HW # 4

1 Show that the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$ has a nonlinear center @ the origin

Let $y = \dot{x}$

so, $\dot{x} = y \to \dot{y} = -x - \varepsilon x^3$

Potential function:

$$V(x) = \frac{x^2}{2} + \frac{\varepsilon x^4}{4}$$

let $E(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4$

$$\frac{dE}{dt} = y\dot{y} + x\dot{x} + \varepsilon x^3 \dot{x}$$
$$= \dot{x}\ddot{x} + x\dot{x} + \varepsilon x^3 \dot{x}$$
$$= \dot{x}(\ddot{x} + x + \varepsilon x^3)$$
$$= 0$$

E is conserved quantity, so system is conservative.

E has hessian $\begin{pmatrix} 3\varepsilon x^2 + 2 & 0 \\ 0 & 1 \end{pmatrix}$ @ $(0,0)$: $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

2nd partial derivative $\to$ $(0,0)$ local minimum of E

Origin is isolated fixed pt of system, so origin is non linear system $\square$

What about other trajectories that are far from origin?
System is conservative

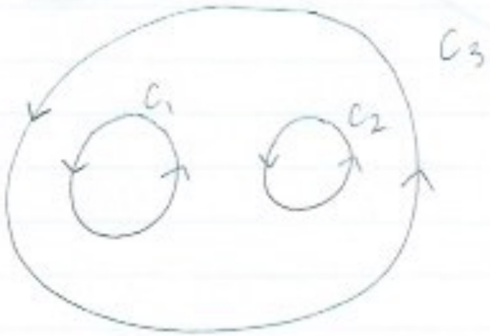
Because the system is conservative, other trajectories that are far from the origin are nonlinear as well

2 smooth vector field, 3 limit cycles

2 of the cycles, C_1 and C_2, lie inside a third cycle C_3

However, C_1 does not lie inside C_2 nor vice versa

a) Sketch arrangement for 3 cycles



b) Show that there must be at least one fixed pt in the region bounded by C_1, C_2, and C_3

assume there is no fixed pt in the bounded region.

Let f = underlying vector field

$C_3 = C_3' \rightarrow$ compressed around C_1 and C_2

$$I(f, C_3) = I(f, C_3')$$

Index theory - index is additive if curve is subdivided

$$I(f, C_3') = I(f, C_1) + I(f, C_2)$$

C_1 and C_2 are closed orbits so,

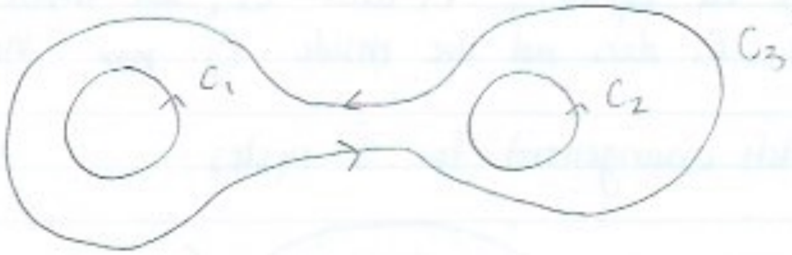
$$I(f, C_1) = 1$$
$$I(f, C_2) = 1$$

so, $I(f, C_3') = 2$

However, C_3 is closed orbit and hence $I(f, C_3)$ should be 1. CONTRADICTION!

So, there must be at least 1 fixed pt in the bounded region. □

c) Sketch trajectories



3 $\ddot{x} + \mu(x^4 - 2)\dot{x} + x^5 = 0$ where $\mu \in \mathbb{R}$

a) Show that if $\mu > 0$, then the system has a unique stable limit cycle

let $f(x) = \mu(x^4 - 2)$

$g(x) = x^5$

$g(-x^5) = -x^5$ $(= -g(x))$

$f(-x) = \mu((-x)^4 - 2)$
$= \mu(x^4 - 2)$
$= f(x)$

$F(x) = \int_0^x f(u)\,du$

$f(u) = \mu(u^4 - 2)$ substitute $\mu(u^4 - 2)$

$F(x) = \int_0^x \mu(u^4 - 2)\,du = \mu\left[\frac{u^5}{5} - 2u\right]_0^x$

$= \mu\left(\frac{x^5}{5} - 2x\right)$

$F(x) = 0 \rightarrow \mu\left(\frac{x^5}{5} - 2x\right) = 0$

$\rightarrow \frac{x^5}{5} - 2x = 0$

$= x\left(\frac{x^4}{5} - 2\right) = 0 = x(x^4 - 10) = 0$

$x(x^4 - 10) = 0$

$x(x^2 - \sqrt{10})(x^2 + \sqrt{10}) =$

$x = 0, \pm i(10)^{1/4}, \pm (10)^{1/4}$

$x = 10^{\frac{1}{4}}$

So, system has a unique stable limit cycle □

b) Does the system still have a limit cycle if $\mu < 0$? Stable or unstable?

let $f(x) = -\mu(x^4 - 2)$, $g(x) = x^5$

$g(x)$ is odd function

$$f(-x) = -\mu((-x)^4 - 2)$$
$$= -\mu(x^4 - 2)$$
$$= f(x) \quad \text{even function}$$

$$F(x) = \int_0^x f(u)\,du$$
$$f(u) = -\mu(u^4 - 2)$$
$$F(x) = \int_0^x -\mu(u^4 - 2)\,du$$
$$= -\mu\left[\frac{u^5}{5} - 2u\right]_0^x = -\mu\left(\frac{x^5}{5} - 2x\right)$$

$$F(x) = 0 \rightarrow -\mu\left(\frac{x^5}{5} - 2x\right) = 0 \rightarrow \frac{x^5}{5} - 2x = 0$$

$$\rightarrow x\left(\frac{x^4}{5} - 2\right) = 0 \Rightarrow x(x^4 - 10) = 0$$

$$x(x^4 - 10) = 0 \rightarrow x(x^2 \sqrt{10})(x^2 + \sqrt{10})$$
$$x = 0, \pm i(10)^{\frac{1}{4}}, \pm (10)^{\frac{1}{4}}$$

$$x = 10^{\frac{1}{4}}$$

So, the limit cycle of the system equation if $\mu < 0$ is stable $\square$

Show that

4 $\begin{cases} \dot{r} = -r(1-r^2)(4-r^2) \\ \dot{\theta} = r^2 - 2 \end{cases}$ has stable limit cycle at $r=2$

let $-r(1-r^2)(4-r^2) = 0 = \dot{r}$

$$-r(4 - 5r^2 + r^4) = 0$$

$$-4r + 5r^3 - r^5 = 0$$

	1	$-r^2$
4	4	$-4r^2$
$-r^2$	$-r^2$	r^4

let $r = 2$

$$-4(2) + 5(2)^3 - 2^5 = 0$$

$$-8 + 40 - 32$$

$$= 0 \checkmark$$

so $r = 2$

$$\dot{\theta} = r^2 - 2 = 0$$

$$r^2 = 2$$

$$r = \pm\sqrt{2}$$

There must be a limit cycle in $r_1 \le r \le r_2$

let $r_1 = \pm\sqrt{2}$ and $r_2 = 2$

so, $\pm\sqrt{2} \le r \le 2$

then for $r = 2$

$\pm\sqrt{2} \le 2 \le 2$ works!

$r=2$ is stable because when plugged into $\dot{r}$, $\dot{r} = 0$, thus making the limit stable. $\square$

Theorem: given a dynamical system, every non-empty compact set of orbit is either

- fixed pt
- periodic orbit
- connected set composed of a finite number of fixed pts together with homoclinic and heteroclinic orbits