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110550062

Assignment 4

1. (10%) Make a divided difference table for $f(x) = 1 + log10\ x$ with data points at $x_0 = 0.15,\ x_1 = 0.21,\ x_2 = 0.23,\ x_3 = 0.27,\ x_4 = 0.32,\ x_5 = 0.35$.

Compute a value for $f'(0.268)$ from a quadratic interpolating polynomial that fits the table at the three points that should give the most accurate answer. Which points are these?

Code and explanation:

```
% Compute the divided difference table
n = length(x);
D = zeros(n, n);
D(:, 1) = fx;
for j = 2:n
  for i = 1:n-j+1
    D(i,j) = (D(i+1,j-1) - D(i,j-1))/(x(i+j-1) - x(i));
  end
end
disp(D)

% Compute the true value
syms t;
tf = 1 + log10(t);
tfd = diff(tf, t);
true = eval(subs(tfd, t, 0.268));

% Compute the value of f prime at x = 0.268 using a quadratic
% interpolating polynomial that fits the table at the three points
% that provide the most accurate answer
est = Inf;
dif = Inf;
start = 1;
s = 0.268;
for idx = 1:4
    tmp = D(idx,2)+D(idx,3)*((s-x(idx))+(s-x(idx+1)));
    tmpd = abs(true - tmp);
    if(tmpd<dif)
        dif = tmpd;
        est = tmp;
        start = idx;
    end
end
```

Results:

```
>> Q1_110550062
    0.1761    2.4355   -5.7505   15.3474  -38.8116   94.5795
    0.3222    1.9754   -3.9088    8.7494  -19.8957         0
    0.3617    1.7409   -2.9464    5.9640         0         0
    0.4314    1.4757   -2.2307         0         0         0
    0.5051    1.2973         0         0         0         0
    0.5441         0         0         0         0         0

From a quadratic polynomial start from x = 0.230000, f'(0.268) = 1.6348
```

Compute with the calculator, we know that the true value of $\boldsymbol{f'(0.268)=1.6205}$ when $\boldsymbol{f(x) = 1 + log10\ x}$. With the three points $\boldsymbol{x_2= 0.23,\ x_3= 0.27,\ x_4= 0.32}$, we get the most accurate value of $\boldsymbol{f'(0.268) = 1.6348}$.

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2. (20%) Make a function difference table for $f(x) = x + \sin(x)/3$ with data points at $x_0 = 0.3$, $x_1 = 0.5$, … , $x_6 = 1.5$. Use it to find

In each part, choose the best starting data points.

Result:

```
>> Q2_110550062
    0.3985    1.3065   -0.0796   -0.0456    0.0089    0.0019   -0.0004
    0.6598    1.2747   -0.1070   -0.0385    0.0108    0.0015         0
    0.9147    1.2318   -0.1301   -0.0299    0.0123         0         0
    1.1611    1.1798   -0.1480   -0.0200         0         0         0
    1.3971    1.1206   -0.1601         0         0         0         0
    1.6212    1.0566         0         0         0         0         0
    1.8325         0         0         0         0         0         0

From a cubic polynomial start from x = 0.500000, f'(0.72) = 1.2505
From a quadratic polynomial start from x = 1.100000, f'(1.33) = 1.0790
From a 4th degree polynomial start from x = 0.300000, f'(0.5) = 1.2925
```

With x = [0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5], choose the estimated value which is the closest to true value:

(a) $f'(0.72)$ from a cubic polynomial.

$f'(0.72) = 1.2505$ has the most accurate value calculated with starting point **$x_1 = 0.5$**.

Third difference approximation formula :

$$P'_3(x_i) = f[x_i, x_{i+1}] + f[x_i, x_{i+1}, x_{i+2}][(x - x_i) + (x - x_{i+1})] + f[x_i, x_{i+1}, x_{i+2}, x_{i+3}][(x - x_i)(x - x_{i+1}) + (x - x_i)(x - x_{i+2}) + (x - x_{i+1})(x - x_{i+2})]$$

```
s = 0.72;
for idx = 1:4
    tmp = D(idx, 2)+D(idx, 3)*((s-x(idx)) + (s-x(idx+1)))+D(idx, 4)*( (s-x(idx))*(s-x(idx+1)) + (s-x(idx+1))*(s-x(idx+2)) + (s-x(idx+2))*(s-x(idx)));
    tmpd = abs(true_a - tmp);
    if(tmpd<dif)
        dif = tmpd;
        est = tmp;
        start = idx;
    end
end
```

(b) $f'(1.33)$ from a quadratic polynomial.

