Area of a Region

We've seen how Green's theorem relates a vector line integral over the boundary of a region to a double integral over the region.

Theorem 1. ***Green's Theorem.*** *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x, y) = (M(x, y), N(x, y))$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

We've used Green's theorem to more easily compute some vector line integrals, and in this section, we'll see how Green's theorem can be used to find the area of a region in $\mathbb{R}^2$.

Area of a region

Suppose we wish to find the area of a region D in $\mathbb{R}^2$.

PICTURE

To find this area, we can instead find a volume with the same value. That is, we can create a solid of height 1 over the region, and find the volume of this solid.

PICTURE

We can use a double integral to compute this volume, giving us the area of the region D.

Proposition 1. *Let D be a region in $\mathbb{R}^2$. Then*

$$\textit{area of } D = \iint_D 1 \, dA.$$

We'll use this fact to find the area of the unit circle. From geometry, we expect this area to be π.

Example 1. *Let D be the unit circle in $\mathbb{R}^2$. To find the area of D, we use*

$$\textit{area of } D = \iint_D 1 \, dA.$$

Learning outcomes: Use Green's theorem to compute the area of a region in the plane.
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In polar coordinates, the unit circle can be described with the inequalities

$$0 \leq r \leq 1,$$
$$0 \leq \theta \leq 2\pi.$$

Changing to polar coordinates to evaluate the double integral, we have

$$\begin{aligned}\iint_D 1 \; dA &= \int_0^{2\pi} \int_0^1 r \; dr d\theta \\ &= \int_0^{2\pi} \frac{1}{2} r^2 |_0^1 \; d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \; d\theta \\ &= \pi.\end{aligned}$$

So, we have confirmed that the area of the unit circle is π*.*

Area using Green's theorem

Now, let's see how we can use Green's theorem to find the area of a region. Suppose we have a curve C enclosing a region D, satisfying the hypotheses for Green's theorem.

PICTURE

Now, suppose we have a vector field $\vec{F}(x,y) = (M(x,y), N(x,y))$ such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$. There are many possible choices for such a vector field; examples include

$$\begin{aligned}\vec{F}(x,y) &= (x, x), \\ \vec{F}(x,y) &= (-y/2, x/2), \\ \vec{F}(x,y) &= \left(y + \sin(x), e^{y^2}\right).\end{aligned}$$

Then, Green's theorem gives us

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_D 1 \; dA.$$

So, we can find the area of D by integrating the vector field F over the boundary of D.

Let's look at an example to see this in action.

Example 2. *We'll find the area enclosed by an ellipse* C*, given by the equation* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

PICTURE

In this example, we'll use the vector field $\vec{F}(x,y) = (-y/2, x/2)$ *to compute this area. By Green's theorem, we have*

$$\begin{aligned} \textit{Area} &= \iint_D 1\, dA \\ &= \oint_C \vec{F} \cdot d\vec{s}. \end{aligned}$$

We can parametrize the ellipse as $\vec{x}(t) = (a\cos t, b\sin t)$ *for* $0 \le t \le 2\pi$*. Notice that this parametrization gives the correct orientation for the ellipse. Now, we evaluate our line integral.*

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} \vec{F}(a\cos t, b\sin t) \cdot (-a\sin t, b\cos t) dt \\ &= \int_0^{2\pi} \left(-\frac{b}{2}\sin t, \frac{a}{2}\cos t\right) \cdot (-a\sin t, b\cos t) dt \\ &= \int_0^{2\pi} \left(\frac{ab}{2}\sin^2 t + \frac{ab}{2}\cos^2 t\right) dt \\ &= \int_0^{2\pi} \left(\frac{ab}{2}\right) dt \\ &= ab\pi \end{aligned}$$

Thus, the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ *is* $ab\pi$*.*