Symmetric functions

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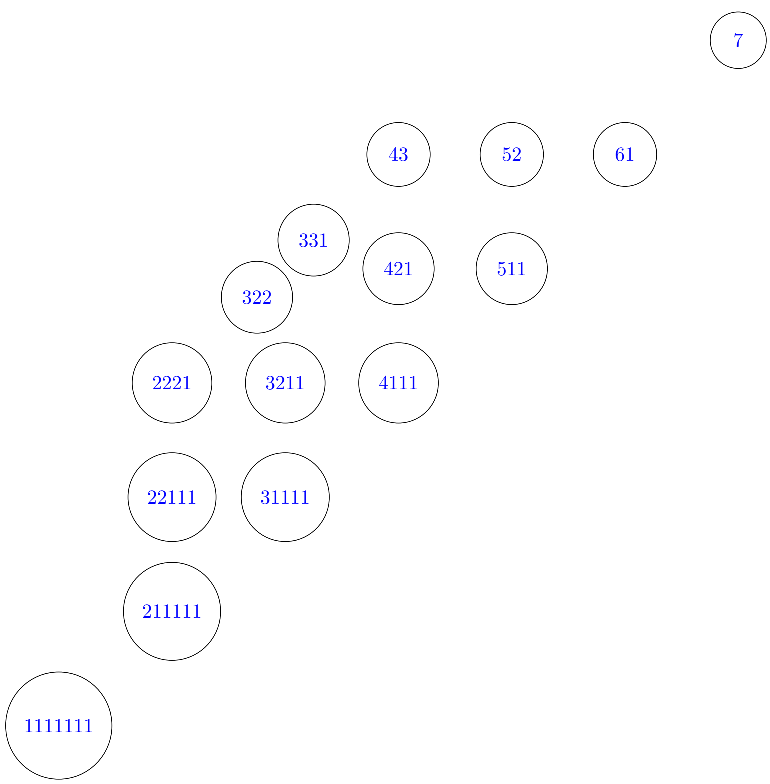
2 Known results

2.1 Partitions

Recall that a partition is a collection of parts of length ℓ with sum of the parts being the **size**.
In essence a partition $\lambda \in \mathbb{Z}^\ell / S_\ell$, as any permutation of a partition is the same.
As a unique representative from $\mathbb{Z}^\ell$, $\lambda = (\lambda_1, ..., \lambda_\ell)$ is chosen to be a weakly decreasing sequence.
This is unique, as the only permutations which keep the sequence decreasing are those which "swap" adjacent equal parts.

To find all the partitions with fixed size, there are multiple methods to do this, although it is difficult to ensure no repetition making counting partitions with fixed size a difficult problem.
As such there are many different ways to plot and connect partition. Below is one where the column and row is determined by the largest part, and the length. This highlights the two main properties under consideration, show the symmetry of a conjugate partition, and that some partitions like 311 and 322 are not uniquely determined by their length and largest part.
In a sense the location of each circle is the bounding box of a young-diagram.



2.2 Symmetric polynomials

A quick way to see that $\texttt{Sym}_n$ is closed under addition is to see that by taking permutations on some shared input to $f, g \in \texttt{Sym}_n$, the output is the same.
This means $f + g$ and $f * g$ are both fixed.

Recall the definition for the jth symmetric polynomial

$$e_j(x_1, ..., x_n) = \sum_{1 \leq i_1 < ... < i_j \leq n} x_{i_1} ... x_{i_j}$$

Most importantly $e_j \in \texttt{Sym}_n$ with degree j.

Theorem 2.1. *[The fundamental theorem of symmetric polynomials] Every $f \in \texttt{Sym}_n$ may be written uniquely as $\mathbb{Q}$-linear combination of finite products of $\{e_1, e_2, ..., e_n\}$.*

Proof. Consider $f = \sum_\lambda c_\lambda x[\lambda]$ with $x[\lambda] = \sum_{\sigma \in S_n} x_{\sigma(1)}^{\lambda_1} ... x_{\sigma(n)}^{\lambda_n}$.
Now consider the terms of largest degree $k = \deg(f)$, meaning $|\lambda| = k$. Consider λ with non-zero c_λ and largest parts.
By taking $f - c_\lambda \prod_{i=1}^k e_{\lambda'_i}$, then the c_λ term will go to 0.
This is because $\prod_{i=1}^k e_{\lambda'_i}$ This also won't effect any terms above, and so by repeating this process the terms in f of degree k go to 0.
Then by repeating this inductively on the degree k, f becomes constant □

2.3 Injectivity

2.3.1 d-ary partitions

Theorem 2.2. *If λ, μ are d-ary partitions with l parts s.t. $1 \leq j \leq l - 1$ and $\texttt{pre}_j(\lambda) = \texttt{pre}_j(\mu)$, then $\lambda = \mu$*

Proof. This proof follows from the work shown by Cimpoease and Tanase [1].

