Ensemble Learning

Overview

In this short lab we will go into some "hands-on" aspects of ensemble learning, covering aspects of the bias-variance decomposition, random forests (bagged decision trees with random subspaces) and boosting algorithms discussed in lectures. The lab is in the form of a walk-though with questions you should try to answer for yourself before reading through the solutions.

Acknowledgement

This notebook contains several excerpts from the Python Data Science Handbook (http://shop.oreilly.com/product/0636920034919.do) by Jake VanderPlas (one of the original sklearn developers); the original content is available on GitHub (https://github.com/jakevdp/PythonDataScienceHandbook).

The text is released under the CC-BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/3.0/us/legalcode), and code is released under the MIT license (https://opensource.org/licenses/MIT). If you find this content useful, please consider supporting the work by buying the book (http://shop.oreilly.com/product/0636920034919.do)!

The boosting example is modified from code from Chapter 13 of Machine Learning: An Algorithmic Perspective (2nd Edition) by Stephen Marsland (http://stephenmonika.net (http://stephenmonika.net))

Introduction

As with previous labs, the objective is to expose some of the methods and to show some visualizations, since although some of the methods appear quite complex, many of the underlying ideas are quite straightforward. We will use the `Seaborn` visualization package which builds on and extends the capabilities of the standard `Matplotlib` package. *You will need to install these if you want to run this notebook locally.*

In [2]:
```
import numpy as np
%matplotlib inline
import matplotlib.pyplot as plt
import seaborn as sns
from ipywidgets import interact
```

Bias-Variance

Recall the usual setting: suppose we train a model on some data, then test on some data *not* used for training, and we evaluate the observed test error as an estimate of the *generalization error* of the model.

Question: how could *bias* be observed in the estimation of error for a learned model?

Question: how could *variance* be observed in the estimation of error for a learned model?

Think of how you would answer these questions before proceeding.

To answer these questions we will attempt firstly to visualize learning outcomes where the effect of bias and variance on error are clearly evident.

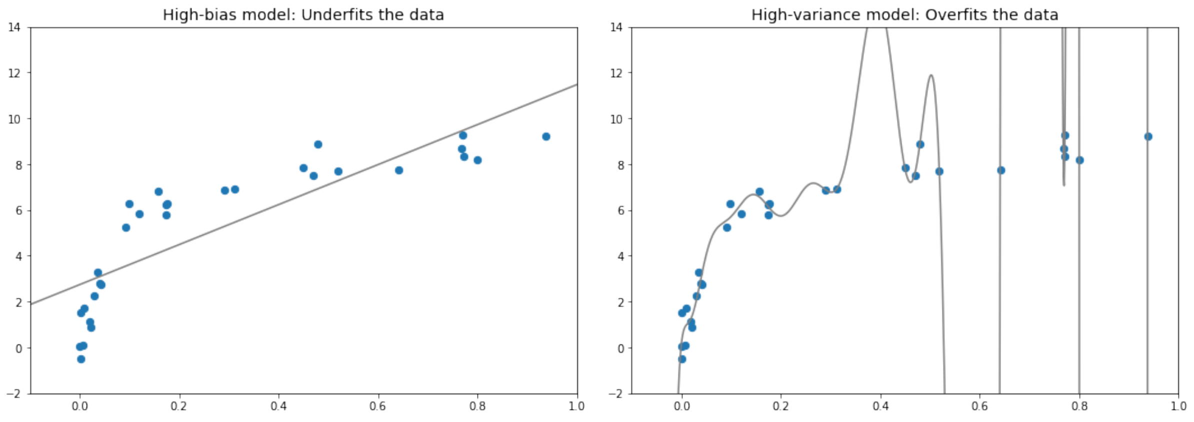
Bias-Variance Tradeoff

We'll start by using regression for this visualization, and compare linear to non-linear models.

In [3]:
```
from sklearn.preprocessing import PolynomialFeatures
from sklearn.linear_model import LinearRegression
from sklearn.pipeline import make_pipeline

def PolynomialRegression(degree=2, **kwargs):
    return make_pipeline(PolynomialFeatures(degree),
                         LinearRegression(**kwargs))
```

Next let's generate some data, and fit the models.

