p-ADIC FAMILIES OF AUTOMORPHIC FORMS NOTES

COURSE BY RICHARD TAYLOR

Introduction

Richard Taylor taught a course on p-adic families of automorphic forms at Stanford in Winter 2020.

Except for the last two lectures, these are "live-TeXed" notes from the course, contributed and maintained by various participants in the course. Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date. The last two chapters were written by the lecturer himself, as course meetings were cancelled due to COVID-19.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are our fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe(s).[1] Please email suggestions to ddore@stanford.edu or lynnelle@stanford.edu.

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[1]This introduction has been adapted from Akhil Mathew's introduction to his notes, with his permission.

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1. 1/7/20: Course overview

Let p be prime. Let $\mathbb{A} = \prod_v' \mathbb{Q}_v = (\hat{\mathbb{Z}} \otimes \mathbb{Q}) \times \mathbb{R}$.

1.1. Modular curves.

Let's start with two examples of the spaces whose cohomologies we will be interested in.

(1) The modular curve for GL_2 of level p^n is

$$Y(p^n) = \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) / \mathrm{GL}_2(\hat{\mathbb{Z}}^p) \times U(p^n) \times (\mathrm{SO}_2 \mathbb{R}_{>0}^\times)$$

where $\hat{\mathbb{Z}}^p = \varprojlim_{p \nmid m} \mathbb{Z}/m\mathbb{Z}$ and

$$U(p^n) = \ker(\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})).$$

This can be written in the form $\coprod_i \Gamma_i \backslash (\mathbb{C} - \mathbb{R})$ where the Γ_i are congruence subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

(2) Let D be a quaternion algebra of center $\mathbb{Q}$ ramified at a prime $r \neq p$ and ∞. We have $D \otimes \mathbb{R} \cong \mathbb{H}$, and $(D \otimes \mathbb{R})^\times / \mathbb{R}^\times$ is compact. The Shimura variety for D of level p^n is

$$Y_D(p^n) = D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / (\hat{\mathscr{O}}_D^p)^\times \times U(p^n)$$

where $\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, noting that $D \otimes \mathbb{Q}_p \cong M_{2\times 2}(\mathbb{Q}_p)$ so the subgroup $U(p^n)$ makes sense.

$Y(p^n)$ is a 1-dimensional $\mathbb{C}$-manifold or 2-dimensional $\mathbb{R}$-manifold; $Y_D(p^n)$ is 0-dimensional. We are interested in the cohomologies $H^i(Y(p^n), A)$ and $H^i(Y_D(p^n), A)$ where A is $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}/p^s\mathbb{Z}$, or $\mathbb{Q}_p/\mathbb{Z}_p$.

For GL_2, only H^0 and H^1 are nontrivial and H^0 is easy to understand because $Y(p^n)$ is the complement of finitely many points in a proper curve. So we'll mainly be concerned with $H^1(Y(p^n), A)$ or $H^1_c(Y(p^n), A)$. For D, we only need to consider $H^0(Y_D(p^n), A) = Map(Y_D(p^n), A)$. These are finite A-modules.

1.2. Limits of cohomologies.

If we take the limit $\varinjlim_n H^1_{(c)}(Y(p^n), A)$, the result has an action by $\mathrm{GL}_2(\mathbb{Q}_p)$, since any $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ takes $U(p^n)$ into $U(p^m)$ for $n \gg m$. This limit also has an action by

$$\mathbb{T}_0 = \mathbb{Z}_p[T_q, S_q^{\pm 1} \mid q \neq r, p \text{ prime}],$$

where T_q, S_q are Hecke operators coming from the maps $Y(p^n, q) \rightrightarrows Y(p^n)$. Finally, there is also an action by $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Similarly, $\varinjlim_n H^0(Y_D(p^n), A) = LC(D^\times \backslash (D \otimes \mathbb{A}^\infty)^\times / (\hat{\mathscr{O}}_D^p)^\times, A)$ (the locally constant functions) has an action by $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathbb{T}_0$.

As representations, these limits are smooth (all stabilizers are open) and admissible (the fixed points of any open subgroup are a finite A-module). We will refer to both of them, where $A = \mathbb{Z}_p$, as "H". We can recover the cohomology at finite level p^n by taking $U(p^n)$-invariants of H.

We will write $\mathbb{T}_{(c)}$ or $\mathbb{T}_D$ for the image of $\mathbb{T}_0$ in $H^1_{(c)}(Y(p^\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ or $H^0(Y_D(p^\infty), \mathbb{Q}_p/\mathbb{Z}_p)$.

We can complete p-adically to get

$$\hat{H}^1_{(c)}(Y(p^\infty), \mathbb{Z}_p) = \varprojlim_s H^1_{(c)}(Y(p^\infty), \mathbb{Z}/p^s\mathbb{Z})$$

$$\hat{H}^0(Y_D(p^\infty), \mathbb{Z}_p) = \varprojlim_s H^0(Y_D(p^\infty), \mathbb{Z}/p^s\mathbb{Z}) = Map_{cts}(Y_D(p^\infty), \mathbb{Z}_p)$$

with the p-adic topology. These have actions by $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathbb{T}_0 \times G_{\mathbb{Q}}$, where the action by $\mathrm{GL}_2(\mathbb{Q}_p)$ is continuous and "unitary", and the $G_{\mathbb{Q}}$ is for the GL_2-case only. We will call these $\hat{H}$. $\hat{H}_{\mathbb{Q}_p} = \hat{H} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a p-adic Banach space. We again can recover $\hat{H}^{U(p^n)} = H^1_{(c)}(Y(p^n), \mathbb{Z}_p)$ or $H^0(Y_D(p^n), \mathbb{Z}_p)$, and $\hat{H}^{sm} = H$ (by which we mean the smooth vectors, that is, those with open stabilizer in $\mathrm{GL}_2(\mathbb{Q}_p)$).

We can instead take the projective limit over the trace maps $\cdots Y(p^n) \to \cdots \to Y(p^2) \to Y(p)$ to get

$$\tilde{H}^1_{(c)} := \varprojlim_n H^1_{(c)}(Y(p^n), \mathbb{Z}_p)$$
$$\tilde{H}^0 := \varprojlim_n H^0(Y_D(p^n), \mathbb{Z}_p).$$

These have actions of $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathbb{T}_{(c)} \times G_{\mathbb{Q}}$ and $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathbb{T}_D$ respectively. We will call them $\tilde{H}$. Then $\tilde{H}$ has an action by

$$\mathbb{Z}_p[\![\mathrm{GL}_2(\mathbb{Z}_p)]\!] = \varprojlim \mathbb{Z}_p[\mathrm{GL}_2(\mathbb{Z}/p^s\mathbb{Z})]$$

and is finite over $\mathbb{Z}_p[\![\mathrm{GL}_2(\mathbb{Z}_p)]\!]$. Note that $\mathbb{Z}_p[\![\mathbb{Z}_p]\!] \cong \mathbb{Z}_p[\![T]\!]$. We have approximately

$$\tilde{H} = \mathrm{Hom}_{cts}(\hat{H}, \mathbb{Z}_p)$$

(for GL_2 we should put $\hat{H}_c$ instead of $\hat{H}$) with the weak topology on the RHS. This is because $H^{blah}(Y(p^m), \mathbb{Z}/p^s\mathbb{Z})$ and $H^{blah}_c(Y(p^m), \mathbb{Z}/p^s\mathbb{Z})$ are dual under Poincare duality and the limits go as expected.

1.3. **Localizations.**

$\mathbb{T}$ has only finitely many maximal ideals and can be written as $\prod_{\mathfrak{m} \text{ max}} \mathbb{T}_{\mathfrak{m}}$. Each $\mathbb{T}_{\mathfrak{m}}$ is a complete noetherian local $\mathbb{Z}_p$-algebra. For each $\mathfrak{m}$ we have a unique continuous semisimple representation

$$\overline{r}_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k(\mathfrak{m}))$$

such that

(1)
$$\overline{r}_{\mathfrak{m}}(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
(2) $\overline{r}_{\mathfrak{m}}$ is ramified only at p and r, and
(3) $\mathrm{tr}(\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_q)) = T_q$, $\det(\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_q)) = qS_q$ for $q \neq p, r$. Furthermore,
(4) in the D case, if φ lifts Frob_r, we have $r(\mathrm{tr}(\overline{r}_{\mathfrak{m}}(\varphi)))^2 = (1+r)^2 \det(\overline{r}_{\mathfrak{m}}(\varphi))$. In the GL_2-case, $\overline{r}_{\mathfrak{m}}$ is unramified at r.

Theorem 1.3.1 (Khare-Wintenberger)**.** *If $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ satisfies (1), (2), and (4), and is irreducible, then there exists $\mathfrak{m}$ and $k(\mathfrak{m}) \hookrightarrow \overline{\mathbb{F}}_p$ such that $\overline{r} \cong \overline{r}_{\mathfrak{m}}$.*

We call $\mathfrak{m}$ Eisenstein if $\overline{r}_{\mathfrak{m}}$ is reducible. Assume $\mathfrak{m}$ is not Eisenstein. Then there is $r_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}})$, a continuous representation satisfying (1)-(4), which is the universal deformation of $\overline{r}_{\mathfrak{m}}$ satisfying (1), (2), and (4). That is to say, if R is any complete noetherian local $\mathbb{Z}_p$-algebra and $r : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ satisfies (1), (2), and (4), then there is a unique $\mathbb{T}_{\mathfrak{m}} \to R$ taking $r_{\mathfrak{m}}$ to r.

Let $V_{\mathfrak{m}}$ be the underlying $\mathbb{T}_{\mathfrak{m}}[G_{\mathbb{Q}}]$-module of $r_{\mathfrak{m}}$. In the GL_2-case, we can write

$$H_{\mathfrak{m}} \cong V_{\mathfrak{m}} \otimes_{\mathbb{T}_{\mathfrak{m}}} H^+_{\mathfrak{m}}$$
$$\hat{H}_{\mathfrak{m}} \cong V_{\mathfrak{m}} \otimes_{\mathbb{T}_{\mathfrak{m}}} \hat{H}^+_{\mathfrak{m}}$$

$$\tilde{H}_{\mathfrak{m}} \cong V_{\mathfrak{m}} \otimes_{\mathbb{T}_{\mathfrak{m}}} \tilde{H}^+_{\mathfrak{m}}$$

for $H^+_{\mathfrak{m}}$ such that $H^+_{\mathfrak{m}} \cong \mathrm{Hom}_{\mathbb{T}_{\mathfrak{m}}[G_{\mathbb{Q}}]}(V_{\mathfrak{m}}, H_{\mathfrak{m}})$. In the D case define $H^+_{\mathfrak{m}} := H_{\mathfrak{m}}$ (similarly with $\hat{H}, \tilde{H}$). Then $H^+_{\mathfrak{m}}$ etc. have actions by $\mathbb{T}_{\mathfrak{m}} \times \mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\theta : \mathbb{T}_{\mathfrak{m}} \to \mathbb{Q}_p$ be a continuous homomorphism. $H^+_{\mathfrak{m}}[\theta]$ is usually 0. For special θ, we may get an irreducible smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. On the other hand $\hat{H}^+_m[\theta]$ (that is, the subspace of $x \in \hat{H}^+_m$ such that $T(x) = \theta(T)x$ for all $T \in \mathbb{T}_{\mathfrak{m}}$) is always nonzero and is a unitary Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Both eigenspaces depend only on $(\theta \circ r_m)|_{G_{\mathbb{Q}_p}}$, but $H^+_{\mathfrak{m}}[\theta]$ does not determine it, and $\hat{H}^+_{\mathfrak{m}}[\theta]$ does, as does $\tilde{H}_{\mathfrak{m}} \otimes_{\mathbb{T}_{\mathfrak{m}},\theta} \mathbb{Q}_p$.

We may now take the subspace $(\hat{H}^+_{\mathfrak{m}})^{la}$ of $\hat{H}^+_{\mathfrak{m}}$, consisting of the locally analytic vectors: $x \in \hat{H}^+_{\mathfrak{m}}$ such that $\mathrm{GL}_2(\mathbb{Q}_p) \to \hat{H}^+_{\mathfrak{m}}$, $g \mapsto gx$, is analytic in some neighborhood of $\mathbb{1}_2$. This has an action by the Lie algebra $\mathfrak{g} = \mathfrak{gl}_2$ such that $((\hat{H}^+_{\mathfrak{m}})^{la})^{\mathfrak{g}} = H^+_m$.

1.4. **Eigenvarieties.** We may now take the Jacquet module of $(\hat{H}^+_{\mathfrak{m}})^{la}$. That is, let

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2;$$

for Π a smooth representation its Jacquet module is $J_N(\Pi) = \Pi_{N(\mathbb{Q}_p)}$, the coinvariants of N, a representation of $T(\mathbb{Q}_p)$. Emerton defined an analogue J_N^{la} which works for locally analytic representations; then $J_N^{la}((\hat{H}^+_{\mathfrak{m}})^{la})$ is a locally analytic representation of $T(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$.

Let $\check{T}$ be the space of all locally analytic characters of $T(\mathbb{Q}_p)$, that is, $(\mathbb{Q}_p^\times)^\vee \times (\mathbb{Q}_p^\times)^\vee$. We can write $\mathbb{Q}_p^\times$ in the form

$$p^{\mathbb{Z}} \times (\text{finite}) \times (1 + p\mathbb{Z}_p),$$

hence write $(\mathbb{Q}_p^\times)^\vee$ in the form

$$\mathbb{G}_m \times (\text{finite}) \times (\text{open disc of radius 1 about 1}).$$

This is a rigid analytic space, and $J_N^{la}((\hat{H}^+_{\mathfrak{m}})^{la})$ gives rise to a coherent sheaf $\mathscr{F}$ on $T^\vee$.

This $\mathscr{F}$ still has an action by $\mathbb{T}_{\mathfrak{m}}$; let $\mathscr{T}$ be the image of $\mathbb{T}_{\mathfrak{m}}$ in $\underline{End}(\mathscr{F})$ over $T^\vee$. Taking $\mathrm{Spa}(\mathscr{T})$ gives the eigenvariety $\mathscr{E}$, which has a finite map to $T^\vee$. Composing with the map from $T^\vee$ to $\mathscr{W} = T(\mathbb{Z}_p)^\vee$ that forgets the values of the character on $\mathbb{Q}_p$ gives the map $\mathscr{E} \to \mathscr{W}$, which has discrete fibers ($\mathscr{W}$ is called "weight space"). We also have a map $\mathscr{E} \to (\mathrm{Spf}(\mathbb{T}_{\mathfrak{m}}))_\eta$, where $(\mathrm{Spf}(\mathbb{T}_{\mathfrak{m}}))_\eta$ is also a rigid space.

(1)
$$\begin{tikzcd} & \mathscr{E} \arrow[dl] \arrow[dr] & \\ (\mathrm{Spf}(\mathbb{T}_{\mathfrak{m}}))_\eta \arrow[dr] & & T^\vee \arrow[dl] \\ & \mathscr{W} & \end{tikzcd}$$

Fixing determinants, $\mathscr{E}, \mathscr{W}$ are 1-dimensional, $(\mathrm{Spf}(\mathbb{T}_{\mathfrak{m}}))_\eta$ is 2-dimensional, and $T^\vee$ is 3-dimensional. $\mathbb{Z}_p[\![T]\!]_\eta$ is the open unit disc. $\mathscr{E} \to (\mathrm{Spf}(\mathbb{T}_{\mathfrak{m}}))_\eta$ has finite fibers and its image is called the infinite fern; it is a sort of space-filling curve.

The image of $\mathscr{E} \to (\mathrm{Spf}(\mathbb{T}_{\mathfrak{m}}))_\eta$ can be characterized in terms of (φ, Γ)-modules. Roughly, points on $\mathscr{E}$ correspond to representations of $G_{\mathbb{Q}_p}$ whose associated rank-2 (φ, Γ)-module is given by an extension of rank-1 (φ, Γ)-modules (that is, locally analytic characters of $\mathbb{Q}_p^\times$).

We will construct/examine these spaces in more detail throughout the quarter. We will start with the p-adic functional analysis necessary to make all of this rigorous.

2. 1/9/20: Introduction to p-adic functional analysis

Fix some notation: let $L/\mathbf{Q_p}$ be a finite extension, $\mathscr{O} = \mathscr{O}_L$ its ring of integers, $\mathscr{O}/(\pi) = \mathbf{F}$ its finite residue field, $\lambda = (\pi)$ its maximal ideal. Letting $v\colon L^\times \twoheadrightarrow \mathbf{Z}$ be the discrete valuation, we define the absolute value on L to be $|\alpha| = (\#\mathbf{F})^{-v(\alpha)}$.

A *topological vector space* over L is a vector space V over L with a topology such that the addition map $+\colon V \times V \to V$ and the scalar multiplication map $L \times V \to V$ are continuous.

An appropriately general class of topological vector spaces consists of *locally convex topological vector spaces*. We may define these either in terms of *lattices* or in terms of *seminorms*.

2.1. Lattices.

A subset $\Lambda \subseteq V$ is called a *lattice* if it is an $\mathscr{O}$-submodule which spans V over L. Equivalently, for any $x \in V$, there is some $\alpha \in L^\times$ with $\alpha x \in \Lambda$.

Note that if Λ is a lattice, then $\alpha\Lambda$ is a lattice for any $\alpha \in L^\times$. If Λ_1, Λ_2 are lattices, then $\Lambda_1 \cap \Lambda_2$ is a lattice.

Two lattices Λ, Λ' are considered equivalent, written $\Lambda \sim \Lambda'$, if there exist $\alpha, \beta \in L^\times$ such that $\alpha\Lambda \subseteq \Lambda' \subseteq b\Lambda$. Note that if $\Lambda_1 \sim \Lambda_1'$ and $\Lambda_2 \sim \Lambda_2'$, then $\Lambda_1 \cap \Lambda_2 \sim \Lambda_1' \sim \Lambda_2'$.

Given a collection of lattices $\mathscr{L}$, we may define a topology on V. This topology $J_{\mathscr{L}}$ is the coarsest topology such that all elements of $\mathscr{L}$ are open and V is a topological vector space.

We define

$$\mathscr{L}' = \{\alpha_1\Lambda_1 \cap \cdots \cap \alpha_r\Lambda_r \colon \alpha_i \in L^\times, \Lambda_i \in L\}.$$

This is the closure of $\mathscr{L}$ under scalar multiplication and finite intersection. Note that $J_{\mathscr{L}} = J_{\mathscr{L}'}$, and that replacing the lattices in $\mathscr{L}$ by equivalent lattices does not change $\mathscr{L}'$. Using this, we can describe $J_{\mathscr{L}}$ more explicitly:

$$J_{\mathscr{L}} = J_{\mathscr{L}'} = \{U \subseteq V \mid \forall x \in U, \exists \Lambda \in \mathscr{L}' \text{ such that } x + \Lambda \subseteq U\}.$$

In other words, we take $\mathscr{L}'$ as a fundamental system of neighborhoods of 0.

To show this claim, it suffices to show that the set described above gives V the structure of a topological vector space. Continuity of scalar multiplication is equivalent to the statement that for any $x \in V, \alpha \in L^\times, \Lambda \in \mathscr{L}$, there is an open neighborhood of (α, x) in $L \times V$ which maps into $\alpha x + \Lambda$. We may find $\beta \in L^\times$ such that $\beta x \in \Lambda$ and $\beta \in \alpha\mathscr{O}$. Then we have:

$$\alpha x + \Lambda \supseteq (\alpha + \beta\mathscr{O})(x + \alpha^{-1}\Lambda) = \alpha x + \mathscr{O}\beta x + \beta\alpha^{-1}\Lambda + \Lambda.$$

This shows continuity of scalar multiplication. The case of addition is similar and easier.

Note that if we define $\widetilde{\mathscr{L}}$ to be the set of lattices equivalent to something in $\mathscr{L}$, we have $J_{\mathscr{L}} = J_{\widetilde{\mathscr{L}}}$.

2.2. Seminorms.

A second approach to describing locally convex topological vector spaces, resembling the archimedean case more closely, is to use *seminorms*. A *seminorm* $\|\cdot\|$ on V is a map $\|\cdot\|\colon V \to \mathbf{R}_{\geq 0}$ such that:

(1). $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in L, x \in V$.
(2). $\|x + y\| \leq \max(\|x\|, \|y\|)$ for all $x, y \in V$.

We call $\|\cdot\|$ a *norm* if in addition $\|x\|=0$ implies $x=0$. We say that two seminorms are equivalent, written $\|\cdot\|\sim\|\cdot\|'$, if there exist $C_1,C_2\in\mathbf{R}_{>0}^\times$ such that $C_1\|\cdot\|\le\|\cdot\|'\le C_2\|\cdot\|$. If $\|\cdot\|,\|\cdot\|'$ are seminorms and $c\in\mathbf{R}_{>0}$, then $C\|\cdot\|$ and $\max(\|\cdot\|,\|\cdot\|')$ are seminorms as well.

Let $\mathscr{V}$ be a set of seminorms. We use this to define the topology $J_{\mathscr{V}}$ on V as the weakest topology which makes V a topological vector space such that each $\|\cdot\|\in\mathscr{V}$ is continuous. This topology has a basis of open sets of the form

$$x+\{y\in V\colon \|y\|_i\le\epsilon_i, i=1,\dots,r\}$$

where $x\in V$, $\|\cdot\|_1,\dots,\|\cdot\|_r\in\mathscr{V}$, and $\epsilon_1,\dots,\epsilon_r\in\mathbf{R}_{>0}$.

If we define $\widetilde{\mathscr{V}}$ to be the set of seminorms equivalent to something in $\mathscr{V}$, we have $J_{\mathscr{V}}=J_{\widetilde{\mathscr{V}}}$.

2.3. **Translating between lattices and seminorms.** These two approaches are equivalent, by the following constructions. Given a lattice Λ, we define a seminorm $\|\cdot\|_\Lambda$:

$$\|x\|_\Lambda=\inf_{\substack{\alpha\in L^\times\\ x\in\alpha\Lambda}}|\alpha|=\begin{cases}0 & Lx\subseteq\Lambda\\ \min\{|\alpha|\colon \alpha\in L^\times, x\in\alpha\Lambda\} & Lx\not\subseteq\Lambda.\end{cases}$$

(There is a general theory for fields like $\mathbb{C}_p$ as well, but then you have to be a bit more careful since you can't say min.)

To show that $\|\cdot\|_\Lambda$ is a seminorm, we check the ultrametric inequality $\|x+y\|_\Lambda\le\max(\|x\|_\Lambda,\|y\|_\Lambda)$. If $Lx\subseteq\Lambda$, i.e. $\|x\|_\Lambda=0$, then $y\in\alpha\Lambda$ if and only if $x+y\in\alpha\Lambda$, and so $\|x+y\|=\|y\|$. If $Lx\not\subseteq\Lambda, Ly\not\subseteq\Lambda$, then there is some α with $|\alpha|$ minimal such that $x\in\alpha\Lambda$ and β with $|\beta|$ minimal such that $y\in\beta\Lambda$. Then

$$x+y\in\alpha\Lambda+\beta\Lambda=\begin{cases}\alpha\Lambda & |\alpha|\ge|\beta|\\ \beta\Lambda & |\beta|\ge|\alpha|\end{cases}$$

Thus, $\|x+y\|_\Lambda\le\max(|\alpha|,|\beta|)=\max(\|x\|_\Lambda,\|y\|_\Lambda)$ as desired.

We have the following compatibilities:

- $\Lambda'\subseteq\Lambda$ implies that $\|\cdot\|_{\Lambda'}\ge\|\cdot\|_\Lambda$.
- $\|\cdot\|_{\alpha\Lambda}=|\alpha|^{-1}\|\cdot\|_\Lambda$.
- If $\Lambda\sim\Lambda'$, then $\|\cdot\|_\Lambda\sim\|\cdot\|_{\Lambda'}$.
- $\|\cdot\|_{\Lambda\cap\Lambda'}=\max(\|\cdot\|_\Lambda,\|\cdot\|_{\Lambda'})$.

On the other hand, if $\|\cdot\|$ is a seminorm, we obtain a lattice $\Lambda(\|\cdot\|)$ via its *unit ball*:

$$\Lambda(\|\cdot\|)=\{x\in V\colon \|x\|\le 1\}.$$

This is always a lattice (this is easy to check).

We have the following compatibilities:

- $\|\cdot\|'\ge\|\cdot\|$ implies that $\Lambda(\|\cdot\|')\subseteq\Lambda(\|\cdot\|)$.
- $\alpha\Lambda(\|\cdot\|)=\Lambda(|\alpha|^{-1}\|\cdot\|)$.
- $\Lambda(\max(\|\cdot\|,\|\cdot\|')=\Lambda(\|\cdot\|)\cap\Lambda(\|\cdot\|')$.
- If $\|\cdot\|\sim\|\cdot\|'$, then $\Lambda(\|\cdot\|)\sim\Lambda(\|\cdot\|')$.
- $\Lambda(\|\cdot\|_\Lambda)=\{x\in V\colon \inf\{|\alpha|\colon \alpha\in L^\times, x\in\alpha\Lambda\}\le 1\}=\Lambda$.
- Note that $x\in\alpha\Lambda(\|\cdot\|)$ if and only if $\|x\|\le|\alpha|$, so $\|x\|_{\Lambda(\|\cdot\|)}=\inf\{|\alpha|\colon \alpha\in L^\times, \|x\|\le|\alpha|\}$. We have $\|\cdot\|\le\|\cdot\|_{\Lambda(\|\cdot\|)}\le|\pi^{-1}|\|\cdot\|$, so these norms are equivalent.
- Given a set of lattices $\mathscr{L}$, let $\mathscr{V}$ be the corresponding set of seminorms $\|\cdot\|_\Lambda$ for $\Lambda\in\mathscr{L}$. Then $J_{\mathscr{L}}=J_{\mathscr{V}}$.

2.4. **Locally convex topological vector spaces.** Now, we define:

Definition 2.4.1. V is a locally convex topological vector space if it is a topological vector space over L whose topology can be defined by a set of lattices, or equivalently by a set of seminorms.

Here are some immediate properties:

- V is a *Hausdorff* topological space if and only if the intersection of all open lattices Λ is 0.
- A seminorm on V is continuous if and only if $\|\cdot\|^{-1}([0,\epsilon))$ is open for all $\epsilon > 0$.

V is called *normable* if the topology can be defined by a single norm. Such V are always Hausdorff (but the converse is not true).

The appropriate notion of morphism between locally convex topological vector spaces is that of a *continuous* linear map. We denote $\mathscr{L}(V,W)$ for the set of all such maps.

Note that $f\colon V \to W$ is continuous if and only if the preimage of any open lattice is open, if and only if for any continuous seminorm $\|\cdot\|$ on W, the seminorm $f^{-1}\|\cdot\| = \|f(\cdot)\|$ is continuous on V.

We let **LCTVS** be the L-linear category of locally convex topological vector spaces over L with continuous linear maps.

This category has all small colimits: given a covariant functor $F\colon J \to \mathbf{LCTVS}$, we have:

$$\varinjlim F = \bigoplus_{A \in \mathrm{ob}(J)} FA / \langle \iota_A(x) - \iota_B(F(f)(x)) \colon f\colon A \to B \in \mathrm{arr}(J) \rangle$$

(here, ι_A etc. denotes inclusion in the direct sum). The topology on the direct sum is the canonical one and we use the quotient topology.

Similarly, we have all small limits: given a covariant functor $F\colon J \to \mathbf{LCTVS}$, we have

$$\varprojlim F = \{(x_A) \in \prod_{A \in \mathrm{ob}(J)} FA \colon \forall f\colon A \to B \in J, F(f)(x_A) = x_B\}.$$

The topology is defined as the subspace topology of the product topology, i.e. a (sub-)basis of open sets is given by sets of the form $(\Lambda \times \prod_{B \neq A} FB) \cap \varprojlim F$ for $A \in \mathrm{ob}(J)$, $\Lambda \subseteq FA$ an open lattice.

In particular, we have products, coproducts, kernels, cokernels, quotients, and finite products.

Some properties:

- For both limits and colimits, the underlying vector space is the corresponding limit/colimit in the category of vector spaces.
- Finite products are equal to finite coproducts.
- $$\bigoplus (V_i/W_i) = (\bigoplus V_i)/(\bigoplus W_i).$$
- Direct sums and products preserve the property of being Hausdorff.
- V/W is Hausdorff if and only if W is closed in V.

On the other hand, **LCTVS** is *not* an abelian category: the continuous identity map $(L, \{\mathscr{O}\}) \to (L, \{L\})$ is an epimorphism which is not a cokernel.

2.5. **More properties of LCTVSs.** If we are dealing with finite-dimensional vector spaces, not much is happening:

Lemma 2.5.1. *If V is a finite dimensional Hausdorff **LCTVS** then V is isomorphic to $L^{\oplus d}$ with the topology defined by $\|x\|_\infty = \max_{i=1,\dots,d}(|x_i|)$. In particular, V is normable.*

(Note that the same is true for finite-dimensional vector spaces over an archimedean field: the Euclidean topology is the unique locally convex topology on $\mathbf{R}^n$).

Proof. Suppose that $\mathscr{L}$ is a collection of open lattices making $L^{\oplus d}$ a Hausdorff LCTVS. Then for $\Lambda \in \mathscr{L}$, define $N_\Lambda = \cap_{\alpha \in L^\times} \alpha\Lambda \subseteq V$. This is a L-vector subspace. Also, we have $N_{\Lambda_1 \cap \Lambda_2} = N_{\Lambda_1} \cap N_{\Lambda_2}$.

Choose Λ such that N_Λ has minimal (*finite*) dimension. Then for all $\Lambda' \in \mathscr{L}$, the inclusion $N_{\Lambda \cap \Lambda'} \subseteq N_\Lambda$ must be an equality. Thus $N_\Lambda = N_{\Lambda\cap\Lambda'} \subseteq N_{\Lambda'}$ for all open lattices $\Lambda' \in \mathscr{L}$. Thus, $N_\Lambda \subseteq \cap_{\Lambda' \in \mathscr{L}} \Lambda' = (0)$, as $(V, \mathscr{L})$ is Hausdorff; therefore $N_\Lambda = 0$.

Now, if $\Lambda' \in \mathscr{L}$ is an open lattice, since V is finite dimensional, we may find some $\alpha \in L^\times$ such that $\alpha e_i \in \Lambda'$ for all i, i.e. that $\alpha \mathscr{O}^{\oplus d} \subseteq \Lambda'$. This implies that all $\Lambda' \in \mathscr{L}$ are open for the $\|\cdot\|_\infty$ topology.

We have to show that $\mathscr{O}^{\oplus d}$ is open in the topology defined by $\mathscr{L}$. It will suffice to show that $\Lambda \subseteq \alpha\mathscr{O}^{\oplus d}$ for some $\alpha \in L^\times$ with Λ as above.

Suppose that this is not the case. Then for each n, there exists some $x_n \in \Lambda$ with $\pi^{n-1}x_n \notin \mathscr{O}^{\oplus d}$, i.e. $\|x_n\|_\infty \geq |\pi|^{-(n-1)}$. Without loss of generality, we may take $\|x_n\|_\infty = |\pi|^{-n}$. Then $\pi^n x_n \in \mathscr{O}^{\oplus d}$, which is a compact metric space with respect to the metric $d(x,y) = \|x-y\|_\infty$. Therefore, by passing to a subsequence, we may assume $\pi^n x_n \to y \in \mathscr{O}^{\oplus d}$. For any $\beta \in L^\times$, we therefore have $\pi^n \beta x_n \to \beta y$. Since $\pi^n\beta x_n \in \Lambda$ for $n \gg 0$, we see $\beta y \in \Lambda$. Thus, $Ly \subseteq \Lambda$, which is a contradiction. □

Lemma 2.5.2. *For a locally convex topological vector space V, the following are equivalent:*

(1) V is metrizable (i.e. the topology can be defined by a metric).
(2) The topology on V can be defined by a translation-invariant metric.
(3) V is Hausdorff and the topology on V can be defined by countably many *lattices (or equivalently countably many seminorms).*

Proof.
- (2) implies (1) trivially.
- (1) implies (3): If the topology on V is defined by a metric d, consider the open neighborhoods of 0 given by $\Omega_n = \{x \mid d(x,0) \leq 1/n\}$ for $n = 1, 2, \dots$. Since V is a LCTVS, we can find an open lattice $\Lambda_n \subseteq \Omega_n$ for each n. Since d defines the topology, the Λ_n give a basis of neighborhoods of 0, so we may take $J = J_{\{\Lambda_n\}}$. Note that $\cap_n \Lambda_n = \{0\}$, so V is Hausdorff.
- (3) implies (2): Suppose the topology on V is defined by a countable family of seminorms $\|\cdot\|_1, \|\cdot\|_2, \dots$. Then we define
$$d(x,y) = \sup_n \frac{\|x-y\|_n}{2^n(1+\|x-y\|_n)}.$$
  - d is well-defined and ≤ 1 always, as the function $r \mapsto \frac{r}{1+r} = 1 - \frac{1}{1+r}$ is a monotonically increasing function from $[0,\infty)$ to $[0,1)$.
  - We clearly have $d(x,y) = d(y,x)$.
  - $d(x,y) = 0$ if and only if $\|x-y\|_n = 0$ for all n if and only if $x = y$ (since V is Hausdorff).

$$d(x,z) \leq \sup_n \max\left(\frac{\|x-y\|_n}{2^n(1+\|x-y\|_n)}, \frac{\|y-z\|_n}{2^n(1+\|y-z\|_n)}\right) = \max(d(x,y), d(y,z)).$$

□

Why do we want this extra generality, instead of just using Banach spaces? Many natural spaces, e.g. locally analytic functions on $\mathbf{Z}^p$, are not Banach and not even Frechet.

3. 1/14/20: Properties of locally convex topological vector spaces

Last time, we discussed when locally convex topological vector spaces are metrizable. Here are some additional facts about metrizable LCTVS's:

Lemma 3.0.1. *(1) A normable LCTVS is metrizable.*
(2) A Hausdorff finite dimensional LCTVS is metrizable.
(3) A countable (inverse) limit of metrizable LCTVS's is again metrizable.

Proof. For (3), we note that the inverse limit has a topology generated by pulling back the seminorms on each V_i. □

3.1. "Convergent" products.

Suppose that $(V_i, \|\cdot\|_i)_{i\in I})$ are normed LCTVS's. There are additional important constructions of normed LCTVS's out of these, given by

$$\perp_{i\in I,0}(V_i, \|\cdot\|) = \left\{(x_i) \in \prod v_i \colon \forall \epsilon > 0, \|x_i\|_i < \epsilon \text{ for all but finitely many } i\right\}$$

and

$$\perp_{i\in I}(V_i, \|\cdot\|_i) = \left\{(x_i) \in \prod v_i \colon \|x_i\|_i \text{ is bounded independently of } i\right\}.$$

We can think of $\perp_0$ as the set of sequences $(x_i) \in \perp$ with $\|x_i\|_i \to 0$. These constructions depend on the choice of norms, not just the topology. Note that these admit a continuous map to $\prod_i V_i$.

We define a norm on these spaces by the supremum $\|(x_i)\|_\infty = \sup_{i\in I} \|x_i\|_i$. We claim that $\perp_0$ is a closed subspace in $\perp$ with respect to this norm topology. Indeed, suppose (x_i^n) is a sequence of elements of $\perp_0$ converging to some $(x_i) \in \perp$, i.e. that $\sup_{i\in I} \|x_i^n - x_i\| \to 0$. Then for any $\epsilon > 0$, we can choose some n such that $\|x_i^n - x_i\| < \epsilon$ for all i. For all but finitely many i, $\|x_i^n\| < \epsilon$, so by the ultrametric inequality $\|x_i\| < \epsilon$ for such i.

Now we give some examples:

Example 3.1.1. Let Ω be a compact topological space, and consider the space $C(\Omega, L)$ of continuous functions from Ω to L. We equip this with a norm by $\|f\| = \sup_{x\in\Omega} |f(x)|$. This is a basic example of a normed LCTVS over L.

Example 3.1.2. Define the ring of convergent power series

$$L\langle T\rangle = \{f(T) = f_0 + f_1 T + \cdots \in L[[T]] \colon |f_i| \to 0 \text{ as } i \to \infty\}$$

This is isomorphic to $\perp_{i=0,0}^{\infty}(L, |\cdot|)$.

We can interpret $L\langle T\rangle$ as analytic functions on the "unit ball" $\mathscr{O}_{\overline{L}} = \cup_{K/L,[K:L]<\infty} \mathscr{O}_K$. Indeed, given $a \in \mathscr{O}_{\overline{L}}$, $f(T) = f_0 + f_1 T + \cdots \in L\langle T\rangle$, then we can define $f(a) \in \mathscr{O}_{L(a)} \subseteq \mathscr{O}_{\overline{L}}$ by

$$f(a) = f_0 + f_1 a + f_2 a^2 + \cdots$$

Note that this infinite sum makes sense by the fact that $|f_i a^i| = |f_i||a|^i \leq |f_i| \to 0$ (so by the ultrametric inequality, the difference between any two partial sums also goes to 0).

Lemma 3.1.3. *For $f \in L\langle T\rangle$, we have*
$$\|f\| = \sup_{t \in \mathscr{O}_{\overline{L}}} |f(t)|.$$

Proof. First of all, we have
$$\sup_{t \in \mathscr{O}_{\overline{L}}} |f(t)| \leq \|f\|.$$
Indeed, applying the ultrametric inequality tells us that
$$|f(t)| = \max_i |f_i||t|^i \leq \max_i |f_i| = \|f\|$$
To see equality, we first assume without loss of generality (by scaling f by an element of $L^\times$) that $\|f\| = 1$, i.e. that $f_i \in \mathscr{O}$ for all i and $f_i \in \mathscr{O}^\times$ for some i. We need to prove that there is some $t \in \mathscr{O}_{\overline{L}}$ such that $|f(t)| = 1$, i.e. that $|f(t)| \in \mathscr{O}_{\overline{L}}^\times$.

Since we have assumed that $f_i \in \mathscr{O}$ for all i, we can take the reduction $\overline{f} = f \pmod{\pi} \in \mathbf{F}[T]$. (Note that it is indeed a polynomial, as $|f_i| < 1$ for all but finitely many i). Since $f_i \in \mathscr{O}^\times$ for at least one i, we see that $\overline{f}$ is a *nonzero* polynomial. Thus, there is some $\bar{t} \in \overline{\mathbf{F}}$ such that $\overline{f}(\bar{t}) \neq 0$. Then, lifting $\bar{t}$ to some $t \in \mathscr{O}_{\overline{L}}$, we have that $\overline{f(t)} = \overline{f}(\bar{t}) \neq 0$, so $|f(t)| = 1$. □

3.2. **Bounded subsets.**

Definition 3.2.1. We say that $B \subseteq V$ is *bounded* if any continuous semi-norm on V is bounded on B.

Lemma 3.2.2. *Let $\mathscr{N}$ be a collection of continuous semi-norms generating the topology of V. Then the following are equivalent for a subset $B \subseteq V$:*
(1) B is bounded.
(2) For any $\|\cdot\| \in \mathscr{N}$, $\|\cdot\|$ is bounded on B.
(3) For any open lattice Λ, there is some $a \in L^\times$ such that $B \subseteq a\Lambda$.

Proof.
- Since the semi-norms in $\mathscr{N}$ generate the topology of V, any continuous semi-norm on V is bounded by the maximum of some finite set of semi-norms inside $\mathscr{N}$. So (1) and (2) are equivalent.
- Assume that B is bounded. If Λ is an open lattice, then $\|\cdot\|_\Lambda$ is a continuous semi-norm, so there is some $a \in L^\times$ such that $\|b\|_\Lambda \leq |a|$ for $b \in B$. Unraveling definitions, this says that $B \subseteq a\Lambda$. Thus (1) implies (3).
- Assume that (3) holds and let $\|\cdot\|$ be a continuous semi-norm. Then its unit ball $\Lambda(\|\cdot\|)$ is an open lattice, so there is some $a \in L^\times$ with $B \subseteq a\Lambda(\|\cdot\|)$, i.e. $\|B\| \leq |a|$. So (3) implies (1).

□

Lemma 3.2.3. *(1) Suppose that $B \subseteq V$ is bounded. Then $\langle B\rangle_{\mathscr{O}}$, the $\mathscr{O}$-module generated by B, is bounded.*
(2) If $B \subseteq V$ is bounded, then its closure $\overline{B}$ is bounded.
(3) If $f\colon V \to W$ is continuous linear and $B \subseteq V$ is bounded, then $f(B)$ is bounded.
(4) If V_i is Hausdorff for all i, then:
- *$B \subseteq \bigoplus_i V_i$ is bounded if and only if $\mathrm{pr}_i B$ is bounded for all i and $\mathrm{pr}_i B = (0)$ for all but finitely many i.*
- *$B \subseteq \prod_i V_i$ is bounded if and only if $\mathrm{pr}_i B$ is bounded for all i.*

Proof. (1) This follows from the ultrametric inequality: $\|ax + by\| \le \max(\|x\|, \|y\|)$ for $a, b \in \mathscr{O}$, $x, y \in V$.
(2) If $\|\cdot\|$ is continuous, then $\|\overline{B}\| \subseteq \overline{\|B\|}$, which is bounded.
(3) If $M \subseteq W$ is an open lattice, then $f^{-1}M \subseteq V$ is an open lattice, so $B \subseteq af^{-1}M$ for some $a \in L^\times$ and thus $f(B) \subseteq aM$.
(4) Note that in $\oplus_i V_i$, the topology is generated by open lattices of the form $\oplus_i \Lambda_i$ where $\Lambda_i \subseteq V_i$ is an open lattice. Thus, the condition that B is bounded is equivalent to saying that for any collection $(\Lambda_i)_i$, there is some $a \in L^\times$ with $\mathrm{pr}_i B \subseteq a\Lambda_i$ for each i. Certainly the condition in the lemma is sufficient.

Conversely, suppose that $\mathrm{pr_i} B$ is non-zero for infinitely many i, and let $i_1, i_2, \dots$ be an infinite sequence of distinct such i. For all n, there exists some $b_n \in B$ such that $b_{n,i_n} \neq 0$. Then since V_{i_n} is Hausdorff, there exists $\Lambda_{i_n} \subseteq V_{i_n}$ an open lattice such that $\pi^n b_{n,i_n} \notin \Lambda_{i_n}$. Consider the open lattice $\Lambda = \prod_{n=1}^\infty \Lambda_{i_n} \times \prod_{j \neq i_n} V_j \subseteq \bigoplus V_i$. Since B is bounded, there exists an m such that $\pi^m B \subseteq \Lambda$, and therefore $\pi^m b_{n,i_n} \in \Lambda_{i_n}$: taking $n \ge m$ gives a contradiction.

In $\prod_i V_i$, the topology is generated by open lattices of the form $\Lambda_i \times \prod_{j \neq i} V_i$ with $\Lambda_i \subseteq V_i$ an open lattice. So B is bounded if for each choice of i, Λ_i, we can choose $a \in L^\times$ such that $\mathrm{pr}_i B \subseteq a\Lambda_i$, i.e. that $\mathrm{pr}_i B$ is bounded. □

Lemma 3.2.4. *If V is metrizable and $\Lambda \subseteq V$ is a lattice, then the following are equivalent:*

- *$\Lambda \subseteq V$ is open.*
- *For all bounded subsets $B \subseteq V$, there exists some $a \in L^\times$ such that $B \subseteq a\Lambda$.*
- *For all bounded closed $\mathscr{O}$-submodules $B \subseteq V$, there is some $a \in L^\times$ such that $B \subseteq a\Lambda$.*

Proof. The second two points are equivalent due to the fact that replacing a bounded subset B by the closure of the $\mathscr{O}$-submodule which it generates preserves the property of being bounded.

Now, since V is metrizable, there exists a countable sequence $\Lambda_1 \supseteq \Lambda_2 \supseteq \Lambda_3 \cdots$ of open lattices generating the topology on V (we can assume they are nested as we can always replace open lattices with finite intersections thereof).

Suppose there is some Λ satisfying the condition as in the lemma but which is not open. Then for all n, there exists some $x_n \in \pi^n \Lambda_n$ with $x_n \notin \Lambda$.

We claim the sequence $\{\pi^{-n} x_n\}$ is bounded: we need to show that for any m, there is some $a \in L^\times$ such that $\pi^{-n} x_n \in a\Lambda_m$ for all n. If $n \ge m$, we have $\pi^{-n} x_n \in \Lambda_n \subseteq \Lambda_m$. Then we just need to choose $a \in L^\times$ such that $\pi^{-i} x_i \in a\Lambda_m$ for $i = 1, \dots, m-1$.

