CS 524: Introduction to Optimization
Lecture 8 : Minimax problems

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A New Case: Demand Uncertainty

- Let's suppose now that we don't know demand with certainty.
- We have a set S of scenarios for the demand: d_{ts} demand for shoes in period t under scenario s
- We must decide on production and hiring decision now
- Inventory and backlogging is allowed to vary with scenario
- Can we give a model that will minimize the maximum cost we pay under any scenario

Cost Structure

$$
\begin{aligned}
c_1 &= \sum_{t \in T} (\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota L_{t1} + \theta S_{t1}) \\
c_2 &= \sum_{t \in T} (\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota L_{t2} + \theta S_{t2}) \\
c_3 &= \sum_{t \in T} (\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota L_{t3} + \theta S_{t3})
\end{aligned}
$$

$$
\min \max\{c_1, c_2, c_3\}
$$

Minimax

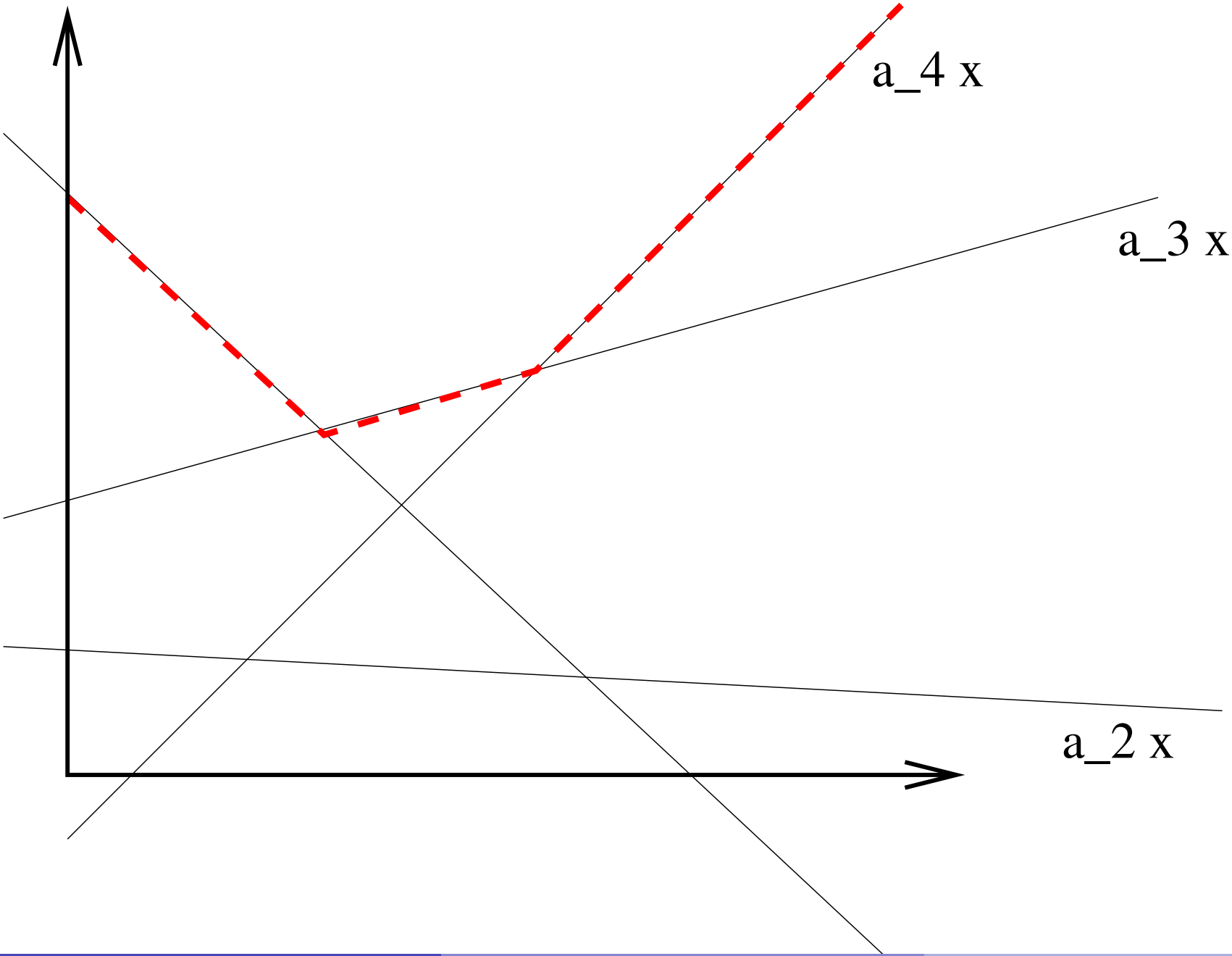
- Minimax: Minimize the maximum of a collection of linear functions.