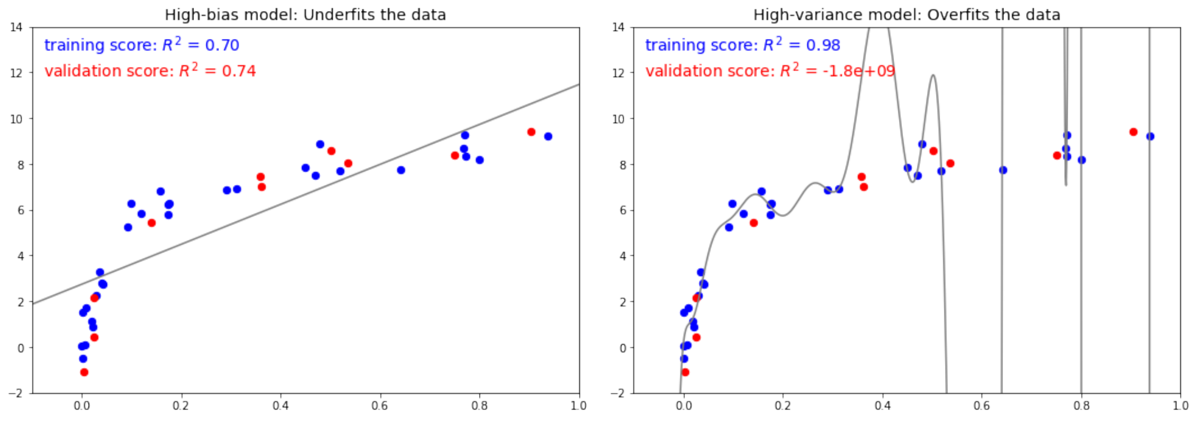
In [4]:
```python
def make_data(N=30, err=0.8, rseed=1):
    # randomly sample the data
    rng = np.random.RandomState(rseed)
    X = rng.rand(N, 1) ** 2
    y = 10 - 1. / (X.ravel() + 0.1)
    if err > 0:
        y += err * rng.randn(N)
    return X, y

X, y = make_data()
xfit = np.linspace(-0.1, 1.0, 1000)[:, None]
model1 = PolynomialRegression(1).fit(X, y)
model20 = PolynomialRegression(20).fit(X, y)

fig, ax = plt.subplots(1, 2, figsize=(16, 6))
fig.subplots_adjust(left=0.0625, right=0.95, wspace=0.1)

ax[0].scatter(X.ravel(), y, s=40)
ax[0].plot(xfit.ravel(), model1.predict(xfit), color='gray')
ax[0].axis([-0.1, 1.0, -2, 14])
ax[0].set_title('High-bias model: Underfits the data', size=14)

ax[1].scatter(X.ravel(), y, s=40)
ax[1].plot(xfit.ravel(), model20.predict(xfit), color='gray')
ax[1].axis([-0.1, 1.0, -2, 14])
ax[1].set_title('High-variance model: Overfits the data', size=14);
```



Looking at these plots can tell us something about the error. The first plot shows a linear model. Since the data appears to be non-linear, the error will have a considerable *bias* component, since the model can never fit all the data points. In the second plot the polynomial regression appears to have fit the data almost completely. However, the error of this model will have a considerable *variance* component due to the complexity of the fitted polynomial.

Bias-Variance Tradeoff Metrics

In [5]:
```
fig, ax = plt.subplots(1, 2, figsize=(16, 6))
fig.subplots_adjust(left=0.0625, right=0.95, wspace=0.1)

X2, y2 = make_data(10, rseed=42)

ax[0].scatter(X.ravel(), y, s=40, c='blue')
ax[0].plot(xfit.ravel(), model1.predict(xfit), color='gray')
ax[0].axis([-0.1, 1.0, -2, 14])
ax[0].set_title('High-bias model: Underfits the data', size=14)
ax[0].scatter(X2.ravel(), y2, s=40, c='red')
ax[0].text(0.02, 0.98, "training score: $R^2$ = {0:.2f}".format(mod
           ha='left', va='top', transform=ax[0].transAxes, size=14,
ax[0].text(0.02, 0.91, "validation score: $R^2$ = {0:.2f}".format(m
           ha='left', va='top', transform=ax[0].transAxes, size=14,

ax[1].scatter(X.ravel(), y, s=40, c='blue')
ax[1].plot(xfit.ravel(), model20.predict(xfit), color='gray')
ax[1].axis([-0.1, 1.0, -2, 14])
ax[1].set_title('High-variance model: Overfits the data', size=14)
ax[1].scatter(X2.ravel(), y2, s=40, c='red')
ax[1].text(0.02, 0.98, "training score: $R^2$ = {0:.2g}".format(mod
           ha='left', va='top', transform=ax[1].transAxes, size=14,
ax[1].text(0.02, 0.91, "validation score: $R^2$ = {0:.2g}".format(m
           ha='left', va='top', transform=ax[1].transAxes, size=14,
```

