Multivariable Calculus II

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Part I

Vector Fields and Line Integrals

Vector Fields

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Such a function takes points (x, y) in the plane, and assigns a real number $f(x, y)$ to each of them. There are a few ways that we can visualize these functions. Perhaps the most common way is to graph the function in $\mathbb{R}^3$. The graph consists of the points $(x, y, f(x, y))$, and we often think of the graph as being height $f(x, y)$ over the point (x, y).

PICTURE

We could also visualize the graph in the plane, using a heat map to indicate the "height" $f(x, y)$ at points (x, y).

PICTURE

We could also represent the function by writing the function values $f(x, y)$ at points (x, y). Of course, we can't write these values at all points, because then we wouldn't be able to read them! But for nicely behaved functions, we can choose a representative sample of points, so that the function values at those points accurately reflect the overall behavior of the function.

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This may remind you of a temperature map used to give the temperature across a region.

Now, suppose instead of having a *value* at each point in the plane, we had a *vector*.

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This might be used to represent windspeed and direction, or the direction and strength of any force, such as gravity or a magnetic field.

Let's think about how we can translate this idea into a function. The inputs are still points in the plane, $\mathbb{R}^2$, but now the outputs are vectors, also in $\mathbb{R}^2$. Thus, we have a function $\mathbb{R}^2 \to \mathbb{R}^2$. When we think of such a function as assigning vectors to points in $\mathbb{R}^2$, we call this a vector field.

Learning outcomes: Understand the definition of vector fields and how to graph them.
Author(s): Melissa Lynn

Vector Fields

We've seen that a vector field consists of vectors placed at each point in some region, and we can think of this as assigning a vector to each point in the region, and we can represent this with a function.

Definition 1. *A* vector field *is a function* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}n$*.*

If $\vec{F}$ *is a continuous function, then we say that* $\vec{F}$ *is a* continuous vector field*.*

In this definition, we're thinking about the inputs as points and the outputs as vectors, even though both are in $\mathbb{R}^n$.

Graphing and Scaling

Let's look at how we can visualize vector fields.

Example 1. *Consider the vector field* $\vec{F}(x, y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$*. We'll plot the vectors* $\vec{F}(x, y)$ *starting at points* (x, y) *in order to graph this vector field. To do this, we'll start by evaluating this function at a few points.*

(x, y)	$\vec{F}(x, y)$
$(1, 0)$	$(0, 1)$
$(0, 1)$	$(1, 0)$
$(1, 1)$	$(1/2, 1/2)$
$(2, 0)$	$(0, 1/2)$
$(0, 2)$	$(1/2, 0)$
$(2, 1)$	$(1/5, 2/5)$
$(1, 2)$	$(2/5, 1/5)$
$(2, 2)$	$(1/4, 1/4)$

Now, we can plot these vectors, and visualize the behavior of our vector field.

PICTURE

If the vectors in our vector field are particularly long, graphing our vector field can quickly turn into a cluttered mess. In these situations, it can be useful to scale the vectors, so that we can more clearly see the behavior of our vector field. In fact, most graphing software will automatically scale vector fields, whether you want them to or not.

Example 2. *Consider the vector field* $\vec{G}(x, y) = (-x, y)$*. We'll begin by evaluating this function at several point.*

(x, y)	$\vec{F}(x, y)$
$(0, 0)$	$(0, 0)$
$(1, 0)$	$(-1, 0)$
$(2, 0)$	$(-2, 0)$
$(0, 1)$	$(0, 1)$
$(1, 1)$	$(-1, 1)$
$(2, 1)$	$(-2, 1)$
$(0, 2)$	$(0, 2)$
$(1, 2)$	$(-1, 2)$
$(2, 2)$	$(-2, 2)$

We can plot these vectors to graph the vector field $\vec{G}$.

PICTURE

Because the vectors are long and overlap with each other, it's a bit difficult to get a sense of the behavior of the vector field. To address this issue, we can scale the vectors, to avoid overlap. Below, we scale the vectors by $1/4$.

PICTURE

Here, it's easier to see the behavior of the function, though it's important to remember that the exact lengths of the vectors are no longer accurate.

Flow Lines

Imagine you have a vector field representing gravitational force in space, and that you have a spaceship floating around in space. The spaceship will move in the direction of the gravitational force, following the vector at its position. As the space ship continues to float through space, it will continue to move in the direction prescribed by the vector field, and trace out a path through space.

PICTURE

A path like this is called a flow line of the vector field. This is the path that "matches" the vectors as it moves through the vector fields. This means that the vectors in the vector field should be tangent to the path, and they will actually be the tangent vectors to the path. This leads us to the definition of the flow lines of a vector field.

Flow Lines

Definition 2. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. Then we say that $\vec{x}$ is a* flow line *of $\vec{F}$ if*

$$\vec{x}'(t) = \vec{F}(\vec{x}(t))$$

for all $t \in I$.

Before we figure out how to find flow lines, we'll give a few examples.

Example 3. *Consider the vector field $\vec{F}(x, y) = (-y, x)$, and the path $\vec{x}(t) = (\cos(t), \sin(t))$. We'll verify algebraically that $\vec{x}$ is a flow line of $\vec{F}$.*

First, we compute $\vec{x}'(t)$.

$$\begin{aligned}\vec{x}'(t) &= \left(\frac{d}{dt}\cos(t), \frac{d}{dt}\sin(t)\right) \\ &= (-\sin(t), \cos(t)).\end{aligned}$$

Next, we find $\vec{F}(\vec{x}(t))$.

$$\begin{aligned}\vec{F}(\vec{x}(t)) &= \vec{F}(\cos(t), \sin(t)) \\ &= (-\sin(t), \cos(t)).\end{aligned}$$

Learning outcomes: Understand the definition and geometry of flow lines of a vector field. Verify algebraically that paths are flow lines.

Author(s): Melissa Lynn

We see that this is equal to $\vec{x}'(t)$, so we have verified that $\vec{x}$ is a flow line of $\vec{F}$.

We can also plot $\vec{F}$ and $\vec{x}$, to see how the path $\vec{x}$ follows the vectors of $\vec{F}$.

PICTURE

Example 4. *Consider the vector field $\vec{F}(x, y) = (x, y)$, and the path $\vec{x}(t) = (e^t, e^t)$. We'll verify that $\vec{x}$ is a flow line of $\vec{F}$.*

First, we'll compute $\vec{x}'(t)$.

$$\vec{x}'(t) = \boxed{(e^t, e^t)}$$

Next, we'll compute $\vec{F}(\vec{x}(t))$.

$$\vec{F}(\vec{x}(t)) = \boxed{(e^t, e^t)}$$

So, we can see that $\vec{x}$ is a flow line of the vector field $\vec{F}$.

Let's also graph $\vec{F}$ and $\vec{x}$, so we can see how the path follows the vectors of the vector field.

How to Find Flow Lines

Although it's relatively straightforward to check if a given path is a flow line for a vector field, it can be difficult to compute the flow lines of a vector field. This is because computing flow lines involves solving a system of differential equations, which is not always possible - even when a solution exists! We'll look at a couple of examples where we can find the flow lines.

Example 5. *Consider the vector field $\vec{F}(x, y) = (-x, -y)$. To find flowlines, we need to find the paths $\vec{x}$ such that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$. If we write $\vec{x}(t) = (x(t), y(t))$, we need to solve*

$$(x'(t), y'(t)) = \vec{F}(x(t), y(t)) = (-x(t), -y(t))$$

for $x(t)$ and $y(t)$. We can write this as a system of differential equations,

$$\begin{cases} x'(t) = -x(t) \\ y'(t) = -y(t) \end{cases}$$

Looking at the first equation, we have the solution $x(t) = Ae^{-t}$ for some constant A. From the second equation, we have the solution $y(t) = Be^{-t}$ for some constant B. Putting these together, we have the flowlines

$$\vec{x}(t) = (Ae^{-t}, Be^{-t}),$$

for constants A and B.

If we graph several flow lines and the vector field $\vec{F}$, we can see how the flow lines follow the vectors of the vector field.

PICTURE

Example 6. *Consider the vector field $\vec{F}(x,y) = (-y,x)$. To find flowlines, we need to find the paths $\vec{x}$ such that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$. If we write $\vec{x}(t) = (x(t), y(t))$, we need to solve*

$$(x'(t), y'(t)) = \vec{F}(x(t), y(t)) = (-y(t), x(t))$$

for $x(t)$ and $y(t)$. We can write this as a system of differential equations,

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

Solve this system of differential equations yields $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$, so we have the flow lines

$$\vec{x}(t) = (r\cos(t), r\sin(t))$$

for real numbers r. Notice that these paths trace circles of radius r. We graph a few flow lines along with the vector field $\vec{F}$, to see how the paths follow the vector field.

PICTURE

Gradient Fields

One common way that vector fields arise is through the gradient of a function. That is, suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$. Then we can find the gradient of f,

$$\nabla f(x_1, ..., x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

This gradient is a function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$, so it can be thought of as a vector field in $\mathbb{R}^n$, which we call the *gradient field* of the function f.

Gradient Fields

Let's look at some examples of gradient fields.

