Review of u-substitution

One of the most commonly used strategies for evaluating single variable integrals is u-substitution. For example, suppose we wish to evaluate the integral $\int_1^3 2xe^{x^2}\,dx$. Here, it's useful to make the substitution $u = x^2$, and this allows us to evaluate the integral.

$$\begin{aligned}\int_1^3 2xe^{x^2}\,dx &= \int_1^9 e^u\,du \\ &= e^u\big|_{u=1}^{u=9} \\ &= e^9 - e\end{aligned}$$

We would like to apply the same ideas to double integrals, in order to make evaluation possible in more cases. Before we do this, we'll review the details of single variable u-substitution, in order to prepare for the more difficult two variable case.

There are three important parts of this process that we wish to highlight here.

- Changing the variable
- Changing the differential
- Changing the interval of integration

Changing the variable

Choosing the change of variable is the most important step of u-substitution. Suppose we are performing a u-substitution on a definite integral $\int_a^b F(x)dx$. Here, we make a choice for u as a function of x, so we write $u = g(x)$ for some function g. This choice is often made in conjunction with planning for the change of differential. That is, we choose $u = g(x)$ in a way so that the integrand can be written as

$$F(x) = f(g(x))g'(x),$$

for some function f.

Learning outcomes: Revisit u-substitution from single variable calculus, in preparation for substitution in double integrals.

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For example, consider the integral $\int_1^3 2xe^{x^2}\ dx$. To evaluate this integral, we choose $u = g(x)$ for $g(x) = x^2$, since then the integrand can be written as

$$2xe^{x^2} = g'(x)e^{g(x)}.$$

Choosing a "good" substitution often requires experience and intuition, built through trying different substitutions, and determining what works well. A common strategy for choosing substitutions is to look for a composition of functions, and taking u to be the "inner" function. In our example, we have the composition e^{x^2}, and took $u = x^2$.

Changing the interval of integration

Once we've made a choice of substitution $u = g(x)$, we start to tranform our integral to be with respect to u. As part of this process, we need to change the bounds of the integral. The new bounds will be $c = g(a)$ and $d = g(b)$. In order to see why this is necessary, let's look at the area represented by the definite integral $\int_a^b F(x)dx$.

PICTURE

Here, we are finding the area under the graph of the function $F(x)$ on the closed interval from $x = a$ to $x = b$. When we make a u-substitution, we are evaluating a different integral $\int_c^d f(u)du$. This represents the area under the function $f(u)$ over the closed interval from c to d. Since our new integral is with respect the u, it doesn't make sense to use the old bounds $x = a$ and $x = b$, since these were bounds for the variable x. Instead, we need to find the bounds on u which correspond to the old bounds on x. When we look at how the function g affects a and b, this gives us our new bounds, $c = g(a)$ and $d = g(b)$. We can also think about this as looking at how g affects the closed interval $[a, b]$. If g is an increasing function, then the image of the interval $[a, b]$ under g is the closed interval $[g(a), g(b)]$.

PICTURE

So, when we're converting an area over the interval $[a, b]$ using the function g, we will be working with an area over the interval $[g(a), g(b)]$.

PICTURE

To phrase this change more generally, when we make a change of variable, we need to consider how this substitution affects the domain of integration.

Changing the differential

The final step necessary for a u-substitution is to change the differential, so we have an integral with respect to u instead of x. We often think about this step as making the replacement

$$du = g'(x)dx,$$

but why is this change necessary? To see this, let's think back to the definition of a definite integral. We approximate the area under a curve with rectangles, and then take the limit as the width of the rectangles goes to zero.

PICTURE

Now, suppose we are approximating the area under a curve with eight rectangles. Then, suppose we make a change of variables $u = g(x)$, and let's look at how this affects the rectangles.

PICTURE

Here, we see that the width of the rectangles are affected by the change $u = g(x)$. This stretches or shrinks the rectangles. The proportion of this stretching or shrinking is determined by the rate of change of the function g, so by $g'(x)$. This is where the change in differential comes in - we make the replacement $du = g'(x)dx$ in order to account for the stretching or shrinking caused by the substitution $u = g(x)$.

To phrase this change more generally, when we make a change of variable, we need to consider how the substitution affects the shape and size of the domain of integration, and account for this with an expansion factor, such as $g'(x)$.

Double Integrals

In the next section, we'll look at how we can change variables in double integrals, and we'll see how this process resembles u-substitution from single variable calculus. When making the change of variables, we will again need to consider how to change the domain of integration, as well as the differential.