$f'(1.33) = 1.0790$ has the most accurate value calculated with starting point **$x_4 = 1.1$**.

Second difference approximation formula:

$$P'_2(x_i) = f[x_i, x_{i+1}] + f[x_i, x_{i+1}, x_{i+2}][(x - x_i) + (x - x_{i+1})]$$

```
s = 1.33;
for idx = 1:5
    tmp = D(idx,2)+D(idx,3)*((s-x(idx))+(s-x(idx+1)));
    tmpd = abs(true_b - tmp);
    if(tmpd<dif)
        dif = tmpd;
        est = tmp;
        start = idx;
    end
end
```

(c)$f'(0.50)$ from a 4th degree polynomial.

f′(0.5) = 1.2925 has the most accurate value calculated with starting point $\boldsymbol{x_0}$**= 0.3**.

4-th difference approximation formula :

$$P'_4(x_i) = f[x_i, x_{i+1}] + f[x_i, x_{i+1}, x_{i+2}][(x - x_i) + (x - x_{i+1})] + f[x_i, x_{i+1}, x_{i+2}, x_{i+3}][(x - x_i)(x - x_{i+1}) + (x - x_i)(x - x_{i+2}) + (x - x_{i+1})(x - x_{i+2})] + f[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}][((x - x_i)(x - x_{i+1})(x - x_{i+2}) + (x - x_i)(x - x_{i+1})(x - x_{i+3}) + (x - x_i)(x - x_{i+2})(x - x_{i+3}) + (x - x_{i+1})(x - x_{i+2})$$

3. (20%) Use the method of underdetermined coefficients to obtain the formula for $f''(x)$ and $f'''(x)$ at x_0 using five evenly spaced points from x_{-2} to x_2 . Please also derive their error terms.

 # For $f''(x_0) = c_{-2}f_{-2} + c_{-1}f_{-1} + c_0f_0 + c_1f_1 + c_2f_2$
$P(u) = au^4 + bu^3 + cu^2 + du + e$

P(u)=1: $f_{-2} = f_{-1} = f_0 = f_1 = f_2 = 1 = P(\mathrm{u})$
$f''(x_0) = c_{-2} + c_{-1} + c_0 + c_1 + c_2 = P''(u) = 0$

P(u)=u: f_{-2}= P(-2h) = -2h, f_{-1}= P(-h) = -h, f_0 = P(0) = 0, f_1 = P(h) = h, f_2 = P(2h) = 2h
$f''(x_0) = c_{-2}(-2h) + c_{-1}(-h) + c_0(0) + c_1(h) + c_2(2h) = P''(u) = 0$

P(u)=u^2: f_{-2}= P(-2h) = $4h^2$, f_{-1}= P(-h) =h^2, f_0 = P(0) = 0, f_1 = P(h) = h^2, f_2 = P(2h) = $4h^2$
$f''(x_0) = c_{-2}(4h^2) + c_{-1}(h^2) + c_0(0) + c_1(h^2) + c_2(4h^2) = P''(u) = 2$

P(u)= u^3: f_{-2}= P(-2h) = -$8h^3$, f_{-1}= P(-h) =$-h^3$, f_0 = P(0) = 0, f_1 = P(h) = h^3, f_2 = P(2h) = $8h^3$

$f''(x_0) = c_{-2}(-8h^3) + c_{-1}(-h^3) + c_0(0) + c_1(h^3) + c_2(8h^3) = P''(u) = 0$

P(u)= u^4: f_{-2}= P(-2h) = $16h^4$, f_{-1}= P(-h) =h^4, f_0 = P(0) = 0, f_1 = P(h) = h^4, f_2 = P(2h) = $16h^4$
$f''(x_0) = c_{-2}(16h^4) + c_{-1}(h^4) + c_0(0) + c_1(h^4) + c_2(16h^4) = P''(u) = 0$

Arrange the above equation as $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, we get $c = \begin{bmatrix} -1/12 \\ 4/3 \\ -5/2 \\ 4/3 \\ -1/12 \end{bmatrix}$

Therefore, $\boldsymbol{f''(x_0) = \frac{-1}{12h^2}f_{-2} + \frac{4}{3h^2}f_{-1} + \frac{-5}{2h^2}f_0 + \frac{4}{3h^2}f_1 + \frac{-1}{12h^2}f_2}$

$$Since\ error = (x - x_{-2})(x - x_{-1})(x - x_0)(x - x_1)(x - x_2)\frac{f^5(\xi)}{5!}$$

$$\frac{d}{dx}\left(\frac{d}{dx}\mathbf{error}\right)\Big|_{x=x_0}$$

$$= \frac{f^5(\xi)}{5!}\left((x_0 - x_{-2})(x_0 - x_{-1})(x_0 - x_1) + (x_0 - x_{-2})(x_0 - x_{-1})(x_0 - x_2) + (x_0 - x_{-2})(x_0 - x_1)(x_0 - x_2) + (x_0 - x_{-1})(x_0 - x_1)(x_0 - x_2)\right)$$

