CS 524: Introduction to Optimization
Lecture 20 : Production Planning

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Production Planning

- PPP – A Production Planning Problem.
- An engineering plant can produce five types of products: $p_1, p_2, \ldots p_5$ by using two production processes: grinding and drilling. Each product requires the following number of hours of each process, and contributes the following amount (in hundreds of dollars) to the net total profit.

	p_1	p_2	p_3	p_4	p_5
Grinding	12	20	0	25	15
Drilling	10	8	16	0	0
Profit	55	60	35	40	20

PPP – More Info

- Each unit of each product take 20 manhours for final assembly.
- The factory has three grinding machines and two drilling machines.
- The factory works a six day week with two shifts of 8 hours/day. Eight workers are employed in assembly, each working one shift per day.

PPPorig - Linear Program

maximize

$$55x_1 + 60x_2 + 35x_3 + 40x_4 + 20x_5 \qquad \text{(Profit/week)}$$

subject to

$$\begin{aligned}
12x_1 + 20x_2 + 0x_3 + 25x_4 + 15x_5 &\leq 288 && \text{(Grinding)}\\
10x_1 + 8x_2 + 16x_3 + 0x_4 + 0x_5 &\leq 192 && \text{(Drilling)}\\
20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 &\leq 384 && \text{(Assembly)}\\
x_i &\geq 0 && \forall i = 1, 2, \ldots 5
\end{aligned}$$

Restrictions on PPP

- Suppose we wish to add the constraint that we wish to make at most two products.
- At most two of the five x_j can be positive.
- $z_j = \begin{cases} 1 & \text{make product } j \\ 0 & \text{otherwise} \end{cases}$
- $(x_j > 0 \Rightarrow z_j = 1)$!!
- Add constraints
    1. $x_j \leq M_j z_j \quad \forall j = 1, 2, \ldots 5.$
    2. $\sum_{j=1}^{5} z_j \leq 2.$

    - z_j is 1 if we use product j, $\sum_{j=1}^{5} z_j$ thus counts the number of products we use
    - Can then impose constraints "at most 2" as simple constraint on this sum!

PPP – Make no more than two model

maximize

$$55x_1 + 60x_2 + 35x_3 + 40x_4 + 20x_5 \qquad \text{(Profit/week)}$$

subject to

$$\begin{aligned}
12x_1 + 20x_2 + 0x_3 + 25x_4 + 15x_5 &\leq 288 \\
10x_1 + 8x_2 + 16x_3 + 0x_4 + 0x_5 &\leq 192 \\
20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 &\leq 384 \\
z_1 + z_2 + z_3 + z_4 + z_5 &\leq 2 \\
x_i &\leq M_i z_i \qquad \forall i = 1, 2, \ldots 5 \\
x_i &\geq 0 \qquad \forall i = 1, 2, \ldots 5 \\
z_i &\in \{0, 1\} \forall i = 1, 2, \ldots 5
\end{aligned}$$

Don't make both product 1 and 2

- Add constraint $z_1 + z_2 \leq 1$

Turn on constraint

- We construct the opposite implication that a "turned-on" indicator variable δ implies a constraint is satisfied.
- We continue to assume an upper bound on $g(x) \leq \mathcal{M}$ for all $x \in X$
- Model:

$$\delta = 1 \Rightarrow g(x) \leq 0$$

  by the inequality:

$$g(x) \leq \mathcal{M}(1 - \delta)$$

- Note that if $\delta = 1$ then $g(x) \leq 0$ is clear, and otherwise $\delta = 0$ so $g(x) \leq \mathcal{M}$ imposes no additional restriction on the problem.
- This is just a rewriting of the contrapositive implication

A Different Logical Condition: Variable Lower Bounds

Model "At least" lower bounds

$$\delta = 1 \Rightarrow x \geq m$$

- Assume that $x \geq 0$ is a given condition
- Model above using "Turn on constraint" with $g(x) = m - x$
- Then $g(x) \leq \mathcal{M}$ becomes $m - \mathcal{M} \leq x$ which is always true if $\mathcal{M} = m$ so $g(x) \leq \mathcal{M}(1 - \delta)$ is just $x \geq m\delta$
  - $\delta = 0 \Rightarrow x \geq 0$ (nothing new)
  - $\delta = 1 \Rightarrow x \geq m$
- GAMS also has `semicont` variables that is an alternative way to model this condition