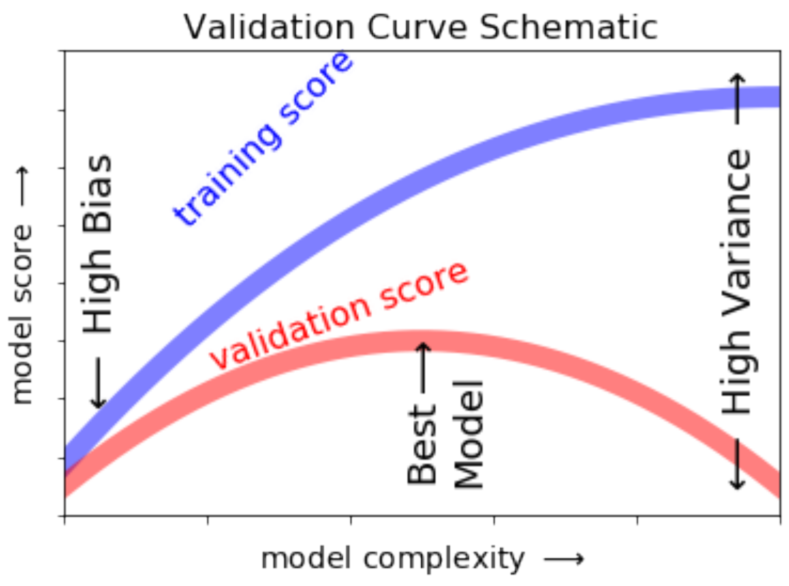


The R^2 metric (coefficient of determination) is an estimate of how well a model fits a dataset.

For the linear regression model, goodness of fit is about the same on training set and validation set. This suggests variance of this model is not high. However, linear regression does not fit the training set as well as polynomial regression, suggesting bias of the linear model is too high. On the other hand, polynomial regression does not fit the validation set well, suggesting the variance of this model is too high.

Validation Curve

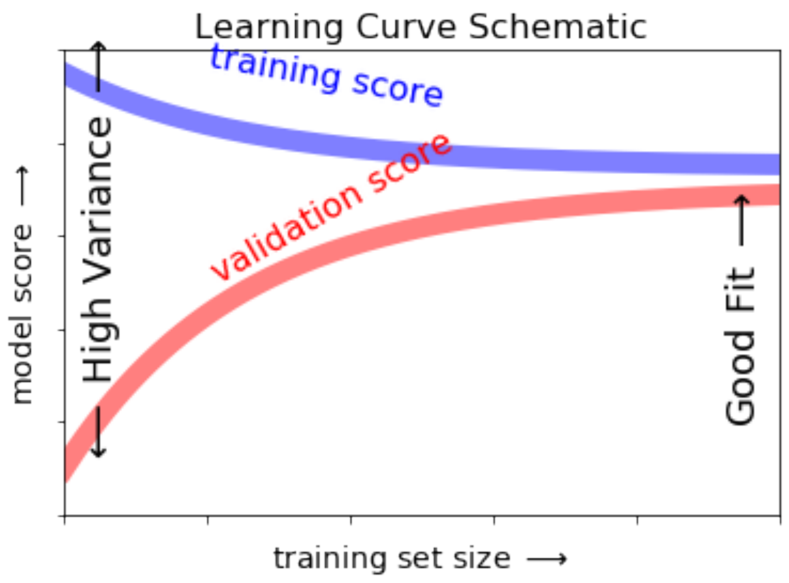
In [6]:
```
x = np.linspace(0, 1, 1000)
y1 = -(x - 0.5) ** 2
y2 = y1 - 0.33 + np.exp(x - 1)

fig, ax = plt.subplots()
ax.plot(x, y2, lw=10, alpha=0.5, color='blue')
ax.plot(x, y1, lw=10, alpha=0.5, color='red')

ax.text(0.15, 0.2, "training score", rotation=45, size=16, color='b
ax.text(0.2, -0.05, "validation score", rotation=20, size=16, color

ax.text(0.02, 0.1, r'$\longleftarrow$ High Bias', size=18, rotation
ax.text(0.98, 0.1, r'$\longleftarrow$ High Variance $\longrightarro
ax.text(0.48, -0.12, 'Best$\\longrightarrow$\nModel', size=18, rota

ax.set_xlim(0, 1)
ax.set_ylim(-0.3, 0.5)

ax.set_xlabel(r'model complexity $\longrightarrow$', size=14)
ax.set_ylabel(r'model score $\longrightarrow$', size=14)

ax.xaxis.set_major_formatter(plt.NullFormatter())
ax.yaxis.set_major_formatter(plt.NullFormatter())

ax.set_title("Validation Curve Schematic", size=16);
```



This plot shows the trade-off of model complexity and model performance using a scoring metric for both training set and validation (off-training-set) data. The falling off in validation score reflects overfitting of the training set.

Learning Curve

In [7]:
```
N = np.linspace(0, 1, 1000)
y1 = 0.75 + 0.2 * np.exp(-4 * N)
y2 = 0.7 - 0.6 * np.exp(-4 * N)

fig, ax = plt.subplots()
ax.plot(x, y1, lw=10, alpha=0.5, color='blue')
ax.plot(x, y2, lw=10, alpha=0.5, color='red')

ax.text(0.2, 0.88, "training score", rotation=-10, size=16, color='
ax.text(0.2, 0.5, "validation score", rotation=30, size=16, color='

ax.text(0.98, 0.45, r'Good Fit $\longrightarrow$', size=18, rotatio
ax.text(0.02, 0.57, r'$\longleftarrow$ High Variance $\longrightarr

ax.set_xlim(0, 1)
ax.set_ylim(0, 1)

ax.set_xlabel(r'training set size $\longrightarrow$', size=14)
ax.set_ylabel(r'model score $\longrightarrow$', size=14)

ax.xaxis.set_major_formatter(plt.NullFormatter())
ax.yaxis.set_major_formatter(plt.NullFormatter())

ax.set_title("Learning Curve Schematic", size=16);
```



This plot shows the effect of increasing the sample size of data used in learning – "big data" can enable complex models to be learned without overfitting the training set.

