Multivariable Calculus II

August 6, 2019

Contents

Part I

Vector Fields and Line Integrals

Vector Fields

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Such a function takes points (x, y) in the plane, and assigns a real number $f(x, y)$ to each of them. There are a few ways that we can visualize these functions. Perhaps the most common way is to graph the function in $\mathbb{R}^3$. The graph consists of the points $(x, y, f(x, y))$, and we often think of the graph as being height $f(x, y)$ over the point (x, y).

PICTURE

We could also visualize the graph in the plane, using a heat map to indicate the "height" $f(x, y)$ at points (x, y).

PICTURE

We could also represent the function by writing the function values $f(x, y)$ at points (x, y). Of course, we can't write these values at all points, because then we wouldn't be able to read them! But for nicely behaved functions, we can choose a representative sample of points, so that the function values at those points accurately reflect the overall behavior of the function.

PICTURE

This may remind you of a temperature map used to give the temperature across a region.

Now, suppose instead of having a *value* at each point in the plane, we had a *vector*.

PICTURE

This might be used to represent windspeed and direction, or the direction and strength of any force, such as gravity or a magnetic field.

Let's think about how we can translate this idea into a function. The inputs are still points in the plane, $\mathbb{R}^2$, but now the outputs are vectors, also in $\mathbb{R}^2$. Thus, we have a function $\mathbb{R}^2 \to \mathbb{R}^2$. When we think of such a function as assigning vectors to points in $\mathbb{R}^2$, we call this a vector field.

Learning outcomes: Understand the definition of vector fields and how to graph them.
Author(s): Melissa Lynn

Vector Fields

We've seen that a vector field consists of vectors placed at each point in some region, and we can think of this as assigning a vector to each point in the region, and we can represent this with a function.

Definition 1. *A* vector field *is a function* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}n$*.*

If $\vec{F}$ *is a continuous function, then we say that* $\vec{F}$ *is a* continuous vector field*.*

In this definition, we're thinking about the inputs as points and the outputs as vectors, even though both are in $\mathbb{R}^n$.

Graphing and Scaling

Let's look at how we can visualize vector fields.

Example 1. *Consider the vector field* $\vec{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$*. We'll plot the vectors* $\vec{F}(x, y)$ *starting at points* (x, y) *in order to graph this vector field. To do this, we'll start by evaluating this function at a few points.*

(x, y)	$\vec{F}(x, y)$
$(1, 0)$	$(0, 1)$
$(0, 1)$	$(1, 0)$
$(1, 1)$	$(1/2, 1/2)$
$(2, 0)$	$(0, 1/2)$
$(0, 2)$	$(1/2, 0)$
$(2, 1)$	$(1/5, 2/5)$
$(1, 2)$	$(2/5, 1/5)$
$(2, 2)$	$(1/4, 1/4)$

Now, we can plot these vectors, and visualize the behavior of our vector field.

PICTURE

If the vectors in our vector field are particularly long, graphing our vector field can quickly turn into a cluttered mess. In these situations, it can be useful to scale the vectors, so that we can more clearly see the behavior of our vector field. In fact, most graphing software will automatically scale vector fields, whether you want them to or not.

Example 2. *Consider the vector field* $\vec{G}(x, y) = (-x, y)$*. We'll begin by evaluating this function at several point.*

(x, y)	$\vec{F}(x, y)$
$(0, 0)$	$(0, 0)$
$(1, 0)$	$(-1, 0)$
$(2, 0)$	$(-2, 0)$
$(0, 1)$	$(0, 1)$
$(1, 1)$	$(-1, 1)$
$(2, 1)$	$(-2, 1)$
$(0, 2)$	$(0, 2)$
$(1, 2)$	$(-1, 2)$
$(2, 2)$	$(-2, 2)$

We can plot these vectors to graph the vector field $\vec{G}$.

PICTURE

Because the vectors are long and overlap with each other, it's a bit difficult to get a sense of the behavior of the vector field. To address this issue, we can scale the vectors, to avoid overlap. Below, we scale the vectors by $1/4$.

PICTURE

Here, it's easier to see the behavior of the function, though it's important to remember that the exact lengths of the vectors are no longer accurate.

Flow Lines

Imagine you have a vector field representing gravitational force in space, and that you have a spaceship floating around in space. The spaceship will move in the direction of the gravitational force, following the vector at its position. As the space ship continues to float through space, it will continue to move in the direction prescribed by the vector field, and trace out a path through space.

PICTURE

A path like this is called a flow line of the vector field. This is the path that "matches" the vectors as it moves through the vector fields. This means that the vectors in the vector field should be tangent to the path, and they will actually be the tangent vectors to the path. This leads us to the definition of the flow lines of a vector field.

Flow Lines

Definition 2. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $\vec{x} : I \subset \mathbb{R} \to \mathbb{R}^n$ be a path in $\mathbb{R}^n$. Then we say that $\vec{x}$ is a* flow line *of $\vec{F}$ if*

$$\vec{x}'(t) = \vec{F}(\vec{x}(t))$$

for all $t \in I$.

Before we figure out how to find flow lines, we'll give a few examples.

Example 3. *Consider the vector field $\vec{F}(x, y) = (-y, x)$, and the path $\vec{x}(t) = (\cos(t), \sin(t))$. We'll verify algebraically that $\vec{x}$ is a flow line of $\vec{F}$.*

First, we compute $\vec{x}'(t)$.

$$\begin{aligned} \vec{x}'(t) &= \left(\frac{d}{dt}\cos(t), \frac{d}{dt}\sin(t)\right) \\ &= (-\sin(t), \cos(t)). \end{aligned}$$

Next, we find $\vec{F}(\vec{x}(t))$.

$$\begin{aligned} \vec{F}(\vec{x}(t)) &= \vec{F}(\cos(t), \sin(t)) \\ &= (-\sin(t), \cos(t)). \end{aligned}$$

Learning outcomes: Understand the definition and geometry of flow lines of a vector field. Verify algebraically that paths are flow lines.

Author(s): Melissa Lynn

We see that this is equal to $\vec{x}'(t)$, so we have verified that $\vec{x}$ is a flow line of $\vec{F}$.

We can also plot $\vec{F}$ and $\vec{x}$, to see how the path $\vec{x}$ follows the vectors of $\vec{F}$.

PICTURE

Example 4. *Consider the vector field $\vec{F}(x, y) = (x, y)$, and the path $\vec{x}(t) = (e^t, e^t)$. We'll verify that $\vec{x}$ is a flow line of $\vec{F}$.*

First, we'll compute $\vec{x}'(t)$.

$$\vec{x}'(t) = \boxed{(e^t, e^t)}$$

Next, we'll compute $\vec{F}(\vec{x}(t))$.

$$\vec{F}(\vec{x}(t)) = \boxed{(e^t, e^t)}$$

So, we can see that $\vec{x}$ is a flow line of the vector field $\vec{F}$.

Let's also graph $\vec{F}$ and $\vec{x}$, so we can see how the path follows the vectors of the vector field.

How to Find Flow Lines

Although it's relatively straightforward to check if a given path is a flow line for a vector field, it can be difficult to compute the flow lines of a vector field. This is because computing flow lines involves solving a system of differential equations, which is not always possible - even when a solution exists! We'll look at a couple of examples where we can find the flow lines.

Example 5. *Consider the vector field $\vec{F}(x, y) = (-x, -y)$. To find flowlines, we need to find the paths $\vec{x}$ such that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$. If we write $\vec{x}(t) = (x(t), y(t))$, we need to solve*

$$(x'(t), y'(t)) = \vec{F}(x(t), y(t)) = (-x(t), -y(t))$$

for $x(t)$ and $y(t)$. We can write this as a system of differential equations,

$$\begin{cases} x'(t) = -x(t) \\ y'(t) = -y(t) \end{cases}$$

Looking at the first equation, we have the solution $x(t) = Ae^{-t}$ for some constant A. From the second equation, we have the solution $y(t) = Be^{-t}$ for some constant B. Putting these together, we have the flowlines

$$\vec{x}(t) = (Ae^{-t}, Be^{-t}),$$

for constants A and B.

If we graph several flow lines and the vector field $\vec{F}$, we can see how the flow lines follow the vectors of the vector field.

PICTURE

Example 6. *Consider the vector field $\vec{F}(x, y) = (-y, x)$. To find flowlines, we need to find the paths $\vec{x}$ such that $\vec{x}'(t) = \vec{F}(\vec{x}(t))$. If we write $\vec{x}(t) = (x(t), y(t))$, we need to solve*

$$(x'(t), y'(t)) = \vec{F}(x(t), y(t)) = (-y(t), x(t))$$

for $x(t)$ and $y(t)$. We can write this as a system of differential equations,

$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$

Solve this system of differential equations yields $x(t) = r\cos(t)$ and $y(t) = r\sin(t)$, so we have the flow lines

$$\vec{x}(t) = (r\cos(t), r\sin(t))$$

for real numbers r. Notice that these paths trace circles of radius r. We graph a few flow lines along with the vector field $\vec{F}$, to see how the paths follow the vector field.

PICTURE

Gradient Fields

One common way that vector fields arise is through the gradient of a function. That is, suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$. Then we can find the gradient of f,

$$\nabla f(x_1, ..., x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}\right).$$

This gradient is a function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$, so it can be thought of as a vector field in $\mathbb{R}^n$, which we call the *gradient field* of the function f.

Gradient Fields

Let's look at some examples of gradient fields.

Example 7. *Consider the function $f(x, y) = x^2 + y^2$. The gradient of this function is*

$$\nabla f(x, y) = (2x, 2y).$$

We graph this gradient field below.

PICTURE

Example 8. *Consider the function $f(x, y) = xy$. The gradient of this function is*

$$\nabla f(x, y) = (y, x).$$

We graph this gradient field below.

PICTURE

Conservative Vector Fields

Taking a slightly different perspective, we can start with a vector field $\vec{F}$, and determine whether we can find a function f such that $\vec{F} = \nabla \vec{f}$, so that $\vec{F}$ is the gradient field of f. If this is possible, we say that $\vec{F}$ is a conservative vector field.

Learning outcomes: Given a scalar valued function, compute and graph its gradient field. Given a vector field, determine if it is conservative, and find a potential function. Use the derivative matrix to show that a vector field is not conservative.

Author(s): Melissa Lynn

Definition 3. *A vector field* $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ *is* conservative *if there is some function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that* $\nabla f = \vec{F}$. *Then* f *is called a* potential function *for* $\mathbf{F}$.

So, one way to show that a vector field $\mathbf{F}$ is conservative is by finding a potential function f.

For simple examples, you might be able to do this by guessing. However, more complicated examples require a more systematic approach. The approach is easiest to understand through examples, so we'll work through a couple before describing the steps for the general case.

Example 9. *Find a potential function for the vector field* $\mathbf{F}(x,y) = (2xy^3 + 1, 3x^2y^2 - y^{-2})$.

Explanation. *First, note that if there there is a function* f *such that* $\nabla f = \mathbf{F}$, *then*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(2xy^3 + 1, 3x^2y^2 - y^{-2}\right)$$

Let's start by looking at the x*-term. We must have* $\dfrac{\partial f}{\partial x} = 2xy^3 + 1$. *Integrating with respect to* x, *we have*

$$\begin{aligned} f(x,y) &= \int 2xy^3 + 1\,dx \\ &= x^2y^3 + x + g(y) \end{aligned}$$

The first part of this expression, $x^2y^3 + x$, *is an antiderivative for* $2xy^3 + 1$ *with respect to* x. *The second part of the expression,* $g(y)$, *is the "constant" for the integral. It's possible that there are some terms which depend only on* y, *hence are constant with respect to* x, *and writing* $g(y)$ *takes these terms into account.*

At this point, we know that f *has the form* $f(x,y) = x^2y^3 + x + g(y)$, *but we still need to figure out what* $g(y)$ *is. For this, we use the* y*-term of the vector field* $\mathbf{F}$. *From this, we have* $\dfrac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$. *Since we know* $f(x,y) = x^2y^3 + x + g(y)$, *we must have*

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2y^3 + x + g(y)\right) \\ &= 3x^2y^2 + g'(y). \end{aligned}$$

Comparing this with $\dfrac{\partial f}{\partial y} = 3x^2y^2 - y^{-2}$, *we must have* $g'(y) = -y^{-2}$. *Then we find that*

$$\begin{aligned} g(y) &= \int -y^{-2}\,dy \\ &= y^{-1} + C \end{aligned}$$

Hence, any potential function would have the form $f(x,y) = x^2y^3+x+y^{-1}+C$. *Choosing* $C=0$*, we obtain a specific potential function* $f(x,y) = x^2y^3+x+y^{-1}$.

We now work through finding a potential function for a three dimensional vector field.

Example 10. *Find a potential function for the vector field* $\mathbf{F}(x,y,z) = (2xy, x^2+z+2y, y+\cos(z))$.

Explanation. *First, note that a potential function* $f(x,y,z)$ *would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\boxed{2xy}, \boxed{x^2+z+2y}, \boxed{y+\cos(z)}\right)$$

We begin by considering the x*-component, noticing that* $\dfrac{\partial f}{\partial x} = 2xy$. *We integrate with respect to* x.

$$\begin{aligned} f(x,y,z) &= \int 2xy\,dx \\ &= \boxed{x^2y} + g(y,z) \end{aligned}$$

Here, $g(y,z)$ *is a function of only* y *and* z*, hence constant with respect to* x. *We now differentiate with respect to* y*, in order to compare to the* y*-component of the vector field.*

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial x}\left(x^2y + g(y,z)\right) \\ &= \boxed{x^2} + \frac{\partial g}{\partial y} \end{aligned}$$

Comparing this with $\dfrac{\partial f}{\partial y} = x^2+z+2y$*, we have* $\dfrac{\partial g}{\partial y} = z+2y$. *We integrate this with respect to* y.

$$\begin{aligned} g(y,z) &= \int z+2y\,dy \\ &= \boxed{yz+y^2} + h(z) \end{aligned}$$

Here, h *is a function of only* z*, hence is constant with respect to* y. *We now know that* f *has the form* $f(x,y,z) = x^2y+yz+y^2+h(z)$. *So, our final task is to find* $h(z)$. *We differentiate* f *with respect to* z *in order to compare with the* z*-component of the vector field.*

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(x^2y+yz+y^2+h(z)\right) \qquad = \boxed{y} + h'(z)$$

Comparing this with $\dfrac{\partial f}{\partial z} = y + \cos(z)$*, we have* $h'(z) = \cos(z)$*. Integrating with respect to* z*, we obtain*

$$\begin{aligned} h(z) &= \int \cos(z)\, dz \\ &= \boxed{\sin(z)} + C \end{aligned}$$

Where C *is a constant. Thus, any potential function would have the form* $f(x, y, z) = \boxed{x^2y + yz + y^2 + \sin(z)} + C$*. Choosing* $C = 0$*, we have a specific potential function* $f(x, y, z) = \boxed{x^2y + yz + y^2 + \sin(z)}$*.*

Summarizing the steps we take in each of the above examples, we have the following process for finding a potential function for a conservative vector field $\mathbf{F}(x_1, x_2, ..., x_n)$.

(a) Integrate the first component of $\mathbf{F}$ with respect to x_1, in order to find the terms of $f(x_1, x_2, ..., x_n)$ which depend on x_1. From this, we can write $f(x_1, x_2, ..., x_n) = (x_1\text{-terms}) + f_1(x_2, ..., x_n)$.

(b) Differentiate $f(x_1, x_2, ..., x_n) = (x\text{-terms}) + f_1(x_2, ..., x_n)$ with respect to x_2. Compare this to the second component of $\mathbf{F}$ in order to determine an expression for $\dfrac{\partial f_1}{\partial x_2}$. Integrate this expression with respect to x_2, so we can write $f_1(x_2, ..., x_n) = (x_2\text{-terms}) + f_2(x_3, ..., x_n)$. Hence we have $f(x_1, x_2, ..., x_n) = (x_1\text{- and } x_2\text{-terms}) + f_2(x_3, ..., x_n)$.

(c) Repeat this process until all components are used.

So far, we've only seen cases where a potential function exists. However, we would also like to be able to show that a vector field is *not* conservative. Let's look at what happens in our process when we have a vector field which is not conservative.

Example 11. *Try (and fail) to find a potential function for the vector field* $\mathbf{F}(x, y) = (-y, x)$*.*

Explanation. *If a potential function existed, it would have to satisfy*

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (\boxed{-y}, \boxed{x})$$

We begin with $\dfrac{\partial f}{\partial x} = -y$*. Integrating with respect to* x*, we have*

$$\begin{aligned} f(x, y) &= \int -y\, dx \\ &= \boxed{-yx} + g(y) \end{aligned}$$

Differentating with respect to y, we then obtain

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-yx + g(y)\right)\\ &= \boxed{-x} + g(y)\end{aligned}$$

When we compare this to the y-component of the vector field $\mathbf{F}$ *in order to determine $g(y)$, we would have to have $x = -x + g(y)$. But this is impossible! Thus we see that our method has broken down, and we are not able to find a potential function.*

Here, we see that the system breaks down, and we aren't able to produce a potential function. This is good, since it turns out the vector field isn't conservative. However, we would an easy way to prove that it isn't conservative. The following theorem gives us a quick way to prove that a vector field is not conservative.

Theorem 1. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If* $\mathbf{F}$ *is conservative, then $D\mathbf{F}$ is symmetric.*

The contrapositive of this theorem states:

Theorem 2. *Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and connected. If $D\mathbf{F}$ is not $\boxed{\textit{symmetric}}$ then* $\mathbf{F}$ *is not $\boxed{\textit{conservative}}$.*

Thus, provided we have a C^1 vector field and the domain is open and connected, we can show a vector field is not conservative by showing that its derivative matrix is not symmetric.

Example 12. *Show that the vector field $\mathbf{F}(x, y) = (-y, x)$ is not conservative.*

Explanation. *First, note that* $\mathbf{F}$ *is a C^1 vector field with domain $\mathbb{R}^2$. Since $\mathbb{R}^2$ is open and connected, our theorem applies. We compute the derivative matrix $D\mathbf{F}$.*

$$\begin{aligned}D\mathbf{F} &= \begin{pmatrix} \frac{\partial}{\partial x}(-y) & \frac{\partial}{\partial y}(-y) \\ \frac{\partial}{\partial x}x & \frac{\partial}{\partial y}x \end{pmatrix}\\ &= \begin{pmatrix} \boxed{0} & \boxed{-1} \\ \boxed{1} & \boxed{0} \end{pmatrix}\end{aligned}$$

Since this matrix is not symmetric, $\mathbf{F}$ *is not a conservative vector field.*

Note how much simpler this is than trying to find a potential function. We now prove our theorem, showing that a conservative C^1 vector field on an open and connected domain has symmetric derivative.

Proof Let $\mathbf{F}$ be a C^1 vector field defined on an open connected domain $X \subset \mathbb{R}^n$. If $\mathbf{F}$ is conservative, then $\mathbf{F} = \nabla f$ for some scalar-valued function f on X. This means

$$\mathbf{F}(x_1, ..., x_n) = \nabla f(x_1, ..., x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right).$$

Then the derivative matrix of $\mathbf{F}$ is

$$D\mathbf{F} = D(\nabla f) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix}$$

By Clairaut's Theorem, the mixed partials are equal, so this matrix is symmetric. ■

For each of the given vector fields $\mathbf{F}$, determine whether or not it's conservative. If it is conservative, find a potential function. If $\mathbf{F}$ is not conservative, compute the derivative matrix of $\mathbf{F}$ in order to prove that it is not conservative.

Problem 1 $\mathbf{F}(x, y) = (2xy + y^2 + e^y, x^2 + 2xy + xe^y)$

Multiple Choice:

(a) *conservative* ✓

(b) *not conservative*

Problem 1.1 $f(x, y) = \boxed{x^2y + y^2x + e^yx}$

Problem 2 $\mathbf{F}(x, y) = (x^2y + e^{x^2}, \sin(x) + y^3)$

Multiple Choice:

(a) *conservative*

(b) *not conservative* ✓

Problem 2.1 $D\mathbf{F}(x, y) = \begin{pmatrix} 2xy + 2xe^{x^2} & \boxed{x^2} \\ \boxed{\cos(x)} & 3y^2 \end{pmatrix}$

We've seen that if a vector field is conservative, then its derivative matrix is symmetric. But is the converse true? That is, if the derivative matrix is symmetric, does that mean that the vector field is conservative? We'll come back to this question later.

Vector Line Integrals

Suppose we have a vector field $\vec{F}$ and a path $\vec{x}$ in $\mathbb{R}^2$. Imagine that the vector field represents some force, such as gravity or a magnetic field. Also imagine that a particle is traveling along the path.

PICTURE

Suppose we would like to measure the total effect of the force on the movement of the particle along the path. In physics, this is called the *work* done by the force on the particle.

If the particle is moving against the vector field, then the force impedes the progress of the particle, so the work done by the force is negative.

PICTURE

If the particle is moving with the vector field, then the force aids the progress of the particle so the work done by the force is positive.

PICTURE

If the particle is moving perpendicular to the vector field, then the force neither impedes nor aids the progress of the particle, so the work done is zero.

PICTURE

In order to compute the work done by a force on the particle, we need to "add up" the microscopic contributions of the force at each point along the path. This leads us to the definition of vector line integrals.

Vector Line Integrals

We can measure the microscopic contribution of a force to the motion of a particle using dot products. That is, if we have a vector field $\vec{F}$ and a path $\vec{x}$ in $\mathbb{R}^n$, we consider the dot product $\vec{F}(x(t)) \cdot \vec{x}'(t)$.

PICTURE

This compares the vector field at the point $\vec{x}(t)$ with the velocity vector $\vec{x}'(t)$ of the path. Notice that if $\vec{F}$ is perpendicular to $\vec{x}'(t)$, then this dot product is zero, which is consistent with our intuition.

In order to find the total contribution of the vector field to the motion along

Learning outcomes: Understand the definition of vector line integrals geometrically, and be able to compute them.

Author(s): Melissa Lynn

the path, we integrate this dot product from the start of the path to the end. This leads us to the definition of a vector line integral.

Definition 4. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $\vec{x} : [a, b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ path in $\mathbb{R}^n$. Then the* vector line integral *of $\vec{F}$ along $\vec{x}$ is*

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt.$$

Let's look at a couple of examples of computing vector line integrals.

Example 13. *Consider the vector field $\vec{F}(x, y) = (x, y)$ and the path $\vec{x}(t) = (t\cos(t), t\sin(t))$ for $t \in [0, 2\pi]$.*

PICTURE

We'll compute the vector line integral, $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$.

$$\begin{aligned}
\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\
&= \int_0^{2\pi} \vec{F}(t\cos(t), t\sin(t)) \cdot (\cos(t) - t\sin(t), \sin(t) + t\cos(t))\ dt \\
&= \int_0^{2\pi} (t\cos(t), t\sin(t)) \cdot (\cos(t) - t\sin(t), \sin(t) + t\cos(t))\ dt \\
&= \int_0^{2\pi} t\cos^2(t) - t^2\cos(t)\sin(t) + t\sin^2(t) + t^2\cos(t)\sin(t)\ dt \\
&= \int_0^{2\pi} t\ dt \\
&= \frac{1}{2}t^2|_0^{2\pi} \\
&= 2\pi^2
\end{aligned}$$

Example 14. *Consider the vector field $\vec{F}(x, y) = (-x, -y)$ and the path $\vec{x}(t) = (\cos(t), \sin(t))$ for $t \in [0, 2\pi]$.*

PICTURE

We'll compute the vector line integral, $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$.

$$\begin{aligned}
\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\
&= \int_0^{2\pi} \vec{F}(\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t))\ dt \\
&= \int_0^{2\pi} (-\cos(t), -\sin(t)) \cdot (-\sin(t), \cos(t))\ dt \\
&= \int_0^{2\pi} (\cos(t)\sin(t) - \sin(t)\cos(t))\ dt \\
&= \int_0^{2\pi} 0\ dt \\
&= 0
\end{aligned}$$

Since the vector field is always perpendicular to this path, it makes sense that the vector line integral should come out to be zero.

Circulation

Now, let's consider the special case where we integrate over a closed curve. In this case, we refer to the value of the vector line integral as circulation, and we use some special notation.

Definition 5. *Let* $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a vector field, and let* $\vec{x} : [a, b] \to \mathbb{R}^n$ *be a* $\mathcal{C}^1$ *path in* $\mathbb{R}^n$*. Suppose further that* $\vec{x}(a) = \vec{x}(b)$*, so that* $\vec{x}$ *is a closed curve. Then the* circulation *of* $\vec{F}$ *along* $\vec{x}$ *is* $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$*, and we write*

$$\oint_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{x}} \vec{F} \cdot d\vec{s},$$

to emphasize that $\vec{x}$ *parametrizes a closed curve.*

Scalar Line Integrals

We've seen how we can integrate vector fields along a path, using vector line integrals. We can also integrate scalar valued functions along a path. For instance, suppose we have a scalar valued function $f : \mathbb{R}^2 \to \mathbb{R}$ and a path $\vec{x} : [a, b] \to \mathbb{R}^2$ in $\mathbb{R}^2$. Suppose we look at the portion of the graph of f lying over the path $\vec{x}$, and drop a "curtain" to the xy-plane.

PICTURE

Integrating f along the path $\vec{x}$ will be equivalent to finding the area of this curtain. We can also describe this as the area between $\vec{x}(t)$ and $f(\vec{x}(t))$.

Scalar line integrals aren't only useful for finding areas of strangely shapes regions. They are also useful throughout physics. For example, if you have the mass density function of a wire, you can compute the scalar line integral of this function to find the total mass of the wire.

Scalar Line Integrals

Suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$ and a $\mathcal{C}'$ path $\vec{x} : [a, b] \to \mathbb{R}^n$, such that the composition $f(\vec{x}(t))$ is defined on $[a, b]$.

PICTURE

In order to find the area under f and over the path $\vec{x}$, we will borrow an important idea from single variable calculus: approximating an area with rectangles.

In order to do this, we'll partition the interval $[a, b]$ into n subintervals, determined by

$$a = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n = b.$$

This partition breaks the path $\vec{x}$ into smaller paths, by restricting to the subintervals.

PICTURE

Our goal will be to approximate the area under f over each of these shorter paths. These approximations will be computed by finding the length of the short path, and multiplying this by a height determined by a test point, t_k^*. We can think of this as a curved rectangle.

PICTURE

Learning outcomes: Understand the definition of scalar line integrals geometrically, and be able to compute them.

Author(s): Melissa Lynn

The specific choice of the test point will be important for simplifying our result - we'll come back to this later.

The height of our curved rectangle will be $f(\vec{x}(t_k^*))$, and the base is the distance Δs_k along the path $\vec{x}$ from t_{k-1} to t_k.

PICTURE

Thinking back to arclength computations, this distance is given by the integral

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt.$$

Now, we need to make a careful choice for our test point t_k^* which will simplify things later. To do this, recall the Mean Value Theorem for Integrals, from single variable calculus.

Theorem 3. *Suppose g is a continuous function on the closed interval $[a, b]$. Then there exists c in $[a, b]$ such that*

$$\int_a^b g(t) dt = (b - a) g(c).$$

Here, we'll take $\|\vec{x}'(t)\|$ for the function $g(t)$. Since $\vec{x}$ is $\mathcal{C}^1$, $\|\vec{x}'(t)\|$ is continuous. Applying the Mean Value Theorem on the interval $[t_{k-1}, t_k]$, there exists c_k such that

$$\Delta s_k = \int_{t_{k-1}}^{t_k} \|\vec{x}'(t)\| dt = (t_k - t_{k-1}) \|\vec{x}'(c_k)\|.$$

We take this c_k to be our test point, so that $t_k^* = c_k$.

Now, the area of the kth curved rectangle is $F(\vec{x}(t_k^*)) \Delta s_k$. We add up these areas and take the limit as the number of rectangles, n, goes to infinity:

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(\vec{x}(t_k^*)) \Delta s_k.$$

Substituting $\Delta s_k = (t_k - t_{k-1}) \|\vec{x}'(c_k)\|$ and writing $\Delta t = t_k - t_{k-1}$, we have

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(\vec{x}(t_k^*)) \|\vec{x}'(c_k)\| \Delta t.$$

We can recognize this as the single variable integral of the function $f(\vec{x}(t)) \|\vec{x}'(t)\|$ over the interval $[a, b]$,

$$\int_a^b f(\vec{x}(t)) \|vecx'(t)\| dt.$$

We take this to be the definition of a scalar line integral.

Definition 6. *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a $\mathcal{C}^1$ path $\vec{x} : [a, b] \to \mathbb{R}^n$. The* scalar line integral *of f along $\vec{x}$ is*

$$\int_{\vec{x}} f \, ds = \int_a^b f(\vec{x}(t)) \|\vec{x}'(t)\| dt.$$

Examples

Let's look at some examples of computing scalar line integrals.

Example 15. *Consider the function* $f(x,y) = x^2 + y^2$ *and the path* $\vec{x}(t) = (\cos t, \sin t)$ *for* $t \in [0,\pi]$. *We compute the scalar line integral of* f *over* $\vec{x}$.

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt \\
&= \int_0^\pi f(\cos t, \sin t)\|(-\sin t, \cos t)\|dt \\
&= \int_0^\pi (\cos^2 t + \sin^2 t)\sqrt{\sin^2 t + \cos^2 t}dt \\
&= \int_0^\pi 1\ dt \\
&= \pi
\end{aligned}$$

Example 16. *Consider the function* $f(x,y) = e^{\sqrt{xy}}$ *and the path* $\vec{x}(t) = (t,t)$ *for* $t \in [0,1]$. *We compute the scalar line integral of* f *over* $\vec{x}$.

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt \\
&= \int_0^1 f(t,t)\|(1,1)\|dt \\
&= \int_0^1 e^{\sqrt{t^2}}\sqrt{2}dt \\
&= \sqrt{2}\int_0^1 e^t dt \\
&= \sqrt{2}(e^1 - e^0) \\
&= \sqrt{2}e - \sqrt{2}
\end{aligned}$$

Line Integrals over Simple Curves

We've defined vector and scalar line integrals over paths.

Definition 7. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and let $\vec{x} : [a,b] \to \mathbb{R}^n$ be a $\mathcal{C}^1$ path in $\mathbb{R}^n$. Then the* vector line integral *of $\vec{F}$ along $\vec{x}$ is*

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt.$$

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a $\mathcal{C}^1$ path $\vec{x} : [a,b] \to \mathbb{R}^n$. The scalar line integral *of f along $\vec{x}$ is*

$$\int_{\vec{x}} f\ ds = \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|dt.$$

Both of these definitions were motivated by geometric considerations. For vector line integrals, we wanted to find the effect of a vector field on a particle moving along a curve. For scalar line integrals, we wanted the find the under the graph of a function over a curve. Although the definitions were motivated by questions about curves, we wound up with definitions that seem to depend on a parametrization of the curve. So, there's a natural follow-up question: do line integrals depend on the parametrization of the curve?

To answer this question, we'll focus on simple curves.

