CS 524: Introduction to Optimization
Lecture 5 : The Simplex Method

Michael Ferris

Computer Sciences Department
University of Wisconsin-Madison

September 15, 2023

Graphically Solving LP's

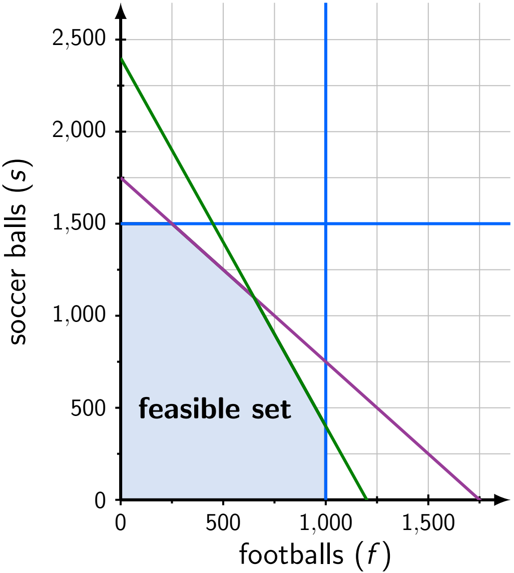
- Not for production use, but gives insight into what the algorithm for solving the problem is doing

---

Identify Feasible Region

- Graph each constraint as an equality
- Note which side is feasible
- Identify the feasible region: The set of all feasible solutions
- Remember to include nonnegativity!

Geometry of Top Brass



$$\max_{f,s} \quad 12f + 9s$$

$$\begin{aligned} \text{s.t.} \quad & 4f + 2s \le 4800 \\ & f + s \le 1750 \\ & 0 \le f \le 1000 \\ & 0 \le s \le 1500 \end{aligned}$$

Each point (f, s) is a possible decision.

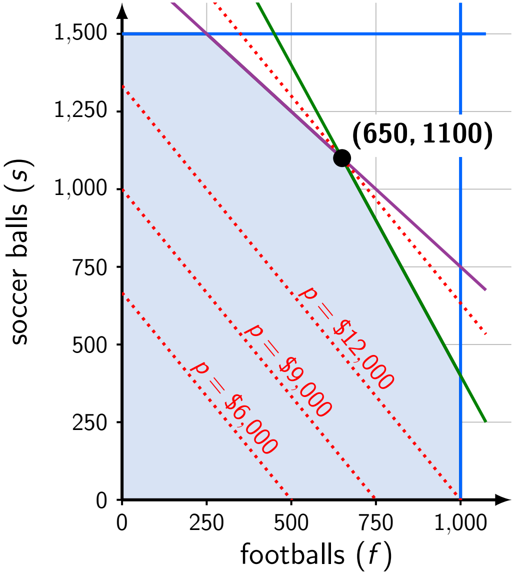
2-12

Graphically Solving LPs

"Move" Objective

- Draw parallel "isoprofit" lines. (All points on each line give the same value of the objective function)
- These are points that are orthogonal to the objective function vector
- Optimal point(s) will be on the highest isoprofit line that touches the feasible region

Geometry of Top Brass



$$\max_{f,s} \quad 12f + 9s$$

$$\text{s.t.} \quad 4f + 2s \le 4800$$

$$f + s \le 1750$$

$$0 \le f \le 1000$$

$$0 \le s \le 1500$$

Which feasible point has the max profit?

$p = 12f + 9s$

2-13

Observations

Let's Think About Geometry

If there exists an optimal solution to a LP instance, then there exists an optimal solution that is at an extreme point of the feasible region.

The Simplex Method

0. Start from an extreme point.
1. Find an improving direction d. If none exists, STOP. The extreme point is an optimal solution.
2. Move along d until you hit a new extreme point. Go to 1.

The Simplex Method

- The simplex method is a systematic way in which to do the algebra necessary to do steps 0, 1, and 2.

Some definitions and facts:

- An inequality $a^T x \leq b$ is *binding* at x if $a^T x = b$.
- An extreme point is the intersection of at least n inequalities in $\mathbb{R}^n$ (sometimes called a vertex).
- *Basis* : The indices of the n inequalities that are "binding" at an extreme point solution. (The solution itself is sometimes called a basic feasible solution).

