Integers

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Computer Systems







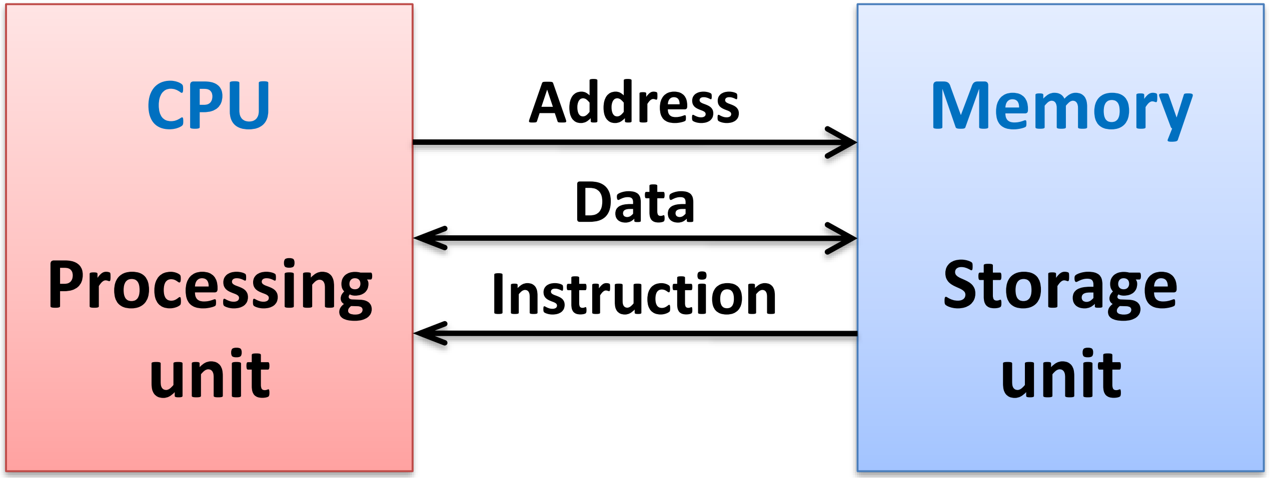





A computer is a $\boxed{\text{programmable}}$ machine.

Von Neumann Architecture

- By John von Neumann, 1945



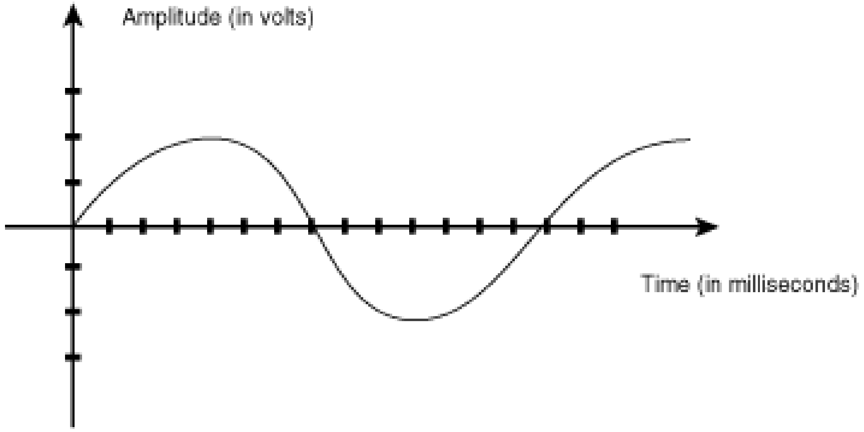
Data movement
Arithmetic & logical ops
Control transfer

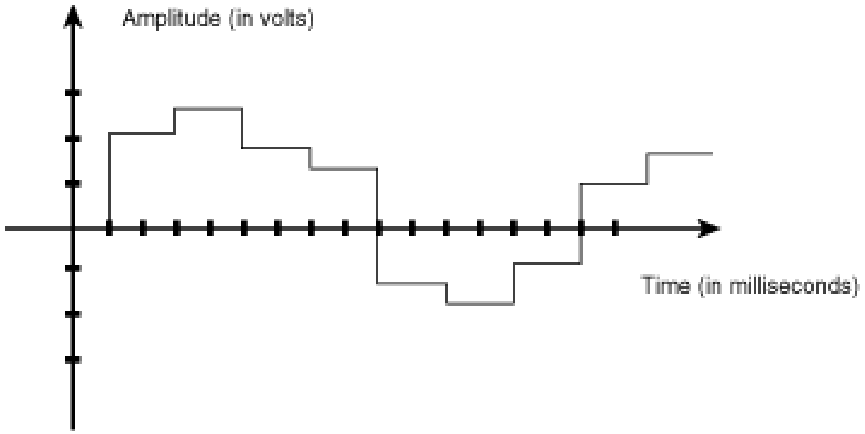
Byte addressable array
Code + data (user program, OS)
Stack to support procedures

The Advent of the Digital Age

- Analog vs. digital?

analog - continuous signal / storing the signal
<-> digital - digit(numbers) everything is converted into the numbers



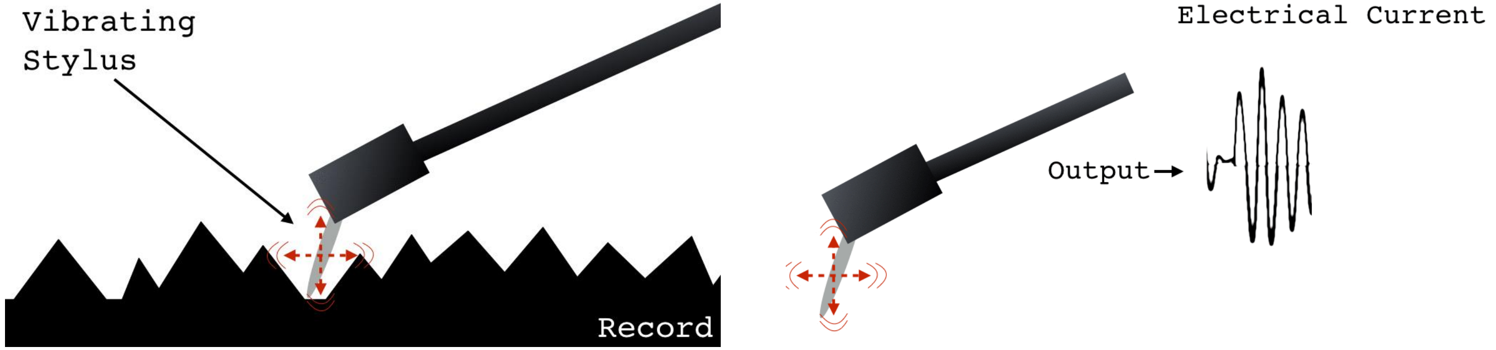


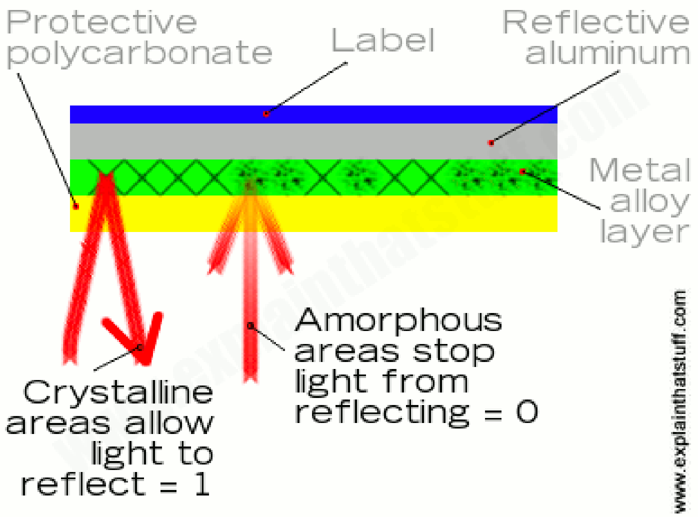
- Compact Disc (CD)
  - 44.1 KHz, 16-bit, 2-channel

frequency(주파수)

- MP3
  - A digital audio encoding with lossy data compression

LP Record vs. Compact Disc

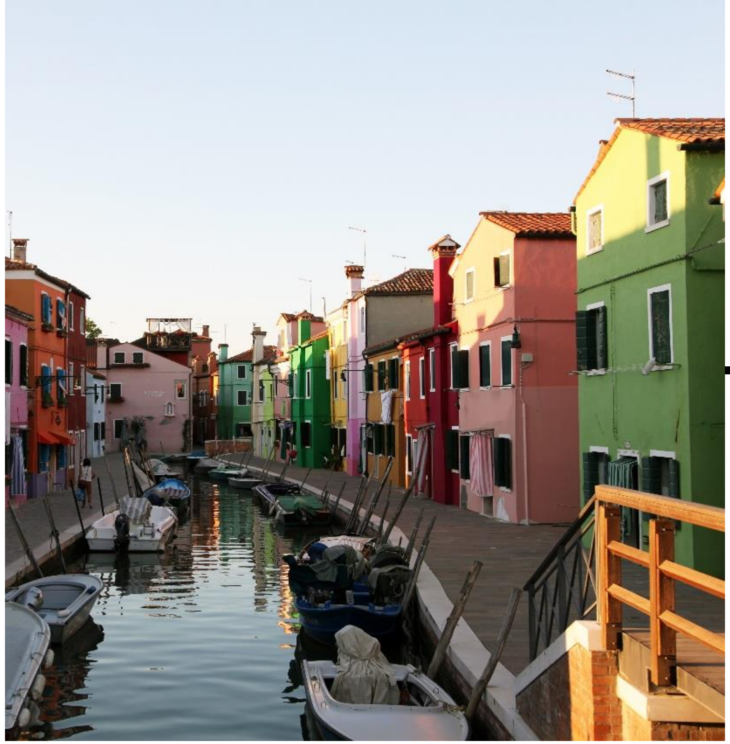


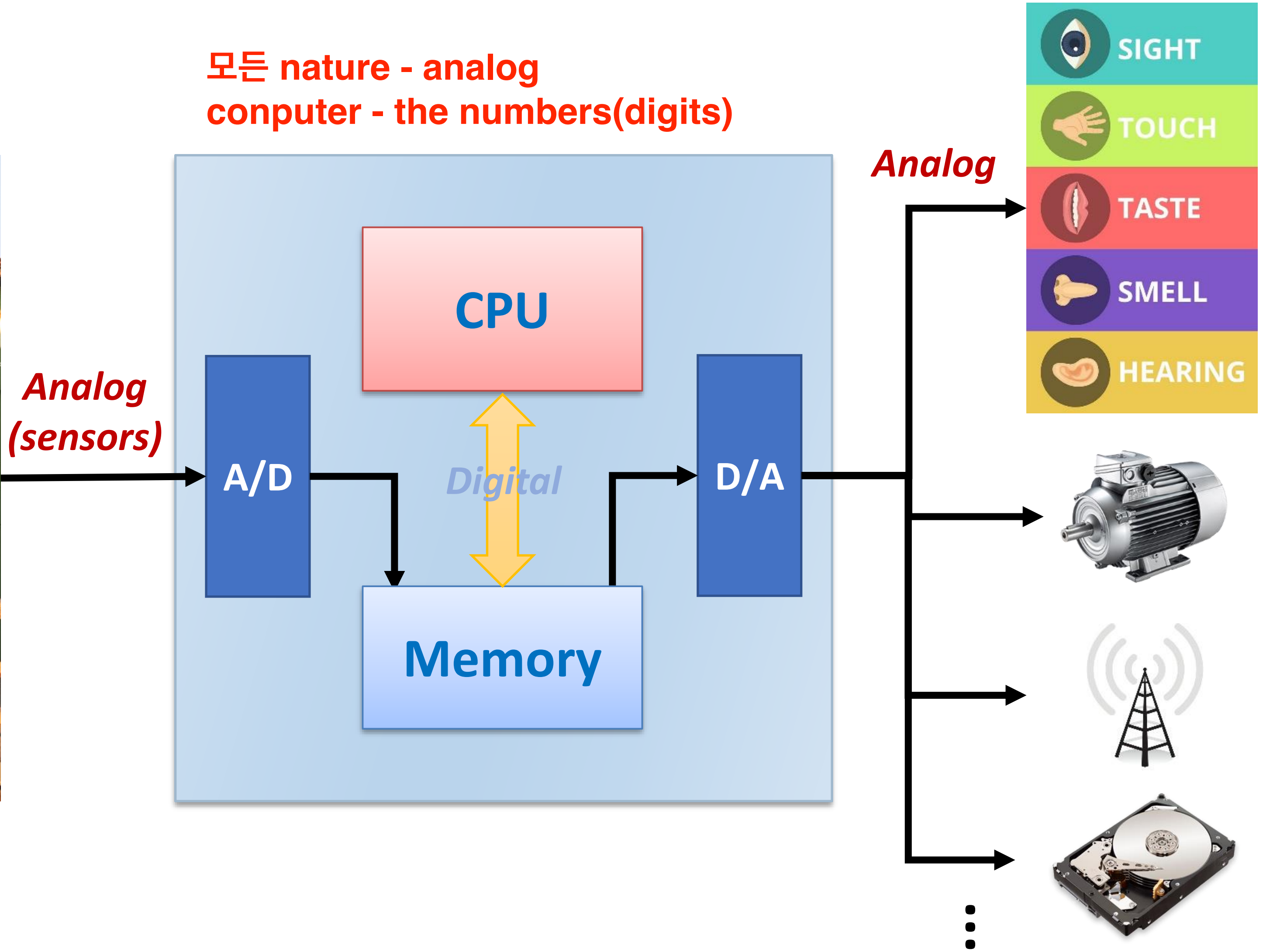


Source: http://www.soundsetal.com/blog-how-do-vinyl-records-work/
http://www.explainthatstuff.com/cdplayers.html

