An introduction to Hodge theory and applications to algebraic geometry

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1 Introduction

In this article, we develop the basic ideas behind the Hodge theorem on harmonic forms and the Hodge identities and decomposition for Kähler manifolds. We then use these ideas to obtain results such as the Kodaira-Nakano vanishing theorem. Our primary source for this material is Principles of Algebraic Geometry by Griffiths and Harris, chapters 0 and 1 of which we follow closely. We begin by recalling some basic definitions and notation.

1.1 Complex manifolds and vector bundles

A *complex manifold* M of complex dimension n is a differentiable manifold given an open cover $\{U_\alpha\}$ and diffeomorphisms φ_α between U_α and open subsets of $\mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β.

As usual, we say that a function $f : U \to \mathbb{C}$ for $U \subset M$ is holomorphic if $f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha)$ for all α, and define holomorphic coordinate charts and holomorphic maps between complex manifolds similarly.

For $z \in U_\alpha \subset M$, let $(z_1, \dots, z_n)$ be holomorphic coordinates for U_α, and let $z_i = x_i + \sqrt{-1}y_i$ with x_i, y_i real. Let

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

We define the *complexified tangent space* $T_{\mathbb{C},z}(M)$ to M at z to be the space of $\mathbb{C}$-linear maps ("derivations") $d : C^\infty(U_\alpha) \to \mathbb{C}$ satisfying $d(fg) = (df)g(z) + f(z)(dg)$. Then $T_{\mathbb{C},z}(M)$ is a $\mathbb{C}$-vector space of dimension $2n$ generated by $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}$, or alternatively by $\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right\}$. We have $T_{\mathbb{C},z}(M) = T_z(M) \oplus \overline{T}_z(M)$, where $T_z(M)$ is the subspace of $T_{\mathbb{C},z}(M)$ generated by $\left\{\frac{\partial}{\partial z_i}\right\}$ (the derivations that vanish on antiholomorphic functions), called the *holomorphic tangent space*, and $\overline{T}_z(M)$ is the subspace generated by $\left\{\frac{\partial}{\partial \bar{z}_i}\right\}$ (the derivations that vanish on holomorphic functions).

A C^∞ *complex vector bundle* E on M of rank r is a disjoint union $E = \sqcup_{z \in M} E_z$, where the E_z are complex vector spaces of rank r, which is given a C^∞ manifold structure such that the projection map $\pi : E \to M$ taking E_z to z is C^∞, and for every $z_0 \in M$ there is an open neighborhood U of z_0 in M and a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ which gives a vector space isomorphism between E_z and $\{z\} \times \mathbb{C}^r$ for all $z \in U$. The map φ_U is called a trivialization of E on U. A vector bundle of rank 1 is a *line bundle*.

We say that a vector bundle $\pi : E \to M$ is *holomorphic* if E is given a complex manifold structure and trivializations $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ around each point of M which are biholomorphic maps of complex manifolds. For example, $T_{\mathbb{C}}(M) = \sqcup T_{\mathbb{C},z}(M)$ is naturally a complex vector bundle on M, and its subbundle $T_{hol}(M) = \sqcup T_z(M)$ is naturally a holomorphic vector bundle.

1.2 De Rham, Dolbeault, and sheaf cohomology

Let $A^k(M)$ be the space of complex-valued differential k-forms on M, that is, the space of sections of $\wedge^k T^\vee_{\mathbb{C}} M$, and $Z^k_d(M)$ the subspace of closed k-forms; the *complex de Rham cohomology space* $H^k_{dR}(M)$ is defined to be $Z^k_d(M)/dA^{k-1}(M)$. This is the same as $H^k_{dR}(M;\mathbb{R})\otimes\mathbb{C}$. Now $A^k(M)$ decomposes as follows: since $T^\vee_{\mathbb{C},z}(M) = T^\vee_z(M) \oplus \overline{T}^\vee_z(M)$, we have

$$\wedge^k T^\vee_{\mathbb{C},z}(M) = \oplus_{p+q=k} \wedge^p T^\vee_z(M) \otimes \wedge^q \overline{T}^\vee_z(M)$$

so

$$A^k(M) = \oplus_{p+q=k} A^{p,q}(M)$$

where

$$A^{p,q}(M) = \{\varphi \in A^k(M) \mid \varphi(z) \in \wedge^p T^\vee_z(M) \otimes \wedge^q \overline{T}^\vee_z(M) \text{ for all } z \in M\}.$$

We write $\pi^{(p,q)} : A^\bullet(M) \to A^{(p,q)}(M)$ for the projection map, and $\varphi^{(p,q)}$ for $\pi^{(p,q)}\varphi$ if $\varphi \in A^\bullet(M)$. If $\varphi \in A^{p,q}(M)$, we say that φ is of type (p,q). In this case, we have

$$d\varphi(z) \in (\wedge^p T^\vee_z(M) \otimes \wedge^q \overline{T}^\vee_z(M)) \wedge T^\vee_{\mathbb{C},z}(M) = (\wedge^p T^\vee_z(M) \otimes \wedge^q \overline{T}^\vee_z(M)) \wedge (T^\vee_z(M) \oplus \overline{T}^\vee_z(M))$$

$$= (\wedge^{p+1} T^\vee_z(M) \otimes \wedge^q \overline{T}^\vee_z(M)) \oplus (\wedge^p T^\vee_z(M) \otimes \wedge^{q+1} \overline{T}^\vee_z(M))$$

for all $z \in M$, so

$$d\varphi \in A^{p+1,q}(M) \oplus A^{p,q+1}(M).$$

Define $\partial = \pi^{(p+1,q)} \circ d$, $\overline{\partial} = \pi^{(p,q+1)} \circ d$, so that $d = \partial + \overline{\partial}$. We call a form φ of type $(q,0)$ holomorphic if $\overline{\partial}\varphi = 0$. The operator $\overline{\partial}$ commutes with maps induced by holomorphic maps of complex manifolds, and $\overline{\partial}^2 = 0$. We write $Z^{p,q}_{\overline{\partial}}(M)$ for the space of $\overline{\partial}$-closed forms of type (p,q), and define the *Dolbeault cohomology spaces* $H^{p,q}_{\overline{\partial}}(M)$ by

$$H^{p,q}_{\overline{\partial}}(M) = \frac{Z^{p,q}_{\overline{\partial}}(M)}{\overline{\partial}A^{p,q-1}(M)}.$$

For future convenience, we define the operator d^c by $d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$. Note that like d, this is a real operator.

We can also define Dolbeault cohomology for a holomorphic vector bundle $E \to M$. Let $\mathscr{A}^k(M;E)$ be the sheaf of C^∞ sections of $\wedge^k T^\vee_{\mathbb{C}}(M) \otimes E$ with space of global sections

$A^k(M;E)$, $\mathscr{A}^{p,q}(M;E)$ the sheaf of C^∞ sections of $\wedge^p T^\vee(M) \otimes \wedge^q \overline{T}^\vee(M) \otimes E$ with space of global sections $A^{p,q}(M;E)$. The operator $\overline{\partial} : \mathscr{A}^{p,q}(M;E) \to \mathscr{A}^{p,q+1}(M;E)$ can be defined by choosing a local holomorphic frame $(e_1, \dots, e_r)$ for E on U and, if $\sigma = \sum \varphi_i \otimes e_i$ for $\varphi_i \in \mathscr{A}^{p,q}(M;E)(U)$, defining $\overline{\partial}\sigma = \sum \overline{\partial}\varphi_i \otimes e_i$. One can check that this is independent of choice of frame and satisfies $\overline{\partial}^2 = 0$; then if $Z^{p,q}_{\overline{\partial}}(M;E)$ is the space of $\overline{\partial}$-closed C^∞ E-valued (p,q)-forms, we can define

$$H^{p,q}_{\overline{\partial}}(M;E) = \frac{Z^{p,q}_{\overline{\partial}}(M;E)}{\overline{\partial}A^{p,q-1}(M;E)}.$$