Thus, there is some $a \in L^\times$ such that $\pi^{-n} x_n \in a\Lambda$ for each n. Therefore, there exists some m such that $x_n \in \pi^{n-m}\Lambda$ for all n. Thus, for $n \ge m$, we have $x_n \in \Lambda$, which is a contradiction. □

Thus, we see that for a metrizable LCTVS, the topology can be described by understanding bounded sets instead of open sets. Since many LCTVS's of interest are not metrizable, we identify a more general class of spaces with this property:

Definition 3.2.5. A LCTVS V is *bornological* if for any lattice $\Lambda \subseteq V$, Λ is open if and only if for any bounded $B \subseteq V$ we may find $a \in L^\times$ with $B \subseteq a\Lambda$.

We just showed that any metrizable LCTVS is bornological.

Lemma 3.2.6. *If V is bornological and $f\colon V \to W$ is linear, then f is continuous if and only if it takes bounded sets to bounded sets.*

Proof. We already have seen that a continuous map preserves boundedness. Conversely, assume that f takes bounded sets to bounded sets. Suppose that $M \subseteq W$ is an open lattice: we want to show that the lattice $f^{-1}M$ is open.

Let $B \subseteq V$ be any bounded set. Since f preserves bounded sets and M is open, there is some $a \in L^\times$ such that $f(B) \subseteq aM$, and therefore $B \subseteq af^{-1}(M)$. Since V is bornological, we see that $f^{-1}M$ is open. □

Lemma 3.2.7. *The following are equivalent for an LCTVS V:*

(1) V is normable.
(2) The topology on V is generated by a single open lattice Λ such that $\cap_{a\in L^\times} a\Lambda = (0)$.
(3) V is Hausdorff and V contains a bounded open lattice.

In this case, the topology is generated by any bounded open lattice.

Proof.

- Let V be normable. Then V is Hausdorff, and the unit ball $\Lambda(\|\cdot\|)$ is a bounded open lattice, so (1) implies (3).
- Suppose that V is Hausdorff and contains a bounded open lattice Λ. Let Λ' be any open lattice. Since Λ is bounded, there is some $a \in L^\times$ such that $\Lambda \subseteq a\Lambda'$, i.e. that $a^{-1}\Lambda \subseteq \Lambda'$. Thus, Λ' is open in the topology generated by Λ, so Λ generates the topology of V. Moreover, let $0 \neq \gamma \in V$ be arbitrary. Since V is Hausdorff, there is some $a \in L^\times$ with $\gamma \notin a\Lambda$ (as Λ generates the topology). Thus, $\cap_{a\in L^\times} a\Lambda = (0)$. Thus, we see that (3) implies (2).
- Finally, assume that the topology on V is generated by an open lattice Λ satisfying $\cap_{a\in L^\times} a\Lambda = (0)$. Then the topology on V is generated by the seminorm $\|\cdot\|_\Lambda$. This seminorm is a norm, since if $\|x\|_\Lambda = 0$ then by definition $x \in a\Lambda$ for all $a \in L^\times$, and therefore $x = 0$. Thus (2) implies (1).

□

3.3. **Completeness.**

Just as in archimedean functional analysis, an essential role will be played by the property of *completeness*. Since we care about potentially non-metrizable (i.e. non-first-countable) LCTVS's, we need to generalize the usual notion of Cauchy sequences to *Cauchy nets*.

Let I be a partially ordered set which is *directed*: i.e. for all $i, j \in I$ there is some $k \in I$ with $k \geq i$ and $k \geq j$. In this setting, we will say "for all $i \gg 0$ ..." to mean "there exists some j such that for all $i \geq j$ ...".

A *Cauchy net* $(x_i)_{i\in I}$ is a collection of elements of V indexed by a directed set I such that for any open lattice Λ, $x_i - x_j \in \Lambda$ whenever $i, j \gg 0$. We say that an $x \in V$ is a *limit* of $(x_i)_{i\in I}$ if for any open lattice Λ, $x - x_i \in \Lambda$ for $i \gg 0$.

Definition 3.3.1. A LCTVS V is *complete* if every Cauchy net $(x_i)_{i\in I}$ in V has a unique limit in V.

4. 1/16/20: Properties of complete LCTVSs

4.1. **General facts about completeness.**

Lemma 4.1.1. *If a LCTVS V is finite-dimensional and Hausdorff, V is automatically complete.*

Proof. We have already seen that $V \simeq L^{\oplus n}$ is normable with topology given by the norm $\|(x_i)\|_\infty = \sup_{i=1,\dots,n} |x_i|$. Since V is normable, we can characterize completeness in terms of Cauchy sequences (rather than Cauchy nets). If $x^{(m)}$ is a Cauchy sequence in V, then for each $i = 1, \dots, n$, $x_i^{(m)}$ is Cauchy as well, as $|x_i^{(m)} - x_j^{(m)}| \le \|x_i - x_j\|$. By completeness of L, the Cauchy sequence $x_i^{(m)}$ converges to a limit $x_i \in L$. Let $x = (x_i)_{i=1}^n \in V$. Since there are only finitely many coordinates, this implies that $x^{(m)} \to x$ in the supremum norm $\|\cdot\|_\infty$. □

Lemma 4.1.2. *Let V be a Hausdorff LCTVS and let $W \subseteq V$ be a linear subspace with the subspace topology. If W is complete, then W is closed.*

Proof. If $x \in \overline{W}$, then for any open lattice $\Lambda \subseteq V$, there is an $x_\Lambda \in W \cap (x + \Lambda)$. The set of open lattices form a directed set, with $\Lambda \ge \Lambda'$ if $\Lambda \subseteq \Lambda'$, so the family (x_Λ) is a net. In fact, it is a Cauchy net: whenever $\Lambda_1, \Lambda_2 \ge \Lambda$, then $x_{\Lambda_1} - x_{\Lambda_2} \in \Lambda$. As W is complete, there is a $y \in W$ with $x_n \to y$. However, since $x_n \to x$ as well, the fact that V is Hausdorff implies that $x = y$. □

Lemma 4.1.3. *If $W \subseteq V$ is a closed subspace and V is complete, then so is W.*

Proof. Left as an exercise. □

Lemma 4.1.4. *If $(V_i)_{i\in I}$ is a collection of complete LCTVS's, then both $\bigoplus_{i\in I} V_i$ and $\prod_{i\in I} V_i$ are complete.*

Proof.

- $\prod_{i\in I} V_i$ is complete: Suppose that (x^μ), $\mu \in U$ is a Cauchy net in $\prod_i V_i$. Then (by e.g. continuity of the projection maps), x_i^μ is a Cauchy net for all i, so $x_i^\mu \to x_i$ for some $x_i \in V_i$. Letting $x = (x_i)_{i\in I} \in V$, we claim that $x^\mu \to x$:

  Let J be a finite subset of I, and let $\Lambda_j \subseteq V_j$ be an open lattice for each $j \in J$. Then a basis of open lattices in $\prod_i V_i$ consists of those of the form $\Lambda = \prod_{j\in J} \Lambda_j \times \prod_{i \in I-J} V_i$, so we must show that for any such Λ, $x^\mu - x \in \Lambda$ for $\mu \gg 0$. This says exactly that $x_j^\mu - x_j \in \Lambda_j$ for $\mu \gg 0$, independently of j. This is true because J is finite.
- $\bigoplus_{i\in I} V_i$ is complete: Let $(x^\mu)_{\mu\in U}$ be a Cauchy net. Each x_i^μ is a Cauchy net in V_i, so there is a limit $x_i \in V_i$ for each i. We claim that $x = (x_i)_{i\in I}$ is in $\oplus V_i$ and that $x^\mu \to x$.

  Indeed, suppose that there is an infinite sequence $i_1, i_2, i_3, \dots$ of distinct indices such that $x_{i_n} \ne 0$ for all n. Thus, there is some open lattice $\Lambda_{i_n} \subseteq V_{i_n}$ with $x_{i_n} \notin \Lambda_{i_n}$ for each n. Consider the open lattice $\Lambda = \bigoplus_n \Lambda_{i_n} \oplus \bigoplus_{j \ne i_n} V_j \subseteq \bigoplus V_i$.

  Suppose that $\Lambda_i \subseteq V_i$ is an open lattice for all i. A basis of open lattices in $\bigoplus_i V_i$ is given by those of the form $\Lambda = \bigoplus_i \Lambda_i$.

  If (x^μ) is a Cauchy net in $\bigoplus V_i$, then for $\mu, \nu \gg_\Lambda 0$, $x^\mu - x^\nu \in \Lambda$, i.e. $x_i^\mu - x_i^\nu \in \Lambda_i$ for all i.

□

Corollary 4.1.5. *Small (inverse) limits of complete LCTVS's are again complete.*

Proof. This follows since $\varprojlim V_i \subseteq \prod_i V_i$ is a *closed* subspace. □

Lemma 4.1.6. *If $(V_i, \|\cdot\|_i)$ are complete normed spaces, then the convergent products $\perp_i (V_i, \|\cdot\|_i)$ and $\perp_{i,0}(V_i, \|\cdot\|_i)$ are complete.*

Proof. The argument is similar to the above. □

Given any LCTVS V, we define its *completion*

$$\widehat{V} = \varinjlim_{\Lambda \subseteq V} V/\Lambda$$

with Λ ranging over open lattices. The inverse limit is taken in the category of $\mathscr{O}$-modules, but it has the structure of a vector space over L: for $a \in L^\times$, we have $a(x_\Lambda + \Lambda) = (ax_{a^{-1}\Lambda} + \Lambda)$.

For any open lattice Λ, we let $\widehat{\Lambda}$ be the kernel of $\widehat{V} \to V/\Lambda$. This is an open lattice in $\widehat{V}$. The following properties hold:

- The set of $\widehat{\Lambda}$ for Λ ranging through all open lattices generates the inverse limit topology.
- The natural map $\widehat{V}/\widehat{\Lambda} \to V/\Lambda$ is an isomorphism.
- The map $V \to \widehat{V}$ sending x to $(x + \Lambda)$ is continuous.
- $\widehat{V}$ is complete: if $x^\mu = (x^\mu_\Lambda + \Lambda)$ is a Cauchy net in $\widehat{V}$, then for $\mu, \nu \gg_\Lambda 0$, we have $x^\mu_\Lambda - x^\nu_\Lambda \in \Lambda$. Thus, $(x^\mu_\Lambda + \Lambda)$ is a constant element of V/Λ for $\mu \gg 0$: denote this value by $x_\Lambda + \Lambda$. Then if we define $x = (x_\Lambda + \Lambda)$, we have $x^\mu \to x$.

Lemma 4.1.7. *If $f\colon V \to W$ is a continuous linear map, and if W is complete, then f factors uniquely through a continuous linear map $\widehat{f}\colon \widehat{V} \to W$.*

Proof. Let $(x_\Lambda + \Lambda) \in \widehat{V}$. Then we define $\widehat{f}(x_\Lambda + \Lambda) = \varprojlim_n f(x_n)$. To see that this makes sense, we must see that $(f(x_n)) \in W$ is a Cauchy net. Indeed, given any open lattice $M \subseteq W$, there is some open lattice $\Lambda \subseteq f^{-1}M$. If $\Lambda_1, \Lambda_2 \subseteq \Lambda$, then $x_{\Lambda_1} - x_{\Lambda_2} \in \Lambda$ and therefore $f(x_{\Lambda_1}) - f(x_{\Lambda_2}) \in M$.

The claims about continuity, uniqueness, etc. are immediate. □

4.2. **Banach and Fréchet spaces.**

Definition 4.2.1.
- A complete normable space is called a *Banach space.*
- A complete metrizable space is called a *Fréchet space.*

Lemma 4.2.2.
(1) A countable (inverse) limit of Fréchet spaces is a Fréchet space.
(2) The convergent products $\perp$ or $\perp_0$ of normed Banach spaces is a Banach space.
(3) The quotient of a Banach (resp. Fréchet) space by a closed subspace is Banach (resp. Fréchet).

Proof.
(1) We have shown that a limit of complete spaces is complete. Then it is metrizable because the norms on the limit can be defined by pulling back norms from the individual pieces. Details left as an exercise.
(2) Again completeness follows from previous work, and the topology is defined by one norm. Details left as an exercise.
(3) Let $\Lambda_1 \supseteq \Lambda_2 \supseteq \Lambda_3 \supseteq \cdots$ be a countable sequence of open lattices generating the topology on V. Their images in V/W generate the quotient topology, so V/W is metrizable. Thus, we can check completeness by looking at Cauchy sequences.

Suppose that $x_n + W$ is a Cauchy sequence in V/W, i.e. for any i, for $n, m \gg_i 0$, $x_n - x_m + W \subseteq \Lambda_i + W$. By passing to a subsequence, we may assume without loss of generality that $x_{n+1} - x_n \in \Lambda_n + W$. Choose $y_n \in \Lambda_n$ with $x_{n+1} - x_n \in y_{n+1} + W$. We define $x'_1 = x_1, x'_2 = x_1 + y_2$, and so on: we have $x'_n = x_1 + y_2 + y_3 + \cdots + y_n$. By construction, we have $x'_n + W = x_n + W$ for all n. Also, (x'_n) is a Cauchy sequence in V, as $x'_{n+1} - x'_n = y_{n+1} \in \Lambda_n$. Thus, it has some limit $x \in V$, and $(x_n + W) = (x'_n + W) \to x + W$ in V/W. □

Lemma 4.2.3. *If V is a Fréchet space and $M \subseteq V$ is a closed lattice, then M is open.*

Proof. Since M is a lattice, we have $V = \cup_{n \in \mathbf{Z}} \pi^{-n} M$. Suppose that M is not open. Then M contains no open set, i.e. M is nowhere dense. The same is true for each $\pi^{-n}M$ (as multiplication by π is an isomorphism). Since V is a complete metric space, the Baire category theorem implies that it cannot be a countable union of closed nowhere dense sets, so we have a contradiction. □

This property is often useful, so we define more generally:

Definition 4.2.4. A LCTVS V is called *barrelled* if any closed lattice in V is also open.

Lemma 4.2.5. *Suppose that V, W are Fréchet spaces and that $f\colon V \to W$ is a linear function,* not *assumed to be continuous.*

1. *(The closed graph theorem): f is continuous if and only if $\Gamma(f) := \{(x, fx) \in V \oplus W : x \in V\} \subseteq V \oplus W$ is a closed subset.*
2. *If f is a continuous bijection, then f is a homeomorphism.*
3. *(The open mapping theorem) If f is continuous and surjective, then f is open.*

Proof. The proof of the closed graph theorem (1) is omitted: see [6] or adapt the well-known proof from functional analysis over $\mathbf{R}$.

To deduce part (2), apply the closed graph theorem to f^{-1}: continuity of f means that $\Gamma(f)$ is closed in $V \oplus W$, and $\Gamma(f^{-1})$ maps bijectively to $\Gamma(f)$ under the isomorphism $W \oplus V \to V \oplus W$.

To deduce the open mapping theorem (3), apply (2) to the induced map $V/\mathrm{ker} f \to W$, which must therefore be a homeomorphism. As f is the composition of this induced map with the open map $V \to V/\mathrm{ker} f$, we see that f is open. □

Lemma 4.2.6. *If $(V_i, \|\cdot\|_i)$ are normed spaces, $(W, \|\cdot\|)$ is a Banach space, $f_i\colon V_i \to W$ is continuous linear for all i, and if there exists some $C \in \mathbf{R}_{>0}$ such that $\|f_i(x)\| \leq C\|x\|_i$ for all i and for all $x \in V_i$, then there exists a unique continuous linear map $f\colon \perp_{i,0}(V_i, \|\cdot\|_i) \to (W, \|\cdot\|)$ whose restriction to each V_i is f_i.*

Proof. f is defined by sending (x_i) to $\sum_i f_i(x_i)$. Since $\|x_i\|_i \to 0$, $\|f_i(x_i)\| \leq C\|x_i\|_i \to 0$, so this sum converges. □

We have the following strong structure theorem for Banach spaces over L:

Lemma 4.2.7. *Let V be a Banach space. Then there is some index set I such that $V \simeq \perp_{i \in I, 0}(L, |\cdot|)$ as topological vector spaces.*

If $x_i \in V$ maps to $e_i \in \perp_{i \in I,0}(L, |\cdot|)$ under some such isomorphism, we say that x_i is a *Banach basis* of V. Note that $\|x_i\|$ is bounded. Any element $y \in V$ can be written uniquely as $\sum_i a_i x_i$ where $a_i \in L$, and $|a_i| \to 0$ 'as $i \to \infty$', i.e. for any $\epsilon > 0$, $|a_i| < \epsilon$ for all but finitely many i.

Proof. Let $\|\cdot\|'$ be a norm on V which defines the topology. Set

$$\|x\| = \min_{\substack{\gamma \in |L| = \{0\} \cup (\#\mathbf{F})^{\mathbf{Z}} \\ \gamma \geq \|x\|'}} \gamma$$

Note that
$$\|x\|' \leq \|x\| \leq (\#\mathbf{F})\|x\|'.$$
$\|\cdot\|$ is a norm which is equivalent to $\|\cdot\|'$, so we may assume without loss of generality that $\|V\| = |L|$. Let $\Lambda = \Lambda(\|\cdot\|)$ be the corresponding unit ball, and consider $\overline{\Lambda} = \Lambda/\pi\Lambda$. This vector space has a basis $\overline{x_i}$ over $\mathbf{F}$: choose some $x_i \in \Lambda$ with $\|x_i\| = 1$ mapping to $\overline{x_i}$. By the above lemma, there is a well-defined continuous linear map $\perp_{i\in I,0}(L, |\cdot|) \to V$ by sending e_i to x_i, and we claim that this is an isometry:

Consider $(a_i) \in \perp_{i,0}(L, |\cdot|)$: we have $\|(a_i)\| = \max_i |a_i| = |a_{i_0}|$ for some i_0. Then we have
$$\sum a_i x_i = a_{i_0} \sum \frac{a_i}{a_{i_0}} x_i =: a_{i_0}\alpha.$$
We have $\frac{a_i}{a_{i_0}} \in \mathscr{O}$, so $\alpha = \sum \frac{a_i}{a_{i_0}} x_i \in \Lambda - \pi\Lambda$, as $\overline{x_i}$ is a basis for $\Lambda/\pi\Lambda$. Since Λ is the unit ball for $\|\cdot\|$ and $\|\cdot\|$ is valued in $|L^\times| = \{0\} \cup |\pi|^{\mathbf{Z}}$, we thus have $\|\alpha\| = 1$, so $\|\sum_i a_i x_i\| = |a_{i_0}| = \|(a_i)\|$.

Now, it suffices to show that $\perp_{i,0}(L, |\cdot|)$ has dense image inside of V: since it is also complete, this implies the image is equal to V. Let $y' \in V$ be arbitrary. We can write $y' = ay$ with $a \in L$ and $|a| = \|y'\|$, so $y \in \Lambda - \pi\Lambda$. Then, since the $\overline{x_i}$ span $\Lambda/\pi\Lambda$, we may find some $x^0 = \sum_i a_i^0 x_i$ with $y = x^0 + \pi y_1$ for some $y_1 \in \Lambda$. Continuing in this manner, we can find $x^n = \sum_i a_i^n x_i$ for each n with $y = \sum_{j=0}^n \pi^j x^j + \pi^{n+1} y_{n+1}$ and $y_{n+1} \in \Lambda$. Thus, we have $\|\sum_{j=0}^n \pi^j x^j - y\| \to 0$, so y and therefore y' is in the closure of the image of $\perp_{i,0}(L, |\cdot|)$. $\square$

Note that a given norm $\|\cdot\|$ on V might not exactly correspond to the supremum norm $\|(a_i)_{i\in I}\|_\infty = \max_i |a_i|$ under the isomorphism above; however, the two norms will be equivalent. (Indeed, the equivalent norm $\|\cdot\|'$ defined in the above proof will coincide with the supremum norm).

5. 1/21/20: Structure and examples of Banach spaces

5.1. **Structure of Banach spaces.** We saw last time that any Banach space V has a Banach basis $\{x_i\}_{i\in I}$, i.e. a set of elements of V with $\|x_i\|$ bounded such that any $x \in V$ may be written uniquely as $x = \sum_i a_i x_i$. Any such set defines a topological isomorphism $\perp_{I,0}(L, |\cdot|) \to V$.

Lemma 5.1.1. *If $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ are Banach bases of V, then the cardinality of I is equal to the cardinality of J.*

Proof. If I is finite, then V has finite dimension $\#I$ and therefore $\#J = \#I$. Thus we may assume that both I and J are infinite.

For $i \in I$, we write $x_i = \sum_{j\in J} a_{i,j} y_j$. For each i, let $J_i = \cup_{n=1}^\infty \{j \in J \colon |a_{ij}| \geq 1/n\}$. This is a countable subset of J such that $a_{ij} = 0$ for $j \notin J_i$. Now, for all $i \in I$, x_i is in the closed subset $\perp_{j\in\cup_i J_i,0}(L, |\cdot|) \subseteq \perp_{J,0}(L, |\cdot|) = V$. Since $\{x_i\}$ is dense in V, this inclusion must be an equality. Thus $J = \cup_{i\in I} J_i$, and therefore $\#J \leq \#I \cdot \aleph_0 = \#I$. $\square$

Lemma 5.1.2. *If V is a Banach space, then the cardinality of any Banach basis is countably infinite if and only if V contains a dense subspace of countably infinite dimension.*

Proof. Write $V = \perp_{I,0}(L, |\cdot|)$. Then if $\#I$ is countably infinite, $\bigoplus_{i\in I} L \subseteq V$ is a dense subspace of countably infinite dimension.

Conversely, let $V_0 \subseteq V$ be a dense subspace with countable basis $x_1, x_2, \ldots$. Then we have $x_n = \sum_{i\in I} a_{ni} e_i$ with $a_{ni} \to 0$. As above, the set $I_n = \{i \in I \colon a_{ni} \neq 0\}$ is countable, and V_0

is contained in the closed subspace $\perp_{\cup_n I_n,0}(L, |\cdot|) \subseteq \perp_{I,0}(L, |\cdot|) = V$. Since V_0 is dense, this is an equality and thus $I = \cup_n I_n$ is a countable set. $\square$

Lemma 5.1.3. *If V is a Banach space and $W \subseteq V$ is a closed subspace, then $V \simeq W \oplus V/W$.*

Note: Here, as everywhere in this course, 'subspace' means "linear subspace with the subspace topology". This is a stronger condition than being the image of a continuous injection.

Proof. Since V/W is Banach, we have an isomorphism $V/W \simeq \perp_{i\in I,0}(L, |\cdot|)$ corresponding to a Banach basis $\{x_i\}$. Let $\pi : V \to V/W$ be the quotient map. We may choose $\widetilde{x_i} \in V$ with $\|\widetilde{x_i}\|$ bounded such that $\pi(\widetilde{x_i}) = x_i$ (as $\|x_i\| = \inf_{y\in\pi^{-1}(\{x_i\})} \|y\|$ is bounded).

Then, we define a map $s\colon V/W \simeq \perp_{i\in I,0}(L, |\cdot|)$ by sending x_i to $\widetilde{x_i}$; we have $\pi \circ s = \mathrm{id}_{V/W}$. Now, sending (y, z) to $y + s(z)$ defines a continuous homomorphism $W \oplus V/W \to V$ with continuous inverse $v \mapsto (v - s\pi v, \pi v)$. $\square$

5.2. **Examples of Banach spaces.** Many important examples of Banach (and Fréchet, locally convex, etc.) spaces are realized as various spaces of functions.

Example 5.2.1. Let Ω be a compact topological space, and define $C(\Omega, L)$ to be the L-algebra of continuous functions from Ω to L. This admits the supremum norm $\|f\| = \sup_{x\in\Omega}(|f(x)|) = \max_{x\in\Omega}(|f(x)|)$, which is well-defined due to compactness of Ω. We may easily see that it is a norm (e.g. deducing the ultrametric inequality from the pointwise ultrametric inequality $|(f + g)(x)| \leq \max\{|f(x)|, |g(x)|\}$).

We claim that $C(\Omega, L)$, topologized with this norm, is complete, and therefore a Banach space. Indeed, let (f_i) be a Cauchy net. Then for all $\epsilon > 0$, for all $i, j \gg_\epsilon 0$, $\|f_i - f_j\| < \epsilon$. This says exactly that $|f_i(x) - f_j(x)| < \epsilon$ for all $x \in \Omega$, so we may define $f(x) = \lim_i f_i(x)$.

We need to check that $f_i \to f$ in $C(\Omega, L)$. Given x, ϵ, we can choose $j \gg_{x,\epsilon}$ such that $|f_j(x) - f(x)| < \epsilon$. But we also have $|f_i(x) - f_j(x)| < \epsilon$ for $i, j \gg_\epsilon 0$, not depending on x. Then by letting j be sufficiently large, we conclude that for all $\epsilon > 0$, we can find $i \gg_\epsilon 0$ such that for all $x \in \Omega$, $|f_i(x) - f(x)| < \epsilon$, as desired.

Finally, we need to check that f is continuous, that is, we need to show that $f^{-1}(a + \pi^n\mathcal{O}_L)$ is open for all $a \in L$ and $n \in \mathbb{Z}$. For $i \gg 0$, for all x, $|f_i(x) - f(x)| < |\pi|^n$. This implies that $f_i(x) \in a + \pi^n\mathcal{O}_L$ if and only if $f(x) \in a + \pi^n\mathcal{O}_L$. Hence $f^{-1}(a + \pi^n\mathcal{O}_L)$ is equal to $f_i^{-1}(a + \pi^n\mathcal{O}_L)$, which is open. Therefore f is continuous.

Consider the compact topological space $\mathbf{Z}_p$. Then we have:

Lemma 5.2.2 (Mahler)**.** *The functions $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n}$ for $n = 0, 1, 2, \ldots$ is a Banach basis for $C(\mathbf{Z}_p, L)$.*

Proof. As they are polynomials in x, the $\binom{x}{n}$ are certainly continuous. For $x \in \mathbf{Z}_{>0}$, we know that $\binom{x}{n} \in \mathbf{Z} \subseteq \mathbf{Z}_p$. Thus by continuity, $\binom{x}{n} \in \mathbf{Z}_p$ for all $x \in \mathbf{Z}_p$, i.e. $\|\binom{x}{n}\| \leq 1$. Now, there exists a unique continuous map $\perp_{\mathbf{Z}_{\geq 0},0}(L, |\cdot|) \to C(\mathbf{Z}_p, L)$ sending e_n to $\binom{x}{n}$. We will see that it is an isomorphism.

Consider the "linear difference" map $D\colon C(\mathbf{Z}_p, L) \to C(\mathbf{Z}_p, L)$ given by

$$D(f)(x) = f(x + 1) - f(x).$$

We have $\|D(f)\| \leq \|f\|$, so D is continuous. By an elementary induction, we see that

$$(D^n f)(x) = \sum_{i=0}^{n} f(x + n - i)(-1)^i \binom{n}{i}$$

We claim that $\|D^n f\| \to 0$ as $n \to \infty$. To see this, apply the above formula with $n = p^k$ for some k:

$$(D^{p^k} f)(x) = f(x+p^k) - f(x) + \sum_{i=1}^{p^k-1} f(x+p^k-i)(-1)^i \binom{p^k}{i} + ((-1)^{p^k}+1)f(x)$$

Since $p^k \to 0$ as $k \to \infty$ and f is continuous, we see that $f(x+p^k) - f(x) \to 0$ as $k \to \infty$. This is uniform in x by compactness of $\mathbf{Z}_p$ (and Dini's theorem). In addition, note that the binomial coefficients $\binom{p^k}{i}$ are divisible by p. Also, the number $((-1)^{p^k}+1)$ (which is 0 unless $p = 2$, in which case it is 2) is divisible by p. Thus, for k large, we have

$$\|D^{p^k} f\| \leq |p|\|f\| < \|f\|$$

Since $\|Dg\| \leq \|g\|$ for any g, the above inequality is true for any $m \geq p^k$ as well. Thus, we see that $\|D^m f\| \to 0$ as $m \to \infty$.

Now, we may define a map $C(\mathbf{Z}_p, L) \to \perp_{\mathbf{Z}_{\geq 0},0}(L, |\cdot|)$ by sending f to $(D^n f)_n(0)$. Note that $(D^n \binom{x}{n})(0) = 1$ and $(D^m \binom{x}{n})(0) = 0$ for $m \neq n$, so the composite map $\perp_{\mathbf{Z}_{\geq 0},0}(L, |\cdot|) \to \perp_{\mathbf{Z}_{\geq 0},0}(L, |\cdot|)$ is the identity and therefore $C(\mathbf{Z}_p, L) \to \perp_{\mathbf{Z}_{>0},0}(L, |\cdot|)$ is surjective.

It is also injective, since $(D^n f)(0) = 0$ implies $f(n) = 0$ for all $n \in \mathbb{Z}_{\geq 0}$ inductively, because $(D^n f)(0) = \sum_{i=0}^n f(x+n-i)(-1)^i \binom{n}{i})$. Then by continuity, $f = 0$ identically.

Since a bijective continuous map between Banach spaces is an isomorphism, we get the desired result. $\square$

Fix some $r \in \mathbf{R}_{>0}$. We define the *closed unit disc of radius r in $\overline{L}$* as

$$D(r) = \{t \in \overline{L} \colon |t| \leq r\}.$$

The ring of analytic functions on this disc is defined as

$$\mathscr{O}(D(r)) = \{f = \sum_{n=0}^{\infty} c_n T^n \in L[[T]] \colon |c_n| r^n \to 0 \text{ as } n \to \infty\}.$$

Note that for $t \in D(r)$ and $f \in \mathscr{O}(D(r))$, the sum $f(t) = \sum c_n t^n$ converges in $L(t) \subseteq \overline{L}$. We define a norm on $\mathscr{O}(D(r))$ for $f = \sum_n c_n T^n \in \mathscr{O}(D(r))$ by

$$\|f\|_r = \max_{n \in \mathbf{Z}_{\geq 0}} |c_n| r^n.$$

We may easily check that this is a norm (e.g. deducing the ultrametric inequality from the ultrametric inequality $|c_n + d_n| \leq \max(|c_n|, |d_n|)$). We claim that $\mathscr{O}(D(r))$ is a *Banach space*.

Indeed, let (f_i) be a Cauchy net. Writing $f_i = \sum_n f_{i,n} T^n$, we see that $(f_{i,n})$ is a Cauchy net for each n, so there is some $f_n \in L$ with $f_{i,n} \to f_n$ for each n. Now define $f(T) = \sum_n f_n T^n$. We must check that $f = \lim_i f_i$ and that $f \in \mathscr{O}(D(r))$.

Indeed, given $\epsilon > 0$, for all $i, j \gg_\epsilon 0$, we have $|f_{i,n} - f_{j,n}| < \epsilon r^{-n}$ uniformly in n. Also, for any fixed n, $|f_{j,n} - f_n| < \epsilon r^{-n}$ for $j \gg_{n,\epsilon} 0$. So by taking j large enough in the uniform-in-n bound $|f_{i,n} - f_{j,n}| < \epsilon r^{-n}$, we see that $|f_{i,n} - f_n| < \epsilon r^{-n}$ for $i \gg_\epsilon 0$ uniformly in n. Thus $f = \lim_i f_i$. Moreover, as for any fixed i, $|f_{i,n}| < \epsilon r^{-n}$ for $n \gg_{\epsilon,i}$, taking i large enough we see that $|f_n| < \epsilon r^{-n}$ for $n \gg_\epsilon 0$.

We can see immediately that $\mathscr{O}(D(1)) = L\langle T \rangle$ has Banach basis $1, T, T^2, \ldots$. More generally, we calculate that a Banach basis for $\mathscr{O}(D(r))$ is given by $1, \pi^{m(1)}T, \pi^{m(2)}T^2, \ldots$ with $m(n) = \left\lceil n \frac{\log r}{\log |\pi|} \right\rceil$.

There is another important characterization of the norm $\|\cdot\|_r$ given above:

Lemma 5.2.3. *If $f \in \mathscr{O}(D(r))$ then $\|f\|_r = \sup_{t\in D(r)} |f(t)|$. If moreover $r \in p^{\mathbf{Q}}$, then $\|f\|_r = \max_{t\in D(r)} |f(t)|$.*

Note that $|f(t)|$ makes sense for $f(t) \in \overline{L}$, as $|\cdot|$ on L has a unique extension to an absolute value on $\overline{L}$.

Proof. We immediately deduce from the ultrametric inequality that

$$|f(t)| = \left|\sum_n c_n t^n\right| \leq \max_n |c_n| r^n = \|f\|_r.$$

To see the converse, we first assume that $r \in p^{\mathbf{Q}}$. This assumption lets us reduce to the case $r = 1$: we replace L by a finite extension L'/L such that there is some $\alpha \in L'$ with $|\alpha| = r$, and replace $f(T)$ by $g(T) = f(\alpha T)$. By construction, we see that $g \in \mathscr{O}(D(1))$, $\|g\|_1 = \|f\|_r$, and $\sup_{t\in D(1)} |g(t)| = \sup_{s\in D(r)} |f(s)|$.

Also, by dividing by an appropriate element of L, we may assume that $\|f\|_1 = 1$, i.e. that $f \in \mathscr{O}[\![T]\!]$. Thus, we may consider the mod-π reduction $\overline{f} \in \mathbf{F}[T]$. Since $\|f\|_1 = 1$, we see that $\overline{f} \neq 0$. Thus, there exists some $\overline{\alpha} \in \overline{F}$ with $\overline{f}(\overline{\alpha}) \neq 0$. Lifting $\overline{\alpha}$ to $\alpha \in \mathscr{O}_{\overline{L}}$, we see that $f(\alpha) \in \mathscr{O}_{\overline{L}}^\times$, i.e. that $|f(\alpha)| = 1$.

For $r \notin p^{\mathbf{Q}}$, note that, as $p^{\mathbf{Q}}$ is dense in $\mathbf{R}_{>0}$, $\|f\|_r = \sup_{\substack{r'<r\\ r'\in p^{\mathbf{Q}}}} \|f\|_{r'}$. Thus, we have

$$\|f\|_r = \sup_{\substack{r'<r\\ r'\in p^{\mathbf{Q}}}} \sup_{t\in D(r')} |f(t)| = \sup_{\substack{t\in\overline{L}\\ |t|\leq r'<r\\ r'\in p^{\mathbf{Q}}}} |f(t)| = \sup_{\substack{t\in\overline{L}\\ |t|\leq r}} |f(t)|$$

The final equality is due to the fact that $|t| \in p^{\mathbf{Q}}$ for any $t \in \overline{L}$. ☐

Remark 5.2.4. The closed disk $D(r)$ with its ring of functions $\mathscr{O}(D(r))$ is the prototypical example of a *rigid-analytic space* over L. These are analogous to complex-analytic spaces, with the role of $\mathbf{C}$ replaced by L. We will see more about these spaces as the course goes on.

5.3. **Duality.** Let V, W be locally convex topological vector spaces, and consider the L-vector space $\mathscr{L}(V,W)$ of continuous linear maps from V to W. It turns out that there are multiple inequivalent ways to topologize this as a LCTVS.

Let $\mathscr{B}$ be a collection of bounded sets in V. We define a locally convex topology $\mathscr{L}_{\mathscr{B}}(V,W)$. This is generated by open lattices $\Lambda(B,M)$ for $B \in \mathscr{B}$ and $M \subseteq W$ an open lattice, defined as

$$\Lambda(B,M) = \{f \in \mathscr{L}(V,W) \colon f(B) \subseteq M\}.$$

To see that this is a lattice, note that for a continuous linear map $f \in \mathscr{L}(V,W)$, $f(B)$ is bounded so $af(B) \subseteq M$ for some $a \in L^\times$, i.e. $af \in \Lambda(B,M)$.

We may state this construction equivalently in terms of seminorms. Indeed, for any $B \in \mathscr{B}$ and any continuous seminorm $\|\cdot\|$ on W, define a seminorm $\|f\|_B = \sup_{b\in B} \|f(b)\|$. Since $f(B)$ is bounded, this is finite.

Now, if $M_{\|\cdot\|}$ is the unit ball for $\|\cdot\|$, the lattice $\Lambda(B, M_{\|\cdot\|})$ is $\Lambda(\|\cdot\|_B)$, the unit ball for $\|\cdot\|_B$. On the other hand, given an open lattice $M \subseteq W$, if $\|\cdot\|_M$ is the associated seminorm, then $\|\cdot\|_{M,B} = \sup_{b\in B} \|f(b)\|_M$ is the seminorm associated to the lattice $\Lambda(B,M)$.

Note that the topology defined on $\mathscr{L}(V,W)$ really does depend on $\mathscr{B}$. The two most common choices for $\mathscr{B}$ are:

- $\mathscr{B} = \{\{b\} \colon b \in V\}$. Then the topology $\mathscr{L}_{\mathscr{B}}(V,W)$ is called the *weak topology*.

- $\mathscr{B} = \{B \subseteq V : B \text{ is bounded}\}$. Then the topology $\mathscr{L}_{\mathscr{B}}(V, W)$ is called the *strong topology.*

6. 1/23/20: Properties of dual spaces

Let V, W be LCTVSs. Last time, we defined the LCTVS $\mathscr{L}_{\mathscr{B}}(V, W)$, where $\mathscr{B}$ is a collection of bounded subsets of V, as the space of linear functionals from V to W where, for each $B \in \mathscr{B}$ and continuous seminorm $\|\cdot\|$ on W, we include the seminorm $\|f\|_B = \sup_{x\in B}\|f(x)\|$. A basis of neighborhoods of the identity is $\Lambda(B, M)$ where $B \in \mathscr{B}$ and $M \subset W$ is an open lattice.

The two most important examples (though others do appear sometimes) are when $\mathscr{B}$ is the collection of singleton points, giving us $\mathscr{L}_s(V, W)$, the "weak topology" on $\mathscr{L}(V, W)$; and when $\mathscr{B}$ is the collection of all bounded subsets, giving us $\mathscr{L}_b(V, W)$, the "strong topology" on $\mathscr{L}(V, W)$.

We write $V^\vee = \mathscr{L}(V, L)$. To specify the topology, we will write $V_s^\vee, V_b^\vee$.

6.1. Some preliminary observations.

Lemma 6.1.1. *If $\mathscr{B} \subset \mathscr{B}'$ then $\mathscr{L}_{\mathscr{B}'}(V, W) \to^{\mathrm{id}} \mathscr{L}_{\mathscr{B}}(V, W)$ is continuous, that is, the $\mathscr{B}'$ topology is finer than the $\mathscr{B}$ topology.*

In particular, the strong topology is stronger than the weak topology.

Lemma 6.1.2. *If W is Hausdorff and $\cup_{B\in\mathscr{B}}B$ is dense in V, then $\mathscr{L}_{\mathscr{B}}(V, W)$ is Hausdorff (so $\mathscr{L}_s$, $\mathscr{L}_b$ are Hausdorff).*

Proof. Let $0 \neq f \in \mathscr{L}(V, W)$. Since $\cup_{B\in\mathscr{B}}B$ is dense, there must be some $b \in B \in \mathscr{B}$ with $f(b) \neq 0$. Since W is Hausdorff, there exists some open lattice $M \subseteq W$ with $f(b) \notin M$. Therefore, f is not contained in the open lattice $\Lambda(B, M)$. $\square$

Lemma 6.1.3. *If $f : V \to W$ is continuous linear, define the dual map $f^\vee : W^\vee \to V^\vee$ by $g \mapsto g \circ f$.*

If $\mathscr{B}$ is a collection of bounded subsets of V, then $f^\vee : W^\vee_{f\mathscr{B}} \to V^\vee_{\mathscr{B}}$ is continuous. In particular, $f^\vee : W_s^\vee \to V_s^\vee$ and $f^\vee : W_b^\vee \to V_b^\vee$ are continuous, since for example the latter factors as

$$W_b^\vee \to W^\vee_{\{fB \mid B\subset V\, bounded\}} \to V_b^\vee.$$

Proof. If $B \subset V$ is bounded then

$$(f^\vee)^{-1}\Lambda(B, \mathscr{O}) = \{g \in W^\vee \mid (g \circ f)(B) \subset \mathscr{O}\} = \Lambda(f(B), \mathscr{O}).$$

$\square$

6.2. The Hahn-Banach theorem and some consequences.

Theorem 6.2.1 (Hahn-Banach)**.** *Suppose $W_0 \subset V_0$ are vector spaces and that $\|\cdot\|$ is a seminorm on V_0. If $f : W_0 \to L$ and $|f|$ is bounded by $\|\cdot\|$ on W_0, then there is $\tilde{f} : V_0 \to L$ such that $|\tilde{f}|_{W_0} = f$, $|\tilde{f}| \le \|\cdot\|$.*

Proof. Consider the poset of pairs (U, f_U) where $W_0 \subset U \subset V_0$, $f_U : U \to L$ satisfies $f_U|_{W_0} = f$, and $|f_U| \le \|\cdot\|$ on U, and we say that $(U, f_U) \ge (U', f_{U'})$ if $U' \subset U$ and $f_U|_{U'} = f_{U'}$. By Zorn's lemma, there is a maximal (U, f_U), so WLOG (W_0, f) is maximal. We need to prove that $W_0 = V$.

Suppose not. Let $x \in V_0 \setminus W_0$. Let $U = W_0 + Lx$. To get a contradiction, we need to find $b \in L$ such that for all $y \in W_0$ and $a \in L$,

$$|f(y) + ab| \leq \|y + ax\|.$$

Actually it suffices to find b such that $|f(y) + b| \leq \|y + x\|$ for all $y \in W_0$, since this implies the above. So for $y \in W_0$, let

$$B(y) = \{b \in L \mid |f(y) + b| \leq \|y + x\|\}.$$

This contains $-f(y)$, so is nonempty, and is closed and compact. It suffices to show that any finite intersection of $B(y)$s is nonempty, since then by compactness we would also have $\cap_{y \in W_0} B(y) \neq \emptyset$. In fact we claim that $B(y_1) \cap B(y_2) = B(y_1)$ if $\|x + y_1\| \leq \|x + y_2\|$, because

$$B(y_1) \cap B(y_2) = \{b \in L \mid |f(y_1) + b| \leq \|y_1 + x\|, |f(y_2) + b| \leq \|y_2 + x\|\}$$

and if $b \in B(y_1)$, then

$$\begin{aligned} |f(y_2) + b| &\leq \max(|f(y_1) - f(y_2)|, |f(y_1) - b|) \\ &\leq \max(\|y_1 - y_2\|, \|y_1 + x\|) \\ &\leq \max(\|y_1 + x\|, \|y_2 + x\|, \|y_1 + x\|) = \|y_2 + x\|. \end{aligned}$$

So we are done. □

Corollary 6.2.2. *If V is a Hausdorff LCTVS and $0 \neq x \in V$, there is $f \in V^\vee$ with $f(x) = 1$.*

Proof. There is a continuous seminorm $\|\cdot\|$ on V with $\|x\| \geq 1$. Let $f : Lx \to L$, $x \mapsto 1$. Then there is $\tilde{f} : V \to L$ with $f(x) = 1$ and $|\tilde{f}| \leq \|\cdot\|$. Then $\tilde{f}$ is also continuous. □

Corollary 6.2.3. *If $W \subset V$ is a subspace of a LCTVS (with the subspace topology) and $f \in W^\vee$, then there is $\tilde{f} \in V^\vee$ such that $\tilde{f}|_W = f$.*

Proof. Since f is continuous, $|f|$ is bounded by some seminorm $\|\cdot\|$. Then f has an extension $\tilde{f}$ bounded by $\|\cdot\|$. So $\tilde{f}$ is also continuous. □

Corollary 6.2.4. *If $M \subset V$ is a closed $\mathcal{O}$-submodule and $x \in V \setminus M$, then there is $f \in V^\vee$ with $fM \subset \mathcal{O}$ and $f(x) \notin \mathcal{O}$.*

Proof. There is an open lattice $\Lambda \subset V$ with $(x + \Lambda) \cap M = \emptyset$, that is, $x \notin \Lambda + M$. So WLOG M is an open lattice. Let $f : Lx \to L$, $x \mapsto \pi^{-1}$. Then $|f| \leq \|\cdot\|_M$ on Lx, so we can find an extension $\tilde{f} : V \to L$ such that $\tilde{f}(x) = \pi^{-1}$ and $|\tilde{f}| \leq \|\cdot\|_M$, so that $|\tilde{f}| \leq 1$ on M. □

Lemma 6.2.5. *If V is a Hausdorff LCTVS and $W \subset V$ is finite dimensional, then $V \cong W \oplus V/W$.*

Proof. Let $x_1, \ldots, x_n$ be a basis of W. Let $f_1, \ldots, f_n$ be the dual basis of $W^\vee$ (note that since W is finite-dimensional, the continuous dual and linear dual are the same thing). Find $\tilde{f}_i \in V^\vee$ extending f_i. Let

$$\begin{aligned} F : V &\to W (\subset V) \\ x &\mapsto \sum_i \tilde{f}_i(x) x_i. \end{aligned}$$

This is continuous and satisfies $F|_W = \mathrm{id}_W$ and $F^2 = F$. So we have an isomorphism $\ker F \xrightarrow{\sim} V/W$. Then we have maps

$$\begin{aligned} V &\cong W \oplus \ker F \\ x &\mapsto (Fx, x - Fx) \\ x + y &\leftarrow\!\shortmid (x, y) \end{aligned}$$

which are mutually inverse, hence isomorphisms. □

6.3. **Double duals.**

Lemma 6.3.1. *If V is a Hausdorff LCTVS then there is a continuous bijection $\delta : V \to (V_s^\vee)_s^\vee$ given by $\delta(x)(f) = f(x)$.*

We define $V_s := (V_s^\vee)_s^\vee$, that is, V with the topology of $(V_s^\vee)_s^\vee$. A generating set of open lattices is given by $f^{-1}\mathscr{O}$ for $f \in V^\vee$; such sets are always open in V, so this topology is weaker than the given topology on V. Thus, V_s called the *weak topology* (or *weak*-topology*) on V.