Bias-Variance in Validation

We can now summarise what we have visualized as follows:

a high bias learner will *tend to* underfit

a high variance learner will *tend to* overfit

Although can only be a general guideline, it gives us at least a partial answer to the questions we asked above of how we can detect bias and variance when attempting to validate machine learning methods in terms of their error.

In [ ]:

```

```

Decision Tree Example

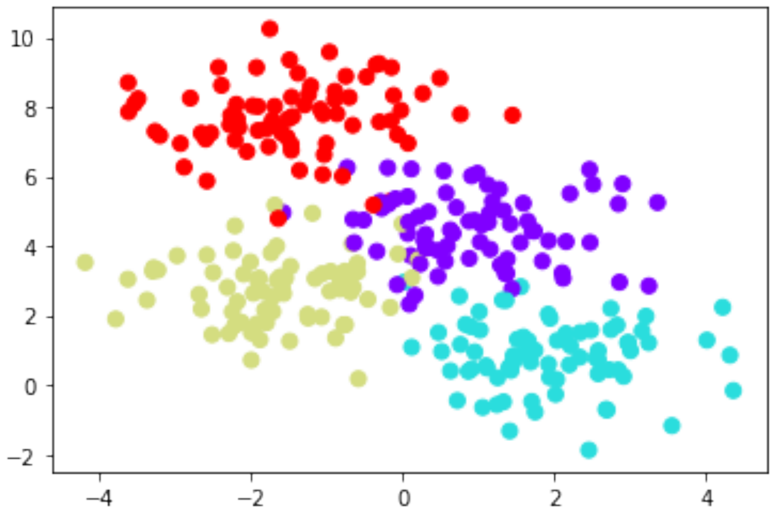
The bias-variance decomposition does not have the same mathematical derivation for classification as it does for regression with squared error. Nonetheless, we can still consider the role of a bias and a variance component in classification error.

Here is an instance space which contains randomly generated examples centred in 4 regions of the 2D area. Each region represents a different class.

In [9]:

```python
from sklearn.tree import DecisionTreeClassifier
from sklearn.datasets import make_blobs
from helpers_ensembles import visualize_tree

X, y = make_blobs(n_samples=300, centers=4,
                  random_state=0, cluster_std=1.0)
plt.scatter(X[:, 0], X[:, 1], c=y, s=50, cmap='rainbow');
```

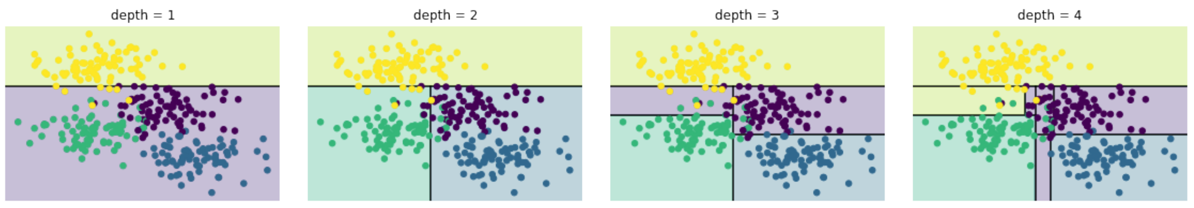


Decision Tree Levels

Here is an example of learning a decision tree at successive depths of refinement, where at each depth a further refinement of the tree is applied to the instance space. At each depth, the decision tree gives a finer partition of the instance space, resulting in a lower-error classification.

In [10]:
```
fig, ax = plt.subplots(1, 4, figsize=(16, 3))
fig.subplots_adjust(left=0.02, right=0.98, wspace=0.1)

for axi, depth in zip(ax, range(1, 5)):
    model = DecisionTreeClassifier(max_depth=depth)
    visualize_tree(model, X, y, ax=axi)
    axi.set_title('depth = {0}'.format(depth))
```