Finally, for any sheaf $\mathscr{F}$ on M, we can define the Cech cohomology $H^k_{Cech}(M;\mathscr{F})$ of $\mathscr{F}$ on M in the usual way, by taking the inverse limit over all locally finite open covers $\{U_\alpha\}$ of the cohomology groups $H^k(\{U_\alpha\};\mathscr{F}) = Z^k(\{U_\alpha\};\mathscr{F})/B^k(\{U_\alpha\};\mathscr{F})$ of the Cech cochain complex associated to $\{U_\alpha\}$. One can show that for the constant sheaf $\underline{\mathbb{R}}$, we have

$$H^k_{Cech}(M;\underline{\mathbb{R}}) \cong H^k_{dR}(M;\mathbb{R})$$

and for the sheaf Ω^p of holomorphic p-forms,

$$H^q(M;\Omega^p) = H^{p,q}_{\overline{\partial}}(M).$$

This last identity is called the Dolbeault theorem.

1.3 Hermitian metrics, connections, and curvature

A *hermitian metric* on M is a positive definite hermitian inner product

$$(\cdot,\cdot)_z : T_z(M) \otimes \overline{T}_z(M) \to \mathbb{C}$$

on the holomorphic tangent space at z for each $z \in M$ which depends smoothly on z. If $(z_1, \dots, z_n)$ are holomorphic coordinates around z and we write $h_{ij}(z) = \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)_z$, then we can write the metric locally as $ds^2 = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z}_j$. Using Gram-Schmidt, we can rewrite this in the form $ds^2 = \sum_i \varphi_i \otimes \overline{\varphi}_i$, where $(\varphi_1, \dots, \varphi_n)$ are forms of type $(1,0)$; we say that the φ_i form a *coframe* for the metric near z. In this case, if we write $\varphi_i = \alpha_i + \sqrt{-1}\beta_i$ where α_i, β_i are real 1-forms, we have

$$ds^2 = \sum_i \varphi_i \otimes \overline{\varphi}_i = \sum_i (\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i) + \sqrt{-1}\sum_i(-\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i).$$

The real part $\Re(ds^2) = \sum_i(\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i)$ is the *induced Riemannian metric* on the underlying real manifold of M. The $(1,1)$-form

$$-\frac{1}{2}\Im(ds^2) = \sum \alpha_i \wedge \beta_i = \frac{\sqrt{-1}}{2}\sum \varphi_i \wedge \overline{\varphi}_i$$

is called the *associated* $(1,1)$-form of the metric and denoted ω. Observe that the metric $\sum \varphi_i \otimes \overline{\varphi}_i$ can be recovered from the associated $(1,1)$-form $\frac{\sqrt{-1}}{2}\sum \varphi_i \wedge \overline{\varphi}_i$, and that an arbitrary $(1,1)$-form ω is the associated $(1,1)$-form of some metric if and only if it can be diagonalized as $\frac{\sqrt{-1}}{2}\sum \varphi_i \wedge \overline{\varphi}_i$, which is true if and only if it is of the form $\frac{\sqrt{-1}}{2}\sum_{i,j} l_{ij}(z) dz_i \wedge d\overline{z}_j$ where $L(z) = (l_{ij}(z))$ is a positive definite hermitian matrix for each z, which is true if and only if

$$-\sqrt{-1}\omega(z)(v, \overline{v}) > 0$$

for all $z \in M$ and $v \in T_z(M)$. Such a $(1,1)$-form is called *positive*.

Let $E \to M$ be a complex vector bundle. We can similarly define hermitian metrics on E. If $E \to M$ is holomorphic and is given a hermitian metric, we call it a *hermitian vector bundle*.

A *connection* is a map $\nabla : \mathscr{A}^0(M;E) \to \mathscr{A}^1(M;E)$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f\nabla(s)$ for all $s \in \mathscr{A}^0(M;E)(U)$ and $f \in C^\infty(U)$. If $e = \{e_1, \ldots, e_r\}$ is a frame for E on U, and $\nabla e_i = \sum_j \theta_{ij} e_j$, then the matrix of 1-forms $\theta_e = (\theta_{ij})$ is called the connection matrix of ∇ with respect to e; e and θ_e together determine ∇, because if $\sigma = \sum_i \sigma_i e_i$, then $\nabla(\sigma) = \sum_j (d\sigma_j + \sum_i \sigma_i \theta_{ij}) e_j$.

We may decompose ∇ as a sum $\nabla_1 + \nabla_2$, where $\nabla_1 : \mathscr{A}^0(M;E) \to \mathscr{A}^{1,0}(M;E)$ and $\nabla_2 : \mathscr{A}^0(M;E) \to \mathscr{A}^{0,1}(M;E)$. We say that ∇ is *compatible with the complex structure* if $\nabla_2 = \overline{\partial}$. If E is hermitian, we say that ∇ is *compatible with the metric* if $d(s,t) = (\nabla(s), t) + (s, \nabla(t))$.

Lemma 1.1. *If E is hermitian, there is a unique connection ∇ on E compatible with both the complex structure and the metric, called the* metric connection.

Proof. Let $(e_1, \ldots, e_r)$ be a holomorphic frame for E. If such a connection ∇ exists, then since $\overline{\partial} e_i = 0$, the coefficients of $\nabla(e_i)$ are of type $(1,0)$, so the matrix θ of ∇ with respect to (e_i) is of type $(1,0)$. Conversely, when θ is of type $(1,0)$, the formula $\nabla(\sigma) = \sum_j (d\sigma_j + \sum_i \sigma_i \theta_{ij}) e_j$ shows that $\nabla_2 = \overline{\partial}$. Now let $(e_i, e_j) = h_{ij}$, $H = (h_{ij})$. By compatibility with the metric we have

$$dh_{ij} = d(e_i, e_j) = \sum_k \theta_{ik} h_{kj} + \sum_k \overline{\theta}_{jk} h_{ik}$$

where the first sum is type $(1,0)$ and the second is type $(0,1)$, so

$$\partial h_{ij} = \sum \theta_{ik} h_{kj}$$

$$\overline{\partial} h_{ij} = \sum \overline{\theta}_{jk} h_{ik}$$

and these are satisfied precisely when $\theta = (\partial H) H^{-1}$. Therefore the unique such connection is given by (e_i), $(\partial H) H^{-1}$. $\square$

Given a connection ∇ on $E \to M$, we can define operators $D : \mathscr{A}^k(M;E) \to \mathscr{A}^{k+1}(M;E)$ by setting $D = \nabla$ on $\mathscr{A}^0(M;E)$ and requiring the Leibniz rule $D(\psi \otimes \eta) = d\psi \otimes \eta + (-1)^k \psi \wedge D\eta$ to hold for $\psi \in \mathscr{A}^k(M;E)(U)$, $\eta \in \mathscr{A}^0(M;E)(U)$. (Of course, D can be decomposed as $D_1 + D_2$ corresponding to $\nabla = \nabla_1 + \nabla_2$.) Then one can check that the operator $D^2 : \mathscr{A}^0(M;E) \to \mathscr{A}^2(M;E)$ is linear over $\mathscr{A}^0(M;E)$, so comes from a map of vector bundles $E \to \wedge^2 T^\vee_{\mathbb{C}}(M) \otimes E$, that is, a global section Θ of $\wedge^2 T^\vee_{\mathbb{C}}(M) \otimes (E^\vee \otimes E)$. If $e = \{e_1, \ldots, e_r\}$ is a frame for E on U, and $D^2 e_i = \sum \Theta_{ij} \otimes e_j$, then the matrix of 2-forms $\Theta_e = (\Theta_{ij})$ is called the curvature matrix of ∇ with respect to e. In fact $\Theta_e = d\theta_e - \theta_e \wedge \theta_e$.