Proof. First, we see that $\delta(x)$ is continuous on $V_s^\vee$, since

$$\delta(x)^{-1}(\mathscr{O}) = \{f \in V^\vee \mid f(x) \in \mathscr{O}\} = \Lambda(\{x\}, \mathscr{O})$$

is open in $V_s^\vee$. We also see that δ itself is continuous, since

$$\delta^{-1}\Lambda(\{f\}, \mathscr{O}) = \{x \in V \mid f(x) \in \mathscr{O}\} = f^{-1}\mathscr{O}$$

is open.

Next, δ is injective because if $0 \neq x \in V$, there is $f \in V^\vee$ with $f(x) \neq 0$ by Hahn-Banach, so $\delta(x)(f) \neq 0$.

Finally, we check that δ is surjective. Let $d \in (V_s^\vee)^\vee$. By definition it is a continuous function $d : V_s^\vee \to L$. We have

$$d^{-1}\mathscr{O} \supset \Lambda(\{x_1, \dots, x_n\}, \mathscr{O})$$

for some $x_1, \dots, x_n \in V$. WLOG $x_1, \dots, x_n$ are linearly independent. Let $W = Lx_1 \oplus \cdots \oplus Lx_n$. By Lemma 6.2.5, we have $V \cong W \oplus U$, hence $V^\vee \cong W^\vee \oplus U^\vee$, where $U^\vee = \{f \in V^\vee \mid f|_W = 0\}$. Again give $W^\vee$ the basis $f_1, \dots, f_n$ where $f_i(x_j) = \delta_{ij}$.

Multiplying the previous condition on the x_is by π^m, we have

$$\pi^m\mathscr{O} \supset d\Lambda(\{x_1, \dots, x_n\}, \pi^m\mathscr{O})$$

for all m, hence

$$d\left(\sum_i \pi^m\mathscr{O} f_i + U^\vee\right) \subset \pi^m\mathscr{O}$$

for all m, hence $dU^\vee \subset \pi^m\mathscr{O}$ for all m, hence $dU^\vee = \{0\}$. But for any $f \in V_s^\vee$, we have $f - \sum_i f(x_i) f_i \in U^\vee$, so

$$d(f) = \sum_i f(x_i) d(f_i) = f\left(\sum d(f_i) x_i\right)$$

for all f. This can be rewritten as $d = \delta\left(\sum_i d(f_i) x_i\right)$, so d has a preimage, as desired. □

Corollary 6.3.2. *$V \to (V_b^\vee)_s^\vee$ has dense image (and likewise if we replace the strong topology with any topology which refines the weak topology). .*

Proof. Suppose not. Then by Hahn-Banach we can find a continuous $0 \neq f : (V_b^\vee)_s^\vee \to L$ such that f is zero on the closure of the image of V, that is, a $0 \neq f \in ((V_b^\vee)_s^\vee)_s^\vee$ that is zero on the image of V. But $((V_b^\vee)_s^\vee)_s^\vee = V_b^\vee$, so we have $0 \neq f \in V_b^\vee$ such that $f(V) = \{0\}$, which is a contradiction. □

Lemma 6.3.3. *If V is Hausdorff and bornological, $\delta : V \to (V_b^\vee)_b^\vee$ gives a topological isomorphism from V to its image. If furthermore V is complete then $\delta(V)$ is closed in $(V_b^\vee)_b^\vee$.*

Note that if V is Banach and $\dim V = \infty$, $V \to (V_b^\vee)_b^\vee$ is not surjective. (However, there are interesting cases where it is.)

6.4. **Duals of direct sums and products.**

Lemma 6.4.1. *(1) $(\bigoplus_i V_i)_s^\vee \xrightarrow{\sim} \prod_i (V_i)_s^\vee$ and $(\bigoplus_i V_i)_b^\vee \xrightarrow{\sim} \prod_i (V_i)_b^\vee$.*
(2) $\bigoplus_i (V_i)_s^\vee \xrightarrow{\sim} (\prod_i V_i)_s^\vee$ and $\bigoplus_i (V_i)_b^\vee \xrightarrow{\sim} (\prod_i V_i)_b^\vee$.

Proof. (1) The fact that these are bijections comes from the universal property of $\oplus$. To get that they are homeomorphisms, first reduce to the case that V_i is Hausdorff for all i by letting $W_i \subset V_i$ be the closure of $\{0\}$ and noting that if the statement is true when everything is Hausdorff, then

$$\left(\bigoplus V_i\right)^\vee \cong \left(\bigoplus V_i / \bigoplus W_i\right)^\vee \cong \left(\bigoplus V_i/W_i\right)^\vee \xrightarrow{\sim} \prod (V_i/W_i)^\vee \cong \prod V_i^\vee.$$

Now assuming everything is Hausdorff, it is easy to check that the natural map is continuous. We now check that the map is open. If $B \subset \bigoplus V_i$ is bounded, $\mathrm{pr}_i B$ is bounded for all i and $\{0\}$ for almost all i. So

$$\Lambda(B, \mathscr{O}) = \left\{ (f_i) \mid \forall b \in B, \sum_i f_i(b_i) \in \mathscr{O} \right\} \supset \prod_i \Lambda(\mathrm{pr}_i B, \mathscr{O}).$$

Since $\Lambda(\mathrm{pr}_i B, \mathscr{O})$ is $V_i^\vee$ for almost all i, $\prod_i \Lambda(\mathrm{pr}_i B, \mathscr{O})$ is open, and thus $\Lambda(B, \mathscr{O})$ is too.

(2) The map in the opposite direction is

$$\left(\prod_i V_i\right)^\vee \to \bigoplus V_i^\vee$$
$$f \mapsto f|_{V_i}.$$

To check that the image really lands in $\bigoplus V_i^\vee$, note that there is some finite I_0 such that

$$\prod_{i \in I_0} \Lambda_i \times \prod_{i \notin I_0} V_i \subset f^{-1}\mathscr{O},$$

so $f \prod_{i \notin I_0} V_i \subseteq \mathscr{O}$. As $\prod_{i \notin I_0} V_i$ is an L-subspace of the product, this implies that $f \prod_{i \notin I_0} V_i \subseteq \cap_{a \in L^\times} a\mathscr{O} = 0$.

This map has the desired properties (composites with the map in the other direction are the identity, and both are continuous) by a similar argument as before. □

7. 1/28/20: More on dual spaces

Last time we defined $V^\vee = \mathscr{L}(V, L)$, or specifically $V^\vee_{\mathscr{B}}$ where $\mathscr{B}$ is a collection of bounded subsets of V. We defined $V^\vee_s$ and $V^\vee_b$, and found that

$$\left(\bigoplus V_i\right)^\vee_{s/b} = \prod V^\vee_{i,s/b}$$
$$\left(\prod V_i\right)^\vee_{s/b} = \bigoplus V^\vee_{i,s/b}.$$

Also, we saw that if V is Hausdorff, $V \to (V^\vee_s)^\vee_s$ is a continuous bijection.

7.1. **Dualizing norms.**

Lemma 7.1.1. *Suppose V is a normable LCTVS with norm $\|\cdot\|$. Then*

$$\|f\|^\vee = \sup_{\substack{x\in V\\ x\neq 0}} \frac{|f(x)|}{\|x\|}$$

defines a norm on $V^\vee$ which generates the strong topology $V^\vee_b$.

Similarly, if W is normed, one can define a norm on $\mathscr{L}(V, W)$ that gives the strong topology $\mathscr{L}_b(V, W)$.

Proof. First note that the supremum exists because $|f|$ is a continuous seminorm on V. For the purpose of the proof, define a new seminorm

$$\|f\|' = \sup_{\substack{v\in V\\ \|x\|\leq 1}} |f(x)|.$$

We can see that this is equivalent to the one in the lemma as follows. Since we can scale any element x by an element of $L^\times$ to lie in the annulus $|\pi| < \|x\| \leq 1$, and scaling x by $L^\times$ does not change the quantity $\frac{|f(x)|}{\|x\|}$, we have

$$\|f\|^\vee = \sup_{\substack{x\in V\\ |\pi|<\|x\|\leq 1}} \frac{|f(x)|}{\|x\|}.$$

Therefore,

$$\|f\|^\vee = \sup_{\substack{x\in V\\ |\pi|<\|x\|\leq 1}} \frac{|f(x)|}{\|x\|} \geq \sup_{\substack{x\in V\\ |\pi|<\|x\|\leq 1}} |f(x)| = \|f\|'$$

where the last equality holds because if $|\pi| < |a|\|x\| \leq 1$ with $a \in L^\times$, $|a| \geq 1$, then $|f(x)| = |a|^{-1}|f(ax)|$, so we might as well replace x with ax. Similarly

$$\|f\|^\vee = \sup_{\substack{x\in V\\ |\pi|<\|x\|\leq 1}} \frac{|f(x)|}{\|x\|} \leq |\pi|^{-1} \sup_{\substack{x\in V\\ |\pi|<\|x\|\leq 1}} |f(x)| = |\pi|^{-1}\|f\|'.$$

So $\|\cdot\|'$ and $\|\cdot\|^\vee$ are equivalent seminorms, and in fact are norms, because if $\|f\|' = 0$ then $f|_{\Lambda(\|\cdot\|)} = 0$ so $f = 0$, so $\|\cdot\|'$ is a norm, so $\|\cdot\|^\vee$ is a norm. To check that $\|\cdot\|'$ generates the strong topology, first we see that

$$\Lambda(\|\cdot\|') = \{f \in V^\vee \mid \text{if } \|x\| \leq 1 \text{ then } |f(x)| \leq 1\} = \Lambda(\Lambda(\|\cdot\|), \mathscr{O})$$

which is indeed open in the strong topology since $\Lambda(\|\cdot\|)$ is bounded. On the other hand if $B \subset V$ is bounded then there is $a \in L^\times$ such that $B \subset a\Lambda(\|\cdot\|)$ and so

$$\Lambda(B, \mathscr{O}) \supset \Lambda(a\Lambda(\|\cdot\|), \mathscr{O}) = a^{-1}\Lambda(\Lambda\|\cdot\|, \mathscr{O}) = a^{-1}\Lambda(\|\cdot\|').$$

□

Lemma 7.1.2.

$$(\perp_0(V_i, \|\cdot\|_i))_b^\vee \cong \perp\left(V_{i,b}^\vee, \|\cdot\|_i^\vee\right).$$

Proof. The desired isomorphism is $\theta : f \mapsto (f|_{V_i})$. Clearly θ is linear. We have

$$\|f|_{V_i}\|_i^\vee = \sup_{\substack{x \in V_i \\ |\pi| \le \|x\| \le 1}} \frac{|f(x)|}{\|x\|_i} \le (\|f\|_\infty)^\vee$$

for all i, so $\|\theta(f)\|_\infty \le \|f\|_\infty^\vee$. On the other hand

$$\begin{aligned}\|f\|_\infty^\vee &\le |\pi|^{-1} \sup_{\|x\|_\infty \le 1} |f(x)| \le |\pi|^{-1} \sup_i \sup_{\|x_i\|_i \le 1} |f(x_i)| \\ &\le |\pi|^{-1} \sup_i \sup_{|\pi| \le \|x_i\|_i \le 1} \frac{f(x_i)}{\|x_i\|_i} = |\pi|^{-1} \sup_i \|f|_{V_i}\|_i^\vee = |\pi|^{-1}\|\theta(f)\|_\infty.\end{aligned}$$

In conclusion the seminorms $\|\theta(\cdot)\|_\infty$ and $\|\cdot\|_\infty^\vee$ are equivalent, so θ is injective and a homeomorphism onto its image.

It remains to show that θ is surjective. If (f_i) is an element of the RHS with $\|f_i\|_i^\vee \le C$ for all i, and $(x_i) \in \perp_0(V_i, \|\cdot\|_i)$, then $|f_i(x_i)| \le C\|x_i\|_i$ goes to 0 as $i \to \infty$, so $\sum_i f_i(x_i)$ converges. So we may define $f(x) = \sum f_i(x_i)$. Then f is linear, $f|_{V_i} = f_i$, and f is continuous because

$$|f(x)| \le \sup_i |f_i(x_i)| \le C \sup_i \|x_i\|_i = C\|(x_i)\|_\infty.$$

□

Note that we can similarly show that if W is Banach, then

$$\perp_i(\mathscr{L}_b(V_i, W), \|\cdot\|_i^\vee) \cong \mathscr{L}_b(\perp_0(V_i, \|\cdot\|_i), W).$$

Lemma 7.1.3. *We have an injection*

$$\perp_0(V_{i,b}^\vee, \|\cdot\|_i^\vee) \hookrightarrow (\perp(V_i, \|\cdot\|_i))_b^\vee$$

and the image is closed (but not in general surjective).

Proof. Similar to before. The map is

$$(f_i) \mapsto \left((x_i) \mapsto \sum f_i(x_i)\right).$$

One checks that $\sum f_i(x_i)$ converges and that this map is a homeomorphism onto its image. The image is closed because $V_{i,b}^\vee$ is complete for all i, which we will prove in a moment. □

7.2. **Stuff about strong duals.**

Lemma 7.2.1. *If V is Hausdorff and bornological (e.g. metrizable), then*

$$\delta : V \to (V_b^\vee)_b^\vee$$

is a homeomorphism onto its image, and if V is complete then its image is closed.

Proof. δ factors as $V \to (V_s^\vee)_s^\vee \subset (V_b^\vee)^\vee$ and the first map is a bijection, so it is well-defined and injective. To show that δ is continuous, we need to show that if $B \subset V_b^\vee$ is bounded, then $\delta^{-1}\Lambda(B, \mathscr{O})$ is open in V. We can rewrite

$$\delta^{-1}\Lambda(B, \mathscr{O}) = \{x \in V \mid \forall f \in B, f(x) \in \mathscr{O}\} = \bigcap_{f \in B} f^{-1}\mathscr{O}.$$

Since V is bornological, it suffices to show that if $C \subset V$ is bounded, then there is $a \in L^\times$ with $aC \subset \bigcap_{f \in B} f^{-1}\mathscr{O}$. But we can find $b \in L$ such that $bB \subset \Lambda(C, \mathscr{O})$, so

$$\bigcap_{f \in B} f^{-1}\mathscr{O} \supset \bigcap_{f \in \Lambda(C, b^{-1}\mathscr{O})} f^{-1}\mathscr{O} \supset bC.$$

To show that δ is open, let $\Lambda \subset V$ be an open lattice. It is enough to show that (1)

$$\Lambda = \delta^{-1}\Lambda(\Lambda(\Lambda, \mathscr{O}), \mathscr{O})$$

and (2) $\Lambda(\Lambda, \mathscr{O}) \subset V_b^\vee$ is bounded. For (1),

$$\Lambda \subset \delta^{-1}\Lambda(\Lambda(\Lambda, \mathscr{O}), \mathscr{O})$$

is clear, and for the other inclusion, if $x \in \delta^{-1}\Lambda(\Lambda(\Lambda, \mathscr{O}), \mathscr{O})$, then for all $f \in \Lambda(\Lambda, \mathscr{O})$ we have $f(x) \in \mathscr{O}$, so $x \in \Lambda$. (As in, if x were not in Λ, Hahn-Banach would give $f \in V^\vee$ such that $f(\Lambda) \subset \mathscr{O}$ but $f(x) \notin \mathscr{O}$.)

For (2), if $B \subset V$ is bounded, we need to show that there is $a \in L^\times$ such that $a\Lambda(\Lambda, \mathscr{O}) \subset \Lambda(B, \mathscr{O})$. But if $b \in L^\times$ is such that $bB \subset \Lambda$, then

$$b^{-1}\Lambda(B, \mathscr{O}) = \Lambda(bB, \mathscr{O}) \supset \Lambda(\Lambda, \mathscr{O}).$$

□

Lemma 7.2.2. *If every closed bounded $\mathscr{O}$-submodule of V is compact then $V \twoheadrightarrow (V_b^\vee)_b^\vee$.*

Proof. We don't have time for this. See Proposition 15.3 of Schneider. □

Note that the above condition usually doesn't apply to Banach spaces. If $V = \perp_{\mathbb{Z}_{>0},0}(L, |\cdot|)$, then $V_b^\vee \cong \perp_{\mathbb{Z}_{>0}}(L, |\cdot|)$ contains $\perp_{\mathbb{Z}_{>0},0}(L, |\cdot|)$ as a proper closed subset. Then if we take a continuous

$$0 \neq f : \perp_{\mathbb{Z}_{>0}}(L, |\cdot|)/\perp_{\mathbb{Z}_{>0},0}(L, |\cdot|) \to L,$$

then $f \notin \operatorname{im}(V \to^\delta (V_b^\vee)^\vee)$, because $f(e_i) = 0$ for all i, and if $f = \delta(x)$ then $\delta(x)(e_i) = x_i = 0$ for all i so $x = 0$.

Remark 7.2.3. If V is any infinite-dimensional Banach space, then $V \to (V_b^\vee)^\vee$ is not surjective. Indeed, choosing a countable Banach sub-basis, we may split $V = W \oplus W'$ with $W \simeq \perp_{\mathbb{Z}_{>0},0}(L, |\cdot|)$, and use the preceding example.

Example 7.2.4. Recall that we have
$$C(\mathbb{Z}_p, L) \cong \perp_{\mathbb{Z}_{\geq 0},0} (L, |\cdot|)$$
via $\binom{x}{n} \mapsto e_n$. From this we have a map
$$A : C(\mathbb{Z}_p, L)_b^\vee \xrightarrow{\sim} \mathscr{O}[\![T]\!] \otimes_{\mathscr{O}} L$$
as follows. Let $\mu \in C(\mathbb{Z}_p, L)_b^\vee$. We want to write μ as a power series $\sum_{n=0}^\infty a_n T^n$ with $a_n \in L$, $|a_n|$ bounded. We should have
$$A(\mu) = \sum_{n=0}^\infty \mu\left(\binom{x}{n}\right) T^n = \mu\left((1+T)^x\right),$$
at least formally. If $t \in L$ with $|t| < 1$ then we can honestly write $A(\mu)(t) = \mu\left((1+t)^x\right)$.

The strong topology on $\mathscr{O}[\![T]\!] \subset C(\mathbb{Z}_p, L)^\vee$ is the p-adic topology, generated by $\pi^r \mathscr{O}[\![T]\!]$. The weak topology on $\mathscr{O}[\![T]\!] \subset C(\mathbb{Z}_p, L)^\vee$ is the (π, T)-adic topology: for a sum of the form $\sum c_n \binom{x}{n}$, $c_n \in L$, $|c_n| \to 0$, the associated open set
$$\Lambda\left(\left\{\sum c_n \binom{x}{n}\right\}, \mathscr{O}\right) = \left\{\sum a_n T^n \in \mathscr{O}[\![T]\!] \mid \sum_{n=1}^N a_n c_n \in \mathscr{O}\right\}$$
(where N is such that $c_n \in \mathscr{O}$ for $n > N$) contains
$$\left\{\sum a_n T^n \in \mathscr{O}[\![T]\!] \mid a_n \in c_n^{-1}\mathscr{O} \text{ for } n = 0, \dots, N\right\} \supset (\pi, T)^{\text{big}}.$$

7.3. **Stuff about completeness.** When is $\mathscr{L}_{\mathscr{B}}(V, W)$ complete?

Definition 7.3.1. We call a subset $H \subset \mathscr{L}(V, W)$ equicontinuous if for all open lattices $M \subset W$, there is an open lattice $\Lambda \subset V$ with $\Lambda \subset f^{-1}M$ for all $f \in H$.

Lemma 7.3.2 (Banach-Steinhaus)**.** *If V is a Fréchet space (or, more generally, barrelled—that is, every closed lattice is open) then any bounded subset of $\mathscr{L}_{s/b}(V, W)$ is equicontinuous. (Note that any bounded subset in $\mathcal{L}_b(V, W)$ is bounded in $\mathcal{L}_s(V, W)$, so we can drop the /b.)*

Proof. Let H be the bounded subset in question. Suppose $M \subset W$ is an open lattice. Then $\bigcap_{f \in H} f^{-1}M$ is a closed $\mathscr{O}$-submodule. Since V is barrelled, it is sufficient to prove that $\bigcap_{f \in H} f^{-1}M$ is a lattice. If $x \in V$ then $H \subset a\Lambda(\{x\}, M)$ for some $a \in L^\times$. That is, if $f \in H$ then $f(a^{-1}x) = a^{-1}f(x) \in M$, so $a^{-1}x \in \bigcap_{f \in H} f^{-1}M$. $\square$

Lemma 7.3.3. *Suppose W is complete and $\mathscr{B}$ contains all singletons and $H \subset \mathscr{L}_{\mathscr{B}}(V, W)$ is closed. Under either of the following conditions, H is complete.*

(1) V is bornological and $\mathscr{B} = b$.
(2) H is equicontinuous.

Corollary 7.3.4. *(1) If V is bornological then $V_b^\vee$ is complete.*
(2) If V, W are Fréchet then $\mathscr{L}_b(V, W)$ is Fréchet.
(3) if V is normable and W is Banach then $\mathscr{L}_b(V, W)$ is Banach.

Proof of Lemma. Let (f_i) be a Cauchy net in H. Then for all $x \in V$, $(f_i(x))$ is a Cauchy net in W. (Reason: if $M \subset W$ is any open lattice then $\Lambda(\{x\}, M)$ is an open lattice in $\mathscr{L}_{\mathscr{B}}(V, W)$, so for $i, j \gg_{x,M} 0$, we have $f_i - f_j \in \Lambda(\{x\}, M)$, meaning $f_i(x) - f_j(x) \in M$.) So $f_i(x)$ converges to a limit $f(x) \in W$, giving us a linear map $f : V \to W$. It suffices to check that f is continuous; then we can conclude that $f_i \to f$ in $\mathscr{L}_{\mathscr{B}}(V, W)$.

To show this, let $B \in \mathscr{B}$ and $M \subset W$ be an open lattice. Then for $i, j \gg_{B,M} 0$ we have $(f_i - f_j)(B) \subset M$, and for $j \gg_x 0$ we have $f_j(x) - f(x) \in M$, so by choosing j large enough depending on $x \in B$, we see that for $i \gg_{B,M} 0$ we have $(f_i - f)(B) \subset M$. We will finish this next time. $\square$

8. 1/30/20: Finishing dual spaces, compact maps, and tensor products

8.1. **Finishing up dual spaces.** We were in the middle of proving the following lemma.

Lemma 8.1.1. *Suppose W is complete, $\mathscr{B}$ is a collection of bounded subsets of V containing all singletons, and $H \subset \mathscr{L}_{\mathscr{B}}(V, W)$ is closed. Under either of the following conditions, H is complete.*

(1) V is bornological and $\mathscr{B} = b$.
(2) H is equicontinuous (e.g. if V is barrelled and H is bounded).

Proof. We took a Cauchy net (f_i) in H, found that $f_i(x)$ converged to some $f(x) \in W$ for all $x \in V$, and thus obtained a linear map $f : V \to W$. It suffices to check that f is continuous: then if $B \in \mathscr{B}$ and $M \subset W$ is an open lattice, we have $(f - f_i)(B) \subset M$ for $i \gg_{B,M} 0$, so $(f - f_i) \in \Lambda(B, M)$, so $f_i \to f$ in $\mathscr{L}_{\mathscr{B}}(V, W)$.

(1) It suffices to prove that if $B \subset V$ is bounded, then $f(B)$ is bounded. Given $M \subset W$, we need to find $a \in L^\times$ such that $f(B) \subset aM$. Choose i such that $(f - f_i)(B) \subset M$. We can find $a \in L^\times$ with $|a| \geq 1$ and $f_i(B) \subset aM$, so $f(B) \subset M + aM = aM$.

Note that we really did need to use the strong topology for this argument to work: to check continuity of a map out of a bornological space, we need to verify that $f(B)$ is bounded for *all* open sets B.

(2) Given $M \subset W$ an open lattice, there is $\Lambda \subset V$ an open lattice with $f_i(\Lambda) \subset M$ for all i. For all $x \in V$ we have $(f - f_i)(x) \in M$ for $i \gg 0$. Since M is closed, we conclude that if $x \in \Lambda$ then $f(x) \in M$, that is, $f(\Lambda) \subset M$.

$\square$

This implies, for example, that if V is bornological then $V_b^\vee$ is complete, and if V, W are Fréchet/Banach then $\mathscr{L}_b(V, W)$ is Fréchet/Banach.

Lemma 8.1.2. *If V is barrelled and Hausdorff (e.g. Fréchet), then $V \xrightarrow{\sim} (V_s^\vee)_b^\vee$.*

Proof. We have seen that $\delta : V \to (V_s^\vee)_b^\vee$ is a linear bijection. To see that it is open, let $\Lambda \subset V$ be an open lattice. Then $\Lambda(\Lambda, \mathscr{O}) \subset V_s^\vee$ is bounded, since it is bounded in $V_b^\vee$. Then $\Lambda(\Lambda(\Lambda, \mathscr{O}), \mathscr{O})$ is an open lattice in $(V_s^\vee)_b^\vee$. But we have seen that $\Lambda(\Lambda(\Lambda, \mathscr{O}), \mathscr{O})$ is precisely the image of Λ under δ.

To see that δ is continuous, suppose $B \subset V_s^\vee$ is bounded. We need to show that $\delta^{-1}\Lambda(B, \mathscr{O})$ is open in V. By definition this is

$$\{x \in V \mid \forall f \in B, f(x) \in \mathscr{O}\} = \bigcap_{f \in B} f^{-1}\mathscr{O}.$$

But the Banach-Steinhaus theorem tells us that B is equicontinuous, so $\bigcap_{f \in B} f^{-1}\mathscr{O}$ is indeed open. $\square$

8.2. **Compact and completely continuous maps.** Now, we isolate a class of continuous linear maps which have particularly well-behaved spectral theory.

Definition 8.2.1. We call $f \in \mathscr{L}(V, W)$ compact if there is an open lattice $\Lambda \subset V$ with $\overline{f\Lambda}$ compact in W. Let

$$\mathscr{C}(V, W) = \{f \in \mathscr{L}(V, W) \mid f \text{ compact}\}.$$

This is an L-subspace of $\mathscr{L}(V, W)$.

Lemma 8.2.2. *(1) If $f : V \to W$ is compact and $B \subset V$ is bounded, then $\overline{fB}$ is compact.*

(2) If $V \xrightarrow{f} W \xrightarrow{g} U$ are continuous and linear, U is Hausdorff, and either f or g is compact, then $g \circ f$ is compact.

Proof. (1) Let $\Lambda \in V$ be an open lattice such that $\overline{f\Lambda}$ is compact. Then $B \subset a\Lambda$ for some $a \in L^\times$, so $\overline{fB} \subset a\overline{f(\Lambda)}$.

(2) First suppose that f is compact, and let $\Lambda \subseteq V$ be an open lattice such that $\overline{f\Lambda}$ is compact. Then $\overline{gf\Lambda} = \overline{g\overline{f\Lambda}}$. Since U is Hausdorff and $\overline{f\Lambda}$ is compact, then $g\overline{f\Lambda}$ is compact and closed, so $\overline{gf\Lambda}$ is compact.

Now suppose instead that g is compact. Let $M \subseteq W$ be an open lattice such that $\overline{gM}$ is compact. By continuity of f, we may pick an open lattice $\Lambda \subseteq V$ with $f\Lambda \subseteq M$. Thus $\overline{gf\Lambda} \subseteq \overline{gM}$, so it is compact.

∎

Note that if $f \in \mathscr{L}(V, W)$ with W Hausdorff and $\operatorname{im} f$ is finite-dimensional, then f is compact. This is because if $M \subset W$ is an open lattice, then $M \cap \operatorname{im} f$ is bounded, and so if $\Lambda \subset V$ is an open lattice with $f\Lambda \subset M$, then $f\Lambda \subset M \cap \operatorname{im} f$ is bounded; so $\overline{f\Lambda}$ is compact.

Lemma 8.2.3. *Let V be normed, W complete, and $\Lambda_0 \subset V$ the unit ball. Then*

(1) $f \in \mathscr{L}(V, W)$ is compact if and only if for any open lattice $M \subset W$,

$$\#(f\Lambda_0 / M \cap f\Lambda_0) < \infty.$$

(2) $\mathscr{C}(V, W) \subset \mathscr{L}_b(V, W)$ is closed.

Proof. (1) In this case f is compact if and only if $\overline{f\Lambda_0}$ is compact (as Λ_0 is an open bounded lattice). If $\overline{f\Lambda_0}$ is compact, then for $M \subset W$ an open lattice, $\overline{f\Lambda_0}/\overline{f\Lambda_0} \cap M$ is also compact. But it is additionally discrete (since for any $M \subseteq W$ an open lattice, the cosets of $\overline{f\Lambda_0} \cap M$ in $\overline{f\Lambda_0}$ form a disjoint open cover), hence finite. In the other direction, assume the finiteness statement and consider the map

$$\overline{f\Lambda_0} \to \varprojlim_M \overline{f\Lambda_0}/\overline{f\Lambda_0} \cap M.$$

This is a bijection because of the completeness of W: if we have a compatible sequence $(x_M + (\overline{f\Lambda_0} \cap M))$, then (x_M) is a Cauchy net, so has a limit x. It is a homeomorphism by definition of the open sets on both sides. Also we have

$$\overline{f\Lambda_0}/\overline{f\Lambda_0} \cap M = (\overline{f\Lambda_0} + M)/M = (f\Lambda_0 + M)/M = f\Lambda_0/M \cap (f\Lambda_0).$$

We conclude that $\overline{f\Lambda_0}$ is profinite, hence compact.

(2) Let $f \in \mathscr{L}(V, W) \setminus \mathscr{C}(V, W)$. Then there is $M \subset W$ an open lattice such that

$$\#(f\Lambda_0/f\Lambda_0 \cap M) = \#((f\Lambda_0 + M)/M) = \infty.$$

For any $g \in \Lambda(\Lambda_0, M)$, we have $(f+g)\Lambda_0 \subset f\Lambda_0 + M$, so

$$((f+g)\Lambda_0 + M)/M = (f\Lambda_0 + M)/M$$

is infinite, so g is not compact. That is, $(f + \Lambda(\Lambda_0, M)) \cap \mathscr{C}(V, W) = \emptyset$.

□

Definition 8.2.4. We say that $f \in \mathscr{L}(V, W)$ has finite rank if $\dim \operatorname{im} f < \infty$. Let $\mathscr{C}\mathscr{C}(V, W)$ be the closure of the space of finite rank elements of $\mathscr{L}_b(V, W)$. We call elements of $\mathscr{C}\mathscr{C}(V, W)$ "completely continuous".

Lemma 8.2.5. *If W is Hausdorff, $\mathscr{C}(V, W) \subset \mathscr{C}\mathscr{C}(V, W)$.*

Corollary 8.2.6. *If W is complete and V is normed, then $\mathscr{C}(V, W)$ contains all finite rank maps $V \to W$ and is closed, so $\mathscr{C}(V, W) = \mathscr{C}\mathscr{C}(V, W)$.*

Proof of Lemma 8.2.5. Let $f \in \mathscr{C}(V, W)$. Given $B \subset V$ bounded and $M \subset W$ an open lattice, we need to find $g \in \mathscr{L}(V, W)$ of finite rank with $f - g \in \Lambda(B, M)$.

Since $\overline{fB}$ is compact, $(\overline{fB} + M)/M = (fB + M)/M$ is finite. Choose $e_1, \ldots, e_d \in f(B)$ such that $\sum_{i=1}^d Le_i + M \supset f(B) + M$. WLOG the e_is are linearly independent mod M, and we can scale them so that $e_i \in \pi^{-1}M - M$ for all i. Let $W_1 = \sum_{i=1}^d Le_i$.

Let $f_1, \ldots, f_d$ be the dual basis of $W_1^\vee$associated to $e_1, \ldots, e_d$. Then $|f_i(x)| \le |\pi| \|x\|_M$ for all i. By Hahn-Banach, we may choose $\tilde{f}_i \in W^\vee$ such that $|\tilde{f}_i| \le |\pi| \| \cdot \|_M$ and

$$\tilde{f}_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} F : W &\to W \\ y &\mapsto \sum \tilde{f}_i(y) e_i. \end{aligned}$$

Note that $\operatorname{im}(F) \subset W_1$, $F|_{W_1} = \mathrm{id}_{W_1}$, and $F^2 = F$, so F is an idempotent projecting onto W_1. We have

$$\|F(y)\|_M \le |\pi| \|y\|_M |\pi|^{-1} = \|y\|_M$$

so if $y \in M$ then $F(y) \in M$. This means that if $y \in W_1 + M$, then $F(y) - y \in M$ (because if $y = y_1 + y_2$ with $y_1 \in W_1$ and $y_2 \in M$, then $F(y) = y_1 + F(y_2) = y + F(y_2) - y_2$, but $F(y_2) - y_2 \in M$). In particular this applies if $y \in fB + M$. So

$$(f - F \circ f)(B) = (\mathrm{id} - F)(fB) \subset M,$$

that is, $f - F \circ f \in \Lambda(B, M)$. But $\operatorname{im}(F \circ f) \subset W_1$, so $F \circ f$ is finite rank, and is our desired finite rank approximating map. □

Lemma 8.2.7. *Suppose V, W are Banach spaces and $f \in \mathscr{L}(V, W)$. Then TFAE:*

(1) f is compact
(2) $f^\vee \in \mathscr{L}(W_b^\vee, V_b^\vee)$ is compact
(3) $f^{\vee\vee} : (V_b^\vee)_b^\vee \to (W_b^\vee)_b^\vee$ factors through the closed inclusion $W \subset (W_b^\vee)_b^\vee$.

Proof. We don't have time for this. See Lemma 16.4 of Schneider. □

8.3. **Tensor products.** There are (at least) two distinct notions of tensor products of LCTVSs, corresponding to the following two notions of continuity for bilinear maps.

Definition 8.3.1. A bilinear map $\beta : V \times W \to U$ is called

(1) separately continuous if for all $v \in V$, the map $W \to U$ given by $w \mapsto \beta(v, w)$ is continuous, and similarly for all $w \in W$, the map $V \to U$ given by $v \mapsto \beta(v, w)$ is continuous.
(2) (jointly) continuous if β is continuous with respect to the product topology on $V \times W$.

Definition 8.3.2. (1) We can define a topology on $V \otimes W$ (the "inductive topology") by declaring a lattice $\Lambda \subset V \otimes W$ to be open if for all $v \in V$, $\{w \in W \mid v \otimes w \in \Lambda\}$ is open in W, and for all $w \in W$, $\{v \in V \mid v \otimes w \in \Lambda\}$ is open in V. We will write $V \otimes_\iota W$ for $V \otimes W$ with this topology.

Note that $V \times W \xrightarrow{\otimes} V \otimes_\iota W$ is separately continuous, and universal among separately continuous bilinear maps on $V \times W$: if $\beta : V \times W \to U$ is separately continuous, the induced map $V \otimes_\iota W \to U$ is continuous.

(2) We can define a different topology on $V \otimes W$ by taking $\Lambda \otimes_{\mathscr{O}} M \hookrightarrow V \otimes_L W$, for open lattices $\Lambda \subset V$ and $M \subset W$, as a generating set of open lattices in $V \otimes W$. This is called the "projective topology". We will write $V \otimes_\pi W$ for $V \otimes W$ with this topology.

Similarly to the previous situation, $V \times W \xrightarrow{\otimes} V \otimes_\pi W$ is (jointly) continuous, and if $\beta : V \times W \to U$ is continuous, then the induced map $V \otimes_\pi W \to U$ is continuous.

Lemma 8.3.3. *(1) If V, W are Hausdorff then $V \otimes_\iota W$, $V \otimes_\pi W$ are Hausdorff.*
(2) The projective topology on $V \otimes W$ is coarser than the inductive topology (that is, $V \otimes_\iota W \xrightarrow{\mathrm{id}} V \otimes_\pi W$ is continuous).
(3) If $f \in \mathscr{L}(V, V')$, $g \in \mathscr{L}(W, W')$, then $f \otimes g : V \otimes_\iota W \to V' \otimes_\iota W'$ and $f \otimes g : V \otimes_\pi W \to V' \otimes_\pi W'$ are continuous.
(4) If $V' \subset V$, $W' \subset W$ are subspaces, then $V' \otimes_\pi W' \hookrightarrow V \otimes_\pi W$ is a subspace.

Proof. Exercise. □

Lemma 8.3.4. *If V and W are normable with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, then $V \otimes_\pi W$ is normable with the norm*

$$\|z\| = \inf \left\{ \max_i \left(\|x_i\|_V \|y_i\|_W \right) \mid z = \sum_{i=1}^m x_i \otimes y_i \right\}.$$

Proof. WLOG $\|V\|_V = |L|$, $\|W\|_W = |L|$. Let $\Lambda_0 = \Lambda(\|\cdot\|_V)$, $M_0 = \Lambda(\|\cdot\|_W)$ be the open unit balls in V and W. Then our proposed norm also satisfies $\|V \otimes W\| = |L|$.

We claim that $\Lambda(\|\cdot\|) = \Lambda_0 \otimes_{\mathscr{O}} M_0$, so that the topologies match. It is clear that $\Lambda(\|\cdot\|) \supset \Lambda_0 \otimes_{\mathscr{O}} M_0$. In the other direction, if $\|z\| \le 1$, then we can write $z = \sum_{i=1}^m x_i \otimes y_i$ so that $\|x_i\|_V \|y_i\|_W \le 1$ for all i. WLOG we can rescale the x_i, y_i so that $\|x_i\|_V = 1$; then $x_i \in \Lambda_0$ for all i. This also implies that $\|y_i\| \le 1$ for all i, so $y_i \in M_0$ for all i, and we have $z \in \Lambda_0 \otimes_{\mathscr{O}} M_0$ as desired. □

9. 2/4/20: More on tensor products, nuclear spaces, and spaces of compact type

9.1. **More on tensor products.** Last time, we defined the conditions of separately continuous and plain/jointly continuous for bilinear maps $\beta : V \times W \to U$. We called the resulting topologies on $V \otimes W$ the injective topology (written $V \otimes_\iota W$) and projective topology (written $V \otimes_\pi W$).

Lemma 9.1.1. *If $\beta : V \times W \to U$ is bilinear, then TFAE:*

(1) β is continuous.
(2) For all open lattices $N \subset U$, there are open lattices $\Lambda \in V$, $M \subset W$ with $\beta(\Lambda, M) \subset N$.
(3) For all continuous seminorms $\|\cdot\|$ on U, there are continuous seminorms $\|\cdot\|_V$ on V and $\|\cdot\|_W$ on W with

$$\|\beta(x,y)\| \le \|x\|_V \|y\|_W$$

for all $x \in V$, $y \in W$.

Proof. (2) implies (1): let $N \subset U$ be any open lattice, and choose Λ, M as in (2). For any $x \in V$ and $y \in W$, for $a, b \in L$, we have

$$\beta(x + a\Lambda, y + bM) = \beta(x,y) + \beta(\Lambda, ay) + \beta(bx, M) + ab\beta(\Lambda, M) \subset \beta(x,y) + N$$

if $ay \in M$, $bx \in \Lambda$, and $ab \in \mathscr{O}$; but we can choose a, b with these properties. So β is continuous.

(3) implies (2): given N, let $\|\cdot\|_N$ be a seminorm associated to N, and choose $\|\cdot\|_V$, $\|\cdot\|_W$ as in (3). Then take $\Lambda = \Lambda(\|\cdot\|_V)$, $M = \Lambda(\|\cdot\|_W)$.

(1) implies (3): since $\|\beta\|$ is continuous, there are continuous seminorms $\|\cdot\|_V$, $\|\cdot\|_W$ on V, W such that if $\|x\|_V \le \epsilon$ and $\|y\|_W \le \epsilon$, then $\|\beta(x,y)\| \le 1$. Rescaling them gives $\|\cdot\|_V$, $\|\cdot\|_W$ satisfying the desired inequality. □

Lemma 9.1.2. *If V, W are Fréchet (or in fact if one is Fréchet and the other metrizable) then*

(1) $V \otimes_\pi W \cong V \otimes_\iota W$.
(2) if $\beta : V \times W \to U$ is bilinear, β being separately continuous implies that it is continuous. (Note that this implies (1).)

Proof. For (2), it suffices to prove that if $N \subset U$ is an open lattice, there are $\Lambda \subset V$ and $M \subset W$ open lattices with $\beta(\Lambda, M) \subset N$. Suppose otherwise. Then since V and W are metrizable, we can find $(x_n, y_n) \in V \times W$ with $(x_n, y_n) \to 0$ but $\beta(x_n, y_n) \notin N$. Note that the map

$$\begin{aligned} W &\to \mathscr{L}_s(V, U) \\ y &\mapsto \beta(-, y) \end{aligned}$$

is continuous: for $x \in V$ and $N' \subset U$ an open lattice, the preimage of $\Lambda(\{x\}, N')$ under this map is

$$\{y \in W \mid \beta(x,y) \in N'\},$$

which is open since β is separately continuous. On the other hand $\{y_n\} \subset W$ is bounded, so $\{\beta(-, y_n)\} \subset \mathscr{L}_s(V, U)$ is bounded. Therefore by Banach-Steinhaus, $\{\beta(-, y_n)\}$ is equicontinuous. Therefore there is an open lattice $\Lambda \subset V$ such that $\beta(\Lambda, y_n) \subset N$ for all n. But $x_n \in \Lambda$ for $n \gg 0$, so $\beta(x_n, y_n) \in N$ for $n \gg 0$, contradiction. □

Definition 9.1.3. Define $V \widehat{\otimes}_\pi W$ to be the completion of $V \otimes_\pi W$.

Lemma 9.1.4. *If V, W are Fréchet/Banach, then $V \widehat{\otimes}_\pi W$ is also Fréchet/Banach.*

Lemma 9.1.5. *Let V, W be Hausdorff. Note that the map*

$$V_b^\vee \otimes W \to \mathscr{L}_b(V,W)$$
$$f \otimes y \mapsto (x \mapsto f(x)y)$$

factors through $\theta : V_b^\vee \otimes_\pi W \to \mathscr{C}\mathscr{C}(V,W)$. Then θ is a homeomorphism onto its image, which is dense in $\mathscr{C}\mathscr{C}(V,W)$.

