Electrical Network Analysis and Design
Assignment 2

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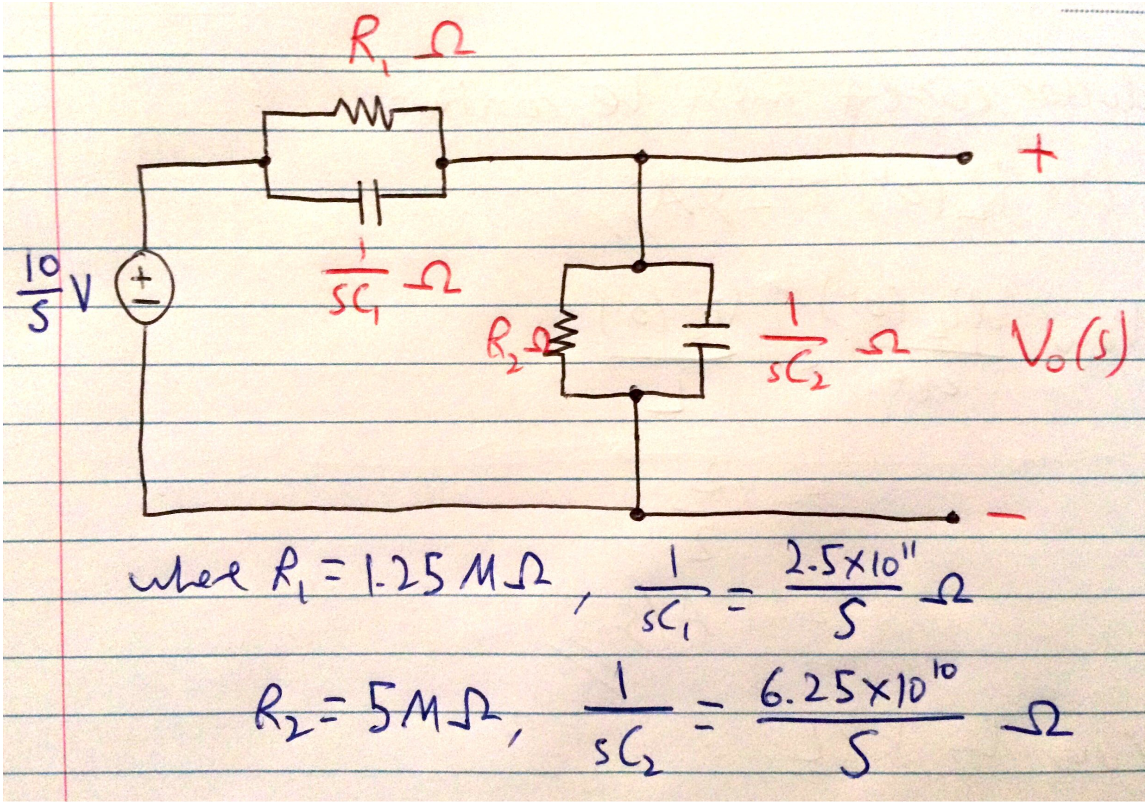
Tuesday 2:15pm - EDS 8

1. (a) The circuit is converted to the s domain:



(b)

$$
\begin{aligned}
Z_{eq1} &= \left(\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}}\right)^{-1} \\
&= \left(\frac{1}{1.25\times10^6} + s\cdot 4\times10^{-12}\right)^{-1} \\
&= \left(\frac{1 + 1.25\times10^6\cdot(s\cdot 4\times10^{-12})}{1.25\times10^6}\right)^{-1} \\
&= \frac{1.25\times10^6}{1 + 1.25\times10^6\cdot(s\cdot 4\times10^{-12})}
\end{aligned}
$$

$$\therefore Z_{eq1} = \frac{2.5\times10^{11}}{s + 2\times10^5}\ \Omega$$

(c)

$$
\begin{aligned}
Z_{eq2} &= \left(\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_2}}\right)^{-1} \\
&= \left(\frac{1}{5 \times 10^6} + s \cdot 16 \times 10^{-12}\right)^{-1} \\
&= \left(\frac{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})}{5 \times 10^6}\right)^{-1} \\
&= \frac{5 \times 10^6}{1 + 5 \times 10^6 \cdot (s \cdot 16 \times 10^{-12})}
\end{aligned}
$$

$$
\therefore Z_{eq2} = \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}\ \Omega
$$