Decision Tree Overfitting

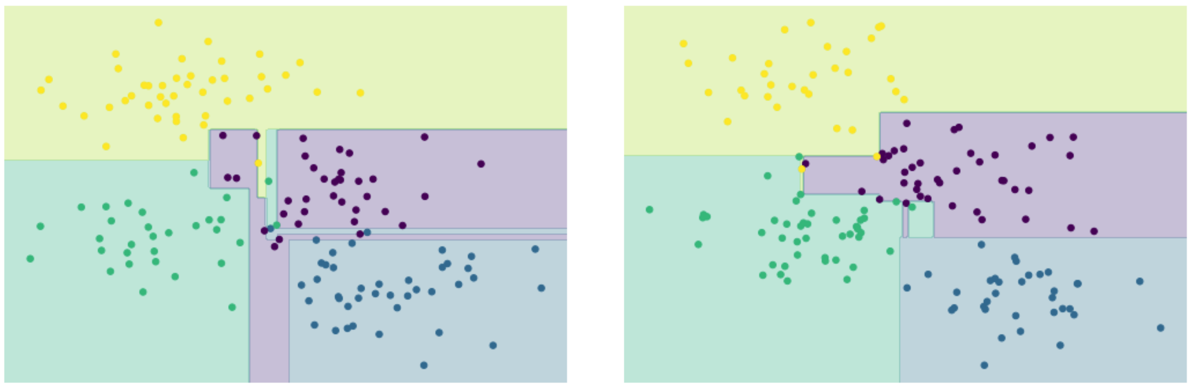
Question: What kind of problem do you think decision tree is likely to have when the depth goes deeper?

The flexibility of the tree in fitting the training data results in the potential to *completely* fit the training data, and hence is very likely to overfit, if this is not controlled in some way.

One way to see the effect of this is to look at models trained on different subsets of the data—for example, in this figure we train two different trees, each on half of the original data:

In [11]:
```
model = DecisionTreeClassifier()

fig, ax = plt.subplots(1, 2, figsize=(16, 6))
fig.subplots_adjust(left=0.0625, right=0.95, wspace=0.1)
visualize_tree(model, X[::2], y[::2], boundaries=False, ax=ax[0])
visualize_tree(model, X[1::2], y[1::2], boundaries=False, ax=ax[1])
```



Question: What do you think is going on in this visualization? What do you think can be done to improve this result?

It is clear that in some places, the two trees produce consistent results (e.g., in the four corners), while in other places, the two trees give very different classifications (e.g., in the regions between any two clusters). The key observation is that the inconsistencies tend to happen where the classification is less certain, and thus by using information from *both* of these trees, we might come up with a better result!

In [ ]:
```
```

Ensembles of Estimators: Random Forests

Question: What could we do to make use of multiple overfitting estimators to reduce the effect of overfitting?

This notion—that multiple overfitting estimators can be combined to reduce the effect of this overfitting—is what underlies an ensemble method called *bagging*. Bagging makes use of an ensemble (a grab bag, perhaps) of parallel estimators, each of which over-fits the data, and averages the results to find a better classification. An ensemble of randomized decision trees is known as a *random forest*.

Let's write a quick utility function to help us visualize the output of the classifier:

In [12]:
```
def visualize_classifier(model, X, y, ax=None, cmap='rainbow'):
    ax = ax or plt.gca()

    # Plot the training points
    ax.scatter(X[:, 0], X[:, 1], c=y, s=30, cmap=cmap,
                        clim=(y.min(), y.max()), zorder=3)
    ax.axis('tight')
    ax.axis('off')
    xlim = ax.get_xlim()
    ylim = ax.get_ylim()

    # fit the estimator
    model.fit(X, y)
    xx, yy = np.meshgrid(np.linspace(*xlim, num=200),
                         np.linspace(*ylim, num=200))
    Z = model.predict(np.c_[xx.ravel(), yy.ravel()]).reshape(xx.sha

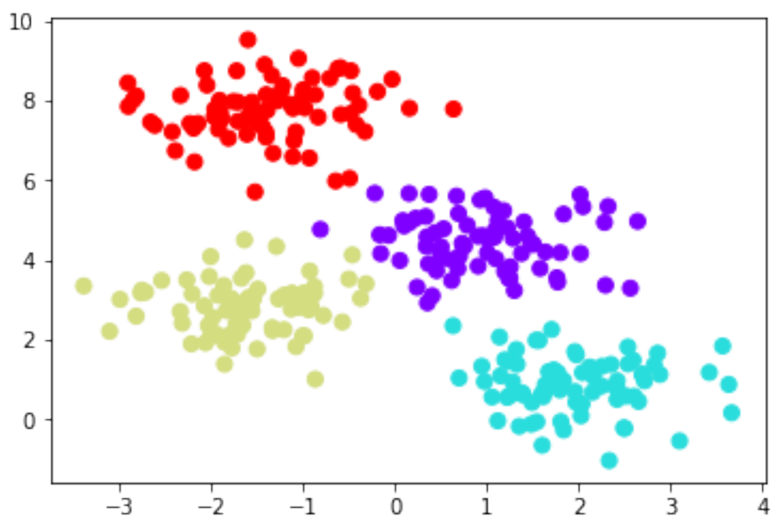
    # Create a color plot with the results
    n_classes = len(np.unique(y))
    contours = ax.contourf(xx, yy, Z, alpha=0.3,
                           levels=np.arange(n_classes + 1) - 0.5,
                           cmap=cmap, zorder=1)

    ax.set(xlim=xlim, ylim=ylim)
```

This type of bagging classification can be done manually using Scikit-Learn's `BaggingClassifier` meta-estimator, as shown here:

In [13]:
```
from sklearn.datasets import make_blobs

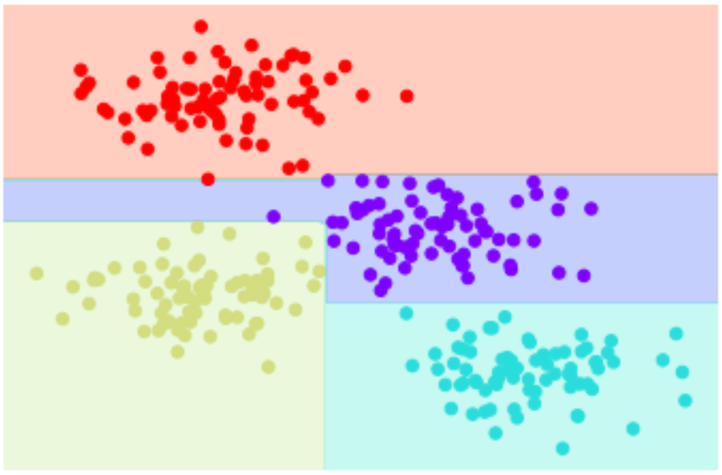
X, y = make_blobs(n_samples=300, centers=4,random_state=0, cluster_
plt.scatter(X[:, 0], X[:, 1], c=y, s=50, cmap='rainbow');
```



In [14]:
```
from sklearn.tree import DecisionTreeClassifier
from sklearn.ensemble import BaggingClassifier

tree = DecisionTreeClassifier()
bag = BaggingClassifier(tree, n_estimators=100, max_samples=0.8,ran

bag.fit(X, y)
visualize_classifier(bag, X, y)
```

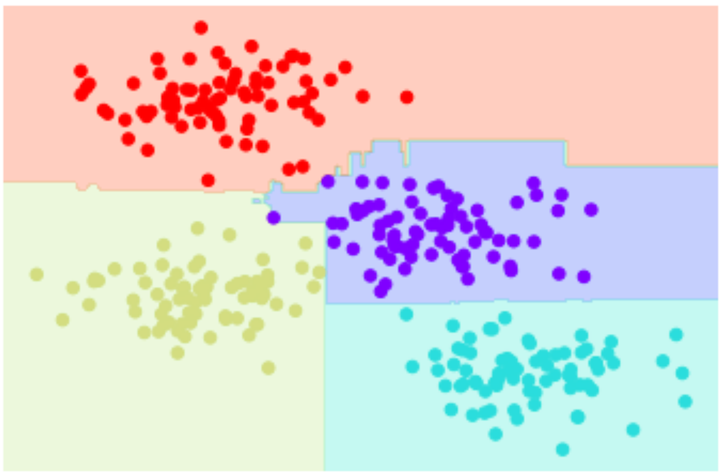


In this example, we have randomized the data by fitting each estimator with a random subset of 80% of the training points. In practice, decision trees are more effectively randomized by injecting some stochasticity in how the splits are chosen: this way all the data contributes to the fit each time, but the results of the fit still have the desired randomness. For example, when determining which feature to split on, the randomized tree might select from among the top several features. You can read more technical details about these randomization strategies in the [Scikit-Learn documentation (http://scikit-learn.org/stable/modules/ensemble.html#forest)](http://scikit-learn.org/stable/modules/ensemble.html#forest) and references within.

In Scikit-Learn, such an optimized ensemble of randomized decision trees is implemented in the `RandomForestClassifier` estimator, which takes care of all the randomization automatically. All you need to do is select a number of estimators, and it will very quickly (in parallel, if desired) fit the ensemble of trees:

In [15]:
```
from sklearn.ensemble import RandomForestClassifier

model = RandomForestClassifier(n_estimators=100, random_state=0)
visualize_classifier(model, X, y);
```



We see that by averaging over 100 randomly perturbed models, we end up with an overall model that is much closer to our intuition about how the parameter space should be split.

Random Forest Regression

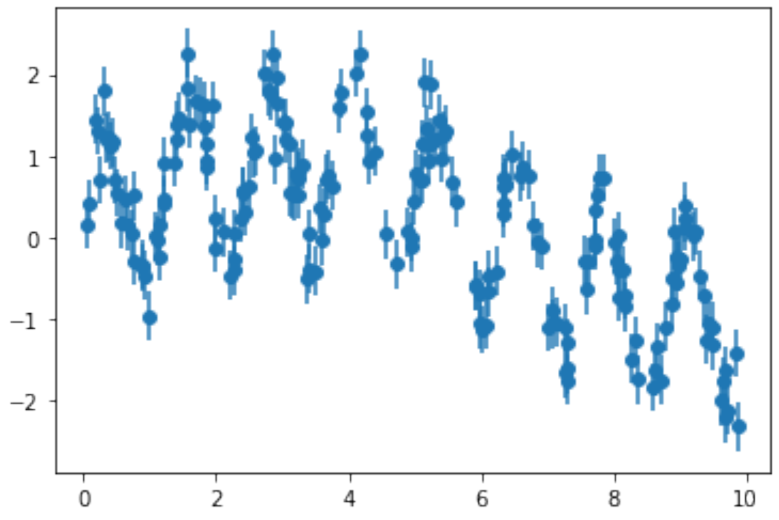
In the previous section we considered random forests within the context of classification. Random forests can also be made to work in the case of regression (that is, continuous rather than categorical variables). The estimator to use for this is the `RandomForestRegressor`, and the syntax is very similar to what we saw earlier.