Proof. Schneider Lemmas 18.1 and 18.9. □

Corollary 9.1.6. *If V is Hausdorff and bornological and W is complete, we have an isomorphism $V_b^\vee \widehat{\otimes}_\pi W \xrightarrow{\sim} \mathscr{C}\mathscr{C}(V,W)$.*

Lemma 9.1.7. *If V is Hausdorff and bornological and W is complete, and either any closed bounded $\mathscr{O}$-submodule of V is compact, or any closed bounded $\mathscr{O}$-submodule of W is compact, then we have an isomorphism $V_b^\vee \widehat{\otimes}_\pi W \xrightarrow{\sim} \mathscr{L}_b(V,W)$.*

Proof. Schneider Corollary 18.8. □

9.2. **Nuclear spaces.** Let $\Lambda \subset V$ be an open lattice, $\|\cdot\|_\Lambda$ the corresponding seminorm on V. Then we can take the completion

$$V \to V_\Lambda^\wedge = V_{\|\cdot\|_\Lambda}^\wedge,$$

giving a Banach space. Furthermore, if M_0 is the open unit ball in some other Banach space W, and $f : V \to W$ is continuous linear, then $f^{-1}M_0$ is an open lattice in V, and f is also continuous in the topology on V generated by $f^{-1}M_0$. So any such f factors through a completion $V \to V_{f^{-1}M_0}^\wedge \to W$ of this form.

We call $V_\Lambda^\wedge = V_{\|\cdot\|_\Lambda}^\wedge$ "a Banach completion of V". If $\Lambda' \subset \Lambda$, then we can factor $V \to V_\Lambda^\wedge$ as $V \to V_{\Lambda'}^\wedge \to V_\Lambda^\wedge$.

Definition 9.2.1. We call V *nuclear* if for all open lattices $\Lambda \subset V$, there is another open lattice $\Lambda' \subset \Lambda$ such that $V_{\Lambda'}^\wedge \to V_\Lambda^\wedge$ is compact.

Lemma 9.2.2. *V is nuclear if and only if for all $f : V \to W$ where f is continuous linear and W is Banach, f is compact.*

Proof. Forward direction: assume V is nuclear. Let $M_0 \subset W$ be a bounded open lattice (i.e. a unit ball). Let $\Lambda \subset f^{-1}M_0$ be an open lattice such that $V_\Lambda^\wedge \to V_{f^{-1}M_0}^\wedge$ is compact. Then $f : V \to W$ factors as $V \to V_\Lambda^\wedge \to V_{f^{-1}M_0}^\wedge \to W$, hence is compact.

Backward direction: let $\Lambda \subset V$ be an open lattice. Then $g : V \to V_\Lambda^\wedge$ is compact, so there is an open lattice $\Lambda' \subset V$ such that $\overline{g\Lambda'}$ is compact. By intersecting with Λ, we can assume that $\Lambda' \subset \Lambda$. Then $V \to V_\Lambda^\wedge$ factors through $V_{\Lambda'}^\wedge \to V_\Lambda^\wedge$, which is compact since it takes $\overline{\Lambda'}$ to $\overline{g\Lambda'}$. □

Lemma 9.2.3. *(1) V_s (i.e. V with the weak-$*$ topology) is nuclear.*

(2) Subspaces of nuclear spaces (with the subspace topology) are nuclear.
(3) Quotients of nuclear spaces by closed nuclear subspaces are nuclear.
(4) Products of nuclear spaces are nuclear.
(5) Countable direct sums of nuclear spaces are nuclear.
(6) If V and W are nuclear, $V \otimes_\pi W$ is nuclear.
(7) If we have a chain $V_1 \hookrightarrow^{\iota_1} V_2 \hookrightarrow^{\iota_2} V_3 \hookrightarrow \cdots$ where each ι_j is a homeomorphism onto its image and V_i is nuclear for all i, then $\varinjlim V_i$ is nuclear.

Remark 9.2.4. An infinite-dimensional Banach space is never nuclear, since a Banach space V is nuclear if and only if $\mathrm{id} : V \to V$ is compact.

Lemma 9.2.5. *(1) If we have a chain $\cdots \to^{\pi_3} V_3 \to^{\pi_2} V_2 \to^{\pi_1} V_1$ where V_i is Hausdorff and π_i is compact for all i, $\varprojlim V_i$ is nuclear.*

(2) If we have a chain $V_i \hookrightarrow^{\iota_1} V_2 \hookrightarrow^{\iota_2} V_3 \hookrightarrow^{\iota_3} \cdots$ with ι_j compact for all j, then $\varinjlim_j V_j$ is nuclear.

We will not prove these; they are all in Schneider.

9.3. LCTVSs of compact type.

Definition 9.3.1. A LCTVS V is said to be "of compact type" if it is a limit of a chain of the form

$$V_1 \hookrightarrow^{\iota_1} V_2 \hookrightarrow^{\iota_2} V_3 \hookrightarrow \cdots$$

with V_n Banach and ι_n compact for all n.

Lemma 9.3.2. *(1) If V is of compact type, then V is complete, bornological, and barrelled.*

(2) If V is of compact type, then $V_b^\vee$ is nuclear and Fréchet and $V \xrightarrow{\sim} (V_b^\vee)_b^\vee$.

(3) If W is nuclear and Fréchet, then $W_b^\vee$ is compact type and $W \xrightarrow{\sim} (W_b^\vee)_b^\vee$.

(4) There is an anti-equivalence of categories

$$\{\textit{nuclear Fréchet spaces}\} \leftrightarrow \{\textit{LCTVSs of compact type}\}$$
$$W \mapsto W_b^\vee$$
$$V_b^\vee \leftarrow\!\shortmid V.$$

Lemma 9.3.3. *(1) If we have $V_1 \hookrightarrow^{\iota_1} V_2 \hookrightarrow^{\iota_2} V_3 \hookrightarrow \cdots$ with each V_n a Hausdorff LCTVS and ι_n compact, then $\varinjlim V_n$ is of compact type.*

(2) A countable direct sum of LCTVSs of compact type is of compact type.

(3) If V is compact type and $W \subset V$ is closed, then

(a) W and V/W are of compact type.

(b) $(V/W)_b^\vee \to V_b^\vee$ is a homeomorphism onto a closed subspace.

(c) $W_b^\vee \cong V_b^\vee/(V/W)_b^\vee$.

So that we don't go too long without any proofs, we will prove a technical lemma that is important for proving many of the above statements.

Lemma 9.3.4. *Let V be a LCTVS of compact type, so that it is a limit $V = \varinjlim V_n$ as above.*

(1) An $\mathscr{O}$-submodule $\Lambda \subset V$ is an open lattice if and only there are bounded open lattices $\Lambda_n \subset V_n$ for all n such that Λ contains

$$\overline{\Lambda}_1 + \overline{\Lambda}_2 + \overline{\Lambda}_3 + \cdots ,$$

where by $\overline{\Lambda}_i$ we mean the closure of Λ_i in V, or equivalently in V_{i+1}.

(2) $B \subset V$ is bounded if and only if $B \subset V_n$ is bounded for some n. In particular, closed bounded subsets in V are compact.

Proof. (1) Forward direction: if Λ is an open lattice then $\Lambda \cap V_n$ is an open lattice, which contains a bounded open lattice Λ_n, which satisfies the desired condition.

Backward direction: $\Lambda \subset V$ is open if and only if $\Lambda \cap V_n$ is open for all n. But since Λ contains $\bigoplus \overline{\Lambda}_i$, $\Lambda \cap V_n$ contains Λ_n, and is indeed open.

(2) Backward direction: easy.

Forward direction: suppose the claim fails for some bounded $B \subset V$. We will recursively find $x_1, x_2, \ldots \in B$ such that $x_i \in V_{n_i}$ for $n_1 < n_2 < \cdots$, and $\Lambda_{n_i} \subset V_{n_i}$ a bounded open lattice, such that

$$x_1, \pi x_2, \pi^2 x_3, \ldots, \pi^{m-1} x_m \notin \overline{\Lambda}_{n_1} + \cdots + \overline{\Lambda}_{n_m}.$$

Then since B is bounded, there should be n such that $\pi^n x_i \in \overline{\Lambda}_{n_1} + \overline{\Lambda}_{n_2} + \cdots$ for all i. But $\pi^n x_{n+1} \notin \overline{\Lambda}_{n_1} + \overline{\Lambda}_{n_2} + \cdots + \overline{\Lambda}_{n_m}$ for any m, so we would have a contradiction.

Suppose $x_1, \ldots, x_{m-1}$ have been chosen. B is not contained in $\pi^{1-m}(\overline{\Lambda}_{n_1} + \cdots + \overline{\Lambda}_{n_{m-1}})$, since otherwise it would be bounded in $V_{n_{m-1}}$. Let $x_m \in B \setminus \pi^{1-m}(\overline{\Lambda}_{n_1} + \cdots + \overline{\Lambda}_{n_{m-1}})$. We have $x_m \in V_{n_m}$ for some $n_m > n_{m-1}$, and $\pi^{m-1} x_m \notin \overline{\Lambda}_{n_1} + \cdots + \overline{\Lambda}_{n_{m-1}}$. There is an open lattice $M \subset V_{n_m+1}$ such that

$$x_1 + M, \ldots, \pi^{m-2} x_{m-1} + M, \pi^{m-1} x_m + M$$

do not meet $\overline{\Lambda}_{n_1} + \cdots + \overline{\Lambda}_{n_{m-1}}$. But we can find a bounded open lattice $\Lambda_{n_m} \subset V_{n_m}$ with $\Lambda_{n_m} \subset M$ and in fact $\overline{\Lambda}_{n_m} \subset M$, so indeed

$$x_1, \pi x_2, \pi^2 x_3, \ldots, \pi^{m-1} x_m \notin \overline{\Lambda}_{n_1} + \cdots + \overline{\Lambda}_{n_m}.$$

□

9.4. **Examples.** If $r_1 > r_2$, the restriction map $\mathscr{O}(D(r_1)) \to \mathscr{O}(D(r_2))$ on the space of analytic functions on the closed disc of radius r_1 is compact. This is because

$$\operatorname{im} \Lambda(\|\cdot\|_{r_1}) = \left\{ \sum_n c_n T^n \mid |c_n| \leq r_1^{-n} \forall n \right\}$$

is in bijection with $\prod_{n \geq 0} \mathscr{O}$ via

$$\begin{aligned} \theta : \prod_{n \geq 0} \mathscr{O} &\to \left\{ \sum_n c_n T^n \mid |c_n| \leq r_1^{-n} \forall n \right\} \\ (a_n) &\mapsto \sum_n \pi^{m(n)} a_n T^n \end{aligned}$$

where $m(n) \in \mathbb{Z}$ is minimal such that $|\pi|^{m(n)} \leq r_1^{-n}$, that is,

$$m(n) \geq -\frac{n \log r_1}{\log |\pi|}.$$

We can check that θ is continuous as follows. The open set

$$\left\{ \sum_n c_n T^n \mid |c_n| \leq r_1^{-n} \forall n, |c_n| \leq \epsilon r_2^{-n} \forall n \right\} \subset \mathscr{O}(D(r_2))$$

can be rewritten as

$$\left\{ \sum_n c_n T^n \mid |c_n| \leq r_1^{-n} \forall n, |c_n| \leq \epsilon r_2^{-n} \forall n = 1, \ldots, N \right\}$$

for some N depending on ϵ (since $r_1 > r_2$, we have $r_1^{-n} < \epsilon r_2^{-n}$ for n sufficiently large). The inverse image of the latter under θ is

$$\left\{(a_n) \in \prod_{n\geq 0} \mathscr{O} \mid |a_n| < \beta_n, n = 1, \dots, N\right\}$$

for some $\beta_n = \beta_n(\epsilon)$, which is open. Since $\prod_{n\geq 0} \mathscr{O}$ is compact, θ is actually a homeomorphism, and $\operatorname{im}\theta$ is compact.

Definition 9.4.1. (1) $\mathscr{O}^+(D(r)) = \varinjlim_{r_1>r} \mathscr{O}(D(r_1))$, called the space of overconvergent analytic functions, is a LCTVS of compact type.
(2) $\mathscr{O}(D^\circ(r)) = \varprojlim_{r_1<r} \mathscr{O}(D(r_1))$, the space of analytic functions on an open disc, is a nuclear Fréchet space.
(3) $\mathscr{O}(\mathbb{A}^1) = \varprojlim_r \mathscr{O}(D(r))$, the space of analytic functions on the affine line, is a nuclear Fréchet space.

10. 2/6/20: Analytic functions and their roots

Last time, we defined spaces of compact type: limits of chains $V_1 \hookrightarrow V_2 \hookrightarrow V_3 \hookrightarrow \cdots$ where the V_is are Banach and the inclusions are compact. We also defined nuclear Fréchet spaces: those such that for any open lattice $\Lambda \subset V$, there is a smaller open lattice $\Lambda' \subset \Lambda \subset V$ such that $V^\wedge_{\|\cdot\|_{\Lambda'}} \to V^\wedge_{\|\cdot\|_\Lambda}$ is compact. We found that these are anti-equivalent notions via taking strong duals.

We also defined

$$\mathscr{O}^+(D(r)) = \varinjlim_{r_1>r} \mathscr{O}(D(r_1)),$$

the space of overconvergent analytic functions on the disc of radius r, and saw that this was of compact type. (These are sometimes useful in rigid geometry, for instance in de Rham cohomology, where overconvergent forms somehow work better than convergent forms.) And we defined

$$\mathscr{O}(D^\circ(r)) = \varprojlim_{r_1<r} \mathscr{O}(D(r_1)),$$

the space of analytic functions on an open disc, and

$$\mathscr{O}(\mathbb{A}^1) = \varprojlim_{r_1} \mathscr{O}(D(r_1)),$$

the space of analytic functions on the affine line; the latter two are nuclear Fréchet spaces.

10.1. Dividing analytic functions.

We call $f = \sum_m f_m T^m \in \mathscr{O}(D(r))$ *n-distinguished* if $\|f\|_r = |f_n| r^n$ and $|f_m| r^m < |f_n| r^n$ for all $m > n$. If $f \neq 0$ then it is n-distinguished for some n, since the terms go to zero.

Lemma 10.1.1 (Weierstrass division). *If $f \in \mathscr{O}(D(1))$ is n-distinguished and $g \in \mathscr{O}(D(1))$, then we can write $g = qf + r$ for a unique $q \in \mathscr{O}(D(1))$ and $r \in L[T]$ with $\deg(r) < n$; moreover*

$$\|g\|_1 = \max(\|q\|_1 \|f\|_1, \|r\|_1).$$

Proof. First suppose that indeed $g = qf + r$ with $q \in \mathscr{O}(D(1))$ and $r \in L[T]$ with $\deg(r) < n$; we will check that we must then have $\|g\|_1 = \max(\|q\|_1\|f\|_1, \|r\|_1)$. We may assume WLOG that $\|f\|_1 = 1$ and $\max(\|q\|_1\|f\|_1, \|r\|_1) = 1$. Then $q, f, r \in \mathscr{O}[\![T]\!]$, and we can reduce them mod π to get $\overline{q}, \overline{f}, \overline{r} \in \mathbb{F}[T]$. We certainly have $\|g\|_1 \leq 1$. If it were the case that $\|g\|_1 < 1$, then we would have $\overline{g} = 0 = \overline{q}\overline{f} + \overline{r}$. But since $\max(\|q\|_1\|f\|_1, \|r\|_1) = 1$, we know that one of $\overline{r}$ or $\overline{q}$ is nonzero, that $\overline{r}$ has degree $< n$, and that $\overline{f}$ is nonzero and has degree n, so it is impossible for $\overline{q}\overline{f} + \overline{r}$ to cancel out to 0.

Now we will find the desired q and r. First, we find $q \in \mathscr{O}(D(1))$ and $r \in L[T]$ with $\deg(r) < n$ such that

$$\|g - (qf + r)\|_1 \leq |\pi|\|g\|_1.$$

To do so, again assume WLOG that $\|f\|_1 = 1$ and $\|g\|_1 = 1$. Then we have $\deg \overline{f} = n$, and by usual polynomial division we can write $\overline{g} = \overline{q}\overline{f} + \overline{r} \in \mathbb{F}[T]$ with $\deg \overline{r} < n$. Choose q, r lifting $\overline{q}, \overline{r}$ such that $r \in \mathscr{O}[T]$ with $\deg(r) < n$. These satisfy the desired condition.

Next we show that for all $N \in \mathbb{Z}_{>0}$, we can find $q \in \mathscr{O}(D(1))$ and $r \in L[T]$ with $\deg(r) < n$ such that

$$\|g - (qf + r)\|_1 \leq |\pi|^N\|g\|_1.$$

We do this by induction on N. Suppose we have found q, r satisfying this condition for $N-1$, so we have

$$\|g - (qf + r)\|_1 \leq |\pi|^{N-1}\|g\|_1.$$

Then applying our base case to $g - (qf + r)$ in place of g, we can find q', r' such that

$$\|g - (qf + r) - (q'f + r')\|_1 \leq |\pi|\|g - (qf + r)\|_1 \leq |\pi|^N\|g\|,$$

from which we see that $q + q', r + r'$ satisfy the condition for N.

Now the map

$$\begin{aligned} \mathscr{O}(D(1)) \times L[T]^{\deg < n} &\to \mathscr{O}(D(1)) \\ (q, r) &\mapsto qf + r, \end{aligned}$$

is an isometry (since $\|qf + r\|_1 = \max(\|q\|_1, \|r\|_1)$), hence injective with closed image, since the space on the left hand side is complete. But we also just proved that it has dense image. So it is an isomorphism. $\square$

10.2. **Factoring analytic functions.**

Lemma 10.2.1 (Weierstrass preparation)**.** *Suppose $r \in p^{\mathbb{Q}}$ and $f \in \mathscr{O}(D(r))$ is n- distinguished. Then $f = gh$ for unique g, h such that $g \in L[T]$ is n-distinguished and satisfies $\|g\|_r = 1$, and $h \in \mathscr{O}(D(r))^\times$.*

Proof. Since the lemma asserts that the factorization is unique, it suffices to prove it after replacing L by a finite extension (then we can use Galois descent). So we may assume WLOG that $r \in |L|$, and then use the change of variables $T \mapsto aT$ for some $a \in L^\times$ with $|a| = r$ to reduce to the case $r = 1$. Then as before, we can assume WLOG that $\|f\|_1 = 1$.

We have $f = gh$ if and only if

$$T^n = h^{-1}f + (T^n - g),$$

where $T^n - g$ is degree $< n$. Then uniqueness follows from uniqueness in Lemma 10.1.1. For existence, use Lemma 10.1.1 to write $T^n = qf + r$ where $q \in \mathscr{O}(D(1))$ and $r \in L[T]$ with $\deg(r) < n$, and $\max(\|q\|_1, \|r\|_1) = 1$. It suffices to prove that $q \in \mathscr{O}(D(1))^\times$, so that we can set $h = q^{-1}$ and $g = T^n - r$. (Note that since $\|r\|_1 \le 1$, this means that g is n-distinguished and $\|g\|_1 = 1$.)

Reducing mod π, we get $T^n = \overline{q}\overline{f} + \overline{r}$ in $\mathbb{F}[T]$. Since $\overline{f}$ is degree n and $\overline{r}$ is degree $< n$, we conclude that $\deg \overline{q} = 0$ but $\overline{q} \neq 0$, which is to say that $q = a - \pi q'$ with $a \in \mathscr{O}^\times$, $q' \in \mathscr{O}(D(1))$, and $\|q'\|_1 \le 1$. Then

$$q^{-1} = a^{-1}\left(a + \frac{\pi q'}{a} + \left(\frac{\pi q'}{a}\right)^2 + \cdots\right)$$

converges in $\mathscr{O}(D(1))$, as desired. □

Corollary 10.2.2. *If $0 \neq f \in \mathscr{O}(D(r))$, then f has only finitely many zeros in $D(r)$. In fact if f is n-distinguished it has at most n zeros in $D(r)$.*

Corollary 10.2.3. *If $r \ge 1$ then $\mathscr{O}(D(r)) \to C(\mathbb{Z}_p, L)$ is injective and continuous.*

Corollary 10.2.4. *If $r \in p^{\mathbb{Q}}$, $\mathscr{O}(D(r))$ is noetherian, in fact a PID.*

Proof. If $I = (f_i)$ is an ideal of $\mathscr{O}(D(r))$, then I can be rewritten as (g_i) where $f_i = g_i h_i$ with $h_i \in \mathscr{O}(D(r))^\times$ and $g_i \in L[T]$. But then $(g_i) \subset L[T]$ is generated by a single polynomial, so I is as well. □

Corollary 10.2.5. *(1) If $r \in p^{\mathbb{Q}}$, $f \in \mathscr{O}(D(r))$ is n-distinguished, and we write $f = gh$ with $h \in \mathscr{O}(D(r))^\times$ and $g \in L[T]$ monic of degree n, then*

$$\mathscr{O}(D(r))/(f) = \mathscr{O}(D(r))/(g) = L[T]/(g).$$

(2) The maximal ideals of $\mathscr{O}(D(r))$ are given by (g) where $g \in L[T]$ is irreducible and all the roots of g lie in $D(r)$. For such g, we have isomorphisms

$$\begin{aligned} L' := L[T]/(g) &\xrightarrow{\sim} \mathscr{O}(D(r))/(g) \\ L[T]/(g^n) &\xrightarrow{\sim} \mathscr{O}(D(r))/(g^n) \\ L'[\![x]\!] \cong L[T]^\wedge_{(g)} &\xrightarrow{\sim} \mathscr{O}(D(r))^\wedge_{(g)}. \end{aligned}$$

Proof. For example, to see that $\mathscr{O}(D(r))/(g)$ isomorphic to $L[T]/(g)$, we take the natural map $L[T]/(g) \to \mathscr{O}(D(r))/(g)$, and observe that it is an isomorphism because any element of either side has a unique representative in $L[T]^{\deg < n}$. □

Lemma 10.2.6 (Hadamard factorization)**.** *Suppose $0 \neq f \in \mathscr{O}(\mathbb{A}^1)$. Then*

$$f = aT^m \prod_{i=1}^{\infty} g_i$$

where $a \in L^\times$, $g_i \in L[T]$ is irreducible, $g_i(0) = 1$, and $g_i \to 1$ in $\mathscr{O}(\mathbb{A}^1)$ (equivalently, if $r_i = |\text{root of } g_i|$, then $r_i \to \infty$ as $i \to \infty$).

Proof. Assume WLOG that $f(0) = 1$. We will recursively define f_n, g_n as follows: $f_1 = f$, and $f_n = f_{n+1} g_n$ is the factorization from Lemma 10.2.1 where f_n is viewed in $\mathscr{O}(D(p^n))$, so that $g_n \in L[T]$ and $f_{n+1} \in \mathscr{O}(D(p^n))^\times$, except rescaled so that $f_{n+1}(0) = 1$ and $g_n(0) = 1$. Note that f_{n+1} is 0-distinguished (because everything in $\mathscr{O}(D(p^n))^\times$ is 0-distinguished), and all roots of g_n lie in $D(p^n) \setminus D(p^{n-1})$.

For the recursion to proceed, we need to show that $f_{n+1} \in \mathscr{O}(\mathbb{A}^1)$, or equivalently that $f_{n+1} \in \mathscr{O}(D(R))$ for any $R \ge p^n$. Assuming inductively that $f_n \in \mathscr{O}(D(R))$, we have by

Lemma 10.1.1 that $f_n = qg_n + r$ where $q \in \mathscr{O}(D(R))$ and $r \in L[T]$ satisfies $\deg(r) < \deg(g_n)$. But we have $f_{n+1}g_n = f_n = qg_n + r$, implying $(f_{n+1} - q)g_n = r$; since $\deg(r) < \deg(g_n)$, this means $r = 0$ and $f_{n+1} = q \in \mathscr{O}(D(R))$ as desired.

Now since f_n is 0-distinguished in $\mathscr{O}(D(p^{n-1}))$, we have for all r that

$$|m\text{th coefficient of } f_n|p^{(n-1)m} \leq 1$$

$$|m\text{th coefficient of } f_n|r^m \leq \left(\frac{r}{p^{n-1}}\right)^m.$$

Then we have $\|f_n - 1\|_r \leq rp^{1-n}$ if $r \leq p^{n-1}$, implying that $f_n \to 1$ in $\mathscr{O}(\mathbb{A}^1)$, hence $\prod_{i=1}^{n-1} g_i \to f$. Also, since $g_i(0) = 1$, the mth coefficient of g_i can be written a sum of m-fold products of inverses of roots of g_i; since the roots of g_i all have size at least p^{i-1}, this means that for all r we have

$$\|g_i - 1\|_r \leq \sup_{m \geq 1} \left(rp^{1-i}\right)^m = rp^{1-i},$$

which goes to 0 as $i \to \infty$. □

The Newton polygon of $f = \sum_m a_m T^m \in \mathscr{O}(\mathbb{A}^1)$, $f(0) = 1$, is drawn on the xy-plane as follows (see Figure 1). First we plot $(m, -\log|a_m|)$ for all m. In $\mathscr{O}(D(r))$, f is i-distinguished if i is the maximal index maximizing

$$|a_i|r^i = p^{\log|a_i| + i\log r}.$$

So if $\log r \leq \frac{-\log|a_m|}{m}$ for all m, then f is 0-distinguished in $\mathscr{O}(D(r))$; but when $\log r$ passes $\min \frac{-\log|a_m|}{m}$, then for i attaining this minimum we have

$$\log|a_i| + i\log r \geq \log|a_i| + i \cdot \frac{-\log|a_i|}{i} = 0,$$

and for the greatest such i and the corresponding $r = r_1$, f is i-distinguished in $\mathscr{O}(D(r_1))$. We conclude that f has exactly i zeros in $D(r_1)$, which must be of valuation exactly $\frac{\log|a_i|}{i}$, and none of lower valuation. We can continue; for a different index j, we have

$$\log|a_j| + j\log r \geq \log|a_i| + i\log r$$

when $\log r$ passes $\frac{-\log|a_j| + \log|a_i|}{j-i}$, so for the greatest $j > i$ minimizing $\frac{-\log|a_j|+\log|a_i|}{j-i}$ and the corresponding choice of $r = r_2$, f is j-distinguished in $\mathscr{O}(D(r_2))$, and thus has $j - i$ additional zeros of valuation exactly $\frac{\log|a_j| - \log|a_i|}{j-i}$. And so on. In conclusion, if we take the lower convex hull of the points in our plot, for every line segment of slope α and horizontal length n in the convex hull, f has n zeros of valuation $-\alpha$. This lower convex hull is the Newton polygon of f. See Figure 1 for an example where $i = 4$ and $j = 7$.

10.3. **Locally analytic functions.** Let

$$\mathrm{LA}_h = \{f : \mathbb{Z}_p \mid L \mid f \text{ continuous}, \forall a \in \mathbb{Z}/p^h\mathbb{Z}, f(a+t) \in \mathscr{O}(D(p^{-h}))\}.$$

This is the space of locally analytic functions of radius of analyticity p^{-h}. It can also be written as

$$\bigoplus_{a \in \mathbb{Z}/p^h\mathbb{Z}} \mathscr{O}(D(p^{-h})).$$

On it we have the norm

$$\|f\|_h = \max_{a \in \mathbb{Z}/p^h\mathbb{Z}} \|f(a+t)\|_{p^{-h}} = \max_{a \in \mathbb{Z}/p^h\mathbb{Z}} \max_{t \in D(p^{-h})} |f(a+t)|.$$

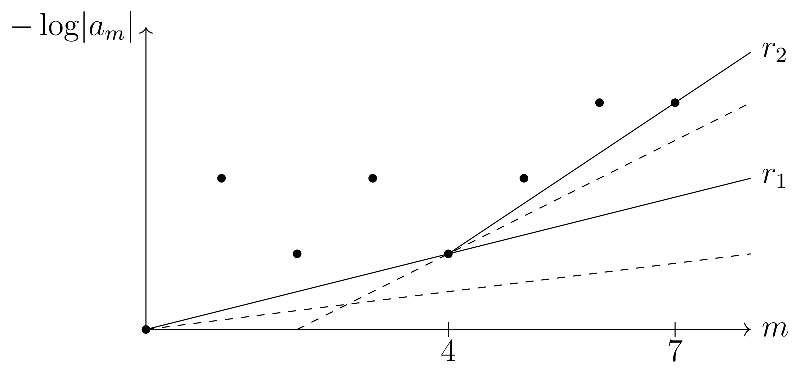

FIGURE 1. Newton polygon of a power series

For $h' > h$, the natural inclusion $\mathrm{LA}_h(\mathbb{Z}_p, L) \to \mathrm{LA}_{h'}(\mathbb{Z}_p, L)$ is compact, and so

$$\mathrm{LA}(\mathbb{Z}_p, L) = \varinjlim_h \mathrm{LA}_h(\mathbb{Z}_p, L)$$

is a LCTVS of compact type. This is the space of locally analytic functions on $\mathbb{Z}_p$. In summary, we have the chain of inclusions

$$\mathscr{O}(D(1)) = \mathrm{LA}_0(\mathbb{Z}_p, L) \hookrightarrow \mathrm{LA}_h(\mathbb{Z}_p, L) \hookrightarrow \mathrm{LA}(\mathbb{Z}_p, L) \hookrightarrow C(\mathbb{Z}_p, L).$$

Next time, we'll see that $\mathrm{LA}(\mathbb{Z}_p, L)$ is dual to $\mathscr{O}(D^\circ(r))$.

11. 2/11/20: LOCALLY CONVEX TOPOLOGICAL L-ALGEBRAS

11.1. **Duals of function spaces.** We have been discussing various spaces of functions on $\mathbf{Z}_p$:

- $\mathscr{O}(D(1))$ consists of the analytic functions on the closed unit disc: these are given globally by a power series with radius of convergence (at least) 1. This is a Banach space.
- For $h \in \mathbf{Z}_{>0}$, we have the space $\mathrm{LA}_h(\mathbf{Z}_p, L)$ of *locally analytic* functions. This is the space of continuous functions $f \colon \mathbf{Z}_p \to L$ such that for any $a \in \mathbf{Z}/p^h\mathbf{Z}$, $f(a+t) \in \mathscr{O}(D(p^{-h}))$. This is a Banach space, isomorphic to the direct sum over $a \in \mathbf{Z}/p^h\mathbf{Z}$ of $\mathbf{O}(D(p^{-h}))$.
- The space of *locally analytic* functions on $\mathbf{Z}_p$ is $\mathrm{LA}(\mathbf{Z}_p, L) := \varinjlim_h \mathrm{LA}_h(\mathbf{Z}_p, L)$. It is a LCTVS of compact type.
- The space of continuous functions $f \colon \mathbf{Z}_p \to L$ with the supremum norm is a Banach space $C(\mathbf{Z}_p, L)$.

We have inclusions of each space in the space below it in the above list.

Lemma 11.1.1. *For any h, the space $\mathrm{LA}_h(\mathbf{Z}_p, L)$ has a Banach basis*

$$\left\{ \left\lfloor \frac{n}{p^h} \right\rfloor ! \binom{x}{n} \mid n = 0, 1, 2, \ldots \right\}.$$

Proof. See e.g. [4, §1.7.2]. (It's very computational.) $\square$

Corollary 11.1.2. *The strong dual of* $\mathrm{LA}_h(\mathbf{Z}_p, L)$ *is isomorphic to the space*

$$\left\{ \sum_{n=0}^{\infty} c_n T^n \colon |c_n| \left| \left\lfloor \frac{n}{p^h} \right\rfloor ! \right| \text{ is bounded } \right\}$$

and this isomorphism restricts to the isomorphism of the strong dual of $C(\mathbf{Z}_p, L)$ *with the space*

$$\left\{ \sum_{n=0}^{\infty} c_n T^n \colon |c_n| \text{ is bounded } \right\}$$

defined by $\left(\sum c_n T^n\right) \binom{x}{n} = c_n$.

Note that the above space $C(\mathbf{Z}_p, L)^\vee$ of power series with bounded coefficients may be identified with the space of bounded analytic functions on the *open* unit disc $D^0(1)$.

Corollary 11.1.3. *The strong dual of* $\mathrm{LA}(\mathbf{Z}_p, L)$ *may be identified with the space*

$$\mathscr{O}(D^0(1)) = \left\{ \sum c_n T^n \colon |c_n| r^n \to 0 \text{ for all } r < 1 \right\}$$

in a manner compatible with the above identifications. Likewise, the strong dual of $\mathscr{O}(D(1))$ *may be compatibly identified with*

$$\left\{ \sum_n c_n T^n \colon |c_n||n!| \text{ is bounded } \right\} \hookrightarrow \mathscr{O}(D(|p|^{1/(p-1)})).$$

Proof. We can prove all of the above statements by studying Banach bases and using the formula

$$\mathrm{val}_p(n!) = \sum_{h \geq 0} \left\lfloor \frac{n}{p^h} \right\rfloor \sim \frac{n}{p} \sum_{h \geq 0} \frac{1}{p^h} = \frac{n}{p-1}$$

which gives

$$\left| \left\lfloor \frac{n}{p^h} \right\rfloor ! \right| \sim \left(|p|^{\frac{1}{p^h(p-1)}} \right)^n,$$

so "$|c_n| r^n \to 0$ as $n \to \infty$ for all $r < 1$" is equivalent to "$|c_n| \left| \left\lfloor \frac{n}{p^h} \right\rfloor ! \right|$ is bounded for all $h \geq 1$". ∎

11.2. Locally convex topological L-algebras.

Definition 11.2.1.
- By a *locally convex topological L-algebra*, abbreviated 'LCTA', we mean a LCTVS A together with the structure of an L-algebra (associative with identity) such that the multiplication map $A \times A \to A$ is continuous.
- A *morphism* of LCTAs is a morphism of L-algebras which is continuous.
- A seminorm $\|\cdot\|$ on a LCTA A is *algebraic* if $\|ab\| \leq C\|a\|\|b\|$ for all $a, b \in A$.

Lemma 11.2.2. *If A is an LCTA whose underlying LCTVS is normable with topology defined by a norm $\|\cdot\|$, then $\|\cdot\|$ is algebraic.*

Proof. By continuity, there exists an $\epsilon > 0$ such that if $\|a\| \leq \epsilon$, $\|b\| \leq \epsilon$, then $\|ab\| \leq 1$. This implies

$$\|ab\| \leq \frac{\|a\|\|b\|}{\epsilon^2 |\pi|^2}$$

for all $a, b \in A$. ∎

Remark 11.2.3. If $\|\cdot\|$ is an algebraic seminorm and $\|ab\| \leq C\|a\|\|b\|$, then $\|\cdot\|$ is equivalent to the seminorm $\|\cdot\|' = C\|\cdot\|$ which satisfies $\|ab\|' \leq \|a\|'\|b\|'$.

Example 11.2.4. (1) If Ω is a compact topological space, then $C(\Omega, L)$ is a commutative Banach L-algebra via the pointwise multiplication map $(f \cdot g)(\omega) = f(\omega)g(\omega)$. Indeed, we have:

$$\|f \cdot g\|_\infty = \sup_{\omega \in \Omega} |f(\omega)||g(\omega)| \leq \|f\|_\infty \|g\|_\infty.$$

(2) $\mathscr{O}(D(r))$ is a commutative Banach L-algebra via multiplication of power series

$$\sum_n f_n T^n \cdot \sum_m g_m T^m = \sum_s \sum_{n+m=s} f_n g_m T^s.$$

Indeed, we have:

$$\sup_s \left| \sum_{n+m=s} f_n g_m \right| r^s \leq \sup_s \sup_{n+m=s} |f_n| r^n |g_m| r^m \leq \| \sum_n f_n T^n \|_r \| \sum_m g_m T^m \|_r.$$

(3) If V is a Banach space over L with norm $\|\cdot\|$, then the Banach space $\mathscr{L}_b(V, V)$ is a Banach L-algebra via composition of linear operators. Recall that the norm on $\mathscr{L}_b(V, V)$ is the operator norm $\|f\| = \sup_{0 \neq x \in V} \frac{\|f(x)\|}{\|x\|}$. Indeed, we have

$$\begin{aligned} \|f \circ g\| &= \sup_{0 \neq x \in V} \frac{\|f(g(x))\|}{\|x\|} \\ &= \sup_{\substack{x \in V \\ g(x) \neq 0}} \frac{\|f(g(x))\|}{\|x\|} \\ &= \sup_{\substack{x \in V \\ g(x) \neq 0}} \frac{\|f(g(x))\|}{\|g(x)\|} \frac{\|g(x)\|}{\|x\|}. \end{aligned}$$

11.3. **Topological modules over LCTAs.** Now that we have algebras, we can also discuss modules.

Definition 11.3.1. Suppose A is a LCTA. By a *locally convex topological A-module* (LCTM), we mean a LCTVS M together with a structure of a (left) A-module such that the scalar multiplication map $A \times M \to M$ is continuous.

Remark 11.3.2. If A and M are normable with norms $\|\cdot\|_A, \|\cdot\|_M$, then the continuity of scalar multiplication $A \times M \to M$ is equivalent to requiring that there exist $C \in \mathbf{R}_{>0}$ such that $\|am\|_M \leq C\|a\|_A\|m\|_M$ for all $a \in A, m \in M$.

In the case of finitely generated modules over noetherian LCTAs, this structure is automatic.

Lemma 11.3.3. *Suppose that A is a (left) noetherian Banach algebra. Then:*

(i) If M is a finitely generated Banach A-module, then any (abstract) A-submodule of M is closed.

(ii) If M is a finitely generated Banach A-module and N is any Banach A-module, then any A-linear map $f\colon M \to N$ is continuous. If N is also finitely generated, then f is strict*, i.e. $M/\mathrm{ker} f \to \mathrm{Im} f$ is an isomorphism.*

(iii) Any finitely generated (abstract) A-module has a unique structure as a Banach A-module.

(iv) The forgetful functor

$$\{\textit{finitely generated Banach } A\textit{-modules}\} \to \{\textit{finitely generated } A\textit{-modules}\}$$

is an equivalence of categories.

Proof. The proofs of all of these statements boil down to the Banach open mapping theorem. Without loss of generality, suppose that the norm on A satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

(i) Let $N \subseteq M$ be an A-submodule, and let $\overline{N}$ be its closure. Since A is noetherian and M is finitely generated, $\overline{N}$ is a finitely generated A-module: choose generators $x_1, \ldots, x_n$. This defines a surjection $A^{\oplus n} \to \overline{N}$ sending (a_i) to $\sum a_i x_i$.

Note that any map from the free module $A^{\oplus n}$ to a LCTM M is continuous. Let $y_1, \ldots, y_n$ be the image of the standard basis. Then by continuity of scalar multiplication on M, for any continuous seminorm $\|\cdot\|_M$ on M, there is a $C > 0$ such that

$$\left\|\sum a_i x_i\right\|_M \leq \max \|a_i x_i\|_M \leq \max C\|a_i\|.$$

Now, consider the open unit ball

$$\mathring{\Lambda} := \{a \in A \colon \|a\| < 1\} \subseteq A.$$

This is an open subset of A which is closed under addition and multiplication. Thus, $\sum \mathring{\Lambda} x_i \subseteq \overline{N}$ is open. Thus, we may write

$$x_i = y_i + \sum_j a_{ij} x_j$$

with $y_i \in N$ and $a_{ij} \in \mathring{\Lambda}$. This can be rearranged to

$$(\mathrm{id}_n - (a_{ij})) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Multiplying this by the matrix series

$$B = 1 + (a_{ij}) + (a_{ij})^2 + \cdots,$$

which converges as $a_{ij} \in \mathring{\Lambda}$, we find that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = B \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

that is, $x_i \in N$ for all i, so $N = \overline{N}$ is closed.

(ii) As M is finitely generated, we may choose generators $x_1, \ldots, x_n \in M$, and thus a continuous surjection $\pi \colon A^{\oplus n} \to M$ which is open by the open mapping theorem. Then the map $f \circ \pi \colon A^{\oplus n} \to N$ sending (a_i) to $\sum a_i f(x_i)$ is continuous. Since π is open, this implies that f is continuous.

Since N is finitely generated, the image of f is closed and Banach. Thus, the continuous bijection $M/\ker f \to \operatorname{im} f$ is a homeomorphism by the open mapping theorem.

(iii) Let M be a finitely generated A-module. Then choosing generators $x_1, \ldots, x_n$, there is a continuous, thus open, surjection $\pi \colon A^{\oplus n} \to M$ sending (a_i) to $\sum a_i x_i$. By part (i), the kernel $\ker \pi$ is closed, so the quotient space $A^{\oplus n}/\ker \pi$ is Banach. Thus, π induces a bijection from the Banach A-module $A^{\oplus n}/\ker \pi$ to M; this gives M the topology of a Banach A-module.

Now, if M has any other Banach A-module topology, then π induces a continuous bijection $A^{\oplus n}/\ker \pi \to M$. This is a homeomorphism by the open mapping theorem.

□

Remark 11.3.4. If A is a LCTA and $\|\cdot\|$ is an *algebraic* seminorm, then $\widehat{A}_{\|\cdot\|}$ is a Banach algebra.

11.4. **Fréchet-Stein algebras.**

Definition 11.4.1 (Schneider-Teitelbaum)**.** Suppose that A is a Fréchet algebra. We call A a *Fréchet-Stein algebra* if there exist algebraic seminorms $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots$ defining the topology on A such that

(1) $\widehat{A}_{\|\cdot\|_n}$ is noetherian for all n.
(2) The map $\widehat{A}_{\|\cdot\|_n} \to \widehat{A}_{\|\cdot\|_{n-1}}$ is flat for all n: i.e. the functor $M \mapsto \widehat{A}_{\|\cdot\|_{n-1}} \otimes_{\widehat{A}_{\|\cdot\|_n}} M$ is exact.

Note that in particular, $A = \varprojlim_n \widehat{A}_{\|\cdot\|_n}$.

Example 11.4.2.
- Any noetherian Banach algebra is a Fréchet-Stein algebra (e.g. $\mathscr{O}(D(1))$).
- $\mathscr{O}(D^0(r))$ and $\mathscr{O}(\mathbf{A}^1)$ are Fréchet-Stein algebras.

To see the latter, we use the following lemma:

Lemma 11.4.3. *(1) If $r_1 < r_2$ are in $p^{\mathbf{Q}}$, then the inclusion map induces an isomorphism $\widehat{\mathscr{O}(D(r_2))}_{\|\cdot\|_{r_1}} \to \mathscr{O}(D(r_1))$.*
(2) If $r_1 < r$, then the inclusion map induces an isomorphism $\widehat{\mathscr{O}(D^0(r))}_{\|\cdot\|_{r_1}} \to \mathscr{O}(D(r_1))$.
(3) If $r_1 < r_2$ are in $p^{\mathbf{Q}}$, then the inclusion map $\mathscr{O}(D(r_2)) \to \mathscr{O}(D(r_1))$ is flat.

Proof. (1) The inclusion $\mathscr{O}(D(r_2)) \hookrightarrow \mathscr{O}(D(r_1))$ is an isometry when we use the r_1-norm on the left side. The induced map $\widehat{\mathscr{O}(D(r_2))}_{\|\cdot\|_{r_1}} \to \mathscr{O}(D(r_1))$ is thus an isometry as well. As both sides are Banach, it is thus injective with closed image. This image is moreover dense, as it includes $L[T]$ and $L[T] \subseteq \mathscr{O}(D(r_1))$ is dense. Thus, it is an isomorphism.
(2) The proof is the same as (1).
(3) Suppose that $M \hookrightarrow N$ is an injection of finitely generated $\mathscr{O}(D(r_2))$-modules. We need to prove that the map $M \otimes_{\mathscr{O}(D(r_2))} \mathscr{O}(D(r_1)) \to N \otimes_{\mathscr{O}(D(r_2))} \mathscr{O}(D(r_1))$ is injective (exercise: it suffices to check flatness on finitely generated modules). Let x be in the kernel. We will show that $\mathrm{Ann}(x)$ is not contained in (g) for any maximal ideal (g) of the principal ideal domain $\mathscr{O}(D(r_1))$; thus, $\mathrm{Ann}(x) = (1)$, so $x = 0$.

Choose some maximal ideal (g); we want to show that x is in the kernel of the localization
$$M \otimes \mathscr{O}(D(r_1)) \to M \otimes \mathscr{O}(D(r_1))_{(g)}.$$

Note that we may take $g \in L[T]$ to be an irreducible monic polynomial whose roots in $\overline{L}$ all have valuation at most r_1. By Krull's intersection theorem, the map

$$M \otimes \mathscr{O}(D(r_1))_{(g)} \to M \otimes \widehat{\mathscr{O}(D(r_1))}_{(g)}$$

is injective. Moreover, we have seen that $\widehat{\mathscr{O}(D(r_1))}_{(g)} \simeq \widehat{L[T]}_{(g)}$. We have a commutative diagram:

$$\begin{tikzcd}
M \otimes \mathscr{O}(D(r_1)) \arrow[r] \arrow[d] & N \otimes \mathscr{O}(D(r_2)) \arrow[d] \\
M \otimes \mathscr{O}(D(r_1))_{(g)} \arrow[r] \arrow[d] & N \otimes \mathscr{O}(D(r_2))_{(g)} \arrow[d] \\
M \otimes \widehat{\mathscr{O}(D(r_1))}_{(g)} \arrow[r] & N \otimes \widehat{\mathscr{O}(D(r_2))}_{(g)}
\end{tikzcd}$$

Thus, to conclude that x is annihilated by an element not contained in (g), it suffices to see that the bottom map

$$M \otimes \widehat{\mathscr{O}(D(r_1))}_{(g)} \to N \otimes \widehat{\mathscr{O}(D(r_2))}_{(g)}$$

is injective. We claim that the map

$$\mathscr{O}(D(r_2)) \to \widehat{\mathscr{O}(D(r_1))}_{(g)} \simeq \widehat{L[T]}_{(g)}$$

is flat. Indeed, note that $g \in L[T] \subseteq \mathscr{O}(D(r_2))$, and we may identify $\widehat{L[T]}_{(g)}$ as the completion of $\mathscr{O}(D(r_2))$ at the maximal ideal (g).