Digital Computer

모든 nature - analog
conputer - the numbers(digits)





Representing Information

- Information = Bits + Context
  - Computers manipulate representations of things
  - Things are represented as binary digits
  - What can you represent with *N* bits?
    - 2^N things
    - context 알아야 information 이해 가능
    - Numbers, characters, pixels, positions, source code, executable files, machine instructions, …
    - Depends on what operations you do on them

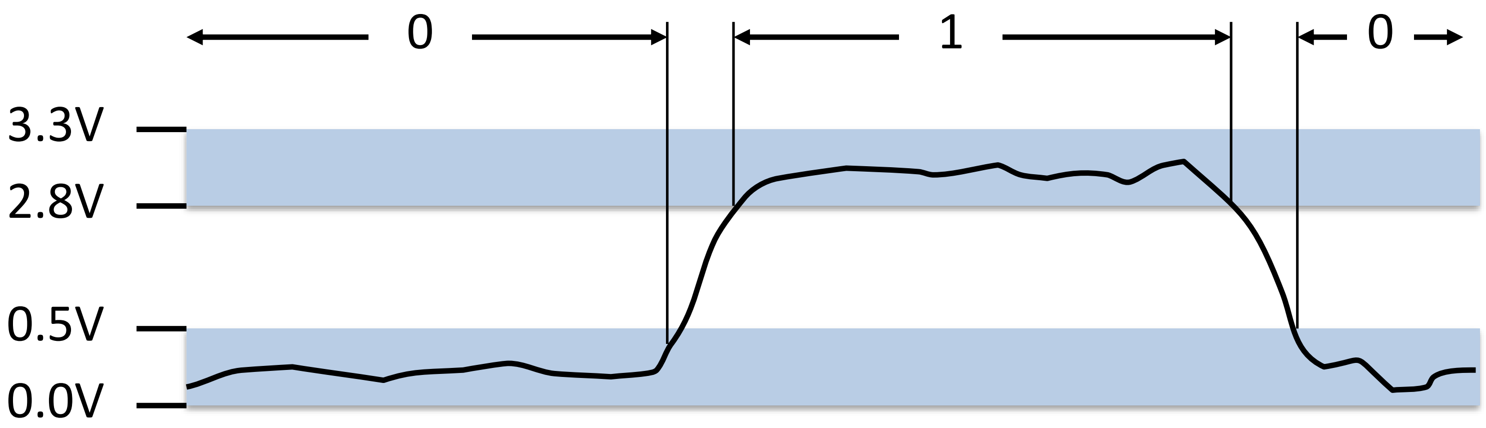
01010011 01001110 01010101 01000001 01000010 01000011 01001000 01001001

(char)	**‘S’**	**‘N’**	**‘U’**	**‘A’**	**‘R’**	**‘C’**	**‘H’**	**‘I’**
(int)	**1096109651**				**1229472594**			
(double)	**1.0821647958380079... x 10^{45}**							

Binary Representations

- Why not base 10 representation?
  - Easy to store with bistable elements   **easy to store compute transfer**
  - Straightforward implementation of arithmetic functions
  - Reliably transmitted on noisy and inaccurate wires

- Electronic implementation



Encoding Byte Values

- Binary: 00000000_2 to 11111111_2
- Octal: 000_8 to 377_8
  - An integer constant that begins with `0` is an octal number in C
- Decimal: 0_{10} to 255_{10}
  - First digit must not be `0` in C
- Hexadecimal: 00_{16} to FF_{16}
  - Base 16 number representation
  - Use characters '0' to '9' and 'A' to 'F'
  - Write $FA1D37B_{16}$ in C as **`0xFA1D37B`** or **`0xfa1d37b`**

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Representing Integers

Unsigned Integers

- Encoding unsigned integers

$$B = [b_{w-1}, b_{w-2}, \ldots, b_0]$$

x = 0001 0000 0101 1110$_2$

$$D(B) = \sum_{i=0}^{w-1} b_i \cdot 2^i$$

D(x) = $2^{12} + 2^{6} + 2^{4} + 2^{3} + 2^{2} + 2^{1}$
= 4096 + 64 + 16 + 8 + 4 + 2
= 4190

- What is the range for unsigned values with *w* bits? **2 to the w - 1 — maximum val**
- Using 64 bits: 0 to +18,446,774,073,709,551,615

Signed Integers

- Encoding positive numbers
  - Same as unsigned numbers

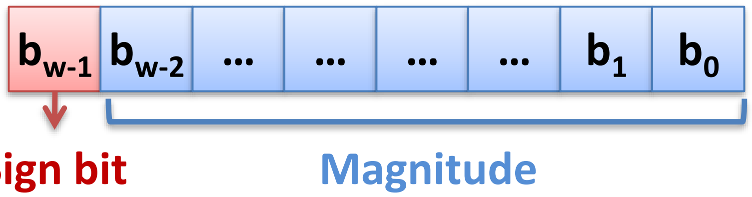
- Encoding negative numbers
  - Sign-magnitude representation
  - Ones' complement representation
  - Two's complement representation

Sign-magnitude Representation

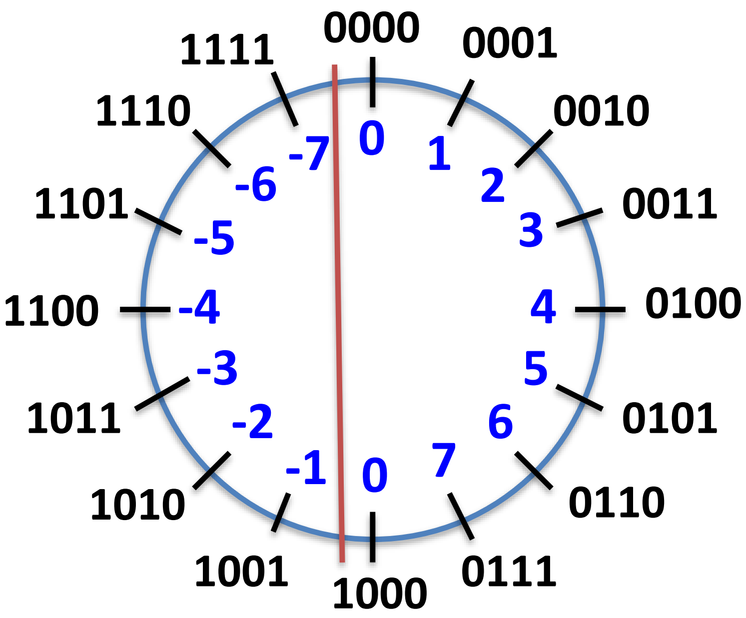
- Two zeros
  - [000…00], [100..00]
- Used for floating-point numbers

2개의 0 0000 +0.   1000 -0.

a-b >0 / a-b == 0
등등 여러 비교 시 +0/-0 사용에 문제



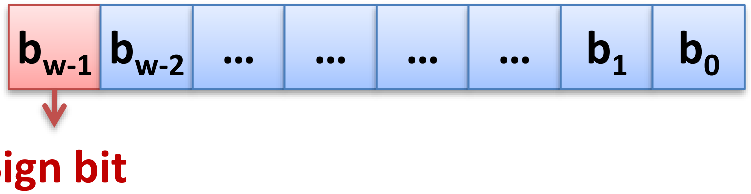
$$S(B) = (-1)^{b_{w-1}} \cdot \left(\sum_{i=0}^{w-2} b_i \cdot 2^i \right)$$

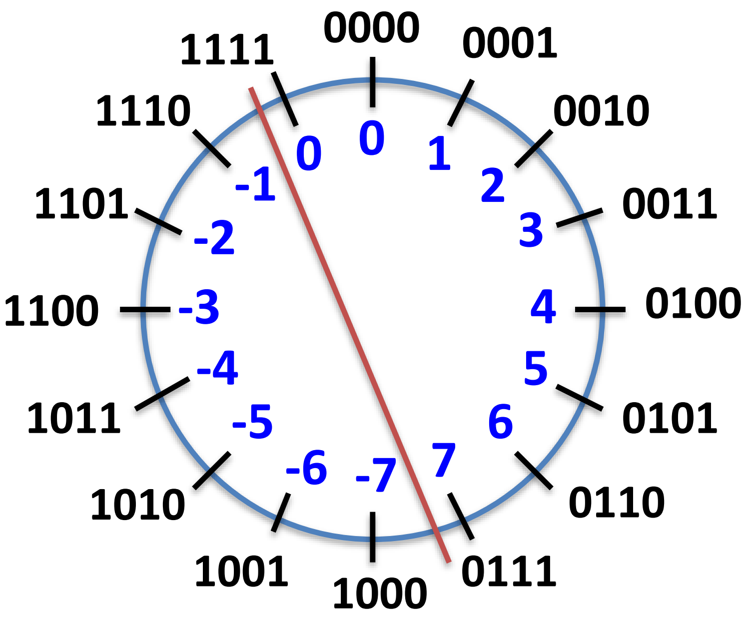


Ones' Complement Representation

- Easy to find $-n$
- Two zeros
  - [000..00], [111..11]
- No longer used

just invert 0 / 1 flip!
2개의 0 문제점



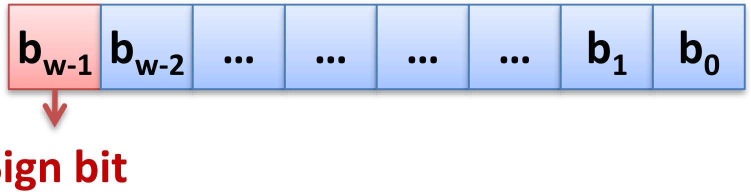


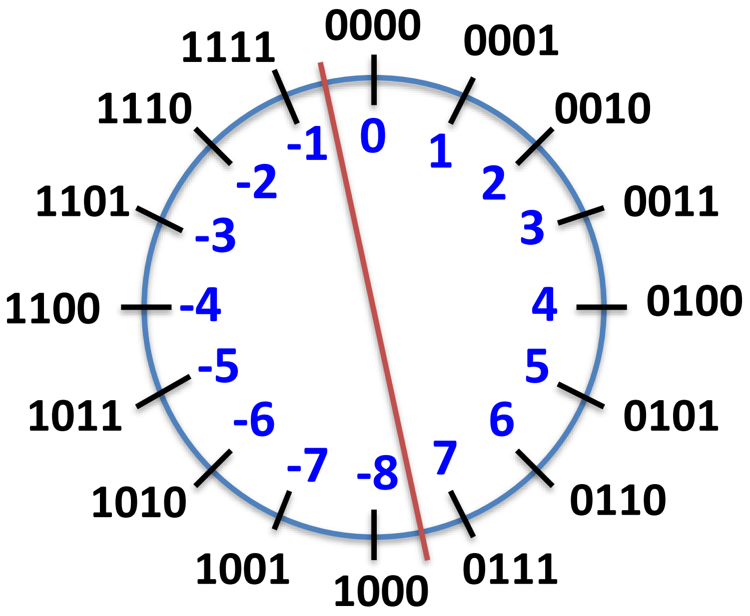
$$O(B) = -b_{w-1}(2^{w-1} - 1) + \left(\sum_{i=0}^{w-2} b_i \cdot 2^i\right)$$

Two’s Complement Representation (1)

- Unique zero
- Easy for hardware
  - leading 0 $\geq$ 0
  - leading 1 < 0
- Used by almost all modern machines

1001(2)
-8. 1.
-8 + 1 = -7





$$O(B) = -b_{w-1} \cdot 2^{w-1} + \left(\sum_{i=0}^{w-2} b_i \cdot 2^i \right)$$

Asymmetric range:

$$-2^{n-1} \sim 2^{n-1} - 1$$

Two's Complement Representation (2)

