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**Numerical Methods, 2023 Spring**

# Assignment 5

**1. (25%) Find the solution to $\frac{dy}{dt} = y^2 + t^2$, $y(1) = 0$ at $t = 2$ by the Euler method, using $h = 0.1$ . Repeat with $h = 0.5$ . From the two results, estimate the accuracy of the second computation. Hints: use the Romberg Integration to estimate the error.**

- $Initial\ condition$: $y(1) = 0$, $h = 0.1$, $t_0 = 1$
- Euler's Method: $y(t+h) = y(t) + h\frac{dy}{dt}$ , $\frac{dy}{dt} = f(y,t) = y^2 + t^2$
- For h=0.1: $y(t+0.1) = y(t) + 0.1 * (y(t)^2 + t^2)$ go through $(2-1)/0.1 = 10$ steps
    - $y(1.1) = 0 + 0.1 * f(0,1) = 0.1$
    - $y(1.2) = 0.1 + 0.1 * f(0.1, 1.1) = 0.222$
    - ……
- For h=0.5: $y(t+0.5) = y(t) + 0.5 * (y(t)^2 + t^2)$ go through $(2-1)/0.05 = 20$ steps
    - $y(1.05) = 0 + 0.05 * f(0,1) = 0.05$
    - $y(1.10) = 0.10525$

```
>> Q1_110550062
---h=0.1---
t0=1.000000, y0=0.000000
t1=1.100000, y1=0.100000
t2=1.200000, y2=0.222000
t3=1.300000, y3=0.370928
t4=1.400000, y4=0.553687
t5=1.500000, y5=0.780344
t6=1.600000, y6=1.066238
t7=1.700000, y7=1.435924
t8=1.800000, y8=1.931112
t9=1.900000, y9=2.628031
t10=2.000000, y10=3.679686
```

Estimated y(2) with h=0.1

```
---h=0.05---
t0=1.000000, y0=0.000000
t1=1.050000, y1=0.050000
t2=1.100000, y2=0.105250
t3=1.150000, y3=0.166304
t4=1.200000, y4=0.233812
t5=1.250000, y5=0.308545
t6=1.300000, y6=0.391430
t7=1.350000, y7=0.483591
t8=1.400000, y8=0.586409
t9=1.450000, y9=0.701603
t10=1.500000, y10=0.831340
t11=1.550000, y11=0.978396
t12=1.600000, y12=1.146384
t13=1.650000, y13=1.340094
t14=1.700000, y14=1.566012
t15=1.750000, y15=1.833132
t16=1.800000, y16=2.154275
t17=1.850000, y17=2.548320
t18=1.900000, y18=3.044142
t19=1.950000, y19=3.687982
t20=2.000000, y20=4.558168
```

Estimated y(2) with h=0.1

- Romberg Integration combines the Composite Trapezoidal Rule(CTR) with Richardson Extrapolation: $\frac{4.55816-3.67969}{2^2-1} = 0.29282$

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**2. (25%) The following equation is the corrector in the Adams-Moulton method. Please derive its coefficients using the undetermined coefficients method.**

$$\tilde{x}_{n+1} = x_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right] - \frac{19}{720}h^5 x^{(5)}(\xi_2)$$

*. Undetermined Coefficents Method for "corrector of Adams-Moulton Method":

$$\int_{t_n}^{t_{n+1}} f(t)\,dt \approx c_0 f_{n-2} + c_1 f_{n-1} + c_2 f_n + c_3 f_{n+1}$$

*. Solving linear system (4th-order):

$f(t)=1$, $f(t)=t$, $f(t)=t^2$, $f(t)=t^3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h \\ 4h^2 & h^2 & 0 & h^2 \\ -8h^3 & -h^3 & 0 & h^3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} h \\ h^2/2 \\ h^3/3 \\ h^4/4 \end{bmatrix}$$

$$\Rightarrow c_0 = \frac{h}{24},\ c_1 = \frac{-5}{24}h,\ c_2 = \frac{19}{24}h,\ c_3 = \frac{9}{24}h$$

