Nonlinear Dynamical Systems and Chaos (AMS 114/214)

Homework 2 - Due Wednesday October 25th (2017)

Exercise 1 Consider the following nonlinear differential equations

$$\frac{dx}{dt} = \arctan(\mu x) - x^2 + x, \tag{1}$$
$$\frac{dx}{dt} = -x\cos(x) + \mu x, \tag{2}$$
$$\frac{dx}{dt} = \mu x - \frac{x}{1+x}. \tag{3}$$

For each case, find the values of the parameter μ at which bifurcations occur (approximately or exactly when possible), and classify those as saddle-node, transcritical, supercritical pitchfork or subcritical pitchfork. In addition, sketch the bifurcation diagram of fixed points versus μ.

Exercise 2 The nonlinear system

$$\frac{dx}{dt} = ax + bx^3 - cx^5, \quad b, c > 0 \tag{4}$$

has a subcritical pitchfork bifurcation at $a = 0$. Show that this equation can be written in a normal form as

$$\frac{dX}{d\tau} = RX + X^3 - X^5 \tag{5}$$

where X, τ and R are to be determined in terms of a, b, and c.

Exercise 3 The equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H\frac{N}{A+N}, \quad H, A > 0 \tag{6}$$

provides a simple model for a fishery. In the absence of fishing, the fish population $N(t)$ is assumed to grow logistically. The effects of fishing are modeled by the term $HN/(A+N)$, which says that fish are caught at a rate that depends on the population N.

1. Show that the system can be rewritten in a dimensionless form as

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} \tag{7}$$

for suitably defined dimensionless quantities x, τ, a and h.

2. Show that the system can have one, two or three fixed points, depending on the values of a and h.

3. Analyze the dynamics near the fixed point $x = 0$ and show that a bifurcation occurs when $h = a$. What type of bifurcation is it?

4. Show that another bifurcation occurs when $h = (a + 1)^2/4$, for $a < a_c$, where a_c is to be determined. Classify this bifurcation.

Exercise 4 (AMS 214 Mandatory, AMS 114 Extra Credit) Write a computer code that returns the bifurcation diagram associated with the ODE

$$\frac{dx}{dt} = e^{-x^2/\mu} - \frac{\sin(x\mu)}{(x^2 + 1)} \tag{8}$$

in the region $\mu \in [1/10, 10]$, $x \in [-10, 10]$. Graph the stable and unstable equilibrium curves with different colors (e.g., blue for stable and red for unstable). Include the plot of bifurcation diagram in your CANVAS submission.