- Following holds for two's complement

$$\sim x + 1 == -x$$

- Complement
  - Observation: $\sim x + x == 1111...11_2 == -1$

- Increment

$\sim x + x == -1$

$\sim x + x + (-x + 1) == -1 + (-x + 1)$

$\sim x + 1 == -x$

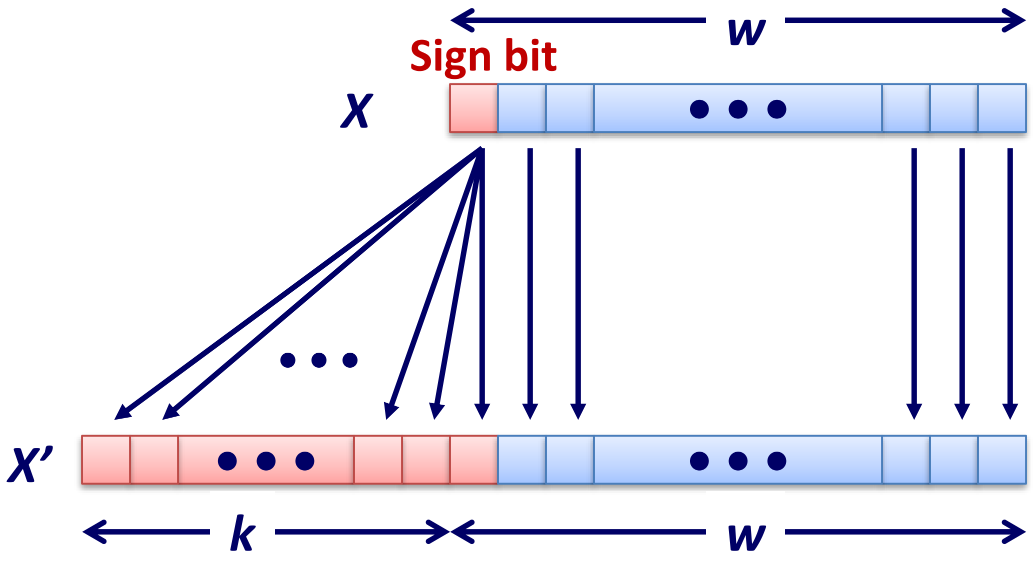
Sign Extension

short s : 16-bit
int i : 32-bit

s = i
i = s (filling with zero but, if it's minus

- Signed: ***w*** bits → ***w+k*** bits
  - Given ***w***-bit signed integer ***x***
  - Convert it to ***w+k***-bit integer with same value
- Replicate the sign bit to the left
  - +2: 0000 0010 → 0000 0000 0000 0010
  - -2: 1111 1110 → 1111 1111 1111 1110

unsigned는 상관없으니 0으로 채운다
type에 따라서 다르게 적용



- In RISC-V instruction set
  - `lb`: sign-extend loaded byte
  - `lbu`: zero-extend loaded byte

Manipulating Integers

Bit-Level Operations in C

- Operations &, |, ~, ^ available in C
  - Apply to any "integral" data type
    - `(unsigned) long, int, short, char`
  - View arguments as bit vectors
  - Arguments applied bit-wise
- Examples (char data type)

`~0x41` → `0xBE`	~01000001_2 → 10111110_2
`~0x00` → `0xFF`	~00000000_2 → 11111111_2
`0x69 & 0x55` → `0x41`	01101001_2 & 01010101_2 → 01000001_2
`0x69 | 0x55` → `0x7D`	01101001_2 & 01010101_2 → 01111101_2
`0x69 ^ 0x55` → `0x3C`	01101001_2 & 01010101_2 → 00111100_2

Logic Operations in C

- &&, ||, !
  - View 0 as "False", anything nonzero as "True"
  - Always return 0 or 1
  - Early termination
- Examples (char data type)

`!0x41 → 0x00`	
`!0x00 → 0x01`	
`!!0x41 → 0x01`	
`0x69 && 0x55 → 0x01`	
`0x69 || 0x55 → 0x01`	
`if (p && *p) …`	`// avoids null pointer access`

Shift Operations

- Left shift: ***x* << *y***
  - Shift bit-vector ***x*** left ***y*** positions
    - Throw away extra bits on left
    - Fill with ***0***'s on right
- Right shift: ***x* >> *y***
  - Shift bit-vector ***x*** right ***y*** positions
    - Throw away extra bits on right
  - Logical shift: fill with ***0***'s on left
  - Arithmetic shift: replicate MSB on right
    - Useful with two's complement integer representation
- Undefined if ***y*** < ***0*** or ***y*** ≥ word size

Argument x	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument x	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

Addition

- Example: 7 + 6

		(0)		*(0)*		*(1)*		*(1)*		*(0)*		*(Carries)*
. . .		0		0		0		1		1		1
. . .		0		0		0		1		1		0
. . .	(0)	0	(0)	0	(0)	1	(1)	1	(1)	0	(0)	1

- Overflow if result out of range
  - Adding +ve and –ve operands: No overflow
  - Adding two +ve operands: Overflow if result sign is 1
  - Adding two –ve operands: Overflow if result sign is 0

Addition: Signed vs. Unsigned

- Signed addition in C
  - Ignores carry output
  - The low-order ***w*** bits are identical to unsigned addition

Examples for 3-bit integer

Mode	x	y	x + y	Truncated x + y
Unsigned	4 [100]	3 [011]	7 [0111]	7 [111]
Two's comp.	-4 [100]	3 [011]	-1 [1111]	-1 [111]
Unsigned	4 [100]	7 [111]	11 [1011]	3 [011]
Two's comp.	-4 [100]	-1 [111]	-5 [1011]	3 [011]
Unsigned	3 [011]	3 [011]	6 [0110]	6 [110]
Two's comp.	3 [011]	3 [011]	6 [0110]	-2 [110]

Subtraction

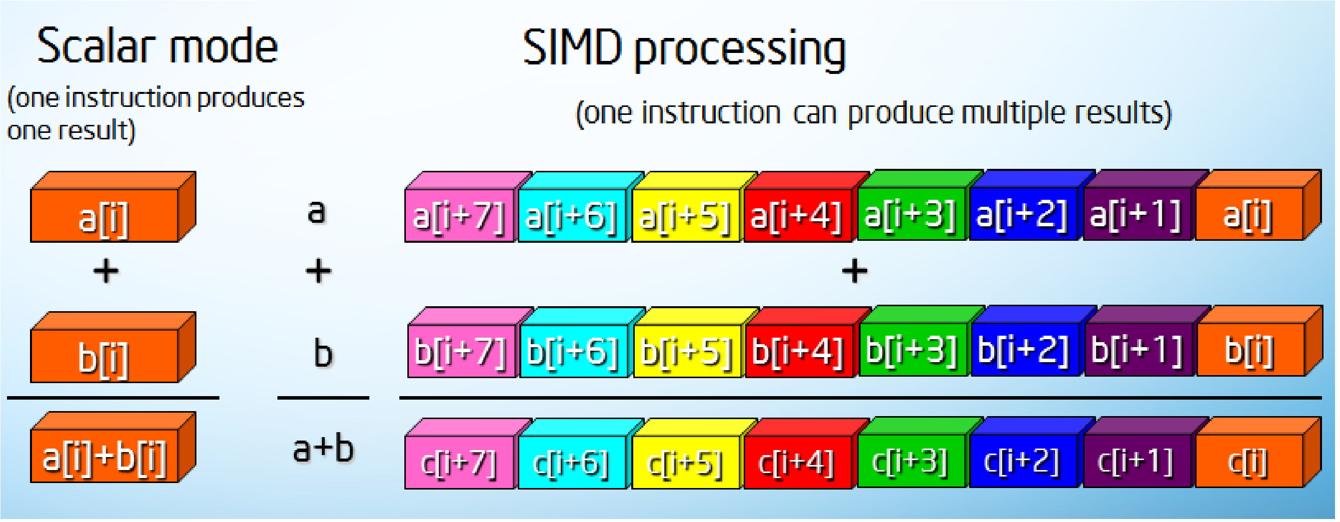
- Add negation of second operand
- Example: 7 – 6 = 7 + (–6)

```
+7:  0000 0000 … 0000 0111
-6:  1111 1111 … 1111 1010
--------------------------
+1:  0000 0000 … 0000 0001
```

- Overflow if result out of range
  - Subtracting two +ve or two –ve operands: No overflow
  - Subtracting +ve from –ve operand: Overflow if result sign is 0
  - Subtracting –ve from +ve operand: Overflow if result sign is 1

Arithmetic for Multimedia

- Graphics and media processing operates on vectors of 8-bit/16-bit data
  - Use 64-bit adder, with partitioned carry chain
  - Operate on 8x8-bit, 4x16-bit, or 2x32-bit vectors
  - SIMD (Single-Instruction, Multiple-Data)

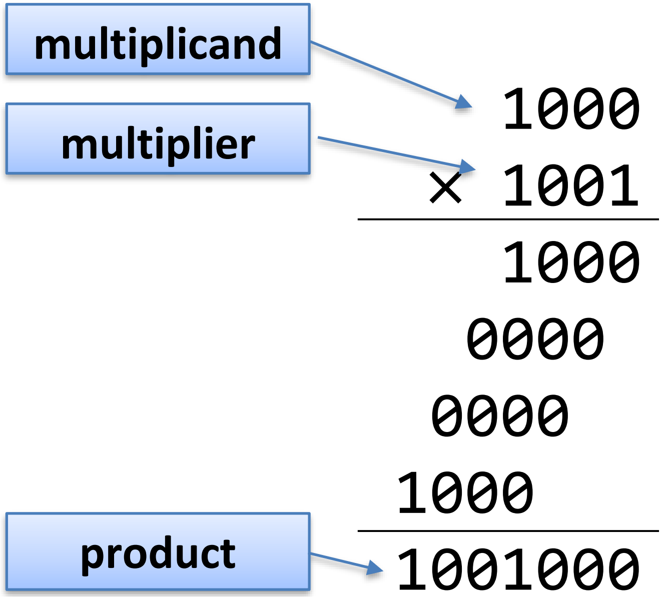


- Saturating operations
  - On overflow, result is largest representable value
  - e.g., clipping in audio, saturation in video

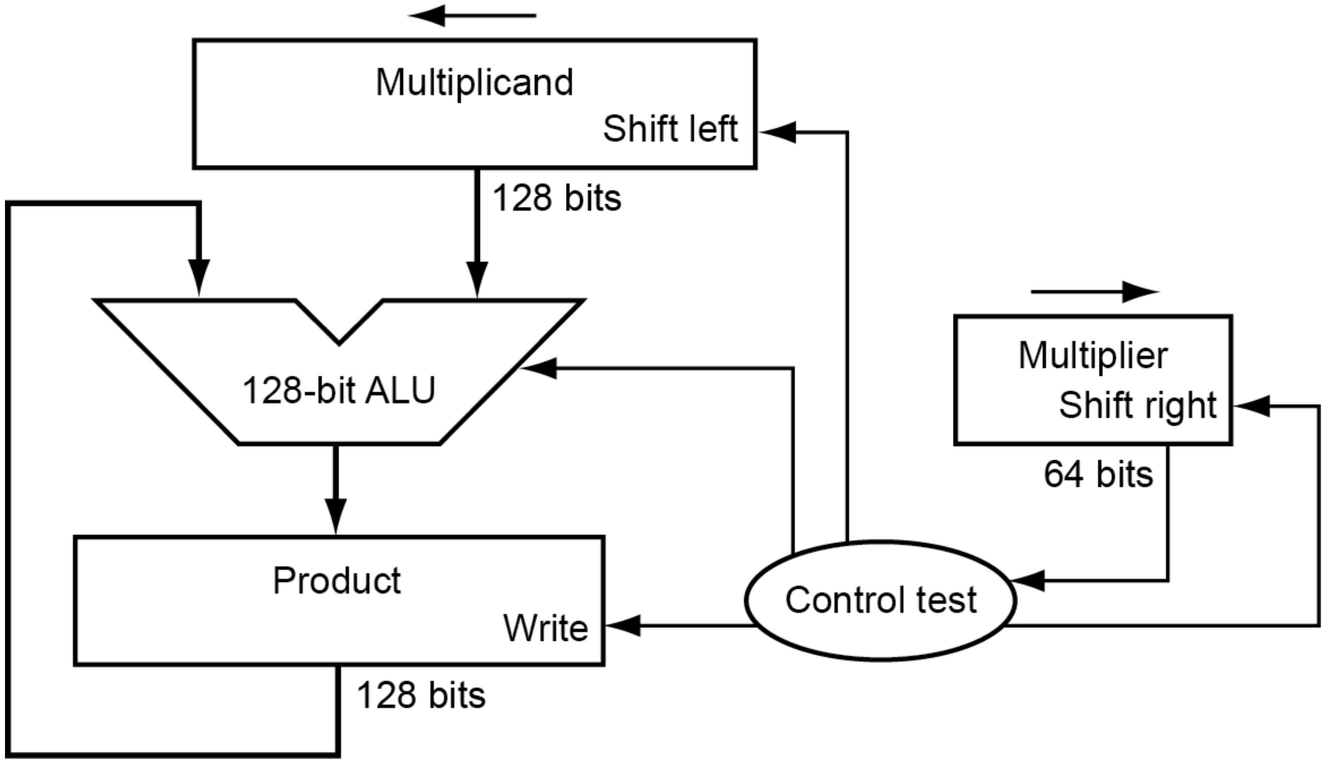
Source: https://software.intel.com/en-us/blogs/2012/01/31/vectorization-find-out-what-it-is-find-out-more

