AMS 114

HW#2

$r = -\left(x + \frac{1}{x}\right)$

Exercise 1

$\frac{dx}{dt} = \arctan(\mu x) - x^2 + x$

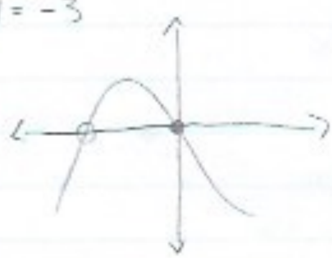
$\arctan(\mu x) - x^2 + x = 0$

$\arctan(\mu x) = x^2 - x$

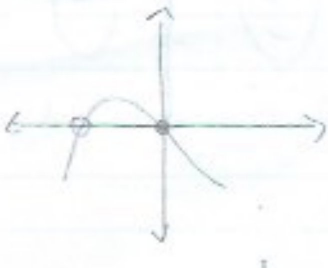
$\mu x = \tan(x^2 - x)$

$\mu = \frac{\tan(x^2 - x)}{x}$

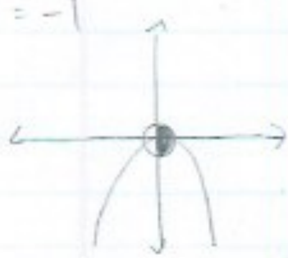
$\mu = -3$



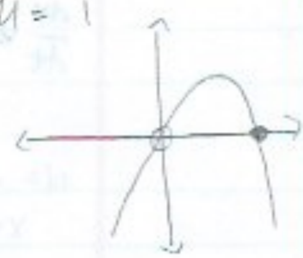
$\mu = -2$



$\mu = -1$

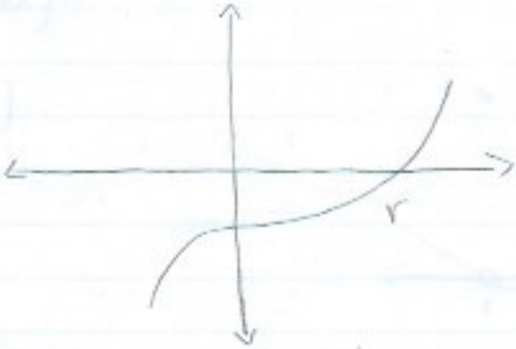


$\mu = 1$



$\mu = -1$ bifurcation occurs ; transcritical bifurcation

bifurcation graph:



$\frac{dx}{dt} = -x\cos(x) + \mu x$

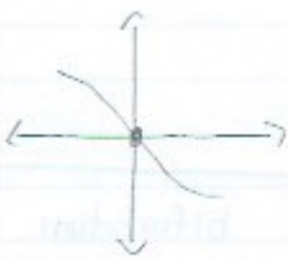
$-x\cos(x) + \mu x = 0$

$\mu x = x\cos(x)$

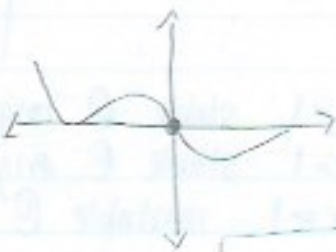
$\mu = \frac{\cancel{x}\cos(x)}{\cancel{x}}$