If $E \to M$ is hermitian and ∇ is compatible with the complex structure, then from $\nabla_2 = \overline{\partial}$ we see that $D_2 \circ \nabla_2 = 0$ and so $\Theta^{(0,2)} = 0$. If ∇ is additionally compatible with the metric and e is a unitary frame, then θ_e is skew-hermitian, and then so is $\Theta_e = d\theta_e - \theta_e \wedge \theta_e$; therefore $\Theta_e^{(2,0)} = 0$ as well. Since the type of Θ is invariant under change of frame, we conclude that the curvature matrix of the metric connection is a hermitian matrix of $(1,1)$-forms.

1.4 Line bundles and Chern classes

Let M be a compact complex manifold of dimension n, and $\mathscr{O}$ the sheaf of holomorphic functions on M. If $\pi : L \to M$ is a line bundle, with trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ for an open cover $\{U_\alpha\}$ of M, define the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ by $g_{\alpha\beta}(z) = (\varphi_\alpha \circ \varphi_\beta^{-1})|_{L_z}$. The collection $(g_{\alpha\beta})$ forms a Cech 1-cocycle on M with coefficients in $\mathscr{O}^\times$, and two such collections give the same line bundle precisely when they differ by a Cech 1-coboundary; in this way we can view a line bundle as an element of $H^1(M, \mathscr{O}^\times)$. The exponential exact sequence

$$0 \to \underline{\mathbb{Z}} \to \mathscr{O} \to^{\exp} \mathscr{O}^\times \to 0$$

gives a boundary map $\delta : H^1(M, \mathscr{O}^\times) \to H^2(M, \mathbb{Z})$. For a line bundle $L \in H^1(M, \mathscr{O}^\times)$, its image $\delta(L) \in H^2(M, \mathbb{Z})$ is called the *first Chern class* of L and is also written $c_1(L)$. We may also use $c_1(L)$ to refer to its image under the map $H^2(M, \mathbb{Z}) \to H^2_{dR}(M)$.

Proposition 1.2. *If L has a metric on it with curvature form Θ, then $\frac{\sqrt{-1}}{2\pi}\Theta$ is in the class of $c_1(L)$ in $H^2_{dR}(M)$.*

Proof idea. This is really just a matter of writing out definitions. If L is given by transition functions $(g_{\alpha\beta})$ for a trivializing open cover $\{U_\alpha\}$, one finds that $c_1(L)$ is represented by the cocycle $(z_{\alpha\beta\gamma}) \in Z^2(\{U_\alpha\}, \underline{\mathbb{Z}})$, where

$$z_{\alpha\beta\gamma} = \frac{1}{2\pi\sqrt{-1}}(\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma}).$$

On the other hand, if ∇ is a connection on L whose connection matrix is θ_α on U_α, then on $U_\alpha \cap U_\beta$ we have

$$\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} = \theta_\beta + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$$

since everything is 1-dimensional; thus

$$\theta_\beta - \theta_\alpha = -g_{\alpha\beta}^{-1} dg_{\alpha\beta} = -d(\log g_{\alpha\beta})$$

and also

$$\Theta = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha.$$

This gives enough information to write out the image of $[\Theta] \in H^2_{dR}(M)$ under the isomorphism to $H^2_{Cech}(M; \underline{\mathbb{C}})$, which turns out to give the same cocycle $(z_{\alpha\beta\gamma})$. □

An *analytic hypersurface* V contained in a complex manifold M is a submanifold of codimension 1 locally cut out as the zero locus of a nontrivial holomorphic function f. V is called *irreducible* if it cannot be written as $V_1 \cup V_2$ where V_1, V_2 are analytic hypersurfaces not equal to V. A *divisor* on M is a locally finite formal linear combination $\sum_i a_i V_i$ of irreducible analytic hypersurfaces of M. If f is a meromorphic function, its corresponding divisor (f) is defined to be $\sum_V \mathrm{ord}_V(f) V$, where $\mathrm{ord}_V(f)$ is the order of vanishing of f along

V, as V runs over all irreducible analytic hypersurfaces of M. If D is a divisor, its associated line bundle L is the one corresponding to the element of $H^1(M, \mathscr{O}^\times)$ represented by the Cech cocycle $(g_{\alpha\beta}) = (f_\alpha/f_\beta)$, where f_α is a function such that $(f_\alpha) = D$ on U_α.

Notice that if a line bundle L with trivializing open cover $\{U_\alpha\}$ and transition functions $(g_{\alpha\beta})$ has a global meromorphic section s, and s_α is the restriction of s to U_α, then $s_\alpha/s_\beta = g_{\alpha\beta}$, so that $L = [(s)]$.

2 The Hodge theorem

We may now proceed to state the Hodge theorem, giving the connection between Dolbeault cohomology and spaces of harmonic forms which forms the basis of this article. The proof of the Hodge theorem relies on standard techniques from analysis which we will not describe in detail, but we will give a sketch just to make it seem doable.

2.1 $\star$, $\overline{\partial}^*$, the Laplacian, and the statement of the theorem

Let M be a connected, compact complex manifold of complex dimension n. Choose a hermitian metric ds^2 with associated $(1,1)$-form $\omega = \frac{\sqrt{-1}}{2}\sum_j \varphi_j \wedge \overline{\varphi}_j$. This induces a hermitian metric on $T^{\vee(p,q)}(M)$, where the inner product on $T_z^{\vee(p,q)}(M)$ is given by setting $\{\varphi_I(z) \wedge \overline{\varphi}_J(z)\}_{|I|=p,|J|=q}$ to be an orthogonal basis with each element of length $2^{(p+q)/2}$. The $(2n)$-form

$$\Phi = \frac{\omega^n}{n!} = \frac{(-1)^{n(n-1)/2}(\sqrt{-1})^n}{2^n}\varphi_1 \wedge \cdots \varphi_n \wedge \overline{\varphi}_1 \wedge \cdots \wedge \overline{\varphi}_n$$

is the volume form on M associated to the metric. The space $A^{p,q}(M)$ has a corresponding inner product

$$(\psi, \eta) = \int_M (\psi(z), \eta(z))\Phi(z).$$

We define the operator $\star : A^{p,q}(M) \to A^{n-q,n-p}(M)$ by setting $\star\eta$ so that

$$(\psi(z), \eta(z))\Phi(z) = \psi(z) \wedge \overline{\star\eta(z)}$$

for all $\psi \in A^{p,q}(M)$. By writing out the $\star$ operator explicitly in local coordinates, one can see that it is metric-preserving, hence that

$$\eta \wedge \overline{\star\psi} = (\eta, \psi)\Phi = (\star\eta, \star\psi)\Phi = \overline{\star\psi} \wedge \star\star\eta$$

$$= (-1)^{(p+q)(2n-p-q)} \star\star\eta \wedge \overline{\star\psi} = (-1)^{p+q} \star\star\eta \wedge \overline{\star\psi}.$$

Therefore $\star\star\eta = (-1)^{(p+q)}\eta$. If we define the operator $\overline{\partial}^* : A^{p,q}(M) \to A^{p,q-1}(M)$ by $-\star\overline{\partial}\star$, then one can use this to check that $(\overline{\partial}^*\psi, \eta) = (\psi, \overline{\partial}\eta)$ for all $\eta \in A^{p,q-1}(M)$, that is, $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$.