Example 7. *Consider the function $f(x, y) = x^2 + y^2$. The gradient of this function is*

$$\nabla f(x, y) = (2x, 2y).$$

We graph this gradient field below.

PICTURE

Example 8. *Consider the function $f(x, y) = xy$. The gradient of this function is*

$$\nabla f(x, y) = (y, x).$$

We graph this gradient field below.

PICTURE

Conservative Vector Fields

Taking a slightly different perspective, we can start with a vector field $\vec{F}$, and determine whether we can find a function f such that $\vec{F} = \nabla \vec{f}$, so that $\vec{F}$ is the gradient field of f. If this is possible, we say that $\vec{F}$ is a conservative vector field.

Learning outcomes: Given a scalar valued function, compute and graph its gradient field. Given a vector field, determine if it is conservative, and find a potential function. Use the derivative matrix to show that a vector field is not conservative.

Author(s): Melissa Lynn

Definition 3. *A vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is* conservative *if there is some function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = \vec{F}$. Then f is called a* potential function *for* $\mathbf{F}$.

So, one way to show that a vector field $\mathbf{F}$ is conservative is by finding a potential function f.

For simple examples, you might be able to do this by guessing. However, more complicated examples require a more systematic approach. The approach is easiest to understand through examples, so we'll work through a couple before describing the steps for the general case.

Example 9. *Find a potential function for the vector field $\mathbf{F}(x, y) = (2xy^3 + 1, 3x^2y^2 - y^{-2})$.*

Explanation. *First, note that if there there is a function f such that $\nabla f = \mathbf{F}$, then*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2xy^3 + 1, 3x^2y^2 - y^{-2}\right)$$

Let's start by looking at the x-term. We must have $\frac{\partial f}{\partial x} = 2xy^3 + 1$. Integrating with respect to x, we have

$$\begin{aligned} f(x, y) &= \int 2xy^3 + 1\, dx \\ &= x^2y^3 + x + g(y) \end{aligned}$$

The first part of this expression, $x^2y^3 + x$, is an antiderivative for $2xy^3 + 1$ with respect to x. The second part of the expression, $g(y)$, is the "constant" for the integral. It's possible that there are some terms which depend only on y, hence are constant with respect to x, and writing $g(y)$ takes these terms into account.

At this point, we know that f has the form $f(x, y) = x^2y^3 + x + g(y)$, but we still need to figure out what $g(y)$ is. For this, we use the y-term of the vector field $\mathbf{F}$. From this, we have $\frac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$. Since we know $f(x, y) = x^2y^3 + x + g(y)$, we must have

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2y^3 + x + g(y)\right) \\ &= 3x^2y^2 + g'(y). \end{aligned}$$

Comparing this with $\frac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$, we must have $g'(y) = -y^{-2}$. Then we find that

$$\begin{aligned} g(y) &= \int -y^{-2}\, dy \\ &= y^{-1} + C \end{aligned}$$

Hence, any potential function would have the form $f(x,y) = x^2y^3+x+y^{-1}+C$. *Choosing* $C=0$*, we obtain a specific potential function* $f(x,y) = x^2y^3+x+y^{-1}$.

We now work through finding a potential function for a three dimensional vector field.

Example 10. *Find a potential function for the vector field* $\mathbf{F}(x,y,z) = (2xy, x^2+z+2y, y+\cos(z))$.

Explanation. *First, note that a potential function* $f(x,y,z)$ *would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\boxed{2xy}, \boxed{x^2+z+2y}, \boxed{y+\cos(z)}\right)$$

We begin by considering the x*-component, noticing that* $\dfrac{\partial f}{\partial x} = 2xy$. *We integrate with respect to* x.

$$\begin{aligned} f(x,y,z) &= \int 2xy\,dx \\ &= \boxed{x^2y} + g(y,z) \end{aligned}$$

Here, $g(y,z)$ *is a function of only* y *and* z*, hence constant with respect to* x. *We now differentiate with respect to* y*, in order to compare to the* y*-component of the vector field.*

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\left(x^2y + g(y,z)\right) \\ &= \boxed{x^2} + \frac{\partial g}{\partial y} \end{aligned}$$

Comparing this with $\dfrac{\partial f}{\partial y} = x^2+z+2y$*, we have* $\dfrac{\partial g}{\partial y} = z+2y$. *We integrate this with respect to* y.

$$\begin{aligned} g(y,z) &= \int z+2y\,dy \\ &= \boxed{yz+y^2} + h(z) \end{aligned}$$

Here, h *is a function of only* z*, hence is constant with respect to* y. *We now know that* f *has the form* $f(x,y,z) = x^2y+yz+y^2+h(z)$. *So, our final task is to find* $h(z)$. *We differentiate* f *with respect to* z *in order to compare with the* z*-component of the vector field.*

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2y+yz+y^2+h(z)\right) \qquad = \boxed{y} + h'(z)$$

Comparing this with $\dfrac{\partial f}{\partial z} = y + \cos(z)$*, we have* $h'(z) = \cos(z)$*. Integrating with respect to* z*, we obtain*

$$\begin{aligned} h(z) &= \int \cos(z)\, dz \\ &= \boxed{\sin(z)} + C \end{aligned}$$

Where C *is a constant. Thus, any potential function would have the form* $f(x,y,z) = \boxed{x^2y + yz + y^2 + \sin(z)} + C$*. Choosing* $C = 0$*, we have a specific potential function* $f(x,y,z) = \boxed{x^2y + yz + y^2 + \sin(z)}$.

Summarizing the steps we take in each of the above examples, we have the following process for finding a potential function for a conservative vector field $\mathbf{F}(x_1, x_2, ..., x_n)$.

(a) Integrate the first component of $\mathbf{F}$ with respect to x_1, in order to find the terms of $f(x_1, x_2, ..., x_n)$ which depend on x_1. From this, we can write $f(x_1, x_2, ..., x_n) = (x_1\text{-terms}) + f_1(x_2, ..., x_n)$.

(b) Differentiate $f(x_1, x_2, ..., x_n) = (x\text{-terms}) + f_1(x_2, ..., x_n)$ with respect to x_2. Compare this to the second component of $\mathbf{F}$ in order to determine an expression for $\dfrac{\partial f_1}{\partial x_2}$. Integrate this expression with respect to x_2, so we can write $f_1(x_2, ..., x_n) = (x_2\text{-terms}) + f_2(x_3, ..., x_n)$. Hence we have $f(x_1, x_2, ..., x_n) = (x_1\text{- and } x_2\text{-terms}) + f_2(x_3, ..., x_n)$.

(c) Repeat this process until all components are used.

So far, we've only seen cases where a potential function exists. However, we would also like to be able to show that a vector field is *not* conservative. Let's look at what happens in our process when we have a vector field which is not conservative.

Example 11. *Try (and fail) to find a potential function for the vector field* $\mathbf{F}(x,y) = (-y, x)$*.*

Explanation. *If a potential function existed, it would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (\boxed{-y}, \boxed{x})$$

We begin with $\dfrac{\partial f}{\partial x} = -y$*. Integrating with respect to* x*, we have*

$$\begin{aligned} f(x,y) &= \int -y\, dx \\ &= \boxed{-yx} + g(y) \end{aligned}$$

Differentating with respect to y, we then obtain

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-yx + g(y)\right) \\ &= \boxed{-x} + g(y)\end{aligned}$$

When we compare this to the y-component of the vector field $\mathbf{F}$ in order to determine $g(y)$, we would have to have $x = -x + g(y)$. But this is impossible! Thus we see that our method has broken down, and we are not able to find a potential function.

Here, we see that the system breaks down, and we aren't able to produce a potential function. This is good, since it turns out the vector field isn't conservative. However, we would an easy way to prove that it isn't conservative. The following theorem gives us a quick way to prove that a vector field is not conservative.

Theorem 1. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $\mathbf{F}$ is conservative, then $D\mathbf{F}$ is symmetric.*

The contrapositive of this theorem states:

Theorem 2. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $D\mathbf{F}$ is not $\boxed{\textit{symmetric}}$ then $\mathbf{F}$ is not $\boxed{\textit{conservative}}$.*

Thus, provided we have a C^1 vector field and the domain is open and connected, we can show a vector field is not conservative by showing that its derivative matrix is not symmetric.

Example 12. *Show that the vector field $\mathbf{F}(x, y) = (-y, x)$ is not conservative.*

Explanation. *First, note that $\mathbf{F}$ is a C^1 vector field with domain $\mathbb{R}^2$. Since $\mathbb{R}^2$ is open and connected, our theorem applies. We compute the derivative matrix $D\mathbf{F}$.*

$$\begin{aligned}D\mathbf{F} &= \begin{pmatrix} \frac{\partial}{\partial x}(-y) & \frac{\partial}{\partial y}(-y) \\ \frac{\partial}{\partial x}x & \frac{\partial}{\partial y}x \end{pmatrix} \\ &= \begin{pmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{pmatrix}\end{aligned}$$

Since this matrix is not symmetric, $\mathbf{F}$ is not a conservative vector field.

