

## Scalar Line Integrals

We've seen how we can integrate vector fields along a path, using vector line integrals. We can also integrate scalar valued functions along a path. For instance, suppose we have a scalar valued function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and a path  $\vec{x} : [a, b] \rightarrow \mathbb{R}^2$  in  $\mathbb{R}^2$ . Suppose we look at the portion of the graph of  $f$  lying over the path  $\vec{x}$ , and drop a "curtain" to the  $xy$ -plane.

PICTURE

Integrating  $f$  along the path  $\vec{x}$  will be equivalent to finding the area of this curtain. We can also describe this as the area between  $\vec{x}(t)$  and  $f(\vec{x}(t))$ .

Scalar line integrals aren't only useful for finding areas of strangely shaped regions. They are also useful throughout physics. For example, if you have the mass density function of a wire, you can compute the scalar line integral of this function to find the total mass of the wire.

## Scalar Line Integrals

Suppose we have a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  and a  $C'$  path  $\vec{x} : [a, b] \rightarrow \mathbb{R}^n$ , such that the composition  $f(\vec{x}(t))$  is defined on  $[a, b]$ .

PICTURE

In order to find the area under  $f$  and over the path  $\vec{x}$ , we will borrow an important idea from single variable calculus: approximating an area with rectangles.

In order to do this, we'll partition the interval  $[a, b]$  into  $n$  subintervals, determined by

$$a = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n = b.$$

This partition breaks the path  $\vec{x}$  into smaller paths, by restricting to the subintervals.

PICTURE

Our goal will be to approximate the area under  $f$  over each of these shorter paths. These approximations will be computed by finding the length of the short path, and multiplying this by a height determined by a test point,  $t_k^*$ . We can think of this as a curved rectangle.

PICTURE

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Learning outcomes: Understand the definition of scalar line integrals geometrically, and be able to compute them.

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The specific choice of the test point will be important for simplifying our result - we'll come back to this later.

The height of our curved rectangle will be  $f(\vec{x}(t_k^*))$ , and the base is the distance  $\Delta s_k$  along the path  $\vec{x}$  from  $t_{k-1}$  to  $t_k$ .

PICTURE

Thinking back to arclength computations, this distance is given by the integral

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt.$$

Now, we need to make a careful choice for our test point  $t_k^*$  which will simplify things later. To do this, recall the Mean Value Theorem for Integrals, from single variable calculus.

**Theorem 1.** Suppose  $g$  is a continuous function on the closed interval  $[a, b]$ . Then there exists  $c$  in  $[a, b]$  such that

$$\int_a^b g(t) dt = (b - a)g(c).$$

Here, we'll take  $\|\vec{x}'(t)\|$  for the function  $g(t)$ . Since  $\vec{x}$  is  $\mathcal{C}^1$ ,  $\|\vec{x}'(t)\|$  is continuous. Applying the Mean Value Theorem on the interval  $[t_{k-1}, t_k]$ , there exists  $c_k$  such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt = (t_k - t_{k-1})\|\vec{x}'(c_k)\|.$$

We take this  $c_k$  to be our test point, so that  $t_k^* = c_k$ .

Now, the area of the  $k$ th curved rectangle is  $f(\vec{x}(t_k^*))\Delta s_k$ . We add up these areas and take the limit as the number of rectangles,  $n$ , goes to infinity:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(\vec{x}(t_k^*))\Delta s_k.$$

Substituting  $\Delta s_k = (t_k - t_{k-1})\|\vec{x}'(c_k)\|$  and writing  $\Delta t = t_k - t_{k-1}$ , we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(\vec{x}(t_k^*))\|\vec{x}'(c_k)\|\Delta t.$$

We can recognize this as the single variable integral of the function  $f(\vec{x}(t))\|\vec{x}'(t)\|$  over the interval  $[a, b]$ ,

$$\int_a^b f(\vec{x}(t))\|\vec{x}'(t)\| dt.$$

We take this to be the definition of a scalar line integral.

**Definition 1.** Let  $f : X \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous function defined on a  $\mathcal{C}^1$  path  $\vec{x} : [a, b] \rightarrow \mathbb{R}^n$ . The scalar line integral of  $f$  along  $\vec{x}$  is

$$\int_{\vec{x}} f ds = \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\| dt.$$

## Examples

Let's look at some examples of computing scalar line integrals.

**Example 1.** Consider the function  $f(x, y) = x^2 + y^2$  and the path  $\vec{x}(t) = (\cos t, \sin t)$  for  $t \in [0, \pi]$ . We compute the scalar line integral of  $f$  over  $\vec{x}$ .

$$\begin{aligned} \int_{\vec{x}} f \, ds &= \int_a^b f(\vec{x}(t)) \|\vec{x}'(t)\| dt \\ &= \int_0^\pi f(\cos t, \sin t) \|(-\sin t, \cos t)\| dt \\ &= \int_0^\pi (\cos^2 t + \sin^2 t) \sqrt{\sin^2 t + \cos^2 t} dt \\ &= \int_0^\pi 1 \, dt \\ &= \pi \end{aligned}$$

**Example 2.** Consider the function  $f(x, y) = e^{\sqrt{xy}}$  and the path  $\vec{x}(t) = (t, t)$  for  $t \in [0, 1]$ . We compute the scalar line integral of  $f$  over  $\vec{x}$ .

$$\begin{aligned} \int_{\vec{x}} f \, ds &= \int_a^b f(\vec{x}(t)) \|\vec{x}'(t)\| dt \\ &= \int_0^1 f(t, t) \|(1, 1)\| dt \\ &= \int_0^1 e^{\sqrt{t^2}} \sqrt{2} dt \\ &= \sqrt{2} \int_0^1 e^t dt \\ &= \sqrt{2}(e^1 - e^0) \\ &= \sqrt{2}e - \sqrt{2} \end{aligned}$$