(d) Using voltage division:

$$
\begin{aligned}
V_o(s) &= V_{in}(s) \cdot \frac{Z_{eq2}}{Z_{eq1} + Z_{eq2}} \\
&= \frac{10}{s} \cdot \frac{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} + \frac{2.5 \times 10^{11}}{s + 2 \times 10^5}} \\
&= \frac{10}{s} \cdot \frac{\frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}}{\frac{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)}{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}} \\
&= \frac{10}{s} \cdot \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4} \cdot \frac{(s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{(6.25 \times 10^{10})(s + 2 \times 10^5) + (2.5 \times 10^{11})(s + 1.25 \times 10^4)} \\
&= \frac{6.25 \times 10^{11} \cdot (s + 1.25 \times 10^4) \cdot (s + 2 \times 10^5)}{3.125 \times 10^{11} \cdot s \cdot (s + 1.25 \times 10^4) \cdot (s + 5 \times 10^4)} \\
&= \frac{2 \cdot (s + 2 \times 10^5)}{s \cdot (s + 5 \times 10^4)}
\end{aligned}
$$

$$
\therefore V_o(s) = \frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s}\ \text{V}
$$

Now perform partial fraction expansion:

$$
\begin{aligned}
\frac{2s + 4 \times 10^5}{s^2 + 5 \times 10^4 \cdot s} &= \frac{A}{s} + \frac{B}{s + 50000} \\
A(s + 50000) + Bs &= 2(s + 200000) \\
(s = 0):\ 50000A &= 400000 \\
\implies A &= 8 \\
(s = -50000):\ -50000B &= 300000 \\
\implies B &= -6
\end{aligned}
$$

$$\therefore V_o(s) = \frac{8}{s} - \frac{6}{s+50000} \text{ V}$$

Perform inverse Laplace transform to find $v_o(t)$:

$$\begin{aligned} v_o(t) &= \mathcal{L}^{-1}[V_o(s)] \\ &= \mathcal{L}^{-1}\left[\frac{8}{s} - \frac{6}{s+50000}\right] \end{aligned}$$

$$\therefore v_o(t) = \left(8 - 6e^{-50000t}\right) u(t) \text{ V}$$

(e) Using Ohm's Law in s domain:

$$\begin{aligned} V_{in}(s) &= Z_{eq} I_o(s) \\ &= (Z_{eq1} + Z_{eq2}) I_o(s) \end{aligned}$$

$$\begin{aligned} \therefore I_o(s) &= \frac{V_{in}(s)}{Z_{eq1} + Z_{eq2}} \\ &= \frac{\frac{10}{s}}{\frac{2.5 \times 10^{11}}{s + 2 \times 10^5} + \frac{6.25 \times 10^{10}}{s + 1.25 \times 10^4}} \\ &= \frac{\frac{10}{s}}{\frac{3.125 \times 10^{11}(s+50000)}{(s+12500)(s+200000)}} \\ &= \frac{(s+12500)(s+200000)}{3.125 \times 10^{10} s(s+50000)} \text{ A} \end{aligned}$$

We note that the order of the numberator polynomial is equal to the order of the denominator polynomial, and so to perform partial fraction expansion we must first perform polynomial division:

$$\frac{(s+12500)(s+200000)}{3.125 \times 10^{10} s(s+50000)} = 3.2 \times 10^{-11} + \frac{5.2 \times 10^{-6} s + 8 \times 10^{-2}}{s(s + 5 \times 10^4)}$$

Now perform partial fraction expansion:

$$\begin{aligned} \frac{5.2 \times 10^{-6} s + 8 \times 10^{-2}}{s(s + 5 \times 10^4)} &= \frac{A}{s} + \frac{B}{s + 5 \times 10^4} \\ \therefore 5.2 \times 10^{-6} s + 8 \times 10^{-2} &= A(s + 5 \times 10^4) + Bs \end{aligned}$$

$$\begin{aligned} s = 0 &\implies 8 \times 10^{-2} = A(5 \times 10^4) \implies A = 1.6 \times 10^{-6} \\ s = -5 \times 10^4 &\implies -0.18 = B(-5 \times 10^4) \implies B = 3.6 \times 10^{-6} \end{aligned}$$

Now the current in the s domain can be represented in partial fraction expanded form:

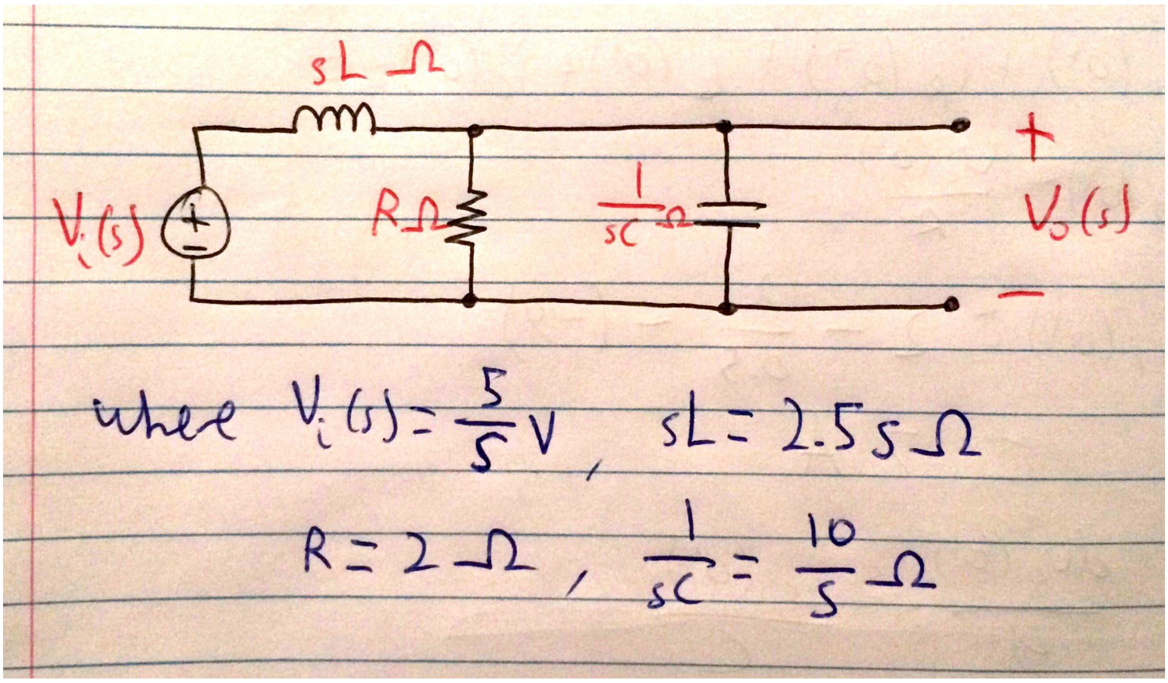
$$\begin{aligned} I_o(s) &= 3.2 \times 10^{-11} + \frac{1.6 \times 10^{-6}}{s} + \frac{3.6 \times 10^{-6}}{s + 5 \times 10^4} \text{ A} \\ &= 3.2 \times 10^{-5} + \frac{1.6}{s} + \frac{3.6}{s + 5 \times 10^4} \ \mu\text{A} \end{aligned}$$

Using inverse Laplace transform to find $i_o(t)$.

$$
\begin{aligned}
i_o(t) &= \mathcal{L}^{-1}\left[I_o(s)\right] \\
&= \mathcal{L}^{-1}\left[3.2\times10^{-5} + \frac{1.6}{s} + \frac{3.6}{s+5\times10^{4}}\right]
\end{aligned}
$$

$$
\therefore i_o(t) = 3.2\times10^{-5}\cdot\delta(t) + \left(1.6 + 3.6\times e^{-50000t}\right)u(t)\ \mu\text{A}
$$

2. (a) Convert circtuit to its s domain equivalent:



The circuit is a voltage divider:

$$
\begin{aligned}
V_o(s) &= V_i(s) \times \frac{Z_R||Z_C}{Z_R||Z_C + Z_L} \\
&= \frac{5}{s} \times \frac{\frac{20s}{s(2s+10)}}{\frac{20s}{s(2s+10)} + 2.5s} \\
&= \frac{5}{s} \times \frac{1}{1 + 0.125(2s^2+10s)} \\
&= \frac{5}{s(0.25s^2+1.25s+1)} \\
&= \frac{20}{s(s+4)(s+1)}
\end{aligned}
$$