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 # For $f'''(x_0) = c_{-2}f_{-2} + c_{-1}f_{-1} + c_0f_0 + c_1f_1 + c_2f_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$, we get $c = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1 \\ 1/2 \end{bmatrix}$

Therefore, $f'''(x_0) = \frac{-1}{2h^4}f_{-2} + \frac{1}{h^4}f_{-1} - \frac{1}{h^4}f_1 + \frac{1}{2h^4}f_2$

$\frac{d}{dx}\frac{d}{dx}(\frac{d}{dx}\text{error})\,|_{x=x_0}$

$= \frac{f^5(\xi)}{5!}[\ \big((x_0 - x_{-2})(x_0 - x_{-1}) + (x_0 - x_{-2})(x_0 - x_1) + (x_0 - x_{-1})(x_0 - x_1)\big) +$

$((x_0 - x_{-2})(x_0 - x_{-1}) + (x_0 - x_{-2})(x_0 - x_2) + (x_0 - x_{-1})(x_0 - x_2)) +$

$\big((x_0 - x_{-2})(x_0 - x_1) + (x_0 - x_{-2})(x_0 - x_2) + (x_0 - x_1)(x_0 - x_2)\big) +$

$((x_0 - x_{-1})(x_0 - x_1) + (x_0 - x_{-1})(x_0 - x_2) + (x_0 - x_1)(x_0 - x_2))]$

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4. (15%) Simpson's 3/8 rule cannot be applied directly to the following table because the number of panels is not divisible by three. Still, you can use it in combination with the 1/3 rule over two panels. There are several choices of where to use the 1/3 rule. Which of these choices gives the most accurate answer?

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.543	1.669	1.811	1.971	2.151	2.352	2.577	2.828	3.107

By Simpson's 1/3 rule, we know $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f_0 + 4f_1 + f_2)$.
By Simpson's 3/8 rule, we know $\int_{x_0}^{x_3} f(x)dx = \frac{8h}{3}(f_0 + 3f_1 + 3f_2 + f_3)$.
There are three combinations using Simpson's 1/3 and 3/8 rules, which is implemented in the code below, with the output:

```
% Define the function and x
fx = [1.543 1.669 1.811 1.971 2.151 2.352 2.577 2.828 3.107];
x = [1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8];

h = (1.1 - 1.0); % panel width
% Combination1: Use Simpson's 3/8 rule over the first two and middle two panels
% [1/3, 3/8, 3/8]
sum13 = h/3 * (fx(1) + 4*fx(2) + fx(3));
sum38 = 3/8 * h * (fx(3) + 3*fx(4) + 3*fx(5) + fx(6));
sum38_2 = 3/8 * h * (fx(6) + 3*fx(7) + 3*fx(8) + fx(9));
approx1 = sum13 + sum38 + sum38_2;
fprintf("Approximate Value of Combination1 [1/3, 3/8, 3/8]:%.9f\n", approx1);

% Combination2: Use Simpson's 3/8 rule over the first two and last two panels
% [3/8, 1/3, 3/8]
sum38 = 3/8 * h * (fx(1) + 3*fx(2) + 3*fx(3) + fx(4));
sum13 = h/3 * (fx(4) + 4*fx(5) + fx(6));
sum38_2 = 3/8 * h * (fx(6) + 3*fx(7) + 3*fx(8) + fx(9));
approx2 = sum13 + sum38 + sum38_2;
fprintf("Approximate Value of Combination2 [3/8, 1/3, 3/8]:%.9f\n", approx2);

% Combination3: Use Simpson's 3/8 rule over the middle and last two panels
% [3/8, 3/8, 1/3]
sum38 = 3/8 * h * (fx(1) + 3*fx(2) + 3*fx(3) + fx(4));
sum38_2 = 3/8 * h * (fx(4) + 3*fx(5) + 3*fx(6) + fx(7));
sum13 = h/3 * (fx(7) + 4*fx(8) + fx(9));
approx3 = sum13 + sum38 + sum38_2;
fprintf("Approximate Value of Combination3 [3/8, 3/8, 1/3]:%.9f\n", approx3);
% Set "the combinations of using Simpson's 1/3 rule over all the panels" as
% true value
sum13_1 =  h/3 * (fx(1) + 4*fx(2) + fx(3));
sum13_2 =  h/3 * (fx(3) + 4*fx(4) + fx(5));
sum13_3 =  h/3 * (fx(5) + 4*fx(6) + fx(7));
sum13_4 =  h/3 * (fx(7) + 4*fx(8) + fx(9));
true = sum13_1 + sum13_2 + sum13_3 + sum13_4;
fprintf("True Value Calculated with Simpson 1/3 rule: %.9f\n", true)

>> Q4_110550062
Approximate Value of Combination1 [1/3, 3/8, 3/8]:1.766945833
Approximate Value of Combination2 [3/8, 1/3, 3/8]:1.766950000
Approximate Value of Combination3 [3/8, 3/8, 1/3]:1.766945833
True Value Calculated with Simpson 1/3 rule: 1.766933333
```

