Elementary Regions

We've defined double integrals over rectangles, and we've used Fubini's theorem to convert them to iterated integrals, which are straightforward to compute. But what if we need to integrate a function over a region that isn't a rectangle?

For example, consider the region below, which is bounded by the curves $y = x^2$, $y = x^2 + 1$, $x = 1$, and $x = 2$.

PICTURE

We can describe this region as the set of points (x, y) such that $1 \leq x \leq 2$ and $x^2 \leq y \leq x^2 + 1$. Here, we can describe the region by bounding the x-coordinate with constants, and bounding the y-coordinate with continuous functions of x. We'll soon see that this kind of description translates easily to integration, and this is our first example of an elementary region.

Elementary regions

We give the definition of elementary regions, which will be the most natural regions to integrate over.

Definition 1. *The following are types of* elementary regions*.*

Suppose a region R can be described as the set of points (x, y) such that

$$\begin{aligned} a \leq & x \leq b, \text{ and} \\ f(x) \leq & y \leq g(x), \end{aligned}$$

*where $f(x)$ and $g(x)$ are continuous functions. Then we say that R is x-*simple*.*

Suppose a region R can be described as the set of points (x, y) such that

$$\begin{aligned} c \leq & y \leq d, \text{ and} \\ f(y) \leq & x \leq g(y), \end{aligned}$$

*where $f(y)$ and $g(y)$ are continuous functions. Then we say that R is y-*simple*.*

We'll now look at some examples of x-simple and y-simple regions. Note that some regions are both x-simple and y-simple.

Example 1. *Consider the region R below, which is bounded by the graphs of $y = x^2$ and $x = y^2$.*

Learning outcomes: Identify and describe elementary regions.
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PICTURE

This region is both x-simple and y-simple. To see that it is x-simple, we can describe R as the set of points (x, y) such that

$$\begin{aligned} 0 \leq & x \leq 1, \text{ and} \\ x^2 \leq & y \leq \sqrt{x}. \end{aligned}$$

To see that R is y-simple, we can describe it as the set of points (x, y) such that

$$\begin{aligned} 0 \leq & y \leq 1, \text{ and} \\ y^2 \leq & x \leq \sqrt{y}. \end{aligned}$$

Example 2. *Consider the region R below, which is bounded by the graphs of $y = \cos x$, $y = 1$, $x = -\pi$, and $x = \pi$.*

PICTURE

This region is x-simple, since R is the set of points (x, y) such that

$$\begin{aligned} -\pi \leq & x \leq \pi, \text{ and} \\ \cos x \leq & y \leq 1. \end{aligned}$$

The region R is not y-simple, since any inequality $f(y) \leq x \leq g(y)$ cannot have a "hole" in the middle. However, R is the union of two y-simple regions R_1 and R_2, where R_1 is the set of points (x, y) such that

$$\begin{aligned} -1 \leq & y \leq 1, \text{ and} \\ -\pi \leq & x \leq -\arccos y, \end{aligned}$$

and R_2 is the set of points (x, y) such that

$$\begin{aligned} -1 \leq & y \leq 1, \text{ and} \\ \arccos y \leq & x \leq 1. \end{aligned}$$