Perform partial fraction expansion:

$$
\frac{20}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1}
$$

$$
\therefore 20 = A(s+4)(s+1) + Bs(s+1) + Cs(s+4)
$$

Now solve for A, B and C:

$$
s = 0 \implies 20 = A(4)(1) \implies A = 5
$$

$$s = -4 \implies 20 = B(-4)(-3) \implies B = \frac{5}{3}$$

$$s = -1 \implies 20 = C(-1)(3) \implies C = -\frac{20}{3}$$

And we arrive at $V_o(s)$ in partial fraction expanded form:

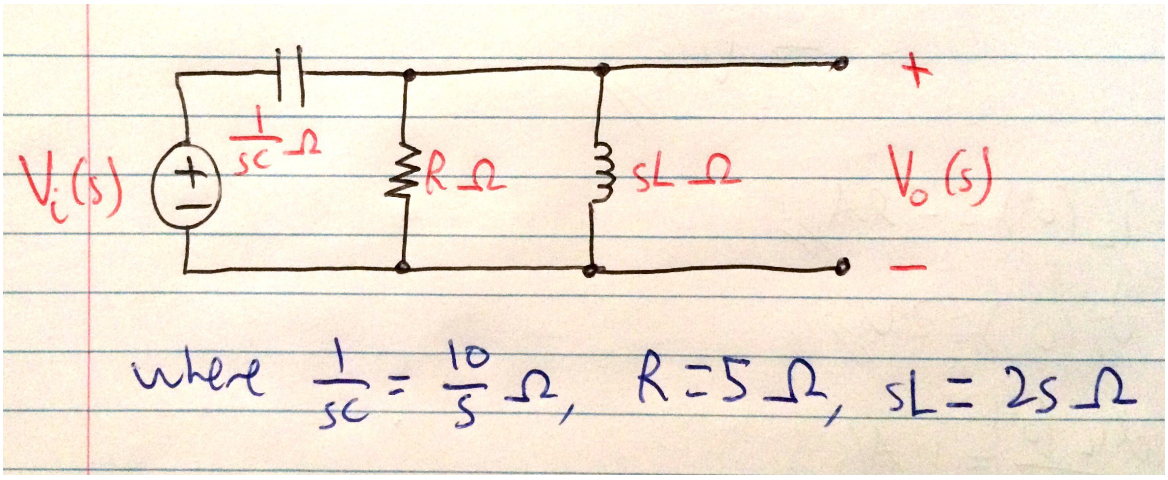
$$V_o(s) = \frac{5}{s} + \frac{5}{3(s+4)} - \frac{20}{3(s+1)} \text{ V}$$

(b) Perform the inverse Laplace transform:

$$\begin{aligned} v_o(t) &= \mathcal{L}^{-1}\left[V_o(s)\right] \\ &= \left(5 + \frac{5}{3}e^{-4t} - \frac{20}{3}e^{-t}\right) u(t) \text{ V} \end{aligned}$$

(c) The circuit elements (R, L and C) remain the same, therefore s_1, s_2 and the form of equation will remain the same.
Input voltage source (forcing function) remains the same, therefore the steady state component of response will remain the same.
One or both of the coefficients of the transient component terms may change, as these are the only parts of the response equation that are determined by the initial conditions.

3. (a) Convert circuit to its s domain equivalent, and assume no energy stored at $t = 0^-$:



Find $V_o(s)$ by recognising circuit is a voltage divider:

$$V_o(s) = V_i(s) \times \frac{\frac{5\times 2s}{5+2s}}{\frac{5\times 2s}{5+2s} + \frac{10}{s}}$$

$$\begin{aligned} \therefore H(s) &= \frac{10s}{10s + \frac{50}{s} + 20} \\ &= \frac{10s^2}{10s^2 + 20s + 50} \end{aligned}$$

$$\therefore H(s) = \frac{s^2}{s^2 + 2s + 5}$$

(b) We note that the steady state response to the sinusoidal input will be given by the following equation:

$$v_{oSS}(t) = 10 \times |H(j20)| \cos(20t + \theta(20))$$

Where $H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}$

Therefore, find $|H(j20)|$ and $\theta(20)$:

$$H(j20) = \frac{-400}{-395 + j40}$$

$$\begin{aligned} \therefore |H(j20)| &= \frac{400}{\sqrt{(-395)^2 + 40^2}} \\ &= 1.008 \end{aligned}$$

$$\begin{aligned} \text{And } \theta(20) &= \arctan\left(\frac{0}{-400}\right) - \arctan\left(\frac{40}{-395}\right) \\ &= 180^\circ - 174.22^\circ \\ &= 5.78^\circ \end{aligned}$$