Example: extended PPP model

- Suppose in PPP that the managers didn't want to make any "small quantities" of product.
  - (Producing 0.01 of product 1 makes little economic sense).
- If you produce product j, produce at least 8 of product j

$$x_j > 0 \Rightarrow x_j \geq 8$$

- Replace by two implications:
- $x_j > 0 \Rightarrow z_j = 1$
- $z_j = 1 \Rightarrow x_j \geq 8$
- Second relationship given by:

$$8 - x_j \leq 8(1 - z_j) \iff x_j \geq 8z_j$$

PPPalt – Produce at least 8

maximize

$$55x_1 + 60x_2 + 35x_3 + 40x_4 + 20x_5$$

subject to

$$\begin{aligned}
12x_1 + 20x_2 + 0x_3 + 25x_4 + 15x_5 &\leq 288 \\
10x_1 + 8x_2 + 16x_3 + 0x_4 + 0x_5 &\leq 192 \\
20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 &\leq 384 \\
x_i &\leq M_i z_i \qquad \forall i = 1, 2, \ldots 5 \\
x_i &\geq 8z_i \qquad \forall i = 1, 2, \ldots 5 \\
x_i &\geq 0 \qquad \forall i = 1, 2, \ldots 5 \\
z_i &\in \{0, 1\} \forall i = 1, 2, \ldots 5
\end{aligned}$$

What about the M's?

- Can we get a bound for M_i by looking at the problem?
- $12x_1 + 20x_2 + 0x_3 + 25x_4 + 15x_5 \leq 288$
- $12x_1 \leq 288 - 20x_2 - 0x_3 - 25x_4 - 15x_5 \leq 288$
- So $x_1 \leq 288/12$
- You can do this for any variable x_j for which $a_j > 0$ and any constraint $a^T x \leq b$ (and other variables have upper and or lower bounds):

$$a_j x_j \leq b - \sum_{k \neq j \mid a_k > 0} a_k \ell_k - \sum_{k \neq j \mid a_k < 0} a_k u_k$$

- Note:

$$\sum_{k \neq j \mid a_k > 0} a_k \ell_k + \sum_{k \neq j \mid a_k < 0} a_k u_k$$

is the *smallest* value that the left hand side can take (excluding variable x_j).

Either-Or Constraints

- In modeling sometimes have situation in which either one constraint holds, or another holds (possibly both). Mathematically:

$$\textit{either } f(x) \leq 0 \textit{ or } g(x) \leq 0.$$

$f(x) > 0 \Rightarrow \delta = 1$

- This is equivalent to $f(x) > 0 \Rightarrow g(x) \leq 0$.

$\delta = 1 \Rightarrow g(x) \leq 0$

- Given a "big $\mathcal{M}$" upper bound on both f and g, can model this by introducing an indicator variable δ with $f(x) > 0 \Rightarrow \delta = 1$ and $\delta = 1 \Rightarrow g(x) \leq 0$. Using "Turn on indicator/constraint" this is enforced by:

$$\underline{f(x) \leq \mathcal{M}_1 \delta}, \quad \underline{g(x) \leq \mathcal{M}_2(1 - \delta)}.$$

- The variable δ indicates whether the first inequality holds ($\delta = 0$) or the second inequality holds ($\delta = 1$)

Example: minimum production levels

- Dorian Auto is considering manufacturing three types of autos: compact, midsize and large. The resources required for, and the profits yielded by, each type of car are shown below. At present 6,000 tons of steel and 60,000 hours of labor are avaiable. In order for production of a type of car to be economically feasible, at least 1,000 cars of that type must be produced.
- Either: produce 0 cars, Or: produce at least 1000 cars
- Never build more than 3000 cars (so $\mathcal{M}_1 = 3000$).
- Have $x = 0$ or $x \geq 1000$. Set $f(x) = x$ and $g(x) = 1000 - x$ to get formulation

either–or $\Longrightarrow$

$$x \leq \mathcal{M}_1\delta, \;\; 1000 - x \leq \mathcal{M}_2(1 - \delta)$$

Since $x \geq 0$, we have $\delta = 0 \Rightarrow x = 0$ and $\delta = 1 \Rightarrow x \geq 1000$.

- Example: `dorian.gms`, could also use `semiint` variables

Disjunctive Constraints

More generally, suppose we have N alternative constraints $f_1(x) \leq 0, \ldots f_N(x) \leq 0$ and want at least one of them to be satisfied—a situation known as *disjunctive constraints*. Enforce this by introducing one binary variable δ_i for each constraint and writing

$$f_i(x) \leq \mathcal{M}_i(1 - \delta_i), \quad i = 1, 2, \ldots, N, \quad \sum_{i=1}^{N} \delta_i = 1.$$

$\delta_i = 1$ turns on the ith constraint (see also sos1 variables). Note: A constraint may still be satisfied even if $\delta_i = 0$. Note that the special case $N = 2$ reduces to the earlier case above (just eliminate $\delta_1 = 1 - \delta_2$). We can enforce "at least m" constraints by replacing the summation constraint with $\sum_{i=1}^{N} \delta_i \geq m$.

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