Note how much simpler this is than trying to find a potential function. We now prove our theorem, showing that a conservative C^1 vector field on an open and connected domain has symmetric derivative.

Proof Let $\mathbf{F}$ be a C^1 vector field defined on an open connected domain $X \subset \mathbb{R}^n$. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some scalar-valued function f on X. This means

$$\mathbf{F}(x_1, ..., x_n) = \nabla f(x_1, ..., x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

Then the derivative matrix of $\mathbf{F}$ is

$$D\mathbf{F} = D(\nabla f) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

By Clairaut's Theorem, the mixed partials are equal, so this matrix is symmetric. ■

For each of the given vector fields $\mathbf{F}$, determine whether or not it's conservative. If it is conservative, find a potential function. If $\mathbf{F}$ is not conservative, compute the derivative matrix of $\mathbf{F}$ in order to prove that it is not conservative.

Problem 1 $\mathbf{F}(x, y) = (2xy + y^2 + e^y, x^2 + 2xy + xe^y)$

Multiple Choice:

(a) *conservative* ✓

(b) *not conservative*

Problem 1.1 $f(x, y) = \boxed{x^2y + y^2x + e^yx}$

Problem 2 $\mathbf{F}(x, y) = (x^2y + e^{x^2}, \sin(x) + y^3)$

Multiple Choice:

(a) *conservative*

(b) *not conservative* ✓

Problem 2.1 $D\mathbf{F}(x, y) = \begin{pmatrix} 2xy + 2xe^{x^2} & \boxed{x^2} \\ \boxed{\cos(x)} & 3y^2 \end{pmatrix}$

We've seen that if a vector field is conservative, then its derivative matrix is symmetric. But is the converse true? That is, if the derivative matrix is symmetric, does that mean that the vector field is conservative? We'll come back to this question later.

Vector Line Integrals

Suppose we have a vector field $\vec{F}$ and a path $\vec{x}$ in $\mathbb{R}^2$. Imagine that the vector field represents some force, such as gravity or a magnetic field. Also imagine that a particle is traveling along the path.

PICTURE

Suppose we would like to measure the total effect of the force on the movement of the particle along the path. In physics, this is called the *work* done by the force on the particle.

If the particle is moving against the vector field, then the force impedes the progress of the particle, so the work done by the force is negative.

PICTURE

If the particle is moving with the vector field, then the force aids the progress of the particle so the work done by the force is positive.

PICTURE

If the particle is moving perpendicular to the vector field, then the force neither impedes nor aids the progress of the particle, so the work done is zero.

PICTURE

In order to compute the work done by a force on the particle, we need to "add up" the microscopic contributions of the force at each point along the path. This leads us to the definition of vector line integrals.

Vector Line Integrals

We can measure the microscopic contribution of a force to the motion of a particle using dot products. That is, if we have a vector field $\vec{F}$ and a path $\vec{x}$ in $\mathbb{R}^n$, we consider the dot product $\vec{F}(x(t)) \cdot \vec{x}'(t)$.

PICTURE

This compares the vector field at the point $\vec{x}(t)$ with the velocity vector $\vec{x}'(t)$ of the path. Notice that if $\vec{F}$ is perpendicular to $\vec{x}'(t)$, then this dot product is zero, which is consistent with our intuition.

In order to find the total contribution of the vector field to the motion along

Learning outcomes: Understand the definition of vector line integrals geometrically, and be able to compute them.

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the path, we integrate this dot product from the start of the path to the end. This leads us to the definition of a vector line integral.

Definition 4. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $\vec{x} : [a,b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ path in $\mathbb{R}^n$. Then the* vector line integral *of $\vec{F}$ along $\vec{x}$ is*

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt.$$

Let's look at a couple of examples of computing vector line integrals.

Example 13. *Consider the vector field $\vec{F}(x,y) = (x,y)$ and the path $\vec{x}(t) = (t\cos(t), t\sin(t))$ for $t \in [0, 2\pi]$.*

PICTURE

We'll compute the vector line integral, $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s}$.

$$\begin{aligned}
\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\
&= \int_0^{2\pi} \vec{F}(t\cos(t), t\sin(t)) \cdot (\cos(t) - t\sin(t), \sin(t) + t\cos(t))\ dt \\
&= \int_0^{2\pi} (t\cos(t), t\sin(t)) \cdot (\cos(t) - t\sin(t), \sin(t) + t\cos(t))\ dt \\
&= \int_0^{2\pi} t\cos^2(t) - t^2\cos(t)\sin(t) + t\sin^2(t) + t^2\cos(t)\sin(t)\ dt \\
&= \int_0^{2\pi} t\ dt \\
&= \frac{1}{2}t^2|_0^{2\pi} \\
&= 2\pi^2
\end{aligned}$$

Example 14. *Consider the vector field $\vec{F}(x,y) = (-x,-y)$ and the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$.*

PICTURE

We'll compute the vector line integral, $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s}$.

$$\begin{aligned}
\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\
&= \int_0^{2\pi} \vec{F}(\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t))\ dt \\
&= \int_0^{2\pi} (-\cos(t), -\sin(t)) \cdot (-\sin(t), \cos(t))\ dt \\
&= \int_0^{2\pi} (\cos(t)\sin(t) - \sin(t)\cos(t))\ dt \\
&= \int_0^{2\pi} 0\ dt \\
&= 0
\end{aligned}$$

Since the vector field is always perpendicular to this path, it makes sense that the vector line integral should come out to be zero.

Circulation

Now, let's consider the special case where we integrate over a closed curve. In this case, we refer to the value of the vector line integral as circulation, and we use some special notation.

Definition 5. *Let* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a vector field, and let* $\vec{x} : [a, b] \to \mathbb{R}^n$ *be a* $\mathcal{C}^1$ *path in* $\mathbb{R}^n$*. Suppose further that* $\vec{x}(a) = \vec{x}(b)$*, so that* $\vec{x}$ *is a closed curve. Then the* circulation *of* $\vec{F}$ *along* $\vec{x}$ *is* $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s}$*, and we write*

$$\oint_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{x}} \vec{F} \cdot d\vec{s},$$

to emphasize that $\vec{x}$ *parametrizes a closed curve.*

Scalar Line Integrals

We've seen how we can integrate vector fields along a path, using vector line integrals. We can also integrate scalar valued functions along a path. For instance, suppose we have a scalar valued function $f : \mathbb{R}^2 \to \mathbb{R}$ and a path $\vec{x} : [a, b] \to \mathbb{R}^2$ in $\mathbb{R}^2$. Suppose we look at the portion of the graph of f lying over the path $\vec{x}$, and drop a "curtain" to the xy-plane.

PICTURE

Integrating f along the path $\vec{x}$ will be equivalent to finding the area of this curtain. We can also describe this as the area between $\vec{x}(t)$ and $f(\vec{x}(t))$.

Scalar line integrals aren't only useful for finding areas of strangely shapes regions. They are also useful throughout physics. For example, if you have the mass density function of a wire, you can compute the scalar line integral of this function to find the total mass of the wire.

Scalar Line Integrals

Suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$ and a $\mathcal{C}'$ path $\vec{x} : [a, b] \to \mathbb{R}^n$, such that the composition $f(\vec{x}(t))$ is defined on $[a, b]$.

PICTURE

In order to find the area under f and over the path $\vec{x}$, we will borrow an important idea from single variable calculus: approximating an area with rectangles.

In order to do this, we'll partition the interval $[a, b]$ into n subintervals, determined by

$$a = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n = b.$$

This partition breaks the path $\vec{x}$ into smaller paths, by restricting to the subintervals.

PICTURE

Our goal will be to approximate the area under f over each of these shorter paths. These approximations will be computed by finding the length of the short path, and multiplying this by a height determined by a test point, t_k^*. We can think of this as a curved rectangle.

PICTURE

Learning outcomes: Understand the definition of scalar line integrals geometrically, and be able to compute them.

Author(s): Melissa Lynn

The specific choice of the test point will be important for simplifying our result - we'll come back to this later.

The height of our curved rectangle will be $f(\vec{x}(t_k^*))$, and the base is the distance Δs_k along the path $\vec{x}$ from t_{k-1} to t_k.

PICTURE

Thinking back to arclength computations, this distance is given by the integral

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt.$$

Now, we need to make a careful choice for our test point t_k^* which will simplify things later. To do this, recall the Mean Value Theorem for Integrals, from single variable calculus.

Theorem 3. *Suppose g is a continuous function on the closed interval $[a, b]$. Then there exists c in $[a, b]$ such that*

$$\int_a^b g(t)dt = (b-a)g(c).$$

Here, we'll take $\|\vec{x}'(t)\|$ for the function $g(t)$. Since $\vec{x}$ is $\mathcal{C}^1$, $\|\vec{x}'(t)\|$ is continuous. Applying the Mean Value Theorem on the interval $[t_{k-1}, t_k]$, there exists c_k such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt = (t_k - t_{k-1})\|\vec{x}'(c_k)\|.$$

We take this c_k to be our test point, so that $t_k^* = c_k$.

