Scalar Line Integrals

We've seen how we can integrate vector fields along a path, using vector line integrals. We can also integrate scalar valued functions along a path. For instance, suppose we have a scalar valued function $f : \mathbb{R}^2 \to \mathbb{R}$ and a path $\vec{x} : [a, b] \to \mathbb{R}^2$ in $\mathbb{R}^2$. Suppose we look at the portion of the graph of f lying over the path $\vec{x}$, and drop a "curtain" to the xy-plane.

PICTURE

Integrating f along the path $\vec{x}$ will be equivalent to finding the area of this curtain. We can also describe this as the area between $\vec{x}(t)$ and $f(\vec{x}(t))$.

Scalar line integrals aren't only useful for finding areas of strangely shapes regions. They are also useful throughout physics. For example, if you have the mass density function of a wire, you can compute the scalar line integral of this function to find the total mass of the wire.

Scalar Line Integrals

Suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$ and a $\mathcal{C}'$ path $\vec{x} : [a, b] \to \mathbb{R}^n$, such that the composition $f(\vec{x}(t))$ is defined on $[a, b]$.

PICTURE

In order to find the area under f and over the path $\vec{x}$, we will borrow an important idea from single variable calculus: approximating an area with rectangles.

In order to do this, we'll partition the interval $[a, b]$ into n subintervals, determined by

$$a = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n = b.$$

This partition breaks the path $\vec{x}$ into smaller paths, by restricting to the subintervals.

PICTURE

Our goal will be to approximate the area under f over each of these shorter paths. These approximations will be computed by finding the length of the short path, and multiplying this by a height determined by a test point, t_k^*. We can think of this as a curved rectangle.

PICTURE

Learning outcomes: Understand the definition of scalar line integrals geometrically, and be able to compute them.

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The specific choice of the test point will be important for simplifying our result - we'll come back to this later.

The height of our curved rectangle will be $f(\vec{x}(t_k^*))$, and the base is the distance Δs_k along the path $\vec{x}$ from t_{k-1} to t_k.

PICTURE

Thinking back to arclength computations, this distance is given by the integral

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt.$$

Now, we need to make a careful choice for our test point t_k^* which will simplify things later. To do this, recall the Mean Value Theorem for Integrals, from single variable calculus.

Theorem 1. *Suppose g is a continuous function on the closed interval $[a, b]$. Then there exists c in $[a, b]$ such that*

$$\int_a^b g(t)dt = (b-a)g(c).$$

Here, we'll take $\|\vec{x}'(t)\|$ for the function $g(t)$. Since $\vec{x}$ is $\mathcal{C}^1$, $\|\vec{x}'(t)\|$ is continuous. Applying the Mean Value Theorem on the interval $[t_{k-1}, t_k]$, there exists c_k such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt = (t_k - t_{k-1})\|\vec{x}'(c_k)\|.$$

We take this c_k to be our test point, so that $t_k^* = c_k$.

Now, the area of the kth curved rectangle is $F(\vec{x}(t_k^*))\Delta s_k$. We add up these areas and take the limit as the number of rectangles, n, goes to infinity:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\vec{x}(t_k^*))\Delta s_k.$$

Substituting $\Delta s_k = (t_k - t_{k-1})\|\vec{x}'(c_k)\|$ and writing $\Delta t = t_k - t_{k-1}$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\vec{x}(t_k^*))\|\vec{x}'(c_k)\|\Delta t.$$

We can recognize this as the single variable integral of the function $f(\vec{x}(t))\|\vec{x}'(t)\|$ over the interval $[a, b]$,

$$\int_a^b f(\vec{x}(t))\|vecx'(t)\| dt.$$

We take this to be the definition of a scalar line integral.

Definition 1. *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a $\mathcal{C}^1$ path $\vec{x} : [a, b] \to \mathbb{R}^n$. The* scalar line integral *of f along $\vec{x}$ is*

$$\int_{\vec{x}} f\, ds = \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\| dt.$$

Examples

Let's look at some examples of computing scalar line integrals.

Example 1. *Consider the function* $f(x,y) = x^2 + y^2$ *and the path* $\vec{x}(t) = (\cos t, \sin t)$ *for* $t \in [0,\pi]$*. We compute the scalar line integral of* f *over* $\vec{x}$*.*

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt \\
&= \int_0^\pi f(\cos t, \sin t)\|(-\sin t, \cos t)\|dt \\
&= \int_0^\pi (\cos^2 t + \sin^2 t)\sqrt{\sin^2 t + \cos^2 t}dt \\
&= \int_0^\pi 1\ dt \\
&= \pi
\end{aligned}$$

Example 2. *Consider the function* $f(x,y) = e^{\sqrt{xy}}$ *and the path* $\vec{x}(t) = (t,t)$ *for* $t \in [0,1]$*. We compute the scalar line integral of* f *over* $\vec{x}$*.*

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt \\
&= \int_0^1 f(t,t)\|(1,1)\|dt \\
&= \int_0^1 e^{\sqrt{t^2}}\sqrt{2}dt \\
&= \sqrt{2}\int_0^1 e^t dt \\
&= \sqrt{2}(e^1 - e^0) \\
&= \sqrt{2}e - \sqrt{2}
\end{aligned}$$