$\square$

12. 2/13/20: Modules over Fréchet-Stein and Banach algebras

Last time we introduced the notion of a Fréchet-Stein algebra: a Fréchet algebra A with a defining set of algebraic seminorms

$$\|-\|_1 \leq \|-\|_2 \leq \|-\|_3 \leq \cdots$$

(implying $A = \varprojlim_i A^\wedge_{\|-\|_i}$) such that

(1) each Banach algebra $A^\wedge_{\|-\|_n}$ is noetherian, and
(2) the canonical map $A^\wedge_{\|-\|_{n+1}} \to A^\wedge_{\|-\|_n}$ is flat for all n.

Any noetherian Banach algebra is an example, as are $\mathscr{O}(D^\circ(r))$ and $\mathscr{O}(\mathbb{A}^1)$. Now we will state some facts about modules over Fréchet–Stein algebras without proofs, due to lack of time. This will allow us to think about sheaves on e.g. $\mathscr{O}(\mathbb{A}^1)$ (note that this is not affinoid, but we can define coherent sheaves on each $\mathscr{O}(D(r))$).

12.1. Coherent sheaves on Fréchet-Stein algebras.

Definition 12.1.1. By a **coherent sheaf** on a Fréchet-Stein algebra $(A, \{\|-\|_n\})$, we mean a sequence of finitely generated $A^\wedge_{\|-\|_n}$-modules M_n together with isomorphisms

$$A^\wedge_{\|-\|_n} \otimes_{A^\wedge_{\|-\|_{n+1}}} M_{n+1} \xrightarrow{\cong} M_n.$$

These form an abelian category $\mathsf{Coh}(A, \{\|-\|_n\})$ where kernels, cokernels, and direct sums are taken termwise. (This follows from flatness of the connecting maps.) If $\mathscr{M} = \{M_n\}$ is a coherent sheaf, we define its **global sections** as

$$\Gamma(\mathscr{F}) = \varprojlim M_n.$$

Lemma 12.1.2. *(1) The collection of A-modules isomorphic to global sections of objects of $\mathsf{Coh}(A, \{\|-\|_n\})$ is independent of the choice of the defining seminorms $\{\|-\|_n\}$. We call such A-modules **coadmissible**.*

(2) The map $\Gamma(\mathscr{M}) \to M_n$ has dense image, and moreover

$$A^\wedge_{\|-\|_n} \otimes_A \Gamma(\mathscr{M}) \xrightarrow{\cong} M_n.$$

(3) The functor

$$\Gamma\colon \mathsf{Coh}(A, \{\|-\|_n\}) \to \{\textit{coadmissible } A\textit{-modules}\}$$

is an equivalence of categories. In particular, it is exact.

Proof. (1) The point is that if we have any other collection of seminorms $\|-\|'_n$ then we have $\|-\|'_n \le C\|-\|_m$ for some m, so we can tensor from $A^\wedge_{\|-\|_m}$ to $A^\wedge_{\|-\|'_n}$ to get a finite module M'_n from M_m. This works the other way as well.

(2) Surjectivity follows from abstract nonsense. Let us now show that the given map is injective. Write $\Gamma(\mathscr{M}) = M$ and suppose that we have

$$\sum_{i=1}^k a_i \otimes x_i \mapsto 0 \in M_n$$

for $a_i \in A^\wedge_{\|-\|_n}$ and $x_i \in M$. We look at the map $A^k \to M$ sending $(a_i) \mapsto \sum a_i x_i$. The kernel of the corresponding map $(A^\wedge_{\|-\|_n})^k \to M_n$ is finitely generated, and using surjectivity to lift the generators, we can form a diagram

$$\begin{tikzcd}
A^r \arrow[r] \arrow[d] & A^k \arrow[r] \arrow[d] & M \arrow[d] \\
\left(A^\wedge_{\|-\|_n}\right)^r \arrow[r] & \left(A^\wedge_{\|-\|_n}\right)^k \arrow[r] & M_n
\end{tikzcd}$$

where the bottom row is exact and the top row has composite equal to 0. Then we can tensor to get a map

$$\begin{tikzcd}
\left(A^\wedge_{\|-\|_n}\right)^r \arrow[r] \arrow[d, "\cong"] & \left(A^\wedge_{\|-\|_n}\right)^k \arrow[r] \arrow[d, "\cong"] & A^\wedge_{\|-\|_n} \otimes_A M \arrow[d, two heads] \\
\left(A^\wedge_{\|-\|_n}\right)^r \arrow[r] & \left(A^\wedge_{\|-\|_n}\right)^k \arrow[r] & M_n.
\end{tikzcd}$$

Then chasing the diagram, you show that the right vertical map is an isomorphism. $\square$

Left-exactness of the global sections functor is obvious from the description as an inverse limit, but it is not so obvious that the derived functor vanishes. To see that Γ is exact, use the Mittag-Leffler condition; since each M_n is noetherian, the dense images of $M_{n+k} \to M_n$ must stabilize.

Lemma 12.1.3. *(1) If $f\colon M \to N$ is an A-linear map of coadmissible A-modules, then $\ker f, \operatorname{im} f, \operatorname{coker} f$ are coadmissible.*
(2) If $M_1, M_2 \subseteq N$ are coadmissible A-modules, so is $M_1 + M_2$.
(3) If N is a coadmissible A-module and $M \subseteq N$ is a finitely generated submodule, then it is coadmissible.
(4) Any finitely generated A-module is coadmissible.
(5) The category of coadmissible A-modules is an abelian subcategory of A-modules.

So far we have been doing just algebra; we are forgetting about the topology. This is because if M is a coadmissible A-module, then $M = \varprojlim M_n$ and each M_n is a uniquely a Banach $A^\wedge_{\|-\|_n}$-module. So M becomes a Fréchet A-module in a natural way. Again, this topology is independent of the choice of defining seminorms $\{\|-\|_n\}$.

Lemma 12.1.4. *Again, let A be a Fréchet–Stein algebra.*

(1) A-linear maps between coadmissible A-modules are automatically continuous with respect to the canonical topology.
(2) Coadmissible A-submodules of coadmissible modules are closed.
(3) If M is coadmissible and $N \subseteq M$ is a closed submodule, then N and M/N are coadmissible.

The upshot is that coadmissible modules over Fréchet-Stein algebras behave exactly the same way as finitely generated modules over noetherian Banach algebras.

12.2. **Banach-free modules.** Now we will go back to Banach algebras, most of the time noetherian and commutative Banach algebras. But we will look at bigger modules. (Richard suspects that the following also extends to Fréchet-Stein algebras, but can't find a reference and didn't have time to check.)

Definition 12.2.1. Let A be a Banach algebra. We say that a Banach A-module M is **Banach-free** (or **potentially ON-able** in the literature following Kevin Buzzard—see [3]) if

$$M \cong \perp_{I,0} (A, \|-\|).$$

We say that M is **Banach-projective** (or has **property (Pr)** in the literature) if M is a direct summand of a Banach-free Banach A-module.

Lemma 12.2.2. *A Banach A-module M is Banach-projective if and only if for every continuous A-linear surjection $N_1 \twoheadrightarrow N_2$ of Banach A-modules, every continuous A-linear map $M \to N_2$ lifts to a continuous A-linear map $M \to N_1$.*

Proof. For the forward direction, choose M' so that

$$M \oplus M' \cong \perp_{I,0} A.$$

Then the map $f\colon M \to N_2$ factors through $M \oplus M'$, so we may as well assume that M is Banach-free. Then there is a Banach basis $\{e_i\} \subset M$, with images $\{f(e_i)\} \in N_2$. The set $\{f(e_i)\}$ is bounded, and the map $N_1 \twoheadrightarrow N_2$ is open by the open mapping theorem. It follows that we can find lifts $y_i \in N_1$ of $f(e_i) \in N_2$ such that $\|y_i\|$ is bounded. Then we can lift the map f to $M \to N_1$ by sending $e_i \mapsto y_i$.

For the reverse direction, given such an M, there is a lift

$$\begin{tikzcd} \perp_{I,0} A \arrow[r, two heads] & M \\ & M. \arrow[u, "\mathrm{id}"] \arrow[ul, dashed] \end{tikzcd}$$

This diagram gives a splitting of $\perp_{I,0} A$ with M as a direct summand. $\square$

From now on, in this lecture, we assume that A is a *noetherian commutative* Banach algebra.

Lemma 12.2.3. *Let A be a noetherian commutative Banach algebra. If M is a finitely generated A-module, then M is projective if and only if M is Banach-projective.*

Proof. If M is projective, then $M \oplus M' \cong A^{\oplus r}$ as finitely generated A-modules. Then since both sides have canonical topologies that agree, M is also Banach-projective.

Now suppose that M is Banach-projective. Since M is finitely generated, we have some surjection

$$A^{\oplus r} \twoheadrightarrow M$$

of A-modules, which is necessarily continuous. This map splits as a map of Banach A-modules, so it splits as a map of A-modules. $\square$

We now want to use finitely generated modules to approximate all modules.

Lemma 12.2.4. *Suppose $M \subseteq \perp_{I,0} A$ is a finitely generated A-submodule. Then*

(1) *there exists a finite subset $J \subseteq I$ such that the projection $\pi_J \colon M \hookrightarrow A^{\oplus J}$ of M onto the J-component is injective,*
(2) *M is automatically closed, and*
(3) *for all $\epsilon > 0$, there exists a finite $J \subseteq I$ such that*

$$\|\pi_J x - x\| \le \epsilon \|x\|$$

for all $x \in M$.

Proof. (1) Let $M^{(1)}, \ldots, M^{(r)}$ generate M. We visualize this as a matrix with r rows and columns corresponding to I. Write $M^{(j)} = \left(m_i^{(j)}\right)_{i \in I}$. For each $i \in I$, let

$$x_i = \left(m_i^{(1)}, \ldots, m_i^{(r)}\right) \in A^r,$$

and let $N \subseteq A^r$ be the submodule generated by the x_i. Then because A is noetherian, N is a finitely generated module, so there exists a finite subset $J \subseteq I$ such that N is generated by the x_j for $j \in J$. We claim that this J works. If

$$\pi_J\left(\sum a_k m^{(k)}\right) = 0,$$

then for all $j \in J$ we have $\sum_{k=1}^r a_k m_j^{(k)} = 0$. This just means that the linear map

$$A^r \to A$$

$$(b_k) \mapsto \sum_k a_k b_k$$

vanishes on x_j for all $j \in J$, and hence on N. That means that $\sum_k a_k m_i^{(k)} = 0$ for all i, so $\sum a_k m^{(k)} = 0$.

(2) Given any $x \notin M$, we need to find a $U \subseteq \perp_{I,0} A$ containing x but disjoint from M. Find a finite $J \subseteq I$ such that

$$(M, x) \hookrightarrow \perp_{I,0} A \xrightarrow{\pi_J} A^J$$

is injective. Then $\pi_J M$ is closed in A^J, so its complement is an open subset of A^J containing $\pi_J(x)$ but disjoint from $\pi_J M$. It follows that the inverse image $\pi_J^{-1}(A^J \setminus \pi_J(M))$ is a neighborhood of x disjoint from M.

(3) Taking generators $m_i \in M$, the map

$$A^r \twoheadrightarrow M$$

is open by the open mapping theorem. This simply means that there exists $\delta > 0$ such that for all $m \in M$ with $\|m\| \leq \delta$, we can write

$$m = \sum_{i=1}^{r} a_i m_i, \quad \|a_i\| \leq 1 \text{ for all } i.$$

Rescaling, we see that for all $m \in M$, there exist $a_i \in A$ such that

$$m = \sum a_i m_i, \quad \|a_i\| \leq \frac{\|m\|}{\delta|\pi|}.$$

Given $\epsilon > 0$, there exists a finite subset $J \subseteq I$ such that

$$\|\pi_J m_i - m_i\| \leq \epsilon\delta\|m_i\|$$

for all i. It follows that for all m, if we write $m = \sum a_i m_i$ as above, then

$$\|\pi_J m - m\| \leq \sup_{i=1,\ldots,r} \|a_i\| \|\pi_J m_i - m_i\| \leq \frac{\|m\|}{\delta|\pi|}\epsilon\delta|\pi| = \epsilon\|m\|.$$

This finishes the proof. □

Let M, N be Banach A-modules. Then there is a closed subspace

$$\mathscr{L}_A(M, N) \subseteq \mathscr{L}_b(M, N)$$

of linear maps that are A-linear. We say that $f \in \mathscr{L}_A(M, N)$ has **finite rank** if its image is a finitely generated A-module. We then define the space of **completely continuous** maps as

$$\mathscr{C}\mathscr{C}_A(M, N) = \text{closure of finite rank maps in } \mathscr{L}_A(M, N).$$

This is consistent with our previous notation in the case $A = L$.

Lemma 12.2.5. *If N is Banach-free, then $\mathscr{C}\mathscr{C}_A(M, N)$ is also the closure of the set of $f \in \mathscr{L}_A(M, N)$ such that* $\mathrm{im}(f)$ *is contained in a finite free A-module.*

Proof. If $f \in \mathscr{L}_A(M, N)$ has finite rank, then we need to see that f is in the closure of those $g \in \mathscr{L}_A(M, N)$ such that $\mathrm{im}(g)$ is contained in a finite free A-module. We first write

$$N = \perp_{I,0} A,$$

and then given $\epsilon > 0$, there exists a finite subset $J \subseteq I$ such that

$$\|\pi_J x - x\| < \epsilon\|x\|$$

for all $x \in \mathrm{im}(f)$. Then $\mathrm{im}(\pi_J \circ f) \subseteq A^{\oplus J}$, and on the other hand

$$\|\pi_J f(x) - f(x)\| \leq \frac{\epsilon \|f(x)\|}{\|f\|} \leq \epsilon \|x\|.$$

So we have $\|\pi_J f - f\| \leq \epsilon$. $\square$

13. 2/18/20: Characteristic power series

13.1. **More on complete continuity.** As before, fix A a noetherian commutative Banach algebra (where we will assume that $\|ab\| \leq \|a\|\|b\|$), and M, N Banach A-modules (where we will assume that $\|am\| \leq \|a\|\|m\|$). Last time we defined

$$\mathscr{C}\mathscr{C}_A(M,N) \subset \mathscr{L}_A(M,W) \subset^{\text{closed}} \mathscr{L}_b(M,N)$$

where the first space is the "completely continuous" maps, the closure of those maps of finite rank (i.e. whose image is finitely generated).

Lemma 13.1.1. *(1)*

$$\mathscr{L}_A(\perp_{I,0}(A, \|\cdot\|), N) = \perp_I(N, \|\cdot\|_N)$$
$$\mathscr{L}_A(\perp_{I,0}(A, \|\cdot\|), \perp_{J,0}(A, \|\cdot\|)) = \perp_I \perp_{J,0}(A, \|\cdot\|).$$

(2)

$$\mathscr{C}\mathscr{C}_A(\perp_{I,0}(A, \|\cdot\|), \perp_{J,0}(A, \|\cdot\|)) = \perp_{J,0}\perp_I(A, \|\cdot\|) \subset^{closed} \perp_{I\times J}(A, \|\cdot\|).$$

That is, if $\{e_i\}_{i\in I}$ is a Banach basis for $\perp_{I,0}(A, \|\cdot\|)$, and $f(e_i) = (f_{ij})_{j\in J}$, then

(1) $f \in \mathscr{L}_A(\cdots)$ if and only if $\|f_{ij}\|$ is bounded and $\lim_{j\to\infty} \|f_{ij}\| = 0$ for all i, and
(2) $f \in \mathscr{C}\mathscr{C}_A(\cdots)$ if and only if $\|f_{ij}\|$ is bounded and $\lim_{j\to\infty} \sup_i \|f_{ij}\| = 0$.

Proof. (1) is basically the same as in the case $A = L$, so we will not do it.

For (2), assume that $\|f_{ij}\|$ is bounded and $\lim_{j\to\infty} \sup_i \|f_{ij}\| = 0$. We need to check that

$$\|f - \pi_{J'} \circ f\| \to 0$$

as $J' \subset J$, $\#J' < \infty$ gets large. But since $\|g\| = \sup_{x\neq 0} \frac{\|gx\|}{\|x\|}$, we have

$$\|f - \pi_{J'} \circ f\| \leq \sup_i \sup_{j\in J\setminus J'} \|f_{ij}\| = \sup_{j\in J\setminus J'} \sup_i \|f_{ij}\| \to 0$$

as desired. For the other direction, as the given condition on f is closed, it suffices to prove that if f is finite rank, then

$$(f_{ij}) \in \perp_{J,0}\perp_I(A, \|\cdot\|).$$

But for all $\epsilon > 0$, there is $J' \subset J$ with $\#J' < \infty$ such that for all $x \in \mathrm{im}\, f$ we have $\|\pi_{J'}x - x\| < \epsilon\|x\|$, and therefore for all $x \in M$, we have $\|\pi_{J'}f(x) - f(x)\| < \epsilon\|f\|\|x\|$. Then for $j \notin J'$, we have $\sup_i \|f_{ij}\| \leq \epsilon\|f\|$. $\square$

13.2. **Functions on the affine line over Banach algebras.** We will see that if f is completely continuous, then $\det(1+Tf)$ makes sense and lies in $\mathscr{O}(\mathbb{A}^1_A)$. (Here

$$\mathscr{O}(D(r)_A) = \left\{ \sum_{n=0}^{\infty} a_n T^n \mid a_n \in A, \|a_n\| r^n \to 0 \text{ as } n \to \infty \right\}$$

$$\mathscr{O}(\mathbb{A}^1_A) = \varprojlim_r \mathscr{O}(D(r)_A) = \left\{ \sum_{n=0}^{\infty} a_n T^n \mid a_n \in A, \forall r > 0, \|a_n\| r^n \to 0 \text{ as } n \to \infty \right\},$$

and the latter has norms $\|\sum a_n T^n\|_r = \sup_n \|a_n\| r^n$.)

Lemma 13.2.1. *Suppose $f(T) \in A[T]$ is a monic polynomial (or more generally the leading coefficient of f is in $A^\times$), and suppose $g \in \mathscr{O}(\mathbb{A}^1_A)$. Then there is a unique $q \in \mathscr{O}(\mathbb{A}^1_A)$ and $r \in A[T]$ with $\deg(r) < \deg(f)$ such that $g = qf + r$.*

Proof. Existence: let $\deg(f) = d$. By Euclidean division for polynomials, we may write $T^n = q_n f + r_n$ for $q_n, r_n \in A[T]$ with $\deg(r_n) < \deg(f)$. Let $g = \sum g_n T^n$. Then define $q = \sum_{n=0}^\infty g_n q_n$ and $r = \sum_{n=0}^\infty g_n r_n$. We need these to converge in $\|\cdot\|_r$ for all $r \in \mathbb{R}_{>0}$.

But the coefficients of q_n are polynomials in the coefficients of f of degree $\leq n$, and similarly for r_n (maybe with degree $\leq n+1$). More precisely, $q_n = T^{n-d} + a_{n-d-1}T^{n-d-1} + a_{n-d-2}T^{n-d-2} + \cdots$ where a_{n-d-1} is linear in the coefficients of f, a_{n-d-2} is quadratic in the coefficients of f, and so on. Consequently

$$\|q_n\|_r \leq \sup_i \|f\|_1^i r^{n-d-i} \leq \max(\|f\|_1^{n-d}, r^{n-d})$$

and

$$\|r_n\|_r \leq \|f\|_1^{n-d+1} r^{d-1}.$$

So

$$\|g_n q_n\|_r \leq \|g_n\| \max(\|f\|_1, r)^{n-d} \to 0$$

as $n \to \infty$, and similarly for $\|g_n r_n\|_r$.

Uniqueness: if uniqueness fails, then we can find $qf = r$ with $\deg r < \deg f$ and one of q and r nonzero. Then we can get a contradiction by arranging that $q, f, r \in \Lambda(\|\cdot\|)[\![T]\!]$ and have nonzero reductions mod π. Since f is monic, we have $\deg(f \pmod{\pi}) = \deg f$, which gives a contradiction mod π. $\square$

The upshot is that these spaces can be thought of as spaces of infinite matrices with some condition on the rows and columns.

Corollary 13.2.2. *If $f, g \in A[T]$ with f monic and $g = qf$ for $q \in \mathscr{O}(\mathbb{A}^1_A)$, then $q \in A[T]$.*

13.3. **The characteristic power series.**

Lemma 13.3.1. *Let A be a commutative noetherian Banach algebra and M a Banach projective Banach A-module. Let $\varphi \in \mathscr{C}\mathscr{C}_A(M,M)$. We can associate to each φ a unique element*

$$1 + \sum_{n=1}^\infty t_n(\varphi) T^n = \det(1 - \varphi T) \in \mathscr{O}(\mathbb{A}^1_A)$$

such that

(1) *$\det(1 - (\cdot)T) : \mathscr{C}\mathscr{C}_A(M,M) \to \mathscr{O}(\mathbb{A}^1_A)$ is continuous.*

(2) *If* $\operatorname{im}\varphi \subset N$ *where* N *is a finite projective* A*-submodule of* M*, then*

$$\det(1-\varphi T) = \det(1-\varphi|_N T)$$

(note that the RHS is the algebraically defined determinant).

(3) *If* $\varphi_i \in \mathscr{C}\mathscr{C}_A(M_i, M_i)$*, so that* $\varphi_1 \oplus \varphi_2 \in \mathscr{C}\mathscr{C}_A(M_1 \oplus M_2, M_1 \oplus M_2)$*, then*

$$\det(1-(\varphi_1\oplus\varphi_2)T) = \det(1-\varphi_1 T)\det(1-\varphi_2 T).$$

Proof. Uniqueness: for M Banach-free, this is implied by properties (1) and (2), because any completely continuous operator can be approximated by operators whose images lie inside finite free submodules. For M Banach-projective, choose N so that $M \oplus N$ is Banach-free and use that $\det(1-(\varphi\oplus 0)T) = \det(1-\varphi T)$.

Existence: again we will reduce to the Banach-free case. If we have already defined $\det(1-(\cdot)T)$ in the Banach-free case, and $\varphi \in \mathscr{C}\mathscr{C}_A(M,M)$ with $M \oplus N$ Banach-free, we can just define $\det(1-\varphi T) := \det(1-(\varphi\oplus 0)T)$. If $M \oplus N'$ is also Banach-free, then $M\oplus N\oplus M\oplus N'$ is Banach-free, and comparing the maps $(\varphi,0,0,0)$ and $(0,0,\varphi,0)$ on this space, which are the same up to an automorphism, we see that this definition is independent of the choice of N.

Now assume M is Banach-free, and write it in the form $\perp_{I,0}(A,\|\cdot\|)$. Then we know that if $\varphi(e_i) = \sum_{j\in I}\varphi_{ij}e_j$, then $\|\varphi_{ij}\|$ is bounded for all i,j (say by $B>1$), and $\lim_{j\to\infty}\sup_i\|\varphi_{ij}\| = 0$. Then we will set $\det(1-\varphi T) = 1+\sum_{n=1}^\infty t_n(\varphi)T^n$ where

$$t_n(\varphi) = (-1)^n \sum_{\substack{S\subset I\\ \#S=n}} \sum_{\sigma \text{ permutation of } S} \operatorname{sign}(\sigma)\prod_{i\in S}\varphi_{i\sigma(i)}.$$

We can find $J_0 \subset J$ with $\#J_0 < \infty$ such that $\|\varphi_{ij}\| \le 1$ if $j \notin J_0$. For all $\epsilon > 0$, we can furthermore find $J_\epsilon \subset J$ with $\#J_\epsilon < \infty$ such that $\|\varphi_{ij}\| \le \frac{\epsilon}{B^{\#J_0}}$ if $j\notin J_\epsilon$. Then if $S \subsetneq J_\epsilon$, we have

$$\left\|\prod_{i\in S}\varphi_{i\sigma(i)}\right\| \le B^{\#J_0}\cdot\frac{\epsilon}{B^{\#J_0}} = \epsilon,$$

so $t_n(\varphi)$ converges. In fact, if $\#S > \#J_\epsilon$, we can strengthen this to

$$\left\|\prod_{i\in S}\varphi_{i\sigma(i)}\right\| \le B^{\#J_0}\cdot\left(\frac{\epsilon}{B^{\#J_0}}\right)^{\#S-\#J_\epsilon} \le \epsilon^{\#S-\#J_\epsilon},$$

so in fact $\|t_n(\varphi)\| \le C\epsilon^n$. We conclude that indeed $\det(1-\varphi T) \in \mathscr{O}(\mathbb{A}^1_A)$. (But note that so far, our definition seems to depend on the choice of Banach basis.)

To show continuity, recall that we can estimate

$$\|a_1a_2\cdots a_n - b_1b_2\cdots b_n\|$$

$$\le \max(\|a_1-b_1\|\|a_2\|\cdots\|a_n\|, \|b_1\|\|a_2-b_2\|\|a_3\|\cdots\|a_n\|,\dots,\|b_1\|\cdots\|b_{n-1}\|\|a_n-b_n\|)$$

and therefore

$$\|\det(1-\varphi T)-\det(1-\varphi' T)\|_r \le \|\varphi-\varphi'\|\sup_n(\epsilon^{n-\#J_\epsilon-1}r^n).$$

Now let's show property (2). Suppose $\operatorname{im}(\varphi)\subset N$ where N is finite projective. We have

$$\det(1-\varphi T) = \lim_{\substack{J\subset I\\ \#J<\infty}} \det(1-\pi_J\circ\varphi T)$$

since for all $\epsilon > 0$ there is $J < I$ with $\#J < \infty$ and $\|\pi_J x - x\| < \epsilon\|x\|$ for all $x \in N$. This limit can also be written as

$$\lim_J \det(1 - \pi_J \circ \varphi|_{A^{\oplus J}} T)$$

since the coefficients of $\det(1 - \pi_J \circ \varphi T)$ are

$$\sum_{S:\#S=n} \sum_\sigma \operatorname{sign}(\sigma) \prod (\pi_J \circ \varphi)_{i\sigma(i)}$$

and $(\pi_J \circ \varphi)_{i\sigma(i)}$ is 0 unless $\sigma(i) \in J$ for all $i \in S$, that is, unless $S \subset J$. The limit can again be rewritten as

$$\lim_J \det(1 - \varphi \circ \pi_J|_N T) = \det(1 - \varphi|_N T)$$

by comparing $A^{\oplus J}$ and N using the maps

$$A^{\oplus J} \xrightarrow{\varphi} N \xrightarrow{\pi_J} A^{\oplus J}$$
$$N \xrightarrow{\pi_J} A^{\oplus J} \xrightarrow{\varphi} N$$

and noting that $\varphi \circ \pi_J|_N \to \varphi|_N$ as $J \to \infty$. Note that from this we see by reducing to the algebraic case that $\det(1 - \varphi T)$ is independent of the choice of Banach basis.

For property (3), we again reduce to the case where $\operatorname{im}(\varphi) \subset N$ and N is finite projective over A as follows: if $\varphi_1^{(i)} \to \varphi_1$ and $\varphi_2^{(1)} \to \varphi_2$, and each $\varphi_j^{(i)}$ has image contained in something finite projective, then $\varphi_1^{(i)} \oplus \varphi_2^{(i)} \to \varphi_1 \oplus \varphi_2$. $\square$

Definition 13.3.2. If $a \in A$, then $\det(1 - a\varphi) = \det(1 - T\varphi)\,|_{T=a}$.

Lemma 13.3.3. *(1) If $\varphi, \psi \in \mathscr{CC}_A(M, M)$, then*

$$\det(1 - \varphi)\det(1 - \psi) = \det(1 - (\varphi + \psi - \varphi\psi))$$

where $\varphi + \psi - \varphi\psi \in \mathscr{CC}_A(M, M)$.

(2) If $\varphi \in \mathscr{CC}_A(M, N)$ and $\psi \in \mathscr{L}_A(N, M)$, then $\varphi \circ \psi \in \mathscr{CC}_A(N, N)$, $\psi \circ \varphi \in \mathscr{CC}_A(M, M)$, and $\det(1 - \varphi \circ \psi T) = \det(1 - \psi \circ \varphi T)$.

(3) If $N \subset M$ is closed and $\varphi \in \mathscr{CC}_A(M, N)$, then $\det(1 - \varphi T) = \det(1 - \varphi|_N T)$.

Proof. For all of these, one uses the same strategy: reduce to the Banach-free case using $M \to M \oplus M'$, reduce to the case where $\operatorname{im}\varphi$ is contained in a finite projective module, and apply the corresponding algebraic result. $\square$

In the next lecture, we will address the following issues:

(1) If $f \in A[T]$ with $f(0) = 0$ and $\varphi \in \mathscr{CC}_A(M, M)$, what is $\det(1 - f(\varphi)T)$?
(2) Do we have compatibility with base change $\psi : A \to B$?

Base change for Banach modules works as follows. Let $\psi : A \to B$ be a continuous algebra map (between commutative noetherian Banach algebras), and assume as usual that

$$\|aa'\| \le \|a\|\|a'\|, \quad \|bb'\| \le \|b\|\|b'\|, \quad \|\psi a\| \le \|a\|.$$

Let M be a Banach module over A with $\|am\| \le \|a\|\|m\|$. Then let $B\widehat{\otimes}_A M$ be the completion of $B \otimes_A M$ with respect to $\Lambda(\|\cdot\|_B) \otimes_{\Lambda(\|\cdot\|_A)} \Lambda(\|\cdot\|_M)$. This has the universal property that if N is a Banach B-module and $\varphi : M \to N$ is continuous and A-linear, then φ factors uniquely through

$$M \to B\widehat{\otimes}_A M$$
$$m \mapsto 1 \otimes m$$

so that the induced map $B\widehat{\otimes}_A M \to N$ is continuous and B-linear.

(Recall that the topology on $B \otimes_A M$ is given by

$$\|x\| = \inf_{x=\sum b_i \otimes y_i, b_i \in B, y_i \in M} \sup_i \|b_i\|\|y_i\|.)$$

14. 2/20/20: More properties of characteristic power series

14.1. **Review and base change.** Let A be a noetherian commutative Banach algebra, M a Banach-projective Banach A-module, and $\varphi \in \mathscr{C}\mathscr{C}_A(MM)$. We saw last time that we can continuously associate to φ a power series $\det(1 - T\varphi) \in \mathscr{O}(\mathbb{A}^1_A)$ such that

$$\det(1 - T(\varphi_1 \oplus \varphi_2)) = \det(1 - T\varphi_1)\det(1 - T\varphi_2),$$

and if $\operatorname{im}\varphi \subset N$ where N is a finite projective A-module then

$$\det(1 - \varphi T) = \det(1 - \varphi|_N T),$$

and if $\varphi_1 \in \mathscr{C}\mathscr{C}_A(M, N)$ and $\varphi_2 \in \mathscr{L}_A(N, M)$ then

$$\det(1 - T\varphi_1 \circ \varphi_2) = \det(1 - T\varphi_2 \circ \varphi_1),$$

and if $N \subset M$ is closed and $\varphi \in \mathscr{C}\mathscr{C}_A(M, N)$, then

$$\det(1 - \varphi T) = \det(1 - \varphi|_N T),$$

and if $\varphi_1, \varphi_2 \in \mathscr{C}\mathscr{C}(M, M)$, then

$$\det(1 - \varphi_1 T)\det(1 - \varphi_2 T) = \det(1 - (\varphi_1 + \varphi_2 - \varphi_1\varphi_2)T).$$

Also, for $a \in A$, we write $\det(1 - a\varphi) = \det(1 - T\varphi)|_{T=a}$.

Lemma 14.1.1. *Let $\psi : A \to B$ be a continuous map of commutative noetherian Banach algebras and M a Banach-projective Banach A-module. Let $\varphi \in \mathscr{C}\mathscr{C}_A(M, M)$. Then*

(1) $B\widehat{\otimes}_A M$ is Banach projective over B (and if M is Banach-free then so is $B\widehat{\otimes}_A M$).
(2) $1 \otimes \varphi \in \mathscr{C}\mathscr{C}_A(B\widehat{\otimes} M, B\widehat{\otimes} M)$.
(3) $\psi(\det(1 - T\varphi)) = \det(1 - T(1 \otimes \varphi))$.

Proof. Once we know (1), then (2) and (3) immediately follow, because the "matrix" of $1 \otimes \varphi$ is the same as the "matrix" of φ. So we only need to check (1). Here, the Banach-projective case immediately reduces to the Banach-free case. So, choosing a Banach basis, we need to show that

$$B \otimes \perp_{I,0} (A, \|-\|) \to \perp_{I,0} (B, \|-\|); \quad 1 \otimes e_i \mapsto e_i'$$

is a homeomorphism. It is well-defined and continuous because the elements $1 \otimes e_i$ are bounded; then one can show that it has dense image and is closed, then that it is surjective, and then that it is an isometry. □

14.2. **Functional calculus for characteristic power series.** If $f(T) \in A[T]$, $f(0) = 0$, what is $\det(1 - f(\varphi)T)$? In the finite-dimensional case, if

$$\det(1 - \varphi T) = \prod (1 - \alpha_i T),$$

then the eigenvalues of $f(\varphi)$ are $f(\alpha_i)$, so

$$\det(1 - f(\varphi)T) = \prod (1 - f(\alpha_i)T).$$

So suppose that $\deg(f) = d$ and $f(T) = F_1T + F_2T^2 + \cdots + F_dT^d$. We can certainly define

$$\prod_{i=1}^{e}(1 - f(A_i)T) \in \mathbb{Z}[F_1, \ldots, F_d, A_1, \ldots, A_e, T]^{S_e} = \mathbb{Z}[F_1, \ldots, F_d, E_1, \ldots, E_e, T]$$

where E_i is the degree i elementary symmetric function in the A_is (so $E_1 = A_1 + \cdots + A_e$, etc.). Let

$$\prod_{i=1}^{e}(1 - f(A_i)T) = G_{d,e}(F_1, \ldots, F_d, E_1, \ldots, E_e, T).$$

Then we have

$$G_{d,e+1}(F_1, \ldots, F_d, E_1, \ldots, E_e, 0, T) = G_{d,e}(F_1, \ldots, F_d, E_1, \ldots, E_e, T)$$
$$G_{d+1,e}(F_1, \ldots, F_d, 0, E_1, \ldots, E_e, T) = G_{d,e}(F_1, \ldots, F_d, E_1, \ldots, E_e, T)$$

via the map

$$\begin{aligned} \mathbb{Z}[E_1, \ldots, E_{e+1}] &\to \mathbb{Z}[E_1, \ldots, E_e] \\ E_i &\mapsto E_i \text{ for } i \leq e \\ E_{e+1} &\mapsto 0 \end{aligned}$$

induced by the map of ambient rings

$$\begin{aligned} \mathbb{Z}[A_1, \ldots, A_{e+1}] &\to \mathbb{Z}[A_1, \ldots, A_e] \\ A_{e+1} &\mapsto 0. \end{aligned}$$

Now let

$$G_{d,e}(\underline{F}, \underline{E}, T) = 1 + \sum_{i=1}^{\infty} T^i G_{d,e}^{(i)}(\underline{F}, \underline{E}).$$

Here $G_{d,e}^{(i)}(\underline{F}, \underline{E})$ is a $\mathbb{Z}$-linear combination of monomials of the form $\prod_{j=1}^{i} F_{i_j} A_j^{i_j}$ which happens to also be a $\mathbb{Z}$-linear combination of expressions of the form $F_{i_1} \cdots F_{i_i} E_1^{m_1} \cdots E_e^{m_e}$. In order for $\prod_{j=1}^{i} F_{i_j} A_j^{i_j}$ to appear in $G_{d,e}^{(i)}(\underline{F}, \underline{E})$ from the expansion of $F_{i_1} \cdots F_{i_i} E_1^{m_1} \cdots E_e^{m_e}$, we must have

$$di \geq \sum i_j = \sum \alpha m_\alpha \geq i.$$

This implies that $G_{d,e}^{(i)}$ cannot involve E_α for $\alpha > di$, so it stabilizes and we can write $G_{d,e}^{(i)} = G_{d,\infty}^{(i)}$ for $e \geq di$.

Furthermore, if $r_1 \geq \cdots \geq r_e$, then $A_1^{r_1} \cdots A_e^{r_e}$ is a monomial of maximal A_1-exponent in the expansion of $E_1^{r_1 - r_2} E_2^{r_2 - r_3} \cdots E_e^{r_e}$. Since $A_1^{r_1} \cdots A_e^{r_e}$ can only appear in $G_{d,e}^{(i)}(\underline{F}, \underline{E})$ if $r_1 \leq d$, we conclude that $d \geq \sum m_\alpha$.

Now assume that A is a Banach algebra, and $f \in A[T]$ with $f(0) = 0$ and $\deg(f) = d$, and $g \in A[T]$ with $g(0) = 1$ and $\deg(g) = e$. Write $g(T) = 1 - g_1T + g_2T^2 - g_3T^3 + \cdots$ and $f(T) = f_1T + f_2T^2 + \cdots$, and let

$$G_{d,e}^{(i)}(f, g) = G_{d,e}^{(i)}(f_1, \ldots, f_d, g_1, \ldots, g_e).$$

Then we must have

$$\|G_{d,\infty}^{(i)}(f, g)\| \leq \|f\|_1^i \|g\|_r^d r^{-i}$$

for all $r > 1$, since in each term $f_{i_1} \cdots f_{i_i} g_1^{m_1} \cdots g_e^{m_e}$, we have

$$\begin{aligned}\|g_1^{m_1} \cdots g_e^{m_e}\| &= r^{-(m_1+2m_2+\cdots)}(\|g_1\|r)^{m_1}(\|g_2\|r)^{2m_2}\cdots \\ &\le r^{-i}\left(\sup_{\alpha}\|g_\alpha\|r^\alpha\right)^{m_1+m_2+\cdots} \le r^{-i}\|g\|_r^d.\end{aligned}$$

Similarly, interpreting $g^{(1)} - g^{(2)}$ as the polynomial with constant coefficient 1 and jth nonconstant coefficient given by $g_j^{(1)} - g_j^{(2)}$, we can bound

$$\|G_{d,\infty}^{(i)}(f, g^{(1)} - g^{(2)})\| \le \|f\|_1^i r^{-i} \max(\|g^{(1)}\|_r, \|g^{(2)}\|_r)^{d-1}\|g^{(1)} - g^{(2)}\|_r.$$

These bounds remain true for $g \in \mathscr{O}(\mathbb{A}_A^1)$. So we have

$$D(f,g) = 1 + \sum_{i=1}^{\infty} G_{d,\infty}^{(i)}(f,g)T^i \in A[\![T]\!]$$

but in fact

$$\|G_{d,\infty}^{(i)}(f,g)\|s^i \le \|g\|_r^d \left(\frac{s\|f\|_1}{r}\right)^i$$

for any $r > 1$ (since we assumed $g \in \mathscr{O}(\mathbb{A}_A^1)$ and $g(0) = 1$), and this goes to 0 as $i \to \infty$ for $r > s\|f\|_1$, so actually $D(f,g) \in \mathscr{O}(\mathbb{A}_A^1)$.

In conclusion, if $f \in A[T]$ with $\deg(f) = d$ and $f(0) = 0$, and $g \in \mathscr{O}(\mathbb{A}_A^1)$ with $g(0) = 1$, then we have found $D(f,g) \in \mathscr{O}(\mathbb{A}_A^1)$ such that $g \mapsto D(f,g)$ is continuous (since

$$\|D(f, g^{(1)} - g^{(2)})\|_s \le \|g^{(1)} - g^{(2)}\|_r \max(\|g^{(1)}\|_r, \|g^{(2)}\|_r)^{d-1} \sup_i \left(\frac{\|f\|_1 s}{r}\right)^i$$

for any $r > 1$, say $r > \|f\|_1 s$), and such that if g is a polynomial of degree e, then

$$D(f,g)(T) = G_{d,e}(f,g,T).$$

Note that if A were an algebraically closed field, we could write $g(T)$ in the form $\prod(1 - \alpha_i T)$ where $\alpha_i \in L$ and $\alpha_i \to 0$ as $i \to \infty$, and then we would have $D(f,g)(T) = \prod(1 - f(\alpha_i)T)$.

Lemma 14.2.1. $D(f, g^{(1)}g^{(2)}) = D(f, g^{(1)})D(f, g^{(2)})$.

Proof. By continuity, we can reduce to the case where $g^{(1)}$ and $g^{(2)}$ are polynomials. Then we can further reduce to the universal case where $A = \mathbb{Z}[F_1, \ldots, F_d, E_1, \ldots, E_e]$ and

$$g^{(1)}(T) = \prod_{i=1}^{e}(1 - A_i T), \quad g^{(2)}(T) = \prod_{i=1}^{e'}(1 - A_i' T).$$

Then since the claim is an equality of polynomials, we can replace A by $\overline{\mathrm{Frac}(A)}$, and thus factorize

$$D(f, g^{(1)})(T) = \prod_{i=1}^{e}(1 - f(A_i)T), \quad D(f, g^{(2)})(T) = \prod_{i=1}^{e'}(1 - f(A_i')T),$$

$$D(f, g^{(1)}g^{(2)})(T) = \prod_{i=1}^{e}(1 - f(A_i)T)\prod_{i=1}^{e}(1 - f(A_i')T).$$

∎

Lemma 14.2.2. *For A, M/A, $\varphi \in \mathscr{C}\mathscr{C}_A(M,M)$, and $f \in A[T]$ with $f(0)=0$ as usual, we have $f(\varphi) \in \mathscr{C}\mathscr{C}_A(M,M)$ and*

$$\det(1-f(\varphi)T) = D(f, \det(1-\varphi T)).$$

The proof is standard given what we already have.

14.3. **Reciprocal polynomials.**

Lemma 14.3.1. *Let $f, g, h \in A[T]$ have constant coefficient 0 and leading coefficient in $A^\times$, with f of degree d. Define $f^*(T) = T^d f(1/T)$. So if*

$$f(T) = f_0 + f_1 T + f_2 T^2 + \cdots + f_d T^d$$

then

$$f^*(T) = f_0 T^d + f_1 T^{d-1} + \cdots + f_d.$$

Then

(1)
$$D\left(1 - \frac{g^*(T)}{g^*(0)}, g(T)\right) = (1-T)^{\deg g}.$$

(2)
$$D\left(1 - \frac{f^*(T)}{f^*(0)}, g + hf\right)|_{T=1} = D\left(1 - \frac{f^*(T)}{f^*(0)}, g\right)|_{T=1}.$$

(3) $D\left(1 - \frac{f^*(T)}{f^*(0)}, g\right)|_{T=1} \in A^\times$ *if and only if* $(f,g) = A[T]$.

Proof. (1) We reduce to the universal case, where $g(T) = \prod(1-A_iT)$ and $g^*(T) = \prod(T-A_i)$. Then

$$D\left(1 - \frac{g^*(T)}{g^*(0)}, g(T)\right) = \prod_i \left(1 - T\left(1 - \frac{\prod_j (A_i - A_j)}{\pm \prod_j A_j}\right)\right) = \prod_i (1-T).$$

(2) This is because

$$D\left(1 - \frac{f^*(T)}{f^*(0)}, g\right)|_{T=1} = f_d^* R_{d,\deg g}(f,g)$$

where $R_{d,\deg g}(f,g)$ is the resolvent of f and g. Looking up how the resolvent works and filling in the details is an exercise.

(3) Same as Part (2).