□

2.3.2 The $j = 2$ case

Theorem 2.3. *If λ, μ are partitions of n s.t. $\texttt{pre}_2(\lambda) = \texttt{pre}_2(\mu)$, then $\lambda = \mu$*

Proof. This proof follows from the work shown by Jiahui Li [2].

□

3 The Homogeneous Analog

From the unpublished preliminary reading [3], a few exercises and definitions were considered. Importantly

Definition 3.1. The pre partition function with repeats is given as

$$\texttt{prh}_j(\lambda) = \{\{\lambda_{i_1}...\lambda_{i_j} : 1 \leq i_1 \leq ... \leq i_j \leq n\}\}$$

Corollary 3.0.1. $\mathfrak{l}(\texttt{prh}_j(\lambda)) = \binom{n+j-1}{j}$

3.1 Injectivity

Proposition 3.1. $\texttt{prh}_j$ is injective for $j > 0$, meaning if $\texttt{prh}_j(\lambda) = \texttt{prh}_j(\mu)$, then $\lambda = \mu$

Proof. Let λ and μ be two arbitrary partitions such that $\texttt{prh}_j(\lambda) = \texttt{prh}_j(\mu)$.
Then by induction on i, it will be shown that $\lambda_i = \mu_i$

Base Case

For the base case $\lambda_1 = (\max(\texttt{prh}_j(\lambda)))^{\frac{1}{j}} = \mu_1$, which requires $j > 0$

Inductive step

Recall the inductive hypothesis that for $i < n : \lambda_i = \mu_i$, and consider the following notation.
Let the elementary degree j monomials be given by

$$e_{(x)}(\lambda) = e_{(x_1,...,x_j)}(\lambda) = \lambda_{x_1}\lambda_{x_2}...\lambda_{x_j} \text{ where } x_1 \leq x_2 \leq ... \leq x_j$$

Notice that the largest term of $\texttt{prh}_j(\lambda)$ with at least one unknown term is $\lambda_1^{j-1}\lambda_n$. This means

$$\lambda_1^{j-1}\lambda_n = \max(\{e_{(x)}(\lambda)\}\backslash\{e_{(x)}(\lambda) : x_j < n\}) = \max(\{e_{(x)}(\mu)\}\backslash\{e_{(x)}(\mu) : x_j < n\}) = \mu_1^{j-1}\mu_n$$

Importantly by the inductive hypothesis $\{e_{(x)}(\lambda) : x_j < n\} = \{e_{(x)}(\mu) : x_j < n\}$
In addition $\{e_{(x)}(\lambda)\} = \texttt{prh}_j(\lambda) = \texttt{prh}_j(\mu) = \{e_{(x)}(\mu)\}$

Overall, this means that by induction $\lambda = \mu$, and $\texttt{prh}_j$ is injective for $j > 0$.
In addition, since this proof contains functions to determine each λ_i, then this serves as a method for an algorithm to compute λ given $\texttt{prh}_j(\lambda)$ and j. □

3.2 Surjectivity

It may not be too surprising that $\texttt{prh}_j$ is not surjective as it "stretches" inputs often making them larger.
Notably, the output length must be $\binom{\mathfrak{l}(\lambda)+j-1}{j}$, and the largest term must be λ_1^j.
Other than this one can look at exactly where the algorithm fails, to see why a certain partition is not an output.
The following are some examples

$j = 2, \mu = (2)$	Reason: $2^{\frac{1}{2}}$ is not an integer
$j = 2, \mu = (1, 1, 1, 1, 1)$	Reason: Improper length
$j = 3, (27, 18, 8)$	Reason: Missing $12 = 3 * 2 * 2$
$j = 3, (27, 12, 4, 2)$	Reason: $\frac{12}{9}$ is not an integer

Interestingly, so of these do exists when the underlying number system is changed. The only problems that wouldn't fix are missing numbers, or improper length.
If the rationals are used then $j = 3, \mu = (27, 12, 4, \frac{64}{27})$ has inverse $(3, \frac{4}{3})$.
A very nice property is that by multiplying the output by some x^3, then the input scales by x, and this problem above is functionally the same as $\texttt{prh}_3((81, 4)) = (729, 324, 108, 64)$.

3.3 New perspectives and definitions

Definition 3.2. A **combination** $(\alpha_1, \alpha_2, ..., \alpha_s)$ is a possibly zero integer-sequence.
A strict combination is one with no zeroes.
Importantly a combination can be taken to a permutation by **sorting** which gives decreasing $(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_n})$ which in these notes removes zero.
Take the set of all compositions of length s be $\texttt{Comp}(s)$

Definition 3.3. Let the monomial given by a composition α be

$$x^\alpha = (x_1, x_2, ..., x_n)^{(\alpha_1, \alpha_2, ..., \alpha_n)} = x_1^{\alpha_1} x_2^{\alpha_2} * ... * x_n^{\alpha_n}$$

Remark. x^α Acts normally under distribution and exponentiation

Definition 3.4. $m_\lambda(x_1, ..., x_n) = \sum_{\alpha \in \texttt{Comp}(n) : \texttt{sort}(\alpha) = \lambda} e_\alpha$

Importantly, 2.1 uses the fact that m_λ must be a linear basis for $\texttt{Sym}_n$.
And the terms of m_λ which are given by $e_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} * ... * x_n^{\alpha_n}$ form the standard basis for $\mathbb{Z}[x_1, x_2, ..., x_n]$.