Multiplication

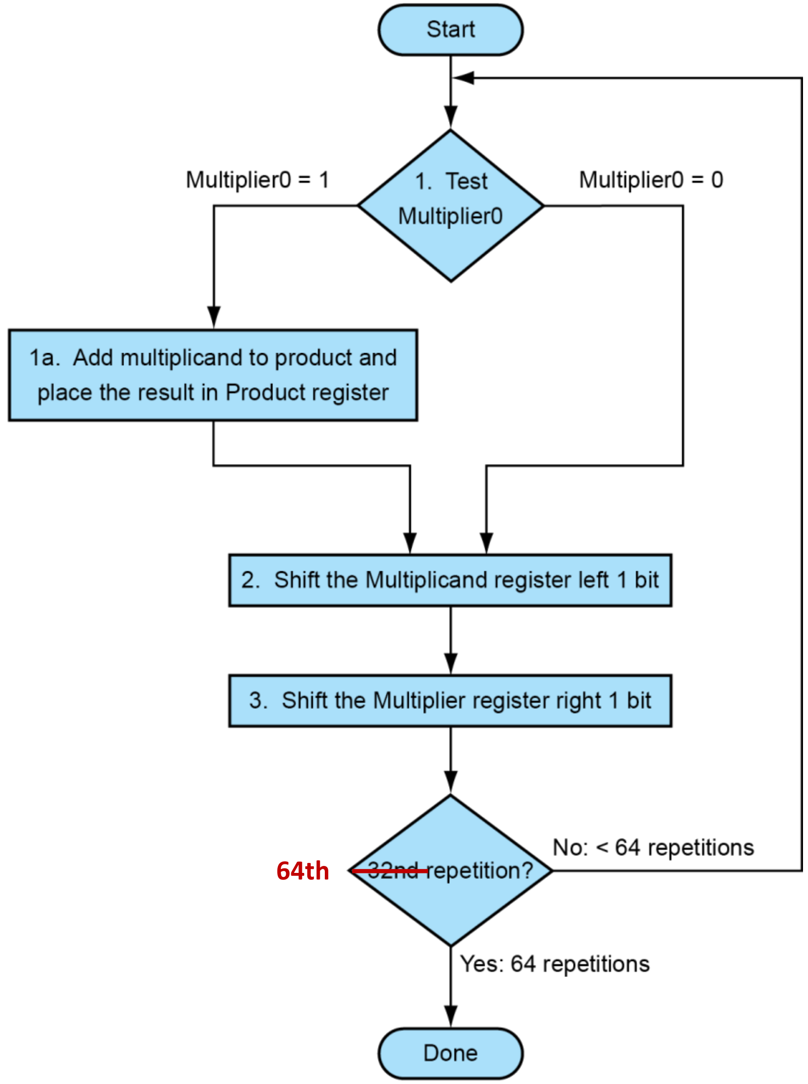
- Long-multiplication approach

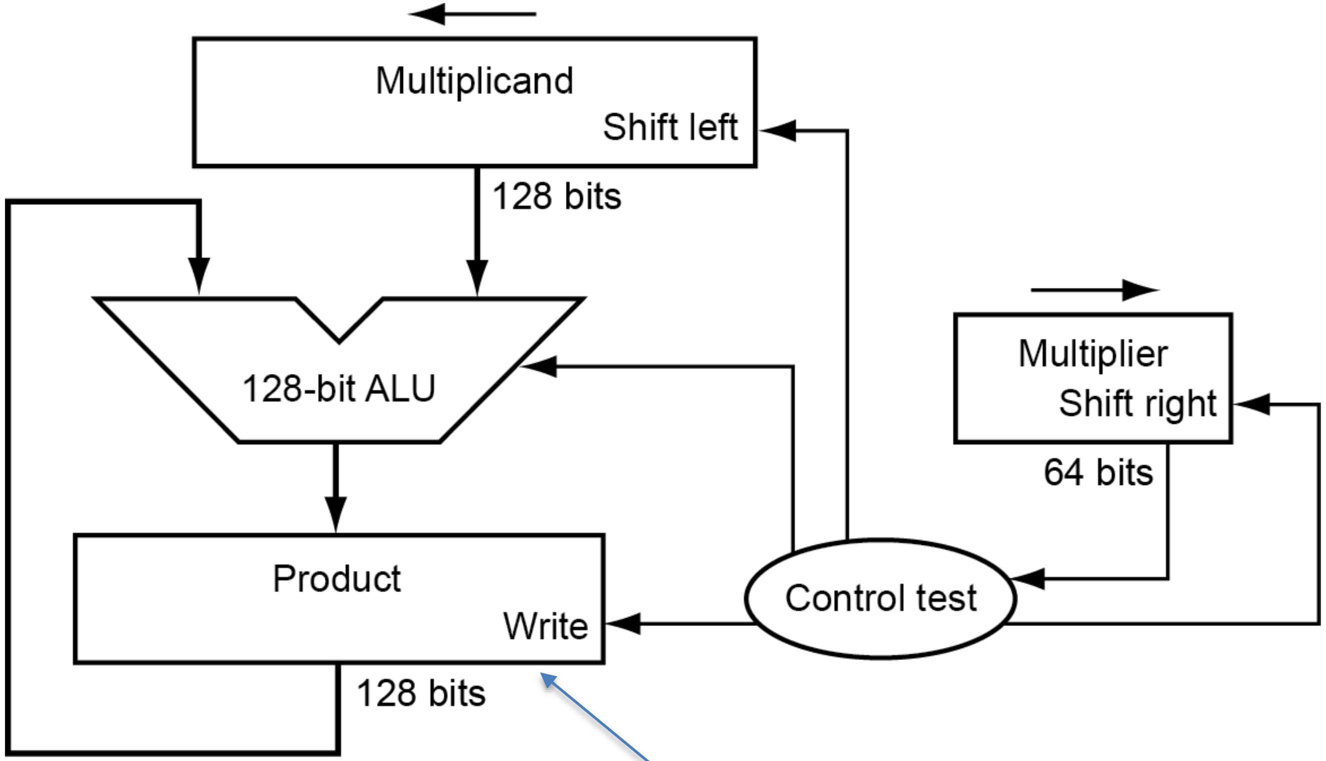


Length of product is the sum of operand lengths



Multiplication Hardware

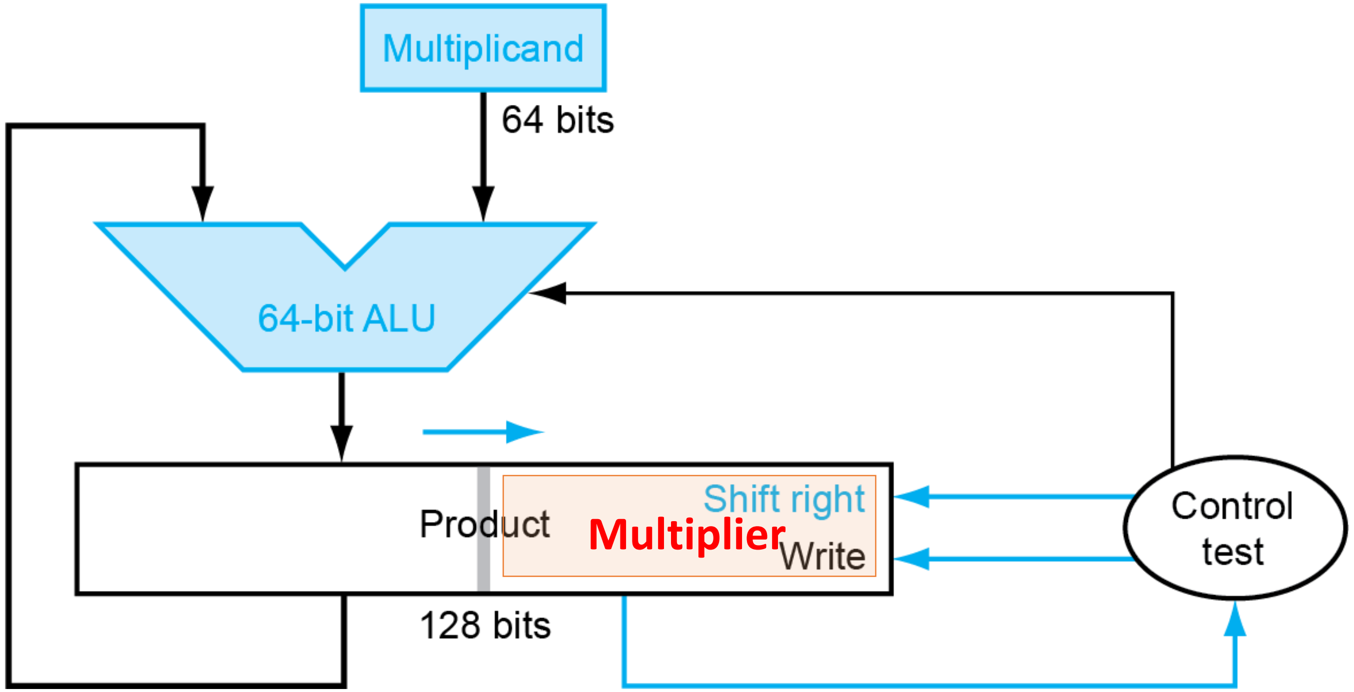




Initially 0

Optimized Multiplier

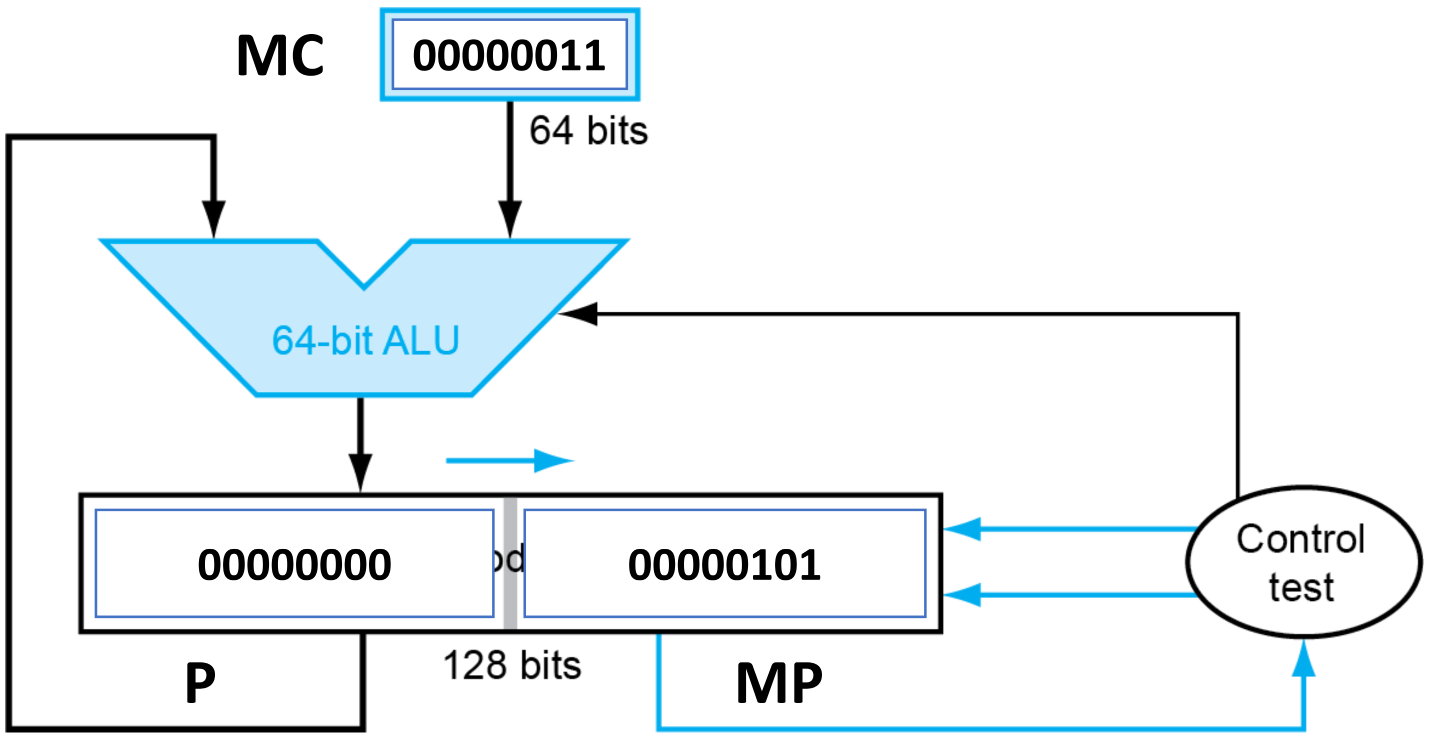
- Perform steps in parallel: add / shift



- One cycle per partial-product addition
  - That's ok, if frequency of multiplication is low