∎

Lemma 14.3.2. *Let A be a commutative Banach algebra, $f(T) \in A[T]$ such that $f(0)=1$ and the leading coefficient of f is in $A^\times$, and $g \in \mathscr{O}(\mathbb{A}^1_A)$ with $g(0)=1$. Then $D\left(1 - \frac{f^*(T)}{f^*(0)}, g\right)|_{T=1}$ is in $A^\times$ if and only if $(f,g) = \mathscr{O}(\mathbb{A}^1_A)$.*

Proof. Let $g = qf + r$ where $r \in A[T]$ with $\deg(r) < \deg(f)$ and $q \in \mathscr{O}(\mathbb{A}^1_A)$. Then by Part (2) of Lemma 14.3.1 and continuity, we have

$$D\left(1 - \frac{f^*(T)}{f^*(0)}, g\right)|_{T=1} = D\left(1 - \frac{f^*(T)}{f^*(0)}, r\right)|_{T=1}$$

and the RHS is in $A^\times$ if and only if $(f,r)=A[T]$, by Part (3) of Lemma 14.3.1. If $(f,r)=A[T]$ then f and r also generate $\mathscr{O}(\mathbb{A}^1_A)$ when viewed in that ring, in which $(f,g)=(f,r)=\mathscr{O}(\mathbb{A}^1_A)$. Conversely, if $(f,g)=\mathscr{O}(\mathbb{A}^1_A)$, then we can find $\lambda,\mu\in\mathscr{O}(\mathbb{A}^1_A)$ with $\lambda f+\mu r=1$, and compute

$$\begin{aligned}1 &= D\left(1-\frac{f^*(T)}{f^*(0)},1\right)=D\left(1-\frac{f^*(T)}{f^*(0)},\lambda f+\mu r\right)|_{T=1}\\ &= D\left(1-\frac{f^*(T)}{f^*(0)},\mu r\right)|_{T=1}=D\left(1-\frac{f^*(T)}{f^*(0)},\mu\right)|_{T=1}D\left(1-\frac{f^*(T)}{f^*(0)},r\right)|_{T=1}\end{aligned}$$

so indeed $D\left(1-\frac{f^*(T)}{f^*(0)},g\right)|_{T=1}\in A^\times$. $\square$

14.4. **Adjugates.**

Lemma 14.4.1. *Let A, M/A, and $\varphi\in\mathscr{C}\mathscr{C}_A(M,M)$ be as before. Let*

$$\det(1-\varphi T)=1+\sum_{n=1}^\infty t_n(\varphi)T^n.$$

Then the formal power series

$$P(T,\varphi)=\sum_{n=0}^\infty T^n\sum_{m=0}^n t_{n-m}(\varphi)\varphi^m,$$

where the coefficient $\sum_{m=0}^n t_{n-m}(\varphi)\varphi^m$ lies in $A[\varphi]\subset\mathscr{L}_A(M,M)$, has the following properties:
1. *If M is finite free over A, then $P(T,\varphi)$ is the adjugate matrix of $1-T\varphi$.*
2. *For all $r\in\mathbb{R}_{>0}$, $\|\sum_{m=0}^n t_{n-m}(\varphi)\varphi^m\|\,r^n\to 0$ as $n\to\infty$.*
3. *$(1-T\varphi)P(T,\varphi)=\det(1-T\varphi)\,\mathrm{id}$.*

Proof. (3) is a formal computation:

$$(1-T\varphi)P(T,\varphi)=\sum_{n=0}^\infty T^n\left(\sum_{m=0}^n t_{n-m}(\varphi)\varphi^m-\sum_{m=0}^{n-1}t_{n-1-m}(\varphi)\varphi^m\right)=\sum_{n=0}^\infty T^n t_n(\varphi)\,\mathrm{id}\,.$$

So this will remain true whenever we know it makes sense. This implies (1), since in that case everything is a polynomial.

For (2), we reduce to when M is Banach-free, with basis $\perp_{I,0}(A,\|\cdot\|)$. Then we have

$$\left\{\sup_i\|\varphi_{ij}\|\right\}=\{r_\alpha\}$$

for a sequence $r_1\geq r_2\geq r_3\geq\cdots\to 0$ (note that all but countably many must equal 0). Choose j_α so that $r_\alpha=\sup_i\|\varphi_{ij_\alpha}\|$. We claim that

$$\left\|\sum_{m=0}^n t_{n-m}(\varphi)\varphi^m\right\|\leq\max(r_1\cdots r_{n-1}\|\varphi\|,\|t_n(\varphi)\|).$$

Proof: Let $J_m=\{j_1,\ldots,j_m\}$. We have $\pi_{J_m}\circ\varphi\to\varphi$ as $m\to\infty$. It suffices to prove the inequality for $\pi_{J_m}\circ\varphi$, so replace φ by $\pi_{J_m}\circ\varphi$. Then

$$\left\|\sum_{m=0}^n t_{n-m}(\varphi)\varphi^m\right\|\leq\max\left(\|t_n(\varphi)\|,\|\varphi\|\left\|\sum_{m=1}^n t_{n-m}(\varphi)(\varphi\circ\pi_{J_m})^{m-1}\right\|\right).$$

Since $\varphi \circ \pi_{J_m} \in \operatorname{End}_A(A^{\oplus J_m})$, we have now reduced to the case of matrices that have finite dimensional domain as well as range. But if $\#I < \infty$ then the adjugate matrix of $(1 - T\varphi)$ is given by determinants of cofactors (Cramer's rule), so the coefficient of T^n is a sum of terms that are products of n "φ_{ij}"s, so indeed

$$\left\| \sum_{m=1}^{n} t_{n-m}(\varphi)(\varphi \circ \pi_{J_m})^{m-1} \right\| = \left\| \sum_{m=0}^{n-1} t_{n-1-m}(\varphi)(\varphi \circ \pi_{J_m})^{m} \right\| \le r_1 \cdots r_{n-1}.$$

□

15. 2/25/20: Eigenspaces, and an introduction to locally analytic p-adic manifolds

Let A be a commutative noetherian Banach algebra, M a Banach-projective Banach A-module, and $\varphi \in \mathscr{C}\mathscr{C}_A(M, M)$. Previously, we defined

$$\det(1 - T\varphi) \in \mathscr{O}(\mathbb{A}^1_A).$$

We also defined the adjugate matrix $P(T, \varphi) \in \mathscr{O}\left(\mathbb{A}^1_{\overline{A[\varphi]}}\right)$, where $\overline{A[\varphi]} \in \mathscr{L}_A(M, M)$, and showed that

$$(1 - T\varphi)P(T, \varphi) = \det(1 - T\varphi) \operatorname{id}_M .$$

15.1. Roots of characteristic power series.

Lemma 15.1.1. *If $\det(1 - a\varphi) \in A^\times$ for a given $a \in A$, then $1 - a\varphi$ is an isomorphism.*

Proof. The inverse of $(1 - a\varphi)$ is $(1 - a\varphi)^{-1} = \det(1 - a\varphi)^{-1}P(a, \varphi)$. □

Corollary 15.1.2. *If $P(T) \in A[T]$ with $P(0) = 1$ and leading coefficient in $A^\times$, and if $(P(T), \det(1 - T\varphi)) = \mathscr{O}(\mathbb{A}^1_A)$, then $P^*(\varphi)$ is an isomorphism.*

Recall that by $P^*(T)$ we mean $T^{\deg P}P(1/T)$, so if $P(T) = 1 + P_1T + \cdots + P_dT^d$ then $P^*(T) = P_d + P_{d-1}T + \cdots + T^d$. The finite-dimensional intuition for this is that if $P(T)$ and $\det(1 - T\varphi)$ are coprime, then the inverses of the eigenvalues of φ are not roots of P, so the eigenvalues of φ are not roots of P^*, so $P^*(\varphi)$ should have no kernel.

Proof. Let $\psi = 1 - \frac{P^*(\varphi)}{P^*(0)}$, a polynomial in φ with no constant term, hence an element of $\mathscr{C}\mathscr{C}_A(M, M)$. Then we have

$$\det(1 - T\psi) = D\left(1 - \frac{P^*(T)}{P^*(0)}, \det(1 - T\varphi)\right).$$

By Lemma 14.3.2 from last time, $(P(T), \det(1 - T\varphi)) = \mathscr{O}(\mathbb{A}^1_A)$ implies that $\det(1 - \psi) \in A^\times$, so $1 - \psi = \frac{P^*(\varphi)}{P^*(0)}$ is invertible, so $P^*(\varphi)$ is an isomorphism. □

Lemma 15.1.3. *Let $a \in A$. Suppose*

$$\left(\frac{d}{dT}\right)^k \det(1 - T\varphi)|_{T=a} = 0$$

for $k = 0, \ldots, h - 1$, but

$$\left(\frac{d}{dT}\right)^h \det(1 - T\varphi)|_{T=a} \in A^\times.$$

(Note that the hth derivative being a unit makes this a stronger condition than saying that $\det(1-T\varphi)$ *has a root at a of multiplicity h.) Then we can find* $e \in \overline{A[\varphi]} \subset \mathscr{L}_A(M,M)$ *such that*

- $e^2 = e$,
- eM *is finite projective over* A *of rank* h,
- $(1-a\varphi)^h eM = (0)$,
- $(1-a\varphi)$ *is an isomorphism on* $(1-e)M$ *and* $\det(1-a\varphi|_{(1-e)M}) \in A^\times$, *and*
- *if* $h > 0$ *then* $a \in A^\times$.

The finite-dimensional intuition for this is that we are isolating the subspace of M of generalized φ-eigenvalue a, that is, finding the Jordan normal form of φ.

Proof. We have already done the case $h = 0$, so assume $h > 0$. We will use the equation

$$(1-T\varphi)P(T,\varphi) = \det(1-T\varphi)\,\mathrm{id}\,.$$

Plugging in a, we get $(1-a\varphi)P(a,\varphi) = 0$. Differentiating both sides k times with respect to T, we get

$$(1-T\varphi)\left(\frac{d}{dT}\right)^k P(T,\varphi) - k\varphi\left(\frac{d}{dT}\right)^{k-1} P(T,\varphi) = \left(\frac{d}{dT}\right)^k \det(1-T\varphi)\,\mathrm{id}\,.$$

For $1 \le k < h$, plugging in a gives

$$(1-a\varphi)\left(\frac{d}{dT}\right)^k P(T,\varphi)|_{T=a} = k\varphi\left(\frac{d}{dT}\right)^{k-1} P(T,\varphi)|_{T=a}.$$

This implies inductively that

$$(1-a\varphi)^{k+1}\left(\frac{d}{dT}\right)^k P(T,\varphi)|_{T=a} = 0.$$

On the other hand, for $k = h$, we get

$$(1-a\varphi)\left(\frac{d}{dT}\right)^h P(T,\varphi)|_{T=a} - h\varphi\left(\frac{d}{dT}\right)^{h-1} P(T,\varphi)|_{T=a} \in A^\times\,\mathrm{id}\,.$$

Rescaling and assigning terms appropriately, this gives us $\psi_1, \psi_2 \in \overline{A[\varphi]}$ such that $(1-a\varphi)\psi_1 + \psi_2 = 1$ and $(1-a\varphi)^h\psi_2 = 0$. Furthermore, we can write

$$(1-a\varphi)^h\psi_1^h + \psi_2\psi_3 = 1$$

for some $\psi_3 \in \overline{A[\varphi]}$ which we can find by writing $\psi_2 = (1-a\varphi)\psi_1 - 1$ and solving for ψ_3 (or by raising $(1-a\varphi)\psi_1 + \psi_2 = 1$ to the hth power). Let $\psi_2\psi_3 = e$. Then indeed $(1-a\varphi)^h e = 0$, and also

$$(1-e)e = (1-a\varphi)^h\psi_1^h\psi_2\psi_3 = 0,$$

that is, $e^2 = e$. This also means that $(1-e)^2 = (1-e)$, or

$$(1-a\varphi)((1-a\varphi)^{h-1}\psi_1^h)(1-e) = 1-e,$$

which is to say that $(1-a\varphi)$ is an isomorphism on $(1-e)M$ with inverse $(1-a\varphi)^{h-1}\psi_1^h$. Also if ψ is such that

$$(1-a\varphi|_{(1-e)M})(1-\psi) = \mathrm{id}_{(1-e)M},$$

then we can solve

$$\begin{aligned} -a\varphi|_{(1-e)M} - \psi + a\varphi|_{(1-e)M}\psi &= 0 \\ \psi = \varphi_{(1-e)M}(a\psi - a) &\in \mathscr{C}\mathscr{C}_A((1-e)M, (1-e)M), \end{aligned}$$

from which we conclude that $\det(1-\psi)$ is a well-defined element of A such that $\det(1-a\varphi|_{(1-e)M})\det(1-\psi) = 1$, and hence $\det(1-a\varphi|_{(1-e)M}) \in A^\times$.

It remains to check that eM is finite projective of rank h and that $a \in A^\times$ if $h > 0$. From $(1-a\varphi)^h e = 0$, we see that $e = ha\varphi e + \cdots$ is φ composed with something, so $e \in \mathscr{C}\mathscr{C}_A(M, M)$. Let f_i be a sequence of finite rank maps such that $f_i \to e$. Then

$$ef_ie \in \mathscr{C}\mathscr{C}_A(eM, eM),$$

and ef_ie is finite rank and goes to id_{eM}. Therefore we can find $g = 1 - ef_ie \in \mathscr{L}_A(eM, eM)$ arbitrarily small with $1-g$ of finite rank. But $1-g$ is invertible with inverse $1+g+g^2+\cdots$ for g sufficiently small, so eM must be finitely generated over A. From $M = eM \oplus (1-e)M$, we see that eM is Banach-projective; by Lemma 12.2.3 we conclude that it is actually projective over A.

If $eM = 0$, we have $\det(1-a\varphi) \in A^\times$, so $h = 0$. If not, we have

$$0 = \mathrm{tr}(1 - a\varphi|_{eM})^h = \mathrm{tr}(1 - a(h\varphi|_{eM} + \cdots)) = \mathrm{rank}(eM) + ab$$

for some $b \in A$; since $\mathrm{rank}(eM) \neq 0$, we conclude that $a \in A^\times$.

Now we can compute $\mathrm{rank}(eM)$. Let $f(T) = \det(1 - T\varphi|_{eM}) \in A[T]$ and $g(T) = \det(1 - T\varphi|_{(1-e)M})$. We have $f(T)g(T) = \det(1-T\varphi)$ and $g(a) \in A^\times$. Since $a \in A^\times$, by polynomial division, we can write $f(T) = (1-a^{-1}T)^{h_1} f_1(T)$ with $f_1(a) \neq 0$. So

$$\det(1-T\varphi) = (1-a^{-1}T)^{h_1} f_1(T) g(T).$$

Therefore

$$\left(\frac{d}{dT}\right)^k \det(1-T\varphi)|_{T=a} = 0$$

for $k = 0, \ldots, h_1 - 1$, and

$$\left(\frac{d}{dT}\right)^{h_1} \det(1-T\varphi)|_{T=a} = h_1!(-a^{-1})^{h_1} f_1(a) g(a),$$

where $h_1!(-a{-}1)^{h_1}$ is a unit, $f_1(a) \neq 0$, and $g(a)$ is a unit. We conclude that $h_1 = h$ and $f_1(a) \in A^\times$.

If A were a field, we would have $\mathrm{charpoly}_{\varphi|_{eM}}(T) = (T-a)^{\mathrm{rank}(eM)}$. Instead, we can write

$$\begin{aligned} \mathrm{charpoly}_{\varphi|_{eM}}(T) &\equiv (T-a)^{\mathrm{rank}(eM)} \pmod{\mathfrak{m}} \\ f(T) &\equiv (1-a^{-1}T)^{\mathrm{rank}(eM)} \pmod{\mathfrak{m}} \end{aligned}$$

for all maximal ideals $\mathfrak{m}$ of A. But we also have $f(T) = (1-a^{-1}T)^{h_1} f_1(T)$, so

$$f_1(T) \equiv (1-a^{-1}T)^{\mathrm{rank}(eM) - h_1} \pmod{\mathfrak{m}}$$

for all maximal ideals $\mathfrak{m}$ of A. But $f_1(a) \in A^\times$, so this is only possible if $\mathrm{rank}(eM) = h_1 = h$, as desired. Note that this means we also have $\det(1 - T\varphi|_{eM}) = (1 - Ta^{-1})^h$. □

15.2. **Polynomial factors of characteristic power series.** Now we would like to generalize the previous discussion beyond powers of linear polynomials.

Lemma 15.2.1. *Let $P(T) \in A[T]$ with $P(0) = 1$ and leading coefficient in $A^\times$. (Remember that the leading coefficient restriction we keep putting in is automatic if A is a field.) Let $Q(T) \in \mathscr{O}(\mathbb{A}^1_A)$ such that*

$$P(T)Q(T) = \det(1 - T\varphi).$$

Suppose that $(P, Q) = \mathscr{O}(\mathbb{A}^1_A)$. Then there is a unique $e \in \overline{A[\varphi]} \subset \mathscr{L}_A(M, M)$ such that

- $e^2 = e$,
- $P^*(\varphi)eM = (0)$, *and*
- $P^*(\varphi)$ *is an isomorphism on* $(1 - e)M$.

Furthermore,

- $\mathrm{rank}(eM) = \deg P$,
- $\det(1 - T\varphi_{eM}) = P(T)$, *and*
- $\det(1 - T\varphi|_{(1-e)M}) = Q(T)$.

Proof. Uniqueness: suppose e_1 and e_2 both satisfy the given conditions. Since both lie in $\overline{A[\varphi]}$, which is commutative, they must commute. Then we can expand

$$M = e_1e_2M \oplus e_1(1 - e_2)M \oplus (1 - e_1)e_2M \oplus (1 - e_1)(1 - e_2)M.$$

Since $P^*(\varphi)$ is both 0 and invertible on $e_1(1 - e_2)M$, we find that $e_1(1 - e_2)M = 0$, so $e_1 = e_1e_2$. Similarly using $(1 - e_1)e_2M$, we find that $e_2 = e_1e_2$ as well, so $e_1 = e_2$.

Existence: again let $\psi = 1 - \frac{P^*(\varphi)}{P^*(0)} = \mathscr{C}\mathscr{C}_A(M, M)$. Then

$$\begin{aligned}
\det(1 - \psi T) &= D\left(1 - \frac{P^*(T)}{P^*(0)}, \det(1 - \varphi T)\right) \\
&= D\left(1 - \frac{P^*(T)}{P^*(0)}, P\right) D\left(1 - \frac{P^*(T)}{P^*(0)}, Q\right) \\
&= (1 - T)^{\deg P} D\left(1 - \frac{P^*(T)}{P^*(0)}, Q\right)
\end{aligned}$$

where we know that $D\left(1 - \frac{P^*(T)}{P^*(0)}, Q\right)$ evaluated at 1 is a unit in A. Then by Lemma 15.1.3 for $a = 1$, we can find $e \in \overline{A[\psi]} \subset \overline{A[\varphi]} \subset \mathscr{L}_A(M, M)$ such that $e^2 = e$, $(1 - \psi)^{\deg P} eM = 0$ (that is, $P^*(\varphi)^{\deg P} eM = 0$), $(1 - \psi)$ is an isomorphism on $(1 - e)M$, $P^*(\varphi)$ is an isomorphism on $(1 - eM)$, $\det(1 - \psi|_{(1-e)M}) \in A^\times$, and $\mathrm{rank}(eM) = \deg P$.

We still need to check that $P^*(\varphi)eM = 0$, $\det(1 - T\varphi|_{eM}) = P(T)$, and $\det(1 - T\varphi|_{(1-e)M}) = Q(T)$. We have

$$\det(1 - \psi|_{(1-e)M}) = D\left(1 - \frac{P^*(T)}{P^*(0)}, \det(1 - T\varphi|_{(1-e)M})\right)(1).$$

Since the LHS is a unit, we conclude that $(P(T), \det(1 - T\varphi|_{(1-e)M})) = \mathscr{O}(\mathbb{A}^1_A)$. So we can find $\lambda, \mu \in \mathscr{O}(\mathbb{A}^1_A)$ with

$$\lambda P + \mu \det(1 - T\varphi|_{(1-e)M}) = 1.$$

Multiplying by $\det(1 - T\varphi|_{eM})$, we get

$$P(\lambda \det(1 - T\varphi|_{eM}) + \mu Q) = \det(1 - T\varphi|_{eM})$$

so P divides $\det(1 - T\varphi|_{eM})$ in $\mathscr{O}(\mathbb{A}^1_A)$, hence in $A[T]$. But $\det(1 - T\varphi|_{eM})$ has degree at most $\operatorname{rank}(eM) = \deg P$ and is nonzero, and both constant terms are 1, so in fact $P(T) = \det(1 - T\varphi|_{eM})$. Then Cayley-Hamilton gives $P^*(\varphi)eM = 0$. Finally, we have

$$P(T)Q(T) = \det(1 - T\varphi) = \det(1 - T\varphi|_{eM}) \det(1 - T\varphi|_{(1-e)M})$$

so $P(T)(Q(T) - \det(1 - T\varphi|_{(1-e)M})) = 0$, so $Q(T) = \det(1 - T\varphi|_{(1-e)M})$. □

Intuitively, if $A = L$ is a field, we can factor

$$\det(1 - T\varphi) = \prod_i P_i(T)^{e_i}$$

where $P_i(T)$ is an irreducible polynomial in $L[T]$ with $P_i(0) = 1$, $P_i \neq P_j$ if $i \neq j$, and if α_i is a root of P_i and $i < j$ then $|\alpha_j| < |\alpha_i|$. In this case we can easily find $M_i \subset M$ such that M_i is preserved by φ and has rank $(\deg P_i)e_i$, $P_i^*(\varphi)^{e_i} = 0$ on M_i, and $M = M_i \oplus M_i'$. For general A, we are trying to interpolate what happens at the various maximal ideals of A, with varying degrees of success.

Lemma 15.2.2. *Let M_1, M_2 be Banach projective Banach A-modules, $\varphi_i \in \mathscr{C}\mathscr{C}_A(M_i, M_i)$ for $i = 1, 2$, and $\psi \in \mathscr{L}_A(M_1, M_2)$ such that $\psi \circ \varphi_1 = \varphi_2 \circ \psi$. Suppose that*

$$\det(1 - \varphi_i T) = P_i(T)Q_i(T)$$

where $P_i(T) \in A[T]$, $P_i(0) = Q_i(0) = 1$, $P_i(T)$ has leading coefficient in $A^\times$, and $(P_1P_2, Q_1Q_2) = \mathscr{O}(\mathbb{A}^1_A)$. Take e_i as in Lemma 15.2.1. Then $\psi \circ e_1 = e_2 \circ \psi$.

Proof sketch. Let $M = M_1 \oplus M_2$ and $\varphi = \varphi_1 \oplus \varphi_2 \in \mathscr{C}\mathscr{C}_A(M, M)$. One may show that $e = e_1 \oplus e_2 \in \overline{A[\varphi]}$. Then one concludes that $\begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix}$ commutes with $\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} = \varphi$, hence with $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. □

15.3. **Locally analytic p-adic manifolds and groups.** References: Bourbaki, *General topology* ([2]) for manifolds and *Lie groups and Lie algebras* ([1]) for groups.

Let X be a second countable (meaning that it has a countable basis) Hausdorff topological space. By a *p-adic locally analytic chart* for X, we mean a pair (U, φ) where $U \subset X$ is open and

$$\varphi : U \xrightarrow{\sim} \varphi(U) \subset \mathbb{Q}_p^n$$

is a homeomorphism from U onto an open subset $\varphi(U)$ of $\mathbb{Q}_p^n$. We call two charts (U, φ) and (U', φ') compatible if

$$\varphi' \circ \varphi^{-1} : \varphi(U) \to \varphi'(U)$$

is locally analytic, meaning that in a neighborhood of any point it is given by a convergent power series.

An n-dimensional *atlas* $\mathscr{A} = \{(U_i, \varphi_i)\}$ is a set of compatible charts with $\bigcup U_i = X$. We call two atlases $\mathscr{A}, \mathscr{A}'$ equivalent if $\mathscr{A} \cup \mathscr{A}'$ is an atlas (i.e. every chart in one is compatible with every chart in the other). This is an equivalence relation.

Definition 15.3.1. A *locally analytic manifold of dimension n over $\mathbb{Q}_p$* is a pair $(X, [\mathscr{A}])$ where $[\mathscr{A}]$ is an equivalence class of atlases.

(Note that we can also do this for L_0^n in place of $\mathbb{Q}_p^n$ for any finite extension $L_0/\mathbb{Q}_p$. We will always work over $\mathbb{Q}_p$, but most things work fine for L_0.)

Such a space X is locally compact and paracompact, so we can find an atlas $\mathscr{A} = \{(U_i, \varphi_i)\}_{i\in I}$ such that I is countable, $X = \coprod_i U_i$, and $\varphi_i U_i = \mathbb{Z}_p^n$ for all i. We will call such an atlas *strict*.

We say that a strict atlas $\mathscr{A}' = \{(V_j, \psi_j)\}$ is a *refinement* of a strict atlas $\mathscr{A} = \{(U_i, \varphi_i)\}$ if for all j there is some i such that $V_j \subset U_i$, and

$$\varphi_i \circ \psi_j^{-1} : \psi_j(V_j) \to \varphi_i(U_i)$$

takes $\underline{x} \in \mathbb{Z}_p^n$ to $\underline{a} + p^m \underline{x}$ for some $\underline{a} \in (\mathbb{Z}/p^m\mathbb{Z})^n$, where m is independent of j. Any strict $\mathscr{A}$ has a proper refinement by cutting up $\mathbb{Z}_p^n$ into residue discs.

Next we will discuss locally analytic functions and maps on such spaces, and their duals.

16. 2/27/20: Functions and distributions on locally analytic manifolds and groups

Last time, we defined p-adic locally analytic manifolds X with charts to $\mathbb{Q}_p^n$. We also defined "strict atlases", those of the form $\mathscr{A} = \{(U_i, \varphi_i)\}_{i\in I}$ such that I is countable, $X = \coprod U_i$, and $\varphi_i : U_i \xrightarrow{\sim} D(1)^n$.

16.1. Locally analytic functions on manifolds.

We wish to define what it means for a map $X \to V$, where V is a LCTVS, to be locally analytic.

Definition 16.1.1. An *augmented atlas* is a collection $\mathscr{A}^* = \{(U_i, \varphi_i, V_i \hookrightarrow V)\}_{i\in I}$ where $\mathscr{A} = \{(U_i, \varphi_i)\}_{i\in I}$ is a strict atlas for X, V_i is a Banach space, and each $V_i \hookrightarrow V$ is continuous linear.

Now let

$$LA_{\mathscr{A}^*}(X, V) = \prod_i \mathscr{O}(D(1))^n \widehat{\otimes}_\pi V_i$$

where, explicitly,

$$\mathscr{O}(D(1))^n \widehat{\otimes}_\pi V_i = \left\{ \sum_{\underline{m}=(m_1,\dots,m_n)} x_{\underline{m}} \underline{T}^{\underline{m}} : x_{\underline{m}} \in V_i \mid \|x_{\underline{m}}\|_{V_i} \to 0 \text{ as } \underline{m} \to \infty \right\}$$

where $\underline{T}^{\underline{m}} := T_1^{m_1} \cdots T_n^{m_n}$, and the norms are given by

$$\left\| \sum x_{\underline{m}} \underline{T}^{\underline{m}} \right\| = \max_{\underline{m}} \|x_{\underline{m}}\|_{V_i}$$

(because we can write

$$\perp_{J,0} (L, |\cdot|) \widehat{\otimes}_\pi V \cong \perp_{J,0} (V, \|\cdot\|_V)$$

if V is a Banach space). We have a map

$$LA_{\mathscr{A}^*}(X, V) \to LA(X, V) := \varinjlim_{\mathscr{A}^*} LA_{\mathscr{A}^*}(X, V).$$

Also, using the maps

$$\mathscr{O}(D(1))^n \widehat{\otimes}_\pi V_i \to C(\mathbb{Z}_p^n, V_i),$$

we get the inclusion

$$LA_{\mathscr{A}^*}(X, V) \to C_{\mathscr{A}}(X, V) = \prod_i C(U_i, V).$$

Here if $\mathscr{A} = \{(U_i, \varphi_i)\}$, so that $X = \coprod U_i$ and $U_i \subset X$ is open compact, the topology on $C_{\mathscr{A}}(X, V) = \prod_i C(U_i, V)$ is given by the norms

$$\|f\|_{\Omega,\|\cdot\|} = \sup_{\omega \in \Omega} \|f(\omega)\|$$

for all $\Omega \subset X$ compact and seminorms $\|\cdot\|$ on V. Actually, $C_{\mathscr{A}}(X, V)$ is independent of $\mathscr{A}$ as an LCTVSs, which you can see by checking that the definition is preserved by refinements. So we get a map

$$LA(X, V) \to C(X, V) := C_{\mathscr{A}}(X, V).$$

If X is compact, $C(X, V)$ recovers our previous definition. Then the spaces $LA(X, V)$ are LCTVSs of compact type, since if $r > 1$ then $\mathscr{O}(D(r))^n \to \mathscr{O}(D(1))^n$ is compact.

16.2. **Locally analytic distributions.** The space of (locally analytic) *distributions* on X (of compact support, though the literature doesn't mention that part) is

$$D_c^{la}(X, L) = LA(X, L)_b^\vee.$$

We have a continuous bijection

$$LA(X, L)_b^\vee \to \varprojlim_{\mathscr{A}} \bigoplus_{(U_i, \varphi_i) \in \mathscr{A}} \mathscr{O}(D(1)^n)_b^\vee.$$

If X is compact, $D_c^{la}(X, L)$ is then a nuclear Fréchet space. We can also define

$$D_c^{cts}(X, L)_{s/b} = C(X, L)_{s/b}^\vee.$$

These are called "measures" (of compact support).

Example 16.2.1. For any $x \in X$, we have the evaluation measure $\delta_x \in D_c^{cts}(X, L)$ given by $\delta_x(f) = f(x)$.

Lemma 16.2.2. *If $X = \coprod_{i \in I} X_i$ where $X_i \subset X$ is compact open, we can write*

$$C(X, V) = \prod_I C(X_i, V)$$

$$LA(X, V) = \prod_I LA(X_i, V)$$

$$D_c^{cts}(X, L)_{b/s} = \bigoplus_I D_c^{cts}(X_i, L)_{b/s}$$

$$D_c^{la}(X, L) = \bigoplus_I D_c^{la}(X_i, L).$$

Corollary 16.2.3. *(1) $D_c^{la}(X, L)$ and $D_c^{cts}(X, L)_b$ are bornological and barrelled.*
(2) $\langle \delta_x \mid x \in X \rangle_L$ is dense in $D_c^{la}(X, L)$ and in $D_c^{cts}(X, L)_s$.

Proof. (1) The properties of being bornological and barrelled are preserved under $\bigoplus$.
(2) Reduce to the case where X is compact. If the claim is false, then by Hahn-Banach, there is $0 \neq \varphi : D \to L$ continuous linear such that $\varphi(\delta_x) = 0$ for all $x \in X$. But in both cases under consideration dualization is reflexive (dualizing again gives back what we started with), so there is $f \in LA(X, L)$ or $C(X, L)_s$ such that $\varphi(\delta) = \delta(f)$ for all δ. Then so $f(x) = 0$ for all $x \in X$, so $f \equiv 0$, so $\varphi \equiv 0$.

□

We have maps

$$D_c^{la}(X, L) \otimes_\iota LA(X, V) \to V$$
$$D_c^{cts}(X, L) \otimes_\iota C(X, V) \to V$$

for V complete. This is not obvious because the definition of LA is complicated, containing both a direct limit and a product, which don't commute. To construct the map for LA (C is easier and left as an exercise), we start with the natural map

$$\mathscr{O}(D(1)^n)_b^\vee \otimes_\pi (\mathscr{O}(D(1)^n)\widehat{\otimes}_\pi V_i) \to V_i$$

where we can put $\widehat{\otimes}$ instead of $\otimes$ because V is complete. This gives a map

$$\mathscr{O}(D(1)^n)_b^\vee \otimes_\pi \prod_{(U_j,\varphi_j,V_j\hookrightarrow V)\in\mathscr{A}^*} \mathscr{O}(D(1)^n)\widehat{\otimes}_\pi V_j \to V_i \hookrightarrow V$$

where $\mathscr{O}(D(1)^n)_b^\vee$ acts by pairing with the ith factor. Using the general fact that

$$(\varinjlim W_i) \otimes_\iota U \cong \varinjlim (W_i \otimes_\iota U),$$

we can combine these to get

$$\left(\bigoplus_{(U_i,\varphi_i)} \mathscr{O}(D(1)^n)_b^\vee\right) \otimes_\iota \left(\prod_i \mathscr{O}(D(1)^n)\widehat{\otimes}_\pi V_i\right) \to V.$$

But we also have a map

$$D_c^{la}(X, L) \to \varprojlim_{\mathscr{A}} \bigoplus_{(U_i,\varphi_i)\in\mathscr{A}} \mathscr{O}(D(1)^n)_b^\vee \to \bigoplus_{(U_i,\varphi_i)} \mathscr{O}(D(1)^n)_b^\vee$$

so putting in $D_c^{la}(X, L)$ and taking limits, we get

$$\varinjlim_{\mathscr{A}^*} \left(D_c^{la}(X, L) \otimes_\iota \left(\prod_i \mathscr{O}(D(1)^n)\widehat{\otimes}_\pi V_i\right)\right) \to V$$
$$D_c^{la}(X, L) \otimes_\iota LA(X, V) \to V$$

which is our desired map.

16.3. **Maps between manifolds, products, and tangent spaces.**

Definition 16.3.1. Let X, Y be locally analytic manifolds. We say that $f : X \to Y$ is locally analytic if for one (therefore all) atlases $\mathscr{B} = \{(V_j, \psi_j)\}$ for Y, the maps

$$\psi_j \circ f : f^{-1}V_j \to \mathbb{Q}_p^n$$

are locally analytic.

Definition 16.3.2. If $f : X \to Y$ is locally analytic, we have a pullback map

$$f^* : LA(Y, V) \to LA(X, V)$$
$$g \mapsto g \circ f.$$

We can similarly define $f^* : C(Y, V) \to C(X, V)$ (even if f is just continuous), $f_* : D_c^{la}(X, L) \to D_c^{la}(Y, L)$, and $f_* : D_c^{cts}(X, L)_{b/s} \to D_c^{cts}(Y, L)_{b/s}$.

If $j : U \hookrightarrow X$ is an open set, we also have the extension-by-0 maps

$$j_! : LA(U,V) \to LA(X,V), \quad j_! : C(U,V) \to C(X,V)$$

where a function on U is extended to be identically 0 outside U, and similarly the restriction maps

$$D_c^{la}(X,L) \to D_c^{la}(U,L), \quad D_c^{cts}(X,L)_{s/b} \to D_c^{cts}(U,L)_{s/b}$$

given by $\delta \mapsto \delta|_U$.

Lemma 16.3.3. *(1) We have an isomorphism*

$$\mathscr{O}(D(1)^n)\widehat{\otimes}_\pi \mathscr{O}(D(1)^m) \xrightarrow{\sim} \mathscr{O}(D(1)^{n+m}).$$

(2) We have maps

$$C(X,L) \otimes_\pi C(Y,L) \to C(X \times Y, L)$$

(which is a isomorphism if X, Y compact) and

$$LA(X,L) \otimes_\iota LA(Y,L) \to LA(X \times Y, L)$$

such that $(f \otimes g)(x,y) = f(x)g(y)$.

(3) We have a map

$$\mathscr{O}(D(1)^n)_b^\vee \widehat{\otimes}_\pi \mathscr{O}(D(1)^m)_b^\vee \to \mathscr{O}(D(1)^{n+m})_b^\vee$$

and if X, Y are compact then we have a map

$$C(X,L)_b^\vee \widehat{\otimes}_\pi C(Y,L)_b^\vee \to C(X \times Y, L)_b^\vee.$$

(4) We have maps

$$\widehat{\otimes} : D_c^{la}(X,L) \otimes_\iota D_c^{la}(Y,L) \to D_c^{la}(X \times Y, L)$$
$$\widehat{\otimes} : D_c^{cts}(X,L)_{b/s} \otimes_\iota D_c^{cts}(Y,L)_{b/s} \to D_c^{cts}(X \times Y, L)_{b/s}$$

such that $(\delta \widehat{\otimes} \epsilon)((x,y) \mapsto f(x)g(y)) = \delta(f)\epsilon(g)$.

Proof. (1) This follows from $\perp_{I,0} (L, |\cdot|)\widehat{\otimes}_\pi V = \perp_{I,0} (V, \|\cdot\|_V)$.

(2) Exercise.

(3) This follows from the general fact that $V_b^\vee \otimes_\pi W_b^\vee \to (V \widehat{\otimes}_\pi W)_b^\vee$.

(4) For the LA case (as usual, C is easier) we have maps

$$\begin{aligned}
D_c^{la}(X,L) \otimes_\iota D_c^{la}(Y,L) &\to \left(\varprojlim_{\mathscr{A}} \bigoplus_i \mathscr{O}(D(1)^n)_b^\vee\right) \otimes_\iota \left(\varprojlim_{\mathscr{B}} \bigoplus_j \mathscr{O}(D(1)^m)_b^\vee\right) \\
&\to \varprojlim_{\mathscr{A},\mathscr{B}} \left(\left(\bigoplus_i \mathscr{O}(D(1)^n)_b^\vee\right) \otimes_\iota \left(\bigoplus_j \mathscr{O}(D(1)^m)_b^\vee\right)\right) \\
&= \varprojlim_{\mathscr{A},\mathscr{B}} \bigoplus_{i,j} (\mathscr{O}(D(1)^n)_b^\vee \otimes_\iota \mathscr{O}(D(1)^n)_b^\vee) \\
&\to \varprojlim_{\mathscr{A},\mathscr{B}} \bigoplus_{i,j} \mathscr{O}(D(1)^{n+m})_b^\vee = \varprojlim_{\mathscr{A},\mathscr{B}} \left(\prod_{i,j} \mathscr{O}(D(1)^{n+m})\right)_b^\vee.
\end{aligned}$$

The last space takes a map from $D_c^{la}(X,Y)$ which is a continuous bijection, but not necessarily a homeomorphism. That gives us a linear map of the desired form; now we have to back up and check that it is separately continuous. So we need to check that if $\delta \in D_c^{la}(X, L)$, then

$$(-\widehat{\otimes}\delta) : D_c^{la}(Y, L) \to D_c^{la}(X \times Y, L)$$

is continuous. Dualizing, this becomes a map $LA(X \times Y, L) \to LA(Y, L)$. The source is a limit of terms of the form $\prod_{i,j} \mathscr{O}(D(1)^{n+m})$, whose map to $LA(Y, L)$ factors through $\prod_j \mathscr{O}(D(1)^m)$, so we need the jth factor of this map to be continuous for all $\mathscr{B}$ and j. But it turns out (check yourself) that the jth factor can be further rewritten as

$$\bigoplus \delta_{\mathscr{A},i} \otimes \mathrm{id} : \delta_{\mathscr{A},i} : \prod_i \mathscr{O}(D(1)^n)\widehat{\otimes}_\pi \mathscr{O}(D(1)^m) = \prod_i \mathscr{O}(D(1)^{n+m}) \to \mathscr{O}(D(1)^m)$$

which is continuous since $\delta_{\mathscr{A}} = \bigoplus \delta_{\mathscr{A},i}$ is continuous.

□

Definition 16.3.4. If X has an atlas $\mathscr{A}$ with a chart (U_i, φ_i), so that $\varphi_i : U_i \hookrightarrow \mathbb{Q}_p^n$, and $x \in U_i$, we define

$$T_{x,(U_i,\varphi_i)}X = \mathbb{Q}_p^n.$$

If (U_j, φ_j) is a compatible chart also containing x, we get a map

$$d(\varphi_i \circ (\varphi_j)^{-1})_x : T_{x,(U_j,\varphi_j)}X \xrightarrow{\sim} T_{x,(U_i,\varphi_i)}X$$

by differentiating

$$\varphi_i \circ (\varphi_j)^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j).$$

Identifying all the $T_{x,(U_i,\varphi_i)}X$s via these maps gives us T_xX, the *tangent space* to X at x. If we have a map $f : X \to V$ where V is a LCTVS, or a locally analytic map $f : X \to Y$, we get corresponding differential maps $df_x : T_xX \to V$ and $df_x : T_xX \to T_{f(x)}Y$.

16.4. **Locally analytic groups.** A locally analytic group G is a locally analytic manifold plus an identity element $e \in G$ and locally analytic inverse and multiplication maps $-1 : G \to G$, $\mu : G \times G \to G$, satisfying the group axioms.

Example 16.4.1. $GL_n(\mathbb{Q}_p)$, or $G(L)$ for any reductive group G over L.

We define $\mathrm{Lie}(G) := T_eG$. There is a bracket

$$[-,-] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$$

making $\mathrm{Lie}(G)$ a Lie algebra, defined as follows. If (U, φ) is a chart with $e \in U$ and $\varphi(e) = 0$, and $V \subset U$ is an open set such that $V^2 \subset U$, then

$$\begin{aligned} m : \varphi(V) \times \varphi(V) &\to \mathbb{Q}_p^n \\ (x, y) &\mapsto \varphi(\varphi^{-1}(x)\varphi^{-1}(y)) \end{aligned}$$

is locally analytic, so given by a power series near 0. If $m = \sum m_{i,j}$, where $m_{i,j}$ is the bidegree i, j term in the power series, then we define

$$[x, y] := m_{1,1}(x, y) - m_{1,1}(y, x) \in T_eG.$$

This is independent of choices.

If $U \subset \mathrm{Lie}(G)$ is a sufficiently small neighborhood of 0, then there is an exponential map $\exp : U \to G$ with $\exp(0) = e$, $d\exp = \mathrm{id} : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$, and $\exp((a + b)x) =$

$\exp(ax)\exp(bx)$ for all $x \in U$ and $a, b \in L$ sufficiently small. Any two such maps agree near 0. Near e, exp has an inverse called log. If $x, y \in \mathrm{Lie}(G)$, we have

$$x + y = \lim_{a\to 0} \frac{\log(\exp(ax)\exp(ay))}{a}$$
$$[x, y] = \lim_{a\to 0} \frac{\log(\exp(ax)\exp(ay)\exp(-ax)\exp(-ay))}{a^2}.$$

The following lemma can be found in Bourbaki.

Lemma 16.4.2. *There are charts (H_m, φ_m) such that*

- *H_m is an open compact subgroup of G,*
- *$\{H_m\}$ is a basis of neighborhoods of $e \in G$,*
- *the group structure on H_m is* analytic*, that is, there are continuous algebra homomorphisms*

$$\mu_m : \mathscr{O}(D(1)^n) \to \mathscr{O}(D(1)^n)\widehat{\otimes}_\pi \mathscr{O}(D(1)^n)$$
$$i_m : \mathscr{O}(D(1)^n) \to \mathscr{O}(D(1)^n)$$

such that for any $g_1, g_2 \in H_m$ and $f \in \mathscr{O}(D(1)^m)$,

$$f(\varphi_m(g_1g_2)) = \mu_m(f)(\varphi_m(g_1), \varphi_m(g_2))$$
$$f(\varphi_m(g^{-1})) = i_m(f)(\varphi_m(g)).$$

Definition 16.4.3. The convolution product

$$* : D_c^{la}(G, L) \otimes_\iota D_c^{la}(G, L) \to D_c^{la}(G, L)$$

is given by the composition

$$D_c^{la}(G, L) \otimes_\iota D_c^{la}(G, L) \to D_c^{la}(G \times G, L) \xrightarrow{m_*} D_c^{la}(G, L).$$

17. 3/3/20: Distributions and representations for locally analytic groups

Last time, we started discussing locally analytic p-adic groups. If G is such a group, we saw that we had maps $\exp : \mathrm{Lie}(G) \to G$ and $\log : G \to \mathrm{Lie}(G)$, and that we can find p-adic analytic subgroups $H_m \subset G$, with $H_m \cong \mathbb{Z}_p^{\oplus n}$, on which the group structure is given by convergent power series.

Example 17.0.1. In $G = GL_n(\mathbb{Q}_p)$, we have neighborhoods $1 + p^N M_{n\times n}(\mathbb{Z}_p) \cong \mathbb{Z}_p^{n^2}$ of the identity, with the following group structure: matrices in $1 + p^N M_{n\times n}(\mathbb{Z}_p)$ multiply as

$$(1 + p^N A)(1 + p^N B) = 1 + p^N(A + B + p^N AB)$$

so the group structure on $\mathbb{Z}_p^{n^2}$ is given by

$$(A, B) \mapsto A + B + p^N AB.$$

Fact: if G is compact, then $G \cong \varprojlim_{H<G} G/H$, so

(1) G is profinite as a topological group, and
(2) G contains an open subgroup H such that H is pro-p.