Lemma 2.1. *A $\overline{\partial}$-closed form $\psi \in Z_{\overline{\partial}}^{p,q}(M)$ is of minimal norm in its Dolbeault cohomology equivalence class $\psi + \overline{\partial}A^{p,q-1}(M)$ if and only if $\overline{\partial}^*\psi = 0$.*

Proof. If $\overline{\partial}^*\psi = 0$, then for any $\eta \in A^{p,q-1}(M)$ we have

$$\|\psi + \overline{\partial}\eta\|^2 = \|\psi\|^2 + \|\overline{\partial}\eta\|^2 + 2\Re(\psi, \overline{\partial}\eta)$$

$$= \|\psi\|^2 + \|\overline{\partial}\eta\|^2 + 2\Re(\overline{\partial}^*\psi, \eta) = \|\psi\|^2 + \|\overline{\partial}\eta\|^2 > \|\psi\|^2$$

so ψ has minimal norm in its class. Conversely, if ψ is of minimal norm, then for any $\eta \in A^{p,q-1}(M)$ we have

$$2\Re(\psi, \overline{\partial}\eta) = \frac{\partial}{\partial t}|_{t=0}\|\psi + t\overline{\partial}\eta\|^2 = 0$$

$$2\Im(\psi, \overline{\partial}\eta) = \frac{\partial}{\partial t}|_{t=0}\|\psi + t\overline{\partial}(\sqrt{-1}\eta)\|^2 = 0$$

so $(\overline{\partial}^*\psi, \eta) = (\psi, \overline{\partial}\eta) = 0$ for all $\eta \in A^{p,q-1}(M)$, so $\overline{\partial}^*\psi = 0$. $\square$

Lemma 2.2. *$\psi \in A^{p,q}(M)$ satisfies the equations $\overline{\partial}\psi = 0$ and $\overline{\partial}^*\psi = 0$ if and only if it satisfies the single equation*

$$\Delta_{\overline{\partial}}\psi := (\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\psi = 0.$$

$\Delta_{\overline{\partial}} : A^{p,q}(M) \to A^{p,q}(M)$ is called the Laplacian *operator.*

Proof. The forward direction is immediate. On the other hand, if $\Delta_{\overline{\partial}}\psi = 0$, then

$$0 = (\Delta_{\overline{\partial}}\psi, \psi) = (\overline{\partial}\overline{\partial}^*\psi, \psi) + (\overline{\partial}^*\overline{\partial}\psi, \psi) = \|\overline{\partial}^*\psi\|^2 + \|\overline{\partial}\psi\|^2$$

so $\overline{\partial}\psi = \overline{\partial}^*\psi = 0$. $\square$

Consequently, we expect each element of $H^{p,q}_{\overline{\partial}}(M)$ to have a unique representative given by a solution of the equation $\Delta_{\overline{\partial}}\psi = 0$; that is, if $\mathscr{H}^{p,q}_{\overline{\partial}}(M)$ is the space of forms ψ of type (p,q) satisfying $\Delta_{\overline{\partial}}\psi = 0$ (these are called *harmonic* forms), there should be an isomorphism $\mathscr{H}^{p,q}_{\overline{\partial}}(M) \cong H^{p,q}_{\overline{\partial}}(M)$. This is indeed true, and is a consequence of the Hodge theorem.

Theorem 2.3 (Hodge theorem)**.** *Let M be a connected, compact complex manifold of complex dimension n. Then $\dim \mathscr{H}^{p,q}_{\overline{\partial}}(M)$ is finite, there is a well-defined orthogonal projection operator $\mathscr{H}_{\overline{\partial}} : A^{p,q}(M) \to \mathscr{H}^{p,q}_{\overline{\partial}}(M)$, and there is a unique operator $G_{\overline{\partial}} : A^{p,q}(M) \to A^{p,q}(M)$, called the Green's operator, such that $G_{\overline{\partial}}(\mathscr{H}^{p,q}_{\overline{\partial}}(M)) = 0$, $\overline{\partial}G_{\overline{\partial}} = G_{\overline{\partial}}\overline{\partial}$, $\overline{\partial}^*G_{\overline{\partial}} = G_{\overline{\partial}}\overline{\partial}^*$, and $\mathrm{id} = \mathscr{H}_{\overline{\partial}} + \Delta_{\overline{\partial}}G_{\overline{\partial}}$ on $A^{p,q}(M)$.*

Corollary 2.4. *The map $\mathscr{H}^{p,q}_{\overline{\partial}}(M) \to H^{p,q}_{\overline{\partial}}(M)$ is an isomorphism.*

Proof. By Lemma 2.1, this map is an injection. By the Hodge theorem, we have

$$\psi = \mathscr{H}_{\overline{\partial}}(\psi) + \overline{\partial}(\overline{\partial}^*G_{\overline{\partial}}\psi) + \overline{\partial}^*(\overline{\partial}G_{\overline{\partial}}\psi)$$

for any $\psi \in A^{p,q}(M)$. If $\psi \in Z^{p,q}(M)$, then $\overline{\partial}G_{\overline{\partial}}\psi = G_{\overline{\partial}}\overline{\partial}\psi = 0$, so

$$\psi = \mathscr{H}_{\overline{\partial}}(\psi) + \overline{\partial}(\overline{\partial}^*G_{\overline{\partial}}\psi)$$

meaning that ψ is in the class of $\mathscr{H}_{\overline{\partial}}(\psi) \in H^{p,q}_{\overline{\partial}}(M)$, and the map is a surjection. $\square$

2.2 Proof sketch for the Hodge theorem

We will give a vague idea of the proof of the Hodge theorem, just enough to explain how $\mathscr{H} = \mathscr{H}_{\overline{\partial}}$ and $G = G_{\overline{\partial}}$ are constructed. Let $\Delta = \Delta_{\overline{\partial}}$, let $\hat{A}_0^{p,q}(M)$ be the Hilbert-space completion of $A^{p,q}(M)$ under the previously defined inner product $(\psi, \eta) = \int_M (\psi(z), \eta(z))\Phi(z)$, and let $\hat{A}_1^{p,q}(M) \subset \hat{A}_0^{p,q}(M)$ be the Hilbert-space completion of $A^{p,q}(M)$ under the Dirichlet inner product $(\cdot, \cdot)_1$ given by

$$(\psi, \eta)_1 = (\psi, \eta) + (\overline{\partial}\psi, \overline{\partial}\eta) + (\overline{\partial}^*\psi, \overline{\partial}^*\eta) = (\psi, (\mathrm{id} + \Delta)\eta).$$

Because the norm induced by $(\cdot, \cdot)_1$ is equivalent to a certain Sobolev norm, by going through various facts about Sobolev spaces, it is possible to show that for any $\psi \in \hat{A}_0^{p,q}(M)$, the linear functional on $A^{p,q}(M)$ given by $\eta \mapsto (\psi, \eta)$ extends to a bounded linear functional on $\hat{A}_1^{p,q}(M)$. By the Riesz representation theorem, this means there is some $T(\psi) \in \hat{A}_1^{p,q}(M) \subset \hat{A}_0^{p,q}(M)$ such that

$$(\psi, \eta) = (T(\psi), \eta)_1 = (T(\psi), (\mathrm{id} + \Delta)\eta)$$

for all $\eta \in A^{p,q}(M)$. Now $T : \hat{A}_0^{p,q}(M) \to \hat{A}_0^{p,q}(M)$ is linear, one-to-one, and self-adjoint (since id and Δ are self-adjoint), and by more facts about Sobolev spaces, it is also compact. Then by the Spectral Theorem, $\hat{A}_0^{p,q}(M)$ splits into a Hilbert space direct sum $\oplus_m E(\rho_m)$ of finite-dimensional eigenspaces $E(\rho_m)$ of T for nonzero eigenvalues ρ_m, $m = 0, 1, \ldots$. Assume $\rho_0 = 1$.

