CS 524: Introduction to Optimization
Lecture 12 : Duality

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October 2, 2023

Guessing Game (`lpduality.gms`)

- Let's try to guess bounds on the optimal solution of

$$\begin{aligned} \min \quad & 5x_1 + 4x_2 + 6x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + x_3 \geq 12 \\ & 2x_1 + x_2 - x_3 \geq 16 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$\pi_1 \geq 0$

$\pi_2 \geq 0$

- The value of any feasible solution provides an upper bound on z^*. $x = (10, 0, 2)$ is feasible $\Rightarrow 5(10) + 4(0) + 6(2) = 62$ is an upper bound. $z^* \leq 62$
- What about a lower bound?

$z^* \geq ?????$

$2\times$ $x_1 + 2x_2 + x_3 \geq 12$

$2x_1 + 4x_2 + 2x_3 \geq 24$

Key Idea

- Use the constraints!

- For example, my multiplying the first constraint by 2, I can show that $z^* \geq 24$

$$z^* = 5x_1 + 4x_2 + 6x_3 \geq 2x_1 + 4x_2 + 2x_3 \geq 2(12) = 24$$

- Can we do better? Take $1 \times$ (First constraint) and $2 \times$ (second constraint) to get an implied inequality: $z^* \geq 44$

$$z^* = 5x_1 + 4x_2 + 6x_3 \geq 5x_1 + 4x_2 - x_3 \geq \boxed{1(12) + 2(16)} = 44$$

Big.

Find the best "multipliers" (π_1, π_2)

$$\begin{aligned} \max \quad & 12\pi_1 + 16\pi_2 \\ \text{s.t.} \quad & \pi_1 + 2\pi_2 \leq 5 \\ & 2\pi_1 + \pi_2 \leq 4 \\ & \pi_1 - \pi_2 \leq 6 \\ & \pi_1, \pi_2 \geq 0 \end{aligned}$$

rows

— x_1

— x_2

— x_3

Woah Nellie!

- Finding the best multipliers is itself a linear program.
- This linear program is called the dual of the original (now called primal) linear program

General Forms

Primal LP

$$z^* = \min c^T x \text{ s.t. } Ax \geq b, x \geq 0$$

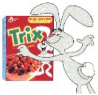
Dual LP

$$w^* = \max b^T \pi \text{ s.t. } A^T \pi \leq c, \pi \geq 0$$

GAMS: "Marginals" or Lagrange multipliers.

- The ".m" field for a primal equation is the dual variable value ($\pi.l$) of that equation.

Some Trix You Should Know



- You can always convert problem to canonical form
- "Reverse" constraints can be multiplied by -1
- You can convert from max objective to min objective by multiplying the objective function by -1

$$Ax = b \equiv \begin{bmatrix} Ax \geq b \\ -Ax \geq -b \end{bmatrix}$$

- Equality constraints can be replaced by two inequality constraints
- Variables $x \leq 0$ can be replaced by $x = -x'$, so $x' \geq 0$
- ~~Variables $\leftrightarrow$ constraints (Columns $\leftrightarrow$ rows)~~

$$\max(\quad) \equiv -\min -(\quad)$$

Solvability relationships

Strong Duality Theorem: There are exactly 4 possible cases

1. Primal and dual feasible, objectives equal at optimality. $z^* = w^*$
2. Primal unbounded, dual infeasible
3. Dual unbounded, primal infeasible
4. Both primal and dual infeasible
5. Takes a considerable amount of math to prove this.
    - Take CS/IE 525 if you care!

Primal/Dual Relationships

Freaky Stuff

- Variables in each problem correspond to constraints in the other
- Objective function coefficients of each problem come from the right-hand side constants of the other
- Technology matrix of each problem comes from that of the other (but transposed)
- The entries in the technology matrix are the coefficients of decision variables in the constraints
- There is a precise relationship between sign patterns
- One problem maximizes; the other minimizes

General form duality

Cheat Sheet

Min problem	Max problem
Nonnegative variable $\geq$	Inequality constraint $\leq$
Nonpositive variable $\leq$	Inequality constraint $\geq$
Free variable	Equality constraint $=$
Inequality constraint $\geq$	Nonnegative variable $\geq$
Inequality constraint $\leq$	Nonpositive variable $\leq$
Equality constraint $=$	Free Variable

You Try (simpledual.gms)

$$\max \quad 4y_1 + 1y_2 + 6y_3$$

$$1y_1 + 2y_2 + 1y_3 \geq 5$$

$$2y_1 - 1y_2 + 3y_3 = 3$$

$$-1y_1 + 10y_2 - 1y_3 \leq 2$$

$$y_1 \geq 0$$

$$y_2 \leq 0$$

y_3 free

$$
\begin{array}{rlcc}
\min & 5x_1 + 3x_2 + 2x_3 & & \\
\text{s.t.} & x_1 + 2x_2 - x_3 \geq 4 & \perp & y_1 \\
& 2x_1 - x_2 + 10x_3 \leq 1 & \perp & y_2 \\
& x_1 + 3x_2 - x_3 = 6 & \perp & y_3 \\
& x_1 \leq 0, x_2 \text{ free }, x_3 \geq 0 & &
\end{array}
$$

