CS 524: Introduction to Optimization
Lecture 33 : Robust LP

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Objective perturbations

Consider a linear program with each linear constraint separately written out:

$$\max_x c^T x$$
$$\text{subject to: } a_i^T x \leq b_i, \quad \text{for } i = 1, \ldots, m$$

The study of perturbations of c and how it affects the optimal x is called sensitivity analysis and is covered for example in CS 525.
For simple ad-hoc tests, can just run multiple problems in a loop in GAMS.

Right hand side perturbations

Consider a linear program with each linear constraint separately written out:

$$\max_x c^T x$$
$$\text{subject to: } a_i^T x \leq b_i, \;\; \text{for } i = 1, \ldots, m$$

- If there is uncertainty in the right hand side b_i, we can optimize the worst case situation, namely $b_i = \min_{k \in \mathcal{K}} b_i^k$ and solve the resulting linear program.
- This will be feasible for all b_i in the set of alternatives $b_i^k, k \in \mathcal{K}$, a so-called robust solution.
- Harder case arises when perturbations in the vectors a_i

Example: robust LP

Consider a linear program with each linear constraint separately written out:

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & a_i^T x \le b_i \quad \text{for } i = 1, \dots, m \end{array}$$

Suppose there is **uncertainty** in some of the a_i vectors. Say for example that $a_i = \bar{a}_i + \rho u$ where $\bar{a}_i$ is a nominal value and u is the uncertainty.

- box constraint: $\|u\|_\infty \le 1$
- ball constraints: $\|u\|_2 \le 1$

Example: robust LP

Substituting $a_i = \bar{a}_i + \rho u$ into $a_i^T x \leq b_i$, obtain:

$$\bar{a}_i^T x + \rho u^T x \leq b_i \quad \text{for all uncertain } u$$

box constraint:

If this must hold for **all** u with $\|u\|_\infty \leq 1$, then it holds for the worst-case u. Therefore:

$$u^T x = \sum_{i=1}^{n} u_i x_i \leq \sum_{i=1}^{n} |u_i||x_i| \leq \sum_{i=1}^{n} |x_i|$$

Then we have

$$\bar{a}_i^T x + \rho \|x\|_1 \leq b_i$$

Robust LP with box constraint

With a box constraint $a_i = \bar{a}_i + \rho u$ with $\|u\|_\infty \leq 1$

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^\mathsf{T} x \\ \text{subject to:} & a_i^\mathsf{T} x \leq b_i \quad \text{for } i = 1, \ldots, m \end{array}$$

Is equivalent to the optimization problem

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^\mathsf{T} x \\ \text{subject to:} & \bar{a}_i^\mathsf{T} x + \rho \|x\|_1 \leq b_i \quad \text{for } i = 1, \ldots, m \end{array}$$

Robust LP with box constraint

With a box constraint $a_i = \bar{a}_i + \rho u$ with $\|u\|_\infty \leq 1$

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & c^T x \\ \text{subject to:} \quad & a_i^T x \leq b_i \quad \text{for } i = 1, \ldots, m \end{aligned}$$

... which is equivalent to the linear program:

$$\begin{aligned} \underset{x,t}{\text{maximize}} \quad & c^T x \\ \text{subject to:} \quad & \bar{a}_i^T x + \rho \sum_{j=1}^{n} t_j \leq b_i \quad \text{for } i = 1, \ldots, m \\ & -t_j \leq x_j \leq t_j \quad \text{for } j = 1, \ldots, n \end{aligned}$$

New case: Robust LP with ball constraint

Substituting $a_i = \bar{a}_i + \rho u$ into $a_i^\mathsf{T} x \le b_i$, obtain:

$$\bar{a}_i^\mathsf{T} x + \rho u^\mathsf{T} x \le b_i \quad \text{for all uncertain } u$$

ball constraint:

If this must hold for **all** u with $\|u\|_2 \le 1$, then it holds for the worst-case u. Using Cauchy-Schwarz inequality:

$$u^\mathsf{T} x \le \|u\|_2 \|x\|_2 \le \|x\|_2$$

Then we have

$$\bar{a}_i^\mathsf{T} x + \rho \|x\|_2 \le b_i$$

(a second-order cone constraint!)