According to the two Simpson's rules, **equation of combination1(3, 4, 4) and combination3 (4, 4, 3) has equal accurate answer compared with true value**. **The equation of combination2 has the worst accuracy**.

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5. (15%) Integrate $f(x) = 1/x^2$ over the interval [0.2, 1] using the trapezoidal rule with an adaptive integration scheme, i.e., halving the panel width h at each iteration until the integral value does not differ more than 0.02 between two iterations. At what value for h do the computation terminate?

First, define function of f, interval of [a, b]. Then run the while loop to check whether integral value does not differ more than 0.02 between two iterations. Therefore, we get h=0.0125.

Code:

```
f = @(x) 1/x^2;
a = 0.2;
b = 1;
n = 2;
integral_sum = 1;
new_integral_sum = 0;

while(abs(new_integral_sum - integral_sum) > 0.02)
    integral_sum = new_integral_sum;
    new_integral_sum = 0;
    h = (b-a)/n;
    for i= 0 : n-1
        new_integral_sum = new_integral_sum + h/2*(f(a + i*h) + f(a + (i+1)*h));
    end
    n = n*2;
end

fprintf("The Value of Integral Sum is %.5f while h = %.4f\n", new_integral_sum, h);
```

Result:

```
>> Q5_110550062
The Value of Integral Sum is 4.00323 while h = 0.0125
```

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6. (20%) Evaluate the following integral using Gaussian quadrature, three-term formulas and h=0.1 in both directions.

$$\int_{-0.2}^{1.4}\int_{0.4}^{2.6} e^x \sin(2y)\,dy\,dx$$

By Gaussian quadrature 3-term rule, we know that

- $\int_b^a f(x)dx = \frac{(b-a)}{2}[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)]$
- $x_1 = \frac{(b-a)}{2}t_1 + \frac{(b+a)}{2}$, $x_2 = \frac{(b-a)}{2}t_2 + \frac{(b+a)}{2}$, $x_3 = \frac{(b-a)}{2}t_3 + \frac{(b+a)}{2}$ (Change the limit to [-1, 1])
- $t_1 = -0.7746$, $t_2 = 0$, $t_3 = 0.7746$
- $w_1 = 0.5555$, $w_2 = 0.8888$, $w_3 = 0.5555$
- $\frac{(b-a)}{2} = h = 0.1$

Main Code:

```
g = @(y) sin(2*y);
f = @(x) exp(x);
for j = 1:2:21
    r = (y(j) + y(j+2))/2;
    g_result = g_result + 0.1 * (5/9 * g(-0.1 * sqrt(3/5) + r) + 8/9 * g(r) + 5/9 * g(0.1 * sqrt(3/5) + r));
end

for j = 1:2:15
    r = (x(j) + x(j+2))/2;
    f_result = f_result + 0.1 * (5/9 * f(-0.1 * sqrt(3/5) + r) + 8/9 * f(r) + 5/9 * f(0.1 * sqrt(3/5) + r));
end

end_result = g_result * f_result;
```

Result:

```
>> Q6_110550062
Integral of f(x) from -0.2 to 1.4 = 3.23646921
Integral of g(y) from 0.4 to 2.6 = 0.11409502
Total integral of f(x)*g(y) = 0.36926502
```

With above knowledge, we can transform the integral into

$$\int_{-0.2}^{1.4}\int_{0.4}^{2.6} e^x \sin(2y)\,dy\,dx = \int_{-0.2}^{1.4} e^x dx \int_{0.4}^{2.6} \sin(2y)\,dy = 3.23646921 * 0.11409502 = 0.36926502$$