Definition 8. *A path $\vec{x} : [a,b] \to \mathbb{R}^n$ is* simple *if $\vec{x}$ is a one-to-one function (except we'll allow $\vec{x}(a) = \vec{x}(b)$).*

A curve is simple *if it can be parametrized by a simple path.*

Essentially, a curve is simple if it doesn't intersect itself. If a curve starts and ends at the same point, but doesn't intersect itself otherwise, we'll still say that the curve is simple.

Let's look at some examples of curves, and determine whether they are simple.

Example 17. *For each of the curves, decide if it is simple.*

PICTURES/MULTIPLE CHOICE

For scalar line integrals, if we have two simple paths parametrizing the same curve, the scalar line integrals will be the same.

Learning outcomes: Understand the independence from parametrization of line integrals for simple curves.

Author(s): Melissa Lynn

Proposition 1. *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function defined on a curve, which has simple $\mathcal{C}^1$ parametrizations $\vec{x} : [a,b] \to \mathbb{R}^n$ and $\vec{y} : [c,d] \to \mathbb{R}^n$. Then*

$$\int_{\vec{x}} f\ ds = \int_{\vec{y}} f\ ds.$$

Let's look at an example to see why it is important to have a simple path.

Example 18. *Consider the function $f(x,y) = x$ and the paths $\vec{x}(t) = (t,t)$ for $t \in [0,1]$ and $\vec{y}(t) = (t^2, t^2)$ for $t \in [-1,1]$. Note that $\vec{x}$ and $\vec{y}$ are both $\mathcal{C}^1$ parametrizations of the line segment connecting $(0,0)$ and $(1,1)$. However, $\vec{y}$ is not a simple parametrization; it traverses the line segment by starting at $(1,1)$, moving along the segment to $(0,0)$, and then returning to $(1,1)$. We'll evaluate the scalar line integrals of f along these paths.*

$$\begin{aligned}
\int_{\vec{x}} f\ ds &= \int_0^1 f(t,t)\|(1,1)\|dt \\
&= \int_0^1 t\sqrt{2}dt \\
&= \sqrt{2}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) \\
&= \frac{\sqrt{2}}{2} \\
\int_{\vec{y}} f\ ds &= \int_{-1}^1 f(t^2,t^2)\|(2t,2t)\|dt \\
&= \int_{-1}^1 t^2\sqrt{4t^2+4t^2}dt \\
&= \sqrt{8}\int_{-1}^1 t^3 dt \\
&= \sqrt{8}\left(\frac{1^4}{4} - \frac{(-1)^4}{4}\right) \\
&= 0
\end{aligned}$$

Next, we'll turn our attention to vector line integrals. For vector line integrals, we also need to consider the orientation of the curve. That is, we need to consider the direction in which we traverse the curve. Notice that a simple curve has exactly two choices of orientation.

PICTURE

The sign of a vector line integral will depend on the orientations of the paths. Let's think about why this is true. Suppose that a vector field impedes the progress of a particle moving along a path. If we reverse the direction of the path, the vector field will contribute to the motion of the particle. Thus, the sign of the vector line integral would change.

PICTURE

Apart from the issue of orientation, vector line integrals will be independent of the parametrization for simple paths.

Proposition 2. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and consider a curve parametrizated by simple $\mathcal{C}^1$ parametrizations $\vec{x} : [a, b] \to \mathbb{R}^n$ and $\vec{y} : [c, d] \to \mathbb{R}^n$.*

If $\vec{x}$ and $\vec{y}$ have the same orientation, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

If $\vec{x}$ and $\vec{y}$ have opposite orientations, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = -\int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

In the following exercise, it's important to consider the orientation of paths as you determine which vector line integrals are equal.

Problem 3 *Consider the following paths, which parametrize the line segment between $(1, 1, 1)$ and $(1, 2, 3)$.*

$$\begin{aligned} \vec{x}(t) &= (1, 1 + t, 1 + 2t) \text{ for } t \in [0, 1] \\ \vec{y}(t) &= (1, 1 + t^2, 1 + 2t^2) \text{ for } t \in [0, 1] \\ \vec{y}(t) &= (1, 1 + t^2, 1 + 2t^2) \text{ for } t \in [-1, 1] \end{aligned}$$

Let $\vec{F}$ be a vector field, and consider the vector line integrals $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$, $\int_{\vec{y}} \vec{F} \cdot d\vec{s}$, and $\int_{\vec{z}} \vec{F} \cdot d\vec{s}$. Which of these line integrals are equal?

Multiple Choice:

(a) *None of them are equal.*

(b) $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$ and $\int_{\vec{y}} \vec{F} \cdot d\vec{s}$ and equal ✓

(c) $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$ and $\int_{\vec{z}} \vec{F} \cdot d\vec{s}$ and equal

(d) $\int_{\vec{y}} \vec{F} \cdot d\vec{s}$ and $\int_{\vec{z}} \vec{F} \cdot d\vec{s}$ and equal

(e) *All of them are equal.*

Path-Connected and Simply Connected Regions

We will soon begin to study properties of line integrals and conservative vector fields. In order to do this, we need to pay attention to the topology of our domains. Recall that we've previously classified sets in $\mathbb{R}^n$ as open, closed, or neither.

We'll begin by recalled the definition of an open set.

Definition 9. *In $\mathbb{R}^n$, we call $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}$ the* open ball *of radius $r > 0$ centered at $\mathbf{x}$.*

A set $U \subset \mathbb{R}^n$ is open *if for every $\mathbf{a} \in U$, there is a radius $r > 0$ such that $B_r(\mathbf{a}) \subset U$.*

In words, for any point $\mathbf{a}$ in U, we can find a radius r small enough that the entire ball of radius r centered at $\mathbf{a}$ is contained in U. We can also restate the definition of an open set as "every point is an interior point."

Example 19. *The following are examples of open sets.*

$\{x : 0 < x < 1\}$ *in* $\mathbb{R}$.

PICTURE

$\{(x, y) : 0 < x < 1\}$ *in* $\mathbb{R}^2$.

PICTURE

$\{(x, y) : x^2 + y^2 < 1\}$ *in* $\mathbb{R}^2$.

PICTURE

We now recall the definition of a closed set, which is given relative to open sets.

Definition 10. *A set $X \subset \mathbb{R}^n$ is* closed *if its complement is open.*

Furthermore, a set is closed if and only if it contains all of its boundary points.

Example 20. *The following are examples of closed sets.*

$\{x : 0 \leq x \leq 1\}$ *in* $\mathbb{R}$.

PICTURE

Learning outcomes: Understand the geometry and definitions of connected and simply connected sets.

Author(s): Melissa Lynn

$\{(x, y) : 0 \le x \le 1\}$ *in* $\mathbb{R}^2$.

PICTURE

$\{(x, y) : x^2 + y^2 \le 1\}$ *in* $\mathbb{R}^2$.

PICTURE

For the results we wish to prove about line integrals and conservative vector fields, we will also need to consider if sets are connected, and how they are connected.

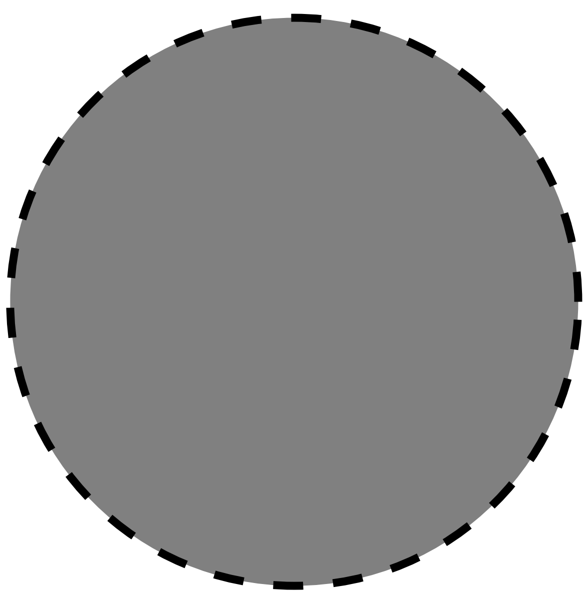
Path-connected sets

You probably have a pretty good intuitive idea of what it should mean for a set to be connected: if you imagine the set to be land, and the complement of the set to be lava, then you can get to the entire set while staying on land, and without jumping. This idea translates to mathematics, using paths.

Definition 11. *A set* $X \subset \mathbb{R}^n$ *is path-connected if any two points can be connected by a path which lies entirely in* X*.*

Notice that we use the word "path-connected" instead of just connected. This is because "connected" actually has a different meaning in topology. However, in some situations, "connected" and "path-connected" are equivalent.

Problem 4 *For each of the following sets, determine whether or not they are path-connected.*

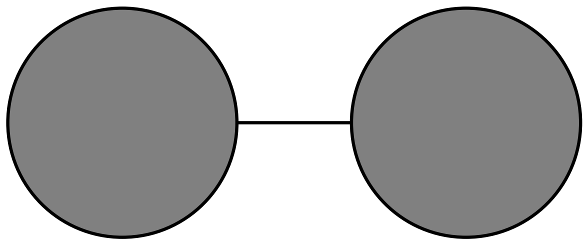


(a)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*

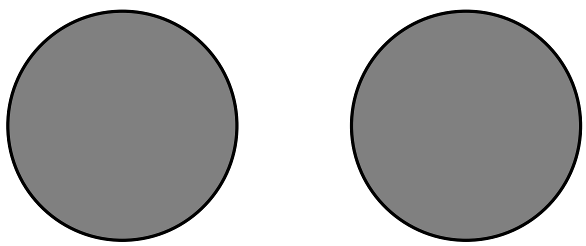


(b)

Multiple Choice:

(i) *path-connected* ✓

(ii) *not path-connected*



(c)

Multiple Choice:

(i) *path-connected*

(ii) *not path-connected* ✓

Simply Connected Sets

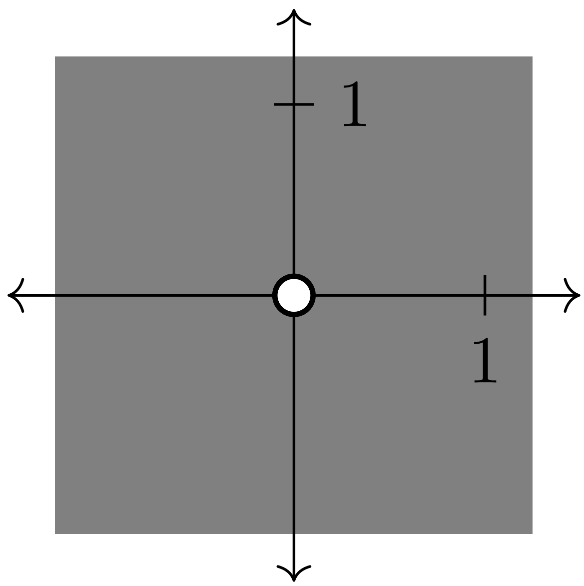
The last topological concept we will cover is when a path-connected set is "simply connected." Intuitively, this depends on whether or not the set has holes. Our definition for simply connected is a bit hand wavy, but this will serve our purpose just fine. It can be made rigorous using continuous maps.

Definition 12. *A path-connected set $X \subset \mathbb{R}^n$ is* simply connected *if any closed path (i.e., loop) can be shrunk to a point, where the shrinking occurs entirely in X.*

Note that a set needs to be path-connected in order to be considered simply connected.

Problem 5 *For each of the following sets, determine whether or not they are simply connected.*

(a) $\mathbb{R}^2 \setminus \{(0,0)\}$

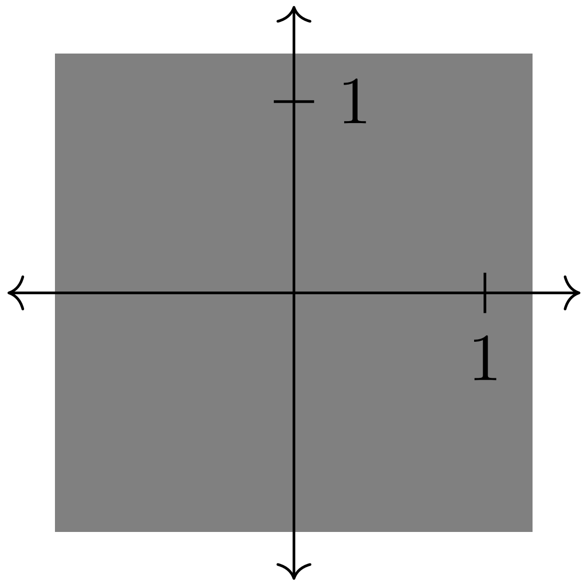


Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

(b) $\mathbb{R}^2$

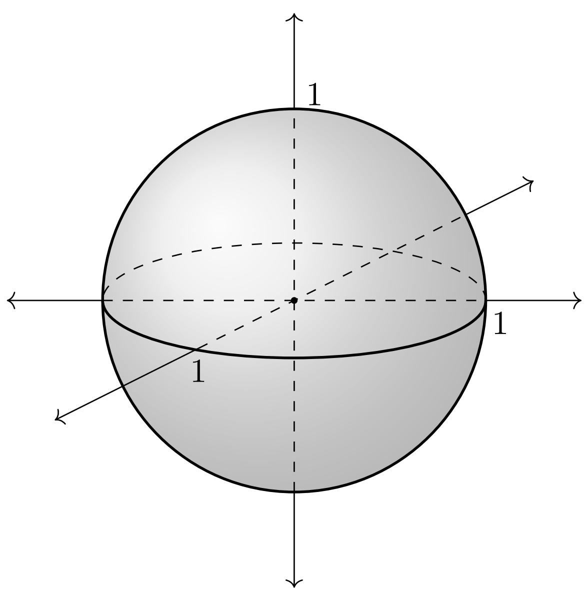


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

(c) $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

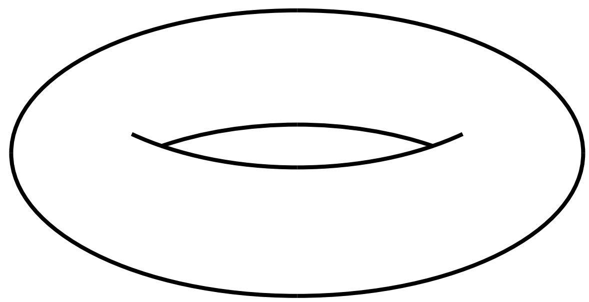


Multiple Choice:

(i) *simply connected* ✓

(ii) *not simply connected*

(d) *A torus (the surface of a donut)*



Multiple Choice:

(i) *simply connected*

(ii) *not simply connected* ✓

Path Independence and FTLI

We've previously seen how vector line integrals are (mostly) independent of the parametrization of the curve, which we restate in the following theorem.

Theorem 4. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and consider a curve parametrizated by simple $\mathcal{C}^1$ parametrizations $\vec{x} : [a,b] \to \mathbb{R}^n$ and $\vec{y} : [c,d] \to \mathbb{R}^n$.*

If $\vec{x}$ and $\vec{y}$ have the same orientation, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

If $\vec{x}$ and $\vec{y}$ have opposite orientations, then

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = -\int_{\vec{y}} \vec{F} \cdot d\vec{s}.$$

Now, let's suppose we have two different curves, but they start and end at the same points. Will vector line integrals over these paths be equal?

PICTURE

Surprisingly, for some vector fields, the answer is "yes." This is called path independence, and we'll explore which vector fields have this property.

Path Independence

A vector field is path independent if its line integrals depend only on the start and end points, and not on the path we take to get between the points.

Definition 13. *A continuous vector field is called* path independent *if* $\int_C \vec{F} \cdot d\boldsymbol{s} = \int_D \vec{F} \cdot d\boldsymbol{s}$ *for any two simple, piecewise C^1, oriented curves C and D with the same start and end points.*

Let's quickly review the meaning of the requirements on the curves C and D.

A curve is *simple* if it (isn't too "bumpy." / doesn't intersect itself, except the start and end point can be the same. ✓/ is smooth.)

Learning outcomes: Understand the definition of path independent. Use the Fundamental Theorem of Line Integrals to compute vector line integrals of conservative vector fields, or to show that a vector field is path independent.

Author(s): Melissa Lynn

A curve is C^1 if it (is continuous. / is differentiable. / has continuous partial derivatives. ✓)

A curve is *oriented* if it (has a specified direction. ✓/ knows which way is North.)

Let's look at some examples.

Example 21. *Consider the vector field* $\vec{F}(x, y) = (y, 0)$. *Let* $\boldsymbol{x}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *along a straight line. Let* $\boldsymbol{y}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *counterclockwise around the unit circle. Compute* $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$. *Are they equal?*

We'll begin by parametrizing our paths.

$$\boldsymbol{x}(t) = (\boxed{1-t}, \boxed{t}) \qquad \text{for } t \in [0,1]$$

$$\boldsymbol{y}(t) = (\boxed{\cos(t)}, \boxed{\sin(t)}) \qquad \text{for } t \in [0, \frac{\pi}{2}]$$

Now, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} = \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Integrating along $\boldsymbol{x}$, *we have*

$$\begin{aligned}
\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\
&= \int_{\boxed{0}}^{\boxed{1}} (t, 0) \cdot (-1, 1)\, dt \\
&= \int_{\boxed{0}}^{\boxed{1}} -t\, dt \\
&= \boxed{-\frac{1}{2}}
\end{aligned}$$

Integrating along $\boldsymbol{y}$, *we have*

$$\begin{aligned}
\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\
&= \int_0^{\pi/2} (\sin(t), 0) \cdot (\boxed{-\sin(t)}, \boxed{\cos(t)})\, dt \\
&= \int_0^{\pi/2} \boxed{-\sin^2(t)}\, dt \\
&= \boxed{-\frac{\pi}{4}}
\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$*, we see that they are*

Multiple Choice:

(a) *Equal.*

(b) *Not equal.* ✓

As a result, we know that the vector field $\vec{F}$*...*

Multiple Choice:

(a) *...is path independent.*

(b) *...is not path independent.* ✓

(c) *...might be path independent. There isn't enough information to tell.*

Now, let's investigate integrating a different vector field along those same paths.

Example 22. *Consider the vector field* $\vec{F}(x, y) = (y, x)$*. Let* $\boldsymbol{x}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *along a straight line. Let* $\boldsymbol{y}(t)$ *be the path from* $(1, 0)$ *to* $(0, 1)$ *counterclockwise around the unit circle. Compute* $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ *and* $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$*. Are they equal?*

Once again, we begin by parametrizing our paths.

$$\boldsymbol{x}(t) = (1 - t, t) \qquad \text{for } t \in [0, 1]$$

$$\boldsymbol{y}(t) = (\cos(t), \sin(t)) \qquad \text{for } t \in [0, \frac{\pi}{2}]$$

Next, we compute these line integrals using the definition

$$\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} = \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt.$$

Integrating along $\boldsymbol{x}$*, we have*

$$\begin{aligned} \int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_0^1 (\boxed{t}, \boxed{1 - t}) \cdot (-1, 1)\, dt \\ &= \int_0^1 \boxed{1 - 2t}\, dt \\ &= \boxed{0} \end{aligned}$$

Integrating along $\boldsymbol{y}$, we have

$$\begin{aligned}\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\ &= \int_0^{\pi/2} (\boxed{\sin(t)}, \boxed{\cos(t)}) \cdot (-\sin(t), \cos(t))\, dt \\ &= \int_0^{\pi/2} \boxed{-\sin^2(t) + \cos^2(t)}\, dt \\ &= \boxed{0}\end{aligned}$$

Comparing $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$ and $\int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s}$, we see that they are

Multiple Choice:

(a) *Equal.* ✓

(b) *Not equal.*

As a result, we know that the vector field $\vec{F}$...

Multiple Choice:

(a) *...is path independent.*

(b) *...is not path independent.*

(c) *...might be path independent. There isn't enough information to tell.* ✓

In fact, it turns out that the vector field $\vec{F}(x, y) = (y, x)$ is path independent. However, in order to check directly that a vector field $\vec{F}$ is path independent, we would need to check the line integrals over any path between any two points in the domain. Of course, this is impossible! We will need to find different methods for showing that a vector field is path independent. Our first result in this direction is the Fundamental Theorem of Line Integrals.

Fundamental Theorem of Line Integrals

We now introduce the Fundamental Theorem of Line Integrals, which gives us a powerful way to compute the integral of a gradient vector field over a piecewise C^1 curve. In particular, note the conditions on the domain X: it must be open and path-connected.

Theorem 5. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and path-connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{B}) - f(\boldsymbol{A})$$

This should look vaguely familiar - it resembles the Fundamental Theorem of Calculus from single variable calculus, also called the evaluation theorem.

We now prove the Fundamental Theorem of Line Integrals in the special case where we have a *simple* parametrization of the curve C.

Proof Let $\mathbf{x}(t)$ be a simple parametrization of C, where $t \in [a, b]$, $\mathbf{x}(a) = \mathbf{A}$, and $\mathbf{x}(b) = \mathbf{B}$ (so the starting point is $\mathbf{A}$, and the ending point is $\mathbf{B}$).

Then we compute the line integral as:

$$\int_C \nabla f \cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s}.$$

By the definition, we can compute this line integral as

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt.$$

The integrand here should look familiar from one of the multivariable versions of the chain rule - it's the derivative of $f(\mathbf{x}(t))$. Making this replacement, we have

$$\int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \frac{d}{dt}(f(\mathbf{x}(t)))\, dt.$$

Now, we can apply the Fundamental Theorem of Calculus (from single variable) to evaluate this integral, since an antiderivative for the integrand will be given by $f(\mathbf{x}(t))$. From this we obtain:

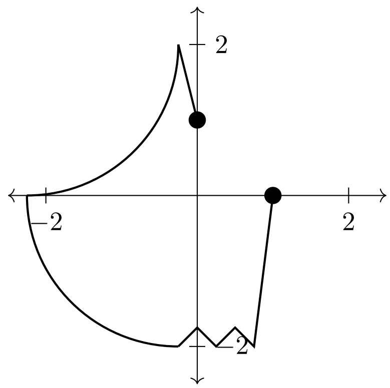
$$\begin{aligned} \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} &= (f(\mathbf{x}(t)))\big|_a^b \\ &= f(\mathbf{x}(b)) - f(\mathbf{x}(a)) \\ &= f(\mathbf{B}) - f(\mathbf{A}) \end{aligned}$$

In the last step, we use that $\mathbf{A}$ and $\mathbf{B}$ are the start an end points of $\mathbf{x}(t)$, respectively.

Thus, we have proven the Fundamental Theorem of Line Integrals (when we have a simple parametrization of the curve C). ■

Before discussing how the Fundamental Theorem of Line Integrals relates to path independence, let's look at how this helps us compute integrals of gradient vector fields.

Example 23. *Let $\vec{F}(x,y) = (y,x)$. Observe that $\vec{F} = \nabla f$, where $f(x,y) = xy$. Compute $\int_C \vec{F} \cdot d\boldsymbol{s}$ for the curve C below, starting at $(0,1)$ and ending at $(1,0)$.*



Explanation. *We certainly would like to avoid parametrizing this curve! So, we will use the Fundamental Theorem of Line Integrals to compute this integral.*

First, let's verify that $\vec{F} = \nabla f$ for $f(x,y) = xy$.

$$\frac{\partial}{\partial x} xy = \boxed{y}$$
$$\frac{\partial}{\partial y} xy = \boxed{x}$$

Thus, we have $\nabla f(x,y) = (y,x) = \vec{F}(x,y)$.

We can then use the Fundamental Theorem of Line Integrals to compute $\int_C \vec{F} \cdot d\boldsymbol{s}$.

$$\begin{aligned}\int_C \vec{F} \cdot d\boldsymbol{s} &= \int_C \nabla f \cdot d\boldsymbol{s} \\ &= f(0,1) - f(1,0) \\ &= \boxed{0}\end{aligned}$$

Now, it turns out that we can use the Fundamental Theorem of Line Integrals to prove the following corollary about the relationship between conservative vector fields and path independence. Note once again that we require the domain to be open and path-connected.

Corollary 1. *If $\vec{F}$ is a conservative vector field defined on an open and connected domain X, then $\vec{F}$ is path independent.*

Proof Let C and D be two curves with starting point $\mathbf{A}$ and ending point $\mathbf{B}$. We will show that $\int_C \vec{F} \cdot d\mathbf{s} = \int_D \vec{F} \cdot d\mathbf{s}$.

Recall that "$\vec{F}$ is conservative" means that $\vec{F} = \nabla f$ for some function f, which will enable us to use the Fundamental Theorem of Line Integrals (FTLI). Then we have:

$$\begin{aligned}
\int_C \vec{F} \cdot d\mathbf{s} &= \int_C \nabla f \cdot d\mathbf{s} \\
&= f(\mathbf{B}) - f(\mathbf{A}) && \text{(by FTLI)} \\
&= \int_D \nabla f \cdot d\mathbf{s} && \text{(also by FTLI)} \\
&= \int_D \vec{F} \cdot d\mathbf{s}.
\end{aligned}$$

Thus, we have shown that $\int_C \vec{F} \cdot d\mathbf{s} = \int_D \vec{F} \cdot d\mathbf{s}$, and so have shown that $\vec{F}$ is path independent. ■

This corollary, with the Fundamental Theorem of Line Integrals, gives us a new tool for computing line integrals.

Strategies for Computing Line Integrals

We now have a few options for computing line integrals:

(a) Using the original definition:

$$\int_C \vec{F} \cdot d\mathbf{s} = \int_a^b \vec{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt$$

(b) If $\vec{F}$ is conservative (so $\vec{F} = \nabla f$ for some f):

   (i) We can use the Fundamental Theorem of Line Integrals:

$$\int_C \vec{F} \cdot d\mathbf{s} = f(\mathbf{B}) - f(\mathbf{A})$$

   (ii) Since the vector field is path independent, we can find an *easier* path with the same start and end points, and integrate over that path.

Let's look at how these different methods can be used in an example.

Example 24. *Let* $\vec{F}(x, y) = (2xy^2, 2x^2y)$, *and consider* $\boldsymbol{x}(t) = (2\cos(\pi t), \sin(\pi t^2))$ *for* $t \in [0, 1]$. *Compute* $\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s}$.

Explanation. *We'll evaluate this integral in three different ways.*

First, let's evaluate using the definition of vector line integrals.

$$\begin{aligned}\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_a^b \vec{F}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}'(t)\, dt \\ &= \int_0^1 (4\cos(\pi t)\sin^2(\pi t^2), 8\cos^2(\pi t)\sin(\pi t^2)) \cdot (\boxed{-2\pi\sin(\pi t)}, \boxed{2\pi t\cos(\pi t^2)})\, dt \\ &= \int_0^1 \boxed{-8\pi\cos(\pi t)\sin(\pi t)\sin^2(\pi t^2) + 16\pi t\cos(\pi t^2)\cos^2(\pi t)\sin(\pi t^2)}\, dt\end{aligned}$$

This is an integral that it might be possible to figure out how to compute, but we certainly do not want to! We can use a computer algebra system to see that the result is 0*, but to compute the integral by hand, we will turn to our other methods.*

For our alternate methods, we need to find a potential function $f(x, y)$ such that $\vec{F} = \nabla f$. It turns out that $f(x, y) = x^2y^2$ works. Let's verify this.

$$\begin{aligned}\frac{\partial}{\partial x}(x^2y^2) &= \boxed{2xy^2} \\ \frac{\partial}{\partial y}(x^2y^2) &= \boxed{2x^2y}\end{aligned}$$

Now that we have our function $f(x, y) = x^2y^2$ such that $\vec{F} = \nabla f$, we will use the Fundamental Theorem of Line Integrals to evaluate. Note the start and end points of our curve

$$\begin{aligned}\boldsymbol{A} &= \boldsymbol{x}(0) = (\boxed{2}, \boxed{0}) \\ \boldsymbol{B} &= \boldsymbol{x}(1) = (\boxed{-2}, \boxed{0})\end{aligned}$$

$$\begin{aligned}\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{x}} \nabla f \cdot d\boldsymbol{s} \\ &= f(\boldsymbol{B}) - f(\boldsymbol{A}) \\ &= \boxed{0}\end{aligned}$$

Note that this is a much easier computation than the integral we had from the first method.

Finally, we compute the line integral using the third method. We have already shown that $\vec{F}$ is a conservative vector field (by finding f such that $\vec{F} = \nabla f$), and *hence we know that* $\vec{F}$ *is path independent. So we can compute this integral by instead integrating over an easier path with the same start and end points,* $(2,0)$ *and* $(-2,0)$*, respectively. Let's choose the straight line from* $(2,0)$ *to* $(-2,0)$*, and parametrize this curve.*

$$\boldsymbol{y}(t) = (\boxed{t}, 0) \qquad \textit{for } t \in [-2,2]$$

Now we can integrate over y *instead, which will be a much easier computation.*

$$\begin{aligned}
\int_{\boldsymbol{x}} \vec{F} \cdot d\boldsymbol{s} &= \int_{\boldsymbol{y}} \vec{F} \cdot d\boldsymbol{s} \\
&= \int_{-2}^{2} \vec{F}(\boldsymbol{y}(t)) \cdot \boldsymbol{y}'(t)\, dt \\
&= \int_{-2}^{2} (\boxed{0}, \boxed{0}) \cdot (1,0)\, dt \\
&= \int_{-2}^{2} \boxed{0}\, dt \\
&= \boxed{0}
\end{aligned}$$

So we've seen that we can compute this line integral in a few different ways, using the fact that the vector field is conservative.

Depending on the particular problem or example, any one of these methods might be easier than the others. You should practice trying these different methods, and see which you prefer! However, remember that for the second and third options, we need to first verify that the vector field $\vec{F}$ is conservative. This usually means finding a potential function f such that $\vec{F} = \nabla f$.

Conservative Vector Fields

In this section, we'll look at several closely related properties of vector fields. We'll begin by recalling these definitions.

Definition 14. *A vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is* conservative *if there is some function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = \vec{F}$.*

A continuous vector field is called path independent *if $\int_C \vec{F} \cdot d\boldsymbol{s} = \int_D \vec{F} \cdot d\boldsymbol{s}$ for any two simple, piecewise $\mathcal{C}^1$, oriented curves C and D with the same start and end points.*

A vector field $\vec{F}$ has no circulation *if $\oint_C \vec{F} \cdot d\vec{s} = 0$ for any simple $\mathcal{C}^1$ curve C.*

A vector field $\vec{F}$ has a symmetric derivative *if the derivative matrix $D\vec{F}$ is symmetric.*

It turns out that these concepts are equivalent in many cases, as long as we have a "nice enough" domain. Let's look at the results we have so far, and how we found them.

Proposition 3. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field defined on an open and path-connected domain X. If $\vec{F}$ is conservative, then $\vec{F}$ is path independent.*

We proved this result using the Fundamental Theorem of Line Integrals. If we have a potential function f for $\vec{F}$, then the vector line integral over any curve can be computed by evaluating f at the endpoints, so it is independent of the path taken between two points.

We have also shown that a conservative vector field has a symmetric derivative matrix.

