Lab03-Greedy Strategy

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2019.

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1. Suppose there is a street with length n, described by an array $A[1...n]$ where $A[i] = 1$ means that there is a house at position i and $A[i] = 0$ means position i is vacant.

   According to some law, every house must be protected by fire hydrant. If a fire hydrant is placed at position i, then all houses at postion $i-1, i, i+1$ will be considered protected. Note that hydrants can be placed at the same place with a house.

   Using what you learnt in class, please design an algorithm that computes the minimum number of hydrants needed to protect all houses. You need to write pseudo code, analyze the time complexity, and prove its correctness.

   Proof.
   We start at position 1 and check each position sequentially. We will place a fire hydrant at position i+1, then the position i and position i+2 will be both under protection. Then we will continue the procedure at the position i+3. The algorithm will shown in Alg.1.

   Algorithm 1: GreedyPlacing

   Input: An array $A[1, \cdots, n]$
   Output: the minimum num of hydrant

   ```
   i ← 1;
   N ← 0;
   for i ← 1 to n do
       if A[i]=0 then
           i ← i + 1
       else
           N ← N + 1;
           i ← i + 3
   return N
   ```

   It is obvious that the time complexity of this algorithm is O(n).

   Correctness:

   We can prove the correctness of the greedy algorithm by contradiction.

   Assume that greedy algorithm does not provide the optimal solution.

   Let $i_1, i_2, ..., i_k (i_1<i_2<...<i_k)$ be the set of positions selected by greedy algorithm.

   Let $j_1, j_2, ..., j_m (j_1<j_2<...<j_m)$ be the set of positions in an optimal solution with $i_1 = j_1, i_2 = j_2, ..., i_r = j_r$ for the largest possible value of r. Then we know position 1 to i_r+1 is protected.

   Now consider the $(r+1)^{th}$ hydrant. Suppose the next closest to i_r house without protection is at position x. To protect the house at position x, we can place the hydrant at position $x-1$, x or $x+1$.

   In our greedy algorithm, we will always place the hydrant at position $x+1$. If the OPT choose the position x or $x-1$, there will be unnecessary waste of space. So the greedy algorithm is optimal.

   □

2. (a) Given a set A containing n real numbers, and you are allowed to choose k numbers from A. The bigger the sum of the chosen numbers is, the better. What is your algorithm to choose? Prove its correctness using **Matroid**.

Remark: This is a very easy problem. Denote $\mathbf{C}$ be the collection of all subsets of A that contains no more than k elements. Try to prove $(A, \mathbf{C})$ is a matroid.

Proof.
Denote $\mathbf{C}$ be the collection of all subsets of A that contains no more than k elements. If $D \subset E$, $E \in \mathbf{C}$, obviously $|D|<|E| \leq k$ and $D \subset A$. Thus $(A, \mathbf{C})$ is an independent system. We apply the Greedy-MAX algorithm to $(A, \mathbf{C})$. We define the associated strictly positive function $c(x)$ to each element $x \in A$. Let $c(x) = x - min\{x|x \in A\} + 1$. The greedy-MAX algorithm is shown in Alg.2.

Algorithm 2: Greedy-MAX

Input: k,the number of elements to choose
Output: A set I

Sort all elements in A into ordering $c(x_1) \geq c(x_2) \geq ... \geq c(x_n)$
$I \leftarrow \emptyset$
for $i \leftarrow 1$ *to* n **do**
  if $|I \cup \{x_i\}| \leq k$ **then**
    $I \leftarrow I \cup \{x_i\}$
return I

Correctness:Firstly we try to prove $(A, \mathbf{C})$ is a matroid.

(i) Hereditary: If $D \subset \mathbf{C}$, $E \in \mathbf{C}$, obviously $|D|<|E| \leq k$ and $D \subset A$, so $D \in \mathbf{C}$ is true.

(ii) Exchange Property: Consider $D, E \in \mathbf{C}$ with $|D|<|E|$. Because $|E| \leq k$, $|D| \leq k-1$. Choose an element x in E that is not in D. Then $|D \cup \{x\}| \leq k$ and $D \cup \{x\}$ is a subset of A. So $D \cup \{x\} \in \mathbf{C}$ is true.

So $(A, \mathbf{C})$ is a matroid. We can prove the correctness of Greedy-MAX algorithm to solve this problem. □

(b) Consider that $B_1, B_2...B_n$ are n disjoint sets, and let d_i be integers with $0 \leq d_i \leq |B_i|$. Define $\mathbf{C}$ is a collection of set $X \subseteq \cup_{i=1}^n B_i$, where X has such property:

$$\forall i \in \{1, 2, 3...n\}, |X \cap B_i| \leq d_i$$

Prove that $(\cup_{i=1}^n B_i, \mathbf{C})$ is a matroid.

Remark: You may easily find that the matroid in (a) is a special case of matroid in (b).

Proof. (i) Hereditary: If $D \subset \mathbf{C}$, $E \in \mathbf{C}$, $\forall i \in 1, 2, ..., n$, $|D \cap B_i| \leq |E \cap B_i| \leq d_i$, so $D \in \mathbf{C}$ is true.

(ii) Exchange Property: Consider $D, E \in \mathbf{C}$ with $|D|<|E|$. As $|D|<|E|$, at least for one set B_x there exists $|D \cap B_x|<|E \cap B_x| \leq d_x$.

Choose an element $x \in B_x$ in E that is not in D. For the set B_x, $|(D \cup \{x\}) \cap B_x|=|D \cap B_x| + 1 \leq d_x$. Since $B_1, B_2...B_n$ are disjoint sets, x does not belong to any other set except for B_x. So $|(D \cup \{x\}) \cap B_i|=|D \cap B_i| \leq d_i$. Therefore we prove $D \cup \{x\} \in \mathbf{C}$.

So $(\cup_{i=1}^n B_i, \mathbf{C})$ is a matroid. □

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