$$\min_{x \in X} f(x) \stackrel{\text{def}}{=} (\max_i \sum_{j \in N} a_{ij} x_j)$$

It's All About The Shape

- If $f(x)$ is convex, then minimizing $f(x)$ is easy, and we can look for a "trick" to model this is a linear program
- (Maximizing Concave functions is also easy)

Minimax

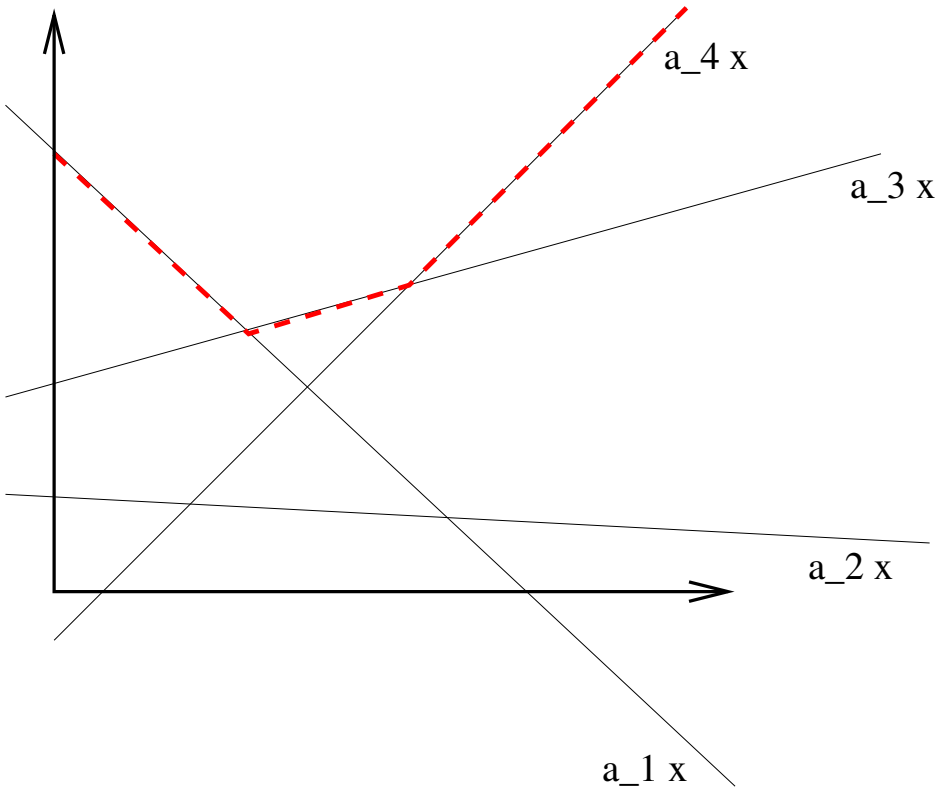


Minimax/Maximin

Key Point

- You *CANNOT* just maximize each of the individual functions and then just take the minimum!

