1

HW # 3

1 $\begin{cases} \dot{x}_1 = -x_1 + 2x_2 \\ \dot{x}_2 = -2x_1 - x_2 \end{cases} \rightarrow \begin{array}{l} \dot{x} = -x+2y \\ \dot{y} = -2x-y \end{array}$ FP @ (0,0)

$J = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$

$(-1-\lambda)(-1-\lambda)+4=0$

$(-1-\lambda)^2+4=0$

$(-1-\lambda)^2=-4$

$-1-\lambda = \pm 2$

$-\lambda = \pm 2 + 1$

$\lambda = -(\pm 2+1)$

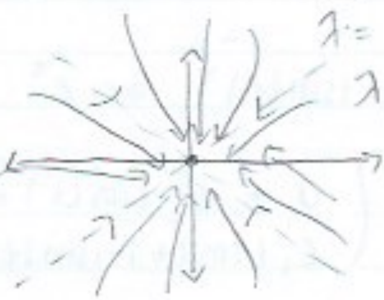
$\lambda_1 = -3 \quad \lambda_2 = 1$

λ_1: $\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_1}$

λ_2: $\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix} = v_2}$



$\begin{cases} \dot{x}_1 = x_1 + 4x_2 \\ \dot{x}_2 = x_2 \end{cases} \rightarrow \begin{array}{l} \dot{x} = x+4y \\ \dot{y} = y \end{array}$ FP @ (0,0)

$J = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$(1-\lambda)(1-\lambda)=0$

$(1-\lambda)^2=0$

$1-\lambda=0$

$\lambda=1$

$\begin{pmatrix} 1-\lambda & 4 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

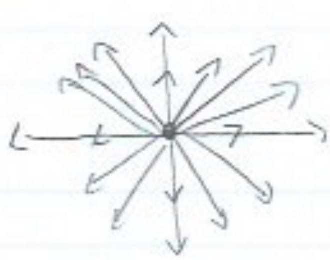
$\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0(v_1) + 4(v_2) = 0$

$0(v_1) + 0(v_2) = 0$

$\boxed{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



2 general solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

1) $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$x(t) = c_1 e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + c_2 e^{t} \\ c_1 e^{-3t} + 0 \end{pmatrix}$$

substitute

$(\cos(t) + i\sin(t))$ for e^{t} and $(\cos(t) - i\sin(t))$ for e^{-3t}

$$\begin{pmatrix} 0 + c_2 (\cos(t) + i\sin(t)) \\ c_1 (\cos(t) - i\sin(t)) + 0 \end{pmatrix}.$$

$$\boxed{x(t) = \begin{pmatrix} c_2 (\cos t) + i\sin(t)) \\ c_1 (\cos(t) - i\sin(t)) \end{pmatrix}}$$

2) $v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \lambda_1 = 1$

$$x(t) = c_1 e^{t} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x(t) = 0$$

3 $A = \begin{bmatrix} a & b & b \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$ 1 eignvalue
algebraic multiplicity 3
geometric multiplicity 1

let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ $\begin{pmatrix} a-\lambda & b & b \\ 0 & a-\lambda & b \\ 0 & 0 & a-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

determinent:

$$(a-\lambda)(a-\lambda) - b(0) + b(0)$$
$$(a-\lambda)^2 = 0$$
$$a-\lambda = 0$$
$$a = \lambda$$ substitute

$$\begin{pmatrix} 0 & b & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

eignvalue : 1
algebraic multiplicity 3
geometric multiplicity 1

$\dot{x} = Ax$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & b & b \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} a(x) + b(y) + b(z) \\ 0(x) + a(y) + b(z) \\ 0(x) + 0(y) + a(z) \end{matrix}$$

$$= \begin{bmatrix} ax + by + bz \\ ay + bz \\ az \end{bmatrix} = \begin{matrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{matrix}$$ let $z = c_3 e^{at}$

$\dot{y} = ay + b c_3 e^{at}$
$\dot{y} - ay = b c_3 e^{at}$
$\frac{d}{dt}(e^{-at} y) = b c_3$
$e^{-at} dy = c_2 b \, dt$

$dy = c_3 b e^{at} dt$
$\int dy = \int e^{at} c_3 b \, dt$
$y = c_3 b e^{at} + c_2 a e^{at}$

$$\boxed{\begin{aligned} z &= c_3 e^{at} \\ y &= c_3 b e^{at} + c_2 a e^{t} \\ x &= c_3 b e^{at^2} + c_2 a e^{t} + c_1 e^{at} \end{aligned}}$$

4 $\begin{cases} \dot{x} = y^2 - x \\ \dot{y} = x^2 - y \end{cases}$ Fixed pts $\begin{cases} y^2 - x = 0 \\ x^2 - y = 0 \end{cases} \rightarrow \begin{cases} x = y^2 \\ y = x^2 \end{cases}$ $\begin{matrix} x = 0,1 \\ y = 0,1 \end{matrix}$

FP @ $(0,0)$ $(1,1)$

Jacobian:

$J = \begin{pmatrix} -1 & 2y \\ 2x & -1 \end{pmatrix}$ $J_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $J_{(1,1)} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

Eigenvalues (vectors)

@ $(0,0)$ $(-1-\lambda)(-1-\lambda) = 0$

$(-1-\lambda)^2 = 0$

$-1-\lambda = 0$

$\lambda = 1$

$\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

@ $(1,1)$ $(-1-\lambda)(-1-\lambda) - 4 = 0$

$(-1-\lambda)^2 = 4$

$-1-\lambda = \pm 2$

$-\lambda = \pm 2 + 1$

$\lambda = -(\pm 2 + 1)$

$\lambda_1 = -3$ $\lambda_2 = 1$

λ_1: $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

λ_2: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

Nullclines:

$0 = y^2 - x$ $0 = x^2 - y$

$x = y^2$ $y = x^2$

$\underline{v}$ vertical $\underline{v}$ horizontal