CS 524: Introduction to Optimization
Lecture 35 : Two stage SPs

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The problem

A furniture maker can manufacture and sell four different dressers. Each dresser requires a certain number t_{cj} of man-hours for carpentry, and a certain number t_{fj} of man-hours for finishing, $j = 1, \ldots, 4$. In each period, there are d_c man-hours available for carpentry, and d_f available for finishing. There is a (unit) profit $\bar{c}_j$ per dresser of type j that's manufactured. The owner's goal is to maximize total profit:

$$\max_{x \geq 0} \; 12x_1 + 25x_2 + 21x_3 + 40x_4 \qquad (\mathit{profit})$$

subject to

$$4x_1 + 9x_2 + 7x_3 + 10x_4 \leq 6000 \qquad (\mathit{carpentry})$$
$$x_1 + x_2 + 3x_3 + 40x_4 \leq 4000 \qquad (\mathit{finishing})$$

Succinctly:

$$\max_x \; c^T x \text{ s.t. } Tx \leq d, x \geq 0$$

Is your time estimate that good?

Solution is $(1333, 0, 0, 67)$, value $\$18, 667$

- The time for carpentry and finishing for each dresser cannot be known with certainty
- Each entry in T takes on four possible values with probability $1/4$, independently
- 8 entries of T are random variables: $s = 65, 536$ different T's each with same probability of occurring
- But decide "now" how many dressers x of each type to build
- Might have to pay for overtime (for carpentry and finishing)
- Can make different overtime decision y_s for each scenario s - recourse!

Creating a Stochastic Model

Here is a general procedure for making a (scenario-based) 2-stage stochastic optimization problem

- For a "nominal" state of nature (scenario), formulate an appropriate LP model
- Decide which decisions are made before uncertainty is revealed (here and now), and which are decided after
- All second stage variables get "scenario" index
- Constraints with scenario indices must hold for all scenarios
- Second stage variables in the objective function should be weighted by the probability of the scenario occurring

Extensive Form (Deterministic Equivalent)

- Assume $\Omega = \{\omega_1, \omega_2, \dots \omega_S\} \subseteq \mathbb{R}^r$, $\mathrm{P}(\omega = \omega_s) = p_s, \forall s = 1, 2, \dots, S$
- $T_s \stackrel{\text{def}}{=} T(\omega_s), d_s \stackrel{\text{def}}{=} d(\omega_s), q_s \stackrel{\text{def}}{=} q(\omega_s), W_S = W(\omega_s)$
- Then can write extensive form:

$$
\begin{array}{ccccccccccc}
c^T x & + & p_1 q_1^T y_1 & + & p_2 q_2^T y_2 & + & \cdots & + & p_s q_s^T y_s & & \\
\text{s.t.} & & & & & & & & & & \\
Ax & & & & & & & & & = & b \\
T_1 x & + & W_1 y_1 & & & & & & & = & d_1 \\
T_2 x & & & + & W_2 y_2 & & & & & = & d_2 \\
\vdots & & & + & & & \ddots & & & & \vdots \\
T_S x & & & & & & & + & W_S y_s & = & d_s \\
x \in X & & y_1 \in Y & & y_2 \in Y & & & & y_s \in Y & &
\end{array}
$$

Extensive Form Problem (`furnrun.gms`)

$$\min_{x,y} -c^T x + \sum_{s=1}^{65{,}536} p_s q^T y_s$$

subject to

$$T^s x - y_s \leq d, \quad s = 1, \ldots, 65, 536$$
$$x, y_s \geq 0$$

- Immediate costs + expected *future costs*
- Stochastic program with recourse
- Solution is $(205, 0, 696, 33)$, value \$18, 128

About the extensive form

- $y_s \equiv y(\omega_s)$ is the recourse action to take if scenario ω_s occurs.
- Pro: It's a linear program.
- Con: It's a BIG linear program.
- Imagine the following (real) problem. A Telecom company wants to expand its network in a way in which to meet an unknown (random) demand.
- There are 86 unknown demands. Each demand is independent and may take on one of seven values.
- $S = |\Omega| = \Pi_{k=1}^{86}(7) = 7^{86} = 4.77 \times 10^{72}$
  - The number of subatomic particles in the universe.
- How do we solve a problem that has more variables and more constraints than the number of subatomic particles in the universe?

But Its Even Worse!

- If Ω is not a countable set (say if it is made up of continuous-valued random variables, our "deterministic equivalent" would have ∞ variables and constraints. :-)
- The answer is we can't!
- We solve an approximating problem obtained through sampling.

Monte Carlo Methods

$$(*) \qquad \min_{x \in X} \{ f(x) \equiv \mathbb{E}_P F(x,\xi) \equiv \int_\Omega F(x,\xi) dP(\xi) \}$$

- Most of the theory presented holds for (*)—A very general SP problem
- Naturally it holds for our favorite SP problem:
  - $X \stackrel{\text{def}}{=} \{x \mid Ax = b, \ x \geq 0\}$
  - $f(x) \equiv c^T x + \mathbb{E}\{Q(x,\omega)\}$
  - $Q(x,\omega) \equiv \min_{y \geq 0} \{q(\omega)^T y \mid Wy = d(\omega) - T(\omega)x\}$

Sampling

- Instead of solving (*), we solve an approximating problem.
- Let $\xi^1, \ldots, \xi^N$ be N realizations of the random variable ξ:

$$\min_{x \in X} \{ \widehat{f}_N(x) \equiv N^{-1} \sum_{j=1}^{N} F(x, \xi^j) \}.$$

- $\widehat{f}_N(x)$ is just the sample average function
- Since ξ^j drawn from P, $\widehat{f}_N(x)$ is an unbiased estimator of $f(x)$
  - $\mathbb{E}[\widehat{f}_N(x)] = f(x)$

What do we learn? (`furnsaa.gms`)

- Deterministic solution: $x^{mvs} = (1333, 0, 0, 67)$
- Expected profit using this solution: $\$16,942$
- Expected (averaged) overtime costs: $\$1,725$
- Extensive form solution: $x^s = (257, 0, 666, 34)$ with expected profit $\$18,051$
- Deterministic solution is not optimal for stochastic program, but more significantly it isn't getting us on the right track!
- Stochastic solution suggests large number of "type 3" dressers, while deterministic solution has none!

Stochastic Programming

A Stochastic Program

$$\min_{x \in X} f(x) \stackrel{\text{def}}{=} \mathbb{E}_\omega[F(x,\omega)]$$

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2 Stage Stochastic LP w/Recourse

$$F(x,\omega) \stackrel{\text{def}}{=} c^T x + Q(x,\omega)$$

- $c^T x$: Pay me now
- $Q(x,\omega)$: Pay me later

The Recourse Problem

$$Q(x,\omega) \stackrel{\text{def}}{=} \min q(\omega)^T y$$

$$\begin{aligned} W(\omega)y &= d(\omega) - T(\omega)x \\ y &\geq 0 \end{aligned}$$

Specialized techniques use this recourse function for decomposition algorithms