For example, $G = GL_n(\mathbb{Z}_p)$ is compact and contains the open pro-p subgroup $1 + pM_{n\times n}(\mathbb{Z}_p)$.

17.1. **Distributions on locally analytic groups.** Last time, we defined algebras $D_c^{la}(G,L)$ and $D_c^{cts}(G,L)_{b/s}$, with multiplication given by convolution, that is,

$$*: D(G,L) \otimes_\iota D(G,L) \xrightarrow{\hat{\hat{\otimes}}} D(G \times G, L) \xrightarrow{m_*} D(G,L).$$

They also have an anti-involution

$$\dagger : D(G,L) \xrightarrow{(-1)_*} D(G,L)$$

(where $(-1) : G \to G$ is $g \mapsto g^{-1}$). We have $\delta_g \delta_h = \delta_{gh}$ and $\delta_g^\dagger = \delta_{g^{-1}}$. Also, $*$ is associative and distributive over addition, with identity δ_1, and we have the identity $(\delta * \epsilon)^\dagger = \epsilon^\dagger * \delta^\dagger$.

If G is compact, $D_c^{la}(G,L)$ is a Fréchet algebra, and $D_c^{cts}(G,L)_b$ is a Banach algebra.

We have a map

$$\begin{aligned} \mathrm{Lie}(G) &\to D_c^{la}(G,L) \\ x &\mapsto \lim_{a \to 0} \frac{\delta_{\exp(ax)} - 1}{a} \end{aligned}$$

which takes $[x,y]$ to $x * y - y * x$, hence extends to a map from the universal enveloping algebra $U(\mathrm{Lie}(G))$.

If $H < G$ is an open subgroup then

$$D_c^{la/cts}(G,L) = \bigoplus_{g \in G/H} \delta_g * D_c^{la/cts}(H,L).$$

(Note that $\delta_g * D_c^{la/cts}(H,L) = D_c^{la/cts}(gH,L)$.) This is useful because it allows one to reduce questions to the case where G is compact.

If G is compact, hence profinite, we can define the completed group algebra

$$\mathscr{O}[\![G]\!] = \varprojlim_{H \lhd G \text{ open normal}} \mathscr{O}[G/H],$$

with the inverse limit topology, and

$$L[\![G]\!] = \mathscr{O}[\![G]\!] \otimes_{\mathscr{O}} L = \bigcup_n \pi^{-n} \mathscr{O}[\![G]\!]$$

(note that this means we are requiring bounded denominators), with the direct limit topology. Usually, $\mathscr{O}[\![G]\!]$ is not open in $L[\![G]\!]$ (certainly not if G is infinite).

Lemma 17.1.1. *If G is compact, $D^{cts}(G,L) \cong L[\![G]\!]$. The strong topology on $D^{cts}(G,L)$ corresponds to the normable topology on $L[\![G]\!]$ with $\mathscr{O}[\![G]\!]$ as a bounded open lattice (*not *with its usual topology, but with the p-adic topology). The weak topology on $D^{cts}(G,L)$ corresponds to the usual topology on $L[\![G]\!]$.*

Proof. Let $\mu \in \mathscr{O}[\![G]\!]$, say

$$\mu = \varprojlim_{H \lhd G} \sum_{g \in G/H} \mu_{gH} gH$$

where $\mu_{gH} \in \mathscr{O}$ is such that if $H_1 \subset H_2$ then $\mu_{gH_2} = \sum_{h \in H_2/H_1} \mu_{ghH_1}$. Given $f \in C(G,L)$, we would like to "integrate f against μ".

Certainly for each H the sum $\sum_{g \in G/H} \mu_{gH} f(g)$ makes sense. We claim that this is a Cauchy sequence as H and the choice of coset representatives vary. Proof: given $\epsilon > 0$, there is an open compact subgroup $H \lhd G$ such that f varies by less than ϵ on each coset of H. If $H > H_1 > H_2$, then

$$\begin{aligned}\sum_{g\in H_1\backslash G} \mu_{gH_1} f(g) - \sum_{g\in H_2\backslash G} \mu_{gH_2} f(g) &= \sum_{g\in H_1\backslash G}\sum_{h\in H_2\backslash H_1} \mu_{ghH_2} f(g) - \sum_{g'\in H_2\backslash G} \mu_{g'H_2} f(g') \\ &= \sum_{g\in H_1\backslash G}\sum_{h\in H_2\backslash H_1} \mu_{ghH_2}(f(g)-f(gh)).\end{aligned}$$

Here $|\mu_{ghH_2}| \le 1$ and $|f(g) - f(g'h)| \le \epsilon$, so the whole thing has norm $\le \epsilon$.

So the sums $\sum \mu_{gH} f(g)$ approach a limit $\mu(f)$. Furthermore, if $\|f\| \le \epsilon$, then $|\mu(f)| \le \epsilon$, so $\mu \in D^{cts}(G, L)$. This gives us a map

$$\mathscr{O}[\![G]\!] \to D_c^{cts}(G, L)$$

which extends naturally to $L[\![G]\!]$. If $\delta \in D_c^{cts}(G, L)$, then it has an inverse image

$$\mu = \varprojlim \sum_{g\in H\backslash G} \delta(\mathbb{1}_{gH}) gH \in L[\![G]\!]$$

(because $\mathbb{1}_{gH}$ has bounded denominator as g, H vary).

The strong topology on $D^{cts}(G, L)$ has open unit ball consisting of the distributions sending $\mathcal{O}$-valued functions into $\mathcal{O}$. This just means that the distribution sends all functions of the form $\mathbb{1}_{gH}$ to $\mathcal{O}$, which is to say that it is in $\mathcal{O}[[G]]$.

For the weak topology, take a typical open set $\Lambda(\{f\}, \mathscr{O})$. If N is such that $\|f\| < |\pi|^{-N}$ and $H \lhd G$ is an open normal subgroup such that f varies by less than 1 on the cosets of H (so if g_1, g_2 are in the same coset then $|f(g_1) - f(g_2)| \le 1$), then the unit ball $\Lambda(\{f\}, \mathscr{O})$ contains the preimage of $\pi^N \mathscr{O}[G/H] \subset \mathscr{O}[G/H]$. On the other hand, that preimage contains $\Lambda(\{\mathbb{1}_{gH}\}_{g\in G/H}, \pi^N \mathscr{O})$. So the two topologies are the same. □

Theorem 17.1.2. *Let G be compact and locally analytic.*

1. *(Lazard) $\mathscr{O}[\![G]\!]$ and $L[\![G]\!]$ are noetherian.*
2. *(Schneider-Teitelbaum [9]) $D_c^{la}(G, L)$ is a Fréchet-Stein algebra and*

$$(D_c^{cts}(G, L) = L[\![G]\!]) \to D_c^{la}(G, L)$$

is faithfully flat.

17.2. **Representations of locally analytic groups.** G is still locally analytic.

Definition 17.2.1. By a Banach representation of G we mean a Banach space V/L with an action of G (in particular a map $G \times V \to V$) which is separately continuous. That is, for all $g \in G$, the map $g : V \to V$ is in $\mathscr{L}(V, V)$, and for all $x \in V$, the map $G \to V$, $g \mapsto gx$, is continuous. One can check that this is equivalent to saying that the map $\pi : G \to \mathscr{L}_s(V, V)$ is continuous (and, of course, satisfies $\pi(gh) = \pi(g)\pi(h)$).

Lemma 17.2.2. *If V is a Banach representation of G, then*

1. *$G \times V \to V$ is actually jointly continuous. That is, $V \to C(G, V)$ is continuous, or equivalently, $G \to \mathscr{L}_b(V, V)$ is continuous.*
2. *There is a unique jointly continuous action of $D_c^{cts}(G, L)_b$ on V such that $\delta_g x = gx$.*

Proof. (1) Let $H < G$ be any compact open subgroup. Then the image of the continuous map $H \to \mathscr{L}_s(V, V)$ is compact, hence bounded. By Banach-Steinhaus, the image of H is equicontinuous, that is, for any open lattice $\Lambda \subset V$, there is a second open lattice $M \subset V$ such that $hM \subset \Lambda$ for all $h \in H$. That is, the image of $H \times M$ is in Λ. This gives us joint continuity at $(1,0)$, which is all we need.

(2) The natural map $V \to C(G,V)$ takes x to $(g \mapsto gx)$. By tensoring, this gives us a map

$$D_c^{cts}(G,L)_b \otimes_\pi V \to D_c^{cts}(G,L)_b \otimes_\pi C(G,V) \to V.$$

It is easy to check that $\delta_g x = gx$ and that $(\delta * \epsilon)(x) = \delta(\epsilon(x))$ (check the latter on δ-functions, which are dense in $D_c^{cts}(G,L)_b$).

□

Remark 17.2.3. G also acts on $V_b^\vee$, which is also a Banach space, by $(gf) = f \circ g^{-1}$, and this action is separately, therefore jointly, continuous. Similarly, $D_c^{cts}(G,L)$ acts on $V_b^\vee$ by $\delta f = f \circ \delta^\dagger$.

Definition 17.2.4. We call a Banach space representation V of G *admissible* if there is a G-invariant bounded open lattice $\Lambda \subset V$ such that if $H < G$ is an open subgroup, then $\mathrm{Hom}_{\mathscr{O}}((V/\Lambda)^H, L/\mathscr{O})$ is a finitely generated $\mathscr{O}$-module. That is,

$$(V/\Lambda)^H \cong (L/\mathscr{O})^h \oplus (\text{a finite cardinality } \mathscr{O}\text{-module}),$$

where $h < \infty$. This property of $(V/\Lambda)^H$ is sometimes called being "cofinite" over $\mathscr{O}$.

Let $\mathscr{M}_{\mathscr{O}}$ be the category of compact Hausdorff linearly topologized torsion-free $\mathscr{O}$-modules ("linearly topologized " means the topology has a basis consisting of translates of $\mathscr{O}$-submodules). The reference for the following results is Schneider-Teitelbaum ([8] and [7]).

Lemma 17.2.5. *The objects of $\mathscr{M}_{\mathscr{O}}$ are all of the form $\prod_I \mathscr{O}$ for some index set I (with the product topology).*

Lemma 17.2.6. *Let Ban_L be the category of Banach spaces over L. We have a contravariant functor*

$$\begin{aligned} \mathscr{M}_{\mathscr{O}} &\to Ban_L \\ M &\mapsto Hom_{\mathscr{O}}^{cts}(M,L) \\ \prod_I \mathscr{O} &\mapsto \perp_{I,0} (L, |\cdot|). \end{aligned}$$

Proof. To be precise about the last line, the isomorphism between $\mathrm{Hom}_{\mathscr{O}}^{cts}(\prod_I \mathscr{O}, L)$ and $\perp_{I,0}(L,|\cdot|)$ is given by $f \mapsto (f(e_i))$, where e_i is the ith standard basis element. This map makes sense because if f is continuous and $\mathscr{O}$-linear then $f^{-1}(\pi^N\mathscr{O})$ contains $\prod_{i \notin J} \mathscr{O}$ for some $J \subset I$ with $\#J < \infty$, so for $i \notin J$, $f(e_i) \in \pi^N \mathscr{O}$. □

This functor induces an anti-equivalence of categories

$$\mathscr{M}_{\mathscr{O}} \otimes L \to Ban_L$$

(where $\mathscr{M}_{\mathscr{O}} \otimes L$ has the same objects as $\mathscr{M}_{\mathscr{O}}$ but the homomorphisms are tensored up to L), whose inverse functor takes $V \in Ban_L$ to the unit ball in $V^\vee$ considered with the weak topology.

Similarly we can define $\mathscr{M}_{\mathscr{O}[\![G]\!]}$, the category of topological $\mathscr{O}[\![G]\!]$-modules (where $\mathscr{O}[\![G]\!]$ has the weak topology) which are compact, Hausdorff, $\mathscr{O}$-torsion-free, and linearly topologized over $\mathscr{O}$. Then we get a functor

$$\begin{aligned} \mathscr{M}_{\mathscr{O}[\![G]\!]} \otimes L &\to \{\text{Banach space representations of } G\} \\ M &\mapsto \operatorname{Hom}_{\mathscr{O},cts}(M, L). \end{aligned}$$

Definition 17.2.7. An $\mathscr{O}[\![G]\!]$- or $L[\![G]\!]$- module M (no topology needed) is called *coadmissible* if for one (hence every) compact open subgroup $H < G$, M is finitely generated over $\mathscr{O}[\![H]\!]$ or $L[\![H]\!]$.

Lemma 17.2.8. *(1) $Mod^{coad}_{\mathscr{O}[\![G]\!]} \otimes L = Mod^{coad}_{L[\![G]\!]}$.*

(2) Any coadmissible $\mathscr{O}[\![G]\!]$-module has a canonical topology, that is, a unique Hausdorff topology such that $\mathscr{O}[\![H]\!] \times M \to M$ is continuous for one (hence every) open compact subgroup H. If M is $\mathscr{O}$-torsion-free, this makes M an object in $\mathscr{M}_{\mathscr{O}[\![G]\!]}$.

(3) There is an anti-equivalence of categories

$$\begin{aligned} \{\textit{coadmissible } L[\![G]\!]\textit{-modules}\} &\to \{\textit{admissible Banach space representations of } G\} \\ M &\mapsto Hom_{L,cts}(M, L) \\ V_s^\vee &\leftarrow\!\shortmid V. \end{aligned}$$

Next time we'll talk about locally analytic representations, and then the eigencurve!

18. 3/5/20: Locally analytic representations and more on compact operators

Last time, we defined admissible Banach representations V of G. We saw that we could equivalently look at V in terms of the coadmissible $L[\![G]\!]$-module $V_s^\vee$.

18.1. Locally analytic representations.

Definition 18.1.1. A *locally analytic representation* V of G is a Hausdorff barrelled LCTVS V (e.g. Fréchet or compact type) together with a separately continuous action $G \times V \to V$ such that for all $x \in V$ the map $G \to V$ given by $g \mapsto gx$ is locally analytic.

Lemma 18.1.2. *(1) $G \times V \to V$ is jointly continuous.*

(2) The map $D_c^{la}(G, L) \times V \to V$ given by $(\delta, x) \mapsto \delta(g \mapsto gx)$ is a separately continuous action on V.

(3) If $X \in D_c^{la}(G, L)$ is in the image of $\mathrm{Lie}(G)$, it acts on $x \in V$ via

$$Xx = \lim_{a \to 0} \frac{\exp(aX)x - x}{a} = \frac{d}{dt} \exp(tX)x.$$

(4) $D_c^{la}(G, L) \to \mathscr{L}_b(V, V)$ is continuous.

Proof. (1) Same proof as before—the conditions suffice to make Banach-Steinhaus apply.

(2) Since $\delta_g x = gx$, we can check that $(\delta * \epsilon)(x) = \delta(\epsilon(x))$ by continuity, so this map is an action. Continuity in $D_c^{la}(G, L)$ follows from $D_c^{la}(G, L) \times LA(G, V) \to V$ being separately continuous by a similar argument as before. Let $\theta : D_c^{la}(G, V) \to \mathscr{L}_s(V, V)$ be the resulting continuous map.

Continuity in V is slightly more complicated because it's not clear that $V \to LA(G, V)$ is continuous. To check it, reduce to the case where G is compact. Fix $\delta \in D_c^{la}(G, V)$. We need to check that the map $V \to V$ given by $x \mapsto \delta x$ is continuous.

Because $D_c^{la}(G, V)$ is metrizable (indeed Fréchet), we can find a sequence $\delta_n \to \delta$ such that δ_n is a finite sum of δ-functions; for these we know that the map $V \to V$

given by $x \mapsto \delta_n x$ is continuous for all n. Since $\delta_n \to \delta$, $\{\delta_n\}$ is bounded, and since $\theta : D_c^{la}(G, V) \to \mathscr{L}_s(V, V)$ is continuous, $\{\theta(\delta_n)\}$ is also bounded. Since V is barrelled, the $\{\theta(\delta_n)\}$ are uniformly continuous. So for any open lattice $\Lambda \subset V$, there is an open lattice $M \subset V$ with $\delta_n M \subset \Lambda$ for all n. So $\delta M \subset \Lambda$ (since Λ is closed), and therefore δ is continuous.

(3) This is because we can write $\exp(X)x = \sum_{n=0}^{\infty} \frac{X^n x}{n!}$ for $X \in \mathrm{Lie}(G)$ sufficiently small.

(4) We know that the composition $D_c^{la}(G, L) \xrightarrow{\theta} \mathscr{L}_b(V, V) \to \mathscr{L}_s(V, V)$ is continuous. Since $D_c^{la}(G, L)$ is bornological, it suffices to check that for any bounded $B \subset D_c^{la}(G, L)$, $\theta(B)$ is bounded. That is, for all bounded $C \subset V$ and open lattices $V \subset \Lambda$, we want to find $a \in L^\times$ such that $\theta(B) \subset a\Lambda(C, \Lambda)$. But we know that $\theta(B) \subset \mathscr{L}_s(V, V)$ is bounded, so there is a lattice $M \subset V$ such that $(\theta(B))(M) \subset \Lambda$ (by equicontinuity, since V is barrelled). Then we can find $b \in L^\times$ with $C \subset bM$. Then $(\theta(B))(C) \subset b\Lambda$, so $\theta(B) \subset \Lambda(C, b\Lambda) = b\Lambda(C, \Lambda)$, so we can take $a = b$.

□

Lemma 18.1.3. *There is an anti-equivalence of categories between {locally analytic representations of G on V of compact type} and {separately continuous actions of $D_c^{la}(G, L)$ on nuclear Fréchet spaces}, sending V to $V_b^\vee$, where the action of $D_c^{la}(G, L) \to \mathscr{L}_b(V, V) \cong \mathscr{L}_b(V_b^\vee, V_b^\vee)$ on the latter is $\delta(f) = f \circ \delta^\dagger$.*

Remark 18.1.4. If G is compact, the action of $D_c^{la}(G, L)$ on $V_b^\vee$ is necessarily jointly continuous.

Definition 18.1.5. (1) A coadmissible $D_c^{la}(G, L)$-module is a module M such that M is coadmissible over $D_c^{la}(H, L)$ for one (hence every) compact open subgroup $H \subset G$.

(Recall that $D_c^{la}(H, L)$ is a Fréchet-Stein algebra, but not usually noetherian, so coadmissible means the following: we can write $D_c^{la}(H, L)$ as the $\varprojlim$ of a sequence $\cdots \to A_2 \to A_1$ where the maps are flat and the A_i are noetherian Banach algebras, and a module M is coadmissible if it is of the form $M = \varprojlim M_i$ where M_i is a finitely generated module over A_i and $A_{i-1} \otimes_{A_i} M_i = M_{i-1}$. Note that we are not sure whether if $M/D_c^{la}(H, L)$ is such that $A_i \otimes_{D_c^{la}(H,L)} M$ is finitely generated for all i, this should imply that M is coadmissible.)

Also recall that any such M has a canonical topology.

(2) A locally analytic representation V of G is called *admissible* if V is of compact type and $V_b^\vee$ is coadmissible over $D_c^{la}(G, L)$.

Corollary 18.1.6. *There is an anti-equivalence of categories between {admissible locally analytic representations of G} and {coadmissible $D_c^{la}(G, L)$-modules} taking V to $V^\vee$.*

Lemma 18.1.7. *The admissible locally analytic representations of G form an abelian category in which*

- *images and kernels take the subspace topologies,*
- *any morphism is strict with closed image (meaning that given $\psi : V \to W$, the bijection $V/\ker\psi \to \operatorname{im}\psi$ is a homeomorphism between the quotient and subspace topologies), and*
- *a closed invariant subspace of an admissible locally analytic representation is again admissible.*

Suppose V is a Banach representation of G. Let

$$V^{la} := \{x \in V \mid G \to V : g \mapsto gx \text{ is locally analytic}\} \subset V.$$

There are different ways of topologizing this in the literature (the subspace topology is a bad idea). We will follow Emerton (see [5]). Let $H \subset G$ be a compact open subgroup with analytic multiplication (i.e. $H \cong D(1)^n$ and multiplication is given by convergent power series). Let

$$\begin{aligned} V^{H-an} &:= \{x \in V \mid h \mapsto hx \text{ is analytic on } H\} \\ &= \{x \in V \mid h \mapsto hx \text{ is analytic on } gH \text{ for all } g \in G\} \\ &\cong \bigoplus_{g \in G/H} \mathscr{O}(D(1)^n) \widehat{\otimes}_\pi V. \end{aligned}$$

Then V^{H-an} is naturally a LCTVS and we have $V^{la} = \bigcup_H V^{H-an}$, and if $H \supset H'$ we have a natural map $V^{H-an} \to V^{H'-an}$. So we can give $V^{la} = \varinjlim V^{H-an}$ the direct limit topology. (This is the same as the Schneider-Teitelbaum definition if V is admissible.)

Lemma 18.1.8. *Suppose V is an admissible Banach representation of G.*

(1) $V^{la} \subset V$ is dense.
(2) V^{la} is an admissible locally analytic representation of G, and $(V^{la})_b^\vee$ is finitely generated over $D_c^{la}(H, L)$ for all compact open subgroups $H \subset G$ (this is sometimes called being "strictly admissible").
(3) We have $(V^{la})_b^\vee = D_c^{la}(G, L) \otimes_{L[\![G]\!]} V_b^\vee$.
(4) The functor from admissible Banach representations to admissible locally analytic representations taking V to V^{la} is exact.

18.2. **More on compact operators.** We have seen, if $u \in \mathscr{C}\mathscr{C}_A(M, M)$ where A is a Banach algebra and M is a Banach projective Banach A-module, how to define $\det(1 - uT)$. If $\det(1 - uT) = Q(T)P(T)$ where $Q \in A[T]$, $P \in \mathscr{O}(\mathbb{A}^1_A)$, $Q(0) = P(0) = 1$, Q has unit leading coefficient, and $(P, Q) = \mathscr{O}(\mathbb{A}^1_A)$, we have seen that we can decompose $M = M_1 \oplus M_2$ so that $Q^*(u) = 0$ on M_1 and $Q^*(u)$ is an isomorphism on M_2.

Now let A be a noetherian Banach algebra and $\|\cdot\|$ a defining norm with $\|ab\| \le \|a\|\|b\|$. If $f(T) = f_0 + f_1 T + f_2 T^2 + \cdots \in A[\![T]\!]$, recall that we obtain the Newton polygon of f as the lower convex hull of the points $(n, -\log_p \|f_n\|)$ in the xy-plane. We refer to the slopes of the sides of the Newton polygon as "the slopes of f". Then $f(T) \in \mathscr{O}(\mathbb{A}^1_A)$ if and only if the slopes of f go to ∞. Furthermore, we have seen that if $A = L$ and $f \in \mathscr{O}(\mathbb{A}^1)$, h is a slope of f if and only if there is $\alpha \in \overline{L}$ such that $f(\alpha) = 0$ and $|\alpha| = p^h$.

We say that f "has slopes $\le h$" or "$> h$" if all slopes of f are $\le h$ or $> h$. We say that an A-module M has a slope $\le h$ decomposition if we can decompose

$$M = M_{\le h} \oplus M_{>h}$$

so that

(1) if $x \in M_{\le h}$, there is $Q \in A[T]$ with unit leading coefficient (we call such Q "multiplicative") and all slopes $\le h$ such $Q^*(u)x = 0$ (recall by definition $Q^*(T) = T^{\deg Q} Q(1/T)$),
(2) $M_{\le h}$ is finitely generated over A, and
(3) for any multiplicative $Q \in A[T]$ with slopes $\le h$, $Q^*(u)|_{M > h}$ is an isomorphism.

If M is a Banach space, $A = L$, and u is completely continuous, then there is a slope $\le h$ decomposition of M for all h.

Remark 18.2.1. (1) If M has a slope $\le h$ decomposition, then it is unique.

(2) Let $D_{\leq h} = \{Q \in A[T] \mid Q$ multiplicative and has slopes $\leq h\} \subset A[T]$. Note that $D_{\leq h}$ contains 1 and is closed under multiplication. Then an $A[T]$-module M has a slope $\leq h$ decomposition if and only if there is a surjection $M \twoheadrightarrow D_{\leq h}^{-1}M$ with finitely generated kernel (over A or equivalently $A[T]$).
(3) If $\psi : M \to N$ is a map of $A[T]$-modules and M, N have slope $\leq h$ decompositions, then ψ takes $M_{\leq h}$ to $N_{\leq h}$ and $M_{>h}$ to $N_{>h}$, and $\operatorname{im}\psi$ and $\ker\psi$ have slope $\leq h$ decompositions.
(4) Taking the $\leq h$ and $> h$ parts is exact when defined.

Lemma 18.2.2. *Suppose $A = \mathscr{O}(D(1)^n)$. Let M be a Banach projective Banach A-module. Let $u \in \mathscr{C}\mathscr{C}_A(M, M)$. Let $x \in \operatorname{Max}(A)$ (by which we approximately mean $x \in D(1)^n$, although really x is a Galois orbit of such). Let $h \in \mathbb{Q}_{\geq 0}$. For an integer N, $d \in \{0, \ldots, N\}$, and $b_0, \ldots, b_N \in A$, we may define the Banach algebra*

$$A' = \mathscr{O}(D(1)_A^{N+2})/(T_i - b_i, b_d T_* - 1)$$

(where $T_0, \ldots, T_N, T_*$ are the variables on $D(1)_A^{N+2}$); this defines an affinoid subdomain (in fact a Laurent subdomain) of A. Then we can choose such an A' such that the resulting map $\operatorname{Max}(A') \hookrightarrow \operatorname{Max}(A)$ has x in its image and $A'\widehat{\otimes}_{A,\pi} M$ has a slope $\leq h$ decomposition for u.

(The reason for the extra condition on the value of b_d is that b_d will be chosen to enforce $|a_d(y)|$ being constant on $\operatorname{Max}(A')$.)

19. 3/10/20: Automorphic forms on definite quaternion algebras

I want to illustrate the theory of p-adic automorphic forms in the simplest possible case. I don't have time to go beyond that.

To that end let D denote a definite quaternion algebra with centre $\mathbb{Q}$. Thus D is characterized by the finite set of places S at which D is not split. The set S will contain ∞ (as D is assumed definite), and any finite set S of places of $\mathbb{Q}$ which contains ∞ and has even cardinality can arise. We will denote by $*$ the anti-involution on D characterized by $d^* + d$ equals the reduced trace $\operatorname{tr} d$ of d for all $d \in D$. Choose a maximal order $\mathcal{O}_D$ of D. It will be stable under $*$ (as $\operatorname{tr} d \in \mathbb{Z}$ for $d \in \mathcal{O}_D$). For $v \notin S$ pick an isomorphism $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_v \cong M_{2\times 2}(\mathbb{Z}_v)$. Under this isomorphism $*$ is carried to the map that takes a matrix to its adjugate. We define an algebraic group G over $\mathbb{Z}$ by setting

$$G(R) = (R \otimes_Z \mathcal{O}_D)^\times.$$

It becomes reductive over $\mathbb{Z}[1/S]$. If $v \notin S$ then we get an isomorphism $G \times \mathbb{Z}_v \cong GL_2/\mathbb{Z}_v$. The topological group $G(\mathbb{R})$ is compact mod centre.

19.1. Classical automorphic forms.

Before discussing p-adic automorphic forms we will quickly review the classical theory of automorphic forms which is particularly simple on G. If $\mathbb{A}^\infty$ denotes the ring of finite adeles then the quotient $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)$ is compact. If R is a ring (commutative with 1) and M is an R-module we will write $\mathcal{A}_M$ for the set of locally constant functions

$$G(\mathbb{Q})\backslash G(\mathbb{A}^\infty) \longrightarrow M.$$

These are the 'weight 2' automorphic forms on G. We could discuss other weights as well, but for lack of time I won't for now. This has a natural smooth action of $G(\mathbb{A}^\infty)$ via

$$(g\varphi)(x) = \varphi(xg).$$

('Smooth' means that the stabilizer of any function in $\mathcal{A}_M$ is open.) If M is finitely generated over R, then this action is also 'admissible' over R in the sense that for any open compact subgroup $U \subset G(\mathbb{A}^\infty)$ the U-fixed points $\mathcal{A}_M^U$ is a finitely generated R-module. In fact

$$\mathcal{A}_M^U = \bigoplus_{G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U} M,$$

and $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U$ is a finite set. We see that

$$\mathcal{A}_M^U = \mathcal{A}_\mathbb{Z}^U \otimes_\mathbb{Z} M$$

and that

$$\mathcal{A}_M = \bigcup_U \mathcal{A}_M^U.$$

The most classical case is the case $M = R = \mathbb{C}$. In this case

$$\mathcal{A}_\mathbb{C} \cong \bigoplus \pi$$

where π runs over irreducible admissible representations π of $G(\mathbb{A}^\infty)$ each of which has a factorization as a restricted tensor product $\otimes_v \pi_v$ with π_v an irreducible admissible representation of $G(\mathbb{Q}_v)$ with $\dim \pi_v^{G(\mathbb{Z}_v)} = 1$ for all but finitely many v (and $= 0$ for the other v). (The tensor product $\otimes'_v \pi_v$ is restricted with respect to the lines $\pi_v^{G(\mathbb{Z}_v)}$.) We say that π is *unramified* at v if $v \notin S$ and $\pi_v^{G(\mathbb{Z}_v)} \neq (0)$. It turns out that either π_v is infinite dimensional (indeed 'generic') for all $v \notin S$, or $\dim \pi_v = 1$ for all v and $\pi = \chi \circ \det$ for some continuous character $\chi : (\mathbb{A}^\infty)^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$. (Here det denotes the reduced norm $G \to \mathbb{G}_m$.) Moreover the strong multiplicity one theorem tells us that if π and π' are two irreducible submodules of $\mathcal{A}_\mathbb{C}$ with $\pi_v \cong \pi'_v$ for all but finitely many v, then $\pi = \pi'$.

If U, V are open compact subgroups of $G(\mathbb{A}^\infty)$ and $g \in G(\mathbb{A}^\infty)$ then we can write

$$UgV = \coprod_i g_i V$$

a finite disjoint union. There is a well defined R-linear map

$$\begin{array}{rcl} [UgV] : \mathcal{A}_M^V & \longrightarrow & \mathcal{A}_M^U \\ \varphi & \longmapsto & \sum_i g_i \varphi, \end{array}$$

called a Hecke operator. In the particular case $V \subset U$ and $g = 1$ then we will write

$$\mathrm{tr}_{U/V} = [UV] = \sum_{g \in U/V} g : \mathcal{A}_M^V \longrightarrow \mathcal{A}_M^U.$$

We call an open compact subgroup U of $G(\mathbb{A}^\infty)$ unramified at v if $v \notin S$ and $U \supset G(\mathbb{Z}_v)$. If U is unramified at v we will write

$$T_v = \left[U \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} U \right] \text{ and } S_v = \left[U \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} U \right].$$

These operators commute as v varies over all unramified primes for U. We will write $\mathbb{T}_R(U)$ for the R-subalgebra of $\mathrm{End}_R(\mathcal{A}_R^U)$ generated by all the Hecke operators T_v and S_v as v runs over unramified primes for U. It is a finite commutative R-algebra, called the *unramified Hecke algebra*. It acts on $\mathcal{A}_M^U$ for any R-module M. If $R \to S$ is a ring morphism then

$$S \otimes_R \mathbb{T}_R(U) \twoheadrightarrow \mathbb{T}_S(U).$$

If S is flat over R this map is an isomorphism. If R is a reduced $\mathbb{Q}$-algebra then $\mathbb{T}_R(U)$ is also reduced. In the case $R = \mathbb{C}$ and $T \in \mathbb{T}_{\mathbb{C}}(U)$ and π is an irreducible constituent of $\mathcal{A}_{\mathbb{C}}$ then T acts on π^U by a scalar, which I will denote $\pi(T)$. There is an isomorpohism

$$\begin{array}{rcl} \mathbb{T}_{\mathbb{C}}(U) & \stackrel{\sim}{\longrightarrow} & \prod_{\pi^U \neq (0)} \mathbb{C} \\ T & \longmapsto & (\pi(T))_\pi, \end{array}$$

where the product is over all irreducible constituents π of $\mathcal{A}_{\mathbb{C}}$ with $\pi^U \neq (0)$.

If π is an irreducible constituent of $\mathcal{A}_{\mathbb{C}}$ and $\imath : \mathbb{C} \cong \overline{\mathbb{Q}}_p$ then there is a continuous semi-simple representation

$$r_{\pi,\imath} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$$

with the following properties:

(1) $r_{\pi,\imath}$ is unramified outside p and the primes where π is ramified, and for such a place v the conjugacy class $r_{\pi,\imath}(\mathrm{Frob}_v)$ has characteristic polynomial
$$X^2 - \pi(T_v)X + v\pi(S_v).$$
(2) If $v \neq l$ then $r_{\pi,\imath}|_{G_{\mathbb{Q}_v}}$ is determined by π_v (and $\imath$).
(3) If $c \in G_{\mathbb{Q}}$ is a complex conjugation then $r_{\pi,\imath}(c)$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
(4) The restriction $r_{\pi,\imath}|_{G_{\mathbb{Q}_p}}$ is de Rham with Hodge-Tate numbers $\{0, 1\}$. Moreover a certain invariant $\mathrm{WD}(r_{\pi,\imath}|_{G_{\mathbb{Q}_p}})$ of $r_{\pi,\imath}|_{G_{\mathbb{Q}_p}}$ is determined by π_p (and $\imath$), however the restriction $r_{\pi,\imath}|_{G_{\mathbb{Q}_p}}$ is not itself in general determined by π_p.
(5) $r_{\pi,\imath}$ is irreducible if and only if π is infinite dimensional.

Finally we need to recall the Petersson pairing on $\mathcal{A}_R^U$. Suppose that $U^* = U$. Then we define a perfect R-bilinear pairing

$$\begin{array}{rcl} \langle\ ,\ \rangle_U : \mathcal{A}_R^U \times \mathcal{A}_R^U & \longrightarrow & R \\ (\varphi_1, \varphi_2) & \longmapsto & \sum_{x \in G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty)/U} \#(xUx^{-1} \cap G(\mathbb{Q}))^{-1} \varphi_1(g)\varphi_2(g^{-*}). \end{array}$$

Note that some mild assumption is needed for this definition to make sense: the term $\#(xUx^{-1} \cap G(\mathbb{Q}))^{-1}$ needs to make sense in R. This will be OK if for instance 6 is invertible in R, or if U is 'sufficiently small'. We will not go into details here. If $V^* = V$ as well then we have

$$\langle [UgV]\varphi_1, \varphi_2 \rangle_U = \langle \varphi_1, [Vg^*U]\varphi_2 \rangle_V.$$

If $V \subset U$ and $g = 1$ we get

$$\langle \mathrm{tr}_{U/V}\, \varphi_1, \varphi_2 \rangle_U = \langle \varphi_1, \varphi_2 \rangle_V.$$

Moreover if v is unramified for U, then T_v and S_v are self-adjoint with respect to $\langle\ ,\ \rangle_U$.

19.2. **p-adic modular forms.** As usual L will denote a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ will denote its ring of integers, λ will denote the maximal ideal of $\mathcal{O}$ and $\mathbb{F} = \mathcal{O}/\lambda$ the residue field of $\mathcal{O}$. We will normalize the absolute value $|\ |$ on L by $|p| = p^{-1}$. *We will assume that $p \notin S$.* Suppose that U is an open compact subgroup of $G(\mathbb{A}^{\infty,l})$.

- $\mathcal{A}_{\mathcal{O}/\lambda^m}^U$ is an admissible smooth representation of $GL_2(\mathbb{Q}_p)$ over $\mathcal{O}/\lambda^m$.
- We will set $\mathcal{A}_{\mathcal{O}}^{\mathrm{cts},U} = C(G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U, \mathcal{O})$ with the p-adic toplogy. We see that it is a separately continuous representation of $GL_2(\mathbb{Q}_p)$ isomorphic to $\varprojlim_m \mathcal{A}_{\mathcal{O}/\lambda^m}^U$.

- $\mathcal{A}_L^{\mathrm{cts},U} = C(G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U, L)$ is a Banach representation of $GL_2(\mathbb{Q}_p)$, which can be identified with $\mathcal{A}_{\mathcal{O}}^{\mathrm{cts},U} \otimes_{\mathcal{O}} L$. If one chooses the defining norm to be the sup norm $\| \|_\infty$ then $\mathcal{A}_L^{\mathrm{cts},U}$ has unit ball $\mathcal{A}_{\mathcal{O}}^{\mathrm{cts},U}$. These spaces are sometimes referred to as 'completed cohomology'.
- Note that
$$\mathcal{A}_{L/\mathcal{O}}^U = \mathcal{A}_L^{\mathrm{cts},U}/\mathcal{A}_{\mathcal{O}}^{\mathrm{cts},U},$$
and so if V is an open compact subgroup of $GL_2(\mathbb{Q}_p)$ we have
$$(\mathcal{A}_L^{\mathrm{cts},U}/\mathcal{A}_{\mathcal{O}}^{\mathrm{cts},U})^V = \mathcal{A}_{L/\mathcal{O}}^{U\times V} \cong (L/\mathcal{O})^{\#G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/UV}.$$
We deduce that $\mathcal{A}_L^{\mathrm{cts},U}$ is an admissible Banach representation of $GL_2(\mathbb{Q}_p)$.

We may sometimes set
$$\mathcal{A}_L^{\mathrm{cts}} = \bigcup_U \mathcal{A}_L^{\mathrm{cts},U}.$$
It has a separately continuous action of $G(\mathbb{A}^\infty)$. The U fixed points of $\mathcal{A}_L^{\mathrm{cts}}$ are identified with $\mathcal{A}_L^{\mathrm{cts},U}$.

We will write $V(p^n)$ for the kernel $GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}/p^n\mathbb{Z})$, and define
$$\widehat{\mathcal{A}}_{\mathcal{O}}^U = \varprojlim_{\mathrm{tr}} \mathcal{A}_{\mathcal{O}}^{U\times V(p^n)}$$
a torsion-free, compact Hausdorff, linearly topologized $\mathcal{O}$-module with a compatible continuous action of $\mathcal{O}[\![GL_2(\mathbb{Q}_p)]\!]$. We will also write
$$\widehat{\mathcal{A}}_L^U = L \otimes_{\mathcal{O}} \widehat{\mathcal{A}}_{\mathcal{O}}^U.$$
At least for n sufficiently large we have
$$\widehat{\mathcal{A}}_{\mathcal{O}}^U \cong \mathcal{O}[\![V(p^n)]\!]^{G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/UV(p^n)},$$
so that $\widehat{\mathcal{A}}_{\mathcal{O}}^U$ and $\widehat{\mathcal{A}}_L^U$ are coadmissible. We have the important observation that
$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cts}}(\widehat{\mathcal{A}}_{\mathcal{O}}^U, L) & \cong & \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cts}}(\widehat{\mathcal{A}}_{\mathcal{O}}^U, \mathcal{O}) \otimes_{\mathcal{O}} L \\ & \xrightarrow{\sim} & (\varprojlim_m \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cts}}(\widehat{\mathcal{A}}_{\mathcal{O}}^U, \mathcal{O}/\lambda^m)) \otimes_{\mathcal{O}} L \\ & \xleftarrow{\sim} & (\varprojlim_m \varinjlim_n \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}_{\mathcal{O}}^{UV(p^n)}, \mathcal{O}/\lambda^m)) \otimes_{\mathcal{O}} L \\ & \cong & (\varprojlim_m \varinjlim_n \mathcal{A}_{\mathcal{O}/\lambda^m}^{UV(p^n)}) \otimes_{\mathcal{O}} L \\ & = & (\varprojlim_m \mathcal{A}_{\mathcal{O}/\lambda^m}^U) \otimes_{\mathcal{O}} L \\ & = & \mathcal{A}_L^{\mathrm{cts},U}. \end{array}$$
[The first isomorphism follows from the fact that $\widehat{\mathcal{A}}_{\mathcal{O}}^U$ is compact. The main point to check about the second is surjectivity, which follows from the completeness of $\mathcal{O}$. The main point to check about the third is again surjectivity, which follows because the kernel of any element of $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cts}}(\widehat{\mathcal{A}}_{\mathcal{O}}^U, \mathcal{O}/\lambda^m)$ is open and hence is the preimage of an open submodule of $\mathcal{A}_{\mathcal{O}}^{UV(p^n)}$ for some n. The fourth isomorphism follows from the duality described at the end of the last section.] Thus we also have that
$$\mathcal{A}_L^{\mathrm{cts},U} = (\widehat{\mathcal{A}}_L^U)_b^\vee$$
and
$$\widehat{\mathcal{A}}_L^U = (\mathcal{A}_L^{\mathrm{cts},U})_s^\vee.$$

For $n \geq 1$ there is a natural map

$$\mathbb{T}_{\mathcal{O}}(U \times V(p^{n+1})) \twoheadrightarrow \mathbb{T}_{\mathcal{O}}(U \times V(p^n))$$

compatible with the maps

$$\mathcal{A}_R^{U \times V(p^n)} \hookrightarrow \mathcal{A}_R^{U \times V(p^{n+1})}$$

and

$$\mathrm{tr} : \mathcal{A}_R^{U \times V(p^{n+1})} \twoheadrightarrow \mathcal{A}_R^{U \times V(p^n)}.$$

It sends T_v (resp. S_v) to T_v (resp. S_v) for all unramified v. Thus if we define

$$\mathbb{T}_{\mathcal{O}}(U) = \lim_{\leftarrow n} \mathbb{T}_{\mathcal{O}}(U \times V(p^n))$$

then there is an action of $\mathbb{T}_{\mathcal{O}}(U)$ on each of the spaces:

$$\mathcal{A}_{\mathcal{O}}^U, \ \widehat{\mathcal{A}}_{\mathcal{O}}^U, \ \mathcal{A}_{\mathcal{O}/\lambda^m}^U, \ \mathcal{A}_{\mathcal{O}}^{\mathrm{cts},U},$$

which commutes with the action of $GL_2(\mathbb{Q}_p)$.

Being finite and free over $\mathcal{O}$ we see that $\mathbb{T}_{\mathcal{O}}(U \times V(p^n))$ has finitely many maximal ideals and

$$\mathbb{T}_{\mathcal{O}}(U \times V(p^n)) \cong \prod_{\mathfrak{m}} \mathbb{T}_{\mathcal{O}}(U \times V(p^n))_{\mathfrak{m}}$$

where the product runs over all maximal ideals $\mathfrak{m}$ of $\mathbb{T}_{\mathcal{O}}(U \times V(p^n))$. Moreover each $\mathbb{T}_{\mathcal{O}}(U \times V(p^n))_{\mathfrak{m}}$ is a complete local noetherian ring finite and free as a $\mathcal{O}$-module. We have a diagram

$$\begin{array}{ccccc} \mathbb{T}_{\mathcal{O}}(U) & \twoheadrightarrow & \mathbb{T}_{\mathcal{O}}(U \times V(p^n)) & \twoheadrightarrow & \mathbb{T}_{\mathcal{O}}(U \times V(p)) \\ & & \downarrow & & \downarrow \\ & & \mathbb{T}_{\mathbb{F}}(U \times V(p^n)) & \twoheadrightarrow & \mathbb{T}_{\mathbb{F}}(U \times V(p)) \end{array}$$

in which all the maps are surjective. It is not hard to see that pull back along the vertical arrows indices a bijection between maximal ideals. The same is true for the horizontal arrows. [To check this it suffices to check it for the lower horizontal arrow. Let us sketch the proof of this. We have an isomorphism

$$\mathcal{A}_{\mathbb{F}}^{UV(p^n)} \cong \{\varphi : G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty)/U \to \mathrm{Ind}_{V(p^n)}^{V(p)} \mathbb{F} : \ \varphi(xu) = u^{-1}\varphi(x) \ \forall u \in V(p)\}.$$

The $V(p)$-module $\mathrm{Ind}_{V(p^n)}^{V(p)} \mathbb{F}$ has a filtration by $V(p)$ invariant submodules such that each graded piece is isomorphic to $\mathbb{F}$ as a $V(p)$-module. Thus the kernel I of

$$\mathbb{T}_{\mathbb{F}}(U \times V(p^n)) \twoheadrightarrow \mathbb{T}_{\mathbb{F}}(U \times V(p))$$

preserves a filtration of $\mathcal{A}_{\mathbb{F}}^{UV(p^n)}$ and acts trivially on each graded piece, from which we conclude that I is nilpotent, and so contained in every maximal ideal.] We deduce that $\mathbb{T}_{\mathcal{O}}(U)$ has only finitely many maximal ideals and that

$$\mathbb{T}_{\mathcal{O}}(U) \cong \prod_{\mathfrak{m}} \mathbb{T}_{\mathcal{O}}(U)_{\mathfrak{m}}$$

where the product runs over all maximal ideals $\mathfrak{m}$ of $\mathbb{T}_{\mathcal{O}}(U)$. Similar considerations show that each $\mathbb{T}_{\mathcal{O}}(U)_{\mathfrak{m}}$ is noetherian, and hence is a complete noetherian local ring.