Now, the area of the kth curved rectangle is $F(\vec{x}(t_k^*))\Delta s_k$. We add up these areas and take the limit as the number of rectangles, n, goes to infinity:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\vec{x}(t_k^*))\Delta s_k.$$

Substituting $\Delta s_k = (t_k - t_{k-1})\|\vec{x}'(c_k)\|$ and writing $\Delta t = t_k - t_{k-1}$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\vec{x}(t_k^*))\|\vec{x}'(c_k)\|\Delta t.$$

We can recognize this as the single variable integral of the function $f(\vec{x}(t))\|\vec{x}'(t)\|$ over the interval $[a, b]$,

$$\int_a^b f(\vec{x}(t))\|vecx'(t)\| dt.$$

We take this to be the definition of a scalar line integral.

Definition 6. *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a $\mathcal{C}^1$ path $\vec{x} : [a, b] \to \mathbb{R}^n$. The* scalar line integral *of f along $\vec{x}$ is*

$$\int_{\vec{x}} f\ ds = \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\| dt.$$

Examples

Let's look at some examples of computing scalar line integrals.

Example 15. *Consider the function $f(x,y) = x^2 + y^2$ and the path $\vec{x}(t) = (\cos t, \sin t)$ for $t \in [0,\pi]$. We compute the scalar line integral of f over $\vec{x}$.*

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt \\
&= \int_0^\pi f(\cos t, \sin t)\|(-\sin t, \cos t)\|dt \\
&= \int_0^\pi (\cos^2 t + \sin^2 t)\sqrt{\sin^2 t + \cos^2 t}dt \\
&= \int_0^\pi 1\ dt \\
&= \pi
\end{aligned}$$

Example 16. *Consider the function $f(x,y) = e^{\sqrt{xy}}$ and the path $\vec{x}(t) = (t,t)$ for $t \in [0,1]$. We compute the scalar line integral of f over $\vec{x}$.*

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt \\
&= \int_0^1 f(t,t)\|(1,1)\|dt \\
&= \int_0^1 e^{\sqrt{t^2}}\sqrt{2}dt \\
&= \sqrt{2}\int_0^1 e^t dt \\
&= \sqrt{2}(e^1 - e^0) \\
&= \sqrt{2}e - \sqrt{2}
\end{aligned}$$

Line Integrals over Simple Curves

We've defined vector and scalar line integrals over paths.

Definition 7. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $\vec{x} : [a, b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ path in $\mathbb{R}^n$. Then the* vector line integral *of $\vec{F}$ along $\vec{x}$ is*

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt.$$

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a $\mathcal{C}^1$ path $\vec{x} : [a, b] \to \mathbb{R}^n$. The scalar line integral *of f along $\vec{x}$ is*

$$\int_{\vec{x}} f\ ds = \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt.$$

Both of these definitions were motivated by geometric considerations. For vector line integrals, we wanted to find the effect of a vector field on a particle moving along a curve. For scalar line integrals, we wanted the find the under the graph of a function over a curve. Although the definitions were motivated by questions about curves, we wound up with definitions that seem to depend on a parametrization of the curve. So, there's a natural follow-up question: do line integrals depend on the parametrization of the curve?

To answer this question, we'll focus on simple curves.

Definition 8. *A path $\vec{x} : [a, b] \to \mathbb{R}^n$ is* simple *if $\vec{x}$ is a one-to-one function (except we'll allow $\vec{x}(a) = \vec{x}(b)$).*

A curve is simple *if it can be parametrized by a simple path.*

Essentially, a curve is simple if it doesn't intersect itself. If a curve starts and ends at the same point, but doesn't intersect itself otherwise, we'll still say that the curve is simple.

Let's look at some examples of curves, and determine whether they are simple.

Example 17. *For each of the curves, decide if it is simple.*

PICTURES/MULTIPLE CHOICE

For scalar line integrals, if we have two simple paths parametrizing the same curve, the scalar line integrals will be the same.

Learning outcomes: Understand the independence from parametrization of line integrals for simple curves.

Author(s): Melissa Lynn

Proposition 1. *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a curve, which has simple $\mathcal{C}^1$ parametrizations $\vec{x} : [a, b] \to \mathbb{R}^n$ and $\vec{y} : [c, d] \to \mathbb{R}^n$. Then*

$$\int_{\vec{x}} f\ ds = \int_{\vec{y}} f\ ds.$$

Let's look at an example to see why it is important to have a simple path.

Example 18. *Consider the function $f(x, y) = x$ and the paths $\vec{x}(t) = (t, t)$ for $t \in [0, 1]$ and $\vec{y}(t) = (t^2, t^2)$ for $t \in [-1, 1]$. Note that $\vec{x}$ and $\vec{y}$ are both $\mathcal{C}^1$ parametrizations of the line segment connecting $(0, 0)$ and $(1, 1)$. However, $\vec{y}$ is not a simple parametrization; it traverses the line segment by starting at $(1, 1)$, moving along the segment to $(0, 0)$, and then returning to $(1, 1)$. We'll evaluate the scalar line integrals of f along these paths.*

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_0^1 f(t, t)\|(1, 1)\|dt \\
&= \int_0^1 t\sqrt{2}dt \\
&= \sqrt{2}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) \\
&= \frac{\sqrt{2}}{2} \\
\int_{\vec{y}} f\ ds &= \int_{-1}^1 f(t^2, t^2)\|(2t, 2t)\|dt \\
&= \int_{-1}^1 t^2\sqrt{4t^2 + 4t^2}dt \\
&= \sqrt{8}\int_{-1}^1 t^3 dt \\
&= \sqrt{8}\left(\frac{1^4}{4} - \frac{(-1)^4}{4}\right) \\
&= 0
\end{aligned}$$

Next, we'll turn our attention to vector line integrals. For vector line integrals, we also need to consider the orientation of the curve. That is, we need to consider the direction in which we traverse the curve. Notice that a simple curve has exactly two choices of orientation.

PICTURE

The sign of a vector line integral will depend on the orientations of the paths. Let's think about why this is true. Suppose that a vector field impedes the progress of a particle moving along a path. If we reverse the direction of the path, the vector field will contribute to the motion of the particle. Thus, the sign of the vector line integral would change.

PICTURE

Apart from the issue of orientation, vector line integrals will be independent of the parametrization for simple paths.

Proposition 2. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and consider a curve parametrizated by simple $\mathcal{C}^1$ parametrizations $\vec{x} : [a, b] \to \mathbb{R}^n$ and $\vec{y} : [c, d] \to \mathbb{R}^n$.*

If $\vec{x}$ and $\vec{y}$ have the same orientation, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

If $\vec{x}$ and $\vec{y}$ have opposite orientations, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = -\int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

In the following exercise, it's important to consider the orientation of paths as you determine which vector line integrals are equal.

Problem 3 *Consider the following paths, which parametrize the line segment between $(1, 1, 1)$ and $(1, 2, 3)$.*

$$\vec{x}(t) = (1, 1 + t, 1 + 2t) \text{ for } t \in [0, 1]$$
$$\vec{y}(t) = (1, 1 + t^2, 1 + 2t^2) \text{ for } t \in [0, 1]$$
$$\vec{y}(t) = (1, 1 + t^2, 1 + 2t^2) \text{ for } t \in [-1, 1]$$

Let $\vec{F}$ be a vector field, and consider the vector line integrals $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$, $\int_{\vec{y}} \vec{F} \cdot d\vec{s}$, and $\int_{\vec{z}} \vec{F} \cdot d\vec{s}$. Which of these line integrals are equal?

Multiple Choice:

(a) *None of them are equal.*

(b) $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$ and $\int_{\vec{y}} \vec{F} \cdot d\vec{s}$ *and equal* ✓

(c) $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$ and $\int_{\vec{z}} \vec{F} \cdot d\vec{s}$ *and equal*

(d) $\int_{\vec{y}} \vec{F} \cdot d\vec{s}$ and $\int_{\vec{z}} \vec{F} \cdot d\vec{s}$ *and equal*

(e) *All of them are equal.*

Path-Connected and Simply Connected Regions

We will soon begin to study properties of line integrals and conservative vector fields. In order to do this, we need to pay attention to the topology of our domains. Recall that we've previously classified sets in $\mathbb{R}^n$ as open, closed, or neither.

We'll begin by recalled the definition of an open set.

Definition 9. *In $\mathbb{R}^n$, we call $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$ the* open ball *of radius $r > 0$ centered at $\mathbf{x}$.*

A set $U \subset \mathbb{R}^n$ is open *if for every $\mathbf{a} \in U$, there is a radius $r > 0$ such that $B_r(\mathbf{a}) \subset U$.*

In words, for any point $\mathbf{a}$ in U, we can find a radius r small enough that the entire ball of radius r centered at $\mathbf{a}$ is contained in U. We can also restate the definition of an open set as "every point is an interior point."

Example 19. *The following are examples of open sets.*

$\{x : 0 < x < 1\}$ *in* $\mathbb{R}$.

PICTURE

$\{(x, y) : 0 < x < 1\}$ *in* $\mathbb{R}^2$.

PICTURE

$\{(x, y) : x^2 + y^2 < 1\}$ *in* $\mathbb{R}^2$.

PICTURE

We now recall the definition of a closed set, which is given relative to open sets.