Proposition 4. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and path-connected. If $\vec{F}$ is conservative, then $D\vec{F}$ is symmetric.*

This is shown by writing $\mathbf{F}$ as ∇f for a potential function f, and then observing that $D\vec{F}$ is the Hessian matrix of f. By Clairaut's theorem, Hessian matrices are symmetric, hence $D\vec{F}$ is symmetric.

Learning outcomes: Understand the relationships between conservative vector fields, path independence, circulation, and symmetric derivatives. Understand the hypotheses under which the equivalences hold.

Author(s): Melissa Lynn

Path independence implies conservative

We've seen that, in the right circumstances, conservative vector fields will be path independent. But will path independent vector fields necessarily be conservative vector fields? In order to show this, the challenge is in constructing a potential function.

Proposition 5. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field, and let X be open and path-connected. If $\vec{F}$ is path-connected, then $\vec{F}$ is conservative.*

Proof In order to prove this, we will construct a potential function $f : X \to \mathbb{R}$ such that $\vec{F} = \nabla f$.

Fix a point $\vec{a}$ in X. Define the function f by

$$f(\vec{x}) = \int_C \vec{F} \cdot d\vec{s},$$

where C is a path from $\vec{a}$ to $\vec{x}$. This is where path-independence is crucial. If $\vec{F}$ were path-dependent, then the definition of $f(\vec{x})$ would depend on the choice of the path C, and so the function f would not be well-defined.

Furthermore, this is also where we require that the domain X be path-connected. If X were not path-connected, there would be points $\vec{x}$ which couldn't be connected to $\vec{a}$ with a path, so f would not be defined on all of X.

Now that we've defined our function f, we need to show that ∇f exists and equals $\vec{F}$. To show that ∇f exists, we need to show that the partial derivatives of f exist. We will show that the first partial derivative of f exists, and the argument for the other partial derivatives is similar.

From the definition of partial derivatives, we have

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{f(x_1 + h, x_2, ..., x_n) - f(x_1, x_2, ..., x_n)}{h}.$$

This is where we need the domain X to be open. Since X is open, there is an open ball around the point $(x_1, x_2, ..., x_n)$ within X. This means that for small enough h, the point $(x_1 + h, x_2, ..., x_n)$ is in X, so that $f(x_1 + h, x_2, ..., x_n)$ is defined.

PICTURE

Now, let's make use of our definition of f, writing $\vec{b} = (x_1, x_2, ..., x_n)$ and $\vec{b}' = (x_1 + h, x_2, ..., x_n)$ to simplify notation. Let C be a path from $\vec{a}$ to $\vec{b}$, and let C' be a path from $\vec{a}$ to $\vec{b}'$. Then,

$$f(x_1, x_2, ..., x_n) = \int_C \vec{F} \cdot d\vec{s}$$
$$f(x_1 + h, x_2, ..., x_n) = \int_{C'} \vec{F} \cdot d\vec{s}$$

Our partial derivative is then

$$\frac{\partial f}{\partial x_1} = \lim_{h \to 0} \frac{\int_C \vec{F} \cdot d\vec{s} - \int_{C'} \vec{F} \cdot d\vec{s}}{h}.$$

Looking at the numerator, $\displaystyle\int_C \vec{F} \cdot d\vec{s} - \int_{C'} \vec{F} \cdot d\vec{s}$ will be equal to $\displaystyle\int_C \vec{D} \cdot d\vec{s}$, where D is any path starting at $\vec{b}$ and ending at $\vec{b}'$.

PICTURE

We'll choose the path D to be parametrized by

$$\vec{x}(t) = (x_1 + ht, x_2, ..., x_n) \text{ for } 0 \leq t \leq 1.$$

Then, substituting this in,

$$\begin{aligned} \int_C \vec{D} \cdot d\vec{s} &= \int_0^1 \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\ &= \int_0^1 \vec{F}(x_1 + ht, x_2, ..., x_n) \cdot (h, 0, ..., 0)\ dt \end{aligned}$$

Let F_1 be the first component of $\vec{F}$. Then $\vec{F}(x_1 + ht, x_2, ..., x_n) \cdot (h, 0, ..., 0) = hF_1(x_1 + ht, x_2, ..., x_n)$.

So, for the partial derivative of f, we have

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \lim_{h \to 0} \frac{\int_C \vec{D} \cdot d\vec{s}}{h} \\ &= \lim_{h \to 0} \frac{\int_0^1 hF_1(x_1 + ht, x_2, ..., x_n)\ dt}{h} \\ &= \lim_{h \to 0} \int_0^1 F_1(x_1 + ht, x_2, ..., x_n)\ dt \end{aligned}$$

At this point, we'll gloss over some details. Essentially, F_1 is a "nice enough" function that we can bring the limit inside of the integral, and arrive at the partial derivative,

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \int_0^1 F_1(x_1, x_2, ..., x_n)\ dt \\ &= F_1(x_1, x_2, ..., x_n). \end{aligned}$$

Thus, the first partial derivative of f is the first component of $\vec{F}$. Following the same argument for the other partial derivatives, we can show that $\nabla f = \vec{F}$. ■

Path independence and circulation

Next, we will show that a vector field is path independent if and only if it has no circulation. For this result, we don't need any special requirements on the domain.

Proposition 6. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field. Then $\vec{F}$ is path independent if and only if $\oint_C \vec{F} \cdot d\vec{s} = 0$ for all closed curves C in X.*

Proof First, let's assume that $\vec{F}$ is path independent, and we'll show that $\vec{F}$ has no circulation. Let C be any closed curve in X, and suppose C starts and ends at the point $\vec{a}$. Let D be the constant curve parametrized by $\vec{x}(t) = \vec{a}$ for $t \in [0, 1]$. By path independence, we have

$$\int_C \vec{F} \cdot d\vec{s} = \int_D \vec{F} \cdot d\vec{s}.$$

Since $\vec{x}(t)$ is constant, $\vec{x}'(t) = \vec{0}$, so we can compute the line integral.

$$\begin{aligned}
\int_D \vec{F} \cdot d\vec{s} &= \int_0^1 \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\
&= \int_0^1 \vec{F}(\vec{a}) \cdot \vec{0}\ dt \\
&= \int_0^1 0\ dt \\
&= 0
\end{aligned}$$

So, $\vec{F}$ has no circulation.

Next, we'll assume that $\vec{F}$ has no circulation, and we'll show that $\vec{F}$ is path independent. Suppose we have two oriented curves C and D, which both start at $\vec{a}$ and end at $\vec{b}$. Let D' be D with the orientation reversed, so that

$$\int_D^{\prime} \vec{F} \cdot d\vec{s} = -\int_D \vec{F} \cdot d\vec{s}.$$

Let E be the oriented curve obtained by first traversing C, then traversing D'. Then E starts and ends at $\vec{a}$, and

$$\begin{aligned}
\int_E \vec{F} \cdot d\vec{s} &= \int_C \vec{F} \cdot d\vec{s} + \int_D^{\prime} \vec{F} \cdot d\vec{s}, \\
&= \int_C \vec{F} \cdot d\vec{s} - \int_D \vec{F} \cdot d\vec{s}.
\end{aligned}$$

Since E starts and ends at $\vec{a}$, it is a closed curve. The vector field $\vec{F}$ has no circulation, so $\int_E \vec{F} \cdot d\vec{s} = 0$. Thus

$$0 = \int_C \vec{F} \cdot d\vec{s} - \int_D \vec{F} \cdot d\vec{s},$$

so $\int_C \vec{F} \cdot d\vec{s} = \int_D \vec{F} \cdot d\vec{s}$. This shows that $\vec{F}$ is path independent. ■

Symmetric derivative implies no circulation

Finally, suppose $\vec{F}$ has a symmetric derivative, then $\vec{F}$ has no circulation. In this case, we require our domain to be simply connected.

Proposition 7. *Let $\vec{F} : X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field, defined on an open and simply connected domain X. If $D\vec{F}$ is a symmetric matrix, then $\vec{F}$ has zero circulation.*

Although this completes our set of equivalences, we don't yet have the tools that we need to prove this result. For this, we will need double integrals and Green's Theorem, so we'll come back to this proof later.

Summary

We summarize the relationship between conservative vector fields, path independence, zero circulation, and symmetric derivatives in the following diagram. Throughout, let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field defined on a set $X \subset \mathbb{R}^n$.

DIAGRAM

Remember that you can often move through more than one box. For example, if $D\vec{F}$ is continuous and symmetric on an open, simply connected set X, then $\vec{F}$ is conservative. That's because an open simply-connected set is also connected, so you can follow arrows all the way up to that box.

Part II

Double Integrals

Double Integrals

We would like to be able to integrate functions $f : \mathbb{R}^2 \to \mathbb{R}$, finding the net volume between the graph of f and the xy-plane. Before we figure out how to do this, let's review how we defined integrals in single variable calculus.

Consider a continuous (or piece-wise continuous) function f defined on the closed interval $[a, b]$. We would like to find the signed area between the graph of f and the x-axis over this interval, where area below the x-axis counts as negative.

PICTURE

We can approximate this area with rectangles. To do this, we divide the interval $[a, b]$ into subintervals, and use a sample point to determine the height of the rectangle over that subinterval.

PICTURE

The area of the ith rectangle will be $f(x_i)\Delta x$, where x_i is the ith sample point, and Δx is the length of each subinterval. To approximate the area under f, we add up the area of the rectangles, $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$. As the number of rectangles increases, the approximation becomes more accurate, so we take the limit as the number of rectangles, n, goes to infinity, and we arrive at the definition of the definite integral.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

To define the integral of a function from $\mathbb{R}^2$ to $\mathbb{R}$, we'll follow this same idea of approximating area with rectangles. Since we'll be dealing with volume instead, we'll approximate with boxes.

Double Integrals

Consider a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ defined on a rectangle $R \subset \mathbb{R}^2$.

Learning outcomes: Understand the geometric motivation behind the definition of a double integral.

Author(s): Melissa Lynn

PICTURE

We'll approximate the signed area between the graph of f and the xy-plane using boxes. To do this, we'll partition R into subrectangles, and take a box over each rectangle. The height of the box will be given by the value of f at a sample point, $\vec{c}_{ij}$, in the subrectangle.

PICTURE

Then, the volume of the box over the (i, j)th rectangle is $f(\vec{c}_{ij})\Delta x \Delta y$, where $\vec{c}_{ij}$ is the sample point, Δx is the width of the rectangle, and Δy is the length of the rectangle.

When we add up the total area volume of the boxes, this approximates the volume under the graph of f.

$$\text{Volume} \approx \sum_{i,j} f(\vec{c}_{ij})\Delta x \Delta y$$

PICTURE

When we increase the number of subrectangles, this approximation becomes more accurate. So, when we take the limit as the number of rectangles, n, goes to infinity, we compute the exact volume, which leads us to the definition of a double integral.

Definition 15. *Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined on a rectangle R in $\mathbb{R}^2$. The* double integral *of f over R is*

$$\iint_R f(x, y)dA = \lim_{n\to\infty} \sum_{i,j} f(\vec{c}_{ij})\Delta x \Delta y,$$

provided this limit exists. Here, n, $\vec{c}_{ij}$, Δx, and Δy are as defined above.

If the double integral above exists, we say that f is integrable *over R.*

The double integral represented the signed volume between the graph of f and the xy-plane. Unsurprisingly, continuous functions are always integrable.

Theorem 6. *Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined on a rectangle R. Then f is integrable.*

Unfortunately, this definition isn't practical for computing double integrals. For computations, we'll use iterated integrals, which will be defined in the next section. Using Fubini's Theorem, we'll see why we can use iterated integrals to compute double integrals.

Although we won't typically use the definition of a double integral for computation, we can approximate double integrals using a finite number of boxes.

Approximating double integrals

We'll now look at a couple of examples of approximating double integrals.

Example 25. *Consider the function* $f(x, y) = x^2 + y^2$ *over the rectangle* $[0, 1] \times [0, 1]$.

PICTURE

We'll partition the rectangle $[0, 1] \times [0, 1]$ *into nine subrectangles, and use the upper right corner of each rectangle for a sample point.*

PICTURE

Next, we'll evaluate f *at each of our sample points.*

$(\vec{c}_{ij})$	$f(\vec{c}_{ij})$
$(1/3, 1/3)$	$2/9$
$(1/3, 2/3)$	$5/9$
$(1/3, 1)$	$10/9$
$(2/3, 1/3)$	$5/9$
$(2/3, 2/3)$	$8/9$
$(2/3, 1)$	$13/9$
$(1, 1/3)$	$10/9$
$(1, 2/3)$	$13/9$
$(1, 1)$	2

The base of each box has area $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$, *so the volume of the* ij*th box is* $\frac{1}{9} f(\vec{c}_{ij})$.

To approximate the double integral $\iint_R f(x, y)dA$, *we add up the volume of all of these boxes.*

$$\begin{aligned}
\iint_R f(x, y)dA &\approx \sum_{i,j} \frac{1}{9} f(\vec{c}_{ij}) \\
&= \frac{1}{9} \sum_{i,j} f(\vec{c}_{ij}) \\
&= \frac{1}{9} \left(\frac{2}{9} + \frac{5}{9} + \frac{10}{9} + \frac{5}{9} + \frac{8}{9} + \frac{13}{9} + \frac{10}{9} + \frac{13}{9} + 2 \right) \\
&= \frac{28}{27}
\end{aligned}$$

So, we have the approximation $\iint_R f(x, y)dA \approx \frac{28}{27}$.

Example 26. *Consider the function* $f(x, y) = e^{xy^2}$ *over the rectangle* $[0, 2] \times [1, 2]$.

PICTURE

We'll partition the rectangle $[0,2] \times [1,2]$ *into sixteen subrectangles, and we'll use the lower right corners as sample points.*

PICTURE

We evaluate f *at each of our sample points.*

$\vec{c}_{ij}$	$f(\vec{c}_{ij})$
$(1/2, 1)$	$e^{1/2}$
$(1, 1)$	e
$(3/2, 1)$	$e^{3/2}$
$(2, 1)$	e^2
$(1/2, 5/4)$	$e^{25/32}$
$(1, 5/4)$	$e^{25/16}$
$(3/2, 5/4)$	$e^{75/32}$
$(2, 5/4)$	$e^{25/8}$
$(1/2, 3/2)$	$e^{9/8}$
$(1, 3/2)$	$e^{9/4}$
$(3/2, 3/2)$	$e^{27/8}$
$(2, 3/2)$	$e^{9/2}$
$(1/2, 7/4)$	$e^{49/32}$
$(1, 7/4)$	$e^{49/16}$
$(3/2, 7/4)$	$e^{147/32}$
$(2, 7/4)$	$e^{49/8}$

The base of each box is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$, so the volume of the ijth box is $\frac{1}{8} f(\vec{c}_{ij})$.

To approximate the double integral $\iint_R f(x,y)dA$, we add up the volume of all of these boxes.

$$\begin{aligned}
\iint_R f(x,y)dA &\approx \frac{1}{8} \sum_{i,j} f(\vec{c}_{ij}) \\
&= \frac{1}{8} \Big(e^{1/2} + e + e^{3/2} + e^2 + e^{25/32} + e^{25/16} + e^{75/32} + e^{25/8} + e^{9/8} + e^{9/4} + e^{27/8} + e^{9/2} + e^{49/32} \\
&\approx 96.27
\end{aligned}$$

So, we have the approximation $\iint_R f(x,y)dA \approx 96.27$.

In the next section, we'll introduce iterated integrals, which we'll be able to use to compute the exact value of double integrals.

Iterated Integrals

We've defined double integrals to compute the signed volume between the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and the xy-plane.

PICTURE

Definition 16. *Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined on a rectangle R in $\mathbb{R}^2$. The* double integral *of f over R is*

$$\iint_R f(x, y)dA = \lim_{n\to\infty} \sum_{i,j} f(\vec{c}_{ij})\Delta x \Delta y,$$

provided this limit exists.

Although this definition has a useful geometric interpretation, and we can use it to approximate double integrals, it isn't practical for computing double integrals. Instead, we'll compute using iterated integrals.

For now, we'll focus on a different geometric interpretation of volume, and see how this can be computed using iterated integrals. Then, in the next section, we'll see how double integrals can be computed using iterated integrals.

Iterated integrals

Back in single variable calculus, we were able to compute the volume of some solids using cross sections.

PICTURE

Once we found the area $A(x)$ of a cross section at x, we were able to compute volume by integrating this function $A(x)$.

$$V = \int_a^b A(x)dx$$

We'll use the same idea for iterated integrals

Suppose we wish to integrate a function $f : \mathbb{R}^2 \to \mathbb{R}$ over a rectangle $R = [a, b] \times [c, d]$.

PICTURE

Learning outcomes: Understand the geometric interpretation of iterated integrals, and compute them.

Author(s): Melissa Lynn

To find the area of the cross section at x, we integrate $f(x, y)$ from $y = c$ to $y = d$. This gives us

$$A(x) = \int_c^d f(x, y)dy.$$

Now that we have the area of a cross section, we can compute the volume of the region by integrating over our x values.

$$\begin{aligned} V &= \int_a^b A(x)dx \\ &= \int_a^b \left(\int_c^d f(x, y)dy \right) dx \\ &= \int_a^b \int_c^d f(x, y)dydx \end{aligned}$$

Example 27. *We'll use such an iterated integral to compute the volume between* $f(x, y) = x^2y$ *and xy-plane, over the rectangle* $[0, 2] \times [1, 2]$.

$$\begin{aligned} V &= \int_0^2 \int_1^2 x^2y \; dydx \\ &= \int_0^2 \left(\int_1^2 x^2y \; dy \right) dx \\ &= \int_0^2 \left(\frac{1}{2}x^2y^2|_{y=1}^{y=2} \right) dx \\ &= \int_0^2 \left(2x^2 - \frac{1}{2}x^2 \right) dx \\ &= \int_0^2 \frac{3}{2}x^2 \; dx \\ &= \frac{1}{2}x^3|_{x=0}^{x=2} \\ &= \frac{1}{2} \cdot 2^3 - \frac{1}{2} \cdot 0^3 \\ &= 4 \end{aligned}$$

Alternatively, we can find the volume by taking cross sections which are constant with respect to y.

PICTURE

Then, we can compute the area of the cross section at y by integrating $f(x, y)$ from $x = a$ to $x = b$. This gives us

$$A(y) = \int_a^b f(x, y)dx.$$

Then, we compute the volume of the region by integrating the function $A(y)$ over our range of y values.

$$\begin{aligned} V &= \int_c^d A(y)dy \\ &= \int_c^d \left(\int_a^b f(x,y)dx \right) dy \\ &= \int_c^d \int_a^b f(x,y)dxdy \end{aligned}$$

Example 28. *We'll use an iterated integral to compute the volume between $f(x,y) = x^2y$ and xy-plane, over the rectangle $[0,2] \times [1,2]$. This time, we'll integrate with respect to y first.*

$$\begin{aligned} V &= \int_1^2 \int_0^2 x^2y \; dxdy \\ &= \int_1^2 \left(\int_0^2 x^2y \; dx \right) dy \\ &= \int_1^2 \left(\frac{1}{3}x^3y|_{x=0}^{x=2} \right) dy \\ &= \int_1^2 \left(\frac{8}{3}y - 0 \right) dy \\ &= \frac{8}{3} \int_1^2 \frac{8}{3}y \; dy \\ &= \frac{8}{3} \left(\frac{1}{2}y^2|_{y=1}^{x=2} \right) \\ &= \frac{8}{3} \cdot \left(4 - \frac{1}{2} \right) \\ &= 4 \end{aligned}$$

Notice that iterated integrals are computed by just computing one integral after another, which is why they are called "iterated."

At this point, you might be wondering why we bothered defining double integrals at all. Iterated integrals are relatively easy to compute, and can also be used to find volume, which makes them seem like a much better choice. In many situations, double integrals and iterated integrals are perfectly interchangeable, and we will explore this in the next section. However, there exist some strange functions where iterated integrals won't be defined, and yet the double integral will exist. Because of this, double integrals are a more general concept than iterated integrals.

Fubini's Theorem

We've now seen two different approaches to finding the signed volume between the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ and the xy-plane over some rectangle $R = [a, b] \times [c, d]$. First, we defined double integrals, motivated by approximating volume with boxes.

$$\iint_R f(x, y) dA = \lim_{n\to\infty} \sum_{i,j} f(\vec{c}_{ij}) \Delta x \Delta y.$$

PICTURE

The second approach we took was using iterated integrals, which computed volume by finding the area of cross sections, and then integrating this area function.

$$V = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

PICTURE

Although both of these methods were designed to compute the volume of the same region, for a function f that isn't "nice enough," they could theoretically give different answers, or one of them might not exist. In this section, we'll explore the relationship between double integrals and iterated integrals through Fubini's Theorem.

Fubini's theorem

Fubini's theorem gives us an equivalence between double integrals and iterated integrals, as we'd expect.

Theorem 7. ***Fubini's Theorem.*** *Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined on a rectangle $R = [a, b] \times [c, d]$. Then*

$$\iint_R f\ dA = \int_c^d \int_a^b f(x, y)\ dx dy = \int_a^b \int_c^d f(x, y)\ dy dx.$$

Notice that Fubini's theorem requires that we have a continuous function. In many cases where the a function has a limited number of discontinuities, we'll be able to piece together some integrals to compute a double integral. We'll return to this idea when we compute double integrals over arbitrary regions.

Learning outcomes: Understand the connection between iterated integrals and double integrals.

Author(s): Melissa Lynn

Now, we'll prove Fubini's theorem.

Proof Since f is a continuous function, it is integrable, so $\iint_R f\, dA$ exists.

BLAAAAAHHH ■

Example 29. *We'll compute the double integral* $\iint_{[-1,1]\times[2,3]} sin(x)e^{y^2} dA$. *Because* $f(x,y) = sin(x)e^{y^2}$ *is a continuous function, we can apply Fubini's theorem and compute the double integral using iterated integrals. Then,*

$$\begin{aligned}\iint_{[-1,1]\times[2,3]} sin(x)e^{y^2} dA &= \int_{-1}^{1}\int_{2}^{3} sin(x)e^{y^2} dydx \\ &= \int_{-1}^{1} sin(x)\left(\int_{2}^{3} e^{y^2} dy\right) dx.\end{aligned}$$

Here, we run into a problem computing $\int_2^3 e^{y^2} dy$. *This integral isn't possible to evaluate precisely using common methods from single variable calculus. Instead, let's compute the double integral using the other iterated integral, where we integrate with respect to* x *first, then with respect to* y.

$$\begin{aligned}\iint_{[-1,1]\times[2,3]} sin(x)e^{y^2} dA &= \int_{2}^{3}\int_{-1}^{1} sin(x)e^{y^2} dxdy \\ &= \int_{2}^{3}\left(-\cos(x)e^{y^2}\right)\Big|_{x=-1}^{x=1} dy \\ &= \int_{2}^{3}\left(-\cos(1)e^{y^2} + \cos(-1)e^{y^2}\right) dy\end{aligned}$$

Since $\cos(1) = \cos(-1)$, *the terms in the integrand cancel, and we're left with*

$$\iint_{[-1,1]\times[2,3]} sin(x)e^{y^2} dA = 0.$$

For this function, it turned out that one of the iterated integrals was easier to compute than the other, and Fubini's theorem allowed us to choose which one to compute.

Elementary Regions

We've defined double integrals over rectangles, and we've used Fubini's theorem to convert them to iterated integrals, which are straightforward to compute. But what if we need to integrate a function over a region that isn't a rectangle?

For example, consider the region below, which is bounded by the curves $y = x^2$, $y = x^2 + 1$, $x = 1$, and $x = 2$.

PICTURE

We can describe this region as the set of points (x, y) such that $1 \leq x \leq 2$ and $x^2 \leq y \leq x^2+1$. Here, we can describe the region by bounding the x-coordinate with constants, and bounding the y-coordinate with continuous functions of x. We'll soon see that this kind of description translates easily to integration, and this is our first example of an elementary region.

Elementary regions

We give the definition of elementary regions, which will be the most natural regions to integrate over.

Definition 17. *The following are types of* elementary regions*.*

Suppose a region R can be described as the set of points (x, y) such that

$$\begin{aligned} a \leq &x \leq b, \text{ and} \\ f(x) \leq &y \leq g(x), \end{aligned}$$

where $f(x)$ and $g(x)$ are continuous functions. Then we say that R is x-simple*.*

Suppose a region R can be described as the set of points (x, y) such that

$$\begin{aligned} c \leq &y \leq d, \text{ and} \\ f(y) \leq &x \leq g(y), \end{aligned}$$

where $f(y)$ and $g(y)$ are continuous functions. Then we say that R is y-simple*.*

We'll now look at some examples of x-simple and y-simple regions. Note that some regions are both x-simple and y-simple.

Example 30. *Consider the region R below, which is bounded by the graphs of $y = x^2$ and $x = y^2$.*

Learning outcomes: Identify and describe elementary regions.
Author(s): Melissa Lynn

PICTURE

This region is both x-simple and y-simple. To see that it is x-simple, we can describe R as the set of points (x, y) such that

$$\begin{aligned} 0 \leq & x \leq 1, \text{ and} \\ x^2 \leq y \leq & \sqrt{x}. \end{aligned}$$

To see that R is y-simple, we can describe it as the set of points (x, y) such that

$$\begin{aligned} 0 \leq & y \leq 1, \text{ and} \\ y^2 \leq & x \leq \sqrt{y}. \end{aligned}$$

Example 31. *Consider the region R below, which is bounded by the graphs of $y = \cos x$, $y = 1$, $x = -\pi$, and $x = \pi$.*

PICTURE

This region is x-simple, since R is the set of points (x, y) such that

$$\begin{aligned} -\pi \leq & x \leq \pi, \text{ and} \\ \cos x \leq & y \leq 1. \end{aligned}$$

The region R is not y-simple, since any inequality $f(y) \leq x \leq g(y)$ cannot have a "hole" in the middle. However, R is the union of two y-simple regions R_1 and R_2, where R_1 is the set of points (x, y) such that

$$\begin{aligned} -1 \leq & y \leq 1, \text{ and} \\ -\pi \leq & x \leq -\arccos y, \end{aligned}$$

and R_2 is the set of points (x, y) such that

$$\begin{aligned} -1 \leq & y \leq 1, \text{ and} \\ \arccos y \leq & x \leq 1. \end{aligned}$$

Double Integrals over Elementary Regions

We've defined double integrals over rectangles, by approximating the volume under a surface with boxes.

PICTURE

Definition 18. *Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined on a rectangle R in $\mathbb{R}^2$. The* double integral *of f over R is*

$$\iint_R f(x,y)dA = \lim_{n\to\infty} \sum_{i,j} f(\vec{c}_{ij})\Delta x \Delta y,$$

provided this limit exists.

We used Fubini's theorem to show that we can evaluate double integrals using iterated integrals.

Now, we would like to define and evaluate double integrals over regions which aren't rectangles. We will start by defining these double integrals.

Double integrals over an arbitrary region

Suppose we wish to integrate a function $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ over a bounded region $D \subset X$ contained in the domain of f. This region, D, may be an elementary region, but this is not required for the definition.

PICTURE

We will define the double integral of f over D, using a double integral over a rectangle. That is, consider a function f^{ext}, which is defined to be equal to f on D, and zero elsewhere.

Definition 19. *Let $f^{ext} : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$f^{ext}(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}.$$

Now, let R be a rectangle containing D. Since D is bounded, such a rectangle exists. Then, we can see that the volume under f and over D is equivalent to the the volume under f^{ext} and over R.

Learning outcomes: Set up and evaluate double integrals over elementary regions.
Author(s): Melissa Lynn

PICTURE

Thus, we define the double integral of f over D using the double integral of f^{ext} over R.

Definition 20. *Let D be a bounded region in $\mathbb{R}^2$, and let R be a rectangle containing D. Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined on D. Then we define the* double integral of f over D *to be*

$$\iint_D f\, dA = \iint_R f^{ext}\, dA.$$

Although this provides us with a definition for double integrals, this definition isn't very useful for evaluating double integrals. Next, we'll look at how we can evaluate double integrals over elementary regions.

Evaluating double integrals over elementary regions

When D is an elementary regions, we can describe it using inequalities, and then use these inequalities for the bounds of integration.

Proposition 8. *Consider an x-simple region D, which can be described as the set of points (x, y) such that*

$$\begin{aligned} a \leq & x \leq b, \text{ and} \\ g(x) \leq & y \leq h(x), \end{aligned}$$

where $g(x)$ and $h(x)$ are continuous functions. Then we can evaluate the double integral of a function $f(x, y)$ over D as

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dydx.$$

Consider a y-simple region D, which can be described as the set of points (x, y) such that

$$\begin{aligned} c \leq & y \leq d, \text{ and} \\ g(y) \leq & x \leq h(y), \end{aligned}$$

where $g(y)$ and $h(y)$ are continuous functions. Then we can evaluate the double integral of a function $f(x, y)$ over D as

$$\iint_D f(x,y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\, dxdy.$$

Let's look at how we can evaluate double integrals over elementary regions.

Example 32. *Consider the x-simple region D below, which is bounded by the curves $x = -1$, $x = 1$, $y = 0$, and $y = x^2$.*

PICTURE

Let's integrate the function $f(x, y) = x^2 y$ over this region. Since D can be described with the inequalities

$$\begin{aligned} -1 \leq & x \leq 1, \text{ and} \\ 0 \leq & y \leq x^2. \end{aligned}$$

We have

$$\begin{aligned} \iint_D f \, dA &= \int_{-1}^{1} \int_0^{x^2} x^2 y \, dy dx \\ &= \int_{-1}^{1} \left(\frac{1}{2} x^2 y^2 |_{y=0}^{y=x^2} \right) dx \\ &= \int_{-1}^{1} \left(\frac{1}{2} x^2 (x^2)^2 - 0 \right) dx \\ &= \int_{-1}^{1} \left(\frac{1}{2} x^6 \right) dx \\ &= \left(\frac{1}{14} x^7 \right) |_{x=-1}^{x=1} \\ &= \left(\frac{1}{14} \cdot 1^7 \right) - \left(\frac{1}{14} \cdot (-1)^7 \right) \\ &= \frac{1}{7}. \end{aligned}$$

When evaluating these double integrals, it can be very useful to start be drawing the region of integration, in order to determine the bounds.

Example 33. *We'll evaluate the double integral $\iint_D f \, dA$, where $f(x, y) = \sin(y^2)$ and D is the region below.*

PICTURE

The region D is y-simple, and can be described with the inequalities

$$\begin{aligned} 0 \leq & y \leq \sqrt{\pi}, \text{ and} \\ -y \leq & x \leq y. \end{aligned}$$

We evaluate the double integral.

$$
\begin{aligned}
\iint_D f\ dA &= \int_0^{\sqrt{\pi}} \int_{-y}^{y} \sin(y^2)\ dxdy \\
&= \int_0^{\sqrt{\pi}} \left(\sin(y^2)x\right) |_{x=-y}^{x=y}\ dy \\
&= \int_0^{\sqrt{\pi}} \left(2y\sin(y^2)\right)\ dy \\
&= \left(-\cos(y^2)\right) |_{y=0}^{y=\sqrt{\pi}} \\
&= -\cos(\sqrt{\pi}^2) + \cos(0^2) \\
&= 2
\end{aligned}
$$

Changing the Order of Integration

Consider the integral $\iint_D \sin(y^2)\, dA$, where D is the region below.

