ELEN30009 - Electrical Network Analysis and Design Assignment 4

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Tuesday - 2:15 PM

1. (a) INSERT DIAGRAM HERE
   Using KVL in loop 1:

$$\begin{aligned} &\Sigma V_{drops} = 0 \\ \implies &V_i + I_i R_i = 0 \\ \implies &V_i = I_i R_i \end{aligned}$$

Using KVL in loop 2:

$$\begin{aligned} &\Sigma V_{drops} = 0 \\ \implies &- V_o + I_o R_o + A_{voc} V_i = 0 \\ \implies &- V_o + I_o R_o + A_{voc} I_i R_i = 0 \\ \implies &I_i = \frac{V_o - I_o R_o}{A_{voc} R_i} \\ \implies &V_i = \frac{V_o - I_o R_o}{A_{voc}} \end{aligned}$$

We know that for a two port network, the general matrix equation written in a parameters is:

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \begin{bmatrix} V_o \\ I_o \end{bmatrix}$$

From the equation obtained before, we can re-write them in matrix form:

$$\begin{bmatrix} V_i \\ I_i \end{bmatrix} = \begin{bmatrix} \frac{1}{A_{voc}} & -\frac{R_o}{A_{voc}} \\ \frac{1}{A_{voc} R_i} & -\frac{R_o}{A_{voc} R_i} \end{bmatrix} \begin{bmatrix} V_o \\ I_o \end{bmatrix}$$

Therefore, the A matrix of this voltage amplifier model is:

$$A = \begin{bmatrix} \frac{1}{A_{voc}} & -\frac{R_o}{A_{voc}} \\ \frac{1}{A_{voc} R_i} & -\frac{R_o}{A_{voc} R_i} \end{bmatrix}$$

   (b) This circuit can be thought of as 3 cascaded two port networks forming a single two port network with a loaded output and a voltage input with source impedance.
   To find the A parameter matrix of the single two port network, first find the A parameter matrices of each amplifier stage and matrix multiply them together to.

$$
\begin{aligned}
A_1 &= \begin{bmatrix} \frac{1}{10} & -\frac{1\times10^3}{10} \\ \frac{1}{101\times10^6} & -\frac{1\times10^3}{101\times10^6} \end{bmatrix} \\
A_2 &= \begin{bmatrix} \frac{1}{100} & -\frac{2\times10^3}{100} \\ \frac{1}{100200\times10^3} & -\frac{2\times10^3}{100200\times10^3} \end{bmatrix} \\
A_3 &= \begin{bmatrix} \frac{1}{2} & -\frac{50\times10^3}{2} \\ \frac{1}{225\times10^3} & -\frac{50\times10^3}{225\times10^3} \end{bmatrix} \\
A &= A_1 A_2 A_3 \\
&= \begin{bmatrix} 457\times10^{-6} & 22.885\ \Omega \\ 457\times10^{-12}\ \mho & 22.885\times10^{-6} \end{bmatrix}
\end{aligned}
$$

2. (a) i. We know that in a Thevenin equivalent circuit, maximum power transfer to the load occurs when $Z_L = Z_{Th}^*$, or for entirely resistive circuits, $R_L = R_{Th}$.

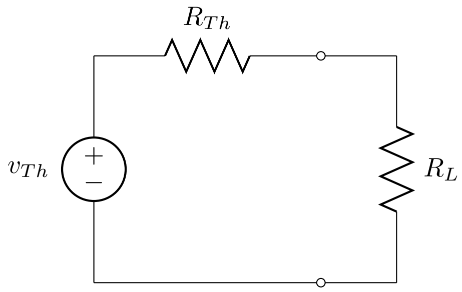


Figure 1: Thevenin equivalent circuit

Now by the formula sheet:

$$
\begin{aligned}
R_{Th} &= \frac{a_{12} + a_{22}\cdot R_g}{a_{11} + a_{21}\cdot R_g} \\
&= \frac{10 + 1.5\cdot 2}{4 + 0.5\cdot 2} \\
&= 2.6\ \Omega
\end{aligned}
$$

Therefore, when $R_L = 2.6\ \Omega$, maximum power is transferred to the load resistor.

ii. Maximum power transferred to load can be found from the formula:

$$
P_{L\ max} = \frac{v_L^2}{R_L}
$$

For the Thevenin equivalent circuit, v_L, the voltage drop across the load, can be found by voltage division. However, since $R_L = R_{Th}$, we know half of v_{Th} will drop across R_L, the other half dropped across R_{Th}.