Now $\psi \in E(\rho_m)$ if and only if $T\psi = \rho_m\psi$, which is true if and only if $(\psi, \eta) = (\rho_m\psi, (\mathrm{id} + \Delta)\eta)$ for all $\eta \in A^{p,q}(M)$, which is true if and only if $\psi = \rho_m\psi + \rho_m\Delta\psi$, which is true if and only if $\Delta\psi = \frac{1-\rho_m}{\rho_m}\psi$ (where we interpret $\Delta\psi$ to mean that this equality holds "weakly"). Therefore Δ has the same eigenspaces $E(\rho_m)$ as T, with eigenvalues $\frac{1-\rho_m}{\rho_m}$, $m = 0, 1, \ldots$.

Hence we can define $\mathscr{H}$ to be the projection of $\hat{A}_0^{p,q}(M)$ onto $E(\rho_0)$ and $G : \hat{A}_0^{p,q}(M) \to \hat{A}_0^{p,q}(M)$ by $G(\psi) = 0$ for $\psi \in E(\rho_0)$ and $G(\psi) = \frac{\rho_m}{1-\rho_m}\psi$ for $\psi \in E(\rho_m)$, $m > 0$. These are the operators whose existence the Hodge theorem asserts.

2.3 Other Laplacians, and vector bundles

Let $\Delta_d = dd^* + d^*d$, and $\Delta_\partial = \partial\partial^* + \partial^*\partial$. Let $\mathscr{H}_d^k(M)$ the space of forms ψ of degree k satisfying $\Delta_d\psi = 0$. Then an exactly analogous statement to the Hodge theorem holds for Δ_d, so that we have $\mathscr{H}_d^k(M) \cong H_{dR}^k(M)$.

For a holomorphic vector bundle $E \to M$, if we are given metrics on both M and E, we have induced metrics on $\wedge^p T^\vee(M) \otimes \wedge^q \overline{T}^\vee(M) \otimes E$. If Φ is the volume form on M, we can define an inner product $(\psi, \eta) = \int_M (\psi(z), \eta(z))\Phi$ for $\psi, \eta \in A^{p,q}(E)$ as before; we have a "wedge product"

$$\wedge : A^{p,q}(M; E) \otimes A^{p',q'}(M; E^\vee) \to A^{p+p',q+q'}(M)$$

given by

$$(\psi \otimes s) \wedge (\psi' \otimes s') = s'(s)\psi \wedge \psi';$$

we then get a star operator

$$\star_E : A^{p,q}(M; E) \to A^{n-q,n-p}(M; E^\vee)$$

satisfying, for $\psi, \eta \in A^{p,q}(M;E)$,

$$(\psi, \eta) = \int_M \psi \wedge \overline{\star_E \eta};$$

and we can construct the adjoint

$$\overline{\partial}^* : A^{p,q}(M;E) \to A^{p,q-1}(M;E)$$

of $\overline{\partial}$ by setting $\overline{\partial}^* = -\star_E \overline{\partial} \star_E$. Then letting $\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ and $\mathscr{H}^{p,q}(M;E) = \ker \Delta$, we get an analogous statement to the Hodge theorem for Δ: $\mathscr{H}^{p,q}(M;E)$ is finite-dimensional; there is an orthogonal projection $\mathscr{H} : A^{p,q}(M;E) \to \mathscr{H}^{p,q}(M;E)$ and an operator $G : A^{p,q}(M;E) \to A^{p,q}(M;E)$ so that $G\mathscr{H} = 0$ and G commutes with $\overline{\partial}$ and $\overline{\partial}^*$, and $\mathrm{id} = \mathscr{H} + \Delta G$; and the map $\mathscr{H}^{p,q}(M;E) \to H^{p,q}_{\overline{\partial}}(M;E)$ is an isomorphism.

3 Kähler manifolds

We now describe manifolds given a special kind of metric, called a Kähler metric, which imposes identifications between the harmonic spaces for ∂, $\overline{\partial}$, and d. We will see that these identifications together with the Hodge theorem provide additional information about the cohomology groups of such manifolds.

3.1 The Kähler condition

Let $ds^2 = \sum_i \varphi_i \otimes \overline{\varphi}_i$ be a Hermitian metric on the complex manifold M.

Lemma 3.1. *The following are equivalent:*

1. The associated $(1,1)$-form $\omega = \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \overline{\varphi}_i$ of ds^2 is d-closed.

2. For every point $z_0 \in M$, we can find a holomorphic coordinate system $(z_1, \dots, z_n)$ in a neighborhood of z_0 for which

$$ds^2 = \sum (\delta_{ij} + g_{ij}) dz_i \otimes d\overline{z}_j$$

where g_{ij} vanishes to order 2 at z_0. (When this is the case, we will also write $ds^2 = \sum(\delta_{ij} + [2]) dz_i \otimes d\overline{z}_j$.)

Proof. 2 implies 1: if we have a coordinate system $(z_1, \dots, z_n)$ around z_0 in which $\omega = \frac{\sqrt{-1}}{2} \sum (\delta_{ij} + [2]) dz_i \wedge d\overline{z}_j$ at z_0, then all first partial derivatives of the coefficients vanish, so $d\omega(z_0) = 0$.

1 implies 2: choose a coordinate system around z_0 so that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} \left(\delta_{ij} + \sum_k (a_{ijk} z_k + b_{ijk} \overline{z}_k) + [2] \right) dz_i \wedge d\overline{z}_j.$$

Since ds^2 is hermitian, $b_{jik} = \overline{a_{ijk}}$. Since $d\omega = 0$, $a_{ijk} = a_{kji}$. From this it is straightforward to compute a change of coordinates $z_k = w_k + \frac{1}{2} \sum_{l,m} c_{klm} w_l w_m$ such that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} (\delta_{ij} + [2]) dw_i \wedge d\overline{w}_j.$$

It turns out that $c_{jki} = -a_{ijk}$ works. $\square$

Definition 3.1. If the conditions in Lemma 3.1 are satisfied, we say that ds^2 is a Kähler metric. If a complex manifold M is given a Kähler metric, we call it a Kähler manifold.

3.2 The Hodge identities

Define $L : A^{p,q}(M) \to A^{p+1,q+1}(M)$ by $L(\eta) = \eta \wedge \omega$; let L^* be its adjoint.

Proposition 3.2. *1.* $[L^*, \partial] = \sqrt{-1}\overline{\partial}^*$.
2. $[L^*, \overline{\partial}] = -\sqrt{-1}\partial^*$.
3. $[L, L^*] = p + q - n$.

