LYNNE
DIEP

HW # 5

1 $\begin{cases} \dot{r} = r - r^2 \\ \dot{\theta} = t \end{cases}$

a) Determine the Poincare map from Σ into itself

$$\dot{r} = r - r^2 \qquad \dot{\theta} = t$$

Solve for $r(t)$

$$\frac{dr}{r-r^2} = dt \rightarrow \frac{dr}{r(1-r)} = dt$$

$$\int_{r_0}^{r}\left(\frac{1}{r} + \frac{1}{1-r}\right) dr = \int_0^{2\pi} dt = 2\pi$$

$$= \ln(r) + (-\ln(1-r))\Big]_{r_0}^{r} = [\ln(r) - \ln(1-r)] - [\ln(r_0) - \ln(1-r_0)]$$

$$\rightarrow \ln\left(\frac{r}{1-r}\right) - \ln\left(\frac{r_0}{1-r_0}\right)$$

$$\rightarrow \ln\left(\frac{\frac{r}{1-r}}{\frac{r_0}{1-r_0}}\right) \qquad \frac{r}{1-r} \cdot \frac{1-r_0}{r_0} = \frac{r - rr_0}{r_0 - rr_0}$$

$$\ln\left(\frac{r - rr_0}{r_0 - rr_0}\right) = 2\pi \rightarrow \frac{r - rr_0}{r_0 - rr_0} = e^{2\pi}$$

$$\rightarrow \frac{r(1-r_0)}{r_0(1-r)} = e^{2\pi} \rightarrow r(t) = \frac{e^{2\pi} r_0}{(e^{2\pi} - 1) r_0 + 1}$$

$\theta(t)$:

$$\int_{\theta_0}^{\theta} t = \int_0^t dt \rightarrow \frac{t^2}{2}\Big]_{\theta_0}^{\theta} = \frac{\theta^2}{2} - \frac{\theta_0^2}{2} = t$$

$$\rightarrow \theta^2 - \theta_0^2 = 2t$$

$$\theta^2 = 2t + \theta_0^2 \rightarrow \theta(t) = \sqrt{2t + \theta_0^2}$$

$$\boxed{r_{n+1} = P(r_n) = \frac{e^{2\pi} r_n}{(e^{2\pi} - 1) r_n + 1}}$$

b) Show system has unique periodic orbit and classify stability by Floquét multiplier

Fixed pts: $r - r^2$ $\quad r = 0, 1$
$r(1-r)$

$$P'(r) = \frac{e^{2\pi}}{((e^{2\pi}-1)r+1)^2} \qquad P'(0) = \frac{e^{2\pi}}{1} = e^{2\pi}$$