If $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}_{\mathcal{O}}(U)$ then there is a continuous semi-simple representation

$$\overline{r}_{\mathfrak{m}} : G_{\mathbb{Q}} \longrightarrow GL_2(k(\mathfrak{m}))$$

such that:

(1) $\overline{r}_{\mathfrak{m}}$ is unramified outside p and the primes where U is ramified, and for such a place v the conjugacy class $\overline{r}_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial
$$X^2 - T_v X + vS_v.$$
(2) If $c \in G_{\mathbb{Q}}$ is a complex conjugation then $\overline{r}(c)$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

We call $\mathfrak{m}$ Eisenstein if $\overline{r}_{\mathfrak{m}}$ is reducible. We think of $\mathfrak{m}$ being non-Eisenstein as the 'generic' situation. If $\mathfrak{m}$ is non-Eisenstein then there is a continuous representation
$$r_{\mathfrak{m}} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{T}_{\mathcal{O}}(U)_{\mathfrak{m}})$$
such that:

(1) $\overline{r}_{\mathfrak{m}}$ is unramified outside l and the primes where U is ramified, and for such a place v the conjugacy class $r_{\mathfrak{m}}(\mathrm{Frob}_v)$ has characteristic polynomial
$$X^2 - T_v X + vS_v.$$
(2) If $c \in G_{\mathbb{Q}}$ is a complex conjugation then $r(c)$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The following theorem is a culmination of the work of many people.

Theorem 19.2.1. *Suppose that*
$$r : G_{\mathbb{Q}} \longrightarrow GL_2(\mathcal{O}) \subset GL_2(L)$$
is unramified outside finitely many primes and that for each finite prime $v \in S$ the semi-simplified restriction $r|_{G_{\mathbb{Q}_v}}^{\mathrm{ss}}$ is unramified and the eigenvalues of $r|_{G_{\mathbb{Q}_v}}^{\mathrm{ss}}(\mathrm{Frob}_v)$ have ratio $v^{\pm 1}$. (This latter condition can be relaxed, but some modified condition is necessary to reflect the fact that we are working with G and not GL_2.) Suppose moreover that for $c \in G_{\mathbb{Q}}$ a complex conjugation
$$r(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(We say r is odd*.) Also suppose that*

(1) $r \bmod \lambda$ is absolutely irreducible.
(2) $p > 2$.
(3) $(r \bmod \lambda)|_{G_{\mathbb{Q}_p}}$ is not the extension of a character χ_2 by a character χ_1 with χ_2/χ_1 either trivial or the cyclotomic character.

(These latter conditions are presumably not necessary, and some progress has been made on relaxing them.) Then for some open compact subgroup $U \subset GL_2(\mathbb{A}^{p,\infty})$ ramified only at primes in S and primes where r ramifies, and some maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{\mathcal{O}}(U)_{\mathfrak{m}}$ there is a continuous homomorphism
$$\theta : \mathbb{T}(U)_{\mathfrak{m}} \longrightarrow \mathcal{O}$$
such that $\theta(r_{\mathfrak{m}}) = r$. Moreover θ factors through some $\mathbb{T}_{\mathcal{O}}(UV(p^n))$ if and only if $r|_{G_{\mathbb{Q}_p}}$ is de Rham with Hodge-Tate numbers $\{0, 1\}$.

We also have the following conjecture:

Conjecture 19.2.2. *Keep the notation and assumptions of the theorem. Then there is a Banach representation $V = V(r|_{G_{\mathbb{Q}_p}})$ of $GL_2(\mathbb{Q}_p)$ which is determined by, and determines, $r|_{G_{\mathbb{Q}_p}}$ such that $\mathcal{A}_L^{\mathrm{cts},U}[\ker\theta] \cong V^{\oplus m}$ for some m. Equivalently*

$$\widehat{\mathcal{A}}_{\mathcal{O}}^U \otimes_{\mathbb{T}_{\mathcal{O}}(U),\theta} L \cong (V_s^\vee)^{\oplus m}.$$

Off the top of my head I am unsure of the status of this conjecture, but Emerton proved the corresponding conjecture when G is replaced by GL_2. Notice that whereas the $GL_2(\mathbb{Q}_p)$ action on $\mathcal{A}_L^U$ did not in general determine $r|_{G_{\mathbb{Q}_p}}$, the action on $\mathcal{A}_L^{\mathrm{cts},U}$ or $\widehat{\mathcal{A}}_L^U$ does. Also note that the $\mathbb{T}_{\mathcal{O}}(U)$ see all odd, irreducible, continuous, two dimensional representations r of $G_{\mathbb{Q}}$ satisfying some conditions at the primes in S, while the $\mathbb{T}_{\mathcal{O}}(U \times V(p^n))$ only see the small subset for which $r|_{G_{\mathbb{Q}_p}}$ is de Rham with Hodge-Tate numbers $\{0, 1\}$.

Finally let us mention the locally analytic versions of these spaces of automorphic forms. For $U \subset G(\mathbb{A}^{\infty,l})$ we will write $\mathcal{A}_L^{\mathrm{la},U}$ for the space of locally analytic functions

$$\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U \longrightarrow L.$$

(Note that $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U$ is a locally analytic manifold.) Thus

$$\mathcal{A}_L^{\mathrm{la},U} = (\mathcal{A}_L^{\mathrm{cts},U})^{\mathrm{la}}$$

is an admissible locally analytic representation of $GL_2(\mathbb{Q}_p)$ with a commuting faithful action of $\mathbb{T}_{\mathcal{O}}(U)$. We have

$$(\mathcal{A}_L^{\mathrm{la},U})_b^\vee \cong \widehat{\mathcal{A}}_L^U \otimes_{L[\![GL_2(\mathbb{Q}_p)]\!]} D_c^{\mathrm{la}}(GL_2(\mathbb{Q}_p), L)$$

and this is a coadmissible $D_c^{\mathrm{la}}(GL_2(\mathbb{Q}_p), L)$-module.

20. 3/12/20: The eigencurve

20.1. **Weight space.** Weight space parametrizes continuous characters of a compact abelian locally analytic group. We will consider noetherian Banach algebras A defined by a norm $\|\ \|$ satisfying

- $\|ab\| \le \|a\|\|b\|$,
- and $\|a^n\| = \|a\|^n$. (We say that $\|\ \|$ is *power multiplicative*.)

An important example is provided by reduced *affinoid algebras*. By an affinoid algebra we mean one of the form $\mathcal{O}(D(1)^n)/I$ for some (necessarily closed) ideal I. If A is an affinoid algebra and $\mathfrak{m}$ is a maximal ideal then $A/\mathfrak{m}$ is a finite extension of L. If A is reduced its topology is defined by the ('spectral') norm

$$\|a\| = \sup_{\mathfrak{m}} |a \bmod \mathfrak{m}|.$$

We will need a small amount of rigid geometry, which I unfortunately don't have time to develop. Hopefully you have had some exposure to these ideas in some other context. I'll briefly summarise some of the concepts we will need. If A is an affinoid algebra we will write $\mathrm{Sp}\, A$ for its set of maximal ideals. For instance $\mathrm{Sp}\, \mathcal{O}(D(1)^n) = \mathrm{Gal}\,(\overline{L}/L)\backslash D(1)^n$. It has various useful topologies. The 'canonical topology' has a basis consisting of the sets of the form

$$\{\mathfrak{m} :\ |a_i \bmod \mathfrak{m}| \le 1 \text{ for } i = 1, ..., n\}$$

for $a_1, ..., a_n \in A$. If $f : A \to B$ is a L-algebra homomorphism between affinoid algebras we get a continuous map $f^* : \mathrm{Sp}\, B \to \mathrm{Sp}\, A$. There is a notion of B being an *affinoid subdomain* of A. In this case $\mathrm{Sp}\, B$ is an open subset of $\mathrm{Sp}\, A$ and B is completely determined (as an

A-algebra) by the open set $\operatorname{Sp} B \subset \operatorname{Sp} A$. One example of affinoid subdomains is provided by the *Laurent domains*

$$A \longrightarrow (A \widehat{\otimes}_\pi \mathcal{O}(D(1)^{n+m}))/(T_i - a_i, a'_j T'_j - 1 : \; i = 1, ..., n; \; j = 1, ..., m).$$

Here $T_1, ..., T_n, T'_1, ..., T'_m$ are the additional $n+m$ variables and the a_i and a'_j lie in A.

One can 'glue' affinoid spaces $\operatorname{Sp} A$ to form *rigid analytic varieties*. We will not explain this in general but consider only the special case of $X = \bigcup A_i$ where

$$\operatorname{Sp} A_1 \subset \operatorname{Sp} A_2 \subset \operatorname{Sp} A_3 ...$$

are affinoid subdomains one of another. (The property of being an affinoid subdomain is transitive.) For example we have

$$(\mathcal{D}(1)^0)^n = \bigcup \operatorname{Sp} \mathcal{O}(D(r)^n)$$

as r runs over $p^{\mathbb{Q}_{<0}}$ and

$$\mathrm{Aff}^n = \bigcup \operatorname{Sp} \mathcal{O}(D(r)^n)$$

as r runs over $p^{\mathbb{Q}}$. On the level of points one has

$$(\mathcal{D}(1)^0)^n = \operatorname{Gal}(\overline{L}/L) \backslash (D(1)^0)^n$$

and

$$\mathrm{Aff}^n = \operatorname{Gal}(\overline{L}/L) \backslash \overline{L}^n.$$

There is a general notion of an *affinoid subdomain* of a rigid analytic variety. In the special case $X = \bigcup A_i$ as above it coincides with being an affinoid subdomain of some $\operatorname{Sp} A_i$. There is also the notion of an *admissible covering* by affinoid subdomains. In the above special case a covering by affinoid subdomains U_j is one in which each $\operatorname{Sp} A_i$ is contained in a union of a finite number of U_j. A rigid analytic variety X has a structure sheaf $\mathcal{O}_X$, and $\mathcal{O}_X(\operatorname{Sp} A) = A$ for any affinoid subdomain $\operatorname{Sp} A \subset X$. This structure sheaf has, in particular, the usual sheaf propery for admissible covers by affinoid subdomains.

Lemma 20.1.1. *For A as above there is a bijection*

$$\begin{array}{ccc} \operatorname{Hom}^{\mathrm{cts}}(\mathbb{Z}_p, A^\times) & \xrightarrow{\sim} & \{a \in A : \; \|a\| < 1\} \\ \chi & \longmapsto & \chi(1) - 1. \end{array}$$

Proof: We sketch the proof. Firstly the map is well defined for if $\|a\| \geq 1$ then

$$\|(1+a)^{p^n} - 1\| = \|a\|^{p^n}$$

which does not tend to 0 as $n \to \infty$. Secondly the map is injective because if χ and χ' have the same image then χ and χ' agree on $\mathbb{Z}$ and hence by continuity are equal. Finally we need to check surjectivity. Given $a \in A$ with $\|a\| < 1$ we get a map $\mathbb{Z} \to A^\times$ which sends n to $(1+a)^n$. As A is complete, it suffices to show that it is continuous with respect to the p-adic topology on $\mathbb{Z}$. Concretely we need to show that $(1+a)^{p^n} \to 1$ as $n \to \infty$. For any $b \in A$ with $\|b\| < 1$ we have

$$\|(1+b)^p - 1\| \leq \max\{|p|\|b\|, \|b\|^p\}.$$

Inductively we conclude that

$$\|(1+a)^{p^n} - 1\| \leq \max\{|p|^n\|a\|, |p|^{n-1}\|a\|^p, ..., \|a\|^{p^n}\},$$

and the desired result follows. $\square$

For $a \in A$ with $\|a\| < 1$ we have

$$\|a^n/n\| \leq \|a\|^n |p|^{-\log_p n} = |p|^{-n\log_p \|a\| - \log_p n} \to 0$$

as $n \to \infty$. Thus

$$\log(1+a) = \sum_{n=1}^{\infty} (-1)^{n-1} a^n/n$$

converges to an element of A. If I didn't make a mistake, we get

$$\|\log(1+a)\| \leq \begin{cases} \|a\| & \text{if } \|a\| \leq 1/e \\ [L:\mathbb{Q}_p]/(-e\log\|a\|) & \text{if } \|a\| \geq 1/e. \end{cases}$$

Thus

$$\exp(p^m t \log(1+a)) = \sum_{n=0}^{\infty} t^n (p^m \log(1+a))^n/n!$$

converges for $t \in \mathbb{Z}_p$ if

$$\log\|a\| < -\min\{1, p^{(1/(p-1)-m)}/e\}.$$

(Again assuming I didn't make a computational error.) We conclude that:

Lemma 20.1.2. *Let A be as above and let $\chi : \mathbb{Z}_p \to A^\times$ be a continuous character. Then χ is locally analytic. More specifically there exists $f_\chi \in \mathbb{Z}_{\geq 0}$ such that*

$$\chi(1 + p^{f_\chi} t) = \sum_{n=0}^{\infty} c_{\chi,n} t^n$$

for all $t \in \mathbb{Z}_p$, where $c_{\chi,n} \in A$ and $\|c_{\chi,n}\| \leq 1$ for all n and $c_{n,\chi} \to 0$ as $n \to \infty$.

More generally consider an abelian compact locally analytic group Γ of the form $\mathbb{Z}_p^d \times \Gamma^{\text{tor}}$, with Γ^{tor} finite. The lemma immediately implies that any continuous character $\chi : \Gamma \to A^\times$ (for A as above) is locally analytic, and on a suitable neighbourhood of the identity is given by a power series in $\mathcal{O}(D(1)^d_A)$ all of whose coefficients have norm ≤ 1.

Now moreover assume that L contains all $\#\Gamma^{\text{tor}}$ roots of unity. Then we set

$$\mathcal{W}_\Gamma = (\mathcal{D}(1)^0)^d \times (\Gamma^{\text{tor}})^\vee.$$

We have

$$\mathcal{O}(\mathcal{W}_\Gamma) = (\varinjlim_{r<1} \mathcal{O}(D(r)^d)) \otimes_L L^{(\Gamma^{\text{tor}})^\vee}.$$

There is a continuous character

$$\chi^{\text{univ}} : \Gamma \longrightarrow \mathcal{O}(\mathcal{W}_\Gamma)^\times,$$

which sends $(\gamma_1, ..., \gamma_d, \gamma_{\text{tor}}) \in \Gamma = \mathbb{Z}_p^d \times \Gamma^{\text{tor}}$ to

$$(1+T_1)^{\gamma_1} ... (1+T_d)^{\gamma_d} \otimes (\chi(\gamma_{\text{tor}}))_\chi.$$

This has the universal property that if A is any reduced affinoid algebra and $\chi : \Gamma \to A^\times$ is a continuous character, then there is a unique map

$$\operatorname{Sp} A \to \mathcal{W}_\Gamma$$

under which χ^{univ} pulls back to χ. Note that χ^{univ} is not itself locally analytic, but its pull back to any affinoid algebra is locally analytic. Loosely speaking the closer you get to the 'boundary' of $\mathcal{W}_\Gamma$ the smaller the radius of analyticity.

20.2. **Finite slope p-adic modular forms.** Let us write B for the subgroup of GL_2 consisting of upper triangular matrices, and T for the subgroup of B consisting of diagonal matrices. Suppose that

$$\chi : T(\mathbb{Z}_p) \longrightarrow L^\times.$$

We will sometimes think of χ as a character of $B(\mathbb{Z}_p)$ and write

$$\chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \chi_1(a)\chi_2(b).$$

There is an $f_\chi \in \mathbb{Z}_{>0}$ such that χ is analytic on $(1 + p^{f_\chi}\mathbb{Z}_p)^2$ and that, more precisely, $\chi_i(1 + p^{f_\chi} t)$ is given by a power series in $\mathcal{O}(D(1))$ all whose coefficients are in $\mathcal{O}$.

We define

$$\mathcal{A}_L^{\mathrm{la},U}(\chi) = \{\varphi \in \mathcal{A}_L^{\mathrm{la},U} : \ \varphi(xu) = \chi(u)^{-1}\varphi(x) \ \forall u \in B(\mathbb{Z}_p)\}.$$

This space is preserved by $\mathbb{T}_{\mathcal{O}}(U)$. Let Iw denote the inverse image of $B(\mathbb{F}_p)$ in $GL_2(\mathbb{Z}_p)$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Iw}$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & (ad-bc)/a \end{pmatrix}.$$

Let $\mathfrak{X}_U$ be a (finite) set of representatives for $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U\mathrm{Iw}$. Then we can write

$$G(\mathbb{A}^\infty) = \coprod_{x \in \mathfrak{X}_U} G(\mathbb{Q})xU \begin{pmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{pmatrix} B(\mathbb{Z}_p).$$

We see that a function $\varphi \in \mathcal{A}_L^{\mathrm{la},U}(\chi)$ is analytic on left cosets of $V(p^n)$ if and only if, for each $x \in \mathfrak{X}_U$ and each $\gamma \in p\mathbb{Z}/p^n\mathbb{Z}$ the function

$$\begin{array}{rcl} f_{x,\gamma} : \mathbb{Z}_p & \longrightarrow & L \\ e & \longmapsto & \varphi\left(x \begin{pmatrix} 1 & 0 \\ \gamma + p^n c & 1 \end{pmatrix}\right) \end{array}$$

is analytic. Moreover a norm on the Banach space $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$ is given by

$$\|\varphi\| = \sup_{x \in \mathfrak{X}_U, \gamma \in \mathbb{Z}/p^n\mathbb{Z}} \|f_{x,\gamma}\|.$$

Moreover $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$ has a Banach basis consisting of the functions $\varphi_{x,\gamma,m}$ for $x \in \mathfrak{X}_U$, $\gamma \in \mathbb{Z}/p^{n-1}\mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$ which is supported on

$$G(\mathbb{Q})xU \begin{pmatrix} 1 & 0 \\ p\gamma + p^n\mathbb{Z}_p & 1 \end{pmatrix} B(\mathbb{Z}_p)$$

and sends

$$x \begin{pmatrix} 1 & 0 \\ p\gamma + p^n t & 1 \end{pmatrix} \longmapsto t^m.$$

Note that $\|\varphi_{x,\gamma,m}\| = 1$.

If $\varphi \in \mathcal{A}_L^{\mathrm{la},U}(\chi)$ we define a magical transformation

$$(U_p\varphi)(x) = \sum_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \varphi\left(x \begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix}\right),$$

where we think of p and α as elements of $\mathbb{Q}_p$. The definition does not depend on the choice of representatives α for classes in $\mathbb{Z}/p\mathbb{Z}$. The image $U_p\varphi$ again lies in $\mathcal{A}_L^{\mathrm{la},U}(\chi)$ because

$$\begin{aligned}(U_p\varphi)\left(x\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right) &= \textstyle\sum_\alpha \varphi\left(x\begin{pmatrix} pa & a\alpha+b \\ 0 & d\end{pmatrix}\right) \\ &= \textstyle\sum_\alpha \chi\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\varphi\left(x\begin{pmatrix} p & (a\alpha+b)/d \\ 0 & 1\end{pmatrix}\right) \\ &= \chi\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}(U_p\varphi)(x).\end{aligned}$$

The operator U_p commutes with the action of $\mathbb{T}_{\mathcal{O}}(U)$.

For $n \geq f_\chi$ we have

$$U_p : (\mathcal{A}_L^{\mathrm{la},U})^{V(p^{n+1})-\mathrm{an}}(\chi) \longrightarrow (\mathcal{A}_L^{\mathrm{la},U})^{V(p^{n})-\mathrm{an}}(\chi),$$

and so U_p defines a completely continuous operator on $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^{n+1})-\mathrm{an}}(\chi)$. To see this note that

(2)

$$\begin{aligned}(U_p\varphi)\left(x\begin{pmatrix} 1 & 0 \\ e & 1\end{pmatrix}\right) &= \sum_\alpha \varphi\left(x\begin{pmatrix} p & \alpha \\ ep & e\alpha+1\end{pmatrix}\right) \\ &= \sum_\alpha \varphi\left(x\begin{pmatrix} p & \alpha \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ ep/(1-\alpha e) & 1\end{pmatrix}\begin{pmatrix} 1-\alpha e & -e\alpha^2/p \\ 0 & 1/(1-\alpha e)\end{pmatrix}\right) \\ &= \sum_\alpha (\chi_1/\chi_2)(1-\alpha e)\varphi\left(x\begin{pmatrix} p & \alpha \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ ep/(1-\alpha e) & 1\end{pmatrix}\right).\end{aligned}$$

We can go further and look at the 'matrix' of U_p on $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^{n})-\mathrm{an}}(\chi)$. Write

$$x\begin{pmatrix} p & \alpha \\ 0 & 1\end{pmatrix} = gx_\alpha u\begin{pmatrix} a_\alpha & b_\alpha \\ pc_\alpha & d_\alpha\end{pmatrix}$$

with $g \in G(\mathbb{Q})$, $u \in U$ and

$$\begin{pmatrix} a_\alpha & b_\alpha \\ pc_\alpha & d_\alpha\end{pmatrix} \in \mathrm{Iw}.$$

Then

$$\begin{aligned}&(U_p\varphi)\left(x\begin{pmatrix} 1 & 0 \\ e & 1\end{pmatrix}\right) \\ &= \sum_\alpha (\chi_1/\chi_2)(1-\alpha e)\varphi\left(x_\alpha\begin{pmatrix} a_\alpha & b_\alpha \\ pc_\alpha & d_\alpha\end{pmatrix}\begin{pmatrix} 1 & 0 \\ ep/(1-\alpha e) & 1\end{pmatrix}\right) \\ &= \sum_\alpha (\chi_1/\chi_2)(1-\alpha e)\varphi\left(x_\alpha\begin{pmatrix} 1 & 0 \\ \frac{p(c_\alpha+(d_\alpha-c_\alpha\alpha)e)}{a_\alpha+(b_\alpha p-a_\alpha\alpha)e} & 1\end{pmatrix}\begin{pmatrix} \frac{a_\alpha+(b_\alpha p-a_\alpha\alpha)e}{1-\alpha e} & b_\alpha \\ 0 & \frac{(a_\alpha d_\alpha-pb_\alpha c_\alpha)(1-\alpha e)}{a_\alpha+(b_\alpha p-a_\alpha\alpha)e}\end{pmatrix}\right) \\ &= \sum_\alpha \chi_2(a_\alpha d_\alpha-pb_\alpha c_\alpha)(\chi_1/\chi_2)(a_\alpha+(b_\alpha p-a_\alpha\alpha)e)\varphi\left(x_\alpha\begin{pmatrix} 1 & 0 \\ \frac{p(c_\alpha+(d_\alpha-c_\alpha\alpha)e)}{a_\alpha+(b_\alpha p-a_\alpha\alpha)e} & 1\end{pmatrix}\right),\end{aligned}$$

and so

$$(3)\quad \begin{array}{rl} & (U_p\varphi_{x',\gamma',m'})\left(x\begin{pmatrix} 1 & 0 \\ p\gamma+p^ne & 1\end{pmatrix}\right) \\ = & \sum_\alpha \chi_2(a_\alpha d_\alpha - pb_\alpha c_\alpha)(\chi_1/\chi_2)(a_\alpha+(b_\alpha p - a_\alpha\alpha)p\gamma + p^n e(b_\alpha p - a_\alpha \alpha)) \\ & \left(\frac{p^{1-n}(c_\alpha+(d_\alpha-c_\alpha\alpha)p\gamma-\gamma'(a_\alpha+(b_\alpha p-a_\alpha\alpha)p\gamma))+pe(d_\alpha-c_\alpha\alpha-\gamma'(b_\alpha p-a_\alpha\alpha))}{a_\alpha+(b_\alpha p-a_\alpha\alpha)p\gamma+p^ne(b_\alpha p-a_\alpha\alpha)}\right)^{m'}\end{array}$$

where the sum is over those α such that $x_\alpha = x'$ and

$$c_\alpha + (d_\alpha - c_\alpha\alpha)p\gamma \equiv \gamma'(a_\alpha+(b_\alpha p - a_\alpha\alpha)p\gamma) \bmod p^{n-1}.$$

We deduce that $\|U_p|_{(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)}\| \leq 1$.

Because $U_p(\mathcal{A}_L^{\mathrm{la},U})^{V(p^{n+1})-\mathrm{an}}(\chi) \subset (\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$ we see that $\det(1-TU_p|_{(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)})$ is independent of n for n large. We will denote it $\det(1-TU_p|_{\mathcal{A}_L^{\mathrm{la},U}(\chi)}) \in \mathcal{O}(\mathrm{Aff}^1_L)$. If $r \in p^{\mathbb{Q}}$ then there is a factorization

$$\det(1-TU_p|_{\mathcal{A}_L^{\mathrm{la},U}(\chi)}) = Q_r(T)P(T)$$

where

- $Q_r(T) \in L[T]$ with $Q_{A,r}(0) = 1$ and all slopes $\leq \log_p r$;
- $P(T) \in \mathcal{O}(\mathrm{Aff}^1_L)$ with $P(0)=1$ and all slopes $> \log_p r$.

For n sufficiently large, there is a slope $\leq \log_p r$ slope decomposition

$$\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{an}} = \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{an}}_{\leq \log_p r} \oplus \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{an}}_{>\log_p r}.$$

This decomposition is preserved by the action of $\mathbb{T}_{\mathcal{O}}(U)$. If $n' > n$ and n is sufficiently large, then

$$\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{an}}_{\leq \log_p r} = \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^{n'})-\mathrm{an}}_{\leq \log_p r},$$

and we will simply denote this $\mathcal{A}_L^{\mathrm{la},U}(\chi)_{\leq \log_p r}$. [The inclusion $\subset$ is clear. For the reverse inclusion use the fact that U_p is an isomorphism on $\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^{n'})-\mathrm{an}}_{\leq \log_p r}$ such that $U_p^{n'-n}\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^{n'})-\mathrm{an}}_{\leq \log_p r} \subset \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{an}}_{\leq \log_p r}$.] Then $\mathcal{A}_L^{\mathrm{la},U}(\chi)_{\leq \log_p r}$ is a $\mathcal{O}(D(r)_L)/(Q_r(T))$-module (where T acts by U_p) which is finite over L and has an action of $\mathbb{T}_{\mathcal{O}}(U)$. We write

$$\mathcal{A}_L^{\mathrm{la},U}(\chi)^{\mathrm{fs}} = \bigcup_r \mathcal{A}_L^{\mathrm{la},U}(\chi)_{\leq \log_p r},$$

the space of *finite slope locally analytic p-adic modular forms*. (Non finite slope locally analytic p-adic modular forms still seem rather mysterious.)

Because $\|U_p\| \leq 1$ we see that $\mathcal{A}_L^{\mathrm{la},U}(\chi)_{\leq h} = (0)$ if $h < 0$.

Suppose now that χ is *locally algebraic* in the sense that there exists $n \in \mathbb{Z}_{>0}$ and $k_1 \geq k_2$ in $\mathbb{Z}$ such that

$$\chi\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix} = a^{k_1}d^{k_2}$$

for all $a, d \in 1+p^n\mathbb{Z}_p$. Then we set

$$\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}$$

to be the set of $\varphi \in \mathcal{A}_L^{\mathrm{la},U}(\chi)$ such that

$$\varphi\left(x\begin{pmatrix} 1 & 0 \\ p^n c & 1 \end{pmatrix}\right)$$

is a polynomial of degree $\leq k_1 - k_2$ in $c \in \mathbb{Z}_p$ for all $x \in G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U$. This space is preserved by $\mathbb{T}_{\mathcal{O}}(U)$ and by U_p. [The first of these assertions is clear. For the latter we use equation 2.]

We will write $V_1(p^n)$ for the subgroup of $GL_2(\mathbb{Z}_p)$ consisting of elements whose reduction modulo p^n is upper triangular with diagonal entries 1. If $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_1(p^n)$ and $e \in p^n\mathbb{Z}_p$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (de+c)/(be+a) & 1 \end{pmatrix}\begin{pmatrix} a+be & b \\ 0 & (ad-bc)/(a+be) \end{pmatrix}.$$

Thus for $\varphi \in \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}$ we have

$$\varphi\left(xv\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}\right) = \det(v)^{k_2}(a+be)^{k_1-k_2}\varphi\left(x\begin{pmatrix} 1 & 0 \\ (de+c)/(be+a) & 1 \end{pmatrix}\right).$$

If W_{k_1,k_2} denotes the space of polynomials of degree $\leq k_1 - k_2$ over L we can make it a representation of $GL_2(\mathbb{Q}_p)$ via the formula

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} P)(T) = (ad-bc)^{k_2}(a+bT)^{k_1-k_2}P((dT+c)/(bT+a)).$$

This is easily checked to be a representation isomorphic to $\det^{-k_1} \otimes \mathrm{Symm}^{\,k_1-k_2}L^2$. We conclude that

$$\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}} \cong \mathcal{A}_L(W_{k_1,k_2})^{U\times V_1(p^n)},$$

where $\mathcal{A}_L(W_{k_1,k_2})^{U\times V_1(p^n)}$ denotes the space of 'classical' automorphic forms consisting of all functions

$$\psi : G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/U \longrightarrow W_{k_1,k_2}$$

such that

$$\psi(xv) = v^{-1}\psi(x)$$

for all $v \in V_1(p^n)$. The map sends $\varphi \in \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V_1(p^n)-\mathrm{alg}}$ to the function that sends

$$x \longmapsto \varphi\left(x\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right)$$

considered as a polynomial in t for $t \in p^n\mathbb{Z}_p$. This identification is equivariant for the action of the unramified Hecke operators T_v and S_v for $v \neq p$. Moreover it is equivariant for the action of U_p if we make U_p act on $\mathcal{A}_L(W_{k_1,k_2})^{U\times V_1(p^n)}$ via the formula

$$(U_p\psi)(x) = p^{k_1}\sum_{\alpha\in\mathbb{Z}/p\mathbb{Z}}\begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix}\psi\left(x\begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix}\right).$$

We deduce in particular that $\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}$ is finite dimensional.

If we fix an isomorphism $\imath : \mathbb{C} \xrightarrow{\sim} \overline{L}$ then $\mathcal{A}_L(W_{k_1,k_2})^{U\times V_1(p^n)} \otimes_{L,\imath^{-1}} \mathbb{C}$ is isomorphic to the space of functions

$$\Psi : G(\mathbb{Q})\backslash G(\mathbb{A})/(U \times V_1(p^n)) \longrightarrow \det^{-k_1} \otimes \mathrm{Symm}^{\,k_1-k_2}\mathbb{C}^2$$

such that
$$\Psi(xh) = h^{-1}\Psi(x)$$
for all $h \in G(\mathbb{R}) \hookrightarrow GL_2(\mathbb{C})$. The map sends ψ to
$$x \longmapsto x_\infty^{-1}(x_p\psi(x^\infty) \otimes 1).$$
This may help explain the adjective 'classical'. It is again equivariant for the action of the unramified Hecke operators T_v and S_v for $v \neq p$; and for the action of U_p if we make U_p act on Ψ as above by the formula
$$(U_p\Psi)(x) = p^{k_1} \sum_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \Psi\left(x\begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix}\right).$$
From the real analytic theory and some p-adic algebraic geometry, in particular Mazur's ('Newton over Hodge') theorem, one can deduce that the slopes of U_p on $\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V_1(p^n)-\mathrm{alg}}$ are all $\leq k_1 - k_2 + 1$.

As $\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}$ and $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$ have weight decompositions, so does
$$(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)/\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}.$$
It follows from equation 3 that
$$\|U_p|_{(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)/\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}}\| \leq |p|^{k_1-k_2+1}.$$
Thus the slopes of U_p on $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)/\mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}$ are all $\geq k_1 - k_2 + 1$. Thus
$$\mathcal{A}_L^{\mathrm{la},U}(\chi)_{<(k_1-k_2+1)} \subset \mathcal{A}_L^{\mathrm{la},U}(\chi)^{V(p^n)-\mathrm{alg}}.$$
This is a version of Coleman's famous 'control theorem': *locally analytic p-adic forms of small slope are classical.*

20.3. **The eigencurve.** A fundamental observation of Coleman and Mazur is that finite slope locally analytic p-adic forms move in p-adic families - these families are called eigenvarieties. If some property of finite slope locally analytic forms can be shown to vary continuously on an eigenvariety, one can hope to use this to prove the property for all such forms from knowing it for some especially easy dense set of points. I'm afraid I don't have time to give examples of this, but will have to content myself with sketching the construction of the eigenvariety for automorphic forms on G. The main point is to copy what we have done over L over an arbitrary affinoid subdomain of weight space.

If A is an affinoid algebra we can define $\mathcal{A}_A^{\mathrm{la},U}$ and $(\mathcal{A}_A^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}$ just as we defined $\mathcal{A}_L^{\mathrm{la},U}$ and $(\mathcal{A}_L^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}$. They still have commuting actions of $\mathbb{T}_{\mathcal{O}}(U)$ and $GL_2(\mathbb{Q}_p)$ (because the ambient space $\mathcal{A}_A^{\mathrm{cts},U} = A\widehat{\otimes}_\pi\mathcal{A}_L^{\mathrm{cts},U}$ does). If $\chi : T(\mathbb{Z}_p) \to A^\times$ is a continuous character we can also define $\mathcal{A}_A^{\mathrm{la},U}(\chi)$ and $(\mathcal{A}_A^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$ just as we did $\mathcal{A}_L^{\mathrm{la},U}(\chi)$ and $(\mathcal{A}_A^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$. These spaces are preserved by $\mathbb{T}_{\mathcal{O}}(U)$.

We can also define
$$U_p : \mathcal{A}_L^{\mathrm{la},U}(\chi) \longrightarrow \mathcal{A}_L^{\mathrm{la},U}(\chi)$$
by
$$(U_p\varphi)(x) = \sum_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \varphi\left(x\begin{pmatrix} p & \alpha \\ 0 & 1 \end{pmatrix}\right),$$

where we think of p and α as elements of $\mathbb{Q}_p$. For n-sufficiently large, this sends $(\mathcal{A}_A^{\mathrm{la},U})^{V(p^{n+1})-\mathrm{an}}(\chi)$ to $(\mathcal{A}_A^{\mathrm{la},U})^{V(p^n)-\mathrm{an}}(\chi)$ and hence defines an A-linear completely continuous endomorphism of $(\mathcal{A}_A^{\mathrm{la},U})^{V(p^{n+1})-\mathrm{an}}(\chi)$. We obtain a characteristic power series

$$\det(1 - TU_p|_{\mathcal{A}_A^{\mathrm{la},U}(\chi)}) \in \mathcal{O}(\mathrm{Aff}^1_A).$$

If $\psi : A \to B$ then

$$\psi \det(1 - TU_p|_{\mathcal{A}_A^{\mathrm{la},U}(\chi)}) = \det(1 - TU_p|_{\mathcal{A}_B^{\mathrm{la},U}(\psi\chi)}).$$

Now write $\mathcal{W}$ for $\mathcal{W}_{T(\mathbb{Z}_p)}$. We get a universal

$$\det(1 - TU_p|_{\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})}) \in \mathcal{O}(\mathcal{W} \times \mathrm{Aff}^1) = \varinjlim \mathcal{O}(\mathrm{Aff}^1_{\mathcal{O}(D(r)^2)}),$$

the limit taken as $r \to \infty$. We can define a rigid analytic subvariety $\mathcal{S}_U \subset \mathcal{W} \times \mathrm{Aff}^1$, the vanishing locus of $\det(1-TU_p|_{\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})})$. We call $\mathcal{S}_U$ the *spectral curve*. Put another way for each affinoid subdomain $\mathrm{Sp}\, A \subset \mathcal{W}$ and each $r \in p^{\mathbb{Q}}$ the power series $\det(1 - TU_p|_{\mathcal{A}_A^{\mathrm{la},U}(\chi^{\mathrm{univ}})})$ defines a closed affinoid rigid analytic subvariety of $\mathrm{Sp}\, A \times D(r)$, namely

$$\mathrm{Sp}\, \mathcal{O}(D(r)_A)/(\det(1 - TU_p|_{\mathcal{A}_A^{\mathrm{la},U}(\chi)})).$$

These affinoid varieties agree on intersections and so provide an admissible covering of the spectral curve $\mathcal{S}_U$.

We will call a pair $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ where $\mathrm{Sp}\, A \subset \mathcal{W}$ is a (necessarily reduced) affinoid domain and $r \in p^{\mathbb{Q}}$ *suitable* if there is a factorization

$$\det(1 - TU_p|_{\mathcal{A}_A^{\mathrm{la},U}(\chi^{\mathrm{univ}})}) = Q_{A,r}(T)P(T)$$

where

- $Q_{A,r}(T) \in A[T]$ is multiplicative (i.e. its leading coefficient is a unit), has $Q_{A,r}(0) = 1$ and has all slopes $\leq \log_p r$;
- $P(T) \in \mathcal{O}(\mathrm{Aff}^1_A)$ with $P(0) = 1$ and all slopes $> \log_p r$.

(The terminology 'suitable' is non-standard - I made up the word.) In this case one can check that

- $P(T) \in \mathcal{O}(D(r)_A)^\times$;
- for any multiplicative $R(T) \in A[T]$ with all slopes $\leq \log_p r$ (including $Q_{A,r}(T)$) we have
$$(P(T), R(T)) = \mathcal{O}(\mathrm{Aff}^1_A).$$

[The first assertion is proved by using the usual power series expansion of $(1 + Tf(T))^{-1}$. To prove the second assertion one can argue by contradiction, and so assume that (P, R) is contained in a maximal ideal $\mathfrak{m}$ of $\mathcal{O}(\mathrm{Aff}^1_A)$. One shows that the residue field of $\mathfrak{m}$ is a finite extension of L and deduces that the preimage $\mathfrak{m}^c$ of $\mathfrak{m}$ in A is also maximal. Working modulo $\mathfrak{m}^c$ one reduces to the special case that A is a finite field extension of L. In this case it follows from our previous analysis of $\mathcal{O}(\mathrm{Aff}^1_A)$ for A a finite extension of $\mathbb{Q}_p$.] Note that if $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ is suitable and if $\mathrm{Sp}\, A' \subset \mathrm{Sp}\, A$ is an affinoid subdomain, then $(\mathrm{Sp}\, A' \subset \mathcal{W}, r)$ is also suitable.

The spaces $D(r)_A$ for which $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ is suitable form an admissible cover of $\mathrm{Aff}^1 \times \mathcal{W}$ by affinoid subdomains, which we will also refer to as 'suitable'. [I alluded to this last week in lectures. Given $x \in \mathcal{W}$ and $r \in p^{\mathbb{Q}}$ one defines a suitable affinoid neighbourhood $\mathrm{Sp}\, A$ of x. Then one constructs a factorisation $\det(1 - TU_p|_{\mathcal{A}_A^{\mathrm{la},U}(\chi^{\mathrm{univ}})}) = Q(T)P(T)$ by a successive approximation. Initially one just constructs $P(T) \in A[\![T]\!]$, and then proves it lies in $\mathcal{O}(\mathrm{Aff}^1_A)$ by looking at its Newton polygon.] The affinoid rings

$$\mathcal{O}(D(r)_A)/(Q_{A,r}(T))$$

for $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ suitable give an admissible covering of $\mathcal{S}_U$ by affinoid domains, which we will call 'suitable affinoid subdomains'.

If $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ is suitable then, for n sufficiently large, there is a slope $\leq \log_p r$ slope decomposition

$$\mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})^{V(p^n)-\mathrm{an}} = \mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})^{V(p^n)-\mathrm{an}}_{\leq \log_p r} \oplus \mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})^{V(p^n)-\mathrm{an}}_{> \log_p r}.$$

(This is not true for general (non-suitable) $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$.) This decomposition is preserved by the action of $\mathbb{T}_{\mathcal{O}}(U)$. If $n' > n$ and n is sufficiently large, then, as in the case $A = L$,

$$\mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})^{V(p^n)-\mathrm{an}}_{\leq \log_p r} = \mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})^{V(p^{n'})-\mathrm{an}}_{\leq \log_p r},$$

and we will simply denote this $\mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})_{\leq \log_p r}$. Then $\mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})_{\leq \log_p r}$ is a module over $\mathcal{O}(D(r)_A)/(Q_{A,r}(T))$ (where T acts by U_p^{-1}) which is finite projective over A. These modules patch to give a coherent sheaf $\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}$ of $\mathcal{O}_{\mathcal{S}_U}$-modules over $\mathcal{S}_U$, which also carries an action of $\mathbb{T}_{\mathcal{O}}(U)$. [If $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ and $(\mathrm{Sp}\, A' \subset \mathcal{W}, r')$ are suitable and if $\mathrm{Sp}\, A'' \subset \mathrm{Sp}\, A \cap \mathrm{Sp}\, A'$, then $(\mathrm{Sp}\, A'' \subset \mathcal{W}, r)$ and $(\mathrm{Sp}\, A'' \subset \mathcal{W}, r')$ are again suitable. So we have two things to check:

(1) If $\mathrm{Sp}\, A' \subset \mathrm{Sp}\, A \subset \mathcal{W}$ and if both $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ and $(\mathrm{Sp}\, A' \subset \mathcal{W}, r)$ are suitable, then

$$A' \otimes_A \mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})_{\leq \log_p r} = \mathcal{A}^{\mathrm{la},U}_{A'}(\chi^{\mathrm{univ}})_{\leq \log_p r}.$$

(2) If $r' \leq r$ are in $p^{\mathbb{Q}}$ and if $(\mathrm{Sp}\, A \subset \mathcal{W}, r)$ and $(\mathrm{Sp}\, A \subset \mathcal{W}, r')$ are both suitable, then

$$\mathcal{O}(D(r')_A)/(Q_{A,r'}(T)) \otimes_{\mathcal{O}(D(r)_A)/(Q_{A,r}(T))} \mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})_{\leq \log_p r} = \mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})_{\leq \log_p r'}.$$

The first property follows from the functoriality of slope decomposition with respect to the Banach algebra A. For the second one shows that there is a factorization

$$Q_{A,r}(T) = Q_{A,r'}(T)P(T)$$

where $P(T) \in A[T]$ and $(P(T), Q_{A,r'}(T)) = A[T]$. This can be done in the same way that we factorize $\det(1 - TU_p|_{\mathcal{A}^{\mathrm{la},U}_A(\chi^{\mathrm{univ}})}) = Q_{A,r}(T)P(T)$.]

The sheaf $\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}$ is finite over $\mathcal{S}_U$. On any suitable affinoid subdomain it is finite projective over the image of that subdomain in $\mathcal{W}$. The fibre $\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}_w$ of $\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}$ at a point $w \in \mathcal{W}$ is identified with $\mathcal{A}^{\mathrm{la},U}_L(\chi^{\mathrm{univ}}_w)^{\mathrm{fs}}$. Thus the sections of $\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}$ are p-adic families of finite slope locally analytic p-adic automorphic forms. As $\mathbb{T}_{\mathcal{O}}(U)$ acts on $\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}$ it generates a sheaf of commutative $\mathcal{O}_{\mathcal{S}_U}$-algebras

$$\mathcal{T}(U) \subset \mathcal{E}nd_{\mathcal{O}_{\mathcal{S}_U}}(\mathcal{A}^{\mathrm{la},U}(\chi^{\mathrm{univ}})^{\mathrm{fs}}).$$

This is a version of the Hecke algebra on finite slope locally analytic p-adic automorphic forms but spread out over weight space. It is finite over $\mathcal{O}_{\mathcal{S}_U}$. One can form the relative Sp of $\mathcal{T}(U)$ over $\mathcal{S}_U$. We get another rigid analytic variety $\mathcal{E}_U \to \mathcal{S}_U$ called the *eigenvariety*. Its points correspond to systems of Hecke eigenvalues on finite slope locally analytic p-adic automorphic forms, but they are organized into rigid analytic families.

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