For the symmetric functions we study, e_α, forms the basis, and in fact exact terms of the output partition up to given coefficients.
This is done by taking $e_\alpha(\lambda_1, ..., \lambda_n) = e_\alpha(\lambda)$.
In addition, if some e_α is present, then $e_{\sigma(\alpha)}$ is present as well since $\texttt{sort}(\sigma(\alpha)) = \texttt{sort}(\alpha)$.

Now consider some arbitrary symmetric function $f(x_1, ..., x_n)$, which in the m_μ basis has coefficients $c_\mu \in \{0, 1\}$, and define

Definition 3.5 (Generic pre symmetric function)**.** For a symmetric function f

$$\texttt{prf}(\lambda) = \{\{c_\alpha \lambda^\alpha : \alpha \in \texttt{Comp}(n), c_\alpha \neq 0\}\} \text{ where } c_\alpha = [x^\alpha](f)$$

4 Ordering

4.1 Setup

To further interrogate $\texttt{prf}(\lambda)$, further structure must be put on the elements λ^α, or more precisely α.

Definition 4.1 (Partial ordering of α)**.** It is said that $\alpha > \beta$ iff $\alpha \neq \beta$ and $\forall \lambda : \lambda^{\hat{\alpha}} \geq \lambda^{\hat{\beta}}$ where $\hat{\alpha} = \frac{\alpha}{\texttt{tot}(\alpha)}$ and the length of λ must be greater than the length of α and β.

For α, β where $\alpha \not> \beta, \beta \not> \alpha, \alpha \neq \beta$, then $\alpha \approx \beta$ giving a full classification of comparisons?

Remark. Notice that not all compositions can be compared
as an example consider $\alpha = (3,3,0)$ and $\beta = (4,0,2)$.
Then for $\lambda = (2,2,1)$ it is the case that $\lambda^{(3,3,0)} \geq \lambda^{(4,0,2)}$ as $2^3 * 2^3 \geq 2^4 * 1^2$
But when $\lambda = (5,2,2)$ then $\lambda^{(3,3,0)} \not\geq \lambda^{(4,0,2)}$ as $5^3 * 2^3 \not\geq 5^4 * 2^2$.
This means $(3,3,0) \approx (4,0,2)$

Definition 4.2. Let $\mathfrak{C}_n$ be the graph with vertex set $\{\hat{\alpha} : \alpha \in \texttt{Comp}(n) \backslash 0^n\}$.
Then $e = \vec{\alpha\beta}$ is a directed edge in $\mathfrak{C}_n$ iff $\alpha > \beta$.

Definition 4.3 (Lattice $\mathbb{N}$-colouring)**.** A lattice $\mathbb{N}$-colouring of G, f, is a function from $\texttt{span}_{\mathbb{N}}(V(G)) \to \mathbb{R}$

- Distributivity: $f(x+y) = f(x) * f(y)$
- Respects ordering: $\forall \vec{\alpha\beta} \in E(G) : f(\alpha) \geq f(\beta)$

Theorem 4.1 (Equivalence of reconstruction)**.** *For* Λ, $\exists \lambda : \texttt{prf}(\lambda) = \Lambda$ *if and only if*

- *There exists a lattice colouring* L *over* $\mathfrak{C}_n$.
- *Agreement* $\{\{c_\alpha L(\alpha) : c_\alpha \neq 0, \alpha \in \texttt{Comp}(n)\}\} = \Lambda$ *where* $c_\alpha = [x^\alpha]f$

Proof. $(\Rightarrow)$
Suppose $\exists \lambda : \texttt{prf}(\lambda) = \Lambda$.
Then define $L(\alpha) = \lambda^\alpha$, which $= \lambda^{\hat{\alpha}}$ over $\mathfrak{C}_n$. Now notice that by construction

- L is distributive over $\mathfrak{C}_n$ as $L(\alpha+\beta) = \lambda^{\alpha+\beta} = \lambda^\alpha * \lambda^\beta = L(\alpha) * L(\beta)$
- For some $\vec{\alpha\beta} \in E(\mathfrak{C}_n)$, then $\alpha > \beta$ and $L(\alpha) \geq L(\beta)$.
- Lastly recall that by the definition of $\texttt{prf}(\lambda)$ it is exactly
  $\{\{c_\alpha \lambda^\alpha : \alpha \in \texttt{Comp}(n), c_\alpha \neq 0\}\} = \{\{c_\alpha L(\alpha) : c_\alpha \neq 0\}\}$, showing L agrees with Λ

This means L is a sufficient lattice colouring.

