1 Polynomials

1.1 What are polynomials?

Definition. A **polynomial** is a mathematical expression that is a finite sum of **terms**. Each term is the product of a non-zero number, and (optionally) some symbols or letters. The symbols are called **variables**.

For example, "$3x^2 + (-2x) + 7$" is a polynomial. It has three terms, and one variable, x. (The first term is the product of 3, x, and x again.)

We can "**evaluate**" a polynomial by picking some numeric value for each symbol, and substituting that value for every occurrence of the symbol.

For example, evaluating $3x^2 + (-2x) + 7$ under the choice $x = 4$, we get

$$3 \cdot 4^2 + (-2 \cdot 4) + 7 = 48 - 8 + 7 = 47.$$

From here on, we will only talk about polynomials in one variable, x.

1.2 The quadratic formula

Definition. A **root** of such a polynomial is a choice of x so that the result of evaluation is 0. (For example, 3 is a root of $x^2 - 9$, since $3^2 - 9 = 0$.)

Definition. A **quadratic** polynomial is one of the form $ax^2 + bx + c$ for some numbers a, b, c where $a \neq 0$.

Theorem. *The roots of $ax^2 + bx + c$ are given by the formula:*

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

Proof. In the equation $ax^2 + bx + c = 0$, divide both sides by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Now add $\left(\frac{b}{2a}\right)^2 - \frac{c}{a}$ to both sides:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}.$$

Recognize the left side as $(x + \frac{b}{2a})^2$ and simplify the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

We know that $p^2 = q$ precisely when $p = \pm\sqrt{q}$. So we conclude:

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}},$$

which we can write as

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

$\square$