Now by the formula sheet:

$$
\begin{aligned}
v_{Th} &= \frac{v_g}{a_{11} + a_{21}\cdot R_g} \\
&= \frac{10}{4 + 0.5\cdot 2} \\
&= 2\ \text{V}
\end{aligned}
$$

$$\therefore v_L = \frac{v_{Th}}{2} = 1 \text{ V}$$

$$\therefore P_{L\ max} = \frac{1^2}{2.6} = 384.62 \text{ mW}$$

The maximum power delivered to the load is $P_{L\ max} = 384.62$ mW.

iii. From (2a ii), $R_L = 2.6\ \Omega \quad \therefore v_L = 1$ V.
From this, i_L can be found by Ohm's law:

$$i_L = \frac{v_L}{R_L} = \frac{1}{2.6} = 384.62 \text{ mA}$$

Now by the formula sheet:

$$\frac{i_2}{i_1} = \frac{-1}{a_{21} \cdot R_L + a_{22}}$$

And from this, i_1 can be found from i_L, where we note $i_2 = -i_L$:

$$\begin{aligned} i_1 &= -i_2 \left(a_{21} \cdot R_L + a_{22}\right) \\ &= i_L \left(a_{21} \cdot R_L + a_{22}\right) \\ &= 384.62 * 10^{-3} \left(0.5 \cdot 2.6 + 1.5\right) \\ &= 1.077 \text{ A} \end{aligned}$$

And so, the current flowing into port 1 is $i_1 = 1.077$ A.

(b) For measurement 1, we have the constraint that $V_2 = 0$ V.
For measurement 2, we have the constraing that $V_1 = 0$ V.

Therefore, we want to find a set of two-port network parameters that have V_1 and V_2 as the independent variables.

We find the y parameters meet this condition:

$$i_1 = y_{11} \cdot v_1 + y_{12} \cdot v_2 \qquad i_2 = y_{21} \cdot v_1 + y_{22} \cdot v_2$$

$$y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2=0} \text{S} \qquad y_{21} = \left.\frac{i_2}{v_1}\right|_{v_2=0} \text{S}$$

$$y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1=0} \text{S} \qquad y_{22} = \left.\frac{i_2}{v_2}\right|_{v_1=0} \text{S}$$

By measurement 1:

$$y_{11} = \frac{1}{10} = 100 \text{ mS} \qquad y_{21} = \frac{-0.5}{10} = -50 \text{ mS}$$

By measurement 2:

$$y_{12} = \frac{-1}{20} = -50 \text{ mS} \qquad y_{22} = \frac{3}{20} = 150 \text{ mS}$$

Now for cascade of two-port networks, $[A_T] = [A_1] \times [A_2]$, where these are the a parameter matrices for the overall network, network 1 and network 2 respectively.

Therefore convert above y parameters to a parameters to make finding overall network parameters easier (note $\Delta[Y]$ is the discriminant of the y parameter matrix):

$$a_{11} = -\frac{y_{22}}{y_{21}} = 3 \qquad a_{12} = -\frac{1}{y_{21}} = 20\ \Omega$$

$$a_{21} = -\frac{\Delta[Y]}{y_{21}} = 0.25\ \mathrm{S} \qquad a_{22} = -\frac{y_{11}}{y_{21}} = 2$$

As stated above, we can find the a parameters of the overall two-port network by matrix multiplying the a parameters of the constituent cascaded two-port networks:

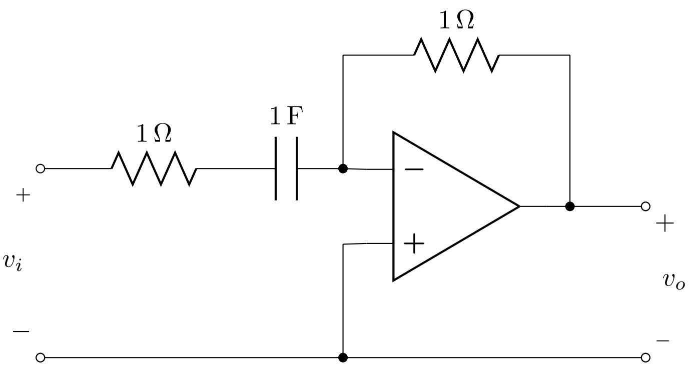
$$\begin{aligned}
[A_T] &= [A_1] \times [A_2] \\
&= \begin{bmatrix} 4 & 10\ \Omega \\ 0.5\ \mathrm{S} & 1.5 \end{bmatrix} \begin{bmatrix} 3 & 20\ \Omega \\ 0.25\ \mathrm{S}2 & \end{bmatrix} \\
&= \begin{bmatrix} 14.5 & 100\ \Omega \\ 1.875\ \mathrm{S} & 13 \end{bmatrix}
\end{aligned}$$

And so we find that the a parameters of the cascaded network are $a_{11} = 14.5$, $a_{12} = 100\ \Omega$, $a_{21} = 1.875$ S and $a_{22} = 13$.

3. (a) Design 2^{nd} order filter by cascading two first order filters together, followed by a gain stage.

Design the filter by starting with prototype stages then frequency and magnitude scaling to get desired properties.

