CS 524: Introduction to Optimization
Lecture 27 : Optimal Control

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Example: hovercraft

We are in command of a hovercraft. We are given a set of k waypoint locations and times. The objective is to hit the waypoints at the prescribed times while minimizing fuel use.

- Discretize time: $t = 0, 1, 2, ..., T$.
- Important variables: position x_t, velocity v_t, thrust u_t.
- Simplified model of the dynamics:

$$x_{t+1} = x_t + v_t \text{ for } t = 0, 1, ..., T-1$$
$$v_{t+1} = v_t + u_t \text{ for } t = 0, 1, ..., T-1$$

- We must choose $u_0, u_1, ..., u_{T-1}$.
- Initial position and velocity: $x_0 = 0$ and $v_0 = 0$.
- Waypoint constraints: $x_{t_i} = w_i$ for $i = 1, ..., k$.
- Minimize fuel use:

$$\|u_0\|^2 + \|u_1\|^2 + \cdots + \|u_{T-1}\|^2$$

Example: hovercraft

First model: hit the waypoints exactly

$$
\begin{aligned}
\min_{x_t, v_t, u_t} \ & \sum_{t=0}^{T-1} \|u_t\|^2 \\
\text{subject to } & x_{t+1} = x_t + v_t && \text{for } t = 0, 1, \ldots, T-1 \\
& v_{t+1} = v_t + u_t && \text{for } t = 0, 1, \ldots, T-1 \\
& x_0 = v_0 = 0 \\
& x_{t_i} = w_i && \text{for } i = 1, \ldots, k
\end{aligned}
$$

Example: `27hovercraft.ipynb`

Example: hovercraft

Second model: allow waypoints misses

$$\begin{aligned}
\min_{x_t, v_t, u_t} \; & \sum_{t=0}^{T-1} \|u_t\|^2 + \lambda \sum_{i=1}^{k} \|x_{t_i} - w_i\|^2 \\
\text{subject to } \; & x_{t+1} = x_t + v_t && \text{for } t = 0, 1, \ldots, T-1 \\
& v_{t+1} = v_t + u_t && \text{for } t = 0, 1, \ldots, T-1 \\
& x_0 = v_0 = 0
\end{aligned}$$

- λ controls the tradeoff between making u small and hitting all the waypoints.

LQR Control: a structured QP

Discuss LQR control. Use example from Bertsekas to motivate:

$$\min \int_0^1 6u(t)^2 + 2x_1(t)^2 + x_2(t)^2 \, \mathrm{d}t,$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix},$$

$$|u(t)| \leq U, \;\; x_1(0) = 15, \;\; x_2(0) = 5,$$

where U is a positive constant. Start with a linear ODE and its solution, introduce concept of an input/control u that influences the evolution of the ODE, objective function and constraints to guide x to a particular goal.

Euler discretization

$$\dot{x} = Ax + Bu$$

$$\frac{x_1^{t+1} - x_1^t}{\delta t} = x_2^t, \quad \frac{x_2^{t+1} - x_2^t}{\delta t} = -x_1^t + u^t$$

leads to

$$\frac{x^{t+1} - x^t}{\delta t} = Ax^t + Bu^t$$

Discuss the structure and detail the QP. Note that Q is diagonal. Also block diagonal form for the constraint matrix.
Discuss example `27lqr.ipynb`. Try different values of the bound U.
Generalizable to different choices of discretization parameter N.