$(\Leftarrow)$
If there is a sufficient colouring L, then $\lambda = L(\{\alpha : \texttt{sort}(\alpha) = (1)\})$.
Then $\Lambda = \{\{c_\alpha L(\alpha) : c_\alpha \neq 0\}\}$ which as above is $\texttt{prf}(\lambda)$. ∎

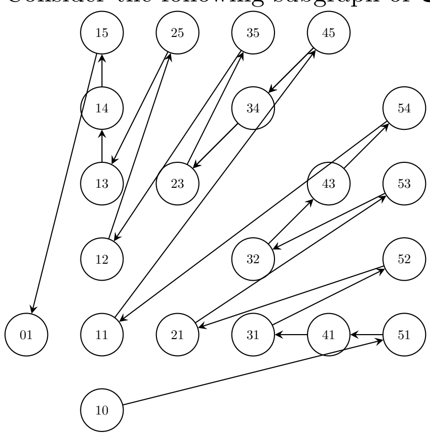
Corollary 4.1.1. L is unique iff λ is unique

Proposition 4.1. If an induced subgraph $\mathfrak{C}_n[V]$ has a unique lattice colouring and V forms a basis for $\texttt{Comp}(n)$, then there is a unique lattice colouring of $\mathfrak{C}_n[V]$

Remark. Notice that $\mathfrak{C}_n$ is an infinite graph, and it is dense in a hyperplane in the sense that $V(\hat{\mathfrak{C}}_n)$ is a subspace of $\mathbb{R}^n$.

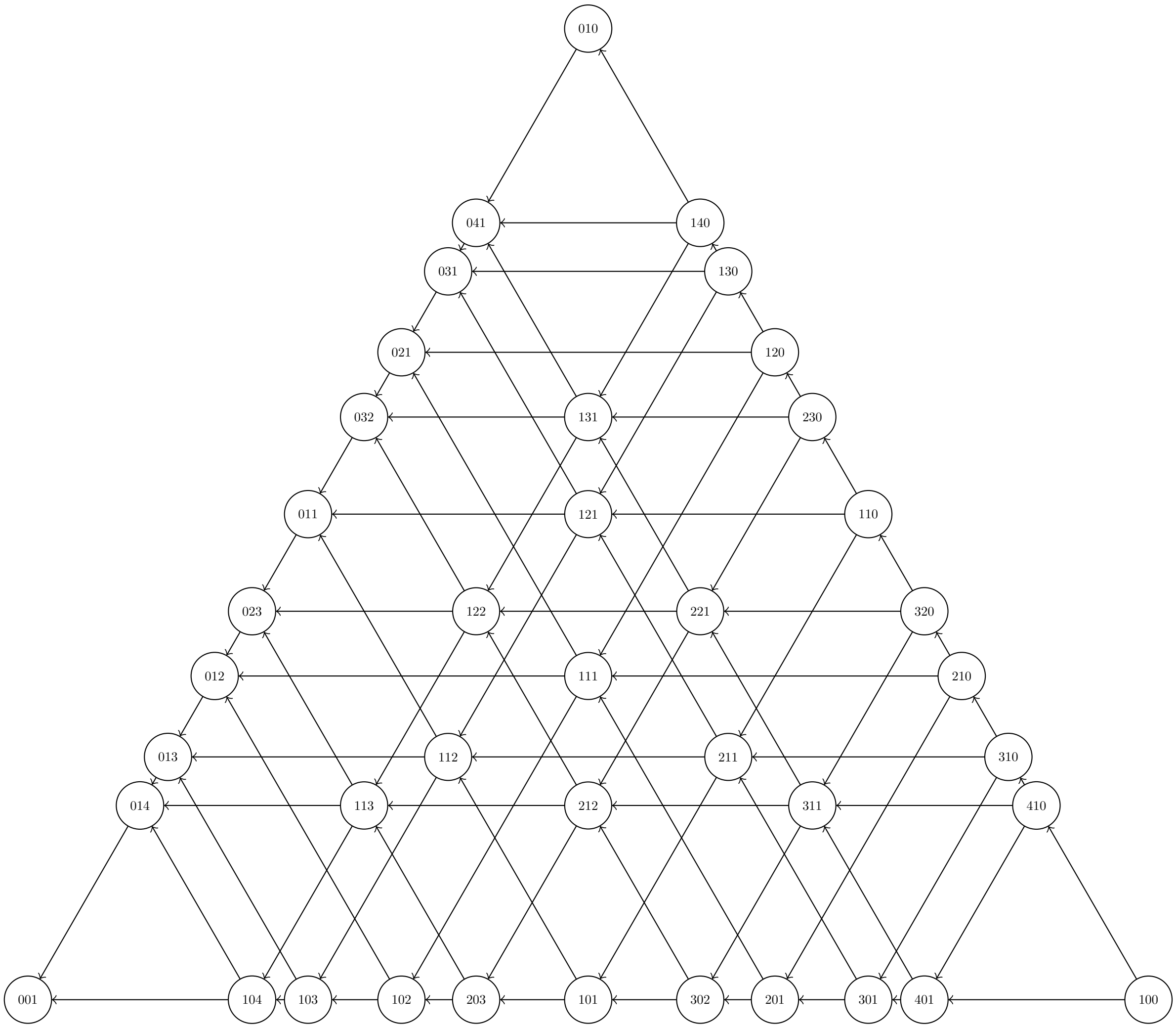
4.2 Examples

Consider the following subgraph of $\mathfrak{C}_2$ induced by vertices with $\texttt{tot}(\alpha) \leq 5$.