Finally sub these values into the equation for v_{oSS}:

$$v_{oSS}(t) = 10.075 \cos(20t + 5.78^\circ) \text{ V}$$

4. Start by converting the output response equation to its s domain equivalent:

$$\begin{aligned} V_o(s) &= \mathcal{L}[v_o(t)] \\ &= \frac{10}{s+3} - \frac{20}{s+4} + \frac{10}{s+5} \\ &= \frac{10(s+4)(s+5) - 20(s+3)(s+5) + 10(s+3)(s+4)}{(s+3)(s+4)(s+5)} \\ &= \frac{20}{(s+3)(s+4)(s+5)} \end{aligned}$$

Now by definition, $V_i(s) = \frac{V_o(s)}{H(s)}$, therefore:

$$\begin{aligned} V_i(s) &= \frac{20}{(s+3)(s+4)(s+5)} \times \frac{(s+3)(s+4)}{2(s+5)} \\ &= \frac{10}{(s+5)^2} \end{aligned}$$

Now convert the input function back to the time domain:

$$\begin{aligned} v_i(t) &= \mathcal{L}^{-1}[V_i(s)] \\ &= 10te^{-5t}u(t) \text{ V} \end{aligned}$$

5. (a)

$$\begin{aligned} F(s) &= \frac{20s^2+141s+315}{s\left(s^2+10s+21\right)} \\ &= \frac{20s^2+141s+315}{s(s+7)(s+3)} \end{aligned}$$

Perform partial fraction expansion:

$$\frac{20s^2+141s+315}{s(s+7)(s+3)} = \frac{A}{s} + \frac{B}{s+7} + \frac{C}{s+3}$$

$$\therefore 20s^2+141s+315 = A(s+7)(s+3) + Bs(s+3) + Cs(s+7)$$

Now solve for A, B and C:

$$s = 0 \implies 315 = A(7)(3) \implies A = 15$$

$$s = -7 \implies 308 = B(-7)(-4) \implies B = 11$$

$$s = -3 \implies 72 = A(-3)(4) \implies C = -6$$

And we arrive at $F(s)$ in partial fraction expanded form:

$$F(s) = \frac{15}{s} + \frac{11}{s+7} - \frac{6}{s+3}$$

Now perform the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \left(15 + 11e^{-7t} - 6e^{-3t}\right) u(t) \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= \frac{14s^2+56s+152}{(s+6)\left(s^2+4s+20\right)} \\ &= \frac{14s^2+56s+152}{(s+6)(s+2-j4)(s+2+j4)} \end{aligned}$$

Perform partial fraction expansion:

$$\frac{14s^2+56s+152}{(s+6)(s+2-j4)(s+2+j4)} = \frac{A}{s+6} + \frac{B}{s+2-j4} + \frac{B^*}{s+2+j4}$$

$$\therefore 14s^2+56s+152 = A(s+2-j4)(s+2+j4) + B(s+6)(s+2+j4) + B^*(s+6)(s+2-j4)$$

Now solve for A and B:

$$s = -6 \implies 320 = A(-4-j4)(-4+j4) \implies A = 10$$

$$s = -2+j4 \implies -128 = B(4+j4)(j8) \implies B = 2+j2$$

And we arrive at $F(s)$ in partial fraction expanded form:

$$\begin{aligned} F(s) &= \frac{10}{s+6} + \frac{2+j2}{s+2-j4} + \frac{2-j2}{s+2+j4} \\ &= \frac{10}{s+6} + \frac{2\sqrt{2}\angle 45^\circ}{s+2-j4} + \frac{2\sqrt{2}\angle -45^\circ}{s+2+j4} \end{aligned}$$

Now perform the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \left(10e^{-6t} + 4\sqrt{2}e^{-2t}\cos(4t + 45^\circ)\right) u(t) \end{aligned}$$