Proof. We first check the identities on $\mathbb{C}^n$ with the Euclidean metric. Define $e_k : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p+1,q}(\mathbb{C}^n)$ by $e_k(\varphi) = dz_k \wedge \varphi$; $\overline{e}_k : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p,q+1}(\mathbb{C}^n)$ by $\overline{e}_k(\varphi) = d\overline{z}_k \wedge \varphi$; $i_k = e_k^*$, $\overline{i}_k = \overline{e}_k^*$. (The c in the subscript is to indicate that we are restricting our attention to compactly supported forms.) For multiindices J, K, L, M such that $k \notin J$, we have

$$(i_k(dz_J \wedge d\overline{z}_K), dz_L \wedge d\overline{z}_M) = (dz_J \wedge d\overline{z}_K, dz_k \wedge dz_L \wedge d\overline{z}_M) = 0$$

whereas

$$(i_k(dz_k \wedge dz_J \wedge d\overline{z}_K), dz_L \wedge d\overline{z}_M) = (dz_k \wedge dz_J \wedge d\overline{z}_K, dz_k \wedge dz_L \wedge d\overline{z}_M) = 2(dz_J \wedge d\overline{z}_K, dz_L \wedge d\overline{z}_M)$$

from which we conclude that, if $k \notin J$, $i_k(dz_J \wedge d\overline{z}_K) = 0$ and $i_k(dz_k \wedge dz_J \wedge d\overline{z}_K) = 2dz_J \wedge d\overline{z}_K$. One can then calculate that $i_k e_k + e_k i_k = 2$ and $e_k i_l + i_l e_k = 0$, and similarly $\overline{i}_k \overline{e}_k + \overline{e}_k \overline{i}_k = 2$. Next define $\partial_k, \overline{\partial}_k : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p,q}(\mathbb{C}^n)$ by

$$\partial_k \left(\sum \varphi_{I,J} dz_I \wedge d\overline{z}_J \right) = \sum \frac{\partial \varphi_{I,J}}{\partial z_k} dz_I \wedge d\overline{z}_J$$

$$\overline{\partial}_k \left(\sum \varphi_{I,J} dz_I \wedge d\overline{z}_J \right) = \sum \frac{\partial \varphi_{I,J}}{\partial \overline{z}_k} dz_I \wedge d\overline{z}_J.$$

$\partial_k, \overline{\partial}_k$ commute with $e_l, \overline{e}_l, i_l, \overline{i}_l$, and each other; one can compute that the adjoint of ∂_k is $-\overline{\partial}_k$. Then $\partial = \sum_k \partial_k e_k$, $\overline{\partial} = \sum_k \overline{\partial}_k \overline{e}_k$, so $\partial^* = -\sum \overline{\partial}_k i_k$, $\overline{\partial}^* = -\sum \partial_k \overline{i}_k$; finally $L = \frac{\sqrt{-1}}{2} \sum e_k \overline{e}_k$, $L^* = -\frac{\sqrt{-1}}{2} \sum \overline{i}_k i_k$. It is then straightforward to check the identity $L^* \partial = \partial L^* + \sqrt{-1}\overline{\partial}^*$ by plugging in for each term the corresponding expression in terms of $\partial_k, \overline{\partial}_k, e_l, \overline{e}_l, i_l, \overline{i}_l$; similarly for the other identities.

We now claim that the first identity holds for any Kähler manifold. This is just because the operators $[L^*, \partial]$ and $\overline{\partial}^* = - \star \partial \star$ only depend on the values and first derivatives of the coefficients of the metric, which agree with the Euclidean one. This also implies $[L^*, \overline{\partial}] = -\sqrt{-1}\partial^*$, since L^* is a real operator. The third identity is proved in the same way. □

Proposition 3.3. *If $\Delta_d = dd^* + d^*d$, then $[L, \Delta_d] = [L^*, \Delta_d] = 0$.*

Proof. Summing parts 1 and 2 of Proposition 3.2 gives $[L^*, d] = -4\pi d^{c*}$. Since ω is closed, we have $d(\omega \wedge \eta) = \omega \wedge d\eta$ and so $[L, d] = [L^*, d^*] = 0$. Then

$$L^*(dd^* + d^*d) = (dL^* - 4\pi d^{c*})d^* + d^*L^*d$$

$$= dd^*L^* + d^*(L^*d - 4\pi d^{c*}) = dd^*L^* + d^*dL^* = (dd^* + d^*d)L^*$$

as desired. □

Proposition 3.4. $\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial$.

Proof. By substituting in $\overline{\partial}^* = -\sqrt{-1}[L^*, \partial]$ from Proposition 3.2, we compute $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. From this we find that

$$\Delta_d = (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) = \Delta_\partial + \Delta_{\overline{\partial}}.$$

Then using $\overline{\partial}^* = -\sqrt{-1}[L^*, \partial]$ again, we can check that $\Delta_\partial = \Delta_{\overline{\partial}}$, which gives the claim. $\square$

We obtain similar results for E a holomorphic vector bundle on the Kähler manifold M: in this case we define

$$L : A^{p,q}(M; E) \to A^{p+1,q+1}(M; E)$$
$$\psi \otimes s \mapsto \omega \wedge \psi \otimes s$$

for $\psi \in A^{p,q}(M)$ and $s \in A^0(M; E)$, where ω is the associated $(1,1)$-form for M. If $\nabla = \nabla_1 + \nabla_2 = \nabla_1 + \overline{\partial}$ is the metric connection on E, the identity corresponding to Part 2 of Proposition 3.2 is:

Proposition 3.5.

$$[L^*, \overline{\partial}] = -\sqrt{-1}D_1^*.$$

3.3 Hodge decomposition

Let $Z_d^{p,q}(M)$ be the space of d-closed forms of type (p,q), and

$$H^{p,q}(M) = \frac{Z_d^{p,q}(M)}{dA^\bullet(M) \cap Z_d^{p,q}(M)}.$$

Let $\mathscr{H}_d^{p,q}(M)$ be the space of forms ψ of type (p,q) satisfying $\Delta_d\psi = 0$.

Theorem 3.6. *For a compact Kähler manifold M, we have*

$$H_{dR}^k(M) \cong \oplus_{p+q=k} H^{p,q}(M)$$
$$H^{p,q}(M) = \overline{H^{q,p}(M)}.$$

Proof. Since $\Delta_{\overline{\partial}}$ takes $A^{p,q}(M)$ to $A^{p,q}(M)$, by Proposition 3.4, so does $\Delta_d = 2\Delta_{\overline{\partial}}$. Thus we have

$$\mathscr{H}_d^k(M) = \oplus_{p+q=k}\mathscr{H}_d^{p,q}(M).$$

The isomorphism $\mathscr{H}_d^k(M) \to H_{dR}^k(M)$ from the Hodge theorem certainly takes $\mathscr{H}_d^{p,q}(M)$ into $H^{p,q}(M) \subset H_{dR}^k(M)$. Conversely, if ψ is a d-closed form of type (p,q), then we have

$$\psi = \mathscr{H}_d(\psi) + \Delta_d G_d(\psi) = \mathscr{H}_d(\psi) + dd^* G_d(\psi).$$

Note that $G_d = \frac{1}{2}G_{\overline{\partial}}$ again by Proposition 3.4, so $\Delta_d G_d(\psi)$ is of type (p,q), and therefore $\mathscr{H}_d(\psi)$ is as well. Thus ψ is in the class of $\mathscr{H}_d(\psi) \in \mathscr{H}_d^{p,q}(M)$, and $\mathscr{H}_d^{p,q}(M)$ surjects onto $H^{p,q}(M)$. So $\mathscr{H}_d^{p,q}(M) \cong H^{p,q}(M)$, which gives the desired decomposition of $H_{dR}^k(M)$.

For the second claim, it suffices to note that since Δ_d is a real operator,

$$\mathscr{H}_d^{p,q}(M) = \overline{\mathscr{H}_d^{q,p}(M)}.$$

$\square$

4 Applications

We now provide a series of results that may be obtained as consequences of the Hodge theorem, identities, and decomposition.