Algorithm of the Century?!?!?!

https://people.sc.fsu.edu/~jburkardt/fun/misc/algorithms_dongarra.html

- Does the simplex algorithm seem like it would be a good algorithm, much less the "algorithm of the century"?
- Depends on how many extreme points – bases – there are?
- Suppose I have m inequalities and n variables.
- The number of extreme points is (roughly)

$$E(m,n) \leq \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}$$

There's Big, and Then There's REALLY Big

$m = 1100, n = 1000$

- 142296717362215353642981626982185928761718752252897324227687612 402698478561102447621700516733667586228081766430153750076191729 4126969341076641376
- 1.42×10^{143}
- (The number of subatomic particles in the universe)^2.

- However, we routinely solve problems orders of magnitude larger than this!
- How is this possible?

A++++++++++++++++

- If you can solve the following problem, you will get a very good grade:
  - Given a polytope of dimension n consisting of m inequalities, there is a path between any two extreme points consisting of at most $m - n$ edges.

- Before you waste your time, this is known as the *Hirsch Conjecture*. It is one of the most famous open problems in discrete mathematics.
- It was recently disproved by Francisco Santos.
  - The smallest counterexample found to date is a $n = 20$ dimensional polytope with $m = 40$ sides for which the shortest path between 2 vertices is 21.
- Moral: Geometric intuition for High-dimensional objects can only take us so far...

GAMS solvers for LP's (and MIP's)

- In this course I suggest you use `cplex`
- If this is not your default solver for LP, then just use the following line before the solve statement
- `option lp=cplex`
- These solvers also have options (interior point, network simplex, etc) - more on that later

---

Other Solvers are also available

- See: Studio $->$ Help $->$ GAMS Licensing $->$ Solvers (tab)
- Gurobi or OSIGurobi (needs separate license setup)
- Mosek or OSIMosek
- XPRESS or COPT
- CBC or HIGHS (open source)

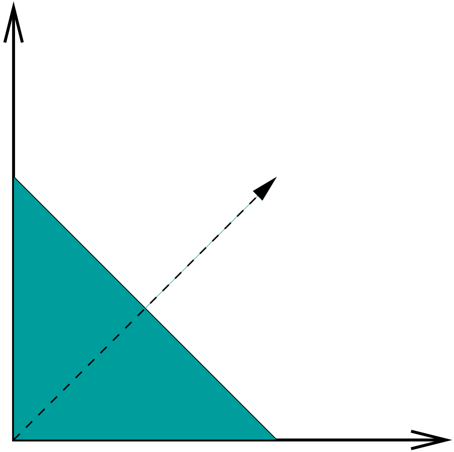
Multiple Optimal Solutions

- What if c is orthogonal to an improving direction d?

maximize

$$x_1 + x_2$$

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- We get an infinite number of optimal solutions.

Simplex Method – What can go wrong?

- Unbounded / Infeasible / Degenerate LPs

Simplex Method: Step 2

Move along d until you hit a new extreme point.

- What if we don't hit an extreme point?

$$\max x_1 + x_2$$

$$\begin{aligned} \text{s.t. } x_1 + 2x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Usually this means you forgot some constraints. Maybe your variable bounds?
- Just because the region is unbounded doesn't mean that the LP is unbounded.

Simplex Method – What can go wrong?

Simplex Method: Step 0

Start from an extreme point

- What if there *are* no extreme points?
  - This (usually) means that the feasible region is empty.
  - The instance is infeasible.
  - $P = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1, x_1 + x_2 \geq 2\}$
- How will we know if an instance is infeasible?
  - "Big-M", "Two-Phase"?
  - The solver will tell us!

Warning!