Notice that the entire $\mathfrak{C}_2$ looks like $\mathbb{Q} > 0$ with the standard ordering.
Interestingly this shows how the true infinite $\mathfrak{C}_n$ has no notion of next in the order, and between any two α, β, there is some γ.

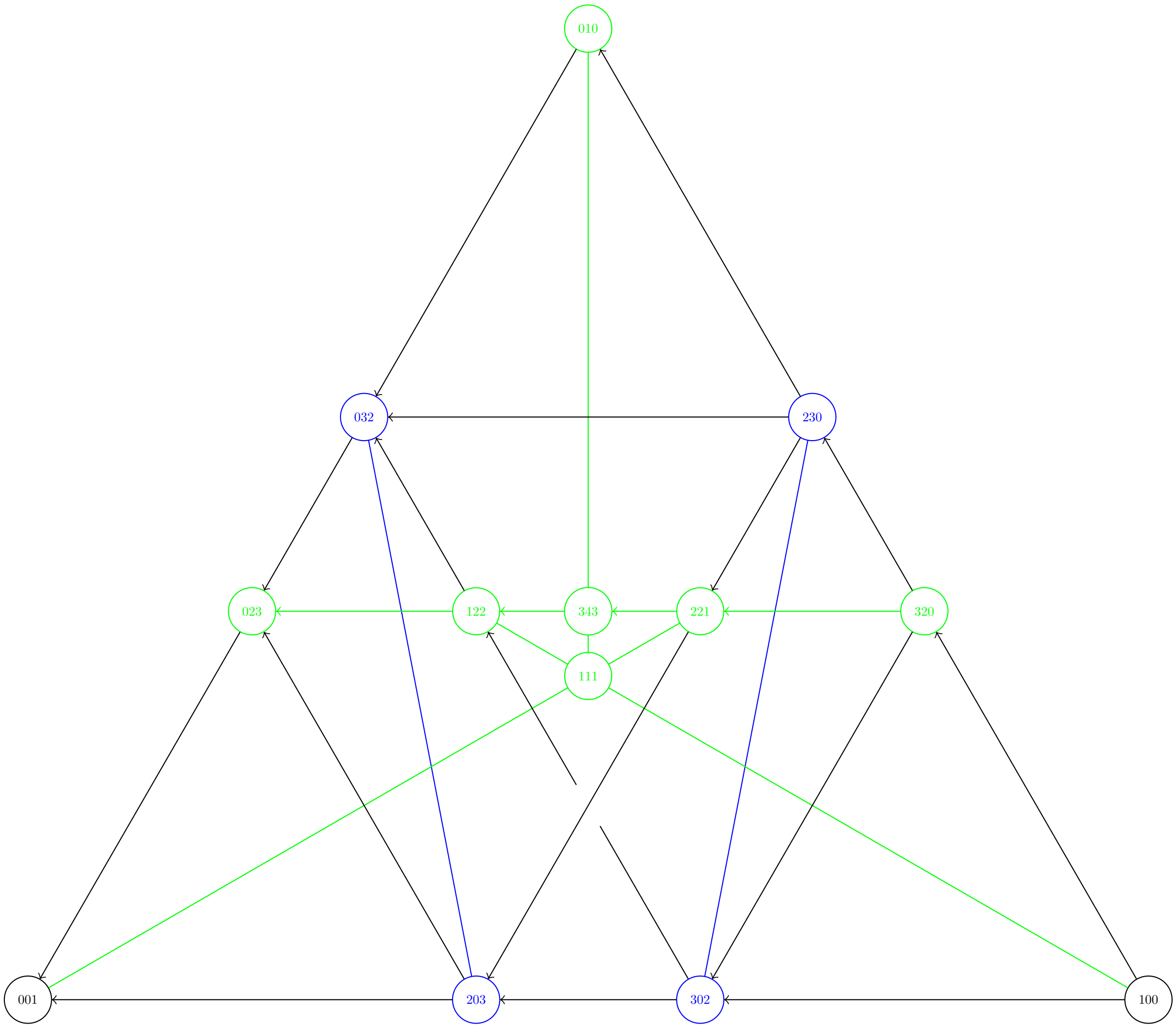
Now for another example, examine $\mathfrak{C}_3$ with vertices having size ≤ 5.



This graph has a lot of symmetries that can be exploited.

4.3 $\texttt{prm}_{(2,3)}$ is injective on most length 3 partitions

Here is a method to investigate $\texttt{prm}_{(2,3)}$.
It seems as though in the order there is a map between 230 and 302 meaning this is not injective.
Although it's not clear.