PICTURE

The region D is x-simple, and can be described with the inequalities

$$\begin{aligned} 0 \leq & x \leq \sqrt{\pi}, \text{ and} \\ 0 \leq & y \leq 2x \end{aligned}$$

Thus, we can write the double integral as

$$\iint_D f\, dA = \int_0^{\sqrt{\pi}} \int_0^{2x} \sin(y^2)\, dy\, dx.$$

Alternatively, the region D is y-simple, and can be described with the inequalities

$$\begin{aligned} 0 \leq & y \leq \frac{1}{2}\sqrt{\pi}, \text{ and} \\ 0 \leq & x \leq \frac{y}{2}. \end{aligned}$$

Thus, we can write the double integral as

$$\iint_D f\, dA = \int_0^{\sqrt{\pi}/2} \int_0^{y/2} \sin(y^2)\, dxdy.$$

Notice that for the first integral, we integrate with respect to y and then with respect to x. For the other integral, we integrate with respect to x and then with respect to y. These integrals can be used to evaluate the double integral of the same function over the same region, so they should give the same value.

Let's see what happens when we evaluate each of the integrals. For the first integral, we have

$$\iint_D f\, dA = \int_0^{\sqrt{\pi}} \int_0^{2x} \sin(y^2)\, dy\, dx.$$

Here, we're stuck right away. We don't have a nice, easy formula for an antiderivative of $\sin(y^2)$.

Maybe we'll have better luck with the second integral.

Learning outcomes: Change the order of integration in double integrals over elementary regions.

Author(s): Melissa Lynn

$$
\begin{aligned}
\iint_D f \, dA &= \int_0^{\sqrt{\pi}/2} \int_0^{y/2} \sin(y^2) \, dx dy \\
&= \int_0^{\sqrt{\pi}/2} \left(\sin(y^2) x\right) |_{x=0}^{x=y/2} \, dy \\
&= \int_0^{\sqrt{\pi}/2} \left(\frac{y}{2} \sin(y^2)\right) \, dy \\
&= \left(-\frac{1}{4} \cos(y^2)\right) |_{y=0}^{y=\sqrt{\pi}} \\
&= -\frac{1}{4} \cos(\sqrt{\pi}^2) + \frac{1}{4} \cos(0^2) \\
&= \frac{1}{2}
\end{aligned}
$$

In this example, evaluate the integral with respect to x and then y was much easier than integrating with respect to y and then x.

Changing the order of integration

As in the example above, sometimes a certain order of integration will be easier than the other, and it can be useful to change the order of integration. When we do this, it's important to pay careful attention to the bounds, and it can be useful to sketch the region of integration.

Let's look at some examples of changing the order of integration.

Example 34. *Consider the integral*

$$\int_0^2 \int_{x^2}^{2x} x^3 \, dy dx.$$

We could evaluate this integral directly, but instead, we'll change the order of integration for this integral, and then evaluate the resulting integral. In order to change the order of integration, we'll start by sketching the region of integration.

PICTURE

We can describe this region using the inequalities

$$
\begin{aligned}
0 \leq & y \leq 4, \text{ and} \\
y/2 \leq & x \leq \sqrt{y}.
\end{aligned}
$$

So, we can change the order of integration, writing

$$\int_0^2 \int_{x^2}^{2x} x^3 \, dy dx = \int_0^4 \int_{y/2}^{\sqrt{y}} x^3 \, dx dy.$$

Evaluating our new integral, we have

$$\begin{aligned}
\int_0^4 \int_{y/2}^{\sqrt{y}} x^3 \, dxdy &= \int_0^4 \left(\frac{1}{4}x^4\right) |_{x=y/2}^{x=\sqrt{y}} dy \\
&= \int_0^4 \left(\frac{1}{4}\sqrt{y}^4 - \frac{1}{4}\left(\frac{y}{2}\right)^4\right) dy \\
&= \int_0^4 \left(\frac{1}{4}y^2 - \frac{1}{64}y^4\right) dy \\
&= \left(\frac{1}{12}y^3 - \frac{1}{320}y^5\right) |_{y=0}^{y=4} \\
&= 32/15.
\end{aligned}$$

Example 35. *Now, consider the integral* $\int_0^2 \int_0^{x+1} x^2 y \, dydx$. *We'll change the order of integration for this integral. To do this, we start by sketching the region of integration.*

PICTURE

We can describe this region as points (x, y) *with* $0 \le y \le 3$, *but the bounds for* x *change. When* $0 \le y \le 1$, *the bounds for* x *are* $0 \le x \le 2$. *When* $1 \le y \le 3$, *the bounds for* x *are* $y-1 \le x \le 2$. *Because of this, we need two integrals. That is, we have*

$$\int_0^2 \int_0^{x+1} x^2 y \, dydx = \int_0^1 \int_0^2 x^2 y \, dydx + \int_1^3 \int_{y-1}^2 x^2 y \, dxdy.$$

We could then evaluate these integrals.

Review of u-substitution

One of the most commonly used strategies for evaluating single variable integrals is u-substitution. For example, suppose we wish to evaluate the integral $\int_1^3 2xe^{x^2}\,dx$. Here, it's useful to make the substitution $u = x^2$, and this allows us to evaluate the integral.

$$\begin{aligned}\int_1^3 2xe^{x^2}\,dx &= \int_1^9 e^u\,du \\ &= e^u\big|_{u=1}^{u=9} \\ &= e^9 - e\end{aligned}$$

We would like to apply the same ideas to double integrals, in order to make evaluation possible in more cases. Before we do this, we'll review the details of single variable u-substitution, in order to prepare for the more difficult two variable case.

There are three important parts of this process that we wish to highlight here.

- Changing the variable
- Changing the differential
- Changing the interval of integration

Changing the variable

Choosing the change of variable is the most important step of u-substitution. Suppose we are performing a u-substitution on a definite integral $\int_a^b F(x)dx$. Here, we make a choice for u as a function of x, so we write $u = g(x)$ for some function g. This choice is often made in conjunction with planning for the change of differential. That is, we choose $u = g(x)$ in a way so that the integrand can be written as

$$F(x) = f(g(x))g'(x),$$

for some function f.

Learning outcomes: Revisit u-substitution from single variable calculus, in preparation for substitution in double integrals.

Author(s): Melissa Lynn

For example, consider the integral $\int_1^3 2xe^{x^2}\ dx$. To evaluate this integral, we choose $u = g(x)$ for $g(x) = x^2$, since then the integrand can be written as

$$2xe^{x^2} = g'(x)e^{g(x)}.$$

Choosing a "good" substitution often requires experience and intuition, built through trying different substitutions, and determining what works well. A common strategy for choosing substitutions is to look for a composition of functions, and taking u to be the "inner" function. In our example, we have the composition e^{x^2}, and took $u = x^2$.

Changing the interval of integration

Once we've made a choice of substitution $u = g(x)$, we start to tranform our integral to be with respect to u. As part of this process, we need to change the bounds of the integral. The new bounds will be $c = g(a)$ and $d = g(b)$. In order to see why this is necessary, let's look at the area represented by the definite integral $\int_a^b F(x)dx$.

PICTURE

Here, we are finding the area under the graph of the function $F(x)$ on the closed interval from $x = a$ to $x = b$. When we make a u-substitution, we are evaluating a different integral $\int_c^d f(u)du$. This represents the area under the function $f(u)$ over the closed interval from c to d. Since our new integral is with respect the u, it doesn't make sense to use the old bounds $x = a$ and $x = b$, since these were bounds for the variable x. Instead, we need to find the bounds on u which correspond to the old bounds on x. When we look at how the function g affects a and b, this gives us our new bounds, $c = g(a)$ and $d = g(b)$. We can also think about this as looking at how g affects the closed interval $[a, b]$. If g is an increasing function, then the image of the interval $[a, b]$ under g is the closed interval $[g(a), g(b)]$.

PICTURE

So, when we're converting an area over the interval $[a, b]$ using the function g, we will be working with an area over the interval $[g(a), g(b)]$.

PICTURE

To phrase this change more generally, when we make a change of variable, we need to consider how this substitution affects the domain of integration.

Changing the differential

The final step necessary for a u-substitution is to change the differential, so we have an integral with respect to u instead of x. We often think about this step as making the replacement

$$du = g'(x)dx,$$

but why is this change necessary? To see this, let's think back to the definition of a definite integral. We approximate the area under a curve with rectangles, and then take the limit as the width of the rectangles goes to zero.

PICTURE

Now, suppose we are approximating the area under a curve with eight rectangles. Then, suppose we make a change of variables $u = g(x)$, and let's look at how this affects the rectangles.

PICTURE

Here, we see that the width of the rectangles are affected by the change $u = g(x)$. This stretches or shrinks the rectangles. The proportion of this stretching or shrinking is determined by the rate of change of the function g, so by $g'(x)$. This is where the change in differential comes in - we make the replacement $du = g'(x)dx$ in order to account for the stretching or shrinking caused by the substitution $u = g(x)$.

To phrase this change more generally, when we make a change of variable, we need to consider how the substitution affects the shape and size of the domain of integration, and account for this with an expansion factor, such as $g'(x)$.

Double Integrals

In the next section, we'll look at how we can change variables in double integrals, and we'll see how this process resembles u-substitution from single variable calculus. When making the change of variables, we will again need to consider how to change the domain of integration, as well as the differential.

Change of Variables in Double Integrals

When performing a u-substitution on a definite integral in single variable calculus, we paid careful attention to the following steps.

- Changing the variable
- Changing the differential
- Changing the interval of integration

For example, making a u-substitution in the integral $\int_1^3 2xe^{x^2}\ dx$, we chose the change of variable $u = x^2$, changed the differential to $du = g'(x)dx$, and changed the bounds of integration to $1^2 = 1$ and $3^2 = 9$. This gave us

$$\int_1^3 2xe^{x^2}\ dx = \int_1^9 e^u\ du,$$

which we could then evaluate.

We'll now turn our attention to change of variables in double integrals, which will be a useful tool for evaluating these integrals. When we used u-substitution in single variable calculus, we often focused on making the *integrand* easier to integrate. That is, we wished to transform the integrand, so that it had a more obvious antiderivative.

In contract, when performing a change of variables on double integrals, we're often more focused on how the change of variables affects the region of integration. Consider the following regions in $\mathbb{R}^2$. The region on the left can most easily be described using polar coordinates, while the region on the right can most easily be described using a linear change of coordinates.

PICTURE

For double integrals, rather than finding complicated combinations of elementary regions to describe a domain, it will be useful to make a change of coordinates in order to describe the domain of integration more simply.

Learning outcomes: Understand how to change variables.
Author(s): Melissa Lynn

A linear change of coordinates

Let's look at what happens when we make a linear change of coordinates. Consider the region D below, and consider the double integral $\iint_D (x+y)\,dxdy$.

PICTURE

The region D can most easily be described with the inequalities

$$\begin{aligned} 1 &\leq 2x+y \leq 3, \\ 0 &\leq x-3y \leq 1. \end{aligned}$$

This motivates the change of coordinates $u = 2x+y$ and $v = x-3y$. This is a linear change of coordinates, which can be represented with a matrix. That is, we can write

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, let's look at how this linear transformation changes the region of integration. In (u,v)-coordinates, our region of integration can be described as

$$\begin{aligned} 1 &\leq u \leq 3, \\ 0 &\leq v \leq 1. \end{aligned}$$

This corresponds to a parallelogram in (x,y)-coordinates.

PICTURE

We can see that the linear transformation changes the domain of integration, and we will need to account for this as we make our change of coordinates.

We'll begin with the simplest step, which is changing our integrand. If we take the equations $u = 2x+y$ and $v = x-3y$, and solve for x and y, we have

$$x = \frac{3}{7}u + \frac{1}{7}v, y \qquad = \frac{1}{7}u - \frac{2}{7}v.$$

So, our integrand becomes

$$\begin{aligned} x+y &= \left(\frac{3}{7}u + \frac{1}{7}v\right) + \left(\frac{1}{7}u - \frac{2}{7}v\right) \\ &= \frac{4}{7}u - \frac{1}{7}v. \end{aligned}$$

Next, let's look at our bounds of integration. Since the region of integration can be described using the bounds

$$\begin{aligned} 1 &\leq u \leq 3, \\ 0 &\leq v \leq 1, \end{aligned}$$

our integral will have the form

$$\int_0^1 \int_1^3 ?\ dudv.$$

Finally, we need to consider how the change of coordinates affects the differential. That is, what will the area expansion factor be?

When we apply the linear transformation $\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$ to any rectangle, the area of the rectangle is multiplied by $\left| \det \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \right| = |2 \cdot -3 - 1 \cdot 1| = 7.$

So, when we change to (u, v)-coordinates, we need to divide by 7, in order to account for this change in area. This gives us the change of coordinates,

$$\iint_D (x+y)\ dxdy = \int_0^1 \int_1^3 \left(\frac{4}{7}u - \frac{1}{7}v \right) \frac{1}{7} dudv.$$

From here, we can evaluate the integral.

$$\begin{aligned}
\int_0^1 \int_1^3 \left(\frac{4}{7}u - \frac{1}{7}v \right) \frac{1}{7} dudv &= \int_0^1 \left(\frac{2}{49}u^2 - \frac{1}{49}vu \right) |_{u=1}^{u=3} dv \\
&= \int_0^1 \left(\frac{2}{49}3^2 - \frac{1}{49}v \cdot 3 \right) - \left(\frac{2}{49}1^2 - \frac{1}{49}v \cdot 1 \right) dv \\
&= \int_0^1 \left(\frac{16}{49} - \frac{2}{49}v \right) dv \\
&= \left(\frac{16}{49}v - \frac{1}{49}v^2 \right) |_{v=0}^{v=1} \\
&= \frac{15}{49}
\end{aligned}$$

Change of variables in general

Now, let's look at change of variables for double integrals in general, where the change of coordinates might not be linear.

Suppose we are computing a double integral $\iint_D f(x,y)dxdy$, and suppose that $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is a change of coordinates which maps some region D^* onto D. Suppose further that T is $\mathcal{C}^1$, and that T is one-to-one on D^*.

PICTURE

We think of the domain of T as being in (u, v)-coordinates, and the codomain of T as being in (x, y)-coordinates. So, the region D^* would be described in terms of u and v, and is mapped to D, which is described in terms of x and y.

We can see that our change of coordinates will have the form

$$\iint_D f(x,y)dxdy = \iint_{D^*} ?dudv,$$

but we still need to determine how the differential will change, by finding the area expansion factor.

Suppose we have a tiny rectangle R in D^*, and we apply the transformation $\vec{T}$ to this rectangle, to obtain a region $\vec{T}(R)$.

PICTURE

We can approximate the transformation $\vec{T}$ with the linear transformation given by its derivative matrix $D\vec{T}$, so the area of $\vec{T}(R)$ can be approximated as

$$\text{area } \vec{T}(R) \approx \left|\det(D\vec{T})\right| (\text{area } R).$$

This idea gives us our area expansion factor, and we change the differential according to

$$dxdy = \left|\det(D\vec{T})\right| \; dudv.$$

Now, we can fully describe how to change variables in double integrals.

Proposition 9. *Let $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function which maps a region $D^* \subset \mathbb{R}^2$ onto a region $D \subset \mathbb{R}^2$, so that $\vec{T}$ restricted to D^* is one-to-one. Suppose $f : D \to \mathbb{R}$ is an integrable function. Then*

$$\iint_D f(x,y)\; dxdy = \iint_{D^*} f(\vec{T}(u,v)) \left| det(D\vec{T}(u,v))\right| \; dudv.$$

Polar coordinates

Consider the double integral $\displaystyle\iint_D (x^2+y^2)\; dxdy$, where D is the region below.

PICTURE

The region D can most easily be described using polar coordinates, so that

$$0 \leq r \leq 2,$$
$$\pi/4 \leq \theta \leq \pi/2.$$

Recall that polar coordinates relate to Cartesian coordinates via

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

Thus, our transformation T is given by $T(r,\theta) = (r\cos\theta, r\sin\theta)$. In order to see how we must change the differential in the double integral, we compute $\left|\det(D\vec{T}(u,v))\right|$.

$$\begin{aligned}
\left|\det(D\vec{T}(u,v))\right| &= \left|\det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\right| \\
&= |\cos\theta \cdot r\cos\theta - \sin\theta \cdot -r\sin\theta| \\
&= |r|
\end{aligned}$$

Since we describe our region using only nonnegative values for r, the absolute value is unnecessary. Thus, our differential will change according to

$$dxdy = r\ drd\theta.$$

Finally, let's look at our integrand. Using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

Putting all of this together, we have our change of variables for the double integral, and we can evaluate our transformed integral.

$$\begin{aligned}
\iint_D (x^2+y^2)\ dxdy &= \int_{\pi/4}^{\pi/2}\int_0^2 r^2 \cdot r\ drd\theta \\
&= \int_{\pi/4}^{\pi/2}\int_0^2 r^3\ drd\theta \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{4}r^4\Big|_{r=0}^{r=2} d\theta \\
&= \int_{\pi/4}^{\pi/2} 4d\theta \\
&= 4\left(\frac{\pi}{2} - \frac{\pi}{4}\right) \\
&= \pi.
\end{aligned}$$

Notice that any time we change to polar coordinates, we will complete the same computation for the area expansion factor. That is, for changing to polar coordinates, we always have

$$dxdy = r\ drd\theta.$$

Additional Examples of Change of Variables

Recall how we can perform a change of variables in a double integral.

Proposition 10. *Let $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function which maps a region $D^* \subset \mathbb{R}^2$ onto a region $D \subset \mathbb{R}^2$, so that $\vec{T}$ restricted to D^* is one-to-one. Suppose $f : D \to \mathbb{R}$ is an integrable function. Then*

$$\iint_D f(x,y)\; dxdy = \iint_{D^*} f(\vec{T}(u,v)) \left| det(D\vec{T}(u,v)) \right| \; dudv.$$

In this section, we'll look at some further examples of changes of variables, and encounter some common challenges along the way.

Examples

Example 36. *We will evaluate the double integral $\iint_D 2xy\; dA$, where D is the region below, bounded by the lines $y = 2x$, $y = -2x$, and $y = x + 3$.*

PICTURE

If we take $u = y - 2x$ and $v = y + 2x$, then the lines $y = 2x$ and $y = -2x$ correspond to $u = 0$ and $v = 0$, respectively. Solving for x and y in terms of u and v, we have

$$x = \frac{u+v}{2},$$
$$y = \frac{v-u}{4},$$

and the change of coordinates is given by $\vec{T}(u,v) = \left(\frac{u+v}{2}, \frac{v-u}{4} \right)$. From this, we compute

$$\left| det(D\vec{T}(u,v)) \right| = \left| det \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix} \right|$$
$$= \frac{1}{4}.$$

Learning outcomes: Solidify the ability to change variables in double integrals.
Author(s): Melissa Lynn

The line $y = x + 3$ *corresponds to* $\frac{v-u}{4} = \frac{u+v}{2} + 3$, *which can be simplified to* $v = -3u - 12$.

PICTURE

So, in uv-coordinates, our region can be described by the inequalities

$$\begin{gathered} -4 \leq u \leq 0, \\ -3u - 12 \leq v \leq 0. \end{gathered}$$

Putting all of this together, we have

$$\begin{aligned}
\iint_D 2xy\,dA &= \int_{-4}^{0}\int_{-3u-12}^{0} 2\frac{u+v}{2}\frac{v-u}{4}\cdot\frac{1}{4}dvdu \\
&= \int_{-4}^{0}\int_{-3u-12}^{0} \frac{v^2-u^2}{16}dvdu \\
&= \int_{-4}^{0} \left(\frac{v^3}{48} - \frac{u^2 v}{16}\right)\Big|_{v=-3u-12}^{v=0} du \\
&= \int_{-4}^{0} \left(\frac{(-3u-12)^3}{48} - \frac{u^2(-3u-12)}{16}\right) du \\
&= \int_{-4}^{0} \left(\frac{-3u^3}{8} - 6u^2 - 27u - 36\right) du \\
&= \left(\frac{-3u^4}{32} - 2u^3 - \frac{27u^2}{2} - 36u\right)\Big|_{u=-4}^{u=0} \\
&= 32
\end{aligned}$$

Sometimes, we may be able to carry out a change of variables without explicitly finding the transformation $\vec{T}$. We see this in the next example.

Example 37. *We will evaluate the double integral* $\iint_D (x+y)dA$, *where* D *is the region below.*

PICTURE

Since D *can be described with the inequalities*

$$\begin{gathered} 1 \leq xy \leq 4, \\ 0 \leq y - x \leq 2, \end{gathered}$$

we will change to the coordinates $u = xy$ *and* $v = y - x$. *Now, in order to find the area expansion factor* $\left|det(D\vec{T}(u,v))\right|$, *we would typically find the transformation* $\vec{T}$. *That is, we would find* x *and* y *in terms of* u *and* v. *However, in this situation, it's difficult to solve for* x *and* y. *Fortunately, we will be able to work around this issue.*

Although we don't have the transformation $\vec{T}(u,v)$*, we do have the inverse transformation,* $\vec{T}^{-1}$*, which is defined by*

$$\vec{T}^{-1}(x,y) = (xy, y-x).$$

Since $\vec{T}$ *and* $\vec{T}^{-1}$ *are inverse transformations, their derivative matrices* $D\vec{T}$ *and* $D\vec{T}^{-1}$ *are also inverses. In order make use of this fact, we compute*

$$\begin{aligned}\left|det(D\vec{T}^{-1}(x,y))\right| &= \left|det\begin{pmatrix} y & x \\ -1 & 1\end{pmatrix}\right| \\ &= |y+x|.\end{aligned}$$

Then, from properties of derivatives and inverse matrices, we have

$$\left|det(D\vec{T}(u,v))\right| = \frac{1}{|y+x|},$$

If we write x *and* y *in terms of* u *and* v*.*

At first glance, this doesn't seem particularly useful, since we'd still need to find x *and* y *in terms of* u *and* v*! However, let's take a look at our integrand, which is* $x+y$*. Since* $x+y \geq 0$ *on our region, we have* $\frac{1}{|y+x|} = \frac{1}{y+x}$*. So, when we make our change of coordinates, the integrand will cancel with the area expansion factor. This gives us*

$$\begin{aligned}\iint_D (x+y)dA &= \int_1^4 \int_0^2 1\ dvdu \\ &= 6.\end{aligned}$$

Green's Theorem

Suppose you are standing at the only door of an initially empty cafeteria, and you keep a count of how many people enter and exit the cafeteria: for each person who enters, you add one, and for each person who leaves, you subtract one. By keeping track of everyone who's entering and exiting, you know the exact count of everyone who's in the cafeteria at any given time.

Green's theorem follows the same idea, but for a curve enclosing a region in $\mathbb{R}^2$.

PICTURE

If we look at how a vector field acts on the boundary curve, this tells us something about what's happening inside of the enclosed region. We'll state this more precisely soon, but we first need to discuss how to orient the boundary.

Orientation

Suppose we have a closed, bounded region D whose boundary consists of finitely many piecewise smooth curves. We write ∂D for the boundary of D.

PICTURE

We say that the boundary ∂D is *positively oriented* if, as you traverse the curve in the indicated direction, the region is on your left.

Example 38. *Each of the boundary curves below is positively oriented.*

PICTURE

Each of the boundary curves below is not *positively oriented.*

PICTURE

Green's Theorem

Now, we are ready to state Green's Theorem.

Theorem 8. *Green's Theorem.* *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as*

Learning outcomes: Understand the statement of Green's theorem, and its geometric justification.

Author(s): Melissa Lynn

$\vec{F}(x,y) = (M(x,y), N(x,y))$. Then

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

Let's look at an example where we apply Green's theorem to simplify computation of a vector line integral.

Example 39. *Let C be the curve below, enclosing the unit square $[0,1] \times [0,1]$, and consider vector field $\vec{F}(x,y) = (xy + e^x, x + y^4)$.*

PICTURE

Suppose we wish to evaluate the vector line integral $\int_C \vec{F} \cdot d\vec{s}$. We could do this directly, but this would involve evaluating four separate line integrals, one on each side of the square. Instead, we'll use Green's Theorem to compute a double integral over the square $D = [0,1] \times [0,1]$, since $\partial D = C$. By Green's theorem, we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{s} &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \\ &= \iint_D (1 - x) \, dA \\ &= \int_0^1 \int_0^1 (1 - x) \, dx dy \\ &= \int_0^1 \left(x - \frac{x^2}{2} \right) |_{x=0}^{x=1} \, dy \\ &= \int_0^1 \left(1 - \frac{1}{2} \right) \, dy \\ &= \frac{1}{2}. \end{aligned}$$

In this example, we saw how Green's theorem can be used to more easily compute some vector line integrals.

Proof of Green's theorem

Now that we've seen how Green's theorem can useful, let's sketch a proof of it. This proof will require some approximations, and a perfectly rigorous proof would require showing that these approximations can be made arbitrarily accurate. However, in the interest of brevity, we will gloss over those details.

Proof In order to prove Green's theorem, we will make use of two approximation. We begin by deriving these approximations.

Consider the path $\vec{x}(t) = (t, y_0)$ for $t \in [x_0, x_0 + \Delta x]$. Then $\vec{x}'(t) = (1, 0)$, and if we write $F(x, y) = (M(x, y), N(x, y))$, we have

$$\begin{aligned} \int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_{x_0}^{x_0+\Delta x} (M(t, y_0), N(t, y_0)) \cdot (1, 0)\ dt \\ &= \int_{x}^{x_0+\Delta x} M(t, y_0)\ dt \end{aligned}$$

For small Δx, the area represented by this definite integral can be approximated with a single rectangle.

PICTURE

That is, the integral can be approximated by $M(x_0, y_0)\Delta x$. Thus, we have

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} \approx M(x_0, y_0)\Delta x.$$

Using a similar argument for the path $\vec{y}(t) = (x_0, t)$ for $t \in [y_0, y_0 + \Delta y]$, we have the approximation

$$\int_{\vec{y}} \vec{F} \cdot d\vec{s} = N(x_0, y_0)\Delta y.$$

Now that we have the approximations that we need, we'll look at the special case of Green's theorem on a tiny rectangle. We'll then use this special case to prove the general case for Green's theorem.

Consider the rectangle R below, with corners (x_0, y_0), $(x_0 + \Delta x, y_0)$, $(x_0 + \Delta x, y_0 + \Delta y)$, and $(x_0, y_0 + \Delta y)$. We label the sides of this rectangle C_1, C_2, C_3, and C_4.

PICTURE

If C is the entire boundary of the rectangle, we have

$$\oint_C \vec{F} \cdot d\vec{s} = \int_{C_1} \vec{F} \cdot d\vec{s} + \int_{C_2} \vec{F} \cdot d\vec{s} + \int_{C_3} \vec{F} \cdot d\vec{s} + \int_{C_4} \vec{F} \cdot d\vec{s}.$$

Using the approximations from above, we then have

$$\oint_C \vec{F} \cdot d\vec{s} \approx M(x_0, y_0)\Delta x + N(x_0 + \Delta x, y_0)\Delta y - M(x_0, y_0 + \Delta y)\Delta x - N(x_0, y_0)\Delta y$$

For small Δx. Regrouping this approximation, we have

$$\oint_C \vec{F} \cdot d\vec{s} \approx \left(\frac{N(x_0 + \Delta x, y_0) - N(x_0, y_0)}{\Delta x} - \frac{M(x_0, y_0 + \Delta y) - M(x_0, y_0)}{\Delta y} \right) \Delta x \Delta y.$$

As Δx and Δy go to 0, this approaches $(N_x(x_0, y_0) - M_y(x_0, y_0))\, \Delta x \Delta y$.

Furthermore, for a small rectangle, the volume $\iint_R N_x - My\ dA$ can be approximated with a single box, with volume $(N_x(x_0, y_0) - M_y(x_0, y_0))\,\Delta x \Delta y$. Putting all of this together, we have the approximation

$$\oint_C \vec{F} \cdot d\vec{s} \approx \iint_R N_x - My\ dA.$$

Now, suppose we have a closed region D, and fill D with tiny rectangles that share common edges.

PICTURE

Notice that if all of the rectangles are positively oriented, the adjoining edges have opposite directions. This means that vector line integrals over these edges will cancel. This gives us

$$\begin{aligned}
\oint_{\partial D} \vec{F} \cdot d\vec{s} &\approx \oint_{\partial B_1} \vec{F} \cdot d\vec{s} + \cdots + \oint_{\partial B_n} \vec{F} \cdot d\vec{s} \\
&\approx \iint_{B_1} N_x - M_y\ dA + \cdots + \iint_{B_n} N_x - M_y\ dA \\
&\approx \iint_D N_x - M_y\ dA.
\end{aligned}$$

As the boxes get smaller, this approximation improves, giving us the conclusion of Green's Theorem. ■

Green's Theorem Examples

We've seen how Green's theorem relates a vector line integral over the boundary of a region to a double integral over the region.

Theorem 9. ***Green's Theorem.*** *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x, y) = (M(x, y), N(x, y))$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

Now, we'll look at several examples of using Green's theorems to simplify some computations.

Green's theorem examples

Example 40. *Consider the integral $\oint_C \vec{F} \cdot d\vec{s}$, where $\vec{F}(x, y) = (x^2, e^y)$, and C is the unit circle, oriented counterclockwise.*

PICTURE

By Green's theorem, this is equivalent to a double integral over the unit disc. That is,

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{s} &= \iint_D \left(\frac{\partial}{\partial x} e^y - \frac{\partial}{\partial y} x^2 \right) \, dA \\ &= \iint_D 0 \, dA \\ &= 0. \end{aligned}$$

Example 41. *Consider the integral $\oint_C \vec{F} \cdot d\vec{s}$, where $\vec{F}(x, y) = (y, -x)$, and C is the unit circle, oriented* clockwise.

PICTURE

Let D be the unit disc, enclosed by the unit circle. Although this curve isn't positively oriented, we can still use Green's theorem to help evaluate our line integral. This will require a sign change.

Learning outcomes: Understand how Green's Theorem can be used to more easily compute integrals.

Author(s): Melissa Lynn

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{s} &= -\oint_{-C} \vec{F} \cdot d\vec{s} \\
&= -\iint_D \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y) \right) \, dA \\
&= -\iint_D (-2) \, dA \\
&= 2 \cdot (\textit{area of } D) \\
&= 2\pi.
\end{aligned}$$

Area of a Region

We've seen how Green's theorem relates a vector line integral over the boundary of a region to a double integral over the region.

Theorem 10. ***Green's Theorem.*** *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x,y) = (M(x,y), N(x,y))$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

We've used Green's theorem to more easily compute some vector line integrals, and in this section, we'll see how Green's theorem can be used to find the area of a region in $\mathbb{R}^2$.

Area of a region

Suppose we wish to find the area of a region D in $\mathbb{R}^2$.

PICTURE

To find this area, we can instead find a volume with the same value. That is, we can create a solid of height 1 over the region, and find the volume of this solid.

PICTURE

We can use a double integral to compute this volume, giving us the area of the region D.

