Geometric Significance of Curl

Consider the vector field $\mathbf{F}(x, y) = (-y, 0)$.

We can compute the curl of this vector field,

$$\nabla \times \mathbf{F} = (0, 0, 1)$$

Imagine that we fix a point (representing a particle) in this vector field, but allow it to rotate. If imagine the vector field acting as a force on this particle, which way will it cause the particle to rotate?

(VECTOR FIELD)

Here, we see that the vector field is applying a greater force to the "top" of the particle than to the "bottom." this will cause the particle to rotate counterclockwise. We describe this type of rotation as *local rotation* or *microscopic rotation*, since it's the rotation when we "zoom in" on the particle.

It turns out that the curl of a vector field provides a measure of this local rotation - but how are these connected? We will answer this question in this section, discussing the geometric significance of the curl.

Geometric Significance of Two-dimensional Curl

Recall that, for a two-dimensional vector field $\mathbf{F}(x, y) = (M, N)$, we can compute the curl as

$$\nabla \times \mathbf{F} = (0, 0, N_x - M_y)$$

where N_x is the partial derivative of N with respect to x, and M_y is the partial derivative of M with respect to y. We'll start by considering how M_y and N_x contribute to local rotation.

First let's consider the case where $M_y < 0$. In this case, the x-component of the vector field $\mathbf{F}$ is *decreasing* as we move in the positive y direction. Select all pictures which match this situation.
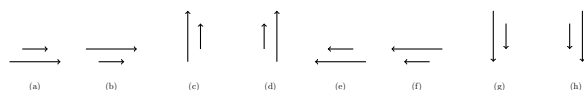
Learning outcomes: Understand the geometric interpretation of the curl as a measure of local rotation.
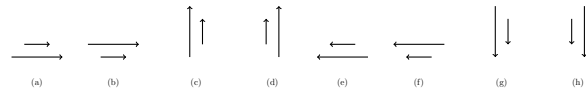
Author(s): Melissa Lynn

Select All Correct Answers:

(a) (a) ✓

(b) (b)

(c) (c)

(d) (d)

(e) (e)

(f) (f) ✓

(g) (g)

(h) (h)

If $M_y < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

Now, let's consider the case where $N_x > 0$. This means that the y-component of the vector field **F** is increasing as we move in the positive x direction. Select all pictures which match this situation.

Select All Correct Answers:

(a) (a)

(b) (b)

(c) (c)

(d) (d) ✓

(e) (e)

(f) (f)

(g) (g) ✓

(h) (h)

If $N_x < 0$, which way will this cause a particle in the vector field to rotate?

Multiple Choice:

(a) Clockwise.

(b) Counterclockwise. ✓

We've seen that the signs of N_x and M_y correspond to the direction of local rotation, with $N_x > 0$ and $M_y < 0$ contributing to counterclockwise rotation.

In general, we have that the sign of $N_x - M_y$ corresponds to the direction of local rotation in the plane. In particular, we have the following correspondences:

$$N_x - M_y > 0 \longleftrightarrow \text{ counterclockwise local rotation}$$
$$N_x - M_y < 0 \longleftrightarrow \text{ clockwise local rotation}$$
$$N_x - M_y = 0 \longleftrightarrow \text{ no local rotation}$$

Remembering that $N_x - M_y$ is the *curl* of the two-dimensional vector field **F**, we now have that the sign of the curl tells us the direction of local rotation for two-dimensional vector fields.

We have a special term for a vector field that never has any local rotation: we call such a vector field *irrotational*.

Furthermore, the length of the curl,

$$\|(0, 0, N_x - M_y)\| = |N_x - M_y|,$$

corresponds to the speed of rotation.

For example, in which case will the particle spin faster?

Multiple Choice:

(a) ✓

(b)

Note that this corresponds to a larger value of N_x (the change in the y-component of $\mathbf{F}$ as we move in the positive x direction).

We now apply our knowledge of the geometric significance of the curl in a couple of examples.

Example 1. *Consider the vector field* $\mathbf{F}(x,y) = (-y, x^2)$. *Compute the curl of* $\mathbf{F}$, *and describe the local rotation of the vector field at the points* $(1,0)$ *and* $(-4,1)$.