4.1 Kodaira-Serre Duality

Theorem 4.1. *There is a nondegenerate pairing*

$$H^q(M,\Omega^p)\otimes H^{n-q}(M,\Omega^{n-p})\to\mathbb{C}.$$

Proof. This is equivalent to giving a nondegenerate pairing

$$H^{p,q}_{\overline{\partial}}(M)\otimes H^{n-p,n-q}_{\overline{\partial}}(M)\to\mathbb{C}.$$

The pairing is just

$$\psi\otimes\eta\mapsto\int_M\psi\wedge\eta$$

which is well-defined by the Leibniz rule and Stokes' Theorem; it is nondegenerate because

$$H^{p,q}_{\overline{\partial}}(M)\cong\mathscr{H}^{p,q}_{\overline{\partial}}(M)$$

and if $\psi\in\mathscr{H}^{p,q}_{\overline{\partial}}(M)$ is nonzero, then

$$\psi\otimes\overline{\star\psi}\mapsto\int_M\psi\wedge\overline{\star\psi}=(\psi,\psi)>0.$$

□

4.2 Kodaira-Nakano Vanishing Theorem

Let M be a compact Kähler manifold.

Proposition 4.2. *If ω is any real closed $(1,1)$-form in the class of $c_1(L)\in H^2_{dR}(M)$, then there is a metric connection on L with curvature form $\Theta=\frac{\sqrt{-1}}{2\pi}\omega$.*

Proof. Fix a metric $(\cdot,\cdot)$ on L and let ∇ be its metric connection, with connection matrix θ—in this case just a single $(1,0)$-form. Let e be a local nonzero holomorphic section of L and let $h(z)=\|e(z)\|^2$. For any other section $s=\lambda e$, we have

$$\begin{aligned}d(s,s)&=(\nabla s,s)+(s,\nabla s)=((d\lambda+\theta\lambda)e,\lambda e)+(\lambda e,(d\lambda+\theta\lambda)e)\\&=h\overline{\lambda}d\lambda+h\lambda d\overline{\lambda}+h\|\lambda\|^2(\theta+\overline{\theta})\end{aligned}$$

but also

$$d(s,s)=d(\lambda\overline{\lambda}h)=h\overline{\lambda}d\lambda+h\lambda d\overline{\lambda}+\|\lambda\|^2dh.$$

As λ varies, we see that in fact

$$\theta+\overline{\theta}=\frac{dh}{h}$$

so

$$\theta = \partial \log h = \partial \log \|e\|^2,$$

$$\Theta = d\theta - \theta \wedge \theta = d\theta = \overline{\partial}\partial \log h = -\partial\overline{\partial} \log h.$$

Suppose we have another metric $(\cdot,\cdot)_1$ with curvature form Θ_1, and let $h_1/h = \|e\|_1^2/\|e\|^2 = e^\rho$, where ρ is a real C^∞ function on M. Then we have

$$\Theta_1 = -\partial\overline{\partial} \log h_1 = -\partial\overline{\partial} \log e^\rho h = -\partial\overline{\partial}\rho - \partial\overline{\partial} \log h = -\partial\overline{\partial}\rho + \Theta.$$

So what we want to show is that if Θ_1 is such that $\frac{2\pi}{\sqrt{-1}}\Theta_1$ is a real closed $(1,1)$-form in the class $c_1(L)$ of $H^2_{dR}(M)$, then we can find a real C^∞ function ρ satisfying the equation $\partial\overline{\partial}\rho = \Theta - \Theta_1$; then the metric $\|s\|_1^2 = e^\rho \|s\|^2$ will have Θ_1 as its curvature form.

We will prove more generally that if η is any (p,q)-form on a compact Kähler manifold which is d-exact, then we can find a $(p-1,q-1)$-form γ with $\eta = \partial\overline{\partial}\gamma$. (Of course this applies in our case since Θ and Θ_1 are in the same de Rham cohomology class, so $\Theta - \Theta_1$ is d-exact.) Recall that $\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\overline{\partial}}$ and $2G_d = G_\partial = G_{\overline{\partial}}$, so that each of $d, \partial, \overline{\partial}, d^*, \partial^*, \overline{\partial}^*$ commutes with each of G_d, G_∂, $G_{\overline{\partial}}$. Since η is exact with respect to d, we have $\mathscr{H}_d(\eta) = \mathscr{H}_\partial(\eta) = \mathscr{H}_{\overline{\partial}}(\eta) = 0$. Therefore, by the Hodge theorem for $\overline{\partial}$,

$$\eta = \mathscr{H}_{\overline{\partial}}(\eta) + \overline{\partial}\overline{\partial}^* G_{\overline{\partial}}\eta + \overline{\partial}^*\overline{\partial} G_{\overline{\partial}}\eta = \overline{\partial}\overline{\partial}^* G_{\overline{\partial}}\eta.$$

In the proof of Proposition 3.4 we saw that $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$. Therefore

$$\partial\overline{\partial}^* G_{\overline{\partial}}\eta = -\overline{\partial}^*\partial G_{\overline{\partial}}\eta = -\overline{\partial}^* G_{\overline{\partial}}\partial\eta = 0$$

which, since $\mathscr{H}^{p,q}_\partial(M) = \mathscr{H}^{p,q}_{\overline{\partial}}(M)$ and so both spaces are orthogonal to $\operatorname{im}\overline{\partial}^*$, implies that

$$\overline{\partial}^* G_{\overline{\partial}}\eta = \mathscr{H}_\partial(\overline{\partial}^* G_{\overline{\partial}}\eta) + \partial\partial^* G_\partial \overline{\partial}^* G_{\overline{\partial}}\eta + \partial^*\partial G_\partial \overline{\partial}^* G_{\overline{\partial}}\eta = \partial\partial^* G_\partial \overline{\partial}^* G_{\overline{\partial}}\eta$$

so that

$$\eta = \overline{\partial}\overline{\partial}^* G_{\overline{\partial}}\eta = \overline{\partial}\partial\partial^* G_\partial \overline{\partial}^* G_{\overline{\partial}}\eta = -\partial\overline{\partial}(\partial^*\overline{\partial}^* G^2_{\overline{\partial}}\eta)$$

and the form $\gamma = -\partial^*\overline{\partial}^* G^2_{\overline{\partial}}\eta$ is, as desired, a solution to the equation. □

Definition 4.1. A line bundle $L \to M$ is called positive if there is a metric on L with curvature form Θ such that $\frac{\sqrt{-1}}{2\pi}\Theta$ is a positive $(1,1)$-form, and L is called *negative* if $L^\vee$ is positive.

Combining Proposition 1.2 and Proposition 4.2, we see that L is positive if and only if its Chern class has a representative which is a positive form in $H^2_{dR}(M)$, that is to say, by the discussion in Section 1.3, a representative which is the associated $(1,1)$-form of a metric on M, which must be Kähler since the form is closed.

Theorem 4.3 (Kodaira-Nakano Vanishing Theorem)**.** *If $L \to M$ is a positive line bundle, then $H^{p,q}_{\overline{\partial}}(M;L) = 0$ for $p+q > n$.*

It is worth remembering here that $H^{p,q}_{\overline{\partial}}(M;L)$ is also isomorphic to $H^q_{Cech}(M;\Omega^p(L))$, where $\Omega^p(L)$ is the sheaf of holomorphic p-forms with coefficients in L.

Proof. Let Θ be the curvature form of the given metric on L, so that $\omega = \frac{\sqrt{-1}}{2\pi}\Theta$ is a positive real closed $(1,1)$-form, hence the associated $(1,1)$-form of a Kähler metric on M. By the Hodge theorem, we have $H^{p,q}_{\overline{\partial}}(M;L) \cong \mathscr{H}^{p,q}(M;L)$ (we drop the subscript since all the harmonic spaces are the same). We will show that there are no nonzero harmonic L-valued forms of degree larger than n, which gives the result.