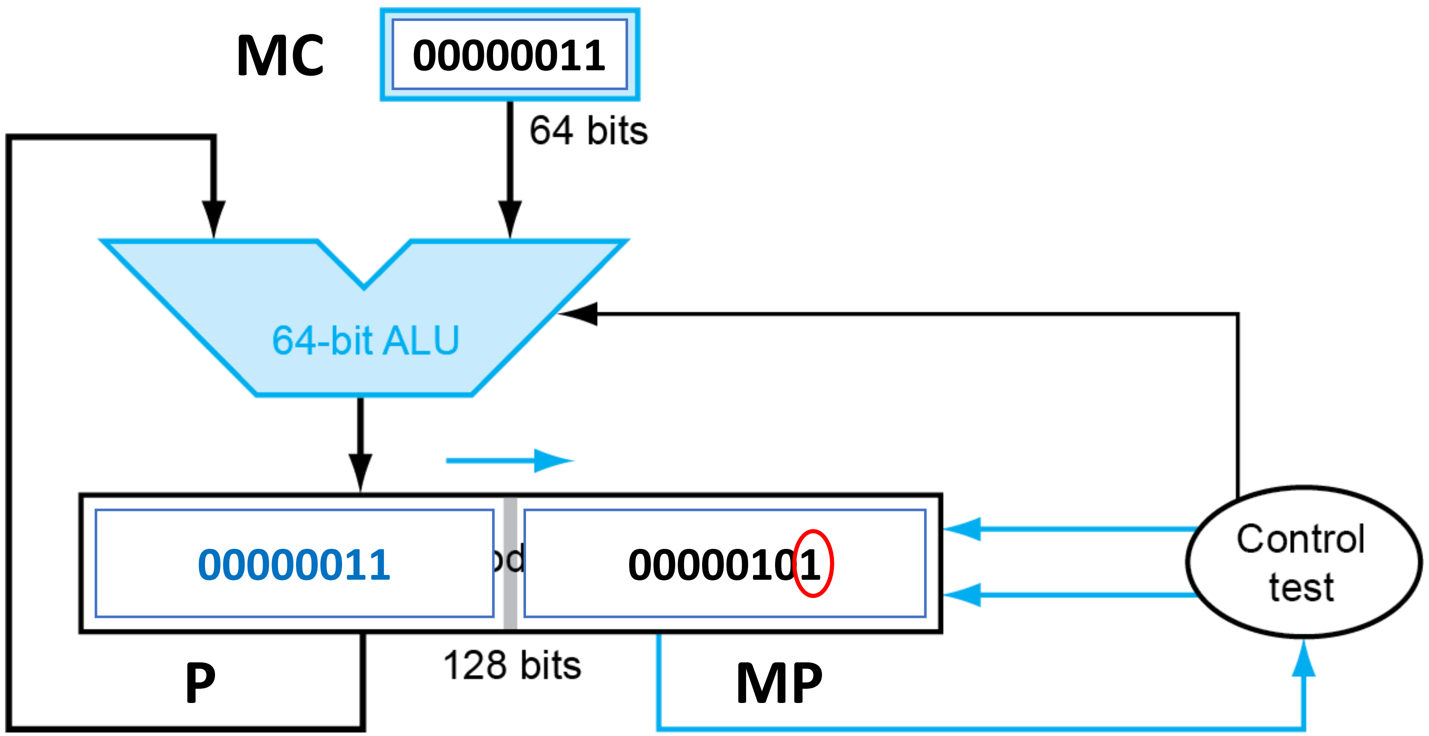
Multiplication Example (1)

- $3_{10} \times 5_{10} = 00000011_2 \times 00000101_2$



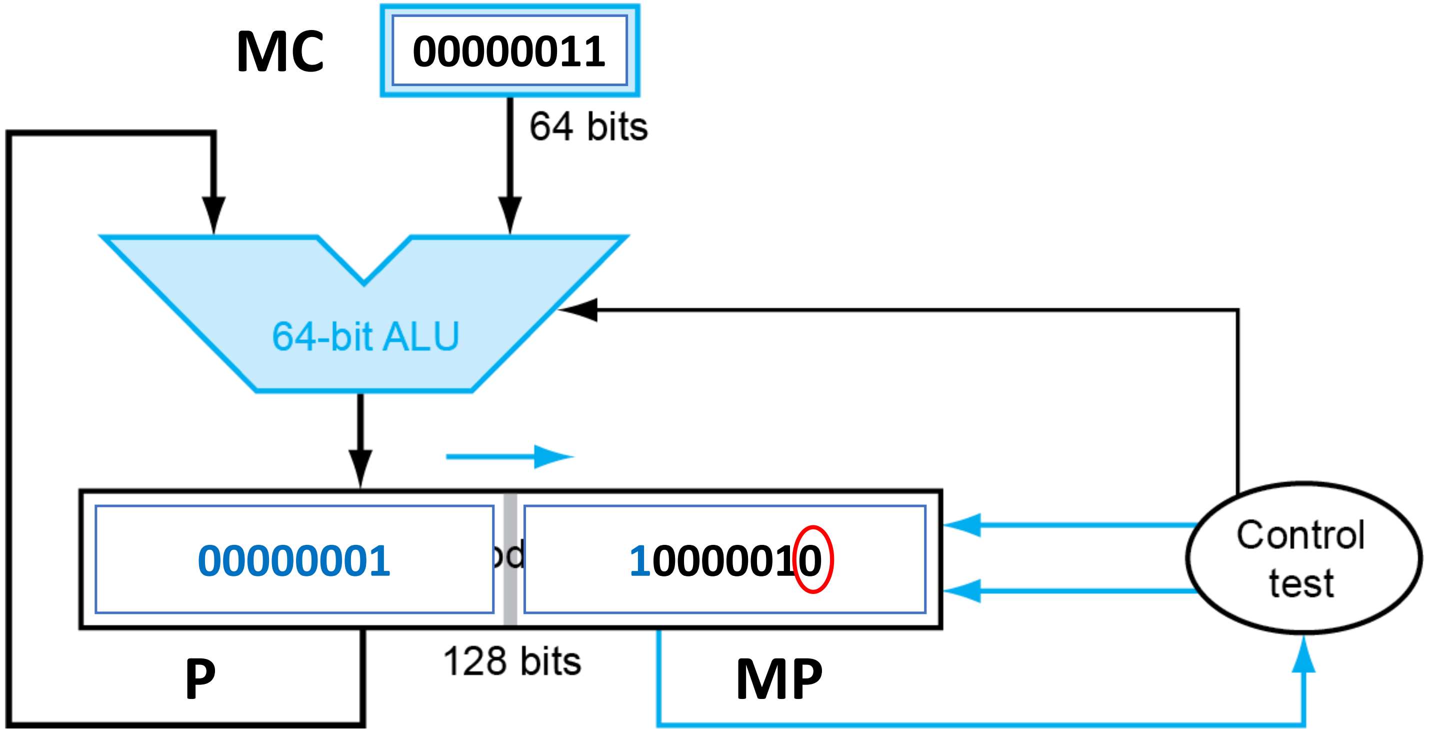
Multiplication Example (2)

- MP_0 = 1: Add MC to P, shift right P and MP by 1 bit



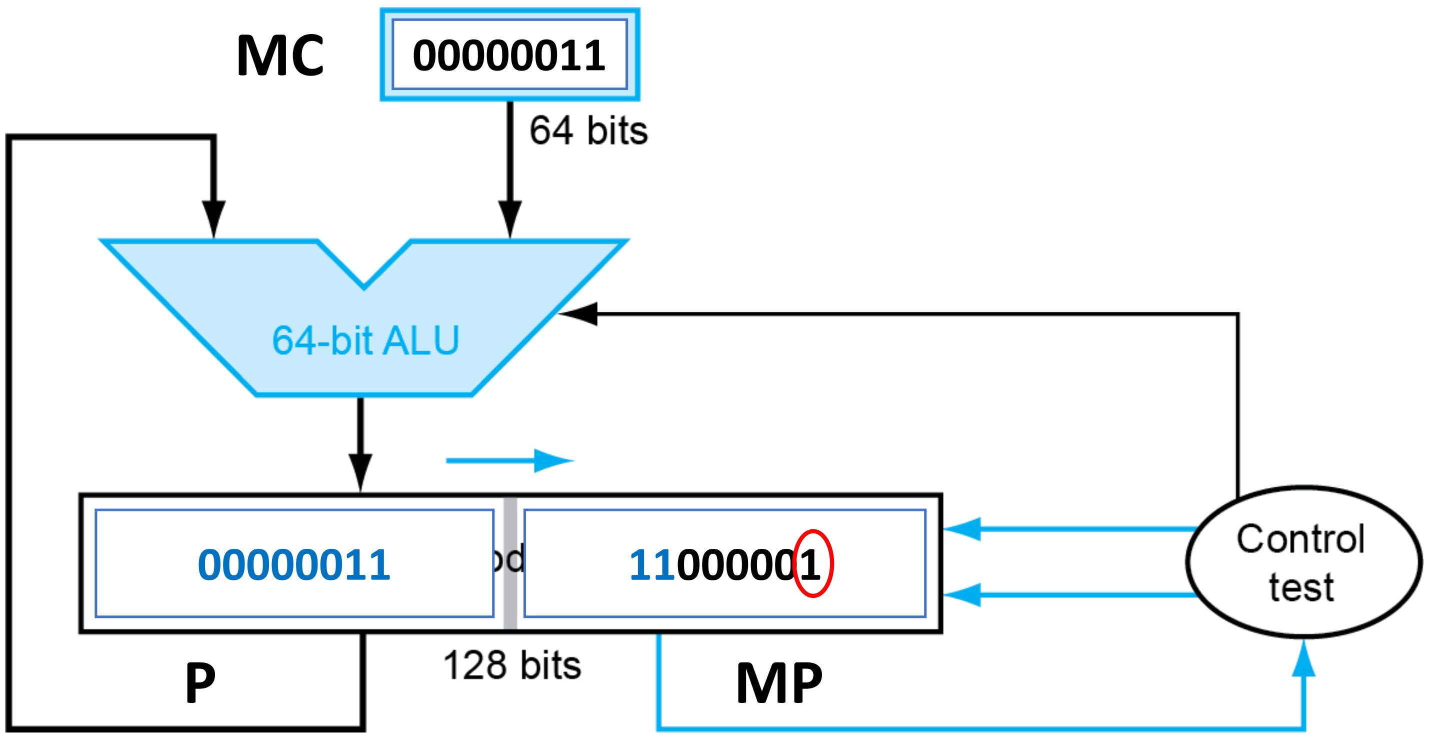
Multiplication Example (3)

- MP_0 = 0: Shift right P and MP by 1 bit



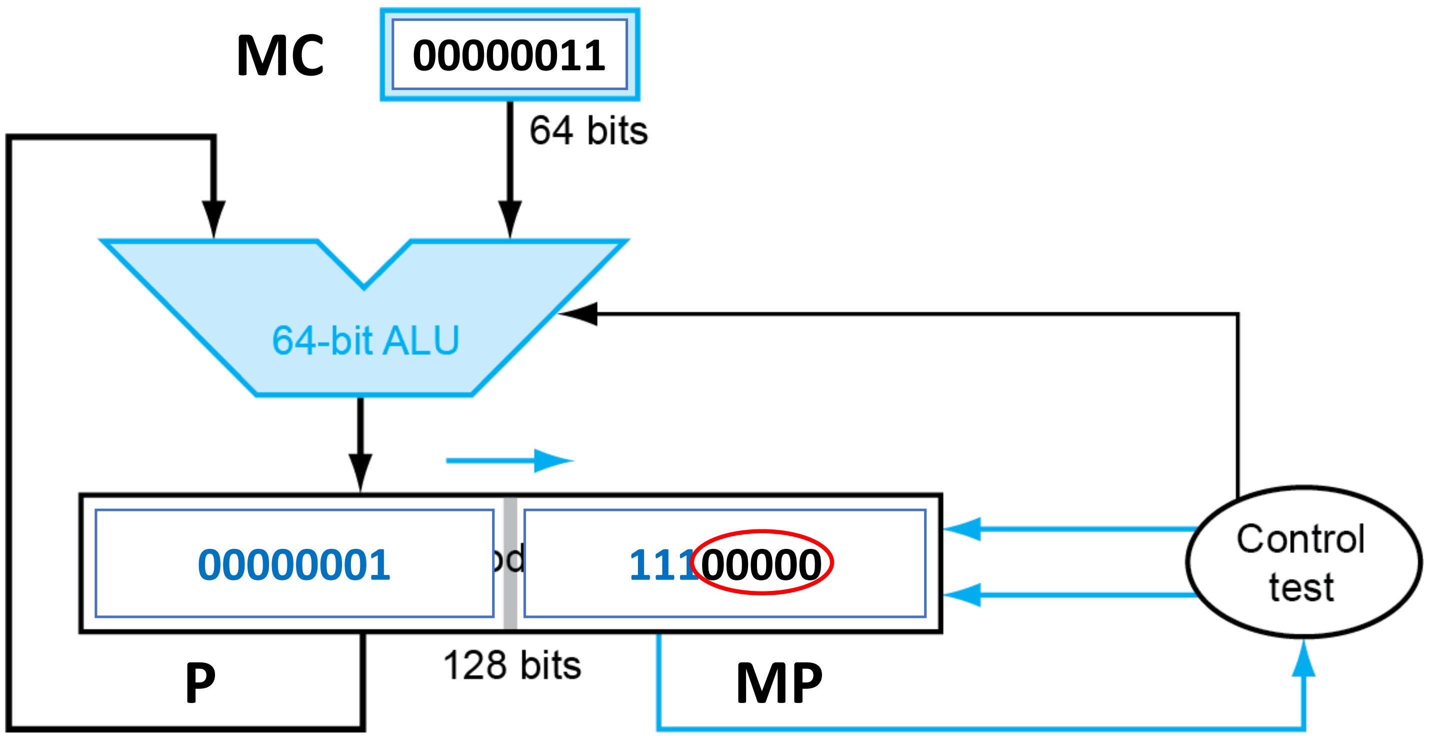
Multiplication Example (4)