(c)

$$F(s) = \frac{25(s+4)^2}{s^2(s+5)^2}$$

Perform partial fraction expansion:

$$\frac{25(s+4)^2}{s^2(s+5)^2} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+5)^2} + \frac{D}{s+5}$$

$$\therefore 25(s+4)^2 = A(s+5)^2 + Bs(s+5)^2 + Cs^2 + Ds^2(s+5)$$

Now solve for A, B, C and D:

$$s = -5 \implies 25 = C(-5)^2 \implies C = 1$$

$$s = 0 \implies 400 = A(5)^2 \implies A = 16$$

$$s = 1 \implies 625 = 36(16) + B(36) + 1 + D(6)$$

$$\begin{aligned} \therefore 36B + 6D &= 48 \\ 6B + D &= 8 \end{aligned} \tag{1}$$

$$s = -1 \implies 225 = 16(16) - B(16) + 1 + D(4)$$

$$\begin{aligned} \therefore -16B + 4D &= -32 \\ -4B + D &= -8 \end{aligned} \tag{2}$$

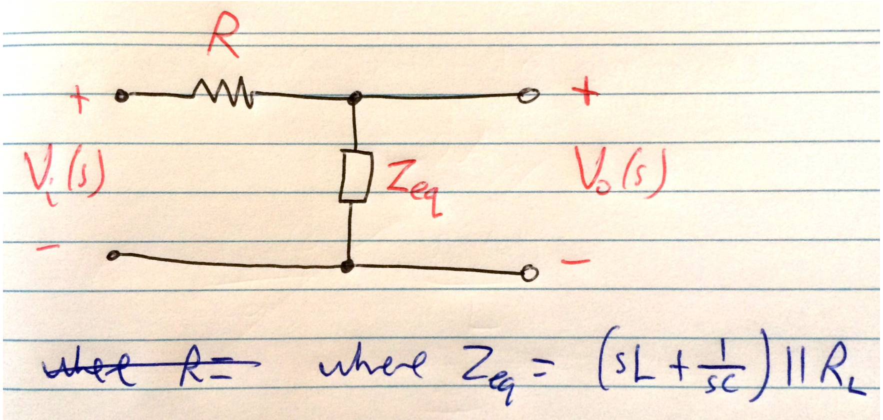
Solving equations (1) and (2) gives $B = \frac{8}{5}$ and $D = -\frac{8}{5}$. We then arrive at $F(s)$ in partial fraction expanded form:

$$F(s) = \frac{16}{s^2} + \frac{8}{5s} + \frac{1}{(s+5)^2} - \frac{8}{5(s+5)}$$

Now perform the inverse Laplace transform:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \left(\frac{8}{5} + 16t + e^{-5t}\left(t - \frac{8}{5}\right)\right) u(t) \end{aligned}$$

6. We convert the circuit to its s domain equivalent and combine the inductor, capacitor and load resistance into one component denoted Z_{eq}:



We note the circuit forms a voltage divider. We start by finding the impedance of the circuit across $V_o(s)$:

$$
\begin{aligned}
Z_{eq} &= (Z_L + Z_C) \mid\mid Z_R \\
&= \left(\frac{1}{sL + \frac{1}{sC}} + \frac{1}{R_L}\right)^{-1} \\
&= \left(\frac{sC}{s^2LC + 1} + \frac{1}{R_L}\right)^{-1} \\
&= \left(\frac{sCR_L + s^2LC + 1}{s^2LCR_L + R_L}\right) \\
&= \frac{R_L s^2 + \frac{R_L}{LC}}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}}
\end{aligned}
$$

$$
\therefore Z_{eq} = \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}}
$$

Now, $V_o(s) = V_i(s) \times \frac{Z_{eq}}{R + Z_{eq}}$, and H(s) can be found from this:

$$
\begin{aligned}
H(s) &= \frac{Z_{eq}}{R + Z_{eq}} \\
&= \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} \times \left(\frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} + R\right)^{-1} \\
&= \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}} \times \left(\frac{s^2(R_L + R) + \frac{R_L R}{L}s + \frac{1}{LC}(R_L + R)}{s^2 + \frac{R_L}{L}s + \frac{1}{LC}}\right)^{-1} \\
&= \frac{R_L\left(s^2 + \frac{1}{LC}\right)}{s^2(R_L + R) + \frac{R_L R}{L}s + \frac{1}{LC}(R_L + R)} \\
&= \frac{\frac{R_L}{R_L + R}\left(s^2 + \frac{1}{LC}\right)}{s^2 + \frac{R_L R}{L(R_L + R)}s + \frac{1}{LC}}
\end{aligned}
$$

$$
\therefore H(s) = \frac{K\left(s^2 + \frac{1}{LC}\right)}{s^2 + K \cdot \frac{R}{L}s + \frac{1}{LC}}, \qquad \text{where } K = \frac{R_L}{R_L + R}
$$