Proposition 11. *Let D be a region in $\mathbb{R}^2$. Then*

$$\text{area of } D = \iint_D 1 \, dA.$$

We'll use this fact to find the area of the unit circle. From geometry, we expect this area to be π.

Example 42. *Let D be the unit circle in $\mathbb{R}^2$. To find the area of D, we use*

$$\text{area of } D = \iint_D 1 \, dA.$$

Learning outcomes: Use Green's theorem to compute the area of a region in the plane.
Author(s): Melissa Lynn

In polar coordinates, the unit circle can be described with the inequalities

$$\begin{aligned} 0 \le r &\le 1, \\ 0 \le \theta &\le 2\pi. \end{aligned}$$

Changing to polar coordinates to evaluate the double integral, we have

$$\begin{aligned} \iint_D 1\, dA &= \int_0^{2\pi} \int_0^1 r\, dr d\theta \\ &= \int_0^{2\pi} \frac{1}{2} r^2 |_0^1\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\, d\theta \\ &= \pi. \end{aligned}$$

So, we have confirmed that the area of the unit circle is π.

Area using Green's theorem

Now, let's see how we can use Green's theorem to find the area of a region. Suppose we have a curve C enclosing a region D, satisfying the hypotheses for Green's theorem.

PICTURE

Now, suppose we have a vector field $\vec{F}(x, y) = (M(x, y), N(x, y))$ such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$. There are many possible choices for such a vector field; examples include

$$\begin{aligned} \vec{F}(x, y) &= (x, x), \\ \vec{F}(x, y) &= (-y/2, x/2), \\ \vec{F}(x, y) &= \left(y + \sin(x), e^{y^2}\right). \end{aligned}$$

Then, Green's theorem gives us

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_D 1\, dA.$$

So, we can find the area of D by integrating the vector field F over the boundary of D.

Let's look at an example to see this in action.

Example 43. *We'll find the area enclosed by an ellipse C, given by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.*

PICTURE

In this example, we'll use the vector field $\vec{F}(x,y) = (-y/2, x/2)$ to compute this area. By Green's theorem, we have

$$\begin{aligned} Area &= \iint_D 1\, dA \\ &= \oint_C \vec{F} \cdot d\vec{s}. \end{aligned}$$

We can parametrize the ellipse as $\vec{x}(t) = (a\cos t, b\sin t)$ for $0 \le t \le 2\pi$. Notice that this parametrization gives the correct orientation for the ellipse. Now, we evaluate our line integral.

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} \vec{F}(a\cos t, b\sin t) \cdot (-a\sin t, b\cos t) dt \\ &= \int_0^{2\pi} \left(-\frac{b}{2}\sin t, \frac{a}{2}\cos t\right) \cdot (-a\sin t, b\cos t) dt \\ &= \int_0^{2\pi} \left(\frac{ab}{2}\sin^2 t + \frac{ab}{2}\cos^2 t\right) dt \\ &= \int_0^{2\pi} \left(\frac{ab}{2}\right) dt \\ &= ab\pi \end{aligned}$$

Thus, the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $ab\pi$.

Part III

Curl and Divergence

Curl of a Vector Field

REWORK THIS TO BUILD DEFINITION FROM GEOMETRY. THEN NEXT SECTION CAN BE EXAMPLE COMPUTATIONS.

Imagine the vector field below represents fluid flow:

Desmos link: https://www.desmos.com/calculator/vhuoyka1ys

If we fix the center point of each + above, which way will they rotate? (clockwise / counter-clockwise ✓)

We can describe this concept as microscopic rotation or local rotation, and we'll soon see that this can be measured by computing the curl of the vector field, which we define in this section.

Definition of Curl

A curl is an example of an *operator*, which is a mathematical object you've seen before. Roughly speaking, it's a "function" on functions. That is, it takes a function as an input, and produces a function as an output. Here, we're using "function" very broadly - a function could be scalar-valued, a path, or even a vector field!

To prove that you've seen operators before, let's look at a specific example:

Problem 6 *What does* $\frac{d}{dt}g(t)$ *mean?*

Multiple Choice:

(a) *Multiply* $g(t)$ *by the fraction* $\frac{d}{dt}$.

(b) *Take the derivative of* g *with respect to* t. ✓

Problem 6.1 *What does* $\frac{d}{dt}$ *mean?*

Learning outcomes: Understand the definition of the curl of a vector field, and be able to compute curl.

Author(s): Melissa Lynn

Multiple Choice:

(a) *The same thing as* $\frac{1}{t}$.

(b) *Take the derivative with respect to* t. ✓

Problem 6.1.1 *It turns out that* $\frac{d}{dt}$ *is an example of an* operator.

To introduce the curl, we need to talk about another operator, ∇ *which we call the* del operator.

What does $\nabla(g(x, y, z))$ *mean?*

Multiple Choice:

(a) *The change in* g.

(b) $\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$ ✓

Problem 6.1.1.1 *From this, we can deduce that* ∇ *should mean* $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. *Note that this is an operator.*

Definition 21. *The* del operator *in* $\mathbb{R}^n$ *is* $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, ..., \frac{\partial}{\partial x_n}\right)$.

There's one more ingredient that we need to review in order to define the curl of a vector field, the cross product.

Problem 7 *If* $\mathbf{v} = (1, 2, 3)$ *and* $\mathbf{w} = (4, 5, 6)$, *what is* $\mathbf{v} \times \mathbf{w}$? $\boxed{(-3, 6, -3)}$.

Problem 7.1 *Note that this is computed as the determinant*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

Problem 7.1.1 *Given a vector field* $\mathbf{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$*, how might we interpret* $\nabla \times \mathbf{F}$*?*

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \boxed{\dfrac{\partial}{\partial x}} & \boxed{\dfrac{\partial}{\partial y}} & \boxed{\dfrac{\partial}{\partial z}} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\end{aligned}$$

Based on this, we give our definition for the curl of a three-dimensional vector field:

Definition 22. *The* curl *of a three-dimensional vector field* $\mathbf{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ *is*

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \boxed{M} & \boxed{N} & \boxed{P} \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\end{aligned}$$

Note that this input is a vector field $\mathbf{F}$ in $\mathbb{R}^3$, and the output is another vector field in $\mathbf{R}^3$.

Problem 8 *Let* $\mathbf{F} = (e^y, xz, 3z)$*. Compute the curl* $\nabla \times \mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{(-x, 0, z - e^y)}$$

Note that we have only defined the curl for three-dimensional vector fields. However, by being a bit clever, we can extend this definition to two-dimensional vector fields.

Definition 23. *If the three-dimensional vector field* $\mathbf{F}$ *has the form* $\mathbf{F}(x,y,z) = (M(x,y), N(x,y), 0)$*, then* $\nabla \times \mathbf{F}$ *is often called the* two-dimensional curl *of* $\mathbf{F}$*.*

Moreover, if $\mathbf{G}(x,y) = (M(x,y), N(x,y))$ *is a vector field in* $\mathbb{R}^2$*, then we define the* curl *of* $\mathbf{G}$ *as the curl of the three-dimensional vector field* $\widetilde{\mathbf{G}}(x,y,z) = (M(x,y), N(x,y), 0)$*.*

It turns out, the curl of a two-dimensional vector field can be written in a simpler form.

Proposition 12. *The two-dimensional curl of* $\mathbf{F}(x,y) = (M(x,y), N(x,y), 0)$ *is*

$$\nabla \times \mathbf{F} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Proof From the definition of the curl, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Since the third component, P, of our vector field is identically 0, we have

$$\nabla \times \mathbf{F} = \left(\boxed{0} - \frac{\partial N}{\partial z}, -\left(\boxed{0} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Both $M(x,y)$ and $N(x,y)$ are constant with respect to z, so we then have

$$\nabla \times \mathbf{F} = \left(\boxed{0}, \boxed{0}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right),$$

as desired. ■

Sometimes we refer to $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ as the curl $\nabla \times \mathbf{F}$ if $\mathbf{F}$ is two-dimensional, instead of writing out the entire vector.

Note that we've only defined the curl of a vector field for two- and three-dimensional vector fields. Why doesn't it make sense to define the curl of a four-dimensional (or higher!) vector field?

Multiple Choice:

(a) We only exist in three dimensions.

(b) The cross product is only defined in $\mathbf{R}^3$. ✓

Problem 9 *Given* $\mathbf{F}(x,y) = (y,0)$*, compute the curl* $\nabla \times \mathbf{F}$*.*

$$\nabla \times \mathbf{F} = \boxed{(0,0,-1)}$$

Problem 10 *Given* $\mathbf{F}(x, y) = (-y, 0)$*, compute the curl* $\nabla \times \mathbf{F}$*.*

$$\nabla \times \mathbf{F} = \boxed{(0, 0, 1)}$$

Let's look at this example, $\mathbf{F}(x, y) = (-y, 0)$. It turns out that this is the vector field from the beginning of this section.

PICTURE

We imagined that the center of the plus signs were fixed, and determined that the vector field would rotate the plus signs counterclockwise. We claimed that this local rotation had something to do with the curl of the vector field, which we computed to be $\nabla \times \mathbf{F} = (0, 0, 1)$.

In the next section, we'll study the geometric significance of the curl, and why the curl measures this "microscopic" rotation.

Geometric Significance of Curl

Consider the vector field $\mathbf{F}(x, y) = (-y, 0)$.

We can compute the curl of this vector field,

$$\nabla \times \mathbf{F} = (0, 0, 1)$$

Imagine that we fix a point (representing a particle) in this vector field, but allow it to rotate. If imagine the vector field acting as a force on this particle, which way will it cause the particle to rotate?

(VECTOR FIELD)

Here, we see that the vector field is applying a greater force to the "top" of the particle than to the "bottom." this will cause the particle to rotate counterclockwise. We describe this type of rotation as *local rotation* or *microscopic rotation*, since it's the rotation when we "zoom in" on the particle.

It turns out that the curl of a vector field provides a measure of this local rotation - but how are these connected? We will answer this question in this section, discussing the geometric significance of the curl.

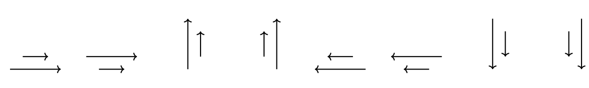
Geometric Significance of Two-dimensional Curl

Recall that, for a two-dimensional vector field $\mathbf{F}(x, y) = (M, N)$, we can compute the curl as

$$\nabla \times \mathbf{F} = (0, 0, N_x - M_y)$$

where N_x is the partial derivative of N with respect to x, and M_y is the partial derivative of M with respect to y. We'll start by considering how M_y and N_x contribute to local rotation.

First let's consider the case where $M_y < 0$. In this case, the x-component of the vector field $\mathbf{F}$ is *decreasing* as we move in the positive y direction. Select all pictures which match this situation.



Learning outcomes: Understand the geometric interpretation of the curl as a measure of local rotation.

Author(s): Melissa Lynn

Select All Correct Answers:

(a) (a) ✓

(b) (b)

(c) (c)

(d) (d)

(e) (e)

(f) (f) ✓

(g) (g)

(h) (h)

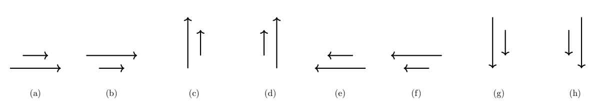
If $M_y < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

Now, let's consider the case where $N_x > 0$. This means that the y-component of the vector field $\mathbf{F}$ is increasing as we move in the positive x direction. Select all pictures which match this situation.



Select All Correct Answers:

(a) (a)

(b) (b)

(c) (c)

(d) (d) ✓

(e) (e)

(f) (f)

(g) (g) ✓

(h) (h)

If $N_x < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

We've seen that the signs of N_x and M_y correspond to the direction of local rotation, with $N_x > 0$ and $M_y < 0$ contributing to counterclockwise rotation.

In general, we have that the sign of $N_x - M_y$ corresponds to the direction of local rotation in the plane. In particular, we have the following correspondences:

$$\begin{aligned} N_x - M_y > 0 &\longleftrightarrow \text{counterclockwise local rotation} \\ N_x - M_y < 0 &\longleftrightarrow \text{clockwise local rotation} \\ N_x - M_y = 0 &\longleftrightarrow \text{no local rotation} \end{aligned}$$

Remembering that $N_x - M_y$ is the *curl* of the two-dimensional vector field $\mathbf{F}$, we now have that the sign of the curl tells us the direction of local rotation for two-dimensional vector fields.

We have a special term for a vector field that never has any local rotation: we call such a vector field *irrotational*.

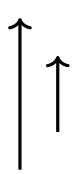
Furthermore, the length of the curl,

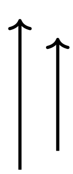
$$\|(0, 0, N_x - M_y)\| = |N_x - M_y|,$$

corresponds to the speed of rotation.

For example, in which case will the particle spin faster?

Multiple Choice:

(a)  ✓

(b) 

Note that this corresponds to a larger value of N_x (the change in the y-component of $\mathbf{F}$ as we move in the positive x direction).

We now apply our knowledge of the geometric significance of the curl in a couple of examples.

Example 44. *Consider the vector field* $\mathbf{F}(x, y) = (-y, x^2)$*. Compute the curl of* $\mathbf{F}$*, and describe the local rotation of the vector field at the points* $(1, 0)$ *and* $(-4, 1)$*.*

Explanation. *We begin by computing the curl of* $\mathbf{F}$*,*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{2x + 1}).$$

At the point $(1, 0)$*, we have* $(\nabla \times \mathbf{F})(1, 0) = (0, 0, \boxed{3})$*. Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(1, 0)$ *is (positive ✓/ negative/ zero). Thus, the local rotation of the vector field at the point* $(1, 0)$ *is (clockwise / counterclockwise ✓/ no rotation).*

At the point $(-4, 1)$*, we have* $(\nabla \times \mathbf{F})(-4, 1) = (0, 0, \boxed{-7})$*. Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(-4, 1)$ *is (positive/ negative ✓/ zero). Thus, the local rotation of the vector field at the point* $(-4, 1)$ *is (clockwise ✓/ counterclockwise/ no rotation).*

Looking at a graph of the vector field, we can see that this local rotation is reflected in the graph.

(ADD GRAPH, WITH ROTATION?)

Example 45. *Let* $\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)$*. Compute the curl* $\nabla \times \mathbf{F}$*, and interpret it geometrically.*

Explanation. *Computing our partial derivatives, we have*

$$N_x = \boxed{\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2}}$$

and

$$M_y = \boxed{\frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2}}.$$

Then, the curl of $\mathbf{F}$ *is*

$$\nabla \times \mathbf{F} = (0, 0, \boxed{0}).$$

Thus, we see that there is no local rotation at any point in the vector field. This is particularly interesting once we look at a graph of the vector field.

(GRAPH)

From the graph of the vector field, there certainly seems to be some larger scale, global rotation of the vector field. However, our computation showed that there *is no local rotation. This example illustrates an important distinction: curl measures* local *rotation of a vector field, which is a different concept from* global *rotation.*

Now that we've found how the curl of a two-dimensional vector field corresponds to local rotation, we'll determine how the curl of a vector field corresponds to local rotation for *three*-dimensional vector fields.

Geometric Significance of Three-dimensional Curl

For a three dimensional vector field $\mathbf{F}(x, y, z) = (M, N, P)$, we can compute the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = (P_y - N_z, \boxed{-P_x + M_z}, \boxed{N_x - M_y}).$$

Here, the situation is more complicated than in two dimensions. In the plane, there are only two possible ways to rotate: clockwise and counterclockwise. In $\mathbb{R}^3$, there are infinitely many different ways to rotate, since we have infinitely many choices of axes. Yikes!

Fortunately, three-dimensional curl still tells about local rotation. In this case, we imagine local rotation as rotation of an infinitesimal (tiny) sphere. This sphere can rotate in infinitely many different ways, depending on which axis we rotate around.

When we look at the components of the curl, this tells us about rotation perpendicular to each of the axes, ignoring rotation in any other direction. Specifically,

$$\begin{aligned}
N_x - M_y \text{ (the } z\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } z\text{-axis)}\\
P_y - N_z \text{ (the } x\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } x\text{-axis)}\\
-P_x + M_z \text{ (the } y\text{-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the } y\text{-axis)}
\end{aligned}$$

Once again, the sign tells us the direction of rotation, with positive sign corresponding to counterclockwise rotation (viewed from the positive axes).

Furthermore, the length of the curl, $\|\nabla \times \mathbf{F}\|$, tells us the speed of rotation, and the direction of $\nabla \times \mathbf{F}$ tells us the axis of rotation.

In $\mathbb{R}^3$, we would like to be able to describe the direction of rotation around a given axis. However, this can be tricky, since it's a matter of perspective. Imagine rotation in the xy-plane. If the rotation is clockwise viewed from above, then it will be counterclockwise from below! Fortunately, curl follows the *right hand rule*:

> If you point your right thumb in the direct of $\nabla \times \mathbf{F}$, then your fingers will curl in the direction of local rotation.

We now put this to use in an example.

Problem 11 *Consider the vector field* $\mathbf{F}(x, y, z) = \left(0, \frac{-z}{(y^2+z^2)^{3/2}}, \frac{y}{(y^2+z^2)^{3/2}}\right)$. *Compute the curl of* $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{\left(\frac{-1}{(y^2+z^2)^{3/2}}, 0, 0\right)}$$

Problem 11.1 *What is the axis of local rotation (at any point)?*

Multiple Choice:

(a) *The x-axis.* ✓

(b) *The y-axis.*

(c) *The z-axis.*

(d) *Some other line.*

Problem 11.1.1 *Viewed from the positive x-axis, what is the direction of local rotation (at any point)?*

Multiple Choice:

(a) *Clockwise.* ✓

(b) *Counterclockwise.*

Problem 11.1.1.1 *How does the speed of local rotation change as we move closer to the origin?*

Multiple Choice:

(a) *Stays the same.*

(b) *Gets slower.*

(c) *Gets faster.* ✓

Connections of Curl with Older Material

We've defined the curl of a two or three dimensional vector field, and we found that this gives a measure of the local rotation of a vector field.

In this section, we discuss connections of the curl to previous topics from the course. In particular, we find the curl of a conservative vector field, and we restate Green's Theorem in terms of curl.

Curl of a Conservative Vector Field

In this section, we prove that the curl of a conservative vector field will always be zero. Thus, conservative vector fields are irrotational.

Theorem 11. *Suppose* $\mathbf{F}$ *is a* C^1 *conservative vector field in* $\mathbb{R}^3$*, so there is a function* $f : \mathbb{R}^3 \to \mathbb{R}$ *such that* $\mathbf{F} = \nabla f$*. Then* $\nabla \times \mathbf{F} = \mathbf{0}$.

Proof Suppose $\mathbf{F}(x, y, z) = (M, N, P)$ is a C^1 conservative vector field, with $\mathbf{F} = \nabla f$. Then we must have

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \boxed{(M, N, P)}.$$

Computing the curl of $\mathbf{F}$, we have

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, -\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right),
\end{aligned}$$

substituting in for the components M, N, and P.

Now, we will use Clairaut's Theorem to simplify this vector. Since $\mathbf{F} = (M, N, P)$ is a C^1 vector field, the partial derivatives of its components ($\frac{\partial M}{\partial y}$, $\frac{\partial M}{\partial z}$, etc.) exist and are continuous. This means that all second-order partial derivatives of f exist and are continuous. Then, by Clairaut's Theorem, the order of differentiation for the second-order mixed partials doesn't matter. In particular, we

Learning outcomes: Understand why the curl of a conservative vector field is zero, and how to state Green's theorem using curl.

Author(s): Melissa Lynn

have

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$
$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}$$
$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

Using this fact in our computation of the curl, we now have

$$\begin{aligned} \nabla \times \mathbf{F} &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, -\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right), \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) \\ &= \boxed{(0,0,0)} \end{aligned}$$

Thus, we have shown that the curl of a conservative vector field is zero. ■

So, conservative vector fields are irrotational. A reasonable follow-up question would be: if the curl of a vector field is zero, is the vector field necessarily conservative? We'll leave this as an open question for the reader, with the suggestion that you think about how you can use past results, and what hypotheses are necessary for this converse to be true.

Curl and Green's Theorem

Now, we find that we've actually already seen the curl of a vector field. It turns out that the curl showed up in Green's Theorem, we just didn't know that it was the curl yet.

Recall the statement of Green's Theorem:

Theorem 12. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x,y) = (M,N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

The integrand of the double integral, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, should now look familiar. This mysterious quantity is actually the two-dimensional curl of the vector field $\mathbf{F}$!

Using this realization, we can now restate Green's Theorem in terms of the curl of $\mathbf{F}$.

Theorem 13. *Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple, closed, piecewise C^1 curves. Orient the boundary ∂D so that D is on the left as one travels along ∂D.*

Let $\mathbf{F}(x, y) = (M, N)$ be a C^1 vector field defined on D. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \nabla \times \mathbf{F} \, dxdy.$$

Now, let's think a bit more about what Green's Theorem is saying here.

The vector line integral, $\oint_C \mathbf{F} \cdot d\mathbf{s}$, computes the global circulation of the vector field around the boundary of the region.

The double integral $\iint_D \nabla \times \mathbf{F} \, dxdy$ is computed by integrating curl over the region D. We can think of this as "adding up" the local rotation of the vector field.

Thus, we can think of Green's Theorem as saying that the global circulation of the vector field around the boundary is equal to the total local rotation across the region. If you think about it, this does make some sense!

Divergence

We've seen how the curl of a vector field measures local rotation, and how we can compute the curl of a vector field as $\nabla \times \vec{F}$.

PICTURE

We'll now look at another local property of a vector field: local expansion or contraction. That is, how can we measure if a vector field is expanding or contracting near a point?

Imagine that we take a tiny ball around a point. If we look at the number and length of vectors entering and leaving this ball, we can decide if there is local expansion or contraction.

In the pictures below, we have some example of local expansion and contraction.

PICTURE

In this section, we'll see how we can compute this local expansion or contraction, similarly to how we computed curl.

Divergence of a vector field

Consider a two-dimensional vector field $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, pictured below. Write $\vec{F}(x, y) = (M(x, y), N(x, y))$.

PICTURE

At the point P, we can see that there is local expansion.

Focusing on the change in the vector field as we move in the positive x-direction near P, we see that the x-coordinates of the vectors are increasing. This means that $\dfrac{\partial M}{\partial x}$ is positive.

Focusing on the change in the vector field as we move in the positive y-direction near P, we can see that the y-coordinates of the vectors are increasing. This means that $\dfrac{\partial N}{\partial y}$ is positive.

When we take the sum $\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$, this gives us a measure of the local expansion of the vector field. Let's look at how we can rewrite this using the operator ∇

Learning outcomes: Understand the definition and geometric intepretation of divergence.
Author(s): Melissa Lynn

and a dot product.

$$\begin{aligned}\frac{\partial M(x,y)}{\partial x} + \frac{\partial N(x,y)}{\partial y} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (M(x,y), N(x,y)) \\ &= \nabla \cdot \vec{F}(x,y).\end{aligned}$$

This leads us to the definition of divergence, for an n-dimensional vector field.

Definition 24. *Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then the* divergence *of $\vec{F}$ is $\nabla \cdot \vec{F}$.*

Example 46. *Consider the vector field $\vec{F}(x,y,z) = (x^2 + yz, y^2 z, x^2 y^3 z^4)$ in $\mathbb{R}^3$. We'll compute the divergence of this vector field at the the point $(1,2,3)$.*

$$\begin{aligned}\nabla \cdot \vec{F}(x,y,z) &= \frac{\partial}{\partial x}(x^2 + yz) + \frac{\partial}{\partial y}(y^2 z) + \frac{\partial}{\partial z}(x^2 y^3 z^4) \\ &= 2x + 2yz + 4x^2 y^3 z^3\end{aligned}$$

Evaluating at the point $(1,2,3)$, we have

$$\begin{aligned}\nabla \cdot \vec{F}(1,2,3) &= 2 \cdot 1 + 2 \cdot 2 \cdot 3 + 4 \cdot 1^2 \cdot 2^3 \cdot 4^3 \\ &= 2062.\end{aligned}$$

Geometric interpretation of divergence

Recall that divergence of a vector field measures the local expansion or contraction of a vector field near a point. With this in mind, for each of the vector fields below, estimate whether the divergence of F at P is positive, negative, or zero.

Example 47. *PICTURE*

Multiple Choice:

(a) *positive* ✓

(b) *negative*

(c) *zero*

Example 48. *PICTURE*

Multiple Choice:

(a) *positive*

(b) *negative*

(c) *zero* ✓

Example 49. *PICTURE*

Multiple Choice:

(a) *positive*

(b) *negative* ✓

(c) *zero*

Further Examples of Divergence

We've defined the divergence of a vector field, and seen how this can be used as a measure of local expansion or contraction.

Definition 25. *Let $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then the* divergence *of $\vec{F}$ is $\nabla \cdot \vec{F}$.*

In this section, we'll explore additional examples of computing divergence, and interpreting our answers geometrically.

Divergence examples

Consider the vector field $\vec{F}(x,y) = \left(\frac{x}{(x^2+y^2)^p}, \frac{y}{(x^2+y^2)^p}\right)$, for various values of p. Graphing this for various values of p gives us a sense of the behavior of the vector field.

PICTURE

Let's compute the divergence, and see what this tells us about the behavior of the vector field.

$$\begin{aligned}
\nabla \cdot \vec{F}(x,y) &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2)^p}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2)^p}\right) \\
&= \frac{(x^2+y^2)^p - 2px(x^2+y^2)^{p-1}x}{(x^2+y^2)^{2p}} + \frac{(x^2+y^2)^p - 2py(x^2+y^2)^{p-1}y}{(x^2+y^2)^{2p}} \\
&= \frac{(x^2+y^2) - 2px^2}{(x^2+y^2)^{p+1}} + \frac{(x^2+y^2) - 2py^2}{(x^2+y^2)^{p+1}} \\
&= \frac{2(x^2+y^2) - 2p(x^2+y^2)}{(x^2+y^2)^{p+1}} \\
&= \frac{2(1-p)}{(x^2+y^2)^p}
\end{aligned}$$

From this, we can see that the sign of the divergence will depend on the value of p. In particular

- If $p < 1$, then the divergence is

Learning outcomes: Explore further examples of computing and understanding the geometry of divergence.
Author(s): Melissa Lynn

Multiple Choice:

(a) positive. ✓
(b) negative.
(c) zero.

- If $p > 1$, then the divergence is

Multiple Choice:

(a) positive.
(b) negative. ✓
(c) zero.

- If $p = 1$, then the divergence is

Multiple Choice:

(a) positive.
(b) negative.
(c) zero. ✓

Let's look at how this manifests itself in the graph of $\vec{F}$, with the local expansion or contraction of the vector field.

For $p = \frac{1}{2}$, we graph the vector field $\vec{F}(x, y) = \left(\frac{x}{(x^2 + y^2)^{1/2}}, \frac{y}{(x^2 + y^2)^{1/2}} \right)$.

PICTURE

Here, if we look at a tiny ball around a point, we see that there are more vectors exiting the ball than coming into it, and the vectors exiting the ball are longer, so we have local expansion. This matches with the divergence being positive.

For $p = 1$, we graph the vector field $\vec{F}(x, y) = \left(\frac{x}{(x^2 + y^2)}, \frac{y}{(x^2 + y^2)} \right)$.

PICTURE

Here, if we look at a tiny ball around a point, we see that there are more vectors exiting the ball than coming into it. However, the vectors exiting the ball are shorter than the vectors entering the ball. Because of this, it's difficult to estimate the sign of the divergence from the graph. As it turns out, there is perfect cancellation, so the divergence of the vector field is zero.

For $p = 2$, we graph the vector field $\vec{F}(x, y) = \left(\frac{x}{(x^2 + y^2)^2}, \frac{y}{(x^2 + y^2)^2} \right)$.

PICTURE

Here, if we look at a tiny ball around a point, we see that there are more vectors exiting the ball than coming into it. However, the vectors exiting the ball are shorter than the vectors entering the ball. Because of this, it's difficult to estimate the sign of the divergence from the graph. In this case, the longer vectors entering the ball more than cancel out the shorter vectors exiting the ball, and the divergence of the vector field turns out to be negative.

Properties of Divergence

We've defined the divergence of a vector field, and seen how this can be used as a measure of local expansion or contraction.

Definition 26. *Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then the* divergence *of $\vec{F}$ is $\nabla \cdot \vec{F}$.*

In this section, we'll explore some properties of divergence, and how it relates to topics we covered previously.

Divergence and curl

Consider a three-dimensional $\mathcal{C}^2$ vector field $\vec{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$. We can compute the curl of $\vec{F}$ as

$$\nabla \times \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right).$$

Notice that we can view $\nabla \times \vec{F}$ as a function from $\mathbb{R}^3$ to $\mathbb{R}^3$. So, we can view it as a vector field, and compute its divergence. This gives us

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{F}) &= \nabla \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\
&= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y}.
\end{aligned}$$

Now, since $\vec{F}$ is $\mathcal{C}^2$, Clairaut's theorem tells us that the order of the mixed partials doesn't matter. That is,

$$\begin{aligned}
\frac{\partial^2 M}{\partial y \partial z} &= \frac{\partial^2 M}{\partial z \partial y}, \\
\frac{\partial^2 N}{\partial x \partial z} &= \frac{\partial^2 N}{\partial z \partial x}, \\
\frac{\partial^2 P}{\partial x \partial y} &= \frac{\partial^2 P}{\partial y \partial x}.
\end{aligned}$$

Learning outcomes: Understand the relationship between divergence and curl, and the divergence theorem in the plane.

Author(s): Melissa Lynn

So, all of the terms above cancel, and we're left with

$$\nabla \cdot (\nabla \times \vec{F}) = 0.$$

Using a similar argument, the same result holds for two-dimensional vector fields.

Proposition 13. *Let $\vec{F}$ be an n-dimensional $\mathcal{C}^2$ vector field, for $n = 2$ or $n = 3$. Then*

$$\nabla \cdot (\nabla \times \vec{F}) = 0.$$

Divergence theorem in the plane

Consider a region D, satisfying the hypotheses of Green's theorem. That is, let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves.

PICTURE

Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D. Since $\nabla \cdot \vec{F}$ is a scalar valued function, we can integrate it over D. Let's think about what the integral

$$\iint_D \nabla \cdot \vec{F}\, dA$$

represents. Since the divergence $\nabla \cdot \vec{F}$ gives the microscopic expansion or contraction of the vector field, the double integral $\iint_D \nabla \cdot \vec{F}\, dA$ represents the total expansion or contraction across the region D. Another way to say this is that $\iint_D \nabla \cdot \vec{F}\, dA$ gives the total net flow across the region D.

PICTURE

Now, let's try to relate this total net flow to something happening on the boundary, analogously to Green's theorem.

Suppose we approximate the region D with a bunch of small rectangles. When we look at the flow across one of the interior rectangle edges, any flow out of one rectangle will flow into another rectangle. So all of the flow happening inside of the region will cancel out in the double integral, and the only flow that matters happens across the edge.