- MP_0 = 1: Add MC to P, shift right P and MP by 1 bit



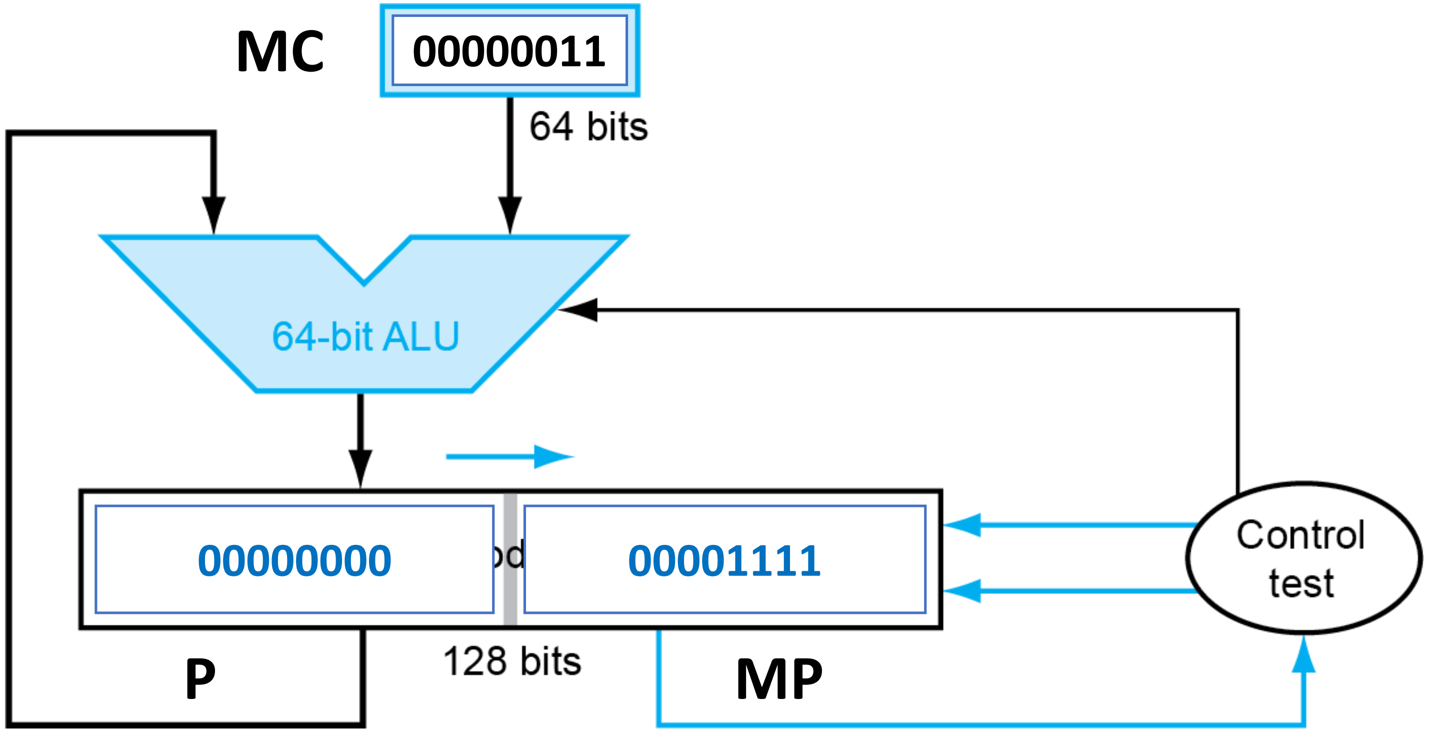
Multiplication Example (5)

- $MP_0 = 0$ (5 times): Shift right P and MP by 1 bit (5 times)



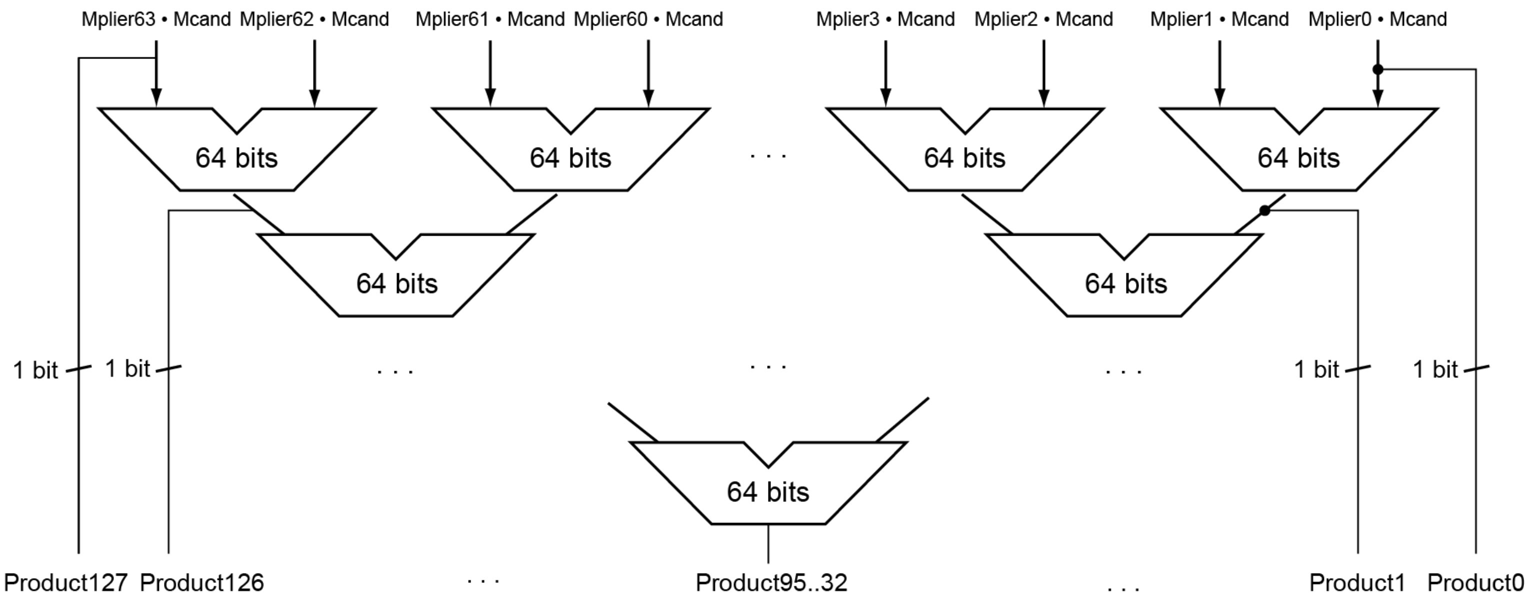
Multiplication Example (6)

- Final product available in P + MP



Faster Multiplier

- Uses multiple adders
  - Cost/performance tradeoff



- Can be pipelined
  - Several multiplication performed in parallel

Multiplication: Signed vs. Unsigned

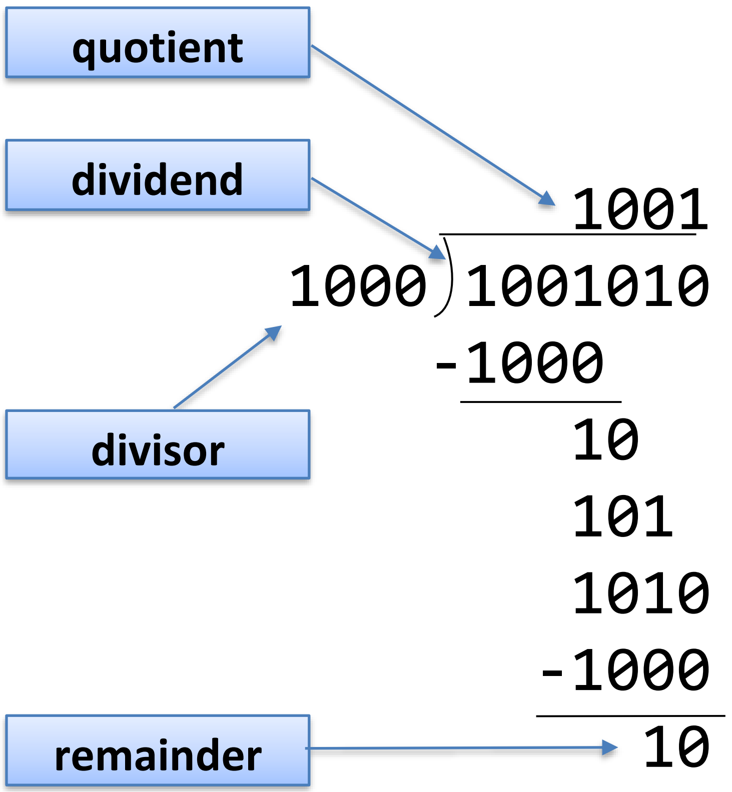
- Signed multiplication in C
  - Ignores high order ***w*** bits
  - The low-order ***w*** bits are identical to unsigned multiplication

Examples for 3-bit integer

Mode	x	y	x · y	Truncated x · y
Unsigned	5 [101]	3 [011]	15 [001111]	7 [111]
Two's comp.	-3 [101]	3 [011]	-9 [110111]	-1 [111]
Unsigned	4 [100]	7 [111]	28 [011100]	4 [100]
Two's comp.	-4 [100]	-1 [111]	4 [000100]	-4 [100]
Unsigned	3 [011]	3 [011]	9 [001001]	1 [001]
Two's comp.	3 [011]	3 [011]	9 [001001]	1 [001]

