CS 524: Introduction to Optimization
Lecture 7 : Convexity and backlog

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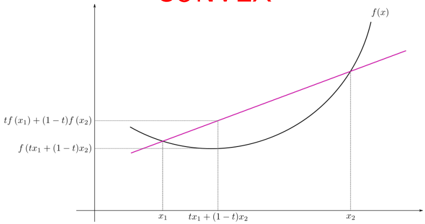
Calculus

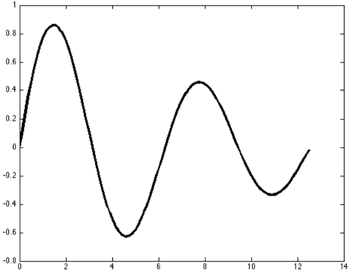
Fundamental Concept

It is *extremely* important to understand the convexity properties of a function you are trying to optimize.

CONVEX



NONCONVEX



Convexity: definitions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if for any two points x and y, the graph of f lies below or on the straight line connecting $(x, f(x))$ to $(y, f(y))$ in $\mathbb{R}^{n+1}$.
  - $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \qquad \forall \alpha \in [0,1]$
- A function $f : \mathbb{R}^n \to \mathbb{R}$ is *concave* if for any two points x and y, the graph of f lies above or on the straight line connecting $(x, f(x))$ to $(y, f(y))$ in $\mathbb{R}^{n+1}$.
  - $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y) \qquad \forall \alpha \in [0,1]$
- A function that is neither convex nor concave, we will call *nonconvex*.

What's the key?

An Optimization Problem

$$(P) \quad \min_{x \in S} f(x)$$

- x^* is a global optimum for (P) if $f(x^*) \leq f(x)\ \forall x \in S$
- x^* is a local optimum for (P) if $f(x^*) \leq f(x)\ \forall x \in S \cap N(x^*)$
  - $N(x^*)$: neighborhood of x^*. (Ball of radius ϵ centered at x^*)

Wal-Mart Theorem

Problems are easy if local optima are also global optima.

Easy/Hard?

- Which of the following do you suspect are easy, which are hard?
  1. Minimize a convex function?
  2. Minimize a concave function?
  3. Minimize a nonconvex function?
  4. Maximize a convex function?
  5. Maximize a concave function?
  6. Maximize a nonconvex function?
- True or False?
  1. Linear Functions are Convex?
  2. Linear Functions are Concave?

Without Drawing Pictures?

- I like drawing pictures nearly as much as my son, but we may need a better way to tell if a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex.
- Remember from calculus: A function $f : \mathbb{R} \to \mathbb{R}$ is convex on a domain $\mathcal{D}$ if and only if $f''(x) \geq 0 \; \forall x \in \mathcal{D}$
  - Think: x^2. $f''(x) = 2 > 0$
- The analogy in multiple dimensions, is a square matrix of second partial derivatives called the Hessian matrix, denoted $\nabla^2 f(x)$
- If f is twice continuously differentiable, then f is convex if and only if the Hessian matrix is postive semi-definite, i.e. $y^T \nabla^2 f(x) y \geq 0, \; \forall x, y$

Constraints

- Recall that our constraints might come in two forms...
  - "Regular": $g_i(x) \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b_i$
  - "Explicit": $x \in X$

Main Categories of Constraints

- Continuous
- Discrete
- None/Empty

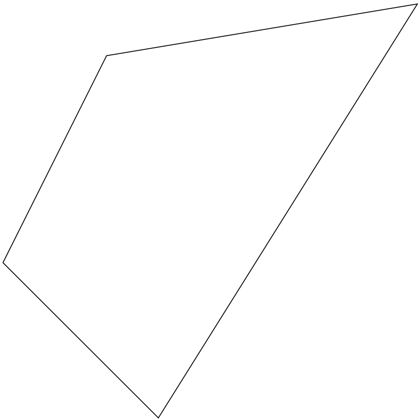
Breaking down the convex sets

- Polyhedral. A Set S is polyhedral if...
  - Formal Definition: it is the intersection of a finite number of half-spaces.
  - Informal Definition: it is "linear".
  - A good working definition: If the explicit constraint set (X) is the whole space (no constraint) and all of the $g_i(x)$ are linear functions.
  - $S = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ is polyhedral.
- Non-polyhedral
  - Anything "curvy"
  - Anything "discrete"

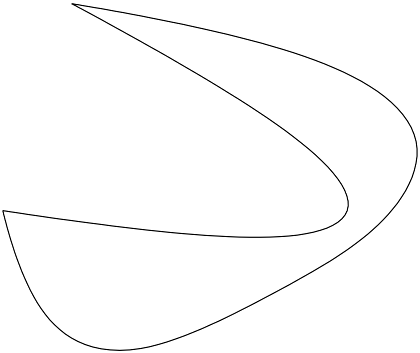
Convexity – Again. Ugh!

- A set S is *convex* if the straight line segment connecting any two points in S lies entirely inside or on the boundary of S.
  - $x, y \in S \Rightarrow \alpha x + (1-\alpha) y \in S \qquad \forall \alpha \in [0,1]$

CONVEX



NONCONVEX

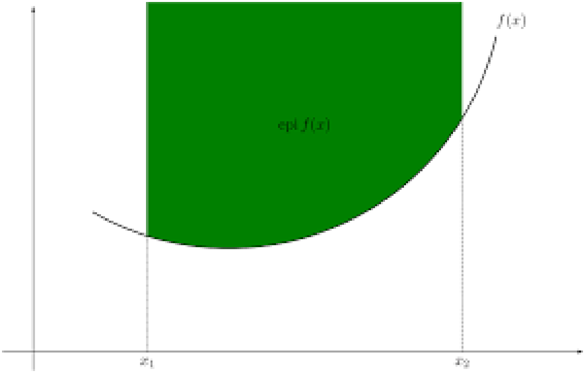


True or False

1. Discrete Constraint Sets are Convex?
2. Empty Constraint Sets are Convex?
3. Continuous Constraint Sets are Convex?

Convex Functions and Sets

- A Confusing Point...
  - Why do they have a *convex function* and a *convex set*? How are they related?
  - f is convex if and only if the *epigraph*, or "over part" of f is a convex set.