Now a prototype first order active high pass filter has unity gain in the passband and a cutoff frequency of $\omega_c = 1$ rad/s, and is of form:



Noting that a circuit of this form has transfer function:

$$H_L(s) = \frac{-s}{s+1}$$

Now when two of these are connected in cascade, the overall transfer function is:

$$\begin{aligned}
H(s) &= H_L(s) \cdot H_L(s) \\
&= \frac{s^2}{(s+1)^2}
\end{aligned}$$

We know this transfer function describes a 2^{nd} order high pass filter with a passband gain of 1 V/V. Now find the cutoff frequency:

$$\omega_c \triangleq \omega : \left|H(j\omega)\right| = \frac{1}{\sqrt{2}}$$

So first find $|H(j\omega)|$:

$$\begin{aligned}H(j\omega) &= H(s)\big|_{s=j\omega}\\ &= \frac{(j\omega)^2}{(j\omega+1)^2}\\ &= \frac{-\omega^2}{(j\omega+1)^2}\end{aligned}$$

$$\begin{aligned}\big|H(j\omega)\big| &= \frac{\big|-\omega^2\big|}{\big|j\omega+1\big|^2}\\ &= \frac{\omega^2}{\left(\sqrt{\omega^2+1^2}\right)^2}\\ &= \frac{\omega^2}{\omega^2+1}\end{aligned}$$

Now find ω_c:

$$\begin{aligned}\big|H(j\omega_c)\big| &= \frac{1}{\sqrt{2}}\\ \therefore \frac{\omega_c^2}{\omega_c^2+1} &= \frac{1}{\sqrt{2}}\\ \therefore \sqrt{2}\cdot\omega_c^2-\omega_c^2 &= 1\\ \therefore \omega_c^2 &= \frac{1}{\sqrt{2}-1}\\ \therefore \omega_c &= \frac{1}{\sqrt{\sqrt{2}-1}}\\ &= 1.554\ \text{rad/s}\end{aligned}$$

Noting that the negative root of ω_c has been rejected, as we define $\omega \geq 0$ rad/s.
We now know this prototype 2^{nd} order active high pass filter has a passband gain of 1 V/V and a cutoff frequency of 1.554 rad/s.

Now we need to modify this prototype design to match the parameters we want. We do this by frequency scaling and magnitude scaling the response, and hence the circuit.

First we will frequency scale to the desired cutoff frequency of 1 kHz. We do this by finding the frequency scaling factor $k_f \triangleq \frac{\omega_c^{'}}{\omega_c}$, where $\omega_c^{'}$ is the desired cutoff frequency and ω_c is the current cutoff frequency:

$$\begin{aligned}k_f &= \frac{2\cdot\pi\cdot 1000}{1.554}\\ &= 4043.8\end{aligned}$$

Now we note that we must use 100 nF capacitors in the final design. Therefore we find the magnitude scaling factor from the following equation:

$$k_m = \frac{C}{C'\cdot k_f}$$

Where C is the prototypical circuit's capacitor value and $C^{'}$ is the desired capacitor value:

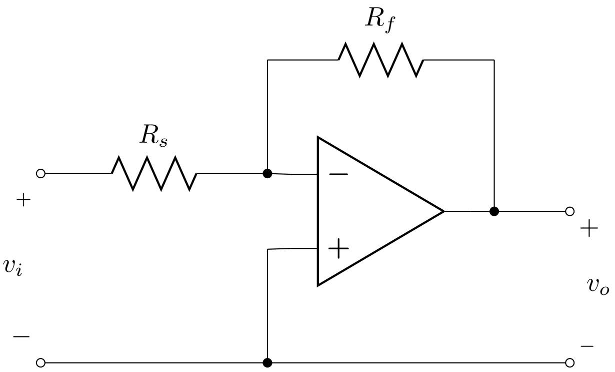
$$\begin{aligned}k_m &= \frac{1}{100\times 10^{-9}\cdot 4043.8}\\ &= 2472.9\end{aligned}$$

Now scale the resistor from the prototypical value (R) to the desired value ($R^{'}$):

$$\begin{aligned} R^{'} &= k_m \cdot R \\ &= 2472.9 \cdot 1 \\ &= 2.47 \text{ k}\Omega \end{aligned}$$

Now we want a gain of 1 V/V in the passband in these filter stages, so we want $R_s^{'} = R_f^{'} =$ 2.47 kΩ.

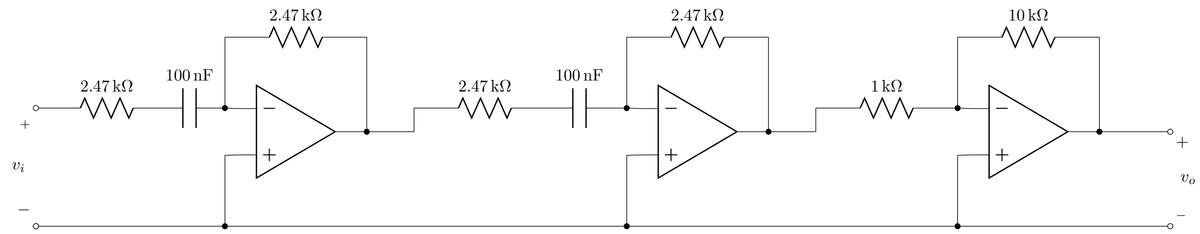
Now we need to design the gain stage, achieve this using an op amp in the inverting amplifier configuration:



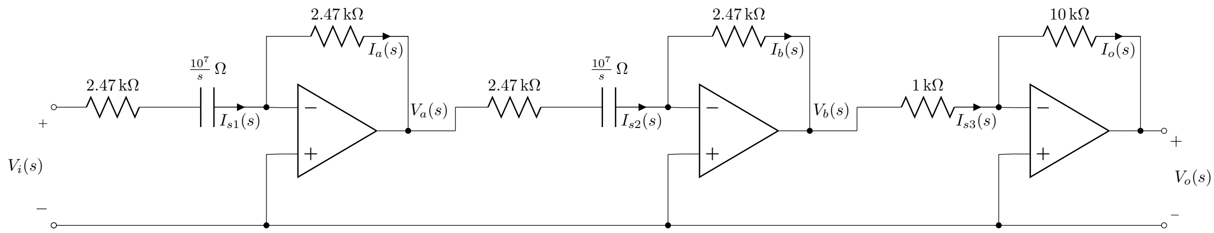
We know this circuit has transfer function $H(s) = -\frac{R_f}{R_s}$
Now we want a passband gain of 10 V/V. Therefore in the gain stage we want:

$$\begin{aligned} &\left|H(s)\right| = 10 \\ &\therefore \frac{R_f}{R_s} = 10, \qquad \text{let } R_s = 1 \text{ k}\Omega, \implies R_f = 10 \text{ k}\Omega \end{aligned}$$

Therefore the overall design is:



(b) Re-draw circuit in s domain:



Now perform KVL from V_a to the common rail via the 2.47 kΩ resistor and the inverting input pin of the first op amp whose voltage is V_n (note we assume the op amp is operating in its linear region):

$$-V_a - I_a \cdot 2.47 \times 10^3 + V_n = 0$$

Now $V_n = 0$ V (virtual ground) due to the virtual short-circuit condition at the op amp's input pins ($V_p - V_n = 0$ V) and $V_p = 0$ V. Therefore:

$$V_a(s) = -I_a \cdot 2.47 \times 10^3 \tag{1}$$

Now find the current I_{s1} by using Ohm's law and the virtual ground at V_n:

$$\begin{aligned} I_{s1}(s) &= \frac{V_i}{2.47 \times 10^3 + \frac{10^7}{s}} \\ &= \frac{s \cdot V_i / \left(2.47 \times 10^3\right)}{s + 4.04 \times 10^3} \end{aligned} \tag{2}$$

Now note that due to current constraint of the input pins of the op amp (no current flows into input pins), $I_{s1} = I_a$. Substitute I_a in equation (1) for I_{s1} in equation (2):

$$\begin{aligned} V_a(s) &= -\frac{s \cdot V_i / \left(2.47 \times 10^3\right)}{s + 4.04 \times 10^3} \cdot 2.47 \times 10^3 \\ &= -\frac{s \cdot V_i}{s + 4.04 \times 10^3} \end{aligned}$$

$$\therefore H_{a-i}(s) = \frac{V_a}{V_i} = -\frac{s}{s + 4.04 \times 10^3}$$

We note that the separate 1st order filter stages will have the same transfer function, therefore the analysis of V_b will be the same as that above:

$$\begin{aligned} V_b(s) &= -\frac{s \cdot V_a}{s + 4.04 \times 10^3} \\ &= -\frac{s}{s + 4.04 \times 10^3} \cdot -\frac{s \cdot V_i}{s + 4.04 \times 10^3} \end{aligned}$$

$$\therefore H_{b-i}(s) = \frac{V_b}{V_i} = \frac{s^2}{\left(s + 4.04 \times 10^3\right)^2}$$

Now we perform KVL from V_o to the common rail via the 10 kΩ resistor and the inverting input pin of the third op amp, making use of the aforementioned virtual ground at the inverting input pin ($V_n = 0$ V)

$$\begin{aligned} -V_o - I_o \cdot 10 \times 10^3 + V_n &= 0 \\ \therefore V_o(s) &= -I_o \cdot 10^4 \end{aligned} \tag{3}$$

Again, due to current constraint of op amp input pins, $I_n = 0$ A. Therefore $I_{s3} = I_o$.
Now by Ohm's law and the virtual ground at V_n:

$$I_{s3}(s) = \frac{V_b}{10^3} \tag{4}$$

Now substitute I_o in equation (2) for I_{s3} in equation (3):

$$\begin{aligned} V_o &= -\frac{V_b}{10^3} \cdot 10^4 \\ &= -10 \cdot V_b \end{aligned}$$

$$\therefore H_{o-b}(s) = \frac{V_o}{V_b} = -10$$

And so we can find the transfer function of the entire circuit:

$$
\begin{aligned}
H(s) &= H_{b-i}(s) \cdot H_{o-b}(s) \\
&= -10 \cdot \frac{s^2}{\left(s + 4.04 \times 10^3\right)^2}
\end{aligned}
$$

(c)