$\mu = \cos(x)$

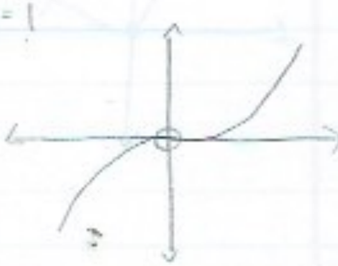
$\mu = -3$



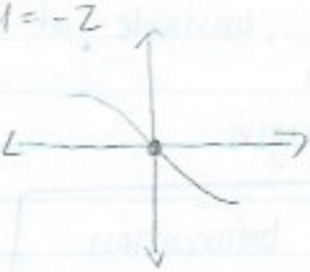
$\mu = -1$



$\mu = 1$



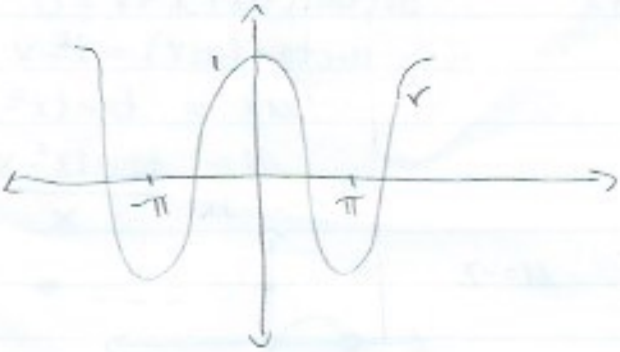
$\mu = -2$



sub critical pitchfork

$\mu = 1$

Bifurcation Graph



$$\frac{dx}{dt} = \mu x - \frac{x}{1+x}$$

$$\mu x - \frac{x}{1+x} = 0$$

$$x\left(\mu - \frac{1}{1+x}\right) = 0$$

$$\boxed{x=0} \qquad \mu - \frac{1}{1+x} = 0$$

$$\mu = \frac{1}{1+x} \Rightarrow \mu + \mu x = 1$$

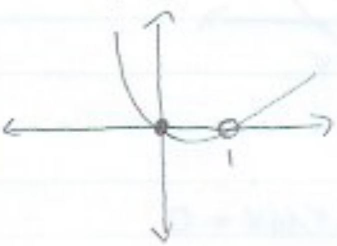
$$\mu(1+v) = 1$$

$$1+x = \frac{1}{\mu}$$

$$y = \frac{1}{\mu} - 1$$

$\mu = .5$

$x = \frac{1}{.5} - 1 = 1$

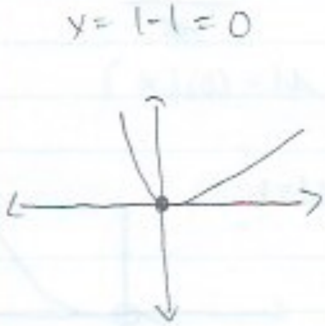


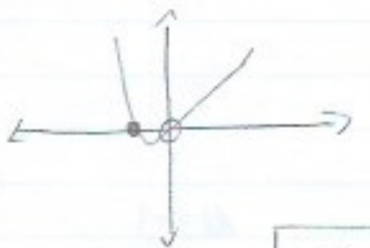
$\mu = 2$

$x = \frac{1}{2} - 1 = -.5$

$\mu = 1$

$y = 1 - 1 = 0$





$\boxed{\mu = 1 \text{ bifurcation occurs}}$

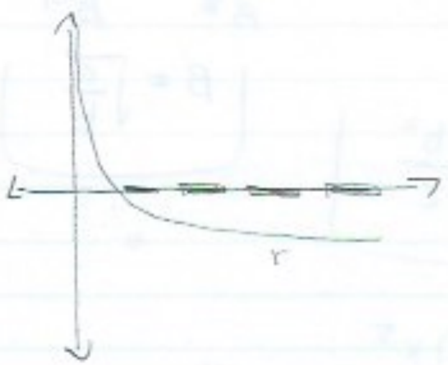
at $r<1$ stable @ origin, unstable at $x = \mu$

at $r=1$ stable @ origin

at $r>1$ unstable @ origin

$\boxed{\mu = 1 \text{ is transcritical bifurcation}}$

bifurcation diagram



Exercise 2

nonlinear system:

$$\frac{dx}{dt} = ax + bx^3 - cx^5 \quad , b, c > 0$$

subcritical pitchfork @ $a = 0$

Show equation can be

$$\frac{dx}{d\tau} = Rx + x^3 - x^5 \quad \text{in normal form}$$

let...