$$P'(1) = \frac{e^{2\pi}}{(e^{2\pi})^2} = \frac{1}{e^{2\pi}}$$

For $P'(0)$, $P'(0) > 1$
so unstable
For $P'(1)$, $P'(1) < 1$
so stable

also has unique periodic orbit

2 $\begin{cases} \dot{x} = xy - x^2 - x \\ \dot{y} = x^2 - y + \mu \end{cases}$

a) Sketch nullclines and determine fixed pts as a func. of μ

$$\dot{x} = xy - x^2 - x$$

$$0 = xy - x^2 - x \rightarrow xy = x^2 + x$$

$$y = \frac{x^2 + x}{x} = \frac{x(x+1)}{x}$$

$$\boxed{y = x + 1}$$

roots above equation give x-nullclines

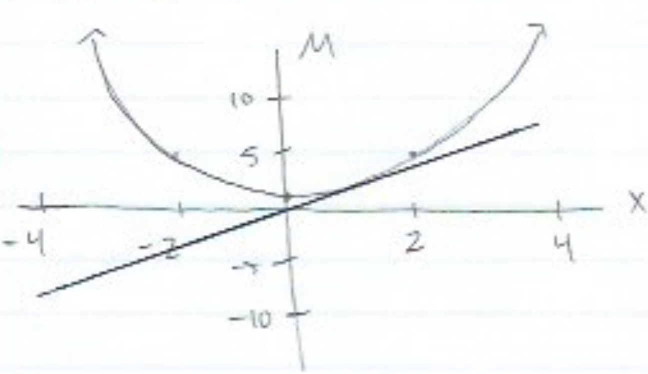
$$\dot{y} = x^2 y + \mu$$

$$0 = x^2 - y + \mu$$

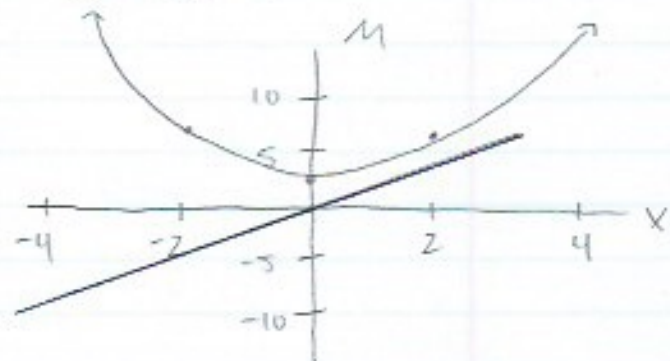
$$\boxed{y = x^2 + \mu}$$

roots above equation give y-nullclines

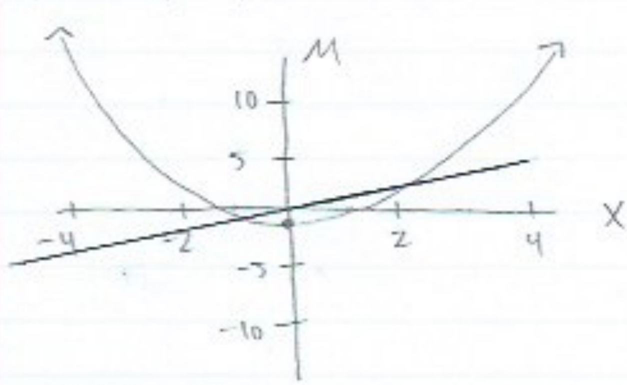
For $\mu = 1$



For $\mu > 1$



For $\mu < 1$



Fixed pts:

$$0 = x^2 + \mu$$

let $x = 1, -1$

$\mu = -1$

$$\boxed{(1,-1) \quad (-1,-1)}$$

b) Find & classify bifurcations

As seen in the graphs, bifurcation occurs at $\mu = 1$
For $\mu < 1$, there are 2 fixed pts and gets wiped out as μ increases.

So, the bifurcation is $\boxed{\text{saddle node}}$

$$\dot{x} = x^2 - x^3 - xy$$
$$\dot{y} = xy - \mu y$$

3 $\begin{cases} \dot{x} = x^2(1-x) - xy \\ \dot{y} = xy - \mu y, \quad \mu \geq 0 \end{cases}$

a) Determine / classify fixed pts

substitute 0 for $\dot{x}$ and $\dot{y}$

$\dot{x} \rightarrow x^2(1-x) - xy = 0$

$x^2(1-x) = xy$

$y = \dfrac{x^2(1-x)}{x} = \boxed{x(1-x) = y}$

$x(x(1-x) - y) = 0$

$\boxed{x = 0}$

similarly,

$0 = xy - \mu y \longrightarrow x - \mu = 0$

$y(x - \mu) \quad \boxed{x = \mu}$

$y = 0$

$\boxed{\text{So, fixed pts: } (0,0)\ (1,0)\ (\mu, \mu(1-\mu))}$

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - \mu \end{pmatrix}$$

@ $(0,0)$: $\begin{pmatrix} 0 & 0 \\ 0 & -\mu \end{pmatrix}$ $\quad \lambda = 0, -\mu$

So, $\boxed{(0,0) \text{ is stable for } \mu \geq 0}$

@ $(1,0)$: $\begin{pmatrix} 2-3-0 & -1 \\ 0 & 1-\mu \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1-\mu \end{pmatrix}$

$\lambda = -1, 1-\mu$

saddle node for $a < 1$
stable for $a > 1$
stable for $a = 1$
} @ $(1,0)$

$$\begin{pmatrix} 2x-3x^2-y & -x \\ y & x-\mu \end{pmatrix}$$

@ $(\mu, \mu(1-\mu))$: $\begin{pmatrix} 2\mu-3\mu^2-(\mu-\mu^2) & -\mu \\ \mu-\mu^2 & \mu-\mu \end{pmatrix}$

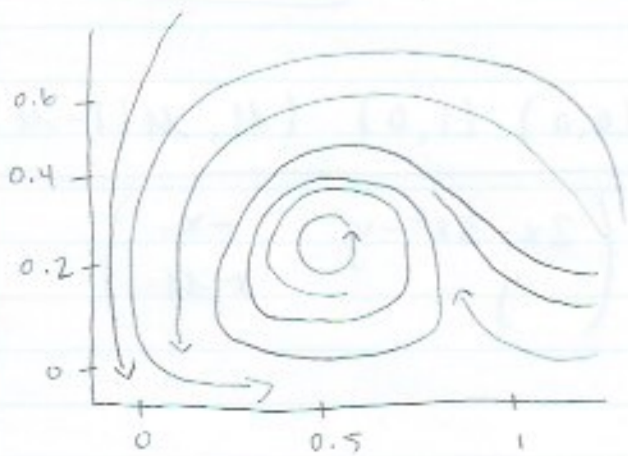
$= \begin{pmatrix} 2\mu-3\mu^2-\mu+\mu^2 & -\mu \\ \mu-\mu^2 & 0 \end{pmatrix} = \begin{pmatrix} \mu-2\mu^2 & -\mu \\ \mu-\mu^2 & 0 \end{pmatrix}$

$\lambda = \dfrac{\mu-2\mu^2 \pm \mu\sqrt{4\mu^2-3}}{2}$

For $\mu > \frac{1}{2}$, stable
$\mu < \frac{1}{2}$, stable

For $4\mu^2-3>0$ or $\mu > \frac{\sqrt{3}}{2}$ it is a node. For $4\mu^2-3<0$ or $\mu < \frac{\sqrt{3}}{2}$, it is a spiral

b) / c) Phase Portrait (rough sketch)



From phase portrait, it is clear that the real part of the eignvalues go through zero @
$\mu = \mu_0 = \frac{1}{2}$

So, Hopf bifurcation occurs.

The fixed pt is losing stability and is surrounded by a stable limit cycle. So, superccritical

phase place : $(\mu, \mu(1-\mu))$