Modeling Absolute Value – Objective

Can We Do It?

$$f(x) = |x|$$

- Is it linear?
- Is it convex?

- Trick: Define *positive* auxiliary variables x_+ and x_-.
  1. Add constraint $x = x_+ - x_-$
  2. Replace $|x|$ with $x_+ + x_-$ in objective
- When will this trick work?
- This tricks works when the optimization prefers small values of x
- When we are minimizing the absolute value of an expression.
- Next time will show an alternative to model $|x| = \max\{x, -x\}$

Recall: Basic ShoeCo Model

$$\min \sum_{t \in T} (\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota I_t)$$

s.t.

$$\begin{aligned}
a x_t &\leq H w_t + o_t \quad \forall t \in T \\
o_t &\leq O w_t \quad \forall t \in T \\
I_t &= I_{t-1} + x_t - d_t \quad \forall t \in T \\
I_0 &= \mathcal{I}_0 \\
w_t &= w_{t-1} + h_t - f_t \quad \forall t \in T \\
w_0 &= \mathcal{W}_0 \\
x_t, I_t, w_t, h_t, f_t, o_t &\geq 0 \quad \forall t \in T
\end{aligned}$$

Exercise: Bad Stuff Happens

- Suppose you don't have to meet forecast demands in every period.
- Meeting demand is often too stringent a requirement for the real-world
- Demand does not have to be met on time, but it must be met *eventually*
- There is a shortage cost $\theta = \$20$ per unit per month backlogged
- \$1 Question: How should the minimum cost compare with cost of earlier model?

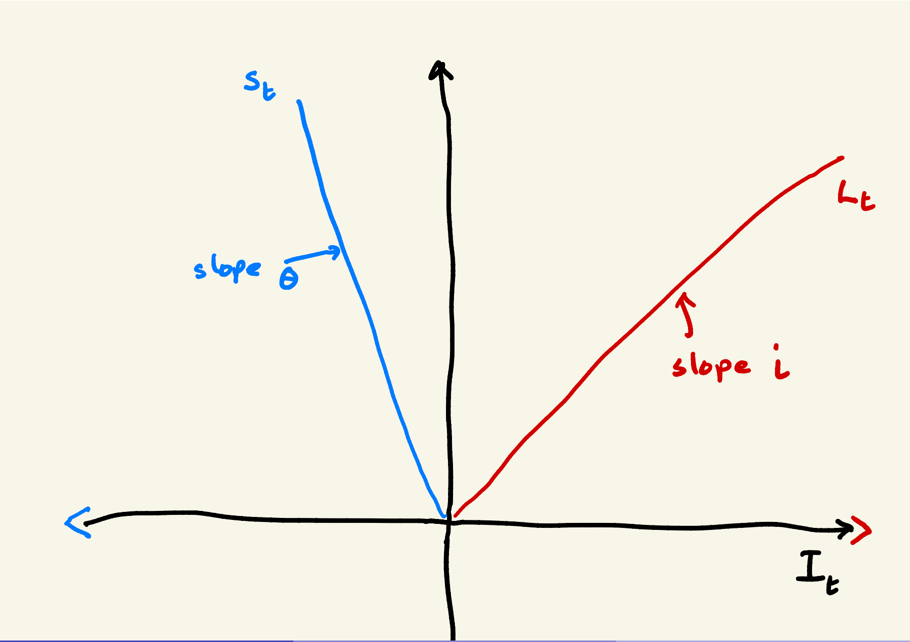
How to Model Backlogging

- Think of inventory being allowed to go negative
- Original picture still makes sense, since if inventory is negative, you need to "make up" for it during one of the next periods
- You can set last period demand $I_{|T|} \geq 0$ to ensure that all demand is *eventually* met.
- Cost function $F(I_t)$:

$$F(I_t) = \begin{cases} \iota I_t & \text{if } I_t \geq 0 \\ -\theta I_t & \text{if } I_t < 0 \end{cases}$$

- Is $F(I_t)$ a linear function of I_t?

Soft Penalties



The Same Nonlinear/Linear Trick

- To model the case where we are minimizing a convex piecewise linear function (like $F(\cdot)$ or $|\cdot|$), we can introduce a variable for each piece
- Write constraints $I_t = L_t - S_t \quad \forall t \in T$
  - Think of this as (Leftover - Shortage)
- Objective gets terms:

$$\sum_{t \in T} (\iota L_t + \theta S_t)$$

- This trick only works if we are *minimizing* costs. Then at most one of L_t and S_t will ever be positive in an optimal solution.

ShoeCo: Backlogged Model

$$\min \sum_{t \in T} (\delta x_t + \alpha w_t + \beta o_t + \eta h_t + \zeta f_t + \iota L_t + \theta S_t)$$

s.t.

$$\begin{aligned}
a x_t &\leq H w_t + o_t \quad \forall t \in T \\
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I_t &= I_{t-1} + x_t - d_t \quad \forall t \in T \\
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w_t &= w_{t-1} + h_t - f_t \quad \forall t \in T, \quad w_0 = \mathcal{W}_0 \\
x_t, w_t, h_t, f_t, o_t, S_t, L_t &\geq 0 \quad \forall t \in T
\end{aligned}$$