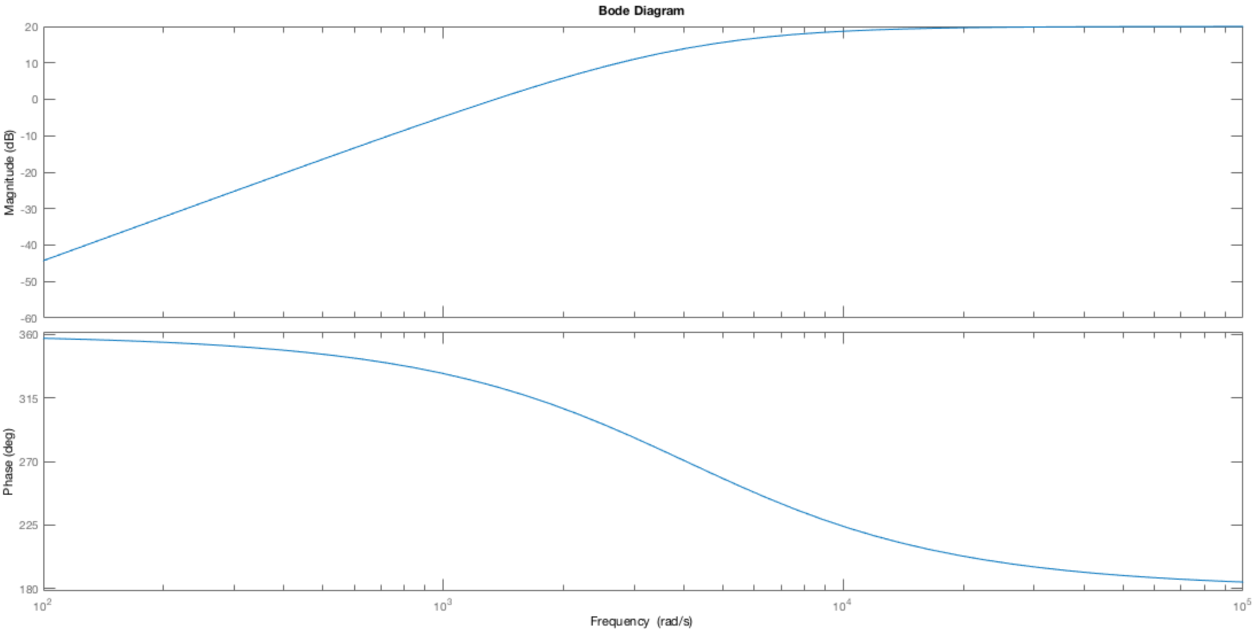


Figure 2: Bode plot of transfer function found in question (3b)

As an aside, we note that the phase response of this circuit is not what may be expected (ours starts at 0° and ends at -180° for increasing frequencies rather than starting at 180° and ending at 0°). This is due to the inverting nature of our filter design as noted as the negative sign in our transfer function from (3b).
However, seeing as we were designing the filter with only the magnitude response in mind, we feel that this is not an issue in answering this question.

4. Design this bandreject filter by connecting a low pass Butterworth filter with a cutoff frequency of 500 Hz and gain of 0 dB in the passband in parallel with a high pass Butterworth filter with a cutoff frequency of 5 kHz and gain of 0 dB in the passband (thereby rejecting frequencies between 500 Hz and 5 kHz), then connecting both to separate inputs of a summing amplifier with gain of 20 dB in the passband.

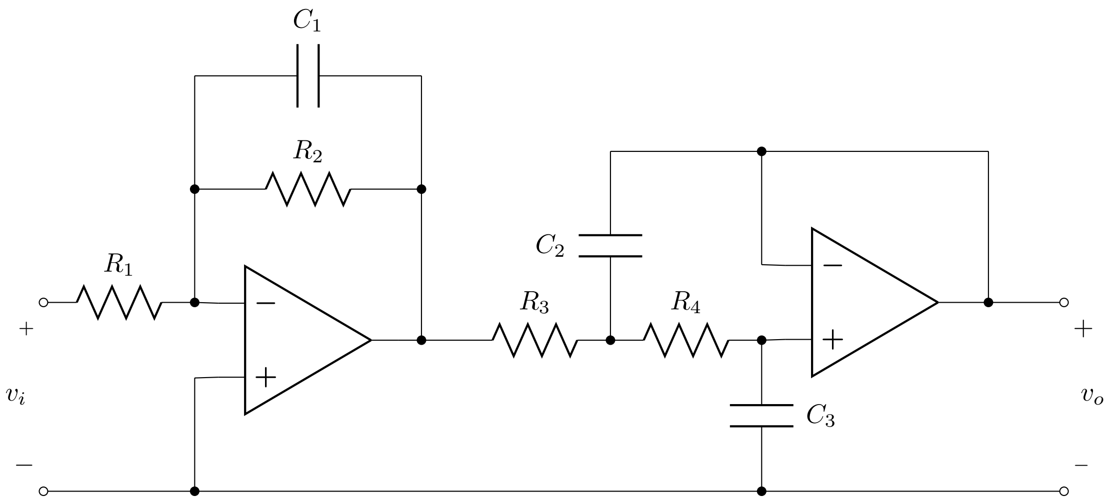
   Start with designing the low pass stage.
   The 3$^{\text{rd}}$ order normalised Butterworth polynomial is:

$$
\left(s + 1\right)\left(s^2 + s + 1\right)
$$

   Therefore the 3$^{\text{rd}}$ order normalised low pass Butterworth filter has the following transfer function:

$$
\frac{1}{\left(s+1\right)\left(s^2+s+1\right)} = \frac{1}{s+1} \cdot \frac{1}{s^2+s+1}
$$

This results in the cascade of a 1st order low pass Butterworth filter with a 2nd order low pass Butterworth filter, both with passband gains of 0 dB and cutoff frequencies of 1 rad/s. This can be achieved with the following circuit:



We need to find resistor and capacitor values for the normalised filters.