Notice that by a geometric construction λ is known.
In addition, it is clear that by changing μ around this geometric property will hold unless, the horizontal line passes through (111).

Remark. As note $\texttt{prm}_{(\alpha_1,\alpha_2)}$ is always injective for length 2 partitions except when $\alpha_1 = \alpha_2$

4.4 when $\texttt{prm}_{(\alpha_1,\alpha_2,\alpha_3)}$ is not injective

The green line passes through (111) exactly when $2y = x + z$.
In the special case where $x = y = z = 1$, then $\texttt{prm}_{(\alpha_1\alpha_2\alpha_3)}(\lambda) = \lambda_1\lambda_2\lambda_3$ and is clearly not injective.
Otherwise: by the ordering property $e_(\alpha)(\lambda_1, \lambda_2, \lambda_3)$ and $e_(\alpha)(\lambda_1, \lambda_2, \lambda_3)$ are known (and distinct).
In addition, but yet to be proved, $e_{(1,1,1)}(\lambda)$ is known and also distinct.
This means $e_{(2,1,0)}(\lambda)$ and $e_{(0,1,2)}(\lambda)$ can be computed.
Then the problem reduces to the case where $(\alpha_1, \alpha_2, \alpha_3) = (2, 1, 0)$.

The $\alpha = (2, 1, 0)$ Case

Label μ_1, μ_2, μ_6 where $\mu = \texttt{prm}_{(2,1)}(\lambda)$.

Then $\mu_3 = \frac{\mu_1\sqrt{\mu_1\mu_6}}{\mu_2}$
Further more $\mu_4 = \frac{\mu_2}{\mu_1\mu_6}$, and $\mu_5 = \frac{\mu_3}{\mu_1\mu_6}$.
To find λ it must be either $((\frac{\mu_1^2}{\mu_2})^{\frac{1}{3}}, (\frac{\mu_2^2}{\mu_1})^{\frac{1}{3}}, (\frac{\mu_1\mu_6^3}{\mu_2^2})^{\frac{1}{6}})$ or $((\frac{\mu_1\mu_2^2}{\mu_6})^{\frac{1}{6}}, (\frac{\mu_1^2\mu_6}{\mu_2^2})^{\frac{1}{3}}, (\frac{\mu_6\mu_2}{\mu_1})^{\frac{1}{3}})$, depending on the ambiguous ordering of μ_2 and μ_3.

To double check if one computes $\mathtt{prm}_{(2,1)}(\lambda)$ for the λs above it gives

	(210)	(120)	(201)	(102)	(021)	(012)
λ_1	μ_1	μ_2	$\frac{\mu_1\sqrt{\mu_1\mu_6}}{\mu_2}$	$\frac{\mu_1\mu_6}{\mu_2}$	$\frac{\mu_2\sqrt{\mu_1\mu_6}}{\mu_1}$	μ_6
λ_2	μ_1	$\frac{\mu_1\sqrt{\mu_1\mu_6}}{\mu_2}$	μ_2	$\frac{\mu_2\sqrt{\mu_1\mu_6}}{\mu_1}$	$\frac{\mu_1\mu_2}{\mu_6}$	μ_6

To ensure that these are integer solution one can take $\mu_1 = x^6, \mu_2 = y^6, \mu_6 = z^6$ for $x \geq y \geq z$ and $x, y^2, z \in \mathbb{Q}$ and $q = \frac{n}{\gcd(x^6y^2z, y^8z, x^3y^2z^4, x^3y^6, x^6z^3, y^6z^3)}$ for $n \in \mathbb{N}$.
This would give $\lambda_1 = q * (x^6y^2z, y^8z, x^3y^2z^4)$ and $\lambda_2 = q * (x^3y^6, x^6z^3, y^6z^3)$.
For some cases λ_1 may equal λ_2 in degenerate cases, but any case where μ is not injective the two λ's for which $\mathtt{prm}_{(2,1)}(\lambda_1) = \mathtt{prm}_{(2,1)}(\lambda_2)$ will take the form above.

Computing a direct example consider $x, y, z = 5, 3, 2$, then $q = 1$.
This gives $\lambda_1 = (281250, 18000, 13122)$ and $\lambda_2 = (125000, 91125, 5832)$.
$\mathtt{prm}_{(2,1)}(\lambda)$ would look as follows (with the ordering from the table)

