Nonlinear Dynamical Systems and Chaos (AMS 114/214)

Homework 4 - Due Sunday November 26th

Exercise 1 Show that the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$ ($\varepsilon \in \mathbb{R}$) has a nonlinear center at the origin for $\varepsilon > 0$. If $\varepsilon < 0$ show that the trajectories near the origin are closed. What about the trajectories that are far from the origin? (Hint: the system is conservative)

Exercise 2 A smooth vector field on the phase plane is known to have exactly three limit cycles. Two of the cycles, say C_1 and C_2 lie inside a third cycle C_3. However, C_1 does not lie inside C_2, nor vice-versa.

a) Sketch the arrangements of the three cycles.

b) Show that there must be at least one fixed point in the region bounded by C_1, C_2 and C_3.

c) Sketch the possible trajectories inside C_1, C_2 and in the region between C_3, C_1 and C_2.

(Hint: Use index theory)

Exercise 3 Consider the equation $\ddot{x} + \mu(x^4 - 2)\dot{x} + x^5 = 0$, where $\mu \in \mathbb{R}$.

a) Show that if $\mu > 0$ then the system has a unique stable limit cycle surrounding the origin.

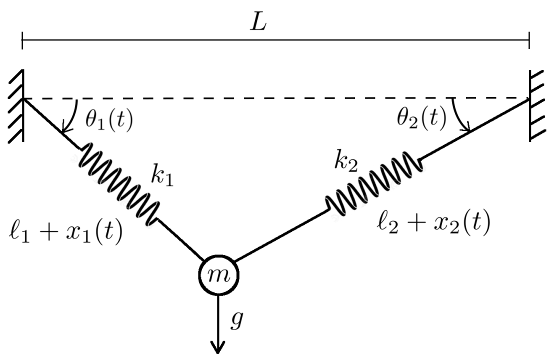
b) Does the system still have a limit cycle if $\mu < 0$? If so, is it stable or unstable?

Exercise 4 Show that the system

$$\begin{cases} \dot{r} = -r(1 - r^2)(4 - r^2) \\ \dot{\theta} = r^2 - 2 \end{cases}$$

where (r, θ) are polar coordinates, has a stable limit cycle at $r = 2$. (Hint: use the Poincaré-Bendixon theorem.)

Exercise 5 (AMS 214 Mandatory, AMS 114 Extra Credit) Consider the following conservative system



where ℓ_1 and ℓ_2 are the lengths of the two (linear) elastic springs at rest, while k_1 and k_2 are the elastic constants. Let $\theta_1(t)$ and $x_1(t)$ be the degrees of freedom (generalized coordinates) identifying the position of the mass m, and let g be the acceleration of gravity.

a) Determine the Lagrangian function and derive the Euler-Lagrange equations.

b) Determine the Hamiltonian function and derive Hamilton's equations.