$$\min f(x) = \max\{3 - 2x, 2x, 1 + x\}, \text{subject to } 0 \leq x \leq 3$$



$f_i(x)$	max	arg max
$3 - 2x$	3	0
$2x$	6	3
$1 + x$	4	3

True Min: $x^* = 2/3, f(x) = 5/3$

Modeling Minimax

Key Equivalences!

$$z \leq \min_{i \in M} a_i^T x \Leftrightarrow z \leq a_i^T x \ \forall i \in M$$

$$z \geq \max_{i \in M} a_i^T x \Leftrightarrow z \geq a_i^T x \ \forall i \in M$$

- $\min_{x \in X} f(x) \stackrel{\text{def}}{=} \max_{i \in M} a_i^T x$
- **Step 1:** $\min_{x \in X, z} z$ s.t. $z \geq f(x)$
- **Step 2:** $\min_{x \in X, z} z$ s.t. $z \geq \sum_{j \in N} a_{ij} x_j \qquad \forall i \in M$
- Special case: $\min_{x \in X} f(x) = |x|$; $\min_{x \in X, z} z$ s.t. $z \geq x, z \geq -x$

Trick: Modeling Absolute Value – Constraint

Modeling $|\cdot|$ in Constraints

We would like to model $|a^T x| \leq b$

The Trick

- Note that $|a^T x| = \max\{a^T x, -a^T x\}$
- Replace $|a^T x| \leq b$ with two inequalities

$$a^T x \leq b \quad \text{and} \quad -a^T x \leq b$$

- Works for any finite maximum of linear functions (see later)

Maximin

- Maximin : $\max_{x \in X} f(x) \stackrel{\text{def}}{=} \min_i \sum_{j \in N} a_{ij} x_j$
- The minimum of linear functions is a concave function, and maximizing such functions is easy: There is a similar LP "trick" we can do:

$$\max_{x \in X, z} z, \quad z \leq \sum_{j \in N} a_{ij} x_j \quad \forall i$$

ShoeCo Minimax: see `shoeco3.gms`

$$\min z \quad \text{s.t.}$$

$$z \geq \sum_{t \in T} \left(\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota L_{ts} + \theta S_{ts} \right), \quad \forall s \in S$$

$$\begin{aligned}
a x_t &\leq H w_t + o_t \quad \forall t \in T \\
o_t &\leq O w_t \quad \forall t \in T \\
I_{ts} &= I_{t-1,s} + x_t - d_{ts} \quad \forall t \in T, \forall s \in S \\
I_{ts} &= L_{ts} - S_{ts} \quad \forall t \in T, \forall s \in S, \quad I_{0,s} = \mathcal{I}_0 \\
w_t &= w_{t-1} + h_t - f_t \quad \forall t \in T, \quad w_0 = \mathcal{W}_0 \\
x_t, w_t, h_t, f_t, &\geq 0 \quad \forall t \in T \\
S_{ts}, L_{ts} &\geq 0 \quad \forall t \in T, s \in S
\end{aligned}$$

Save and restart

- GAMS allows the state of a model to be saved to a file for later use: `gams shoeco s=shoe`
- This "saves" the state of the shoeco model into a file `shoe.g00`. Note command line argument "s=shoe" can be put into the box that is to the right of the "run" button in studio
- You can then "restart" with new command (just as if you had typed them in at the end of `shoeco.gms`) using a new file `shoeco3.gms`: `gams shoeco3 r=shoe`
- Note also the files `shoerun.gms` (that runs all three models) and `shoecoI.gms` (that incorporates initial conditions in a different way)

GAMS – Pretty Display

- "The Display Statement" in GUG.
- `option decimals = 1;`
- `option ident:d-value:r-value:c-value ;`
  - Displays `ident` with `d-value` decimals, putting `r-value` rows together and `c-value` columns together
- Putting `r-value` $= 0$, produces a list format for display
- See file `display.gms`