*. Calculate error term (5th-order):

$f(t)=1$, $f(t)=t$, $f(t)=t^2$, $f(t)=t^3$, $f(t)=t^4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} h \\ h^2/2 \\ h^3/3 \\ h^4/4 \\ h^5/5 \end{bmatrix}$$

$$\Rightarrow c_0 = \frac{-19}{720}h,\ c_1 = \frac{106}{720}h,\ c_2 = \frac{-264}{720}h,\ c_3 = \frac{646}{720}h,\ c_4 = \frac{251}{720}h$$

Therefore, we have the conclusion that the corrector of Adams-Moulton method is

$$\tilde{x}_{n+1} = x_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right] - \frac{19}{720}h^5 x^{(5)}(\xi)$$

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**3. (25%) For the third-order equation**

$$y''' + ty' - 2y = t, \qquad y(0) = y''(0) = 0, \qquad y'(0) = 1$$

**(a) Solve for $y(0.2)$, $y(0.4)$, $y(0.6)$, by RKF (you can use ode45() in Matlab to run RKF, but you need to list the system of ODE equations and explain how you solve this equation. Please upload you code to E3).**

- The system of ODE equations: $y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, f = y' = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 2x_1 - tx_2 + t \end{bmatrix}$
- Matlab Code and Result:

```
%% (a)
fprintf("---(a)---\n");
y = [0 1 0];

[t1,y1] = ode45('vdp', [0 0.2], y);
[t2,y2] = ode45('vdp', [0 0.4], y);
[t3,y3] = ode45('vdp', [0 0.6], y);

fprintf("y(0.2) = %f\n", y1(end,1));
fprintf("y(0.4) = %f\n", y2(end,1));
fprintf("y(0.6) = %f\n", y3(end,1));
```

```
>> Q3_110550062
---(a)---
y(0.2) = 0.200133
y(0.4) = 0.402132
y(0.6) = 0.610778
```

**(b) Advance the solution to $t = 1.0$ with the Adams-Moulton method and results obtained in (a).**

- **Compute y(0.8) using y(0), y(0.2), y(0.4), y(0.6) and h=0.2**
  - **Predictor:** $y_{0.8} = y_{0.6} + \frac{0.2}{24}(55f_{0.6} - 59f_{0.4} + 37f_{0.2} - 9f_0) = \begin{bmatrix} 0.8340 \\ 1.1694 \\ 0.6296 \end{bmatrix}$
  - $f_{0.8} = \begin{bmatrix} 1.1694 \\ 0.6296 \\ 1.5325 \end{bmatrix}$
  - **Corrector:** $y_{0.8} = y_{0.6} + \frac{0.2}{24}(9f_{0.8} - 19f_{0.6} - 5f_{0.4} + f_{0.2}) = \begin{bmatrix} 0.8340 \\ 1.1692 \\ 0.6291 \end{bmatrix}$
- **Compute y(1.0) using y(0.2), y(0.4), y(0.6), y(0.8)**
  - **Predictor:** $y_{1.0} = y_{0.8} + \frac{0.2}{24}(55f_{0.8} - 59f_{0.6} + 37f_{0.4} - 9f_{0.2}) = \begin{bmatrix} 1.0827 \\ 1.3284 \\ 0.9679 \end{bmatrix}$
  - $f_{1.0} = \begin{bmatrix} 1.3284 \\ 0.9679 \\ 1.8370 \end{bmatrix}$
  - **Corrector:** $y_{1.0} = y_{0.8} + \frac{0.2}{24}(9f_{1.0} - 19f_{0.8} - 5f_{0.6} + f_{0.4}) = \begin{bmatrix} 1.0826 \\ 1.3281 \\ 0.9675 \end{bmatrix}$