Let $R_1 = R_2 = R_3 = R_4 = 1\Omega$.
For the normalised 1st order low pass Butterworth filter, we note that the transfer function is the same as the transfer function of the prototypical 1st order active low pass filter. Therefore, the component values for this normalised Butterworth filter should match those of the 1st order prototype filter, and so $C_1 = 1$ F.

For the normalised 2nd order low pass Butterworth filter, we note that the transfer function is:

$$\begin{aligned} H_n(s) &= \frac{1/(C_2\,C_3)}{s^2 + \frac{2}{C_2}\cdot s + \frac{1}{C_2\,C_3}} \\ &= \frac{1}{s^2 + s + 1} \end{aligned}$$

$$\begin{aligned} \therefore \frac{2}{C_2} &= 1 \\ \therefore C_2 &= 2\,\mathrm{F} \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{C_2\,C_3} &= 1 \\ \therefore C_3 &= \frac{1}{C_2} \\ &= 0.5\,\mathrm{F} \end{aligned}$$

With these above values, we have a 3rd order low pass Butterworth filter with a gain of 0 dB in the passband and a cutoff frequency of 1 rad/s.

We now need to adjust the filter to get the desired cutoff frequency of 500 Hz (desired gain will be achieved by the gain stage).

First off we perform frequency scaling by finding the frequency scaling factor k_f:

$$\begin{aligned} k_f &\triangleq \frac{\omega_c^{'}}{\omega_c} \\ &= \frac{2 \cdot \pi \cdot 500}{1} \\ &= 3141.6 \end{aligned}$$

Now we must use 1 kΩ resistors in the final design of the low pass filter section, so we perform magnitude scaling by finding the magnitude scaling factor k_m:

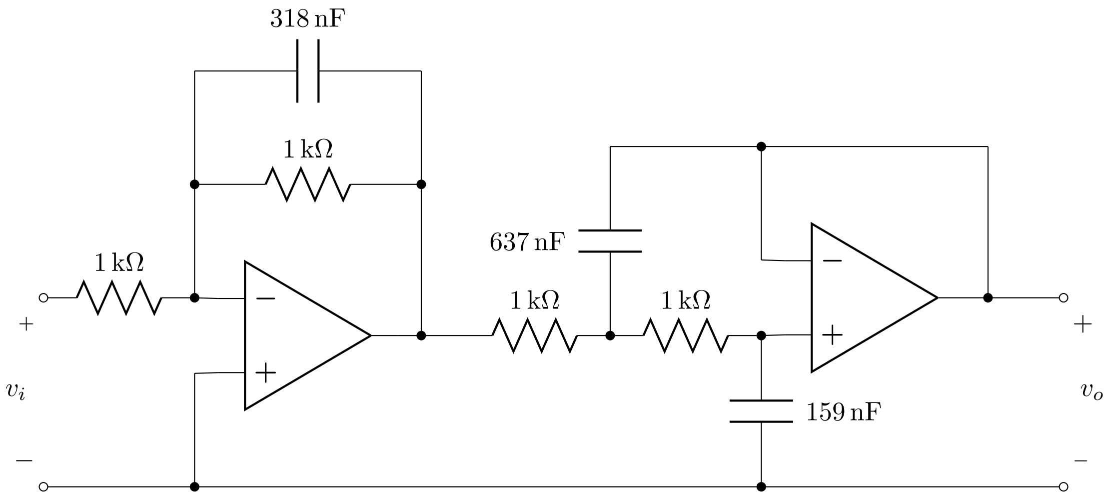
$$\begin{aligned} k_m &\triangleq \frac{R^{'}}{R} \\ &= \frac{1000}{1} \\ &= 1000 \end{aligned}$$

We can now scale the capacitors to new values to achieve the desired cutoff frequency:

$$C_n^{'} = \frac{C_n}{k_m\, k_f}$$

$$\begin{aligned} \therefore C_1^{'} &= \frac{1}{1000 \cdot 3141.6} = 318\,\mathrm{nF} \\ \therefore C_2^{'} &= \frac{2}{1000 \cdot 3141.6} = 637\,\mathrm{nF} \\ \therefore C_3^{'} &= \frac{0.5}{1000 \cdot 3141.6} = 159\,\mathrm{nF} \end{aligned}$$