Definition 10. *A set $X \subset \mathbb{R}^n$ is* closed *if its complement is open.*

Furthermore, a set is closed if and only if it contains all of its boundary points.

Example 20. *The following are examples of closed sets.*

$\{x : 0 \leq x \leq 1\}$ *in* $\mathbb{R}$.

PICTURE

Learning outcomes: Understand the geometry and definitions of connected and simply connected sets.

Author(s): Melissa Lynn

$\{(x, y) \,:\, 0 \le x \le 1\}$ *in* $\mathbb{R}^2$.

PICTURE

$\{(x, y) \,:\, x^2 + y^2 \le 1\}$ *in* $\mathbb{R}^2$.

PICTURE

For the results we wish to prove about line integrals and conservative vector fields, we will also need to consider if sets are connected, and how they are connected.

Path-connected sets

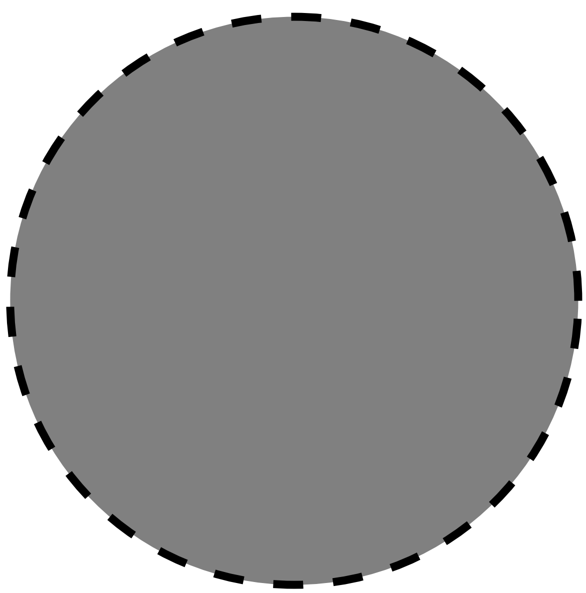
You probably have a pretty good intuitive idea of what it should mean for a set to be connected: if you imagine the set to be land, and the complement of the set to be lava, then you can get to the entire set while staying on land, and without jumping. This idea translates to mathematics, using paths.

Definition 11. *A set $X \subset \mathbb{R}^n$ is path-connected if any two points can be connected by a path which lies entirely in X.*

Notice that we use the word "path-connected" instead of just connected. This is because "connected" actually has a different meaning in topology. However, in some situations, "connected" and "path-connected" are equivalent.

Problem 4 *For each of the following sets, determine whether or not they are path-connected.*

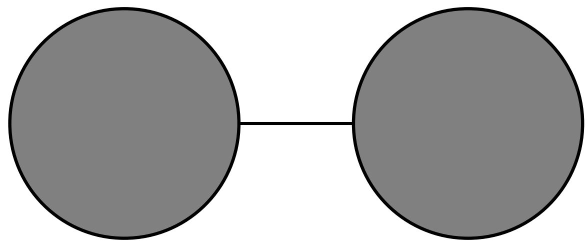
(a)



Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*

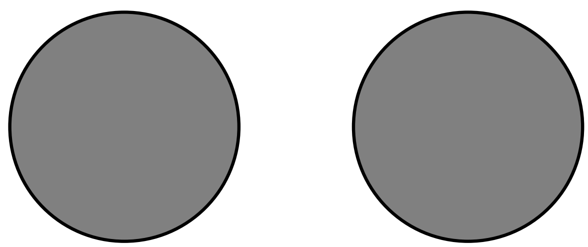


(b)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*



(c)

Multiple Choice:

(i) *path-connected*

(ii) *not path-connected* ✓

Simply Connected Sets

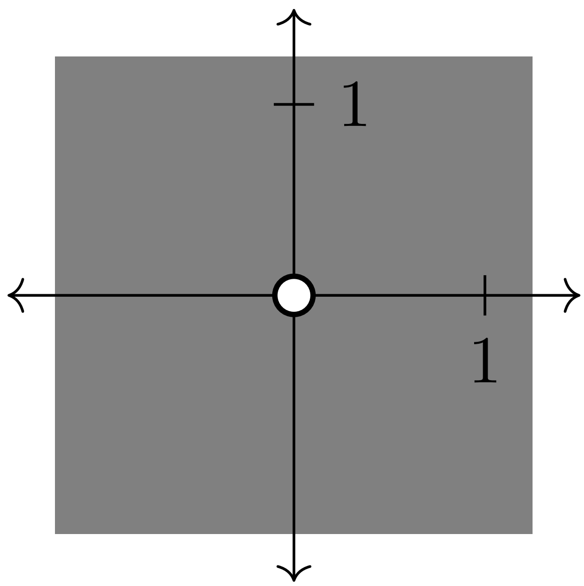
The last topological concept we will cover is when a path-connected set is "simply connected." Intuitively, this depends on whether or not the set has holes. Our definition for simply connected is a bit hand wavy, but this will serve our purpose just fine. It can be made rigorous using continuous maps.

Definition 12. *A path-connected set $X \subset \mathbb{R}^n$ is* simply connected *if any closed path (i.e., loop) can be shrunk to a point, where the shrinking occurs entirely in X.*

Note that a set needs to be path-connected in order to be considered simply connected.

Problem 5 *For each of the following sets, determine whether or not they are simply connected.*

(a) $\mathbb{R}^2 \setminus \{(0,0)\}$

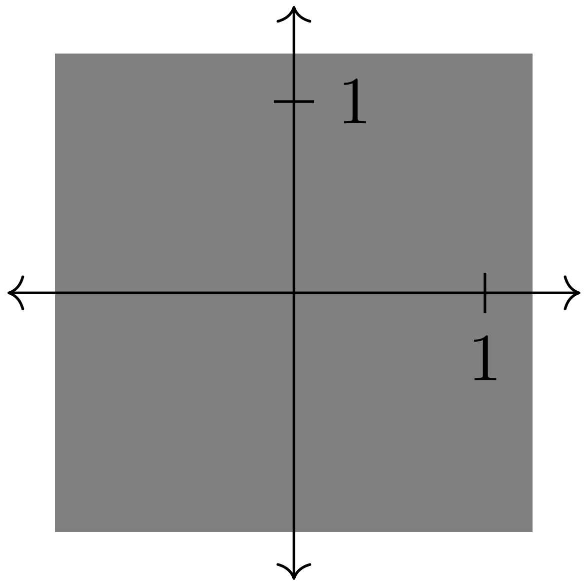


Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

(b) $\mathbb{R}^2$

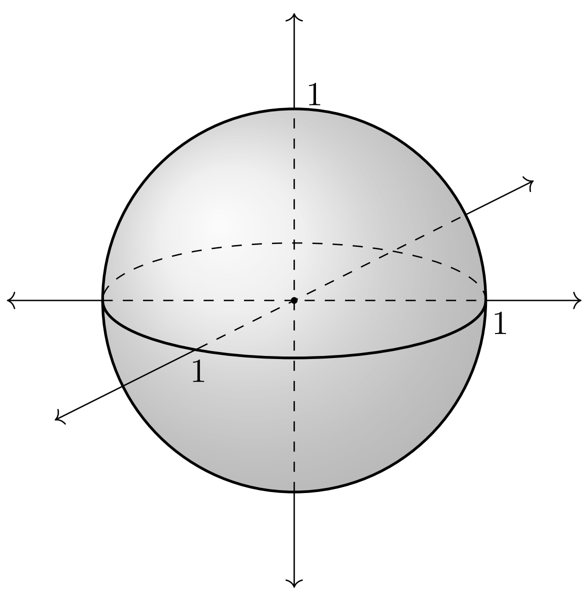


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

(c) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

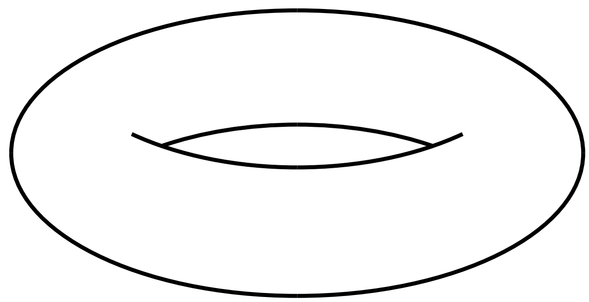


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

(d) *A torus (the surface of a donut)*



Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

Path Independence and FTLI

We've previously seen how vector line integrals are (mostly) independent of the parametrization of the curve, which we restate in the following theorem.

Theorem 4. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and consider a curve parametrizated by simple $\mathcal{C}^1$ parametrizations $\vec{x} : [a,b] \to \mathbb{R}^n$ and $\vec{y} : [c,d] \to \mathbb{R}^n$.*

If $\vec{x}$ and $\vec{y}$ have the same orientation, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

If $\vec{x}$ and $\vec{y}$ have opposite orientations, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = -\int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

Now, let's suppose we have two different curves, but they start and end at the same points. Will vector line integrals over these paths be equal?

PICTURE

Surprisingly, for some vector fields, the answer is "yes." This is called path independence, and we'll explore which vector fields have this property.