- Matlab Code and Result

```
%% (b)
fprintf("\n---(b)---\n");
% compute f
f0 = vdp(0,y)';
f1 = vdp(0.2,y1(end,:))';
f2 = vdp(0.4,y2(end,:))';
f3 = vdp(0.6,y3(end,:))';

% first, compute y0.8 using predictor
y4_p = y3(end,:) + 0.2/24*(55*f3-59*f2+37*f1-9*f0);
% compute derivative f0.8
f4 = vdp(0.8,y4_p(end,:))';
% recompute y0.8 using corrector
y4_c = y3(end,:) + 0.2/24*(9*f4+19*f3-5*f2+f1);

fprintf("y0.8 from predictor: (%f, %f, %f)\n", y4_p);
fprintf("f0.8                : (%f, %f, %f)\n", f4);
fprintf("y0.8 from corrector: (%f, %f, %f)\n", y4_c);

% first, compute y1.0 using predictor
y5_p = y4_p(end,:) + 0.2/24*(55*f4-59*f3+37*f2-9*f1);
% compute derivative f1.0
f5 = vdp(1.0,y5_p(end,:))';
% recompute y1.0 using corrector
y5_c = y4_p(end,:) + 0.2/24*(9*f5+19*f4-5*f3+f2);

fprintf("\ny1.0 from predictor: (%f, %f, %f)\n", y5_p)
fprintf("f1.0                : (%f, %f, %f)\n", f5);
fprintf("y1.0 from corrector: (%f, %f, %f)\n", y5_c);

function dy=vdp(t,y)
    dy = [y(2) y(3) y(1)*2-t*y(2)+t]';
end
```

```
---(b)---
y0.8 from predictor: (0.834004, 1.169388, 0.629579)
f0.8               : (1.169388, 0.629579, 1.532497)
y0.8 from corrector: (0.833975, 1.169232, 0.629126)

y1.0 from predictor: (1.082738, 1.328445, 0.967912)
f1.0               : (1.328445, 0.967912, 1.837031)
y1.0 from corrector: (1.082645, 1.328103, 0.967529)
```

**(c) Estimate the accuracy of $y(1.0)$ in part (b).**

Assuming the fifth-derivative term (as error term in the Adam-Moulton method) are equal, we can estimate the error in the corrector and predictor value of the coefficient:

$$\text{error} = (\tilde{y}(1.0) - y(1.0)) * \left(\frac{19}{251+19}\right) = |1.082738\text{-}1.082645| * \frac{19}{270} = 6.54 * 10^{-6}$$

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**4. (25%) Given the boundary-value problem:**

$$\frac{d^2y}{d\theta^2} + \frac{y}{4} = 0, y(0) = 0, y(\pi) = 2$$

**which has the solution** $y = 2\sin\frac{\theta}{2}$

**(a) Solve using finite difference approximations to the derivative with** $h = \frac{\pi}{4}$ **and tabulate the errors.**

**(b) Solve again by finite differences but with a value of h small enough to reduce the maximum error to 0.5% .**

**(c) Solve again by the shooting method. Find how large h can be to have a maximum error of 0.5% . (Please use the secant method to find y'(0))**

$$\frac{d^2y}{d\theta^2} + \frac{y}{4} = 0 \ , \ y(0)=0, \ y(\pi)=2 \ , \ y = 2\cdot\sin\frac{\theta}{2}$$

(a) Finite Difference Approximation

$$y_{i+1} - 2y_i + y_{i-1} = h^2 y_i'' \ , \ h = \frac{\pi}{4} \ ,$$

points : $-\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi$

$$\frac{16}{\pi^2} y_{i-1} + \left(\frac{\pi^2-128}{4\pi^2}\right) y_i + \frac{16}{\pi^2} y_{i+1} = 0 \ , \ i = 0 \sim 4$$

$$\begin{bmatrix} \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} \end{bmatrix} \begin{bmatrix} y_{-1} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

| | $y_{-1}$ | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |
|---|---|---|---|---|---|---|---|
| Estimated | -0.7902 | 0 | 0.7902 | 1.4215 | 1.8537 | 2 | 1.8399 |
| True | -0.7654 | 0 | 0.7654 | 1.4142 | 1.8478 | 2 | 1.8478 |
| Error | 0.0048 | 0 | -0.0048 | -0.0073 | -0.0059 | 0 | 0.0099 |