Shadow Prices

- Dual Variables are sometimes called shadow prices

Shadow Price

- The rate of change in the (optimal) objective value with a change in associated right-hand side element

- Economically: How much should you (the primal problem decision maker) be willing to pay for an additional unit of the resource?
- Mathematically, like a derivative

$$\pi_i^* = \frac{\partial z^*}{\partial b_i}$$

Reduced Costs

Reduced Costs

- The slack in the dual constraint: $\bar{c}_j = c_j - \sum_{i=1}^m a_{ij}\pi_i$
- The amount by which the objective function coefficient would need to change for the corresponding variable to not be at its bound in an optimal solution.
- For non-basic variables, tells the rate of change of the optimal objective value if the bound was changed.
- Like a derivative

$$\bar{c}_j = \frac{\partial z^*}{\partial \{\ell, u\}_j}$$

More on Reduced Costs

- If $x = x^*$ is optimal, reduced costs say so. In a minimization problem,
  - If variable is at its lower bound, reduced cost ≥ 0
  - If variable is at its upper bound, reduced cost ≤ 0
  - If variable is strictly between bounds, reduced cost $= 0$
- Note: In a maximization problem, these inequalities are reversed

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- We can see reduced costs as shadow prices on the activity level bounds

GAMS: "Marginals" or Lagrange multipliers.

- The ".m" field for an equation is the dual variable value of that equation.
- The ".m" field for a variable is that variable's reduced cost

Another Small Example

- The WorldLight Company produces two types of light fixtures (products 1 and 2) that require both metal frame parts and electrical components.
- Management wants to determine how many units of each product to produce so as to maximize profit.
- For each unit of product 1, 1 unit of frame parts and 2 units of electrical components are required.
- For each unit of product 2, 3 units of frame parts and 2 units of electrical components are required.
- The company has 200 units of frame parts and 300 units of electrical components.
- Each unit of product 1 gives a net profit of $1, and each unit of product 2, gives a profit of $2.

WorldLight (`worldlight.gms`)

$$\max x_1 + 2x_2 \text{ s.t.}$$

$$\begin{aligned}
x_1 + 3x_2 &\leq 200 && \text{Frame Part Units} \\
2x_1 + 2x_2 &\leq 300 && \text{Electrical Components} \\
x_1 &\geq 0 && \text{The immutable laws of physics} \\
x_2 &\geq 0 && \text{The immutable laws of physics}
\end{aligned}$$

$$\max c^T x \text{ s.t.}$$

$$\begin{aligned}
Ax &\leq b \\
x &\geq 0
\end{aligned}$$

$$\min b^T \pi \text{ s.t.}$$

$$\begin{aligned}
A^T \pi &\geq c \\
\pi &\geq 0
\end{aligned}$$

World Light Dual

- You're looking to purchase frame parts and electrical components to meet your needs at minimum costs:

$$\min 200\pi_1 + 300\pi_2$$

What prices would they have to have?

- One frame part + 2 electrical components has a value of at least 1 to you: $\pi_1 + 2\pi_2 \geq 1$
- Three frame parts plus two electrical components are worth at least 2 to you: $3\pi_1 + 2\pi_2 \geq 2$
- $\pi_1, \pi_2 \geq 0$

Extending the Small Example (`worldlight2.gms`)

- The WorldLight Company produces two types of light fixtures (products 1 and 2) that require both metal frame parts and electrical components.
- Management wants to determine how many units of each product to produce so as to maximize profit.
- For each unit of product 1, 1 unit of frame parts and 2 units of electrical components are required.
- For each unit of product 2, 3 units of frame parts and 2 units of electrical components are required.
- The company has 200 units of frame parts and 300 units of electrical components.
- Each unit of product 1, up to 120 units, gives a net profit of $1, and each unit of product 2 gives a profit of $2.
- Any excess over 120 units of product 1 brings no profit, so such an excess has been ruled out explicity.

WorldLight

maximize

$$x_1 + 2x_2$$

subject to

$$\begin{array}{rcll}
x_1 + 3x_2 & \leq & 200 & \text{Frame Part Units} \\
2x_1 + 2x_2 & \leq & 300 & \text{Electrical Components} \\
x_1 & \leq & 120 & \text{Rule out production over 120 units} \\
x_1 & \geq & 0 & \text{The immutable laws of physics} \\
x_2 & \geq & 0 & \text{The immutable laws of physics}
\end{array}$$

World Light Dual

minimize

$$200\pi_1 + 300\pi_2 + 120\pi_3$$

subject to

$$\begin{aligned} \pi_1 + 2\pi_2 + \pi_3 &\geq 1 \\ 3\pi_1 + 2\pi_2 \quad\quad &\geq 2 \\ \pi_1, \pi_2, \pi_3 &\geq 0 \end{aligned}$$