Green's Theorem Examples

We've seen how Green's theorem relates a vector line integral over the boundary of a region to a double integral over the region.

Theorem 1. ***Green's Theorem.*** *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x,y) = (M(x,y), N(x,y))$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

Now, we'll look at several examples of using Green's theorems to simplify some computations.

Green's theorem examples

Example 1. *Consider the integral $\oint_C \vec{F} \cdot d\vec{s}$, where $\vec{F}(x,y) = (x^2, e^y)$, and C is the unit circle, oriented counterclockwise.*

PICTURE

By Green's theorem, this is equivalent to a double integral over the unit disc. That is,

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{s} &= \iint_D \left(\frac{\partial}{\partial x} e^y - \frac{\partial}{\partial y} x^2 \right) \, dA \\
&= \iint_D 0 \, dA \\
&= 0.
\end{aligned}$$

Example 2. *Consider the integral $\oint_C \vec{F} \cdot d\vec{s}$, where $\vec{F}(x,y) = (y, -x)$, and C is the unit circle, oriented* clockwise*.*

PICTURE

Let D be the unit disc, enclosed by the unit circle. Although this curve isn't positively oriented, we can still use Green's theorem to help evaluate our line integral. This will require a sign change.

Learning outcomes: Understand how Green's Theorem can be used to more easily compute integrals.

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$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{s} &= -\oint_{-C} \vec{F} \cdot d\vec{s} \\
&= -\iint_D \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) \right) \, dA \\
&= -\iint_D (-2) \, dA \\
&= 2 \cdot (\textit{area of } D) \\
&= 2\pi.
\end{aligned}$$