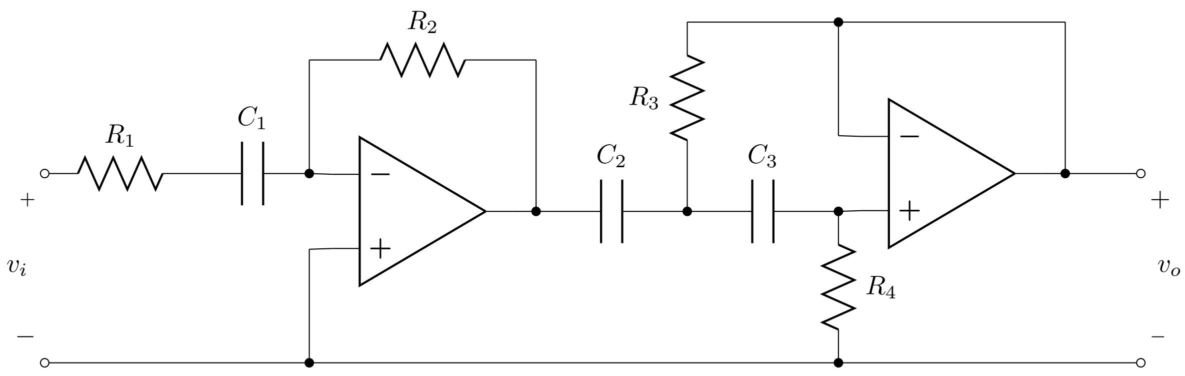
And so we have finished the design of the low pass section, whose circuit is shown below:



We now design the high pass stage.
We will use the same normalised Butterworth polynomial as from the low pass design section. Therefore, the transfer function of the 3rd order normalised high pass Butterworth filter is:

$$\frac{s^3}{(s+1)\big(s^2+s+1\big)} = \frac{s}{s+1} \cdot \frac{s^2}{s^2+s+1}$$

This results in the cascade of a 1st order high pass Butterworth filter with a 2nd order high pass Butterworth filter, both with passband gains of 0 dB and cutoff frequencies of 1 rad/s. This can be achieved with the following circuit:



We need to find resistor and capacitor values for the normalised filters.

Let $C_1 = C_2 = C_3 = 1$ F.
For the normalised 1st order high pass Butterworth filter, we note that the transfer function is the same as the transfer function of the prototypical 1st order active high pass filter. Therefore, the component values for this normalised Butterworth filter should match those of the 1st order prototype filter, and so $R_1 = R_2 = 1\,\Omega$.

For the normalised 2nd order high pass Butterworth filter, we note that the transfer function is:

$$\begin{aligned} H_n(s) &= \frac{s^2/(R_3\,R_4)}{s^2 + \frac{2}{R_3}\cdot s + \frac{1}{R_3\,R_4}} \\ &= \frac{s^2}{s^2 + s + 1} \end{aligned}$$

$$\begin{aligned} \therefore \frac{2}{R_3} &= 1 \\ \therefore R_3 &= 2\,\Omega \end{aligned}$$

$$\begin{aligned} \therefore \frac{1}{R_3\,R_4} &= 1 \\ \therefore R_4 &= \frac{1}{R_3} \\ &= 0.5\,\Omega \end{aligned}$$

With these above values, we have a 3rd order high pass Butterworth filter with a gain of 0 dB in the passband and a cutoff frequency of 1 rad/s.

We now need to adjust the filter to get the desired cutoff frequency of 5 kHz.
First off we perform frequency scaling by finding the frequency scaling factor k_f:

$$\begin{aligned} k_f &= \frac{\omega_c^{'}}{\omega_c} \\ &= \frac{2\cdot\pi\cdot 5000}{1} \\ &= 31,415.9 \end{aligned}$$

Now we must use 10 nF capacitors in the final design of the high pass filter section, so we perform magnitude scaling by finding the magnitude scaling factor k_m:

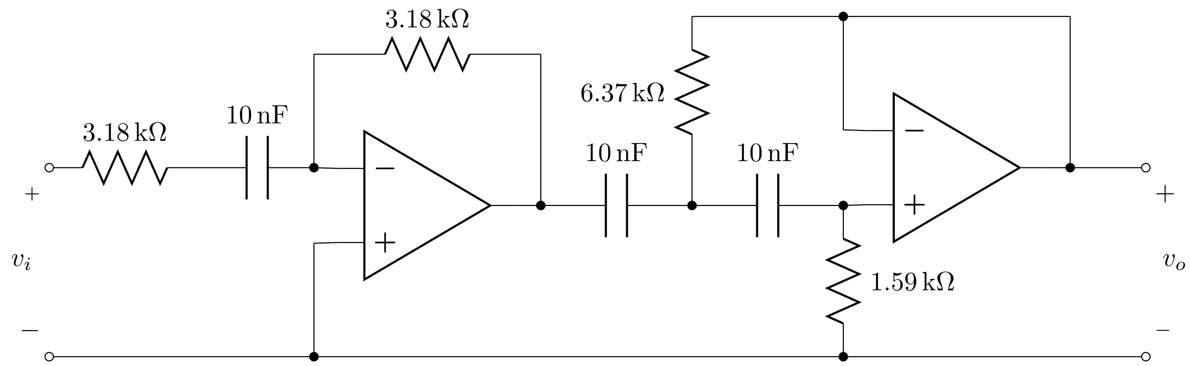
$$\begin{aligned} k_m &\triangleq \frac{C}{C'\,k_f} \\ &= \frac{1}{10^{-8}\cdot 31415.9} \\ &= 3183.1 \end{aligned}$$

We can now scale the resistors to new values to achieve the desired cutoff frequency:

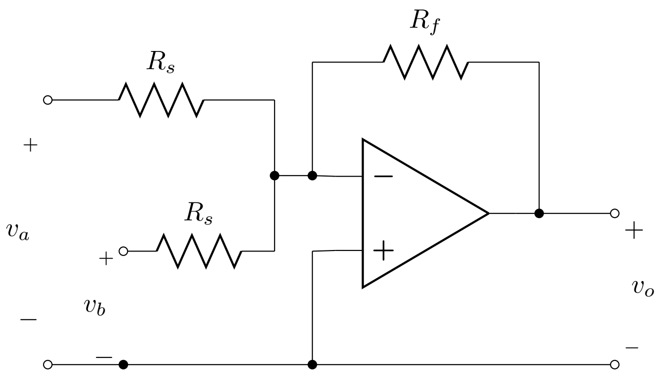
$$R_n^{'} = R_n \cdot k_m$$

$$\begin{aligned} \therefore R_1^{'} &= 1\cdot 3183.1 = 3.18\,\mathrm{k\Omega} \\ \therefore R_2^{'} &= 1\cdot 3183.1 = 3.18\,\mathrm{k\Omega} \\ \therefore R_3^{'} &= 2\cdot 3183.1 = 6.37\,\mathrm{k\Omega} \\ \therefore R_4^{'} &= 0.5\cdot 3183.1 = 1.59\,\mathrm{k\Omega} \end{aligned}$$

And so we have finished the design of the high pass section, whose circuit is shown below:



We now design the gain stage.
We are designing based off a desired magnitude response, with no attention paid to the phase response. Therefore, we will choose to implement the gain using an inverting summing op-amp amplifier circuit.



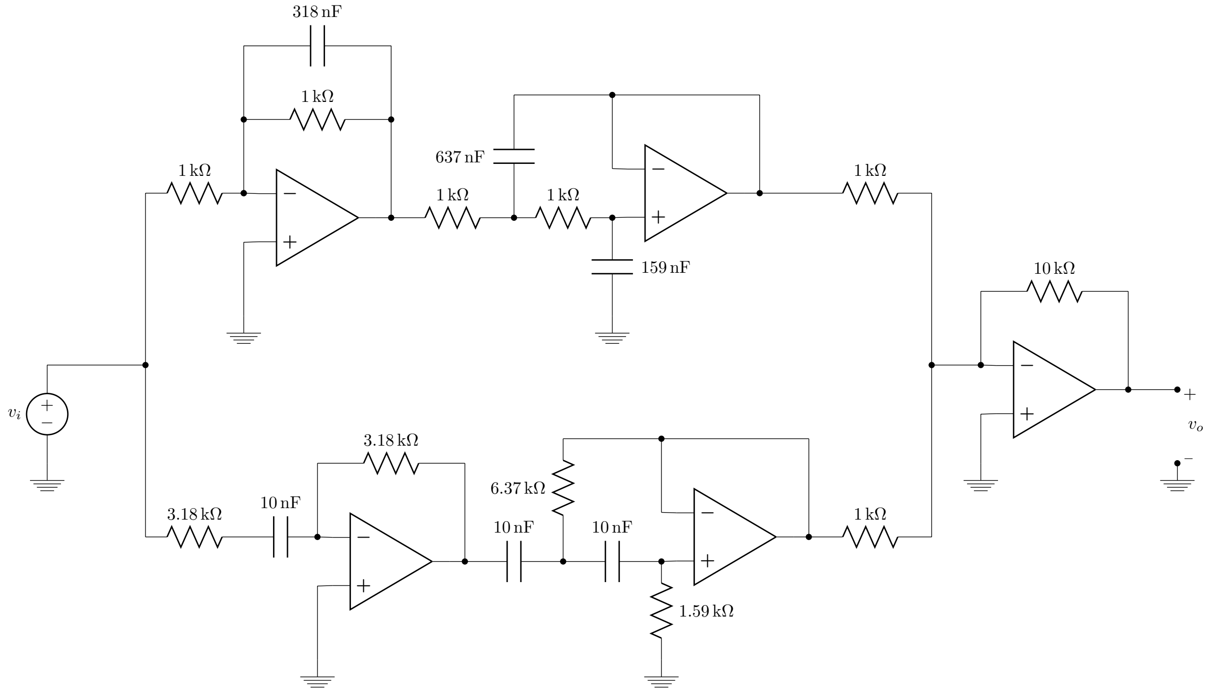
We note that there is equal weighting to both inputs, and so the transfer function for either input is:

$$H(s) = -\frac{R_f}{R_s}$$

Now we want a passband gain of 20 dB = 10 V/V. And so we want:

$$\begin{aligned} \big|H(s)\big| &= 10 \\ \therefore\ \frac{R_f}{R_s} &= 10, \qquad \text{let } R_s = 1\ \text{k}\Omega,\ \Longrightarrow\ R_f = 10\ \text{k}\Omega \end{aligned}$$

We now have all of the separate stages designed, so we bring them together as described already:



And so we have a 3$^{\text{rd}}$ order bandreject Butterworth filter with passgand gain of 20 dB, a lower cutoff frequency of 500 Hz and an upper cutoff frequency of 5 kHz, using only 1 kΩ resistors in the low pass stage and 10 nF capacitors in the high pass stage.

5.