(b) To reduce the maximum error to 0.5% , set $h = \frac{\pi}{5}$

| | $y_{-1}$ | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |
|---|---|---|---|---|---|---|---|---|
| points : | $-\frac{\pi}{5}$, | 0, | $\frac{\pi}{5}$, | $\frac{2}{5}\pi$, | $\frac{3}{5}\pi$, | $\frac{4}{5}\pi$, | $\pi$, | $\frac{6}{5}\pi$ |
| Estimated | -0.6205 | 0 | 0.6205 | 1.1798 | 1.6229 | 1.9504 | 2 | 1.8973 |
| True | -0.6180 | 0 | 0.6180 | 1.1956 | 1.6180 | 1.9021 | 2 | 1.9021 |
| Error | 0.0025 | 0 | -0.0025 | -0.0042 | -0.0046 | -0.0032 | 0 | 0.0049 |

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## (c) Shooting method with secant method

```
% Demonstration of shooting method.
% Given a BVP: y" + y/4=0, y(0)=0, y(pi)=2
% Let U = [u1
%          u2];
%    dU/dx = [du1/dx    = [ y2
%              du2/dx ]    -y/4];
clear all;
% Secant method for finding the root of P(v) = y(pi,v) - 2 = 0
% where y(x,v) is the solution curve y(x) obtained with initial velocity v
SHOW_PLOT = 0;
U3 = secant(@shooting, [0, 0], [0, 1], 1e-5, SHOW_PLOT);
% In this root finding problem, the root is found in one step.
fprintf("U3(2):%f\n", U3(2));
% Plot results:
% Ux1: solution u(x) with U1 = [u, u'] = [0, 2/pi]
% Ux2: solution u(x) with U2 = [u, u'] = [0, 1]
% Ux3: solution u(x) with U3 = [u, u'] = [0, root]
% the first output argument of shooting() is only used for root-finding, see shooting.m for the details.
%[dummy, x, Ux1] = shooting([0, 0]);
%[dummy, x, Ux2] = shooting([0, 1]);
max_h = 0;
U = @(t) 2*sin(t/2);
tol = 0.005;
for h=0.2:-0.01:0.1
    [dummy, x, Ux3] = shooting([0, U3(2)], h);
    flag = 0;
    for i=0:h:pi
        if abs(Ux3(i)-U(i))>tol
            flag = 1;
            break;
        end
    end
    if flag==0
        if h>max_h
            max_h=h;
        end
    end
end
```

Revised shooting_demo.m

```
% function [x, u]=shooting(u0)

% Secant method for finding the root of P(v) = u(3,v) + 1 = 0
% where u(x,v) is the solution curve u(x) obtained with initial veloci
% u(3,v) is then a curve when x=3.
% The first output argument is the function valuae P(v)
function [P, x, U] = shooting(U0, h)
[x, U] = rkf(U0, h);
% P = u(3, v) + 1; v = U0(2);
P = U(length(x),1) + 1;

function [x, U] = rkf(U0, h)
    tspan = 0:h:3.2;
    [x, U] = ode45(@F, tspan, U0);
% Set U = [u
%          u'];
% dU/dx = f(x,U)
function dUdx = F(x,U)
dUdx = [ U(2)
        -U(1)/4];
```