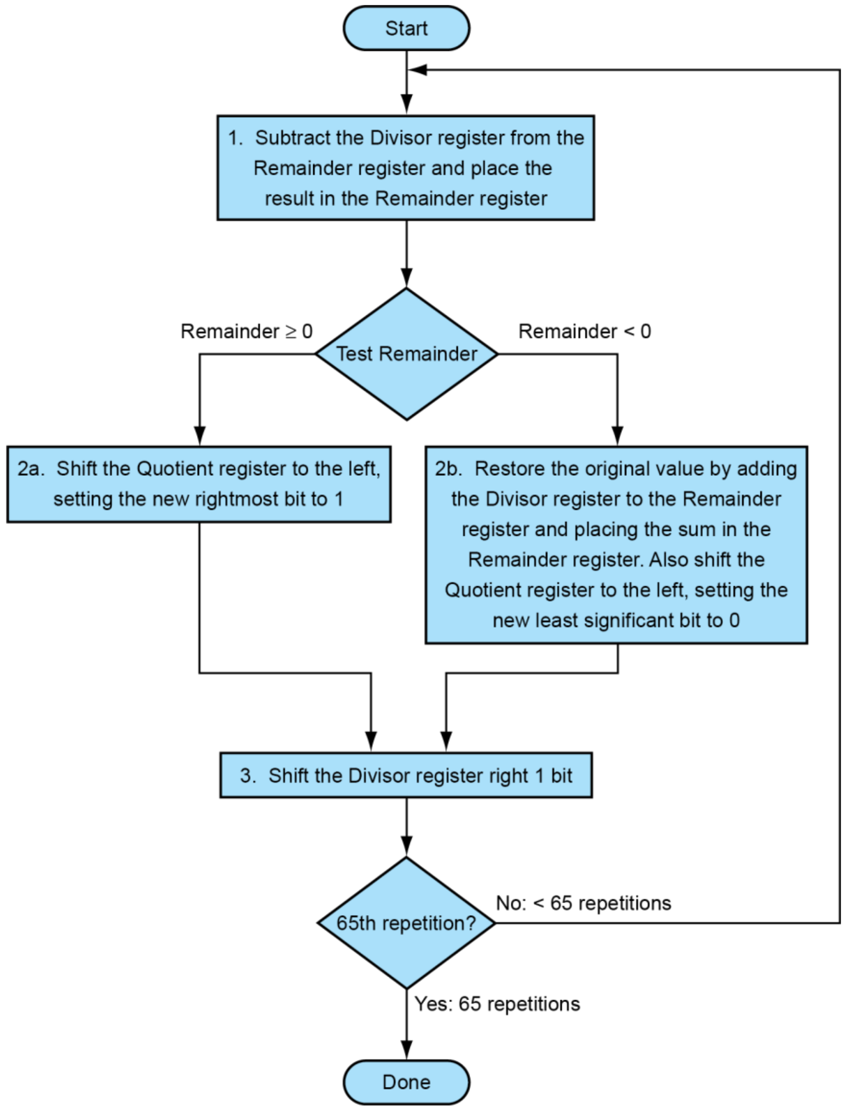
Division

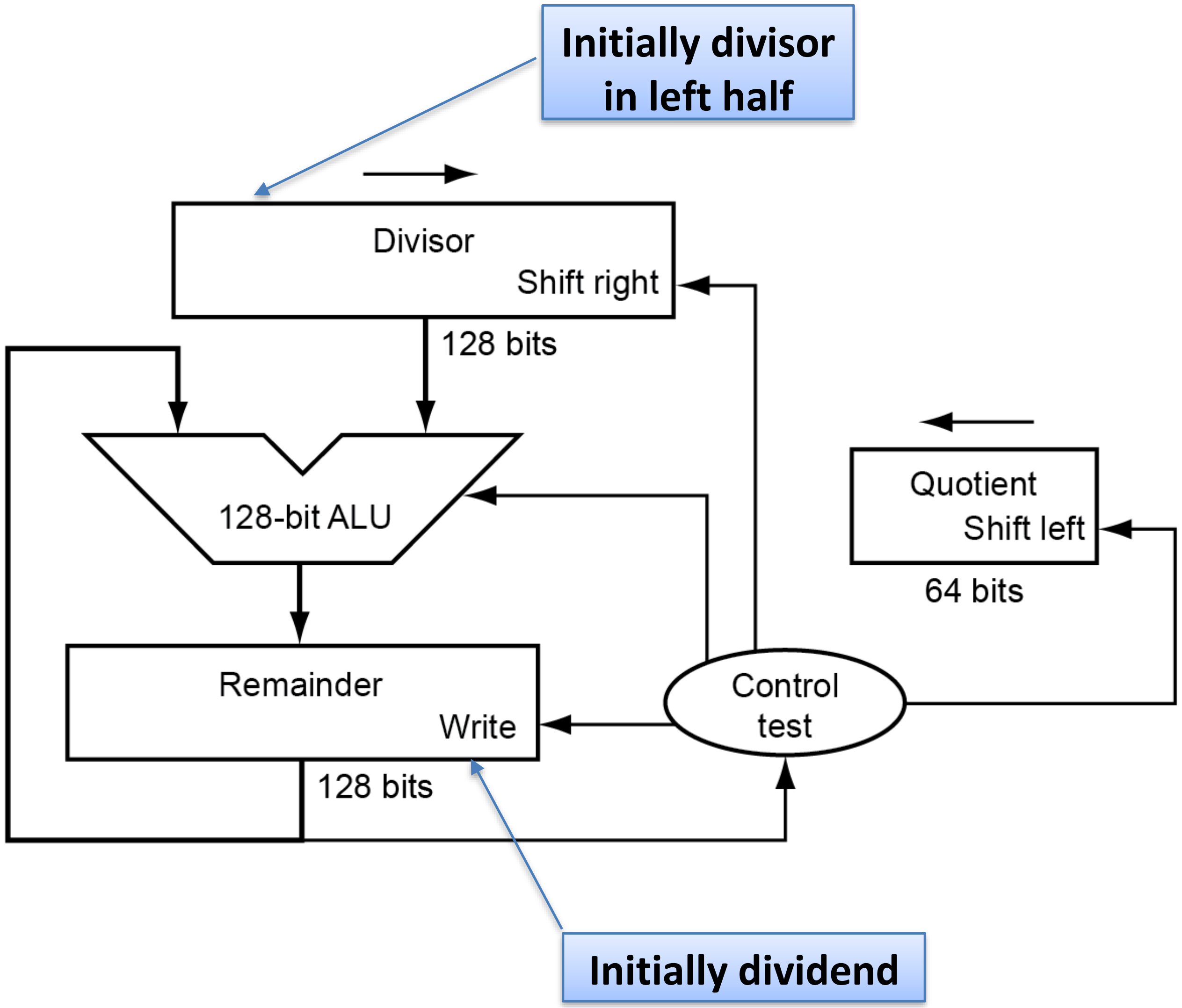
- Check for 0 divisor
- If divisor ≤ dividend bits:
  1 bit in quotient, subtract
- Otherwise: 0 bit in quotient, bring down next dividend bit
- Restoring division
  - Do the subtract, and if remainder goes < 0, add divisor back



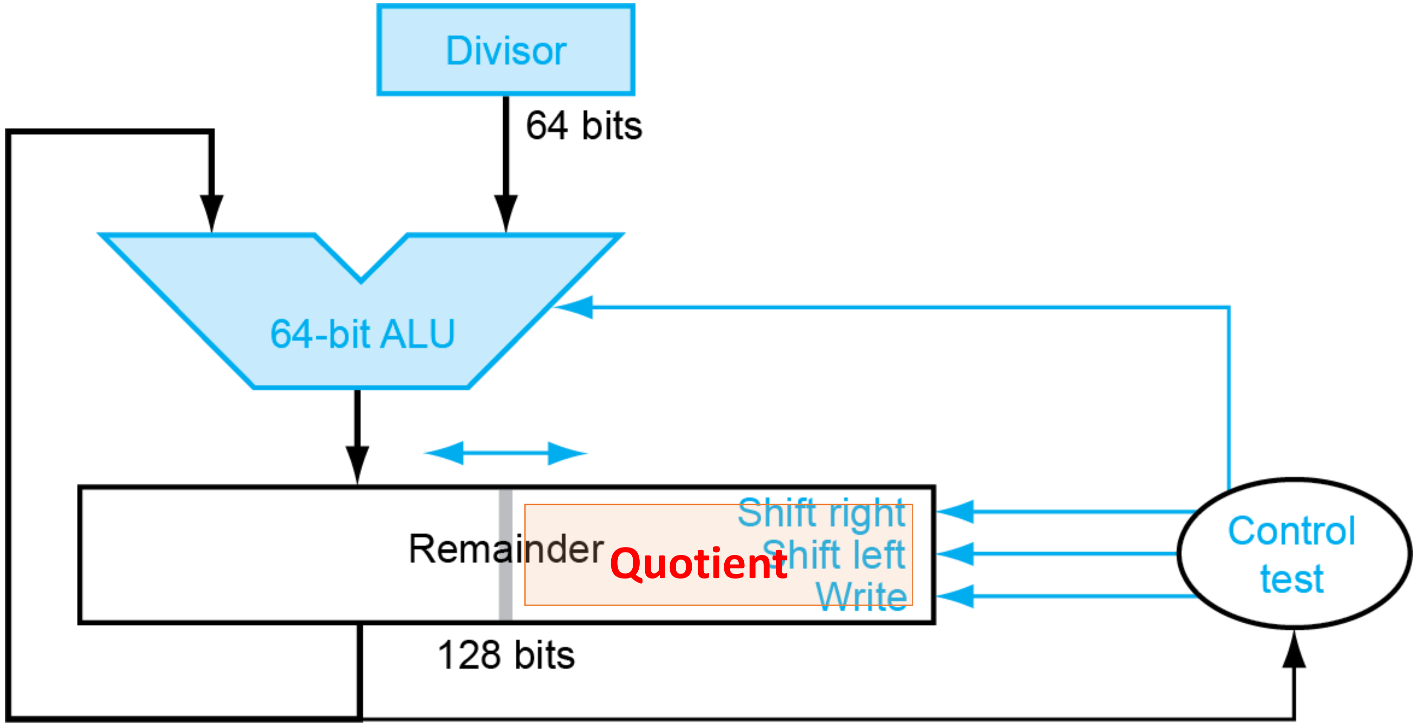
***n*-bit operands yield *n*-bit quotient and remainder**

Division Hardware





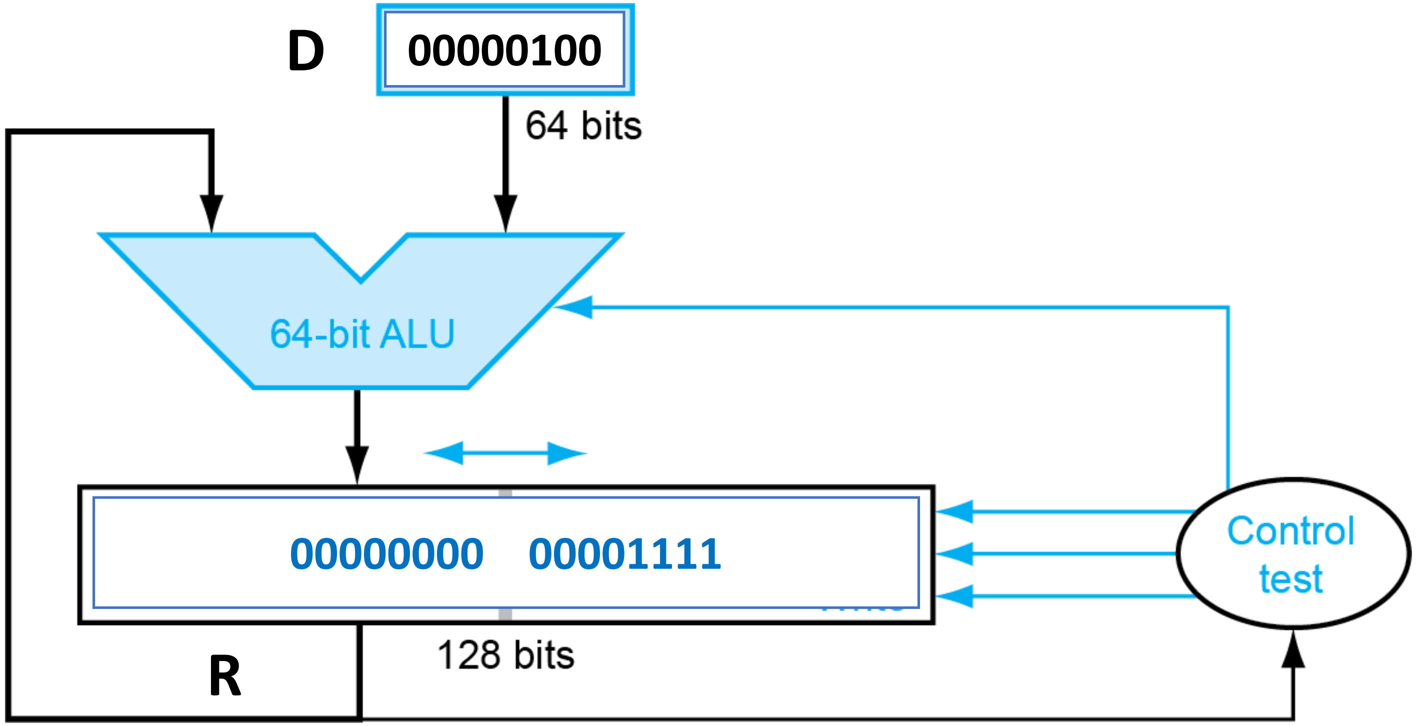
Optimized Divider



- One cycle per partial-remainder subtraction
- Looks a lot like a multiplier!
  - Same hardware can be used for both

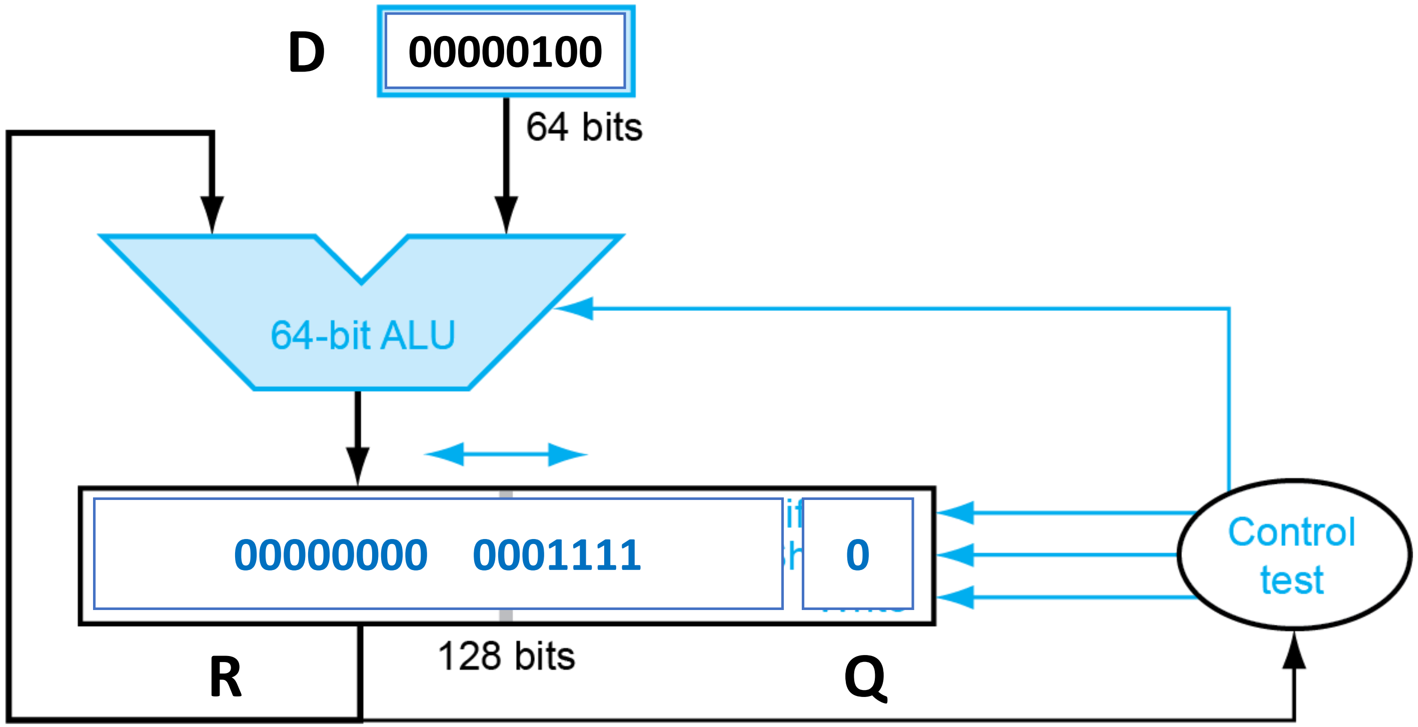
Division Example (1)

