Additional Examples of Change of Variables

Recall how we can perform a change of variables in a double integral.

Proposition 1. *Let $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function which maps a region $D^* \subset \mathbb{R}^2$ onto a region $D \subset \mathbb{R}^2$, so that $\vec{T}$ restricted to D^* is one-to-one. Suppose $f : D \to \mathbb{R}$ is an integrable function. Then*

$$\iint_D f(x, y)\; dxdy = \iint_{D^*} f(\vec{T}(u, v)) \left| det(D\vec{T}(u, v)) \right| \; dudv.$$

In this section, we'll look at some further examples of changes of variables, and encounter some common challenges along the way.

Examples

Example 1. *We will evaluate the double integral $\iint_D 2xy\; dA$, where D is the region below, bounded by the lines $y = 2x$, $y = -2x$, and $y = x + 3$.*

PICTURE

If we take $u = y - 2x$ and $v = y + 2x$, then the lines $y = 2x$ and $y = -2x$ correspond to $u = 0$ and $v = 0$, respectively. Solving for x and y in terms of u and v, we have

$$\begin{aligned} x &= \frac{u+v}{2}, \\ y &= \frac{v-u}{4}, \end{aligned}$$

and the change of coordinates is given by $\vec{T}(u, v) = \left(\frac{u+v}{2}, \frac{v-u}{4} \right)$. From this, we compute

$$\begin{aligned} \left| det(D\vec{T}(u, v)) \right| &= \left| det \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix} \right| \\ &= \frac{1}{4}. \end{aligned}$$

Learning outcomes: Solidify the ability to change variables in double integrals.
Author(s): Melissa Lynn

The line $y = x + 3$ *corresponds to* $\frac{v-u}{4} = \frac{u+v}{2} + 3$*, which can be simplified to* $v = -3u - 12$.

PICTURE

So, in uv-coordinates, our region can be described by the inequalities

$$-4 \leq u \leq 0,$$
$$-3u - 12 \leq v \leq 0.$$

Putting all of this together, we have

$$\begin{aligned}
\iint_D 2xy\, dA &= \int_{-4}^{0} \int_{-3u-12}^{0} 2\frac{u+v}{2}\frac{v-u}{4} \cdot \frac{1}{4} dvdu \\
&= \int_{-4}^{0} \int_{-3u-12}^{0} \frac{v^2 - u^2}{16} dvdu \\
&= \int_{-4}^{0} \left(\frac{v^3}{48} - \frac{u^2 v}{16} \right) |_{v=-3u-12}^{v=0}\, du \\
&= \int_{-4}^{0} \left(\frac{(-3u-12)^3}{48} - \frac{u^2(-3u-12)}{16} \right)\, du \\
&= \int_{-4}^{0} \left(\frac{-3u^3}{8} - 6u^2 - 27u - 36 \right)\, du \\
&= \left(\frac{-3u^4}{32} - 2u^3 - \frac{27u^2}{2} - 36u \right) |_{u=-4}^{u=0} \\
&= 32
\end{aligned}$$

Sometimes, we may be able to carry out a change of variables without explicitly finding the transformation $\vec{T}$. We see this in the next example.

Example 2. *We will evaluate the double integral* $\iint_D (x+y)dA$*, where* D *is the region below.*

PICTURE

Since D *can be described with the inequalities*

$$1 \leq xy \leq 4,$$
$$0 \leq y - x \leq 2,$$

we will change to the coordinates $u = xy$ *and* $v = y - x$*. Now, in order to find the area expansion factor* $\left| det(D\vec{T}(u,v)) \right|$*, we would typically find the transformation* $\vec{T}$*. That is, we would find* x *and* y *in terms of* u *and* v*. However, in this situation, it's difficult to solve for* x *and* y*. Fortunately, we will be able to work around this issue.*

Although we don't have the transformation $\vec{T}(u, v)$, we do have the inverse transformation, $\vec{T}^{-1}$, which is defined by

$$\vec{T}^{-1}(x, y) = (xy, y - x).$$

Since $\vec{T}$ and $\vec{T}^{-1}$ are inverse transformations, their derivative matrices $D\vec{T}$ and $D\vec{T}^{-1}$ are also inverses. In order make use of this fact, we compute

$$\begin{aligned} \left| det(D\vec{T}^{-1}(x, y)) \right| &= \left| det \begin{pmatrix} y & x \\ -1 & 1 \end{pmatrix} \right| \\ &= \left| y + x \right| . \end{aligned}$$

Then, from properties of derivatives and inverse matrices, we have

$$\left| det(D\vec{T}(u, v)) \right| = \frac{1}{|y + x|},$$

If we write x and y in terms of u and v.

At first glance, this doesn't seem particularly useful, since we'd still need to find x and y in terms of u and v! However, let's take a look at our integrand, which is $x + y$. Since $x + y \geq 0$ on our region, we have $\dfrac{1}{|y + x|} = \dfrac{1}{y + x}$. So, when we make our change of coordinates, the integrand will cancel with the area expansion factor. This gives us

$$\begin{aligned} \iint_D (x + y) dA &= \int_1^4 \int_0^2 1 \; dv du \\ &= 6. \end{aligned}$$