Path Independence

A vector field is path independent if its line integrals depend only on the start and end points, and not on the path we take to get between the points.

Definition 13. *A continuous vector field is called* path independent *if* $\int_C \vec{F} \cdot d\boldsymbol{s} = \int_D \vec{F} \cdot d\boldsymbol{s}$ *for any two simple, piecewise C^1, oriented curves C and D with the same start and end points.*

Let's quickly review the meaning of the requirements on the curves C and D.

A curve is *simple* if it (isn't too "bumpy." / doesn't intersect itself, except the start and end point can be the same. ✓/ is smooth.)

Learning outcomes: Understand the definition of path independent. Use the Fundamental Theorem of Line Integrals to compute vector line integrals of conservative vector fields, or to show that a vector field is path independent.

Author(s): Melissa Lynn

A curve is C^1 if it (is continuous. / is differentiable. / has continuous partial derivatives. ✓)

A curve is *oriented* if it (has a specified direction. ✓/ knows which way is North.)

Let's look at some examples.

Example 21. *Consider the vector field* $\vec{F}(x, y) = (y, 0)$*. Let* $\boldsymbol{x}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *along a straight line. Let* $\boldsymbol{y}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *counterclockwise around the unit circle. Compute* $\displaystyle\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\displaystyle\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$*. Are they equal?*

We'll begin by parametrizing our paths.

$$\boldsymbol{x}(t) = (\boxed{1-t}, \boxed{t}) \qquad \textit{for } t \in [0, 1]$$

$$\boldsymbol{y}(t) = (\boxed{\cos(t)}, \boxed{\sin(t)}) \qquad \textit{for } t \in [0, \frac{\pi}{2}]$$

Now, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} = \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Integrating along $\boldsymbol{x}$*, we have*

$$\begin{aligned}
\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\
&= \int_{\boxed{0}}^{\boxed{1}} (t, 0) \cdot (-1, 1)\, dt \\
&= \int_{\boxed{0}}^{\boxed{1}} -t\, dt \\
&= \boxed{-\frac{1}{2}}
\end{aligned}$$

Integrating along $\boldsymbol{y}$*, we have*

$$\begin{aligned}
\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\
&= \int_0^{\pi/2} (\sin(t), 0) \cdot (\boxed{-\sin(t)}, \boxed{\cos(t)})\, dt \\
&= \int_0^{\pi/2} \boxed{-\sin^2(t)}\, dt \\
&= \boxed{-\frac{\pi}{4}}
\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$*, we see that they are*

Multiple Choice:

(a) *Equal.*

(b) *Not equal.* ✓

As a result, we know that the vector field $\vec{F}$*...*

Multiple Choice:

(a) *...is path independent.*

(b) *...is not path independent.* ✓

(c) *...might be path independent. There isn't enough information to tell.*

Now, let's investigate integrating a different vector field along those same paths.

Example 22. *Consider the vector field* $\vec{F}(x, y) = (y, x)$*. Let* $\boldsymbol{x}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *along a straight line. Let* $\boldsymbol{y}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *counterclockwise around the unit circle. Compute* $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$*. Are they equal?*

Once again, we begin by parametrizing our paths.

$$\boldsymbol{x}(t) = (1 - t, t) \qquad \textit{for } t \in [0, 1]$$

$$\boldsymbol{y}(t) = (\cos(t), \sin(t)) \qquad \textit{for } t \in [0, \frac{\pi}{2}]$$

Next, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} = \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Integrating along $\boldsymbol{x}$*, we have*

$$\begin{aligned}
\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\
&= \int_0^1 (\boxed{t}, \boxed{1 - t}) \cdot (-1, 1)\, dt \\
&= \int_0^1 \boxed{1 - 2t}\, dt \\
&= \boxed{0}
\end{aligned}$$

Integrating along $\boldsymbol{y}$, we have

$$\begin{aligned}\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\ &= \int_0^{\pi/2} (\boxed{\sin(t)}, \boxed{\cos(t)}) \cdot (-\sin(t), \cos(t))\, dt \\ &= \int_0^{\pi/2} \boxed{-\sin^2(t) + \cos^2(t)}\, dt \\ &= \boxed{0}\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$*, we see that they are*

Multiple Choice:

(a) *Equal.* ✓

(b) *Not equal.*

As a result, we know that the vector field $\vec{F}$*...*

Multiple Choice:

(a) *...is path independent.*

(b) *...is not path independent.*

(c) *...might be path independent. There isn't enough information to tell.* ✓

In fact, it turns out that the vector field $\vec{F}(x, y) = (y, x)$ is path independent. However, in order to check directly that a vector field $\vec{F}$ is path independent, we would need to check the line integrals over any path between any two points in the domain. Of course, this is impossible! We will need to find different methods for showing that a vector field is path independent. Our first result in this direction is the Fundamental Theorem of Line Integrals.

Fundamental Theorem of Line Integrals

We now introduce the Fundamental Theorem of Line Integrals, which gives us a powerful way to compute the integral of a gradient vector field over a piecewise C^1 curve. In particular, note the conditions on the domain X: it must be open and path-connected.

Theorem 5. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and path-connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{B}) - f(\boldsymbol{A})$$

This should look vaguely familiar - it resembles the Fundamental Theorem of Calculus from single variable calculus, also called the evaluation theorem.

We now prove the Fundamental Theorem of Line Integrals in the special case where we have a *simple* parametrization of the curve C.

Proof Let $\mathbf{x}(t)$ be a simple parametrization of C, where $t \in [a, b]$, $\mathbf{x}(a) = \mathbf{A}$, and $\mathbf{x}(b) = \mathbf{B}$ (so the starting point is $\mathbf{A}$, and the ending point is $\mathbf{B}$).

Then we compute the line integral as:

$$\int_C \nabla f \cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s}.$$

By the definition, we can compute this line integral as

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

The integrand here should look familiar from one of the multivariable versions of the chain rule - it's the derivative of $f(\mathbf{x}(t))$. Making this replacement, we have

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \frac{d}{dt}(f(\mathbf{x}(t)))\, dt.$$

Now, we can apply the Fundamental Theorem of Calculus (from single variable) to evaluate this integral, since an antiderivative for the integrand will be given by $f(\mathbf{x}(t))$. From this we obtain:

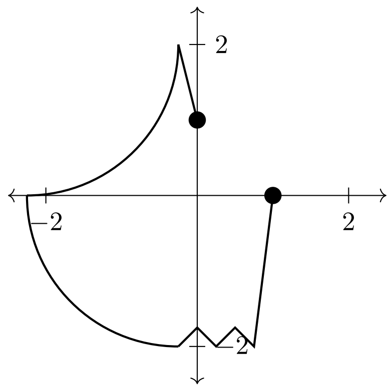
$$\begin{aligned}\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} &= (f(\mathbf{x}(t)))\big|_a^b \\ &= f(\mathbf{x}(b)) - f(\mathbf{x}(a)) \\ &= f(\mathbf{B}) - f(\mathbf{A})\end{aligned}$$

In the last step, we use that $\mathbf{A}$ and $\mathbf{B}$ are the start an end points of $\mathbf{x}(t)$, respectively.

Thus, we have proven the Fundamental Theorem of Line Integrals (when we have a simple parametrization of the curve C). ■

Before discussing how the Fundamental Theorem of Line Integrals relates to path independence, let's look at how this helps us compute integrals of gradient vector fields.

Example 23. *Let $\vec{F}(x,y) = (y,x)$. Observe that $\vec{F} = \nabla f$, where $f(x,y) = xy$. Compute $\int_C \vec{F} \cdot d\boldsymbol{s}$ for the curve C below, starting at $(0,1)$ and ending at $(1,0)$.*



Explanation. *We certainly would like to avoid parametrizing this curve! So, we will use the Fundamental Theorem of Line Integrals to compute this integral. First, let's verify that $\vec{F} = \nabla f$ for $f(x,y) = xy$.*

$$\frac{\partial}{\partial x} xy = \boxed{y}$$
$$\frac{\partial}{\partial y} xy = \boxed{x}$$

Thus, we have $\nabla f(x,y) = (y,x) = \vec{F}(x,y)$.

We can then use the Fundamental Theorem of Line Integrals to compute $\int_C \vec{F} \cdot d\boldsymbol{s}$.

$$\begin{aligned}\int_C \vec{F} \cdot d\boldsymbol{s} &= \int_C \nabla f \cdot d\boldsymbol{s} \\ &= f(0,1) - f(1,0) \\ &= \boxed{0}\end{aligned}$$

Now, it turns out that we can use the Fundamental Theorem of Line Integrals to prove the following corollary about the relationship between conservative vector fields and path independence. Note once again that we require the domain to be open and path-connected.

Corollary 1. *If $\vec{F}$ is a conservative vector field defined on an open and connected domain X, then $\vec{F}$ is path independent.*

Proof Let C and D be two curves with starting point $\mathbf{A}$ and ending point $\mathbf{B}$. We will show that $\int_C \vec{F} \cdot d\mathbf{s} = \int_D \vec{F} \cdot d\mathbf{s}$.