Revised shooting.m,  left secant.m unchaged

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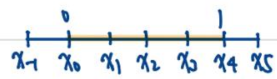
**5. (25%) The most general form of boundary condition normally encountered in second-order boundary-value problems is a linear combination of the function and its derivatives at both ends of the region. Solve through finite difference approximations with four subintervals:**

$$x'' - tx' + t^2 x = t^3,$$
$$x(0) + x'(0) - x(1) + x'(1) = 3,$$
$$x(0) - x'(0) + x(1) - x'(1) = 2.$$

*. Since there are 4 subintervals, we set $h = 1/4$. t starts from 0 to 1

*. $x'' - tx' + t^2 x - t^3 = 0$




$$\left(\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}\right) - t_i\left(\frac{x_{i+1} - x_i}{2h}\right) + t_i^2 x_i = t_i^3$$

$$\left(\frac{1}{h^2} + \frac{t_i}{2h}\right) x_{i-1} + \left(t_i^2 - \frac{2}{h^2}\right) x_i + \left(\frac{1}{h^2} - \frac{t_i}{2h}\right) x_{i+1} = t_i^3, \quad i = 0 \sim 4$$

$i=0$: $(16+2t_0)x_{-1} + (t_0^2 - 32)x_0 + (16 - 2t_0)x_1 = t_0^3$ — ①

$i=1$: $(16+2t_1)x_0 + (t_1^2 - 32)x_1 + (16 - 2t_1)x_2 = t_1^3$ — ②

$i=2$: $(16+2t_2)x_1 + (t_2^2 - 32)x_2 + (16 - 2t_2)x_3 = t_2^3$ — ③

$i=3$: $(16+2t_3)x_2 + (t_3^2 - 32)x_3 + (16 - 2t_3)x_4 = t_2^3$ — ④

$i=4$: $(16+2t_4)x_3 + (t_4^2 - 32)x_4 + (16 - 2t_4)x_5 = t_4^3$ — ⑤

*. $\begin{cases} x(0) + x'(0) - x(1) + x'(1) = 3 \\ x(0) - x'(0) + x(1) - x'(1) = 2 \end{cases}$ / $x(0) = \frac{5}{2}$

$x_0 + \frac{(x_1 - x_{-1})}{2h} - x_4 + \frac{(x_5 - x_3)}{2h} = 3 \rightarrow x_0 + 2(x_1 - x_{-1}) - x_4 + 2(x_5 - x_3) = 3$ — ⑥

$x_0 - \frac{(x_1 - x_{-1})}{2h} + x_4 - \frac{(x_5 - x_3)}{2h} = 2 \rightarrow x_0 - 2(x_1 - x_{-1}) + x_4 - 2(x_5 - x_3) = 2$ — ⑦

*. From ①~⑦. we can build a linear equation system in matrix form.

$$\begin{bmatrix} 16+2t_0 & t_0^2-32 & 16-2t_0 & 0 & 0 & 0 & 0 \\ 0 & 16+2t_1 & t_1^2-32 & 16-2t_1 & 0 & 0 & 0 \\ 0 & 0 & 16+2t_2 & t_2^2-32 & 16-2t_2 & 0 & 0 \\ 0 & 0 & 0 & 16+2t_3 & t_3^2-32 & 16-2t_3 & 0 \\ 0 & 0 & 0 & 0 & 16+2t_4 & t_4^2-32 & 16-2t_4 \\ -2 & 1 & 2 & 0 & -2 & -1 & 2 \\ 2 & 1 & -2 & 0 & 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} t_0^3 \\ t_1^3 \\ t_2^3 \\ t_3^3 \\ t_4^3 \\ 3 \\ 2 \end{bmatrix}$$

$t_0 = 0$
$t_1 = 1/4$
$t_2 = 1/2$
$t_3 = 3/4$
$t_4 = 1$

put $t_{0\sim4}$ into formula, simplify the equations and we can get $x = \begin{bmatrix} 1.7446 \\ 2.5000 \\ 3.2554 \\ 4.0474 \\ 4.8858 \\ 5.7374 \\ 6.4938 \end{bmatrix}$