Explanation. *We begin by computing the curl of* $\mathbf{F}$,

$$\nabla \times \mathbf{F} = (0, 0, \boxed{2x+1}).$$

At the point $(1,0)$, *we have* $(\nabla \times \mathbf{F})(1,0) = (0,0,\boxed{3})$. *Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(1,0)$ *is (positive* ✓*/ negative/ zero). Thus, the local rotation of the vector field at the point* $(1,0)$ *is (clockwise / counterclockwise* ✓*/ no rotation).*

At the point $(-4,1)$, *we have* $(\nabla \times \mathbf{F})(-4,1) = (0,0,\boxed{-7})$. *Looking at the third component, we see that the sign of* $N_x - M_y$ *at* $(-4,1)$ *is (positive/ negative* ✓*/ zero). Thus, the local rotation of the vector field at the point* $(-4,1)$ *is (clockwise* ✓*/ counterclockwise/ no rotation).*

Looking at a graph of the vector field, we can see that this local rotation is reflected in the graph.

(ADD GRAPH, WITH ROTATION?)

Example 2. *Let* $\mathbf{F}(x,y) = \left(\dfrac{y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$. *Compute the curl* $\nabla \times \mathbf{F}$, *and interpret it geometrically.*

Explanation. *Computing our partial derivatives, we have*

$$N_x = \boxed{\dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2}}$$

and

$$M_y = \boxed{\dfrac{-x^2-y^2+2x^2}{(x^2+y^2)^2}}.$$

Then, the curl of $\mathbf{F}$ *is*

$$\nabla \times \mathbf{F} = (0,0,\boxed{0}).$$

Thus, we see that there is no local rotation at any point in the vector field. This is particularly interesting once we look at a graph of the vector field.

(GRAPH)

From the graph of the vector field, there certainly seems to be some larger scale, global rotation of the vector field. However, our computation showed that there

is no local rotation. This example illustrates an important distinction: curl measures local *rotation of a vector field, which is a different concept from* global *rotation.*

Now that we've found how the curl of a two-dimensional vector field corresponds to local rotation, we'll determine how the curl of a vector field corresponds to local rotation for *three*-dimensional vector fields.

Geometric Significance of Three-dimensional Curl

For a three dimensional vector field $\mathbf{F}(x, y, z) = (M, N, P)$, we can compute the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = (P_y - N_z, \boxed{-P_x + M_z}, \boxed{N_x - M_y}).$$

Here, the situation is more complicated than in two dimensions. In the plane, there are only two possible ways to rotate: clockwise and counterclockwise. In $\mathbb{R}^3$, there are infinitely many different ways to rotate, since we have infinitely many choices of axes. Yikes!

Fortunately, three-dimensional curl still tells about local rotation. In this case, we imagine local rotation as rotation of an infinitesimal (tiny) sphere. This sphere can rotate in infinitely many different ways, depending on which axis we rotate around.

When we look at the components of the curl, this tells us about rotation perpendicular to each of the axes, ignoring rotation in any other direction. Specifically,

$$
\begin{aligned}
N_x - M_y \text{ (the z-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the z-axis} \\
P_y - N_z \text{ (the x-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the x-axis} \\
-P_x + M_z \text{ (the y-component of } \mathbf{F}) &\longleftrightarrow \text{ rotation perpendicular to the y-axis}
\end{aligned}
$$

Once again, the sign tells us the direction of rotation, with positive sign corresponding to counterclockwise rotation (viewed from the positive axes).

Furthermore, the length of the curl, $\|\nabla \times \mathbf{F}\|$, tells us the speed of rotation, and the direction of $\nabla \times \mathbf{F}$ tells us the axis of rotation.

In $\mathbb{R}^3$, we would like to be able to describe the direction of rotation around a given axis. However, this can be tricky, since it's a matter of perspective. Imagine rotation in the xy-plane. If the rotation is clockwise viewed from above, then it will be counterclockwise from below! Fortunately, curl follows the *right hand rule*:

> If you point your right thumb in the direct of $\nabla \times \mathbf{F}$, then your fingers will curl in the direction of local rotation.

We now put this to use in an example.

Problem   **1**   *Consider the vector field* $\mathbf{F}(x, y, z) = \left(0, \dfrac{-z}{(y^2 + z^2)^{3/2}}, \dfrac{y}{(y^2 + z^2)^{3/2}} \right).$
Compute the curl of $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \boxed{\left(\dfrac{-1}{(y^2 + z^2)^{3/2}}, 0, 0 \right)}$$

Problem   **1.1**   *What is the axis of local rotation (at any point)?*

Multiple Choice:

(a) *The x-axis.* ✓

(b) *The y-axis.*

(c) *The z-axis.*

(d) *Some other line.*

Problem   **1.1.1**   *Viewed from the positive x-axis, what is the direction of local rotation (at any point)?*

Multiple Choice:

(a) *Clockwise.* ✓

(b) *Counterclockwise.*

Problem   **1.1.1.1**   *How does the speed of local rotation change as we move closer to the origin?*

Multiple Choice:

(a) *Stays the same.*

(b) *Gets slower.*

(c) *Gets faster.* ✓