Robust LP with ball constraint

With a ball constraint $a_i = \bar{a}_i + \rho u$ with $\|u\|_2 \leq 1$

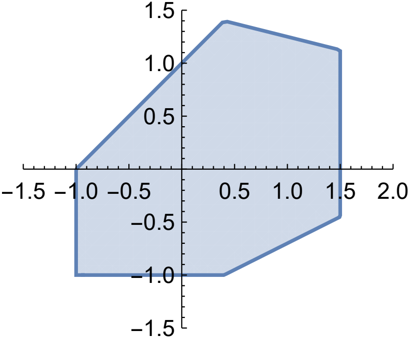
$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & a_i^T x \leq b_i \quad \text{for } i = 1, \ldots, m \end{array}$$

Is equivalent to the optimization problem

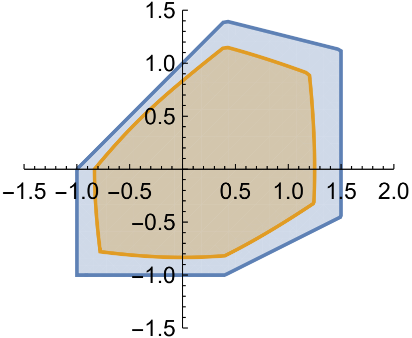
$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to:} & \bar{a}_i^T x + \rho \|x\|_2 \leq b_i \quad \text{for } i = 1, \ldots, m \end{array}$$

which is an SOCP

Ball constraint example



$a_i^T x \leq b_i$



$a_i^T x + 0.2\|x\|_2 \leq b_i$

- New region is smaller, no longer a polyhedron
- More robust to uncertain constraints

Example: `robustlp0.gms`

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \; & -x_1 \le 1 \\
& -x_1 + x_2 \le 1 \\
& x_1/3 + 2x_2 \le 3 \\
& x_1 \le 1.5 \\
& 0.5x_1 - x_2 \le 1.25 \\
& -x_2 \le 1
\end{aligned}
$$

Example: Cattle feeding

Mix a number of raw materials so that resulting cattle feed satisfies specified nutritive and other requirements at minimum cost for the raw materials. If the nutritive contents, unit costs of raw materials and the requirements for nutrients are known, the problem is:

$$\min_x \sum_f p_f x_f \text{ s.t. } \sum_f x_f = 1$$

and

$$\sum_f a_{r,f} x_f \geq d_r, \forall r$$

The percentage protein content and percentage fat content of the raw materials (barley, oats, sesame flakes, and groundnut meal) are given, with the required percentage content of protein and fat. Cost per ton of the four raw materials is also given. The problem is to determine a mix with minimum cost per ton that satisfies the nutritive requirements.

Instance: `chance.gms`

One problem that arises is that the nutritive content of the raw materials varies randomly, so that the solution given by linear programming using expected values, for instance, does not always satisfy the requirements. The chance constraint model differs in two respects.

1. The protein content of the four raw materials used for one batch of the mixture is constant but subject to variation for different batches of the mixture. The distribution of protein control of each raw material is normal and independent of the other raw materials, with mean equal to the values given in the LP and variance given separately.
2. The specification is made that the probability of achieving a protein content of 21 percent is at least 0.95.

Robust LP alternative

Here we suppose that the parameters a_i are independent Gaussian random vectors, with mean $\bar{a}_i$ and covariance Σ_i. We require that each constraint $a_i^T x \leq b_i$ should hold with a probability (confidence) exceeding $\eta \geq 0.5$, i.e.,

$$\text{Prob}(a_i^T x \leq b_i) \geq \eta$$

Letting $u = a_i^T x$, with σ denoting its variance, this constraint can be written as

$$\text{Prob}\left(\frac{u - \bar{u}}{\sqrt{\sigma}} \leq \frac{b_i - \bar{u}}{\sqrt{\sigma}}\right) \geq \eta$$

Since $(u - \bar{u})/\sqrt{\sigma}$ is a zero mean unit variance Gaussian variable, the probability above is simply $\Phi((b_i - \bar{u})/\sqrt{\sigma})$, where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp(-t^2/2) dt$$

is the CDF of a zero mean unit variance Gaussian random variable.

Conversion to SOCP

Thus the probability constraint above can be expressed as

$$\frac{b_i - \bar{u}}{\sqrt{\sigma}} \geq \Phi^{-1}(\eta)$$

or, equivalently,

$$\bar{u} + \Phi^{-1}(\eta)\sqrt{\sigma} \leq b_i$$

From $\bar{u} = \bar{a}_i^T x$ and $\sigma = x^T \Sigma_i x$ we obtain

$$\bar{a}_i^T x + \Phi^{-1}(\eta) \left\| \Sigma_i^{\frac{1}{2}} x \right\| \leq b_i$$

Now, provided $\eta \geq 1/2$ (i.e. $\Phi^{-1}(\eta) \geq 0$) this is a SOCP constraint. (Can look up $\Phi^{-1}(\eta)$ in a table or use the `stolib`!)