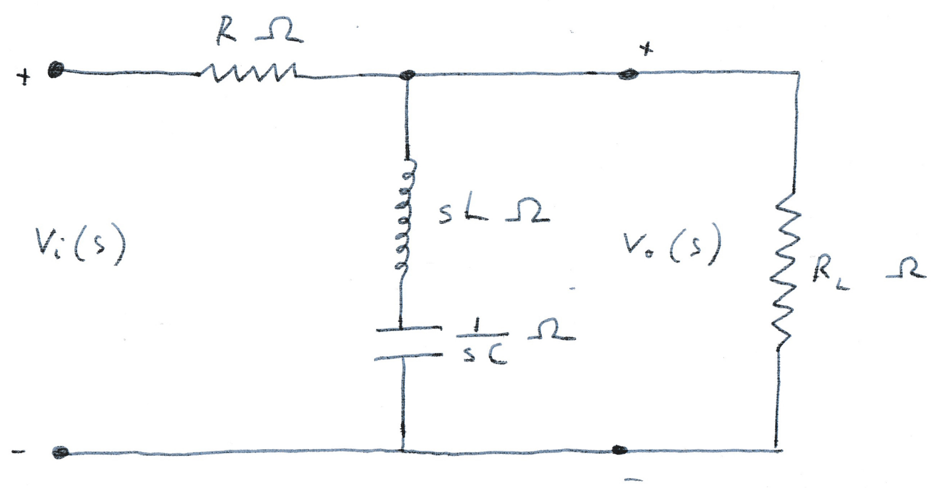
Equating co-efficients gives:

$$
\begin{aligned}
A + C &= 0 \\
10A + B + 5C + D &= 25 \\
25A + 10B &= 200 \\
25B &= 400 \\
\implies B &= 16 \\
\implies A &= \frac{200 - 160}{25} \\
&= \frac{8}{5} \\
\implies C &= -\frac{8}{5} \\
\implies D &= 1 \\
\implies F(s) &= \frac{8}{5s} + \frac{16}{s^2} + \frac{-8}{5(s+5)} + \frac{1}{(s+5)^2}
\end{aligned}
$$

Taking the inverse Laplace transform:

$$
\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
f(t) &= \mathcal{L}^{-1}\left[\frac{8}{5s} + \frac{16}{s^2} + \frac{-8}{5(s+5)} + \frac{1}{(s+5)^2}\right] \\
&= \left(\frac{8}{5} + 16t - \frac{8}{5}e^{-5t} + te^{-5t}\right) u(t)
\end{aligned}
$$

6. Transforming circuit to the frequency (s) domain:



Taking Z_{eq} to be the equivalent impedance of the resistor R_L, the capacitor and the inductor.

$$
\begin{aligned}
Z_{eq} &= \left(\frac{1}{R_L} + \frac{1}{sL + \frac{1}{sC}} \right)^{-1} \\
&= \left(\frac{1}{R_L} + \frac{1}{\frac{s^2LC+1}{sC}} \right)^{-1} \\
&= \left(\frac{1}{R_L} + \frac{sC}{s^2LC+1} \right)^{-1} \\
&= \left(\frac{s^2LC+1+sR_LC}{R_L(s^2LC+1)} \right)^{-1} \\
&= \frac{R_L(s^2LC+1)}{s^2LC+1+sR_LC} \\
Z_{eq} + R &= \frac{R_L(s^2LC+1)}{s^2LC+1+sR_LC} + R \\
&= \frac{R_L(s^2LC+1) + R(s^2LC+1+sR_LC)}{s^2LC+1+sR_LC} \\
H(s) &= \frac{V_o(s)}{V_i(s)} \\
&= \frac{Z_{eq}}{Z_{eq}+R} \\
&= \frac{R_L(s^2LC+1)}{s^2LC+1+sR_LC} \div \frac{R_L(s^2LC+1) + R(s^2LC+1+sR_LC)}{s^2LC+1+sR_LC} \\
&= \frac{R_L(s^2LC+1)}{s^2LC+1+sR_LC} \times \frac{s^2LC+1+sR_LC}{R_L(s^2LC+1) + R(s^2LC+1+sR_LC)} \\
&= \frac{R_L(s^2LC+1)}{R_L(s^2LC+1) + R(s^2LC+1+sR_LC)} \\
&= \frac{R_L(s^2LC+1)}{s^2LC(R+R_L) + sRR_LC + R + R_L} \\
&= \frac{\frac{R_L}{R+R_L}(s^2 + \frac{1}{LC})}{s^2 + s\frac{RR_L}{L(R+R_L)} + \frac{1}{LC}}
\end{aligned}
$$