I have read and agree to the collaboration policy. Lynne Diep.

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Homework Heavy Grading
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Homework 2-1
Due: May 1, 2017

Solution:

a. $H_n [i,j] = \frac{1}{\sqrt{2}} (-1)^{i \circ j}$

Show $H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$ We are showing that $i \circ j$ is even or odd

Let $i \circ j = k$

In a $2^n \times 2^n$ matrix, divide matrix into 4 quadrants.

So top-most left indices are:

i) $i = 0, j = 0 \rightarrow k = 0 =$ even

ii) $i = 0, j = \frac{2^n}{2} \rightarrow k = 0 =$ even

iii) $i = \frac{2^n}{2}, j = 0 \rightarrow k = 0 =$ even

iv) $i = \frac{2^n}{2}, j = \frac{2^n}{2} \rightarrow k = (2^{n-1})(2^{n-1}) =$ odd because 2^{n-1} is a power of 2, so there is only one "1" in its binary representation. Additionally, the dot product of the same number will always be 1.

Thus, $H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$

b. $\|x\| = \sqrt{x_1^2 \ldots x_n^2}$ $H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

row 0 = column 0 $\rightarrow \sqrt{\left(\frac{1}{\sqrt{2}^0}\right)^2 + \left(\frac{1}{\sqrt{2}^0}\right)^2} = 1$

row 1 = column 1 $\rightarrow \sqrt{\left(\frac{1}{\sqrt{2}^1}\right)^2 + \left(-\frac{1}{\sqrt{2}^1}\right)^2} = 1$

for any row and any column, $\|x\| = \sqrt{x_1^2 \ldots x_n^2} = \sqrt{\left(\pm\frac{1}{\sqrt{2}^n}\right)^2 + \left(\pm\frac{1}{\sqrt{2}^n}\right)^2} = \frac{2^n}{2^n} = 1$

Essentially, you add all the elements of the matrix squared and it will equal 1. Thus, the Eucliden norm of every row and column is 1.

c. Induction Claim –

Show p(n) = any 2 columns of H_n forms an othonormal basis, which is the dot product of 2 columns = 0 and the Euclidean norm is 1, assuming part b is true.

Base Case: Let n = 1. So, $H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow (1)(1) + (1)(-1) = 1 + (-1) = 0$

So base case is true when n = 1.

Induction Hypothesis (IH): for $k > 1$, $p(k)$ = any 2 columns of H_k, the dot product is 0. We show $p(k+1)$.

Proof:

$$\begin{array}{cc} & 2^k \\ 2^k & [\quad] \end{array} \rightarrow \begin{array}{cc} & 2^{k+1} \\ 2^{k+1} & \begin{bmatrix} 2^k & 2^k \\ 2^k & 2^k \end{bmatrix} \end{array}$$

By definition of dot product, we can add the dot product of the two submatrices, which equals to the dot product of the entire column combined.

$(1)(1) +(1)(-1) + (1)(1) + (1)(-1) = 0$

Thus, the columns of H_n form an orthonormal basis, i.e., the dot-product of any two columns of H_n equals to zero, and every column of H_n has Euclidean norm of one as claimed.

d. Algorithm –

if $n = 0$

multiple the element with the vector element of H_n

if $n > 0$

divide matrix into 4 quadrants of size $2^{n-1} \times 2^{n-1}$

vector into v_1,v_2 each with size 2^{n-1}

call algorithm on $\begin{bmatrix} A(Q_1, V_1) \\ A(Q_2, V_2) \\ A(Q_3, V_1) \\ A(Q_4, V_2) \end{bmatrix}$

Conquer

Add $Q_1 + Q_2$

Add $Q_3 + Q_4$

Prove Runtime:

$O(n \log n)$ - case 2

$T(n) \geq 4T(n/4) + n$

$\log_b a = \log_4 4 = 1$

$n^{\log_b a} = n^1 = n$

$f(n) = \theta(n^{\log_b a})$

$n = \theta(n^1) = \theta(n)$

$\therefore$ case

$T(n) = O(n \log n)$