How can we measure the flow across the edge? If $\vec{n}$ is the unit normal vector pointing outside of the boundary curve, then we can measure the flow through a boundary point with $\vec{F} \cdot \vec{n}$.

PICTURE

This works because if $\vec{F}$ and $\vec{n}$ are in the same direction, there is positive flow out of the region. If $\vec{F}$ and $\vec{n}$ are perpendicular, there is zero flow across the boundary at that point.

So, we can find the total flow across the boundary by computing the scalar line integral

$$\oint_{\partial D} \vec{F} \cdot \vec{n} ds.$$

The above argument provides a sketch of a proof for the following theorem, sometimes called the divergence theorem in the plane.

Theorem 14. *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F} : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ vector field, and suppose that $\vec{n}$ is the outward pointing normal vector to the boundary of D. Then*

$$\oint_{\partial D} \vec{F} \cdot \vec{n} \, ds = \iint_D \nabla \cdot \vec{F} \, dA.$$

This theorem has applications similar to Green's theorem; we can use it to more easily compute some integrals.

Part IV

Surface Integrals

Surface Area

Suppose we have a parametrized surface, $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, and suppose we wish to find the surface area of this surface.

PICTURE

Recall that by fixing one parameter, we get a coordinate curve. Doing this for several values for each parameter, we obtain a "grid" on the surface.

PICTURE

This grid consists of many small regions, which we'll describe as curvy rectangles. If we can approximate the area of the curvy rectangles, and add all of these areas together, we'll approximate the surface area. This provides us with the idea for our surface area computation.

To implement this, we need to begin by approximating the area of a small curvy rectangle. We can think of the curvy rectangle as the image of a rectangle in (s,t)-coordinates, under a transformation $\vec{X}$.

PICTURE

In (s,t)-coordinates, suppose that the rectangle has base Δs and height Δt, so the area is $\Delta s \Delta t$.

We can approximate the sides of the curvy rectangle with the vectors $\vec{X}_s(s_0,t_0)\Delta s$ and $\vec{X}_t(s_0,t_0)\Delta t$.

PICTURE

This is a reasonable approximation because these vectors are tangent to the grid curves, and are scaled to the appropriate length by the rectangle in (s,t)-coordinates.

Then, we can approximate the curvy rectangle with the rectangle determined by the vectors $\vec{X}_s(s_0,t_0)\Delta s$ and $\vec{X}_t(s_0,t_0)\Delta t$.

PICTURE

From linear algebra, we know that this rectangle has area

$$\left\| \vec{X}_s(s_0,t_0)\Delta s \times \vec{X}_t(s_0,t_0)\Delta t \right\|,$$

Learning outcomes: Compute the surface area of a parametric surface.
Author(s): Melissa Lynn

and we can rewrite this as

$$\left\|\vec{X}_s(s_0,t_0) \times \vec{X}_t(s_0,t_0)\right\| \Delta s \Delta t.$$

Now, if we add up the areas of all of these approximations, and let $\Delta s \to 0$ and $\Delta t \to 0$ (so that the rectangles get very small), we obtain the double integral,

$$\iint_D \|\vec{X}_s \times \vec{X}_t\| \, dsdt.$$

This provides us with a formula to compute the surface area of a parametrized surface.

Proposition 14. *Let $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametric surface in $\mathbb{R}^3$. Then the surface area of $\vec{X}$ is given by*

$$\mathit{area}(\vec{X}) = \iint_D \|\vec{X}_s \times \vec{X}_t\| \, dsdt.$$

Surface area computations

We'll now use our formula for the surface area of a parametric surface to confirm several well-known surface area formulas from geometry.

Example 50. *We'll find the surface area of a sphere of radius R, which can be parametrized by*

$$\vec{X}(s,t) = (R\cos s \sin t, R \sin s \sin t, R\cos t),$$

for $0 \le t \le \pi$ and $0 \le s \le 2\pi$. First, let's find the derivatives $\vec{X}_s$ and $\vec{X}_t$.

$$\begin{aligned} \vec{X}_s(s,t) &= \left(\frac{\partial}{\partial s} R\cos s \sin t, \frac{\partial}{\partial s} R\sin s \sin t, \frac{\partial}{\partial s}\cos t\right) \\ &= (-R\sin s \sin t, R\cos s \sin t, 0)\vec{X}_t(s,y) \qquad = \left(\frac{\partial}{\partial t} R\cos s \sin t, \frac{\partial}{\partial t} R\sin s \sin t, \frac{\partial}{\partial t} R\cos t\right) \\ &= (R\cos s\cos t, R\sin s \cos t, -R\sin t) \end{aligned}$$

Next, we'll find the cross product, $\vec{X}_s \times \vec{X}_t$.

$$\begin{aligned} \vec{X}_s(s,t) \times \vec{X}_t(s,t) &= (-R\sin s \sin t, R\cos s \sin t, 0) \times (R\cos s \cos t, R \sin s \cos t, -R\sin t) \\ &= \mathit{det}\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin s \sin t & R\cos s \sin t & 0 \\ R\cos s\cos t & R\sin s \cos t & -R\sin t \end{pmatrix} \\ &= (R\cos s \sin t)(-R\sin t)\vec{i} - (-R\sin s \sin t)(-R\sin t)\vec{j} + (-R\sin s\sin t)(R\sin s\cos t)\vec{k} - (R\,c \\ &= (-R^2\cos s \sin^2 t, -R^2 \sin s \sin^2 t, -R^2\sin^2 s\cos t \sin t - R^2\cos^2 s \cos t\sin t) \\ &= (-R^2\cos s\sin^2 t, -R^2\sin s \sin^2 t, -R^2\cos t \sin t) \end{aligned}$$

The length of this vector is

$$\begin{aligned}
\|\vec{X}_s(s,t) \times \vec{X}_t(s,t)\| &= \left\|(-R^2\cos s\sin^2 t, -R^2\sin s\sin^2 t, -R^2\cos t\sin t)\right\| \\
&= \sqrt{R^4\cos^2 s\sin^4 t, R^4\sin^2 s\sin^4 t, R^4\cos^2 t\sin^2 t} \\
&= \sqrt{R^4\sin^4 t + R^4\cos^2 t\sin^2 t} \\
&= \sqrt{R^4\sin^2 t} \\
&= R^2|\sin t|
\end{aligned}$$

Since $0 \le t \le \pi$, *this is* $R^2\sin t$.

Then the surface area is

$$\begin{aligned}
\mathit{area} &= \int_0^\pi \int_0^{2\pi} \|\vec{X}_s(s,t) \times \vec{X}_t(s,t)\|\, dsdt \\
&= \int_0^\pi \int_0^{2\pi} R^2\sin t\, ds\, dt \\
&= \int_0^\pi 2\pi R^2 \sin t\, dt \\
&= 2\pi R^2(-\cos t)_{t=0}^{t=\pi} \\
&= 2\pi R^2(-(-1)-(-1)) \\
&= 4\pi R^2.
\end{aligned}$$

Thus, we have shown that the surface area of a sphere of radius R *is* $4\pi R^2$.

Example 51. *We'll compute the surface area of a right circular cone of radius* R *and height* H.

PICTURE

In order to do this, we'll need to split the surface into two pieces: the top disc, and the rest of the cone.

PICTURE

We'll start by finding the surface area of the top disc. We can parametrize the disc as

$$\vec{X}(s,t) = (s\cos t, s\sin t, H),$$

for $0 \le s \le R$ *and* $0 \le t \le 2\pi$. *The partial derivatives are then*

$$\begin{aligned}
\vec{X}_s(s,t) &= (\cos t, \sin t, 0), \\
\vec{X}_t(s,t) &= (-s\sin t, s\cos t, 0).
\end{aligned}$$

Taking the cross product, we have

$$\begin{aligned}\vec{X}_s(s,t)\times\vec{X}_t(s,t) &= det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos t & \sin t & 0\\ -s\sin t & s\cos t & 0\end{pmatrix}\\ &= (s\cos^2 t + s\sin^2 t)\vec{k}\\ &= s\vec{k}\end{aligned}$$

Since s is nonnegative, the length of this vector is

$$\|\vec{X}_s(s,t)\times\vec{X}_t(s,t)\| = s.$$

Now, we can find the surface area of the disc.

$$\begin{aligned}area &= \int_0^{2\pi}\int_0^R s\ dsdt\\ &= \int_0^{2\pi}\left(\frac{s^2}{2}\right)_{s=0}^{s=R} dt\\ &= \int_0^{2\pi}\frac{R^2}{2}\ dt\\ &= \pi R^2\end{aligned}$$

Unsurprisingly, this is the area of a circle of radius R.

Next, we'll find the surface area of the remaining portion of the cone. This can be parametrized as

$$\vec{Y}(s,t) = \left(s\cos t, s\sin t, \frac{H}{R}s\right),$$

for $0 \le s \le R$ and $0 \le t \le 2\pi$.

The partial derivatives are

$$\begin{aligned}\vec{Y}_s(s,t) &= (\cos t, \sin t, H/R),\\ \vec{Y}_t(s,t) &= (-s\sin t, s\cos t, 0).\end{aligned}$$

Taking the cross product of these vectors, we have

$$\begin{aligned}\vec{Y}_s(s,t)\times\vec{Y}_t(s,t) &= det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos t & \sin t & H/R\\ -s\sin t & s\cos t & 0\end{pmatrix}\\ &= (-(H/R)s\cos t)\vec{i} + (-(H/R)s\sin t)\vec{j} + (s\cos^2 t + s\sin^2 t)\vec{k}\\ &= (-(H/R)s\cos t)\vec{i} + (-(H/R)s\sin t)\vec{j} + s\vec{k}.\end{aligned}$$

The length of this vector is

$$\begin{aligned}\|\vec{Y}_s(s,t)\times\vec{Y}_t(s,t)\| &= \sqrt{(-(H/R)s\cos t)^2 + (-(H/R)s\sin t)^2 + s^2}\\ &= \sqrt{H^2/R^2\cdot s^2 + s^2}\\ &= s\sqrt{H^2/R^2+1},\end{aligned}$$

since s is nonnegative.

Now, we can compute the surface area.

$$\begin{aligned}
area &= \int_0^{2\pi} \int_0^R s\sqrt{H^2/R^2+1}\, dsdt \\
&= \sqrt{H^2/R^2+1} \int_0^{2\pi} \left(\frac{s^2}{2}\right)_{s=0}^{s=R} dt \\
&= \sqrt{H^2/R^2+1} \int_0^{2\pi} \frac{R^2}{2}\, dt \\
&= \sqrt{H^2/R^2+1} \cdot \pi R^2
\end{aligned}$$

So, the total surface area of the cone is

$$\pi R^2 + \sqrt{H^2/R^2+1} \cdot \pi R^2 = \pi R\left(R + \sqrt{H^2+R^2}\right).$$

Scalar Surface Integrals

We've seen how we can compute the surface area of a parametrized surface using a double integral.

Proposition 15. *Let $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametric surface in $\mathbb{R}^3$. Then the surface area of $\vec{X}$ is given by*

$$area(\vec{X}) = \iint_D \|\vec{X}_s \times \vec{X}_t\| \, dsdt.$$

We found this by looking at how grid curves divided the surface up into tiny curved rectangles, and then approximating the areas of these rectangles, and adding them up.

PICTURE

Now, suppose we have some thin surface, and we want to find its mass. If the density of the surface is constant, this is simple - we multiply the density by the surface area, and we're done. For example, if we have a hollow sphere of radius 1 cm and density 3 kg/cm^2, then the sphere has surface area $\frac{4}{3}\pi$ cm^2, and mass

$$3 \cdot \frac{4}{3}\pi = 4\pi \text{ kg}.$$

But what if the density of the sphere isn't constant, and instead varies across the surface? That is, if the density is given by some function $\rho : S \to \mathbb{R}$ defined on the sphere S, how can we find the total mass across the entire surface?

More generally, how do we integrate a scalar-valued function over a surface? This brings us to scalar surface integrals.

Scalar surface integrals

Let's think back to how we found the surface area of a parametrized surface, $\vec{X} : D \to \mathbb{R}^3$. We did this by approximating the area of curvy rectangles along the surface, with $\|\vec{X}_s \times \vec{X}_t\|$.

PICTURE

Now, suppose the density across this curvy rectangle is a constant, ρ. Then the mass of the curvy rectangle is $\rho\|\vec{X}_s \times \vec{X}_t\|$. Even if the density is not constant,

Learning outcomes: Compute and understand the geometric meaning of scalar surface integrals.

Author(s): Melissa Lynn

but is given by some function $\rho(\vec{X})$ that takes points on the surface and returns the density, we can still approximate the mass using the density at a sample point, $\vec{X}^*$, in the curvy rectangle.

PICTURE

That is, the mass of the curvy rectangle will be approximately $\rho(\vec{X}^*)\rho\|\vec{X}_s \times \vec{X}_t\|$. When we add up the masses of curvy rectangles across the entire surface, and take the limit as the rectangles get arbitrarily small, we obtain a double integral,

$$\iint_S \rho(\vec{X}(s,t)\|\vec{X}_s \times \vec{X}_t\| \; dsdt.$$

Thus, we arrive at the definition of a scalar surface integral.

Definition 27. *Let $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization for a surface S in $\mathbb{R}^3$. Let $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous function which is defined on the surface S. We define the* surface integral *of f over $\vec{X}$ to be*

$$\iint_{\vec{X}} f \; dS = \iint_D f(\vec{X}(s,t))\|\vec{X}_s \times \vec{X}_t\| \; dsdt.$$

Scalar surface integral computations

Let's look at some examples of computing scalar surface integrals.

Example 52. *Let's integrate the function $f(x,y,z) = z^2$ over the sphere of radius 2 centered at the origin, which can be parametrized as*

$$\vec{X}(s,t) = (2\cos s \sin t, 2\sin s \sin t, 2\cos t),$$

for $0 \leq s \leq 2\pi$ and $0 \leq t \leq \pi$.

First, let's find the derivatives $\vec{X}_s$ and $\vec{X}_t$.

$$\vec{X}_s(s,t) = (-2\sin s \sin t, 2\cos s \sin t, 0)\vec{X}_t(s,y) \quad = (2\cos s \cos t, 2\sin s \cos t, -2\sin t)$$

Next, we'll find the cross product, $\vec{X}_s \times \vec{X}_t$.

$$\begin{aligned}
\vec{X}_s(s,t) \times \vec{X}_t(s,t) &= (-2\sin s \sin t, 2\cos s \sin t, 0) \times (2\cos s \cos t, 2\sin s \cos t, -2\sin t) \\
&= det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin s \sin t & 2\cos s \sin t & 0 \\ 2\cos s \cos t & 2\sin s \cos t & -2\sin t \end{pmatrix} \\
&= (2\cos s \sin t)(-2\sin t)\vec{i} - (-2\sin s \sin t)(-2\sin t)\vec{j} + (-2\sin s \sin t)(2\sin s \cos t)\vec{k} - (2\cos s \\
&= (-4\cos s \sin^2 t, -4\sin s \sin^2 t, -4\sin^2 s \cos t \sin t - 4\cos^2 s \cos t \sin t) \\
&= (-4\cos s \sin^2 t, -4\sin s \sin^2 t, -4\cos t \sin t)
\end{aligned}$$

The length of this vector is

$$\begin{aligned}
\|\vec{X}_s(s,t) \times \vec{X}_t(s,t)\| &= \left\|(-4\cos s \sin^2 t, -4\sin s \sin^2 t, -4\cos t \sin t)\right\| \\
&= \sqrt{16\cos^2 s \sin^4 t, 16\sin^2 s \sin^4 t, 16\cos^2 t \sin^2 t} \\
&= \sqrt{16\sin^4 t + 16\cos^2 t \sin^2 t} \\
&= \sqrt{16\sin^2 t} \\
&= 4|\sin t|
\end{aligned}$$

Since $0 \le t \le \pi$*, this is* $4\sin t$*.*

Now, we evaluate the scalar surface integral.

$$\begin{aligned}
\iint_{\vec{X}} f\ dS &= \iint_D f(\vec{X}(s,t))\|\vec{X}_s \times \vec{X}_t\|\ dsdt \\
&= \int_0^{2\pi}\int_0^{\pi} f(2\cos s \sin t, 2\sin s \sin t, 2\cos t) \cdot 4\sin t\ dtds \\
&= \int_0^{2\pi}\int_0^{\pi} 4\cos^2 t) \cdot 4\sin t\ dtds \\
&= \int_0^{2\pi} \left(-\frac{16}{3}\cos^3 t\right)_{t=0}^{t=\pi} ds \\
&= \int_0^{2\pi} (16/3 + 16/3)\ ds \\
&= \frac{64}{3}\pi
\end{aligned}$$

So, we have $\displaystyle\iint_{\vec{X}} f\ dS = \frac{64}{3}\pi$*.*

Example 53. *Suppose the graph of the function* $g(x,y) = xy$ *for* $0 \le x \le 2$ *and* $0 \le y \le 2$ *represents a small (very) snowy region, and the depth of the snow at a point* (x,y,xy) *is given by* $4xy$*, where* x *and* y *are in meters. Suppose also that we wish to find the total amount of snow over the region.*

First, we need to figure out how to set this problem up using a scalar surface integral. Our surface is the graph of the function $g(x,y) = xy$*, and we can parametrize this surface as*

$$\vec{X}(s,t) = (s,t,st)$$

for $0 \le s \le 2$ *and* $0 \le t \le 2$*. If we define the function* $f(x,y,z) = 4xy$*, then at a point* (s,t,st) *on the surface, we have* $f(\vec{X}(s,t)) = 4st$*. So, with this set up, we wish to evaluate the integral*

$$\iint \vec{X} f\ dS.$$

First, we find the partial derivatives $\vec{X}_s$ and $\vec{X}_t$.

$$\vec{X}_s(s,t) = (1, 0, t)$$
$$\vec{X}_t(s,t) = (0, 1, s)$$

We take the cross product of these vectors, to obtain

$$\begin{aligned}\vec{X}_s(s,t) \times \vec{X}_t(s,t) &= det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & t \\ 0 & 1 & s \end{pmatrix} \\ &= (-t)\vec{i} + (-s)\vec{j} + \vec{k}.\end{aligned}$$

The length of this vector is

$$\|\vec{X}_s(s,t) \times \vec{X}_t(s,t)\| = \sqrt{t^2 + s^2 + 1}.$$

Now, we evaluate our scalar surface integral.

$$\begin{aligned}\iint \vec{X} f \; dS &= \int_0^2 \int_0^2 f(s,t,st)\sqrt{t^2 + s^2 + 1} \; dtds \\ &= \int_0^2 \int_0^2 4st\sqrt{t^2 + s^2 + 1} \; dtds \\ &= \int_0^2 \left(2s\sqrt{t^2 + s^2 + 1}^3 \right)_{t=0}^{t=2} \; ds \\ &= \int_0^2 \frac{4}{3} s\sqrt{s^2 + 5}^3 \; ds \\ &= \left(\frac{4}{15}\sqrt{s^2 + 5}^5 \right)_{s=0}^{s=2} \\ &= \frac{324}{5}\end{aligned}$$

So, the total volume of snow is $324/5 \text{ m}^3$.

Orientatation of a Surface

Suppose you have water flowing through some filter, and you'd like a measure of how much water is going through the filter. If we represent the water flow with a vector field and the filter with a surface in $\mathbb{R}^3$, we can conceptualize this question as measuring the flow of the vector field through a surface.

PICTURE

Now, depending on your perspective, you might view the flow as either positive flow or negative flow. In the picture above, if you view the left of the surface as "inside" and the right of the surface as "outside," the water is leaving, so we'd view this as negative flow.

PICTURE

However, if you view the left of the surface as "outside" and the right of the surface as "inside," the water is entering, so we'd view this as positive flow.

PICTURE

In order to define integrals to compute the flow of a vector field through a surface, we first need to define mathematically what it means to choose an "inside" and an "outside" for a surface. This brings us to orientation.

Orientation

Mathematically, a choice of orientation for a surface means a consistent choice of normal vector across the entire surface. We can think of the normal vector as giving the direction of positive flow through the surface.

Definition 28. *Let $S \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$. An* orientation *on S is a continuous function $\vec{n} : S \to \mathbb{R}^3$ such that $\vec{n}(\vec{x})$ is a unit vector and is perpendicular S at the point $\vec{x}$.*

We say that the surface S is orientable *if such a function $\vec{n}$ exists. We say that the surface S is* oriented *if such a function $\vec{n}$ has been chosen.*

Let's look at some examples of orientations.

Example 54. *Consider the plane S defined by $x = 0$ in $\mathbb{R}^3$.*

PICTURE

Learning outcomes: Understand how to choose an orientation of a surface, and that some surfaces are not orientable.

Author(s): Melissa Lynn

There are only two possible orientations for this plane, given by

$$\vec{n}(\vec{x}) = (1, 0, 0)$$

for all $\vec{x}$ in S, and by

$$-\vec{n}(\vec{x}) = (-1, 0, 0)$$

for all $\vec{x}$ in S.

PICTURE

Because there are only two choices for orientation, we can indicate the chosen orientation with just a single vector.

PICTURE

Example 55. *Consider the unit sphere S defined by $x^2 + y^2 + z^2$ in $\mathbb{R}^3$.*

PICTURE

There are only two possible orientations for this sphere, given by

$$\vec{n}(\vec{x}) = \frac{\vec{x}}{\|\vec{x}\|}$$

for all $\vec{x}$ in $\vec{S}$, and by

$$-\vec{n}(\vec{x}) = \frac{-\vec{x}}{\|\vec{x}\|}$$

for all $\vec{x}$ in $\vec{S}$.

PICTURE

Because there are only two choices for orientation, we can indicate the chosen orientation with just a single vector.

Based on the examples above, you might conjecture that every surface has exactly two orientations. This is almost true: any orientable, connected surface has exactly two orientations. However, there are some surfaces which are not orientable, as we will now see.

Non-orientable surfaces

For some surfaces S, it's impossible to define a continuous function $\vec{n} : S \to \mathbb{R}^3$ such that $\vec{n}(\vec{x})$ is a unit vector and is perpendicular S at the point $\vec{x}$. We say that these surfaces are non-orientable.

Our first example of a non-orientable surface is a möbius strip.

Example 56. *A möbius strip is a looped surface with a single twist, as pictured below.*

PICTURE

Let's see what happens when we try to chose an orientation on the möbius strip. If we choose a normal vector at a point, in order for $\vec{n}$ to be continuous, this forces the choice of normal vector at nearby points.

PICTURE

Continuing this around the surface, we have a problem: we eventually get two normal vectors at the same point, pointing in opposite directions.

PICTURE

This is because it is impossible to choose unit normal vectors continuously on the möbius strip. Thus, the möbius strip is non-orientable.

You might also hear this described as "the möbius strip only has one side."

Example 57. *Another, trickier example of a non-orientable surface is the Klein bottle. The Klein bottle actually exists in four dimensions, and can't be embedded in three dimensions without intersecting itself. This makes it difficult to visualize! Below, we have a three-dimensional representation of the Klein bottle. When the Klein bottle is embedded in $\mathbb{R}^4$, the highlighted self-intersection does not occur.*

PICTURE

The Klein bottle has the same issue as the möbius strip. If we try to continuously choose a normal vector across the entire surface, we contradict ourselves, because the Klein bottle only has one side.

Fortunately, we will focus on orientable surfaces.

Finding an orientation

Now that we've defined an orientation for a surface, we will determine how to find an orientation in practice. This means finding a function $\vec{n} : S \to \mathbb{R}^3$ such that $\vec{n}(\vec{x})$ is a unit vector and is perpendicular S at the point $\vec{x}$. The function $\vec{n}$ gives a unit normal vector at each point on the surface.

Suppose we have a surface S parametrized by $\vec{X}(s,t)$ in $\mathbb{R}^3$. If we find the tangent vectors $\vec{X}_s$ and $\vec{X}_t$ to the grid curves, these vectors will be parallel to the surface S. Then, if we take the cross product of these vectors, we obtain a vector $\vec{X}_s \times \vec{X}_t$ which is perpendicular to the surface S.

PICTURE

Now, it could be that $\vec{X}_s \times \vec{X}_t$ is the zero vector. Let's assume that our parametrization is such that $\vec{X}_s \times \vec{X}_t$ is never zero, and is a continuous function of s and t. Then, we can normalize $\vec{X}_s \times \vec{X}_t$ by dividing by its length. This gives us a unit vector which is perpendicular the surface. So, we can define

$$\vec{n}(s,t) = \frac{\vec{X}_s \times \vec{X}_t}{\|\vec{X}_s \times \vec{X}_t\|},$$

and this gives us an orientation on S.

Let's use this idea to show that a surface is orientable.

Example 58. *Consider the paraboloid parametrized by* $vecX(s,t) = (s, t, s^2 + t^2)$ *for* $0 \leq s, t \leq 1$.

PICTURE

We will show that this surface is orientable, by finding an orientation.

First, let's find the partial derivatives $\vec{X}_s$ *and* $\vec{X}_t$.

$$\vec{X}_s(s,t) = (1, 0, 2s)$$
$$\vec{X}_t(s,t) = (0, 1, 2t)$$

Next, we find the cross product $\vec{X}_s \times \vec{X}_t$.

$$\begin{aligned}
\vec{X}_s(s,t) \times \vec{X}_t(s,t) &= det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2s \\ 0 & 1 & 2t \end{pmatrix} \\
&= (-2s)\vec{i} + (-2t)\vec{j} + \vec{k} \\
&= (-2s, -2t, 1)
\end{aligned}$$

Notice that this is never the zero vector, no matter the values of s *and* t. *So, we can normalize* $\vec{X}_s \times \vec{X}_t$, *to obtain a unit vector. This gives us our orientation, giving a normal vector to the surface at all points.*

$$\begin{aligned}
\vec{n}(s,t) &= \frac{\vec{X}_s \times \vec{X}_t}{\|\vec{X}_s \times \vec{X}_t\|} \\
&= \frac{(-2s, -2t, 1)}{\sqrt{4s^2 + 4t^2 + 1}} \\
&= \left(\frac{-2s}{\sqrt{4s^2 + 4t^2 + 1}}, \frac{-2t}{\sqrt{4s^2 + 4t^2 + 1}}, \frac{1}{\sqrt{4s^2 + 4t^2 + 1}} \right)
\end{aligned}$$

Vector Surface Integrals

Suppose you have water flowing through some filter, and you'd like a measure of how much water is going through the filter. If we represent the water flow with a vector field and the filter with a surface in $\mathbb{R}^3$, we can conceptualize this question as measuring the flow of the vector field through a surface.

PICTURE

We will find a way to integrate a vector field over a surface, so that the integral will represent the flow of the vector field through the surface.

We've discussed how the orientation of the surface will affect whether we view the flow as positive or negative. An orientation on a surface is a continuous choice of unit normal vector on the surface, and the normal vector indicates the direction of positive flow.

PICTURE

As we begin to construct our definition of vector surface integrals, let's start with the special case where our surface is the plane $x = 0$ in $\mathbb{R}^3$, and our vector field is constant, $\vec{F} = \vec{c}$.

If $\vec{c}$ is parallel to the plane $x = 0$, there is no flow through the surface, so it makes sense that the vector line integral should be zero.

PICTURE

We can also see that the flow through the surface will be largest when $\vec{c}$ is perpendicular to the plane.

PICTURE

So, the flow is zero when the vector field is parallel to the surface, and the flow is largest when the vector field is perpendicular to the surface. This indicates that we should be comparing the vector field $\vec{F}$ to a normal vector $\vec{n}$ to the surface, and considering the dot product $\vec{F} \cdot \vec{n}$.

This brings us to the definition of vector surface integrals.

Vector surface integrals

Let $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface S in $\mathbb{R}^3$, so that $\vec{X}_s \times \vec{X}_t$ is continuous and nonzero. Let $\vec{F}$ be a vector field defined on S. We would like

Learning outcomes: Compute and understand the geometric meaning of vector surface integrals.

Author(s): Melissa Lynn

to find the flow of $\vec{F}$ through the surface S, and we have seen that this can be done by integrating $\vec{F} \cdot \vec{n}$ over S, where $\vec{n} = \dfrac{\vec{X}_s \times \vec{X}_t}{\|\vec{X}_s \times \vec{X}_t\|}$ is the normal vector giving the orientation on $\vec{X}$. That is, we have the integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS.$$

Evaluating a vector surface integral as a scalar surface integral isn't ideal; let's take a look at this integral to see if we can simplify things. By the definition of scalar surface integrals, we have

$$\begin{aligned}
\iint_{\vec{X}} \vec{F} \cdot d\vec{S} &= \iint_S \vec{F} \cdot \vec{n}\, dS \\
&= \iint_D (\vec{F}(\vec{X}(s,t)) \cdot \vec{n}(s,t)) \|\vec{X}_s \times \vec{X}_t\|\, dsdt \\
&= \iint_D \left(\vec{F}(\vec{X}(s,t)) \cdot \frac{\vec{X}_s \times \vec{X}_t}{\|\vec{X}_s \times \vec{X}_t\|} \right) \|\vec{X}_s \times \vec{X}_t\|\, dsdt \\
&= \iint_D \vec{F}(\vec{X}(s,t)) \cdot (\vec{X}_s \times \vec{X}_t)\, dsdt.
\end{aligned}$$

We make this our definition of vector surface integrals.

Definition 29. *Let $\vec{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of a surface S in $\mathbb{R}^3$, so that $\vec{X}_s \times \vec{X}_t$ is continuous and nonzero. Let $\vec{F}$ be a vector field defined on S. The* vector line integral *of $\vec{F}$ over $\vec{X}$ is*

$$\iint_{\vec{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{X}(s,t)) \cdot (\vec{X}_s \times \vec{X}_t)\, dsdt.$$

Computing vector surface integrals

We'll now look at an example of computing vector surface integrals.

Example 59. *Let's evaluate the vector surface integral of $\vec{F}(x,y,z) = (yz, xz, xy)$ over the paraboloid parametrized by*

$$\vec{X}(s,t) = (s, t, s^2 + t^2),$$

for $0 \le s, t \le 1$.