- It may be hard to "blame" one constraint for being infeasible.
- When building models for the real world determining what is "causing" the infeasibility may be tough.
- Whose "fault" is this?

$$x_1 - x_2 \geq 1, x_2 - x_3 \geq 1, -x_1 + x_3 \geq 1$$

See "GAMS Output" Section in GUG

```
scalar mstat; mstat = MODNAME.modelstat; display mstat;
```

Model Status

1. OPTIMAL
2. LOCALLY OPTIMAL
3. UNBOUNDED
4. INFEASIBLE
5. LOCALLY INFEASIBLE
6. INTERMEDIATE INFEASIBLE
7. FEASIBLE SOLUTION
8. INTEGER SOLUTION
9. INTERMEDIATE NON-INTEGER
10. INTEGER INFEASIBLE

See "GAMS Output" Section in GUG

```
scalar sstat; sstat = MODNAME.solvestat; display sstat;
```

Solver Status

1. Normal Completion
2. Iteration Interrupt (`option iterlim`)
3. Resource Interrupt (`option reslim`)
4. Terminated By Solver
5. Evaluation Interrupt
6. Capability Problems (solver not appropriate)
7. Licensing Problems
8. User Interrupt

I Will Gladly Pay You Tuesday...

- I really like hamburgers.
- Let's suppose in the diet problem, I decide to maximize the number of hamburgers I eat
- Let $B \subset F$

$$B = \{\text{QP, MD, BM}\}$$

- My new objective is to

$$\max \sum_{j \in B} x_b$$

- `mcgreasy1.gms`, beef1 model

GAMS tip: inferring sets from parameters

- When we have a table or a parameter defined over a set we can populate the elements of the set from the parameter definition
- In `mcgreasy1.gms` we can do this for the sets food and nutr using the table defining a

```
set food, nutr;

table a(nutr<,food<)  per unit nutrients
        QP   MD   BM   FF   MC   FR   SM   1M    OJ
Prot    28   24   25   14   31    3   15    9     1
...
```

Subsets

- It is often necessary to define a set whose members must *all* be members of a larger set. The syntax to do this is

```
set S(I) ;
```

  which will declare a set $S \subseteq I$
- We can populate S either with data statements (as in next slide) or via assignment statements `S('soccer') = yes;`
- Defining subsets allows for domain checking.
  - Domain checking is a good idea
- We will demonstrate...

Mmmmmmmmmmm. Beef

- Always check the solver status in the solution report when you are done running GAMS.

```
**** SOLVER STATUS     1 NORMAL COMPLETION
**** MODEL STATUS      3 UNBOUNDED
**** OBJECTIVE VALUE               50.0000
```

- Obviously, I haven't constrained the number of hamburgers I get to eat.
- Add some constraints to do this! (see beef2 model)

My Wife Loves Me!

- In the interest of extending my life, my wife has requested that I obey the following constraints:

1. Don't eat more than 3 sandwiches per day

$$x_{QP} + x_{MD} + x_{BM} + x_{FF} + x_{MC} + x_{SM} \leq 3$$

2. Don't drink too much: $x_{1M} + x_{OJ} \leq 3$
3. Only two french fries per day: $x_{FF} \leq 2$

- But with these constraints, the problem is infeasible!

Handling Infeasibility

Our First Trick

- Introduce slack/surplus variables and try to minimize the slack/surplus.

- Suppose I think that the "too much drinking" constraint is the one causing the problem to be infeasible[1]
- New decision variable s: Number of extra drinks (over three) that I must drink in order to get a feasible solution

$$x_{1M} + x_{OJ} \leq 3 + s, s \geq 0$$

- New Objective: $\min s$

[1] You can never drink too much!

Simplex Method – What else can go wrong?

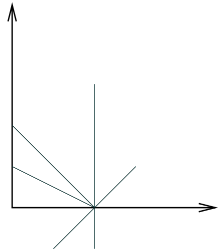
Simplex Method: Step 2

Move along d until you hit a new extreme point.

- What if moving in our "improving direction" doesn't take us anywhere!

$$\max x_1 + x_2$$

$$\begin{array}{lrcl}
\text{s.t.} & x_1 + 2x_2 & \leq & 1 \\
& x_1 & \leq & 1 \\
& x_1 + 3x_2 & \leq & 1 \\
& 2x_1 - 4x_2 & \leq & 2 \\
& x_1, x_2 & \geq & 0
\end{array}$$



I'm a Degenerate!

- The previous case is known as the LP being degenerate
- Degeneracy is what happens when more than n inequalities intersect at a point.
- This doesn't seem likely to happen, but BELIEVE ME it does happen in nearly all practical problems.
- This is not really a modeling problem, but it can lead to computational difficulties.
- What do solvers do?
  - Perturb the inequalities so they don't intersect
  - Smallest subscript rule.[2]

[2] They don't really do this