Let $\eta \in \mathscr{H}^{p,q}(M;L)$, so that $\overline{\partial}\eta = 0$ and $\overline{\partial}^*\eta = 0$. Now

$$D^2\eta = (D_1 + D_2) \circ (D_1 + D_2)\eta = (D_1\overline{\partial} + \overline{\partial}D_1)\eta = \overline{\partial}D_1\eta$$

so, using Proposition 3.5,

$$\sqrt{-1}(L^*D^2\eta, \eta) = 2\sqrt{-1}(L^*\overline{\partial}D_1\eta, \eta)$$

$$= \sqrt{-1}((\overline{\partial}L^* - \sqrt{-1}D_1^*)D_1\eta, \eta) = \sqrt{-1}(\overline{\partial}L^*D_1\eta, \eta) + (D_1^*D_1\eta, \eta)$$
$$= \sqrt{-1}(L^*D_1\eta, \overline{\partial}^*\eta) + (D_1\eta, D_1\eta) = (D_1\eta, D_1\eta) \geq 0$$

and by a similar computation,

$$\sqrt{-1}(D^2L^*\eta, \eta) = -(D_1^*\eta, D_1^*\eta) \leq 0$$

so

$$\sqrt{-1}([L^*, D^2]\eta, \eta) \geq 0.$$

Now write $\eta = \eta_1 \otimes e$, where $\eta_1 \in \mathscr{H}^{p,q}(M)$ and e is a section of L. It is a standard fact that by applying the Leibniz rule repeatedly, we get

$$D^2\eta = D^2(\eta_1 \otimes e) = \eta_1 \wedge D\nabla e = \eta_1 \wedge \Theta \otimes e = \eta \wedge \Theta.$$

This means that $D^2 = \frac{2\pi}{\sqrt{-1}}L$, so

$$0 \leq \sqrt{-1}([L^*, D^2]\eta, \eta) = 2\pi([L^*, L]\eta, \eta) = 4\pi(n - p - q)\|\eta\|^2.$$

Thus if $p + q > n$, we must have $\eta = 0$. □

Using essentially the same method, one may also prove the following generalization:

Theorem 4.4. *Let M be a compact complex manifold and $L \to M$ a positive line bundle. Then for any holomorphic vector bundle E, there exists μ_0 such that*

$$H^q(M, \mathscr{O}(L^\mu \otimes E)) = 0$$

for $q > 0$ and $\mu \geq \mu_0$.

4.3 All line bundles come from divisors

Theorem 4.5. *Let $M \subset \mathbb{P}^N$ be a submanifold. Then every line bundle on M is of the form $[D]$ for some divisor D.*

Proof. We need to show that every line bundle L on M has a global meromorphic section. Let H be the restriction to M of the hyperplane bundle on $\mathbb{P}^N$ (the bundle associated to a divisor coming from a hyperplane), with some global holomorphic section t. We will show that for sufficiently large μ, $L \otimes H^\mu$ has a nontrivial global holomorphic section s; then s/t^μ is a global meromorphic section of L.

We proceed by induction on the dimension of M; the claim is trivial if this is 0. Assume as the inductive hypothesis that for every submanifold $V \subset \mathbb{P}^N$ of dimension less than n and every line bundle $K \to V$, we have $\dim H^0(V, \mathscr{O}(K \otimes H^\mu)) > 0$ for μ sufficiently large. By Bertini's theorem, we can choose a hyperplane $H_1 \subset \mathbb{P}^N$ so that $V = H_1 \cap M$ is smooth. Then we have an exact sequence of sheaves

$$0 \to \mathscr{O}_M(L \otimes H^{\mu-1}) \to \mathscr{O}_M(L \otimes H^\mu) \to \mathscr{O}_V(L \otimes H^\mu) \to 0$$

where the first map is given by tensoring with a section of H cutting out H_1, and the second by restriction. This gives a long exact sequence

$$\cdots H^0(M, \mathscr{O}(L \otimes H^\mu)) \to H^0(V, \mathscr{O}(L \otimes H^\mu)) \to H^1(M, \mathscr{O}(L \otimes H^{\mu-1})) \cdots .$$

For μ sufficiently large, we have $\dim H^0(V, \mathscr{O}(L \otimes H^\mu)) > 0$ by the inductive hypothesis and $H^1(M, \mathscr{O}(L \otimes H^{\mu-1})) = 0$ by Theorem 4.4, so that $H^0(M, \mathscr{O}(L \otimes H^\mu))$ surjects onto $H^0(V, \mathscr{O}(L \otimes H^\mu))$, hence is positive-dimensional as desired. $\square$

4.4 Lefschetz theorem on hyperplane sections

Theorem 4.6 (Lefschetz theorem on hyperplane sections)**.** *Let M be an n-dimensional compact complex manifold and $V \subset M$ a smooth hypersurface such that $[V]$ is a positive line bundle. Then the map $H^k(M, \mathbb{C}) \to H^k(V, \mathbb{C})$ induced by the inclusion $\iota : V \hookrightarrow M$ is an isomorphism for $k \leq n-2$ and an injection for $k = n-1$.*

Proof. We have the Hodge decomposition

$$H^k(M, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M)$$

and

$$H^{p,q}(M) \cong H^{p,q}_{\bar{\partial}}(M) \cong H^q(M, \Omega^p_M)$$

by the Hodge and Dolbeault theorems, and similarly for V. So it suffices to check that

$$H^q(M, \Omega^p_M) \to H^q(V, \Omega^p_V)$$

is an isomorphism if $p + q \leq n-2$ and an injection for $p + q = n-1$. This map is induced by the composition

$$\Omega^p_M \to^r \Omega^p_M|_V \to^i \Omega^p_V.$$

The kernel of r is the sheaf of holomorphic p-forms on M vanishing along V, that is, $\Omega_M^p(-V)$; thus we have an exact sequence

$$0 \to \Omega_M^p(-V) \to \Omega_M^p \to^r \Omega_M^p|_V \to 0$$

yielding the long exact sequence

$$\cdots H^q(M, \Omega_M^p(-V)) \to H^q(M, \Omega_M^p) \to^{r^*} H^q(M, \Omega_M^p|_V) \to H^{q+1}(M, \Omega_M^p(-V)) \cdots .$$

Since $[V]$ is positive, $[V]^\vee$ is negative; by dualizing the Kodaira-Nakano vanishing theorem, we see that $H^q(M, \Omega_M^p(-V)) = 0$ for $p+q < n$. Therefore r^* is an isomorphism for $p+q \leq n-2$ and an injection for $p+q = n-1$. Similarly, for each $z \in M$, the conormal exact sequence

$$0 \to N_{V,z}^\vee \to T_z^\vee(M) \to T_z^\vee(V) \to 0,$$

where N_V is the normal bundle of V in M, gives

$$0 \to N_{V,z}^\vee \otimes \wedge^{p-1} T_z^\vee(V) \to \wedge^p T_z^\vee(M) \to \wedge^p T_z^\vee(V) \to 0$$

and thus, using the adjunction formula $N_V^\vee = [V]^\vee|_V$, an exact sequence of sheaves

$$0 \to \Omega_V^{p-1}(-V) \to \Omega_M^p|_V \to^i \Omega_V^p \to 0.$$

Again taking the corresponding long exact sequence and applying dualized Kodaira-Nakano vanishing, we find that i^* is also an isomorphism for $p+q \leq n-2$ and an injection for $p+q = n-1$. Combining our results about r^* and i^* shows what we wanted. $\square$

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