I have read and agree to the collaboration policy. Lynne Diep.

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Homework Heavy Grading
Collaborators: None

CMPS 102: HW 1-3
Due: April 21, 2017

Problem 3:
Part a -
UniformShuffle(A):
1) for i ← n down to 1
2) do j ← random integer such that 1 ≤ j ≤ i
3) exchange A[i] and A[j]
4) return A

Prove that the algorithm indeed generates a uniform random shuffle of A. What is the running time of the algorithm, given that generating random integer takes time O(1)? Hint: Start by thinking of what a uniform shuffle means in terms of probability.

Answer:
Running Time - $O(n)$

Claim: A uniform shuffle of A is a sequence of n random elements from A, such that the probability of selecting any such sequence is the same.

Proof: We proceed with induction on the number of random elements, n.

Base Case: If n=1, there is only one element in the sequence. Since the probability of choosing a sequence is $\frac{1}{n}$, then when n = 1, the probability is 1. If n=2, then the probability of choosing each sequence is $\frac{1}{2}$. Thus, base case is proven true.

Induction step: Assume n-1 elements of A are completely shuffled after n-1 iterations of UniformShuffle. To insert one last element n to A, to make sure all elements in A are uniformly shuffled, we will look for index k, such that $a_k \leq n < a_{k+1}$, and then insert n between elements k and k+1. Since the rest of A is shuffled, index k will be unique. Once n is inserted, then the whole A is shuffled. Thus, the probability of selecting any such sequence is the same for the uniform shuffle of A.

Part b -
Point out the error in the following proof by induction.
Claim: Given any set of b buses, all buses lead to the same destination.
Proof: We proceed by induction on the number of buses, b.

Base case: If $b = 1$, then there is only one bus in the set, and so all buses in the set lead to the same destination.
Induction step: For $k \geq 1$, we assume that the claim holds for $b = k$ and prove that it is true for $b = k + 1$. Take any set B of $b + 1$ buses. To show that all buses lead to the same destination, we take the following approach. Remove one bus from this set to obtain the set B1 with just b buses. By the induction hypothesis, all the buses in B1 lead to the same destination. Now go back to the original set and remove a different bus to obtain a the set B2. By the same argument, all the buses in B2 lead to the same destination. Therefore all the buses in $B = B1 \cup B2$ must lead to the same destination, and the proof is complete.

Error: The base case fails for when $b = 2$. When $b = 2$, we cannot conclude that the removed buses lead to the same destination, because there aren't any buses remaining in B to apply the transitivity of "leading to the same direction". There are no "middle" buses so the argument does not make sense.