PICTURE

First, let's find the partial derivatives $\vec{X}_s$ and $\vec{X}_t$.

$$\begin{aligned}
\vec{X}_s(s,t) &= (1, 0, 2s) \\
\vec{X}_t(s,t) &= (0, 1, 2t)
\end{aligned}$$

Next, we find the cross product $\vec{X}_s \times \vec{X}_t$.

$$\begin{aligned}
\vec{X}_s(s,t) \times \vec{X}_t(s,t) &= det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2s \\ 0 & 1 & 2t \end{pmatrix} \\
&= (-2s)\vec{i} + (-2t)\vec{j} + \vec{k} \\
&= (-2s, -2t, 1)
\end{aligned}$$

Now, we evaluate the vector surface integral,

$$\begin{aligned}
\iint_{\vec{X}} \vec{F} \cdot d\vec{S} &= \iint_D \vec{F}(\vec{X}(s,t)) \cdot (\vec{X}_s \times \vec{X}_t)\ dsdt \\
&= \int_0^1 \int_0^1 \vec{F}(s,t,s^2+t^2) \cdot (-2s,-2t,1)\ dsdt \\
&= \int_0^1 \int_0^1 (s^2t+t^3, s^3+st^2, st) \cdot (-2s,-2t,1)\ dsdt \\
&= \int_0^1 \int_0^1 -2s^3t - 2st^3 + -2s^3t - 2st^3 + st\ dsdt \\
&= \int_0^1 \int_0^1 -4s^3t - 4st^3 + st\ dsdt \\
&= \int_0^1 \left(-s^4t - 2s^2t^3 + \frac{1}{2}s^2t \right)_{s=0}^{s=1} dt \\
&= \int_0^1 -t - 2t^3 + \frac{1}{2}t\ dt \\
&= \int_0^1 -\frac{1}{2}t - 2t^3\ dt \\
&= \left(-\frac{1}{4}t^2 - \frac{1}{2}t^4 \right)_{t=0}^{t=1} \\
&= -\frac{1}{4} - \frac{1}{2} \\
&= \frac{3}{4}.
\end{aligned}$$

Choice of orientation

When we were figuring out how to define an orientation on a surface, we noticed that the orientation will affect the sign of a vector surface integral. That is, if the normal vector is in approximately the same direction as the vector field, the flow will be positive. However, the opposite orientation should give negative flow.

PICTURE

Let's see how this manifests itself in our definition of a vector surface integral,

$$\iint_{\vec{X}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{X}(s,t)) \cdot (\vec{X}_s \times \vec{X}_t)\, dsdt.$$

This uses the orientation $\vec{n}(s,t) = \dfrac{\vec{X}_s \times \vec{X}_t}{\|\vec{X}_s \times \vec{X}_t\|}$. Every orientable closed surface has exactly two orientations, and the other orientation is given by $-\vec{n}(s,t) = -\dfrac{\vec{X}_s \times \vec{X}_t}{\|\vec{X}_s \times \vec{X}_t\|}$. When we replace the normal vector $\vec{X}_s \times \vec{X}_t$ in the vector surface integral with the opposite normal vector, $-(\vec{X}_s \times \vec{X}_t)$, we get

$$\iint_D \vec{F}(\vec{X}(s,t)) \cdot -(\vec{X}_s \times \vec{X}_t)\, dsdt = -\iint_D \vec{F}(\vec{X}(s,t)) \cdot (\vec{X}_s \times \vec{X}_t)\, dsdt,$$

as we'd hope.

Thus, if we're attempting to evaluate a vector surface integral, and we find that our parametrization gives the wrong orientation, we can fix this by changing the sign of the normal vector. We'll see how this works in an example.

Example 60. *Consider the vector field $\vec{F}(x,y,z) = (x,y,z)$, and let S be the unit sphere, oriented with the outward pointing normal vector.*

PICTURE

The sphere can be parametrized as

$$\vec{X}(s,t) = (\cos s \sin t, \sin s \sin t, \cos t)$$

for $0 \le s \le 2\pi$ and $0 \le t \le \pi$.

We start by computing the partial derivatives, $\vec{X}_s$ and $\vec{X}_t$.

$$\begin{aligned}\vec{X}_s(s,t) &= (-\sin s \sin t, \cos s \sin t, 0)\\ \vec{X}_t(s,t) &= (\cos s \cos t, \sin s \cos t, -\sin t)\end{aligned}$$

Now, we compute the cross product

$$\begin{aligned}\vec{X}_s(s,t) \times \vec{X}_t(s,t) &= det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin s \sin t & \cos s \sin t & 0 \\ \cos s \cos t & \sin s \cos t & -\sin t \end{pmatrix}\\ &= (-\cos s \sin^2 t)\vec{i} + (-\sin s \sin^2 t)\vec{j} + (-\sin^2 s \sin t \cos t - \cos^2 s \sin t \cos t)\vec{k}\\ &= (-\cos s \sin^2 t, -\sin s \sin^2 t, -\sin t \cos t).\end{aligned}$$

Notice that this is the inward pointing normal vector, but the surface was oriented with the outward pointing normal vector. We can salvage our computation by using the normal vector $-\vec{X}_s \times \vec{X}_t$ instead.

Now, we can evaluate the vector surface integral.

$$\begin{aligned}
\iint_{\vec{X}} \vec{F} \cdot d\vec{S} &= \iint_D \vec{F}(\vec{X}(s,t)) \cdot -(\vec{X}_s \times \vec{X}_t)\ dsdt \\
&= \int_0^{\pi} \int_0^{2\pi} \vec{F}(\cos s \sin t, \sin s \sin t, \cos t) \cdot (\cos s \sin^2 t, \sin s \sin^2 t, \sin t \cos t)\ dsdt \\
&= \int_0^{\pi} \int_0^{2\pi} (\cos s \sin t, \sin s \sin t, \cos t) \cdot (\cos s \sin^2 t, \sin s \sin^2 t, \sin t \cos t)\ dsdt \\
&= \int_0^{\pi} \int_0^{2\pi} \cos^2 s \sin^3 t + \sin^2 s \sin^3 t + \sin t \cos^2 t\ dsdt \\
&= \int_0^{\pi} \int_0^{2\pi} \sin^3 t + \sin t \cos^2 t\ dsdt \\
&= \int_0^{\pi} \int_0^{2\pi} \sin t\ dsdt \\
&= \int_0^{\pi} 2\pi \sin t\ dt \\
&= (-2\pi \cos t)_{t=0}^{t=\pi} \\
&= 2\pi + 2\pi \\
&= 4\pi.
\end{aligned}$$

Stokes Theorem

Recall that Green's theorem gave us a relationship between a double integral over a region in $\mathbb{R}^2$, and the line integral of a vector field over the boundary.

Theorem 15. ***Green's Theorem.*** *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x,y) = (M(x,y), N(x,y))$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

For Green's theorem to hold, it was important that the boundary be oriented so that if you traverse the boundary in the given direction, the region is on your left.

PICTURE

Green's theorem applies to regions in $\mathbb{R}^2$, and we will now work towards generalizing Green's theorem to surfaces in $\mathbb{R}^3$. This result will be called Stokes theorem, and our first step towards Stokes theorem will be determining how to orient the boundary of a surface in $\mathbb{R}^3$.

Orientation

Consider a surface in $\mathbb{R}^3$.

PICTURE

We could try to orient this surface according to the same rule as Green's theorem, but that would be ambiguous. Whether or not the surface is on the left of the boundary would depend on the direction from which you viewed the surface.

PICTURE

In order to orient the boundary consistently with the surface, we need to be given an orientation for the surface. We say that normal vector points in the direction of the positive side of the surface. Then, we will determine how to orient the boundary by looking at the surface from the positive side.

PICTURE

Learning outcomes: Understand the statement of Stokes theorem, and its geometric interpretation.

Author(s): Melissa Lynn

As with Green's theorem, if you traverse the boundary with your head pointing in the positive direction, the surface will be on your left. Here are a couple of other ways to think about this:

- The boundary is oriented counterclockwise if you view it from the positive side.
- The boundary follows the right hand rule with the orientation of the surface: if you point your right thumb in the direction of the normal vector, your fingers curl in the direction of the orientation on the boundary.

PICTURES/VIDEO

Example 61. *For each of the surfaces below, determine the direction in which the boundary should be oriented.*

EXAMPLES/PICTURES

Stokes theorem

We're now ready to state Stokes theorem.

Theorem 16. ***Stokes Theorem.*** *Suppose S is a smooth and bounded surface in $\mathbb{R}^3$, and that ∂S consists of finitely many closed, simple, and piecewise $\mathcal{C}^1$ curves. Suppose further that S and ∂S are consistently oriented. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field, which is defined on S. Then*

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{S},$$

where $\nabla \times \vec{F}$ denotes the curl of $\vec{F}$.

The proof of Stokes theorem follows the same idea as Green's theorem: when we integrate the curl of $\vec{F}$ over S, we are "adding up" all of the microscopic rotation of the vector field over the surface, and this is equal to the global circulation over the boundary, since circulation on the interior of the surface will cancel. Because of the similarity to the proof of Green's theorem, we will omit a complete proof of Stokes theorem.

Instead, let's look at some examples of Stokes theorem in action.

Example 62. *Consider the upper unit hemisphere pictured below, oriented with the upward pointing normal vector.*

PICTURE

Consider the vector field $\vec{F}(x, y, z) = (y^2, -xy, z^2)$. We'll verify Stokes theorem for this surface and vector field, by computing both the line integral over the boundary and the double integral over the surface.

Let's begin with the line integral over the boundary. In order to be consistent with the orientation of the surface, we orient the boundary as indicated below.

PICTURE

Then, we can parametrize the boundary as

$$\vec{x}(t) = (\cos t, \sin t, 0),$$

for $0 \leq t \leq 2\pi$*. Integrating* $\vec{F}$ *over this path, we have*

$$\begin{aligned}
\oint_{\partial S} \vec{F} \cdot d\vec{s} &= \int_0^{2\pi} \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\ dt \\
&= \int_0^{2\pi} \vec{F}(\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0)\ dt \\
&= \int_0^{2\pi} (\sin^2 t, -\cos t \sin t, 0) \cdot (-\sin t, \cos t, 0)\ dt \\
&= \int_0^{2\pi} -\sin^3 t - \cos^2 t \sin t\ dt \\
&= \int_0^{2\pi} -\sin t\ dt \\
&= (\cos t)_{t=0}^{2\pi} \\
&= 0.
\end{aligned}$$

For comparison, we integrate $\nabla \times \vec{F}$ *over the surface* S*. We can parametrize* S *as*

$$\vec{X}(s,t) = (\cos s \sin t, \sin s \sin t, \cos t),$$

for $0 \leq s \leq 2\pi$ *and* $0 \leq t \leq \pi/2$*. We'll now compute the normal vector,* $\vec{X}_s \times \vec{X}_t$*.*

$$\begin{aligned}
\vec{X}_s(s,t) \times \vec{X}_t(s,t) &= (-\sin s \sin t, \cos s \sin t, 0) \times (\cos s \cos t, \sin s \cos t, -\sin t) \\
&= (-\cos s \sin^2 t, -\sin s \sin^2 t, -\sin t \cos t)
\end{aligned}$$

Notice that this is the downward pointing normal vector, so does not represent the correct orientation. So, we will work with $(\cos s \sin^2 t, \sin s \sin^2 t, \sin t \cos t)$ *as the normal vector instead.*

Now, let's integrate $\nabla \times \vec{F}$ over S.

$$\begin{aligned}\iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_S (0,0,-3y) \cdot d\vec{S} \\ &= \int_0^{\pi/2} \int_0^{2\pi} (0,0,-3\sin s \sin t) \cdot (-\cos s \sin^2 t, -\sin s \sin^2 t, -\sin t \cos t) \, dsdt \\ &= \int_0^{\pi/2} \int_0^{2\pi} -3\sin s \sin^2 t \cos t \, dsdt \\ &= \int_0^{\pi/2} (3\cos s \sin^2 t \cos t)_{s=0}^{s=2\pi} \, dt \\ &= \int_0^{\pi/2} 0 \, dt \\ &= 0.\end{aligned}$$

So, we have verified that Stokes theorem holds in this case.

Example 63. *Consider the surface S pictured below, and the vector field $\vec{F}(x,y,z) = (x,y,z)$.*

PICTURE

Suppose we wish to integrate $\vec{F}$ over the boundary of S. We could try to parametrize S and evaluate this line integral directly, but this would be very difficult! Instead, let's try applying Stokes theorem. In this case, Stokes theorem is particularly effective, since

$$\begin{aligned}\nabla \times \vec{F}(x,y,z) &= \nabla \times (x,y,z) \\ &= (0,0,0).\end{aligned}$$

So, we have

$$\begin{aligned}\oint_{\partial S} \vec{F} \cdot d\vec{s} &= \iint_S \nabla \times \vec{F} \cdot d\vec{S} \\ &= \iint_S (0,0,0) \cdot d\vec{S} \\ &= 0.\end{aligned}$$

More on Stokes Theorem

We've seen how Stokes theorem relates the line integral of a vector field over the boundary of a surface with the double integral of curl over the entire surface, given consistent orientations.

PICTURE

Theorem 17. ***Stokes Theorem.*** *Suppose S is a smooth and bounded surface in $\mathbb{R}^3$, and that ∂S consists of finitely many closed, simple, and piecewise $\mathcal{C}^1$ curves. Suppose further that S and ∂S are consistently oriented. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field, which is defined on S. Then*

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{S},$$

where $\nabla \times \vec{F}$ denotes the curl of $\vec{F}$.

We'll now look at another way to use Stokes theorem, as well as some ways Stokes theorem can be used to understand earlier material.

Replacing a surface

In the following example, we'll see how Stokes theorem can be used to evaluate a double integral over a surface, by replacing the surface with a different surface that has the same boundary.

Example 64. *Consider the paraboloid S defined by the equation $z = 1-x^2-y^2$, for $z \geq 0$, oriented with the upward pointing normal vector.*

PICTURE

Also consider the vector field $\vec{F} = (x^2, z^2, y^3, z)$, and notice that the curl of $\vec{F}$ is

$$\nabla \times \vec{F} = (3y^2 z - 2z, 0, 0).$$

Suppose we wish to evaluate the vector surface integral $\iint_S (3y^2 z - 2z, 0, 0)\, d\vec{S}$. We have a few options for how we could approach evaluating this surface integral. We could parametrize S and evalute directly, or we could apply Stokes theorem, and integral $\vec{F}$ over the boundary of S instead.

Learning outcomes: Understand additional ways that Stokes theorem can be used, and learn some connections to earlier material.

Author(s): Melissa Lynn

Notice that the boundary of S is the unit circle in the xy-plane, pictured below.

PICTURE

Although this approach is possible by Stokes theorem, it doesn't necessarily seem any easier than evaluating the surface integral directly. However, notice that ∂S is also the boundary of a simpler surface, the unit disc D in the xy-plane.

PICTURE

Because of this, we can apply Stokes theorem twice, to get

$$\iint_S \nabla \times \vec{F}\, d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \iint_D \nabla \times \vec{F}\, d\vec{S}.$$

So, we'll evaluate $\displaystyle\iint_D \nabla \times \vec{F}\ d\vec{S}$. *Now, D lies in the xy-plane, and* $\nabla \times \vec{F}(x, y, z) = (3y^2z - 2z, 0, 0)$. *So, when $z = 0$, $\nabla \times \vec{F}(x, y, z) = (0, 0, 0)$. This means that the vector field $\nabla \times \vec{F}$ is zero on the disc D. Putting all of this together,*

$$\begin{aligned}\iint_S \nabla \times \vec{F}\, d\vec{S} &= \oint_{\partial S} \vec{F} \cdot d\vec{s}\\ &= \iint_D \nabla \times \vec{F}\, d\vec{S}\\ &= \iint_D (0, 0, 0)\, d\vec{S}\\ &= 0.\end{aligned}$$

In this case, we were able to use Stokes theorem to replace our surface integral with a different surface integral, over a surface with the same boundary. This gave us a much easier integral to evaluate.

Stokes theorem and circulation

Now, we'll use Stokes theorem to show that if the curl of a vector field $\vec{F}$ is zero, then $\vec{F}$ has no circulation over any (reasonably nice) closed curve.

Proposition 16. *Let $\vec{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ vector field defined on an open and simply connected domain D, and suppose that $\nabla \times \vec{F} = (0, 0, 0)$. Let C be a closed, simple, and piecewise $\mathcal{C}^1$ curve contained in D. Then*

$$\oint_C \vec{F} \cdot d\vec{s} = 0.$$

We'll give a sketch of a proof of this proposition, glossing over some of the details.

Proof Since D is open and simply connected, there is some smooth surface S in D such that C is the boundary of D. A rigorous proof of this assertion is very involved, so we'll skip over this detail.

Furthermore, we can orient D so that Stokes theorem will apply. Then we have

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}.$$

Since the curl of $\vec{F}$ is zero, this gives us

$$\oint_C \vec{F} \cdot d\vec{s} = 0,$$

as desired. ■

Curl as microscopic rotation

We've previously discussed how curl represents microscopic rotation, and we've given some geometric justification for why this should be the case. We'll now use Stokes theorem to provide further justification that curl represents microscopic rotation.

Consider a point $\vec{a}$ in $\mathbb{R}^3$, and let S_r be a small disc of radius r, centered at $\vec{a}$, with unit normal vector $\vec{n}$.

PICTURE

Applying Stokes theorem to this disc, we have

$$\int_{\partial S_r} \vec{F} \cdot d\vec{s} = \iint_{S_r} (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

Since we are integrating over a very small surface, we can approximate the integrand with a constant,

$$(\nabla \times \vec{F}(x, y, z)) \cdot \vec{n} \approx (\nabla \times \vec{F})(\vec{a}) \cdot \vec{n}.$$

Then our integral can be approximated as

$$\begin{aligned} \int_{\partial S_r} \vec{F} \cdot d\vec{s} &= \iint_{S_r} (\nabla \times \vec{F}) \cdot \vec{n}\, dS \\ &\approx \iint_{S_r} (\nabla \times \vec{F})(\vec{a}) \cdot \vec{n}\, dS \\ &= \pi r^2\, (\nabla \times \vec{F})(\vec{a}) \cdot \vec{n}. \end{aligned}$$

This means that

$$(\nabla \times \vec{F})(\vec{a}) \cdot \vec{n} \approx \frac{1}{\pi r^2} \int_{S_r} \vec{F} \cdot d\vec{s}.$$

As r approaches zero, this approximation becomes arbitrarily good, so we have

$$(\nabla \times \vec{F})(\vec{a}) \cdot \vec{n} = \lim_{r \to 0^+} \frac{1}{\pi r^2} \int_{S_r} \vec{F} \cdot d\vec{s}.$$

The quantity on the right represents the circulation or rotation at the point $\vec{a}$ and in the plane perpendicular to $\vec{n}$, so we see that $(\nabla \times \vec{F})(\vec{a}) \cdot \vec{n}$ measures this rotation.

Furthermore, notice that $(\nabla \times \vec{F})(\vec{a}) \cdot \vec{n}$ is largest when $\vec{n}$ and $\nabla \times \vec{F}(\vec{a})$ point in the same direction. This means that the curl gives the axis of rotation.

Putting this together, we have that $(\nabla \times \vec{F})(\vec{a})$ gives the axis of rotation induced by $\vec{F}$ at $\vec{a}$, and provides a measure of this microscopic rotation.

Part V

Triple Integrals

Triple Integrals

We've defined single variable integrals, by approximating an area with rectangles, and taking the limit as the width of the rectangles goes to zero.

PICTURE

In order to do this, we split the domain, $[a, b]$ into subintervals, and pick a sample point in each subinterval.

PICTURE

We've also defined double integrals over rectangles, by approximating a volume with boxes, and taking the limit as the base of the boxes goes to zero.

PICTURE

In order to do this, we split the domain, $[a, b] \times [c, d]$ into subrectangles, and pick a sample point in each subrectangles.

PICTURE

In this section, we'll define triple integrals.

Triple integrals

Consider a function $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ defined over a box $B = [m, n] \times [p, q] \times [r, s]$. We can think of the triple integral as representing a four-dimensional hypervolume, but this is hard to visualize, since it's a four-dimensional object. We can, however, visualize the domain of integration.

PICTURE

We divide this box up into small boxes, and choose a sample point (x_i, y_j, z_k) in each box.

PICTURE

Then, we evaluate the function f at each of the sample points, and multiply by the area of a box. We then add these up, and take the limit as the size of the

Learning outcomes: Understand the geometric ideas behind the definition of a triple integral.

Author(s): Melissa Lynn

boxes goes to zero.

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \sum f(x_i, y_j, z_k)\ \Delta x \Delta y \Delta z.$$

This gives us our definition of a triple integral.

Definition 30. *Let $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ be a function defined on the box $B = [m,n] \times [p,q] \times [r,s]$ in $\mathbb{R}^3$. Let Δx, Δy, Δz, x_i, y_j, and z_k be as above. The* triple integral *of f over B is defined to be*

$$\iiint_B f\ dV = \sum_{i,j,k} f(x_i, y_j, z_k) \Delta x \Delta y \Delta z.$$

Now, we'll look at we use this definition to approximate triple integrals.

Approximating triple integrals

Example 65. *We'll approximate the triple integral of $f(x,y,z) = x^2 + y^2 + z^2$ over the box $B = [0,1] \times [0,1] \times [0,1]$.*

To approximate this triple integral, we'll split B into eight subboxes of equal size. For sample points, we take the center of each box. These boxes, listed with their sample points, are:

$$\begin{aligned}
&[0,1/2] \times [0,1/2] \times [0,1/2], \quad &&(1/4,1/4,1/4);\\
&[0,1/2] \times [0,1/2] \times [1/2,1], \quad &&(1/4,1/4,3/4);\\
&[0,1/2] \times [1/2,1] \times [0,1/2], \quad &&(1/4,3/4,1/4);\\
&[0,1/2] \times [1/2,1] \times [1/2,1], \quad &&(1/4,3/4,3/4);\\
&[1/2,1] \times [0,1/2] \times [0,1/2], \quad &&(3/4,1/4,1/4);\\
&[1/2,1] \times [0,1/2] \times [1/2,1], \quad &&(3/4,1/4,3/4);\\
&[1/2,1] \times [1/2,1] \times [0,1/2], \quad &&(3/4,3/4,1/4);\\
&[1/2,1] \times [1/2,1] \times [1/2,1], \quad &&(3/4,3/4,3/4).
\end{aligned}$$

Evaluating the function f at each of the test points, we have

$$\begin{aligned}
f(1/4, 1/4, 1/4) &= 3/16 \\
f(1/4, 1/4, 3/4) &= 11/16 \\
f(1/4, 3/4, 1/4) &= 11/16 \\
f(1/4, 3/4, 3/4) &= 19/16 \\
f(3/4, 1/4, 1/4) &= 11/16 \\
f(3/4, 1/4, 3/4) &= 19/16 \\
f(3/4, 3/4, 1/4) &= 19/16 \\
f(3/4, 3/4, 3/4) &= 27/16.
\end{aligned}$$

The volume of each subbox is $1/2 \cdot 1/2 \cdot 1/2 = 1/8$, so we can approximate the triple integral as

$$\begin{aligned}
\iiint_B f \, dV &\approx \frac{1}{8}\left(\frac{3}{16} + \frac{11}{16} + \frac{11}{16} + \frac{19}{16} + \frac{11}{16} + \frac{19}{16} + \frac{19}{16} + \frac{27}{16}\right) \\
&= \frac{15}{16}.
\end{aligned}$$

Of course, we would like to be able to compute the exact values of triple integrals, and we would like to do this without using the limit definition. In the next section, we'll prove a version of Fubini's theorem for triple integrals, which will allow us to evaluate triple integrals more easily.

Fubini's Theorem for Triple Integrals

When we studied double integrals, we found that we could most easily compute double integrals as iterated integrals. This was possible because of Fubini's theorem.

Theorem 18. ***Fubini's Theorem.*** *Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined on a rectangle $R = [a, b] \times [c, d]$. Then*

$$\iint_R f\, dA = \int_c^d \int_a^b f(x, y)\, dxdy = \int_a^b \int_c^d f(x, y)\, dydx.$$

In this section, we'll see that we can prove a version of Fubini's theorem for triple integrals, so we can evaluate triple integrals using iterated integrals.

Fubini's theorem for triple integrals

The statement of Fubini's theorem for triple integrals is unsurprisingly similar to the statement for double integrals. Notice that there are six different ways that the iterated integrals can be ordered.

Theorem 19. ***Fubini's Theorem.*** *Let $f : B \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous function defined on a box $B = [m, n] \times [p, q] \times [r, s]$ in $\mathbb{R}^3$. Then*

$$\begin{aligned}\iiint_B f\, dV &= \int_r^s \int_p^q \int_m^n f(x, y, z)\, dxdydz\\ &= \int_p^q \int_r^s \int_m^n f(x, y, z)\, dxdzdy\\ &= \int_p^q \int_m^n \int_r^s f(x, y, z)\, dzdxdy\\ &= \int_r^s \int_m^n \int_p^q f(x, y, z)\, dydxdz\\ &= \int_m^n \int_r^s \int_p^q f(x, y, z)\, dydzdx\\ &= \int_m^n \int_p^q \int_r^s f(x, y, z)\, dzdydx.\end{aligned}$$

Learning outcomes: Evaluate triple integrals using Fubini's theorem.
Author(s): Melissa Lynn

The proof of this theorem echoes the proof for double integrals, so we will omit it here. We'll now look at an example of using iterated integrals to evaluate triple integrals, which is possible as a result of Fubini's theorem.

Example 66. *We'll integrate the function $f(x, y, z) = x + 2y + 3z$ over the box $B = [0, 1] \times [0, 2] \times [0, 3]$. Since f is a continuous function, Fubini's theorem applies, and we can evaluate the triple integral as an iterated integral.*

$$
\begin{aligned}
\iiint_B f\ dV &= \int_0^1 \int_0^2 \int_0^3 (x + 2y + 3z)\ dzdydx \\
&= \int_0^1 \int_0^2 \left(xz + 2yz + \frac{3}{2}z^2 \right)_{z=0}^{z=3} dydx \\
&= \int_0^1 \int_0^2 \left(3x + 6y + \frac{27}{2} \right) dydx \\
&= \int_0^1 \left(3xy + 3y^2 + \frac{27}{2}y \right)_{y=0}^{y=2} dx \\
&= \int_0^1 (6x + 12 + 27)\ dx \\
&= \left(3x^2 + 39x\right)_{x=0}^{x=1} \\
&= 42
\end{aligned}
$$

Alternatively, we could use a different order of integration for the iterated integral, and we should see that we'll get the same result. Here, we'll choose the order $dydxdz$.

$$
\begin{aligned}
\iiint_B f\ dV &= \int_0^3 \int_0^1 \int_0^2 (x + 2y + 3z)\ dydxdz \\
&= \int_0^3 \int_0^1 (xy + y^2 + 3zy)_{y=0}^{y=2}\ dxdz \\
&= \int_0^3 \int_0^1 (2x + 4 + 6z)\ dxdz \\
&= \int_0^3 (x^2 + 4x + 6zx)_{x=0}^{x=1}\ dz \\
&= \int_0^3 (1 + 4 + 6z)\ dz \\
&= (5z + 3z^2)_{z=0}^{z=3} \\
&= 42
\end{aligned}
$$

Indeed, we have the same result.

Triple Integrals over Elementary Regions

In this section, we'll look at how we can evaluate triple integrals over elementary regions. This will be similar to integrating double integrals over elementary regions, so we'll start with a quick review.

Review of double integrals over elementary regions

To define a double integral over a region D, we used an extension function in order to leverage our definition of double integrals over rectangles. This extension function was defined as

$$f^{\text{ext}}(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}.$$

We were then able to define the double integral of f over D using a double integral of f^{ext} over a rectangle containing D.

Definition 31. *Let D be a bounded region in $\mathbb{R}^2$, and let R be a rectangle containing D. Let $f : X \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined on D. Then we define the* double integral of f over D *to be*

$$\iint_D f \, dA = \iint_R f^{ext} \, dA.$$

We describe elementary regions in $\mathbb{R}^2$ using inequalities, which then provide our bounds of integration.

Proposition 17. *Consider an x-simple region D, which can be described as the set of points (x, y) such that*

$$\begin{aligned} a \leq & x \leq b, \text{ and} \\ g(x) \leq & y \leq h(x), \end{aligned}$$

Learning outcomes: Evaluate triple integrals over elementary regions, and change the order of integration. Focus on visualizing regions.

Author(s): Melissa Lynn

where $g(x)$ and $h(x)$ are continuous functions. Then we can evaluate the double integral of a function $f(x,y)$ over D as

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dydx.$$

Consider a y-simple region D, which can be described as the set of points (x,y) such that

$$\begin{aligned} c \leq & y \leq d, \text{ and} \\ g(y) \leq & x \leq h(y), \end{aligned}$$

where $g(y)$ and $h(y)$ are continuous functions. Then we can evaluate the double integral of a function $f(x,y)$ over D as

$$\iint_D f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dxdy.$$

When we wished to change the order of integration, it was important to visualize the region of integration, in order to correctly transform the inequalities.

Triple integrals over elementary regions

We'll now define the triple integral of a function f over a bounded region D in $\mathbb{R}^3$.

Definition 32. *Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a function defined on a bounded region D in $\mathbb{R}^3$. We define an extension function by*

$$f^{ext}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \in D \\ 0, & (x,y,z) \notin D \end{cases}.$$

Let B be a box containing D. We define the triple integral of f over D *to be*

$$\iiint_D f\,dV = \iiint_B f^{ext}\,dV.$$

As with double integrals, we could most easily evaluate integrals over elementary regions, which we now define in $\mathbb{R}^3$.

Definition 33. *An* elementary region D *in $\mathbb{R}^3$ is the set of points (x,y,z) which can be described as one of the following:*

- *$a \leq x \leq b$, $f_1(x) \leq y \leq f_2(x)$, and $g_1(x,y) \leq z \leq g_2(x,y)$, where f_1, f_2, g_1 and g_2 are continuous functions. In this case,*

$$\iiint_D f\,dv = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f\,dzdydx.$$

- *$a \le x \le b$, $f_1(x) \le z \le f_2(x)$, and $g_1(x,z) \le y \le g_2(x,z)$, where f_1, f_2, g_1 and g_2 are continuous functions. In this case,*

$$\iiint_D f\,dv = \int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,z)}^{g_2(x,z)} f\,dydzdx.$$

- *$a \le y \le b$, $f_1(y) \le x \le f_2(y)$, and $g_1(x,y) \le z \le g_2(x,y)$, where f_1, f_2, g_1 and g_2 are continuous functions. In this case,*

$$\iiint_D f\,dv = \int_a^b \int_{f_1(y)}^{f_2(y)} \int_{g_1(x,y)}^{g_2(x,y)} f\,dzdxdy.$$

- *$a \le y \le b$, $f_1(y) \le z \le f_2(y)$, and $g_1(y,z) \le x \le g_2(y,z)$, where f_1, f_2, g_1 and g_2 are continuous functions. In this case,*

$$\iiint_D f\,dv = \int_a^b \int_{f_1(y)}^{f_2(y)} \int_{g_1(y,z)}^{g_2(y,z)} f\,dxdzdy.$$

- *$a \le z \le b$, $f_1(z) \le x \le f_2(z)$, and $g_1(x,z) \le y \le g_2(x,z)$, where f_1, f_2, g_1 and g_2 are continuous functions. In this case,*

$$\iiint_D f\,dv = \int_a^b \int_{f_1(z)}^{f_2(z)} \int_{g_1(x,z)}^{g_2(x,z)} f\,dydxdz.$$

- *$a \le z \le b$, $f_1(z) \le y \le f_2(z)$, and $g_1(y,z) \le x \le g_2(y,z)$, where f_1, f_2, g_1 and g_2 are continuous functions. In this case,*

$$\iiint_D f\,dv = \int_a^b \int_{f_1(z)}^{f_2(z)} \int_{g_1(y,z)}^{g_2(y,z)} f\,dxdydz.$$

Since there are so many different types of elementary regions in $\mathbb{R}^3$, we will not bother to distinguish between them.