Recall that "$\vec{F}$ is conservative" means that $\vec{F} = \nabla f$ for some function f, which will enable us to use the Fundamental Theorem of Line Integrals (FTLI). Then we have:

$$\begin{aligned}
\int_C \vec{F} \cdot d\mathbf{s} &= \int_C \nabla f \cdot d\mathbf{s} \\
&= f(\mathbf{B}) - f(\mathbf{A}) && \text{(by FTLI)} \\
&= \int_D \nabla f \cdot d\mathbf{s} && \text{(also by FTLI)} \\
&= \int_D \vec{F} \cdot d\mathbf{s}.
\end{aligned}$$

Thus, we have shown that $\int_C \vec{F} \cdot d\mathbf{s} = \int_D \vec{F} \cdot d\mathbf{s}$, and so have shown that $\vec{F}$ is path independent. ■

This corollary, with the Fundamental Theorem of Line Integrals, gives us a new tool for computing line integrals.

Strategies for Computing Line Integrals

We now have a few options for computing line integrals:

(a) Using the original definition:

$$\int_C \vec{F} \cdot d\mathbf{s} = \int_a^b \vec{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$$

(b) If $\vec{F}$ is conservative (so $\vec{F} = \nabla f$ for some f):

   (i) We can use the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\mathbf{s} = f(\mathbf{B}) - f(\mathbf{A})$$

   (ii) Since the vector field is path independent, we can find an *easier* path with the same start and end points, and integrate over that path.

Let's look at how these different methods can be used in an example.

Example 24. *Let $\vec{F}(x, y) = (2xy^2, 2x^2y)$, and consider $\boldsymbol{x}(t) = (2\cos(\pi t), \sin(\pi t^2))$ for $t \in [0, 1]$. Compute $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$.*

Explanation. *We'll evaluate this integral in three different ways.*

First, let's evaluate using the definition of vector line integrals.

$$\begin{aligned}\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_0^1 (4\cos(\pi t)\sin^2(\pi t^2), 8\cos^2(\pi t)\sin(\pi t^2)) \cdot (\boxed{-2\pi \sin(\pi t)}, \boxed{2\pi t \cos(\pi t^2)})\, dt \\ &= \int_0^1 \boxed{-8\pi \cos(\pi t)\sin(\pi t)\sin^2(\pi t^2) + 16\pi t \cos(\pi t^2)\cos^2(\pi t)\sin(\pi t^2)}\, dt\end{aligned}$$

This is an integral that it might be possible to figure out how to compute, but we certainly do not want to! We can use a computer algebra system to see that the result is 0*, but to compute the integral by hand, we will turn to our other methods.*

For our alternate methods, we need to find a potential function $f(x,y)$ *such that* $\vec{F} = \nabla f$*. It turns out that* $f(x,y) = x^2y^2$ *works. Let's verify this.*

$$\begin{aligned}\frac{\partial}{\partial x}(x^2y^2) &= \boxed{2xy^2} \\ \frac{\partial}{\partial y}(x^2y^2) &= \boxed{2x^2y}\end{aligned}$$

Now that we have our function $f(x,y) = x^2y^2$ *such that* $\vec{F} = \nabla f$*, we will use the Fundamental Theorem of Line Integrals to evaluate. Note the start and end points of our curve*

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{x}(0) = (\boxed{2}, \boxed{0}) \\ \boldsymbol{B} &= \boldsymbol{x}(1) = (\boxed{-2}, \boxed{0})\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{x}} \nabla f \cdot d\boldsymbol{s} \\ &= f(\boldsymbol{B}) - f(\boldsymbol{A}) \\ &= \boxed{0}\end{aligned}$$

Note that this is a much easier computation than the integral we had from the first method.

Finally, we compute the line integral using the third method. We have already shown that $\vec{F}$ *is a conservative vector field (by finding* f *such that* $\vec{F} = \nabla f$*), and* *hence we know that $\vec{F}$ is path independent. So we can compute this integral by instead integrating over an easier path with the same start and end points,* $(2, 0)$ *and* $(-2, 0)$*, respectively. Let's choose the straight line from* $(2, 0)$ *to* $(-2, 0)$*, and parametrize this curve.*

$$\boldsymbol{y}(t) = (\boxed{t}, 0) \qquad \textit{for } t \in [-2, 2]$$

Now we can integrate over y instead, which will be a much easier computation.

$$\begin{aligned}
\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s} \\
&= \int_{-2}^{2} \vec{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\
&= \int_{-2}^{2} (\boxed{0}, \boxed{0}) \cdot (1, 0)\, dt \\
&= \int_{-2}^{2} \boxed{0}\, dt \\
&= \boxed{0}
\end{aligned}$$

So we've seen that we can compute this line integral in a few different ways, using the fact that the vector field is conservative.

Depending on the particular problem or example, any one of these methods might be easier than the others. You should practice trying these different methods, and see which you prefer! However, remember that for the second and third options, we need to first verify that the vector field $\vec{F}$ is conservative. This usually means finding a potential function f such that $\vec{F} = \nabla f$.

Conservative Vector Fields

In this section, we'll look at several closely related properties of vector fields. We'll begin by recalling these definitions.

Definition 14. *A vector field* $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ *is* conservative *if there is some function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that* $\nabla f = \vec{F}$.

A continuous vector field is called path independent *if* $\int_C \vec{F} \cdot d\boldsymbol{s} = \int_D \vec{F} \cdot d\boldsymbol{s}$ *for any two simple, piecewise* $\mathcal{C}^1$*, oriented curves* C *and* D *with the same start and end points.*

A vector field $\vec{F}$ *has* no circulation *if* $\oint_C \vec{F} \cdot d\vec{s} = 0$ *for any simple* $\mathcal{C}^1$ *curve* C.

A vector field $\vec{F}$ *has a* symmetric derivative *if the derivative matrix* $D\vec{F}$ *is symmetric.*

It turns out that these concepts are equivalent in many cases, as long as we have a "nice enough" domain. Let's look at the results we have so far, and how we found them.

Proposition 3. *Let* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a* $\mathcal{C}^1$ *vector field defined on an open and path-connected domain* X. *If* $\vec{F}$ *is conservative, then* $\vec{F}$ *is path independent.*

We proved this result using the Fundamental Theorem of Line Integrals. If we have a potential function f for $\vec{F}$, then the vector line integral over any curve can be computed by evaluating f at the endpoints, so it is independent of the path taken between two points.

We have also shown that a conservative vector field has a symmetric derivative matrix.

Proposition 4. *Let* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^1 *vector field, and let* X *be open and path-connected. If* $\vec{F}$ *is conservative, then* $D\vec{F}$ *is symmetric.*

This is shown by writing $\mathbf{F}$ as ∇f for a potential function f, and then observing that $D\vec{F}$ is the Hessian matrix of f. By Clairaut's theorem, Hessian matrices are symmetric, hence $D\vec{F}$ is symmetric.

Learning outcomes:
Author(s): Melissa Lynn

Path independence implies conservative

We've seen that, in the right circumstances, conservative vector fields will be path independent. But will path independent vector fields necessarily be conservative vector fields? In order to show this, the challenge is in constructing a potential function.

Proposition 5. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and path-connected. If $\vec{F}$ is path-connected, then $\vec{F}$ is conservative.*

Proof In order to prove this, we will construct a potential function $f : X \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

Fix a point $\vec{a}$ in X. Define the function f by

$$f(\vec{x}) = \int_C \vec{F} \cdot d\vec{s},$$

where C is a path from $\vec{a}$ to $\vec{x}$. This is where path-independence is crucial. If $\vec{F}$ were path-dependent, then the definition of $f(\vec{x})$ would depend on the choice of the path C, and so the function f would not be well-defined.

Furthermore, this is also where we require that the domain X be path-connected. If X were not path-connected, there would be points $\vec{x}$ which couldn't be connected to $\vec{a}$ with a path, so f would not be defined on all of X.

Now that we've defined our function f, we need to show that ∇f exists and equals $\vec{F}$. To show that ∇f exists, we need to show that the partial derivatives of f exist. We will show that the first partial derivative of f exists, and the argument for the other partial derivatives is similar.

From the definition of partial derivatives, we have

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, ..., x_n) - f(x_1, x_2, ..., x_n)}{h}.$$

This is where we need the domain X to be open. Since X is open, there is an open ball around the point $(x_1, x_2, ..., x_n)$ within X. This means that for small enough h, the point $(x_1 + h, x_2, ..., x_n)$ is in X, so that $f(x_1 + h, x_2, ..., x_n)$ is defined.

