CS 524: Introduction to Optimization
Lecture 13 : Max flow/min cut

Michael Ferris

Computer Sciences Department
University of Wisconsin-Madison

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Some network models

- Introduced the minimum cost network flow (MCNF) problem and special cases: assignment and shortest path problems
- What is the dual of a shortest path problem
- Then introduce the max flow problem to ship as much as possible from a source to a sink
- What is the dual of a max flow problem
- Duality theory: max flow $=$ min cut
- Note: separation of data, model and post-processing

$$\min_{x_{ij} \geq 0} \sum_{(i,j)} c_{ij} x_{ij}$$

st. $\underbrace{\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji}}$ $= b_i$ $\quad \forall i \in N \quad \perp \; \pi_i$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$\forall (i,j) \in A$. $\quad 0 = 1^T A x = 1^T b$

$$\begin{array}{l} i\text{th} \\ j\text{th} \end{array} \begin{bmatrix} & 1 & \\ & -1 & \end{bmatrix} x = \begin{bmatrix} (x_{ij}) \end{bmatrix}$$

x_{ij}

$$\boxed{\lambda^T A = 0}$$

$$(1 \; \cdots \; 1) A = 0 \qquad \neq 0 \qquad = \sum_i b_i$$

$$\boxed{\begin{array}{l} \max \sum_{i \in N} b_i \pi_i \\ \text{st.} \quad \boxed{\pi_i - \pi_j \leq c_{ij}} \\ \pi_i \leq c_{ij} + \pi \qquad \pi_i \text{ free.} \end{array}}$$

$= \pi_1 - \pi_8$

$\perp \; \underline{x_{ij}}$

Dual of min cost flow (short1.gms)

$$(P): \min \sum_{(i,j)\in\mathcal{A}} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{k:(i,k)\in\mathcal{A}} x_{ik} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji} = b_i, \;\; x_{ij} \geq 0$$

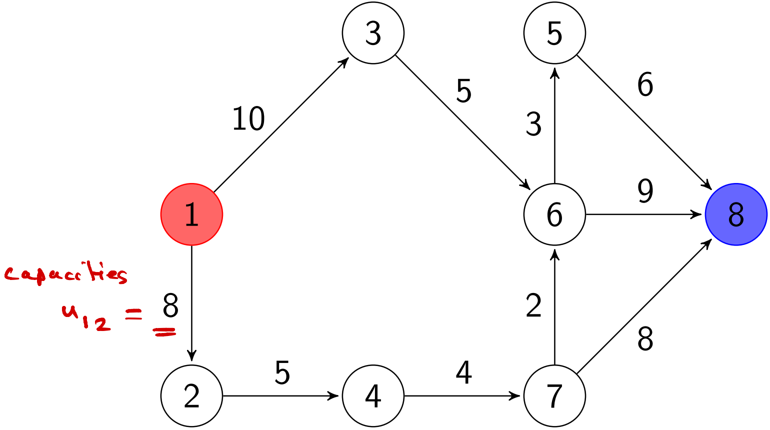
$$(D): \max \sum_i b_i\pi_i$$

$$\text{s.t.} \;\; \pi_i - \pi_j \leq c_{ij}, \quad \forall (i,j) \in \mathcal{A}$$

- Interpretation for SPP data: π_i is the minimum path length from i to sink.
- Can choose one of the distance labels arbitrarily (set sink distance to 0).

Maximum Flow from source to sink (`simpmaxflow.gms`)

- Data: nodes, arcs (obvious), source, sink
- u_{ij} are the arc capacities (not costs)



Max Flow

Maximum Flow Problem

Given a network capacitated $G = (N, A)$, with capacities $u \in \mathbb{R}_+^{|A|}$, a source node $s \in N$, and a sink node $t \in V$, what is the maximum flow that can be sent from s to t.

- Model "what goes in = what come out".

$$\sum_{k:(i,k)\in A} x_{ik} - \sum_{j:(j,i)\in A} x_{ji} = 0$$

- This constraint holds for all nodes except the source and sink: $(\forall i \in N \setminus \{s, t\})$

Max Flow Problem

$$\max \sum_{j:(s,j)\in A} x_{sj}$$

$$\begin{aligned}
\text{s.t.} \sum_{k:(i,k)\in A} x_{ik} - \sum_{j:(j,i)\in A} x_{ji} &= 0 \qquad \forall i \in N \setminus \{s,t\} \\
x_{ij} &\leq u_{ij} \qquad \forall (i,j) \in A \\
x_{ij} &\geq 0 \qquad \forall (i,j) \in A
\end{aligned}$$

- Note that due to flow balance, objective is the same as

$$\sum_{j:(j,t)\in A} x_{jt}$$

- Constraint (node-arc incidence) matrix of max flow problem is totally unimodular

Max flow (s and t represent source and sink nodes)

$$(P): \max_{x_{ij}: x_{ij} \in [0, u_{ij}]} \sum_{(s,j) \in \mathcal{A}} x_{sj}$$

$$\text{s.t.} \sum_{j:(i,j) \in \mathcal{A}} x_{ij} - \sum_{j:(j,i) \in \mathcal{A}} x_{ji} = 0, \; \forall i \in \mathcal{N} \setminus \{s, t\}$$