$$\frac{du}{dt} = au + bu^3 - cu^5 \qquad \tau = (\alpha)(t)$$
$$x = (\beta)(u)$$

$$\frac{du}{dt} = \left(\frac{d\tau}{dt}\right)\left(\frac{du}{d\tau}\right) \to \frac{d\tau}{dt} = \frac{d(\alpha t)}{dt} = \alpha$$

$$\frac{du}{dt} = \frac{d\left(\frac{x}{\beta}\right)}{dt} = \frac{1}{\beta}\left(\frac{dx}{d\tau}\right)$$

$$\to \frac{du}{dt} = \frac{\alpha}{\beta}\left(\frac{dx}{d\tau}\right) \to \frac{\alpha}{\beta}\left(\frac{dx}{d\tau}\right) = \frac{ax}{\beta} + \frac{bx^3}{\beta^3} - \frac{cx^5}{\beta^5}$$

$$\to \alpha\left(\frac{dx}{d\tau}\right) = ax + \frac{bx^3}{\beta^2} - \frac{cx^5}{\beta^4} \quad : \quad x = \frac{u}{\beta} \ *$$

$$\to \frac{dx}{d\tau} = \frac{ax}{\alpha} + \frac{bx^3}{\alpha\beta^2} - \frac{cx^5}{\alpha\beta^4}$$

let $\frac{b}{\alpha\beta^2}=1$ and $\frac{c}{\alpha\beta^4}=1$ $\rightarrow \frac{b}{\beta^2}=\frac{c}{\beta^4}$

$$\boxed{\beta=\sqrt{\frac{c}{b}}}$$

$$\boxed{\alpha=\frac{c}{\frac{c}{b}^{\frac{1}{2}(4)}}=\frac{c(b^2)}{c^2}=\frac{b^2}{c}}$$

$$\frac{dx}{dT}=\frac{ac}{b^2}x+(1)(x^3)-(1)x^5$$

let $r=\frac{ac}{b^2}$

$$\boxed{\text{so, } \frac{dx}{dT}=rx+x^3-x^5}$$

Exercise 3

$$\frac{dN}{dt}=rN\left(1-\frac{N}{K}\right)-H\left(\frac{N}{A+N}\right),\quad H,A>0$$

1) Show the equation can be $\frac{dx}{dT}=x(1-x)-h\left(\frac{x}{a+x}\right)$

* Let $T=\frac{1}{r}$ and $\tau=\frac{t}{T}$

$$\frac{dx}{dt}=\frac{dx}{d\tau}\frac{d\tau}{dt}=\frac{1}{T}\frac{dx}{d\tau}\Rightarrow r\frac{dx}{d\tau}=rx'$$

substitute rx' to $\frac{dN}{dt}$

$$rN'=rN\left(1-\frac{N}{K}\right)-H\left(\frac{N}{A+N}\right)$$

$$N'=N\left(1-\frac{N}{K}\right)-\frac{H}{r}\frac{N}{A+N}$$

let $x=\frac{N}{K}$ then $N=Kx$ $\rightarrow$ $Kx'=Kx(1-x)-\frac{H}{r}\left(\frac{Kx}{A+Kx}\right.$

$$x' = x(1-x) - \frac{H}{r}\left(\frac{x}{A+Kx}\right) = x(1-x) - \left(\frac{H}{rK}\right)\left(\frac{x}{\frac{A}{K}+x}\right)$$