Consider the following data, drawn from the combination of a fast and slow oscillation:

In [16]:
```
rng = np.random.RandomState(42)
x = 10 * rng.rand(200)

def model(x, sigma=0.3):
    fast_oscillation = np.sin(5 * x)
    slow_oscillation = np.sin(0.5 * x)
    noise = sigma * rng.randn(len(x))
    return slow_oscillation + fast_oscillation + noise

y = model(x)
plt.errorbar(x, y, 0.3, fmt='o');
```

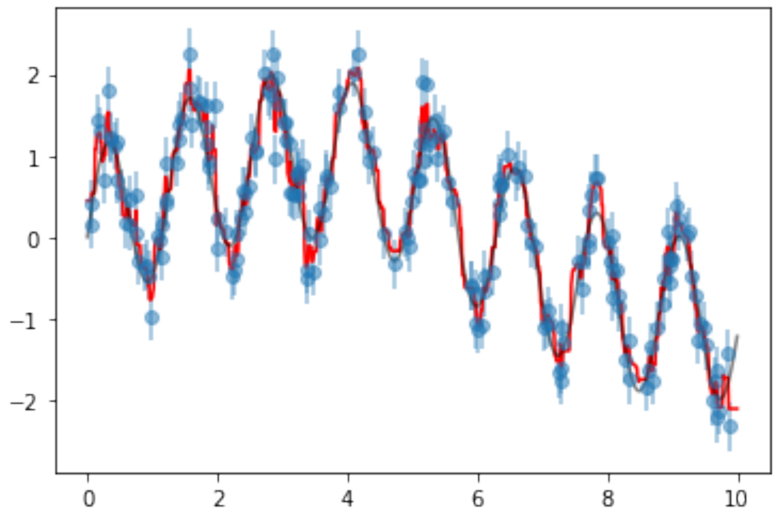


Using the random forest regressor, we can find the best fit curve as follows:

In [17]:
```
from sklearn.ensemble import RandomForestRegressor
forest = RandomForestRegressor(200)
forest.fit(x[:, None], y)

xfit = np.linspace(0, 10, 1000)
yfit = forest.predict(xfit[:, None])
ytrue = model(xfit, sigma=0)

plt.errorbar(x, y, 0.3, fmt='o', alpha=0.5)
plt.plot(xfit, yfit, '-r');
plt.plot(xfit, ytrue, '-k', alpha=0.5);
```



Here the true model is shown in the smooth gray curve, while the random forest model is shown by the jagged red curve. As you can see, the non-parametric random forest model is flexible enough to fit the multi-period data, without us needing to specifying a multi-period model!

Example: Random Forest for Classifying Digits

We previously looked at the hand-written digits data, and we use that again here to see how the random forest classifier can be used in this context.

In [18]:
```
from sklearn.datasets import load_digits
digits = load_digits()
digits.keys()
```