We'll now look at an example of evaluating triple integrals over elementary regions in $\mathbb{R}^3$.

Example 67. *Consider the region D in $\mathbb{R}^3$ bounded by the surfaces $x = 1$, $y = -x^2$, $y = x^2$, $z = -1$, and $z = xy^2$, pictured below.*

PICTURE

This region can be described with the inequalities

$$\begin{aligned} 0 \le& x \le 1, \\ -x^2 \le& y \le x^2, \\ -1 \le& z \le xy^2. \end{aligned}$$

So, we can integrate the function $f(x, y, z) = xz$ over D as

$$\begin{aligned}
\iiint_D f\ dV &= \int_0^1 \int_{-x^2}^{x^2} \int_{-1}^{xy^2} xz\ dzdydx \\
&= \int_0^1 \int_{-x^2}^{x^2} \left(\frac{1}{2}xz^2\right)_{z=-1}^{z=xy^2} dydx \\
&= \int_0^1 \int_{-x^2}^{x^2} \left(\frac{1}{2}x^3y^4 - \frac{1}{2}x\right) dydx \\
&= \int_0^1 \left(\frac{1}{10}x^3y^5 - \frac{1}{2}xy\right)_{y=-x^2}^{y=x^2} dx \\
&= \int_0^1 \left(\left(\frac{1}{10}x^{13} - \frac{1}{4}x^3\right) - \left(-\frac{1}{10}x^{13} + \frac{1}{4}x^3\right)\right) dx \\
&= \int_0^1 \left(\frac{1}{5}x^{13} - \frac{1}{2}x^3\right) dx \\
&= \left(\frac{1}{70}x^{14} - \frac{1}{8}x^4\right)_{x=0}^{x=1} \\
&= \frac{1}{70} - \frac{1}{8} \\
&= -\frac{31}{280}.
\end{aligned}$$

Changing the order of integration

Now let's look at how we can change the order of integration in a triple integral. The most challenging part of this process is accurately visualizing the domain of integration from different perspectives. It can be helpful to think of the "shadow" that the region casts on the xy-, xz-, and yz-planes, in order to more effectively visualize the region.

Example 68. *Consider the region D in the first octant, bounded by the plane $x + 2y + 3z = 6$. This region is pictured below, along with its projections onto the xy-, xz-, and yz-planes.*

PICTURE

We'll set up integrals over this region, with three different orders of integration.

First, let's set up a $dzdydx$ integral. Here, it's helpful to look at the projection of D onto the xy-plane.

PICTURE

We see that this projection can be described with the inequalities

$$\begin{aligned} 0 \leq & x \leq 6, \\ 0 \leq & y \leq \frac{6-x}{2}. \end{aligned}$$

Then, we also have $0 \leq z \leq \dfrac{6-2y-x}{3}$*. So we can set up an integral*

$$\iiint_D f\ dV = \int_0^6 \int_0^{(6-x)/2} \int_0^{(6-2y-x)/3} f\ dzdydx.$$

Next, let's set up a $dzdxdy$ *integral. Again, we'll look at the projection of* D *onto the* xy*-plane.*

PICTURE

From a different perspective, this projection can be described with the inequalities

$$\begin{aligned} 0 \leq & y \leq 3, \\ 0 \leq & x \leq 6-2y. \end{aligned}$$

We also have $0 \leq z \leq \dfrac{6-2y-x}{3}$*, so we can set up an integral*

$$\iiint_D f\ dV = \int_0^3 \int_0^{6-2y} \int_0^{(6-2y-x)/3} f\ dzdxdy.$$

Finally, we'll set up a $dxdydz$ *integral. Here, it will be helpful to look at the projection of* D *onto the* yz*-plane.*

PICTURE

This projection can be described with the inequalities

$$\begin{aligned} 0 \leq & y \leq 3, \\ 0 \leq & z \leq \frac{6-2y}{3}. \end{aligned}$$

We also have $0 \leq x \leq 6-2y-3z$*, so we can set up an integral*

$$\iiint_D f\ dV = \int_0^3 \int_0^{(6-2y)/3} \int_0^{6-2y-3z} f\ dxdydz.$$

Example 69. *Consider the integral*

$$\int_1^3 \int_{-x}^0 \int_0^{x+y} e^{x^2} y\ dzdydx.$$

We'll change the order of integration to $dxdydz$. To do this, we first need to figure out what the domain of integration looks like. From the bounds on x and y, we have the inequalities

$$\begin{aligned} 1 \leq & x \leq 3, \\ -x \leq & y \leq 0. \end{aligned}$$

These determine the following region in the xy-plane.

PICTURE

Putting this together with the bounds $0 \leq z \leq x + y$, we can visualize our region of integration in $\mathbb{R}^3$.

PICTURE

Below, we have the projection of this region onto the yz-plane.

PICTURE

We can describe the projection with the inequalities

$$\begin{aligned} -3 \leq & y \leq 0, \\ 0 \leq & z \leq y + 3. \end{aligned}$$

Over this projection, the x-coordinates in our region are bounded by the planes $z = x + y$ and $x = 3$. In inequalities, this can be written

$$y - z \leq x \leq 3.$$

So, we have

$$\int_1^3 \int_{-x}^0 \int_0^{x+y} e^{x^2} y \; dzdydx = \int_{-3}^0 \int_0^{y+3} \int_{y-z}^3 e^{x^2} y \; dxdydz$$

Volume of an elementary region

When working with double integrals, we discovered that we could find the area of a region in $\mathbb{R}^2$ by integrating 1 over the region. That is,

$$\text{area}(D) = \iint_D 1 \; dA.$$

Similarly, we can find the volume of a region in $\mathbb{R}^3$ by integrating 1 over the region.

Proposition 18. *If D is an elementary region in $\mathbb{R}^3$, then*

$$\mathit{volume}(D) = \iiint_D 1 \; dV.$$

Let's look at an example of using this to compute volume.

Example 70. *Let D be the region in the first octant bounded by the plane $x+y+z=1$. We'll find the volume of this region.*

PICTURE

The region D can be described with the inequalities

$$\begin{aligned} 0 \le x &\le 1, \\ 0 \le y &\le 1-x, \\ 0 \le z &\le 1-x-y. \end{aligned}$$

So we have

$$\begin{aligned} \mathit{volume}(D) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \; dzdydx \\ &= \int_0^1 \int_0^{1-x} (z)_0^{1-x-y} \; dydx \\ &= \int_0^1 \int_0^{1-x} 1-x-y \; dydx \\ &= \int_0^1 \left(y - xy - \frac{1}{2}y^2 \right)_0^{1-x} \; dx \\ &= \int_0^1 \left((1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right) \; dx \\ &= \int_0^1 \left(1 - x - x + x^2 - \frac{1}{2} + x - \frac{1}{2}x^2 \right) \; dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2 \right) \; dx \\ &= \left(\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3 \right)_{x=0}^{x=1} \\ &= \frac{1}{6}. \end{aligned}$$

Thus, the volume of the region D is $\frac{1}{6}$.

Change of Variables in Triple Integrals

When working with double integrals, we found that it was sometimes advantageous to change coordinates. This was often used to make the region of integration easier to describe.

When we change from one coordinate system to another, this can change volumes and areas. To account for this, we incorporated a scaling factor into our differential.

More precisely, if we have a change of coordinates given by a function $T : \mathbb{R}^2 \to \mathbb{R}^2$, we need to consider how this transformation affects the area of a small rectangle. We can some linear algebra, along with the derivative matrix of T, and see that area is scaled by approximately $\det(D\vec{T}(u,v))$.

PICTURE

Below, we repeat the entire statement for change of variables in double integrals.

Proposition 19. *Let $\vec{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function which maps a region $D^* \subset \mathbb{R}^2$ onto a region $D \subset \mathbb{R}^2$, so that $\vec{T}$ restricted to D^* is one-to-one. Suppose $f : D \to \mathbb{R}$ is an integrable function. Then*

$$\iint_D f(x,y)\, dxdy = \iint_{D^*} f(\vec{T}(u,v)) \left| det(D\vec{T}(u,v)) \right| \, dudv.$$

In this section, we'll see how we can change coordinates in triple integrals.

Change of variables in triple integrals

Suppose we have a transformation $\vec{T} : \mathbb{R}^3 \to \mathbb{R}^3$ giving our desired change of coordinates. As with double integrals, we need to consider how this transformation scales. That is, if we have a parallelepiped in $\mathbb{R}^3$, how does applying T affect the volume?

PICTURE

We can approximate the transformation $\vec{T}$ using its derivative matrix, and the absolute value of the determinant of $D\vec{T}$ then gives the scaling factor. So, we have the following result for change of variables in triple integrals.

Proposition 20. *Let $\vec{T} : \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^1$ function which maps a region $D^* \subset \mathbb{R}^3$ onto a region $D \subset \mathbb{R}^3$, so that $\vec{T}$ restricted to D^* is one-to-one.*

Learning outcomes: Understand how to change coordinates in triple integrals.
Author(s): Melissa Lynn

Suppose $f : D \to \mathbb{R}$ is an integrable function. Then

$$\iiint_D f(x,y,z)\, dxdydz = \iiint_{D^*} f(\vec{T}(u,v,w)) \left| det(D\vec{T}(u,v,w)) \right| \, dudvdw.$$

We'll look at examples of this, where we convert to cylindrical and spherical coordinates.

Cylindrical coordinates

Example 71. *Consider the cylinder D pictured below, bounded by $x^2 + y^2 = 1$, $z = 0$, and $z = 2$.*

PICTURE

Consider the integral $\iiint_D x \, dxdydz$. We'll evaluate this integral by converting to cylindrical coordinates.

The cylinder D can be described in cylindrical coordinates by

$$\begin{aligned} 0 \leq &\theta \leq 2\pi \\ 0 \leq &r \leq 1 \\ 0 \leq z &\leq 2, \end{aligned}$$

and these with provide the bounds for our transformed integral.

Recall the relationship between Cartesian coordinates and cylindrical coordinates,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

Then if $\vec{T}(\theta, r, z) = (r\cos\theta, r\sin\theta, z)$ is the transformation taking cylindrical coordinate to Cartesian coordinates, we have

$$\begin{aligned} det(D\vec{T}) &= det \begin{pmatrix} \frac{\partial}{\partial\theta} r\cos\theta & \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial z} r\cos\theta \\ \frac{\partial}{\partial\theta} r\sin\theta & \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial z} r\sin\theta \\ \frac{\partial}{\partial\theta} z & \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial z} z \end{pmatrix} \\ &= det \begin{pmatrix} -r\sin\theta & \cos\theta & 0 \\ r\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= -r\sin^2\theta - r\cos^2\theta \\ &= -r. \end{aligned}$$

Since r is nonnegative, the absolute value of this is r. Now, we can complete our change of variables.

$$\iiint_D x\ dxdydz = \int_0^2 \int_0^1 \int_0^{2\pi} (r\cos\theta) r\ d\theta dr dz$$

We can then evaluate the integral.

$$\begin{aligned}\int_0^2 \int_0^1 \int_0^{2\pi} (r\cos\theta) r\ d\theta dr dz &= \int_0^2 \int_0^1 (r^2 \sin\theta)_0^{2\pi}\ dr dz \\ &= \int_0^2 \int_0^1 0\ dr dz \\ &= 0\end{aligned}$$

In general, the change in differential from cylindrical coordinates to Cartesian coordinates is given by

$$dxdydz = r\ dr d\theta dz.$$

Spherical coordinates

Example 72. *Consider the solid sphere D of radius 1 centered at the origin.*

PICTURE

The volume of this sphere is given by

$$volume(D) = \iiint_D 1\ dV.$$

We'll compute this volume by changing to spherical coordinates. In spherical coordinates, the sphere D can be described with the inequalities

$$\begin{aligned}0 \le &\rho \le 1, \\ 0 \le &\theta \le 2\pi, \\ 0 \le &\phi \le \pi.\end{aligned}$$

These will provide the bounds for our iterated integral.

Now, consider the transformation

$$\vec{T}(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi),$$

which converts from spherical coordinates to Cartesian coordinates. We find the *scaling factor for this transformation.*

$$\begin{aligned}
det(D\vec{T}) &= det\begin{pmatrix} \frac{\partial}{\partial\rho}\rho\cos\theta\sin\phi & \frac{\partial}{\partial\theta}\rho\cos\theta\sin\phi & \frac{\partial}{\partial\phi}\rho\cos\theta\sin\phi \\ \frac{\partial}{\partial\rho}\rho\sin\theta\sin\phi & \frac{\partial}{\partial\theta}\rho\sin\theta\sin\phi & \frac{\partial}{\partial\phi}\rho\sin\theta\sin\phi \\ \frac{\partial}{\partial\rho}\rho\cos\phi & \frac{\partial}{\partial\theta}\rho\cos\phi & \frac{\partial}{\partial\phi}\rho\cos\phi \end{pmatrix} \\
&= det\begin{pmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} \\
&= -\cos\phi\left(-\rho^2\sin^2\theta\sin\phi\cos\phi - \rho^2\cos^2\theta\sin\phi\cos\phi\right) + \rho\sin\theta\left(\rho\cos^2\theta\sin^2\phi + \rho\sin^2\theta\sin^2\phi\right) \\
&= -\cos\phi\left(-\rho^2\sin\phi\cos\phi\right) + \rho\sin\theta\left(\rho\sin^2\phi\right) \\
&= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi \\
&= \rho^2\sin\phi
\end{aligned}$$

Notice that this is nonnegative, since $0 \leq \phi \leq \pi$. Now, we can complete our change of variables.

$$\iiint_D 1\ dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho^2\sin\phi)\ d\rho d\phi d\theta$$

We then evaluate our integral to find the volume of the sphere.

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho^2\sin\phi)\ d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{3}\rho^3\sin\phi\right)_0^1\ d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{3}\sin\phi\right)\ d\phi d\theta \\
&= \int_0^{2\pi}\left(-\frac{1}{3}\cos\phi\right)_0^{\pi}\ d\theta \\
&= \int_0^{2\pi}\left(\frac{2}{3}\right)\ d\theta \\
&= \frac{4}{3}\pi
\end{aligned}$$

Thus, we have that the volume of the unit sphere is $\frac{4}{3}\pi$.

In general, the change in differential from spherical to Cartesian coordinates is

$$dxdydz = \rho^2\sin\phi\ d\rho d\theta d\phi.$$

The Divergence Theorem

Given a region D and vector field $\vec{F}$ in $\mathbb{R}^2$, we found that the double integral of the curl of $\vec{F}$ over D is equal to the vector line integral of $\vec{F}$ over the boundary of D. This result is called Green's theorem.

Theorem 20. ***Green's Theorem.*** *Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x,y) = (M(x,y), N(x,y))$. Then*

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

In order to apply Green's theorem, we require that the boundary be positively oriented with respect to the region. This means that, as we traverse the boundary in the indicated direction, the region will be on our right.

PICTURE

We'll now work towards a version of Green's theorem for solid regions in $\mathbb{R}^3$, and this result will be called the divergence theorem.

Requirements on the boundary

As with Green's theorem, we need to be careful about orientation.

Suppose we have a solid region W in $\mathbb{R}^3$, whose boundary ∂W is an orientable surface. We say that ∂W is *positively oriented* if its normal vector points away from the region W.

For regions which don't have any holes, this always means the outward pointing normal vector.

PICTURE

For a region with a hole, this can be a bit trickier, but we still choose the normal vector(s) to point away from the region W.

PICTURE

Example 73. *Which of the boundaries below are positively oriented?*

Learning outcomes: Understand the statement and geometric idea of the divergence theorem.

Author(s): Melissa Lynn

Multiple Choice:

(a) *Positively oriented*

(b) *Not positively oriented*

In addition to being positively oriented, we will require that the boundary surfaces be closed. A surface is *closed* if it has no boundary curves.

For example, the following surfaces are closed.

PICTURE

The following surfaces are not closed, since they have boundary curves.

PICTURE

Example 74. *Which are of surfaces below are closed?*

Multiple Choice:

(a) *Closed*

(b) *Not closed*

Notice that saying a surface is a closed surface means something different from saying that a surface is closed as a set. This is an example of very confusing terminology in math, where a single word is used to mean two different things!

Divergence theorem

Now, suppose we have a vector field $\vec{F}$ defined on a solid region W with a positively oriented boundary. Now, consider the triple integral

$$\iiint_W \nabla \cdot \vec{F}\, dV.$$

Let's figure out what this integral represents. Recall that the divergence of a vector field measures the local expansion or contraction of a vector field. When we integrate the divergence over a region, we obtain the total net expansion or contraction of the vector field over that region.

PICTURE

However, if we look at points on the interior of the region, any expansion or contraction stays within the region. So, in the integral, it would be canceled by expansion or contraction at nearby points. This means that everything in the interior of the region cancels, and we are left with considering what happens on the boundary. The total net expansion or contraction on the boundary is equivalent to the flow of the vector field across the boundary, which leads us to a flux integral.

PICTURE

This is the geometric intuition behind the divergence theorem, which we now state.

Theorem 21. ***Divergence Theorem.*** *Let W be a solid region in $\mathbb{R}^3$, with boundary ∂W. Suppose that ∂W consists of finitely many orientable, piecewise smooth, and closed surfaces, which are positively oriented with respect to W. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on W. Then*

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F}\, dV.$$

We'll now prove the divergence theorem.

Proof INSERT PROOF HERE. ■

More on the Divergence Theorem

We've seen that the divergence theorem related a flux integral over a the boundary of a region in $\mathbb{R}^3$ with a triple integral of the divergence over the region.

Theorem 22. ***Divergence Theorem.*** *Let W be a solid region in $\mathbb{R}^3$, with boundary ∂W. Suppose that ∂W consists of finitely many orientable, piecewise smooth, and closed surfaces, which are positively oriented with respect to W. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on W. Then*

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F}\, dV.$$

We'll now look at some examples of how the divergence theorem can be applied to simplify computation.

Divergence theorem examples

Example 75. *Let S be the unit sphere in $\mathbb{R}^3$, oriented with the outward pointing normal vector.*

PICTURE

Let $\vec{F}(x, y, z) = \left(\cos(ye^z), x^{100}z^{1000}, e^{\sin(xy^2)}\right)$, and we'll consider the flux integral

$$\iint_S \vec{F} \cdot d\vec{S}.$$

If we tried to compute this flux integral directly, it would be very difficult to work with the vector field $\vec{F}$. Instead, we'll apply the divergence theorem.

Notice that S is the boundary of the solid unit sphere D, and S is positively oriented relative to D. So, we can apply the divergence theorem, and we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F}\, dV.$$

Before we begin to compute the triple integral, we need to find the divergence of

Learning outcomes: Understand how the divergence theorem can be used.
Author(s): Melissa Lynn

$\vec{F}$.

$$\begin{aligned}\nabla\cdot\vec{F}(x,y,z) &= \nabla\cdot\left(\cos(ye^z), x^{100}z^{1000}, e^{\sin(xy^2)}\right)\\ &= \frac{\partial}{\partial x}\cos(ye^z) + \frac{\partial}{\partial y}x^{100}z^{1000} + \frac{\partial}{\partial z}e^{\sin(xy^2)}\\ &= 0\end{aligned}$$

Since the divergence of $\vec{F}$ is zero, we have

$$\iiint_D \nabla\cdot\vec{F}\;dV = 0,$$

so $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = 0$ as well.

Example 76. *Consider the solid unit cube $D = [0,1]\times[0,1]\times[0,1]$ in $\mathbb{R}^3$, and let $\vec{F}(x,y,z) = \left(x^2+\sin(yz), e^z, xz\right)$. Consider the flux integral*

$$\iint_{\partial D} \vec{F}\cdot d\vec{S},$$

where ∂D is oriented with the outward pointing normal vector.

Evaluating this flux integral directly would require splitting it up into six integrals, one for each face of the unit cube. Instead, it will be much easier to use the divergence theorem. From this, we have

$$\iint_{\partial D} \vec{F}\cdot d\vec{S} = \iiint_D \nabla\cdot\vec{F}\;dV.$$

We'll start by finding the divergence of $\vec{F}$.

$$\begin{aligned}\nabla\cdot\vec{F}(x,y,z) &= \nabla\cdot\left(x^2+\sin(yz), e^z, xz\right)\\ &= \frac{\partial}{\partial x}(x^2+\sin(yz)) + \frac{\partial}{\partial y}(e^z) + \frac{\partial}{\partial z}(xz)\\ &= 2x + 0 + x\\ &= 3x\end{aligned}$$

Now, we'll integrate the divergence over the solid unit cube.

$$\begin{aligned}\iiint_D \nabla\cdot\vec{F}\;dV &= \int_0^1\int_0^1\int_0^1 3x\;dzdydx\\ &= \int_0^1\int_0^1 3x\;dydx\\ &= \int_0^1 3x\;dx\\ &= \left(\frac{3}{2}x^2\right)_{x=0}^{x=1}\\ &= \frac{3}{2}\end{aligned}$$

So, using the divergence theorem, we have

$$\iint_{\partial D} \vec{F} \cdot d\vec{S} = \frac{3}{2}.$$

Example 77. *Let S be the upper half of the unit sphere, oriented with the downward pointing normal vector.*

PICTURE

Let $\vec{F} = \big(y\sin(z), xz^10, z+1\big)$, and consider the surface integral $\iint_S \vec{F} \cdot d\vec{S}$. This isn't the simplest vector field, so we would like to find an alternative approach. However, S is not a closed surface, so it isn't immediately obvious how we could apply the divergence theorem.

If we let W be the solid upper hemisphere, then the boundary of W is $S \cup D$, where D is the unit disc in the xy-plane.

PICTURE

We do still have an issue here - S was oriented with the downward *pointing normal vector, so is not positively oriented relative to W. Fortunately, we can account for this by writing $\partial W = -S \cup D$, where D is oriented with the downward pointing normal vector.*

PICTURE

Now, we can apply the divergence theorem over the region W and it's boundary. This gives us

$$\iint_{-S\cup D} \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F} \, dV.$$

Let's start with the right side of this equation, and we'll compute the divergence of $\vec{F}$.

$$\begin{aligned}
\nabla \cdot \vec{F} &= \nabla \cdot \big(y\sin(z), xz^10, z+1\big) \\
&= \frac{\partial}{\partial x} y\sin(z) + \frac{\partial}{\partial y} xz^10 \frac{\partial}{\partial z}(z+1) \\
&= 0 + 0 + 1 \\
&= 1
\end{aligned}$$

When we integrate this over W, we have

$$\begin{aligned}
\iiint_W 1 \, dV &= (\textit{volume of } W) \\
&= \frac{2}{3}\pi,
\end{aligned}$$

since W is half of the solid unit sphere.

Next, let's look at the left side of the equation, $\iint_{-S\cup D} \vec{F}\cdot d\vec{S}$*. Splitting this up into two surface integrals, we have*

$$\iint_{-S\cup D} \vec{F}\cdot d\vec{S} = -\iint_S \vec{F}\cdot d\vec{S} + \iint_D \vec{F}\cdot d\vec{S}.$$

We'll now compute the surface integral $\iint_D \vec{F}\cdot d\vec{S}$ *directly. At first glance, this might not seem like much of an improvement over our original problem. However, here we are integrating over the unit disc in the xy-plane, where* $z = 0$*. When* $z = 0$*, the vector field* $\vec{F}$ *becomes much simpler. This will make the computation reasonable.*

We can parametrize D as $\vec{X}(s,t) = (s\cos t, s\sin t, 0)$ *for* $0 \le s \le 1$ *and* $0 \le t \le 2\pi$*. To compute the surface integral, we compute the normal vector,* $X_s \times X_t$*.*

$$\begin{aligned}
X_s(s,t)\times X_t(s,t) &= (\cos t, \sin t, 0)\times(-s\sin t, s\cos t, 0)\\
&= det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k}\\ \cos t & \sin t & 0\\ -s\sin t & s\cos t & 0\end{pmatrix}\\
&= (s\cos^2 t + s\sin^2 t)\vec{k} \qquad\qquad = (0,0,s)
\end{aligned}$$

This is the upward pointing normal vector, so we need to adjust with a sign change. We now compute the surface integral.

$$\begin{aligned}
\iint_D \vec{F}\cdot d\vec{S} &= \int_0^1\int_0^{2\pi} \vec{F}(s\cos t, s\sin t, 0)\cdot(0,0,-s)\ dtds\\
&= \int_0^1\int_0^{2\pi} (0,0,1)\cdot(0,0,-s)\ dtds\\
&= \int_0^1\int_0^{2\pi} -s\ dtds\\
&= \int_0^1 -2\pi s\ ds\\
&= \left(-\pi s^2\right)_{s=0}^{s=1}\\
&= -\pi
\end{aligned}$$

Now we have enough information to find $\iint_S \vec{F}\cdot d\vec{S}$*. Putting all of this together, we have*

$$\begin{aligned}
\iint_S \vec{F}\cdot d\vec{S} &= \iint_D \vec{F}\cdot d\vec{S} - \iiint_W \nabla\cdot\vec{F}\ dV\\
&= -\pi - \frac{2}{3}\pi\\
&= -\frac{5}{3}\pi,
\end{aligned}$$

Completing our computation.

In this example, we were able to evaluate a surface integral, by creating a closed surface to which we could apply the divergence theorem.

Geometric interpretation of divergence

We'll now use the divergence theorem for an extra check on our geometric understanding of divergence, as local expansion or contraction of a vector field.

Let B_r be a small, solid ball of radius r centered at a point $\vec{a}$ in $\mathbb{R}^3$. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined near $\vec{a}$. By the divergence theorem, we have

$$\iint_{\partial B_r} \vec{F} \cdot d\vec{S} = \iiint_{B_r} \nabla \cdot \vec{F}\, dV$$

Since B_r is a small ball, we can approximate the triple integral $\iiint_{B_r} \nabla \cdot \vec{F}\, dV$ with

$$(\text{volume of } B_r)\left(\nabla \cdot \vec{F}\right) = \pi r \left(\nabla \cdot \vec{F}\right).$$

So, we have

$$\iint_{\partial B_r} \vec{F} \cdot d\vec{S} \approx \pi r \left(\nabla \cdot \vec{F}\right),$$

which gives us

$$\nabla \cdot \vec{F} \approx \frac{1}{\pi r} \iint_{\partial B_r} \vec{F} \cdot d\vec{S}.$$

When we take $r \to 0$, this approximation because more and more accurate, so we have

$$\nabla \cdot \vec{F} = \lim_{r \to 0} \frac{1}{\pi r} \iint_{\partial B_r} \vec{F} \cdot d\vec{S}.$$

The flux integral on the right measures the flow across the small sphere centered at $\vec{a}$, so by taking the limit, we are measuring how much the vector field is flowing "into" or "out of" the point $\vec{a}$, giving us local contraction or expansion.

Thus, we see that the divergence of a vector field measures the local expansion or contraction of a vector field.

Generalized Stokes

Throughout calculus, we've seen many different theorems which relate integrals over some region with behavior on the boundary of the region. The earliest example of this is in the fundamental theorem of calculus, also called the evaluation theorem.

Theorem 23. ***Fundamental Theorem of Calculus***

Let f be a continuous function on a closed interval $[a, b]$, and let F be an antiderivative of f. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Here, we integrate over the closed interval $[a, b]$, and the boundary of this region consists of two points, a and b. Also, the integrand f is the derivative of the function F, which is evaluated at the endpoints.

We saw similar behavior for line integrals of conservative vector fields, in the fundamental theorem of line integrals.

Theorem 24. ***Fundamental Theorem of Line Integrals***

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be C^1, where X is open and path-connected. Then if C is any piecewise C^1 curve from $\boldsymbol{A}$ to $\boldsymbol{B}$, then

$$\int_C \nabla f \cdot d\boldsymbol{s} = f(\boldsymbol{B}) - f(\boldsymbol{A})$$

Here, we integrate over the curve C, and the boundary of this curve is the two points $\vec{A}$ and $\vec{B}$. The integrand, ∇f, is the gradient of the function f, which is evaluated at the endpoints.

Next, we went up a dimension, to consider regions in $\mathbb{R}^2$, through Green's theorem.

Theorem 25. ***Green's Theorem***

Let D be a closed an bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple and piecewise smooth curves. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on D, written in components as $\vec{F}(x, y) = (M(x, y), N(x, y))$. Then

$$\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

Learning outcomes: See the similarities between the various special cases of Stokes theorem we've covered.

Author(s): Melissa Lynn

We have a double integral over a region D, compared to a line integral over the boundary of D. When we look at the integrand of the double integral, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$ is the two-dimensional curl of the vector field $\vec{F}$.

Considering surfaces in $\mathbb{R}^3$, we next arrived at Stokes theorem.

Theorem 26. ***Stokes Theorem***

Suppose S is a smooth and bounded surface in $\mathbb{R}^3$, and that ∂S consists of finitely many closed, simple, and piecewise $\mathcal{C}^1$ curves. Suppose further that S and ∂S are consistently oriented. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field, which is defined on S. Then

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \iint_S \nabla \times \vec{F} \cdot d\vec{S},$$

where $\nabla \times \vec{F}$ denotes the curl of $\vec{F}$.

Similar to Green's theorem, we have a double integral over a surface S, compared to a line integral over the boundary of S. The integrand of the double integral is the curl of the vector field $\vec{F}$.

Finally, the divergence theorem related a triple integral over a region in $\mathbb{R}^3$ with a flux integral over the boundary of the region.

Theorem 27. ***Divergence Theorem***

Let W be a solid region in $\mathbb{R}^3$, with boundary ∂W. Suppose that ∂W consists of finitely many orientable, piecewise smooth, and closed surfaces, which are positively oriented with respect to W. Let $\vec{F}$ be a $\mathcal{C}^1$ vector field defined on W. Then

$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \nabla \cdot \vec{F} \, dV.$$

The integrand of the triple integral is the divergence of the vector field $\vec{F}$.

Generalized Stokes theorem

In all of these theorems, we equated an integral over a region with an integral over its boundary. In each case, the integrand of the higher-dimensional integral was some sort of derivative.

There is a theorem which in encompasses all of these results, as well as similar results for even higher dimensions. This result appears in an area of math called differential geometry, and is called the generalized Stokes theorem. Without going into too much detail, we give an approximate statement of this theorem here.

Theorem 28. ***Generalized Stokes Theorem***

Let Ω *be an* n*-dimensional* manifold. *Essentially, this means that* Ω *is a nice, smooth region in some* $\mathbb{R}^n$. *The sphere is an example of a two-dimensional manifold in* $\mathbb{R}^3$.

Let $\partial\Omega$ *be the boundary of* Ω*, positively oriented.*

Let ω *be a* differential form. *Roughly speaking, a differential form is something you integrate, and it consists of both the integrand and the differential from an integral. An example of a one-dimensional differential form is* $x^2\,dx$.

Let d *be a differential operator. Roughly, this means taking some sort of derivative. It could be the single variable operator* $\dfrac{d}{dx}$*, or the del operator* ∇.

Under the right conditions, we have

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega.$$

So, integrating $d\omega$ *over the region* Ω *is equivalent to integrating* ω *over the boundary* $\partial\Omega$.