Then let $h = \frac{H}{rK}$ and $a = \frac{A}{K}$

$$\boxed{\text{So, } \frac{dx}{d\tau} = x(1-x) - h\left(\frac{x}{a+x}\right)}$$

2) Show that the system can have 1, 2, or 3 fixed pts depending on a and h

$$f(x) = x(1-x) - \frac{hx}{a+x}$$

fixed pt at $x=0$

$$f'(x) = 1-2x - \frac{h(a+x-x)}{(a+x)^2} = 1-2x - \frac{ha}{(a+x)^2}$$

$$f'(0) = 1-0-\frac{ha}{a^2} = 1-\frac{h}{a} = \frac{1}{a}(a-h)$$

for $x=0$,
stable if $h>a$, unstable if $h<a$

let $f(x)=0$

$$x(1+x) = \frac{hx}{a+x}$$

$$\Rightarrow 1+x = \frac{h}{a+x}$$

$$\Rightarrow (1+x)(a+x) = h$$

$$\Rightarrow a + (1-a)x - x^2 = h$$

$$\Rightarrow x^2 + (a-1)x + h - a = 0$$

quadratic equation form

$$\Delta = (a-1)^2 - 4(h-a)$$
$$= (a^2-2a+1) - 4h + 4a$$
$$= (a+1)^2 - 4h$$

if $\Delta < 0$,

$$(a+1)^2 < 4h$$
$$h < \frac{1}{4}(a+1)^2$$

only equilibrium pt is @ $x = 0$

if $\Delta = 0$,

then there are 2 roots at $x = \frac{1-a}{2}$

if $h = \frac{1}{4}(a+1)^2$ and $0 < a < 1$,

there is a positive double root

consider roots λ_1 and λ_2

$$(x-\lambda_1)(x-\lambda_2) = x^2(\lambda_1+\lambda_2)x + \lambda_1\lambda_2$$

if $\lambda_1 \lambda_2 > 0$

$$a-1 = -(\lambda_1+\lambda_2) < 0 \quad\Big|\quad h-a = \lambda_1\lambda_2 > 0$$
$$a < 1 \qquad\qquad h > a$$

$\boxed{\text{2 positive roots when } \Delta > 0,\ a < 1,\ h > a}$

if $\lambda_1 \lambda_2 < 0$

$$a-1 = -(\lambda_1+\lambda_2) > 0 \quad\Big|\quad h-a \ \lambda_1\lambda_2 > 0$$
$$h > a$$

$\boxed{\text{No positive roots when } \Delta > 0,\ 1 < a < h}$

if $\lambda_1 > 0$ and $\lambda_2 < 0$

$$h-a = \lambda_1\lambda_2 < 0$$
$$h < a$$

$\boxed{\text{1 positive root when } \Delta < 0 \text{ and } h < a}$

3 Show bifurcation occurs at $h=a$. type? $x=0$

expand $f(x) = x(1-x) - h\left(\frac{x}{a+x}\right)$

$$= x(1-x) - hx\frac{1}{a(1+\frac{x}{a})} = x(1-x) - \frac{h}{a}\left(1-\frac{x}{a}+\dots\right)$$

$$= x - x^2 - \frac{h}{a}x + \frac{h}{a^2}x^2 + O(x^3)$$

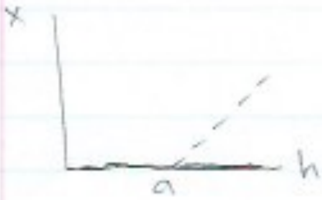
$$f(x) = \left(1-\frac{h}{a}\right)x - \left(1-\frac{h}{a^2}\right)x^2 + O(x^3)$$

close to $x=0$, if $1-\frac{h}{a^2} \neq 0$, we can neglect $O(x^3)$

If $h=a<1$, then $\frac{h}{a^2} = \frac{1}{a} > 1$ and $1-\frac{h}{a^2} < 0$

Compare with $\dot{x} = rx + x^2$

Let $r = 1-\frac{h}{a}$. $h>a \rightarrow r<0$



If $h=a>1$ then $\frac{h}{a^2} = \frac{1}{a} < 1$

and $1-\frac{h}{a^2} > 0$

Compare with $\dot{x} = rx - x^2$

Let $r = 1-\frac{h}{a}$

$h<a \rightarrow r>0$

If $h=a=1$, have to include the $O(x^3)$ terms

$\boxed{\text{transcritical bifurcation}}$

4. Show bifurcation occurs @ $h = \frac{(a+1)^2}{4}$ for $a < a_c$. Classify

The discriminant of the quadratic equation is 0 when $h = \frac{1}{4}(a+1)^2$.

When the discriminant is 0, a pair of roots collide and then become complex for discriminant < 0.

For $a > 1$, the real roots are negative; outside of the domain.

$$\boxed{\text{So, a saddle node bifurcation occurs when } h = \tfrac{1}{4}(a+1)^2 \text{ for } a < a_c}$$