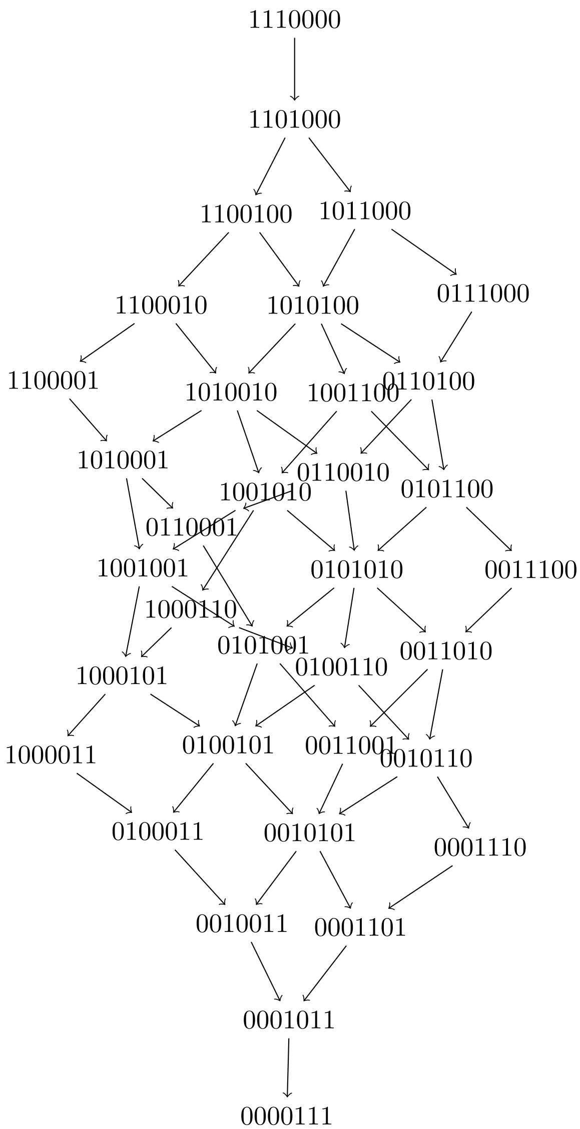
$$1423828125000000+91125000000000+1037970703125000+48427561125000+4251528000000+3099363912000$$

$$1423828125000000+1037970703125000+91125000000000+4251528000000+48427561125000+3099363912000$$

Which is exactly expected.
This shows that $\mathtt{prm}_{(2,1)}(281250, 18000, 13122) = \mathtt{prm}_{(2,1)}(125000, 91125, 5832)$

4.5 The $\texttt{pre}_3(\lambda)$ length 7 cases

Unfortunately the geometric interetation becomes difficult with higher length as in this case the nodes are the consecutive faces of a 7-simplex. A full graph and subgraph are shown below



Notice that as usual μ_1, μ_2 and μ_{34}, μ_{35} are known.
In addition this means (0001000) is known, and $(0010000), (0000100)$ are known.
So $(1100100), (0011100), (0010011)$ are known. Notice

$$f(1110000) = \mu_1 \qquad f(1101000) = \mu_2$$

$$f(1111111) = (\mu_1 ... \mu_{35})^{\frac{1}{15}}$$

$$f(0000111) = \mu_{35} \qquad f(0001011) = \mu_{34}$$

Then $f(aaxyzbb)$ is known

Case 1 (determinable)

If $\mu_3 \neq f(1100100)$, then $\mu_3 \in \{(1011000, 0111000)\}$ and is the larger of the two (1011000).

Then $f(1000000) = \frac{\mu_3}{f(0011000)}$. Then by removing $(*****00)$ from μ and taking the smallest, this will be (1100010) giving (0000010) and (0000001).

This means that for this case λ is unique.

Case 2

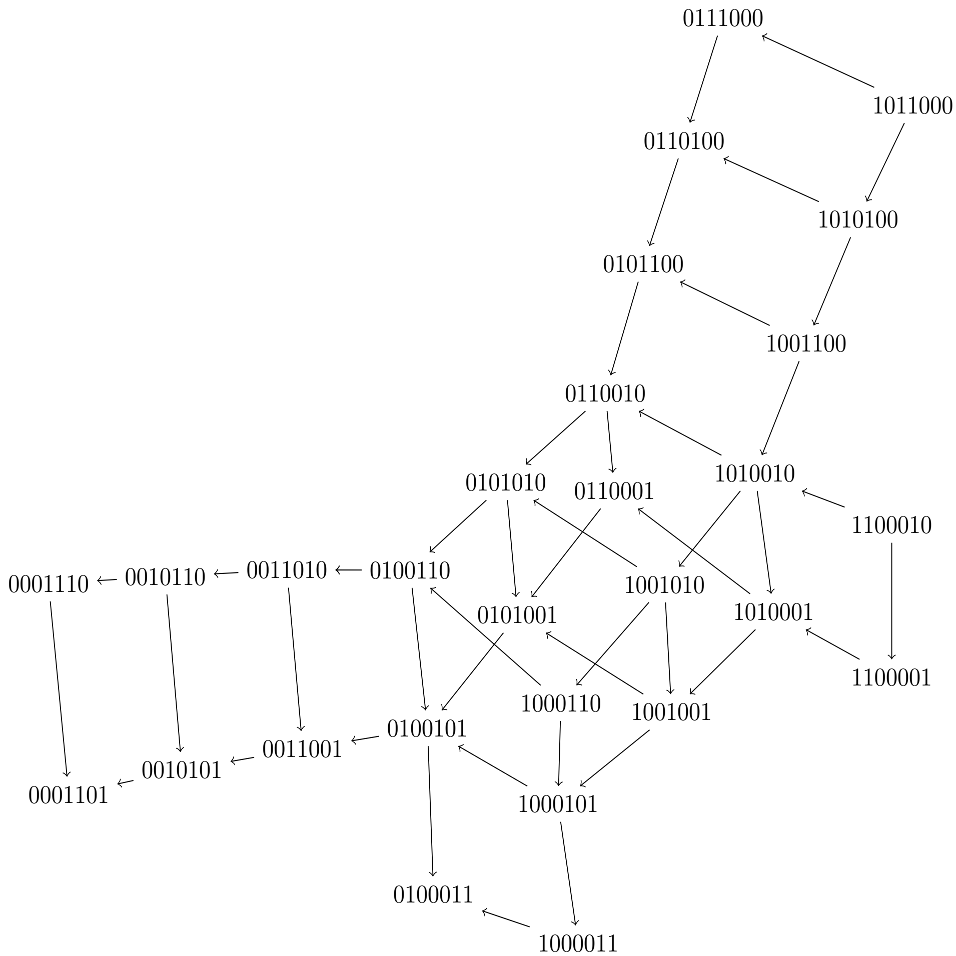
Otherwise if $\mu_{33} \neq f(0010011)$, then by the same reasoning as above λ is fixed.

