Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2019.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof.

   (i) **Claim 1.1:** For any integer n >2, $n! - 1$ >n.

   Since n >2, $n! \geq 2n$ >n+1. So we have that for any integer >2, $n! - 1$ >n.

   Claim 2.2: $\forall$ $n \in \mathbb{N}$ with n $\geq$ 2, it has prime factorization.

   Define $P(n)$ be the statement that "n is either prime or product of two or more primes". We only have to prove that $P(n)$ is true for every n $\geq$ 2.

   First, $P(2)$ is true because 2 is a prime. Assume $P(k)$ is true for $k \geq 2$, then our goal is to prove $P(k+1)$ is true.

   According to Strong Principle, for $k \geq 2$ and $2 \leq n \leq$ k, $P(n)$ is true.

   If $P(k+1)$ is a prime, it's obviously that $P(k+1)$ is true.

   If $P(k+1)$ is not a prime, by definition of a prime, $k+1 = r \times s$, where r and s are positive integers greater than 1 and less than $k+1$. It follows that $2 \leq r \leq$ k and $2 \leq s \leq$ k. Thus by induction hypothesis, both r and s are either prime or the product of two or more primes. Thus, $P(k+1)$ is true.

   (ii)It obviously that $n!$ is not a prime,so from Claim 2.2, we can know that $n!$ has prime factorization, so there must be a prime p that satisfies p <$n!$. To show p must be greater than n, we can first assume that $p \leq n$. From Claim 1.1 and 1.2 we know that $n! - 1$ has prime factor, we can assume that p is a factor of $n! - 1$ which is smaller than n.

   Then p must also be a factor of $n!$, because p is both the factor of $n!$ and $n! - 1$, p can only be 1, which contradicts the fact that p is a prime.

   Therefore, the assumption is true. □

2. Use the minimal counterexample principle to prove that for any integer $n > 17$, there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   Proof. First, we will show some base case when 17 <n <32.

$$18 = 1 \times 4 + 2 \times 7$$
$$19 = 3 \times 4 + 1 \times 7$$
$$20 = 5 \times 4 + 0 \times 7$$
$$21 = 0 \times 4 + 3 \times 7$$

   Definition 2.1: $P(n)$:there exist integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 4 + j_n \times 7$.

   We have shown that $P(n)$ is true for 17 <n <32. Assume that $P(k)$ is true. Then $k + 4 = i_k \times 4 + j_k \times 7 + 4 = (i_k + 1) \times 4 + j_k \times 7$, let $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$, so $P(k+1)$ is true.

   Therefore, the assumption is true. □

3. Suppose $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, and $a_k = a_{k-1} + a_{k-2} + a_{k-3}$ for $k \geq 3$. Use the strong principle of mathematical induction to prove that $a_n \leq 2^n$ for any integer $n \geq 0$.

Proof.

$$\text{When n=0, } a_0 = 1 \leq 2^0.$$
$$\text{When n=1, } a_1 = 2 \leq 2^1.$$
$$\text{When n=2, } a_2 = 2 \leq 2^2.$$

Assume that $a_k \leq 2^k$, then $a_{k+1} = a_k + a_{k-1} + a_{k-2} \leq 2^k + 2^{k-1} + 2^{k-2} \leq 2^k + 2^{k-1} + 2^{k-1} \leq 2^{k+1}$, so the assumption is true when $n = k + 1$.

Therefore, the assumption is true. □

4. Prove, by mathematical induction, that

$$(n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots + (2n)^2 = \frac{n(2n+1)(7n+1)}{6}$$

is true for any integer $n \geq 1$.

Proof.
First we can show that when $n = 1$,the equation is true.

$$(1+1)^2 = \frac{1 \times 3 \times 8}{6}$$

Definition 4.1: $P(n)$: $(n+1)^2 + (n+2)^2 + (n+3)^2 + \cdots + (2n)^2 = \dfrac{n(2n+1)(7n+1)}{6}$

Assume $P(k)$ is true, $k \geq 1$, then when $n = k + 1$, the right side of the equation is $\dfrac{k(2k+1)(7k+1)}{6} + (2k+1)^2 + (2k+2)^2 - (k+1)^2 = \dfrac{(k+1)(2(k+1)+1)(7(k+1)+1)}{6}$.
So, the equation is true for any integer $n \geq 1$. □

Remark: You need to include your .pdf and .tex files in your uploaded .rar or .zip file.