PICTURE

Now, let's make use of our definition of f, writing $\vec{b} = (x_1, x_2, ..., x_n)$ and $\vec{b}' = (x_1 + h, x_2, ..., x_n)$ to simplify notation. Let C be a path from $\vec{a}$ to $\vec{b}$, and let C' be a path from $\vec{a}$ to $\vec{b}'$. Then,

$$f(x_1, x_2, ..., x_n) = \int_C \vec{F} \cdot d\vec{s}$$
$$f(x_1 + h, x_2, ..., x_n) = \int_{C'} \vec{F} \cdot d\vec{s}$$

Our partial derivative is then

$$\frac{\partial f}{\partial x_1} = \lim_{h\to 0} \frac{\int_C \vec{F}\cdot d\vec{s} - \int_{C'} \vec{F}\cdot d\vec{s}}{h}.$$

Looking at the numerator, $\displaystyle\int_C \vec{F}\cdot d\vec{s} - \int_{C'} \vec{F}\cdot d\vec{s}$ will be equal to $\displaystyle\int_C \vec{D}\cdot d\vec{s}$, where D is any path starting at $\vec{b}$ and ending at $\vec{b}'$.

PICTURE

We'll choose the path D to be parametrized by

$$\vec{x}(t) = (x_1 + ht, x_2, ..., x_n) \text{ for } 0 \le t \le 1.$$

Then, substituting this in,

$$\begin{aligned} \int_C \vec{D}\cdot d\vec{s} &= \int_0^1 \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\ &= \int_0^1 \vec{F}(x_1 + ht, x_2, ..., x_n) \cdot (h, 0, ..., 0)\ dt \end{aligned}$$

Let F_1 be the first component of $\vec{F}$. Then $\vec{F}(x_1 + ht, x_2, ..., x_n) \cdot (h, 0, ..., 0) = hF_1(x_1 + ht, x_2, ..., x_n)$.

So, for the partial derivative of f, we have

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \lim_{h\to 0} \frac{\int_C \vec{D}\cdot d\vec{s}}{h} \\ &= \lim_{h\to 0} \frac{\int_0^1 hF_1(x_1 + ht, x_2, ..., x_n)\ dt}{h} \\ &= \lim_{h\to 0} \int_0^1 F_1(x_1 + ht, x_2, ..., x_n)\ dt \end{aligned}$$

At this point, we'll gloss over some details. Essentially, F_1 is a "nice enough" function that we can bring the limit inside of the integral, and arrive at the partial derivative,

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \int_0^1 F_1(x_1, x_2, ..., x_n)\ dt \\ &= F_1(x_1, x_2, ..., x_n). \end{aligned}$$

Thus, the first partial derivative of f is the first component of $\vec{F}$. Following the same argument for the other partial derivatives, we can show that $\nabla f = \vec{F}$. ■

Path independence and circulation

Next, we will show that a vector field is path independent if and only if it has no circulation. For this result, we don't need any special requirements on the domain.

Proposition 6. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field. Then $\vec{F}$ is path independent if and only if $\oint_C \vec{F} \cdot d\vec{s} = 0$ for all closed curves C in X.*

Proof First, let's assume that $\vec{F}$ is path independent, and we'll show that $\vec{F}$ has no circulation. Let C be any closed curve in X, and suppose C starts and ends at the point $\vec{a}$. Let D be the constant curve parametrized by $\vec{x}(t) = \vec{a}$ for $t \in [0, 1]$. By path independence, we have

$$\int_C \vec{F} \cdot d\vec{s} = \int_D \vec{F} \cdot d\vec{s}.$$

Since $\vec{x}(t)$ is constant, $\vec{x}'(t) = \vec{0}$, so we can compute the line integral.

$$\begin{aligned} \int_D \vec{F} \cdot d\vec{s} &= \int_0^1 \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\ &= \int_0^1 \vec{F}(\vec{a}) \cdot \vec{0}\ dt \\ &= \int_0^1 0\ dt \\ &= 0 \end{aligned}$$

So, $\vec{F}$ has no circulation.

Next, we'll assume that $\vec{F}$ has no circulation, and we'll show that $\vec{F}$ is path independent. Suppose we have two oriented curves C and D, which both start at $\vec{a}$ and end at $\vec{b}$. Let D' be D with the orientation reversed, so that

$$\int_D' \vec{F} \cdot d\vec{s} = -\int_D \vec{F} \cdot d\vec{s}.$$

Let E be the oriented curve obtained by first traversing C, then traversing D'. Then E starts and ends at $\vec{a}$, and

$$\begin{aligned} \int_E \vec{F} \cdot d\vec{s} &= \int_C \vec{F} \cdot d\vec{s} + \int_D' \vec{F} \cdot d\vec{s}, \\ &= \int_C \vec{F} \cdot d\vec{s} - \int_D \vec{F} \cdot d\vec{s}. \end{aligned}$$

Since E starts and ends at $\vec{a}$, it is a closed curve. The vector field $\vec{F}$ has no circulation, so $\int_E \vec{F} \cdot d\vec{s} = 0$. Thus

$$0 = \int_C \vec{F} \cdot d\vec{s} - \int_D \vec{F} \cdot d\vec{s},$$

so $\int_C \vec{F} \cdot d\vec{s} = \int_D \vec{F} \cdot d\vec{s}$. This shows that $\vec{F}$ is path independent. ■

Symmetric derivative implies no circulation

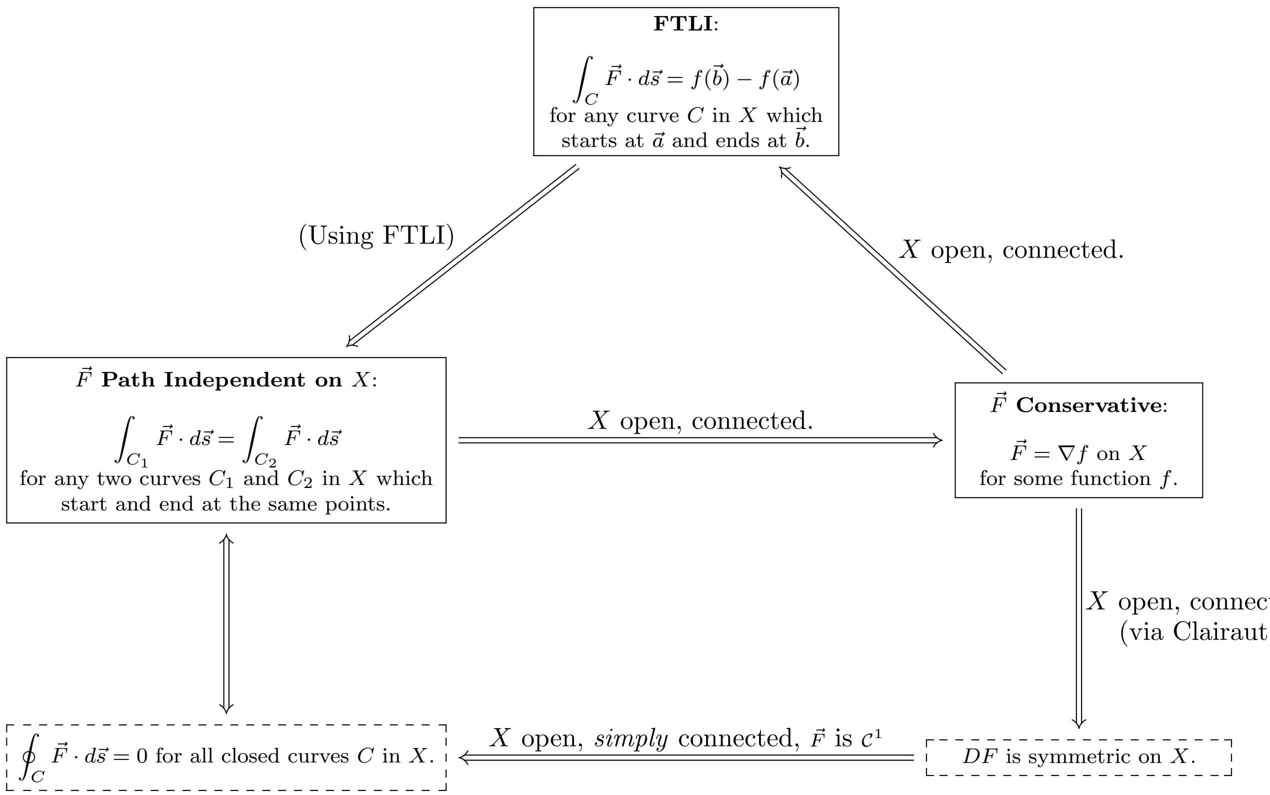
Finally, suppose $\vec{F}$ has a symmetric derivative, then $\vec{F}$ has no circulation. In this case, we require our domain to be simply connected.

Proposition 7. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field, defined on an open and simply connected domain X. If $D\vec{F}$ is a symmetric matrix, then $\vec{F}$ has zero circulation.*

Although this completes our set of equivalences, we don't yet have the tools that we need to prove this result. For this, we will need double integrals and Green's Theorem, so we'll come back to this proof later.

Summary

We summarize the relationship between conservative vector fields, path independence, zero circulation, and symmetric derivatives in the following diagram. Throughout, let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field defined on a set $X \subset \mathbb{R}^n$.



Remember that you can often move through more than one box. For example, if $D\vec{F}$ is continuous and symmetric on an open, simply connected set X, then $\vec{F}$ is conservative. That's because an open simply-connected set is also connected, so you can follow arrows all the way up to that box.

Part II
Double Integrals

Part III
Curl and Divergence

Part IV
Surface Integrals

Part V
Triple Integrals