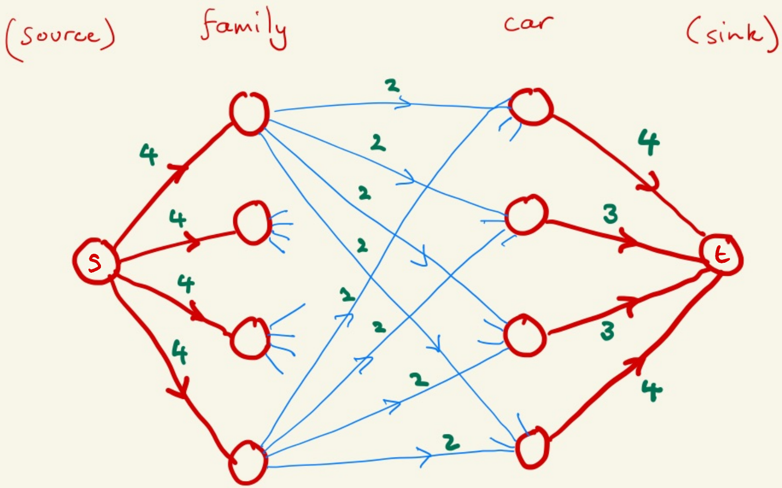
- Rewrite problem using variables f_p where $p \in \mathcal{P}$ represents an enumeration of paths from s to t. (see `simpmaxflow.gms`)

$$(P): \max_{f_p: f_p \geq 0} \sum_{p \in \mathcal{P}} f_p$$

$$\text{s.t.} \sum_{p \in \mathcal{P}:(i,j) \in p} f_p \leq u_{ij}, \; \forall (i,j) \in \mathcal{A}$$

Let's Have a Picnic! (see `picnic.gms`)

- The Hatfields, Montagues, McCoys and Capulets are going on their annual family picnic.
- Four cars are available to transport the families to the picnic.
- The cars can carry the following numbers of people: car 1, 4; car 2, 3; car 3, 3; car 4, 4.
- There are four people in each family, and no car can carry more than two people from any one family.
- Determine the maximum number of people that can be transported to the picnic.



picnic!

General form duality

Min problem	Max problem
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

$$\underline{\max} \quad \begin{array}{c} (s,f_1)(s,f_2) \\ \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \end{array} \underset{\sim}{x}$$

$$\begin{array}{cc} \phi_i \perp & \begin{bmatrix} 1 & 1 & & 1 & & & (i,t) \\ -1 & & & & & & \\ & -1 & & -1 & & & \\ & & & & & & 1 \\ & & & & & 1 & \\ & & & & -1 & -1 & -1 \end{bmatrix} x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \forall i \notin \{s,t\} \\ \pi_{ij} \perp & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} x_{ij} \leq \begin{bmatrix} u_{ij} \\ \\ \\ \end{bmatrix} \end{array}$$

$$\boxed{x_{ij} \geq 0}$$

$$\underline{\min} \quad \sum_{(i,j)\in A} u_{ij} \pi_{ij}$$

$$\phi_i \text{ free}$$

$$\boxed{\pi_{ij} \geq 0}$$

$$\rightarrow -\phi_j + \pi_{sj}$$

$$\rightarrow -\phi_j + \phi_i + \pi_{ij}$$

$$\rightarrow \phi_i + \pi_{it}$$

$$\begin{matrix} \geq \\ \\ \geq \end{matrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} i = s \\ i \neq s,t \\ i = t \end{matrix}$$

Standard dual of max flow (picnicdual.gms)

$$
\begin{aligned}
(P):\ \max_{x_{ij}:x_{ij}\geq 0} \quad & \sum_{(s,j)\in\mathcal{A}} x_{sj} \\
\text{s.t.} \quad & \sum_{k:(i,k)\in\mathcal{A}} x_{ik} - \sum_{j:(j,i)\in\mathcal{A}} x_{ji} = 0,\ \forall i \in \mathcal{N} \setminus \{s,t\} \\
& x_{ij} \leq u_{ij},\ \forall (i,j) \in \mathcal{A}
\end{aligned}
$$

$$
\begin{aligned}
(D):\ \min_{\phi_i,\pi_{ij}:\pi_{ij}\geq 0} \quad & \sum_{(i,j)\in\mathcal{A}} u_{ij}\pi_{ij} \\
\text{s.t.} \quad & \underline{-\phi_j + \pi_{sj} \geq 1},\ \forall j : (s,j) \in \mathcal{A} \\
& \underline{\phi_i - \phi_j + \pi_{ij} \geq 0},\ \forall (i,j) \in \mathcal{A}, i \neq s, j \neq t \\
& \phi_i + \pi_{it} \geq 0,\ \forall i : (i,t) \in \mathcal{A}
\end{aligned}
$$

Dual of max flow (picnicpath.gms)

$$
\begin{aligned}
(D): \min_{\pi_{ij}:\pi_{ij}\geq 0} \quad & \sum_{(i,j)\in\mathcal{A}} u_{ij}\pi_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in p} \pi_{ij} \geq 1, \ \forall p \in \mathcal{P}
\end{aligned}
$$

- $\mathcal{P}$ can be "labelled" using (family,car) tuples
- π_{ij} are weights on the arcs
- Along each $s-t$ path, weights sum to at least 1
- Identifies a set of arcs (with positive values) whose removal disconnects s from t - a cut
- Duality gives max flow = min cut capacity

Dual of max flow (picnicpath.gms)

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