Case 3

This means that $\mu_3 = f(1100100)$ and $\mu_{33} = f(0010011)$.

More importantly this means $f(0011000) < f(0100100)$ and $f(0010010) < f(0001100)$

By using these to reorder the graph, as well as removing nodes that are known this gives



Case 3.1

Suppose that $\mu_4 = 1011000$, then $f(1000000) = \frac{\mu_4}{f(0011000)}$.

Then since $f(*****00)$ are known, consider largest element in $\Lambda \backslash f(*****00)$, which can be used to figure out $f(0000010)$ and fixing λ.

Case 3.2

If instead $\mu_4 = 1100010$, then $\mu_4 > 1011000$ and $f(0100010) > f(0011000)$.

Notice that this fixes 0001101 as μ_{32}, this allows to solve for λ.

Letting $m_i = \mu_i^{\frac{1}{3}}$, then these give exact equations for λ which are as follows.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	Condition
Case 1	$\frac{m_3}{\lambda_3\lambda_4}$	$\frac{m_1\lambda_4}{m_3}$	$\frac{m_1 \ldots m_{35}}{m_2^{15} m_{35}^{15}}$	$\frac{m_1 \ldots m_{35}}{m_1^{15} m_{35}^{15}}$	$\frac{m_1 \ldots m_{35}}{m_1^{15} m_{34}^{15}}$	$q\frac{\lambda_3}{m_1}$	$q\frac{m_1 m_{35}}{\lambda_3\lambda_5}$	$m_3 \neq m_1(\frac{m_2 m_{35}}{m_1 m_{34}})^{15}, q = \max(\Lambda \backslash f(* * * * * \ 0 \ 0))$
Case 2	$\frac{q m_1 \lambda_3}{m_{35}\lambda_3}$	$\frac{q\lambda_5}{m_{35}}$	"	"	"			$m_{33} \neq m_{35}(\frac{m_1 m_{34}}{m_2 m_{35}})^{15}, q = \min(\Lambda \backslash f(0 \ 0 \ * * * * *))$
Case 3.1	$\frac{m_4}{\lambda_3\lambda_4}$	$\frac{m_1\lambda_4}{m_4}$	"	"	"	$q\frac{\lambda_3}{m_1}$	$q\frac{m_1 m_{35}}{\lambda_3\lambda_5}$	$q = \max(\Lambda \backslash f(* * * * * \ 0 \ 0))$
Case 3.2	$\frac{q m_1 \lambda_3}{m_{35}\lambda_3}$	$\frac{q\lambda_5}{m_{35}}$	"	"	"	$\frac{m_{35}\lambda_4}{m_{32}}$	$\frac{m_{32}}{\lambda_4\lambda_5}$	$q = \min(\Lambda \backslash f(0 \ 0 \ * * * * *))$

In theory by picking values for μ on at a time to force case 3. one would get two different λ which give the same μ.

5 Formalism

These examples above show a robust process for computing algebraic expressions for λ given case analysis.

This section hopes to provide a rigorous method to interrogate specific problems, and relationships between solutions.

References

[1] M. Cimpoeas and R. Tanase, “Remarks on d-ary partitions and an application to elementary symmetric partitions,” 2025. [Online]. Available: https://arxiv.org/abs/2506.04459

[2] S. J. Li, “A note on multiset reconstruction from sum and pairwise products,” 2025. [Online]. Available: https://arxiv.org/abs/2508.00971

[3] H. Niergarth, “Fall 2025 wim drp: Plugging partitions into symmetric functions,” 2025.