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CMPS 130 HW 1

0.1

a) $\{1, 3, 5, 7 \ldots\}$

A set of odd numbers

b) $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$

A set of all even integers

c) $\{n \mid n = 2m$ for some m in $\mathcal{N}\}$

A set of all even natural numbers

d) $\{n \mid m = 2m$ for some m in $\mathcal{N}$, and $n = 3k$ for some k in $\mathcal{N}\}$

A set of natural numbers divisible by 3 and 2.

e) $\{w \mid w$ is a string of 0s and 1s and w equals the reverse of $w\}$

The set of all strings being a palindrome and made up of 0s and 1s.

f) $\{n \mid n$ is an integer and $n = n+1\}$

The set of all integers that are equal to one added to that number

0.2

a) containing numbers 1, 10 and 100

$\{n \mid n = 10^m$ for some $m \in \{0, 1, 2\}\}$

b) integers greater than 5

$\{n \mid n$ is an integer and $n > 5\}$

c) All natural numbers that are less than 5

$\{n \mid n$ is a natural number and $n < 5\}$

d) set containing the string aba

$\{aba\}$

e) set containg empty string

$\{\varepsilon\}$

f) containing nothing at all

$\emptyset$

0.3

Let $A = \{x, y, z\}$

$B = \{x, y\}$

a) A subset of B?

No, because B does not contain z

b) B subset of A?

Yes, because x, y is in set A, $\{x, y, z\}$

c) What is $A \cup B$

$= \{x, y, z\} = A$

d) $A \cap B$

$\{x, y\} = B$

e) $A \times B$

$= \{(x,x), (x,y), (z,x), (y,x), (y,y), (z,y)\}$

f) Power set of B

$= \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$

0.4

If A has a elements and B has b elements, how many elements are in $A \times B$?

There are $a \times b$ elements in $A \times B$ because the cartesian product states that the product set of A and set B, is the set that contains all ordered pairs (a, b) for which a belongs to A and b belongs to B. So $a \times b$ is the number of elements.

0.5

If C is a set with c elements, how many elements are in the power set of C?

There are 2^c elements in the power set of C because the the number of elements in the set S is n, then its power set consists of 2^n elements. In set C, just replace n with c, thus there are 2^c elements in power set of C.

Example, let $C = \{1, 2, 3\}$
So, there should be $2^3 = 8$ elements in power set of C.

$P(C) = \{ \{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

There are 8 elements as claimed.

0.6

n	f(n)
1	6
2	7
3	6
4	7
5	6

g	6	7	8	9	10
1	10	10	10	10	10
2	7	8	9	10	6
3	7	7	8	8	9
4	9	8	7	6	10
5	6	6	6	6	6

$X = \{1,2,3,4,5\}$
$Y = \{6,7,8,9,10\}$

a) value of $f(2)$

$f(2) = 7$

b) range / domain of f

range = $\{6, 7\}$
domain = $\{1,2,3,4,5\}$

c) value of $g(2,10)$

$g(2,10) = 6$

d) range / domain of g

range = $\{6,7,8,9,10\}$
domain = $X \times Y$

$= \{(1,6), (1,7), (1,8), (1,9), (1,10)$
$(2,6)\ (2,7)\ (2,8)\ (2,9)\ (2,10)$
$(3,6)\ (3,7)\ (3,8)\ (3,9)\ (3,10)$
$(4,6)\ (4,7)\ (4,8)\ (4,9)\ (4,10)$
$(5,6)\ (5,7)\ (5,8)\ (5,9)\ (5,10)\}$

e) value of $g(4, f(4))$

$g(4, f(4)) = 8$

0.7

Give relation that satisfies the condition

a) Reflexive and symmetric but not transitive

Let $A = \{1,2,3,4\}$

relation $A_1 = \{(1,1), (2,2), (3,3),$
$(2,1)\ (1,2)\ (3,2)\ (2,3)\}$

b) Reflexive and transitive but not symmetric

let $A = \{1,2,3,4\}$

relation $A_2 = \{(1,1), (2,2)\ (3,3), (1,2)\}$

c) Symmetric and transitive but not reflexive

let $A = \{1,2,3,4\}$

relation $A_3 = \{(1,1)\ (2,2)\}$

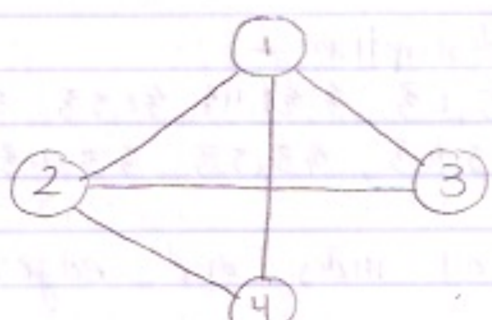
0.8

Undirected graph $G = (V, E)$

$V = \{1,2,3,4\}$ nodes

$E = \{\{1,2\}, \{2,3\}, \{1,3\}, \{2,4\}, \{1,4\}\}$ edges

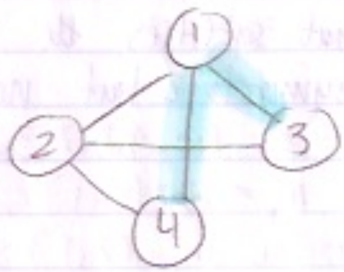
Draw Graph G



Degree of Nodes:

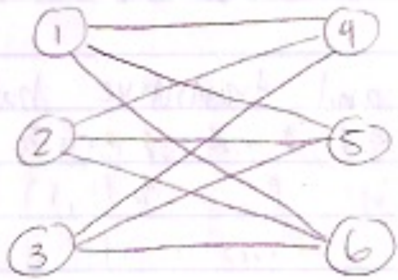
Node	Degree
1	3
2	3
3	2
4	2

Indicate path from node 3 to node 4 :



0.9

Formal description of graph



V, set of nodes = $\{1,2,3,4,5,6\}$

E, edges = $\{\{1,4\}, \{1,5\}, \{1,6\}$
$\{2,4\}, \{2,5\}, \{2,6\}$
$\{3,4\}, \{3,5\}, \{3,6\}\}$

Formal description -
$(\{1,2,3,4,5,6\}, \{\{1,4\}, \{1,5\}, \{1,6\}, \{2,4\}, \{2,5\},$
$\{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}\})$

Shows all nodes and edges

0.10

To Prove : 2=1

Given Proof -

1 consider $a=b$

2 Multiply both sides by a to obtain $a^2=ab$

3 Subtract b^2 from both sides to get $a^2-b^2=ab-b^2$

4 factor each side, $(a+b)(a-b)=b(a-b)$

5 Divide each side by $(a-b)$ to get $a+b=b$

6 let $a=1$, $b=1$

7 Shows $2=1$

Error:

The error is at line 6. Since $a=1$, $b=1$ $(a-b)$ equals 0. Division by 0 is undefined, so the argument of the proof is not valid.

0.12

Find Error

Claim - any set of h horses, all horses are same color

Proof - induction on h

Error: Base case fails for when $h=2$.
When $h=2$, one horse can have a color and horse 2 can have a peculiar color. Therefore, the is no meaning to conclude that the horses have the same color.

0.13

Show every graph with 2 or more nodes contain 2 nodes that have equal degrees.

Proof by contradiction:

Prove that 2 nodes contain unique degrees. Let the degrees for a graph with n vertices be:

$$0, 1, \ldots, n-1$$

Assign degree 0 to one vertex, which means that vertex is not connected to any other vertex.

Assign degree $n-1$ to a vertex, and it is connected to every other vertex in the graph.

This is a contradiction because, that vertex cannot be connected to every other vertex, including a vertex that is not connected to anyone (degree 0). So by contradiction, there are at least 2 nodes that have equal degree in a graph with 2 or more nodes. □

Prove

$$(\overline{A \cap B}) = (\bar{A} \cup \bar{B})$$

Proof: $(\overline{A \cap B}) = (\bar{A} \cup \bar{B})$

if and only if

$\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Show $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$:

Assume x is an element of $\overline{A \cap B}$, so

$x \in \overline{A \cap B}$, $x \notin A \cap B$.

By De Morgan's Law, $x \notin A$ and $x \notin B$

Hence, $x \in \bar{A} \cup \bar{B}$

So, $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

Show $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$:

Assume x is an element of $\bar{A} \cup \bar{B}$. Then $x \notin A$ and $x \notin B$. By De Morgan's Law, $x \notin A \cap B$.

This shows $x \in \overline{A \cap B}$.

So, $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

So, this proves $(\overline{A \cap B}) = (\bar{A} \cup \bar{B})$ $\square$

Prove set of odd numbers is countable

Proof:

Let $2n-1$ be odd positive integers

Prove injectivity –

$f(a) = f(b) \rightarrow a = b$

So, $2a - 1 = 2b - 1$

This implies $a = b$, so injectivity is established

Prove surjectivity –

Function is surjective for every $b \in B$, there exists $a \in A$, such that $f(a) = b$

So, lets prove by contradiction:

suppose there is some odd integer b such that $\forall x \in \mathbb{Z}$ $2x - 1 \neq b$

This implies $\frac{b+1}{2}$ is not an integer

Since b is odd, b+1 is even
which means b+1 is divisible by 2,
yielding a contradiction.

We showed there is a bijection, so the set of odd integers is countably infinite.

Prove by induction on n

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof:

i) Base case - when $i, n = 1 \Rightarrow 1^2 = \frac{1(1+1)(2+1)}{6}$

$\Rightarrow 1 = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$ ✓

So base case is true.

ii) By hypothesis, $\sum_{i=1}^{n+1} i^2 = \frac{n+1(n+2)(2n+3)}{6}$,

let's prove it.

Induction hypothesis is $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

So, $\sum_{i=1}^{n+1} i^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}$$

$$= \frac{(n+1)[2n^2 + n + 6n + 6]}{6}$$

$$= \frac{(n+1)\left[2n^2+7n+6\right]}{6} = \frac{(n+1)\left[(n+2)(2n+3)\right]}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

By induction, our conclusion of $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$ holds true. Hence, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all $n \in \mathbb{N}$.

$\square$