- $15_{10} / 4_{10} = 00001111_2 / 00000100_2$



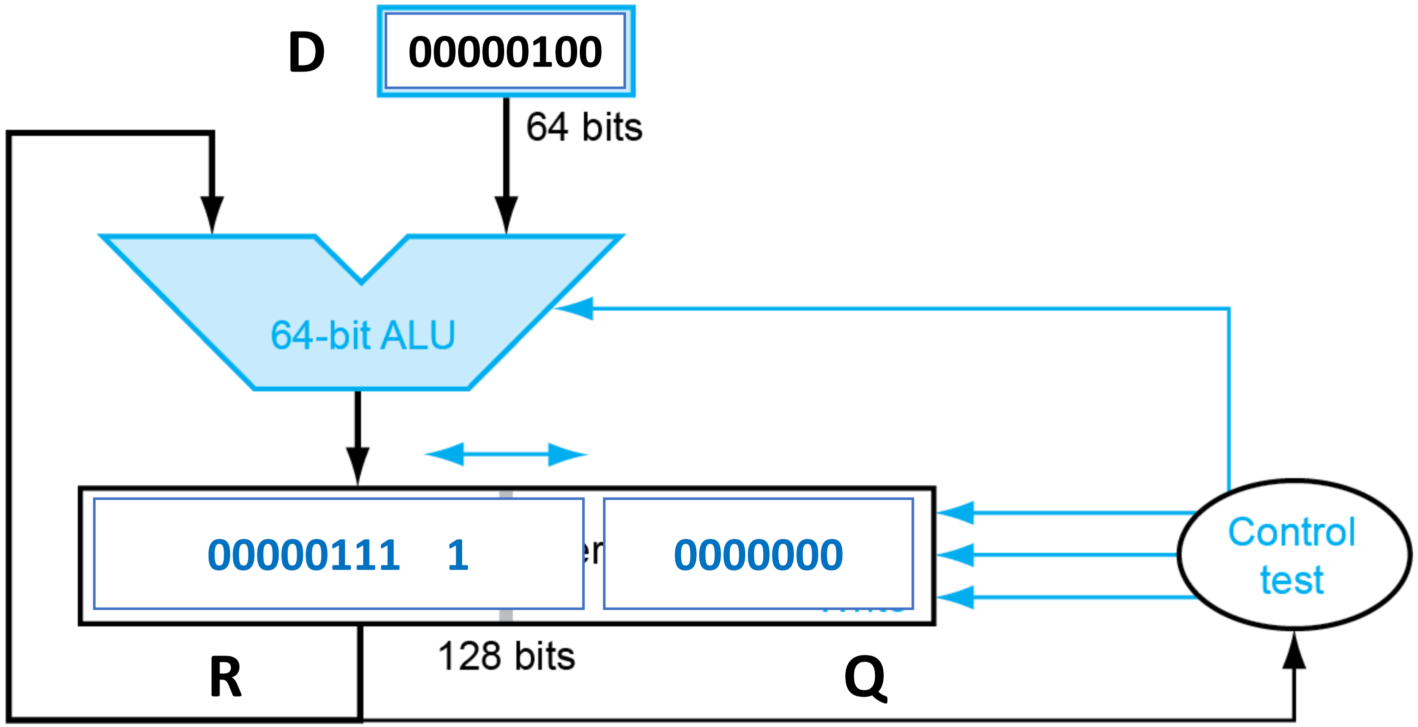
Division Example (2)

- R – D < 0: Shift left R and Q by 1 bit, $Q_0 = 0$



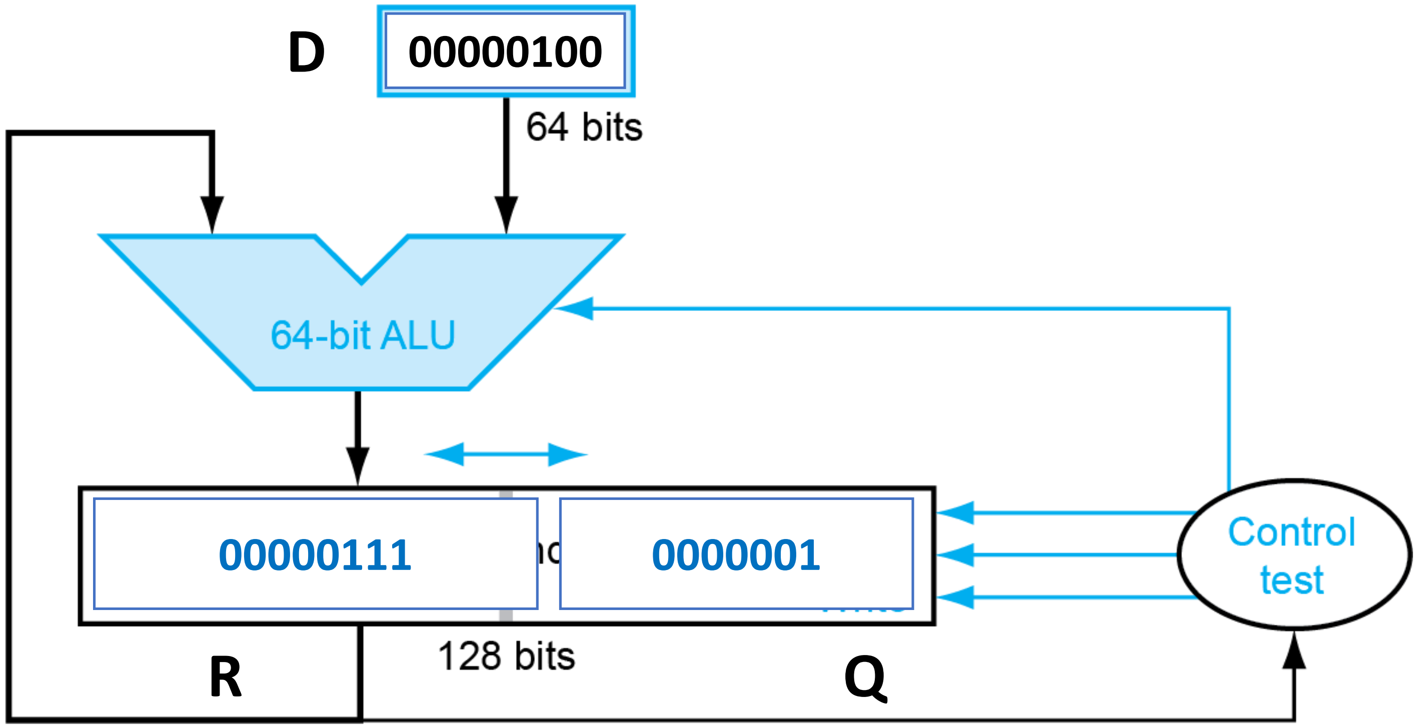
Division Example (3)

- R – D < 0: Shift left R and Q by 1 bit, $Q_0 = 0$ (6 times)



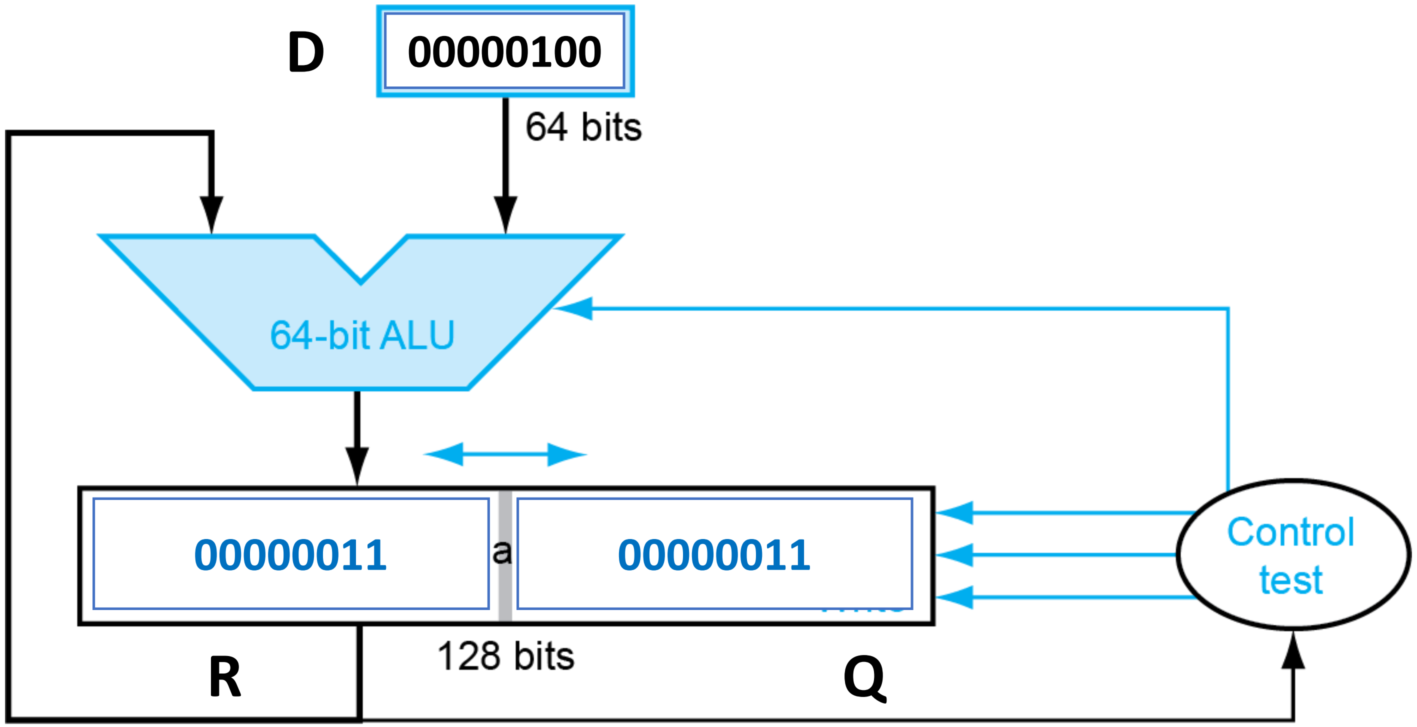
Division Example (4)

- $R - D >= 0$: $R = R - D$, shift left R and Q by 1 bit, $Q_0 = 1$



Division Example (5)

- R – D >= 0: R = R – D, shift left Q by 1 bit, Q_0 = 1



More on Division

- Signed division
  - Divide using absolute values
  - Adjust sign of quotient and remainder as required
    - (e.g.) 7 / 2: Q = 3, R = 1
      (-7) / 2: Q = -3, R = -1
      7 / (-2): Q = -3, R = 1
      (-7) / (-2): Q = 3, R = -1

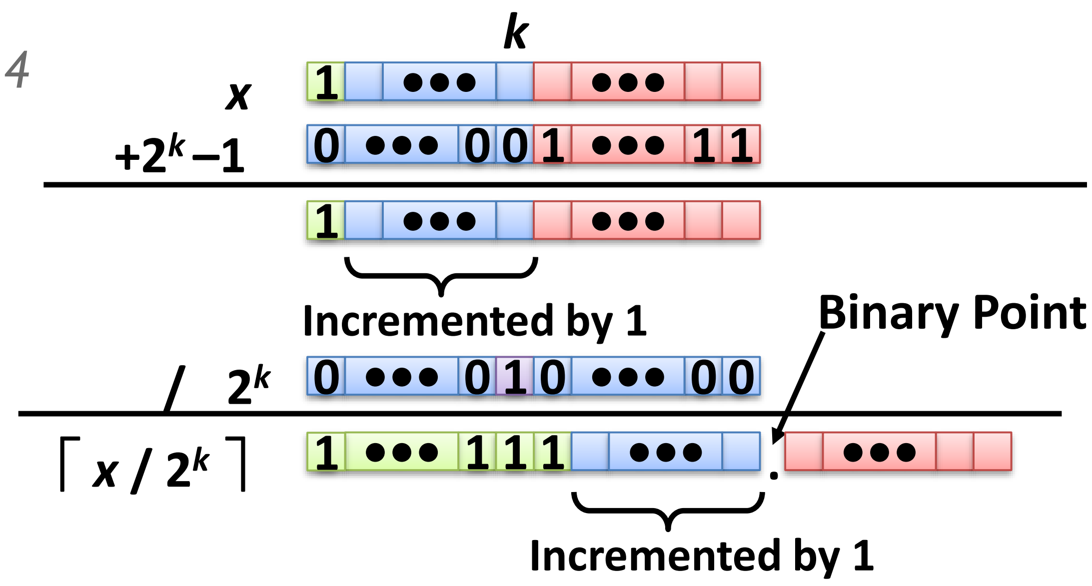
$$-\left(\frac{x}{y}\right) == \frac{(-x)}{y}$$

$$\boldsymbol{Dividend} = \boldsymbol{Quotient} \times \boldsymbol{Divisor} + \boldsymbol{Remainder}$$

- Faster division
  - Can’t use parallel hardware as in multiplier – Subtraction is conditional on sign of remainder
  - Faster dividers (e.g. SRT division) generate multiple quotient bits per step, but still require multiple steps

Shift for Multiplication/Division

- Power-of-2 multiplication with left shift
  - ***u*** **<<** ***k*** gives ***u*** ***** $\mathbf{2^{k}}$
    - e.g. *u << 3 == u * 8, (u << 5) – (u << 3) == u * 24*
  - Both signed and unsigned
  - Faster than multiply
- Power-of-2 division with right shift
  - ***u*** **>>** ***k*** gives $\lfloor \mathbf{u / 2^{k}} \rfloor$
  - Only for unsigned integers
    - e.g. $10_{10} / 4_{10} = 2_{10}$, $00001010_2 >> 2 = 00000010_2 = 2_{10}$
      $-10_{10} / 4_{10} = -2_{10}$, $11110110_2 >> 2 = 11111101_2 = -3_{10}$
  - For negative numbers, should be computed as (***u*** + (1 << ***k***) – 1) >> ***k***
    - e.g. $11110110_2 + 00000011_2 = 11111001_2$, $11111001_2 >> 2 = 11111110_2 = -2_{10}$



Summary

- Two's complement for representing negative numbers
- Sign extension preserves the value
- Same hardware can be used for both signed and unsigned addition. This is also true for multiplication. But their overflow conditions are different.
- Remember?

```
int x = 50000;
printf (“%s\n”, (x*x >= 0)? “Yes” : “No”);
```

- Using shift right for power-of-2 division requires correction for negative numbers