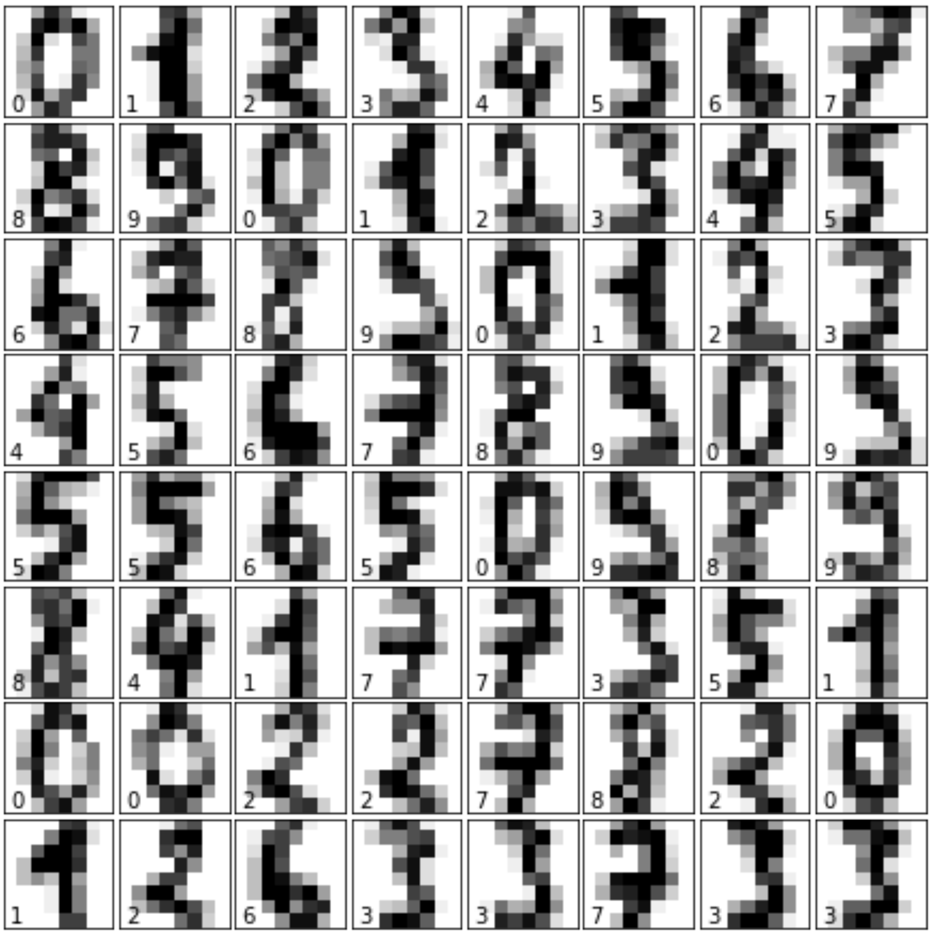
Out[18]:
```
dict_keys(['data', 'target', 'target_names', 'images', 'DESCR'])
```

To remind us what we're looking at, we'll visualize the first few data points:

In [19]:
```
# set up the figure
fig = plt.figure(figsize=(6, 6))  # figure size in inches
fig.subplots_adjust(left=0, right=1, bottom=0, top=1, hspace=0.05,

# plot the digits: each image is 8x8 pixels
for i in range(64):
    ax = fig.add_subplot(8, 8, i + 1, xticks=[], yticks=[])
    ax.imshow(digits.images[i], cmap=plt.cm.binary, interpolation='

    # label the image with the target value
    ax.text(0, 7, str(digits.target[i]))
```



We can quickly classify the digits using a random forest as follows:

In [20]:
```
# from sklearn.cross_validation import train_test_split
from sklearn.model_selection import train_test_split

Xtrain, Xtest, ytrain, ytest = train_test_split(digits.data, digits
                                                random_state=0)
model = RandomForestClassifier(n_estimators=1000)
model.fit(Xtrain, ytrain)
ypred = model.predict(Xtest)
```

We can take a look at the classification report for this classifier:

In [23]:
```
print (Xtest.shape)
```

```
(450, 64)
```

In [24]:
```
from sklearn import metrics
print(metrics.classification_report(ypred, ytest))
```

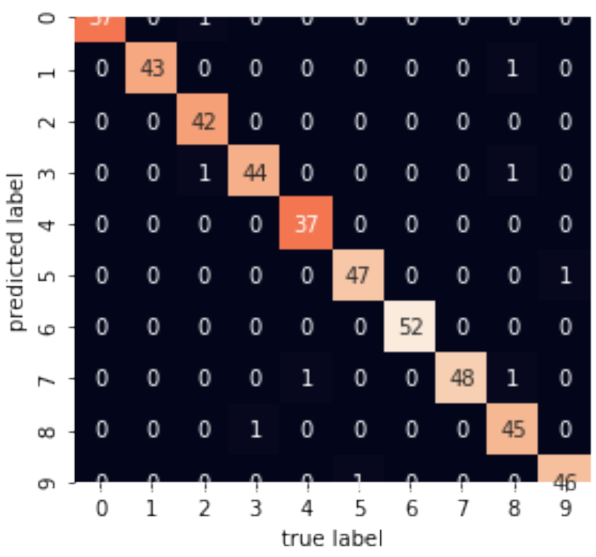
```
              precision    recall  f1-score   support

           0       1.00      0.97      0.99        38
           1       1.00      0.98      0.99        44
           2       0.95      1.00      0.98        42
           3       0.98      0.96      0.97        46
           4       0.97      1.00      0.99        37
           5       0.98      0.98      0.98        48
           6       1.00      1.00      1.00        52
           7       1.00      0.96      0.98        50
           8       0.94      0.98      0.96        46
           9       0.98      0.98      0.98        47

    accuracy                           0.98       450
   macro avg       0.98      0.98      0.98       450
weighted avg       0.98      0.98      0.98       450
```

And for good measure, plot the confusion matrix:

In [25]:
```
from sklearn.metrics import confusion_matrix
mat = confusion_matrix(ytest, ypred)
sns.heatmap(mat.T, square=True, annot=True, fmt='d', cbar=False)
plt.xlabel('true label')
plt.ylabel('predicted label');
```



We find that a simple, untuned random forest results in a very accurate classification of the digits data.

Summary of Random Forests

This section contained a brief introduction to the concept of *ensemble estimators*, and in particular the random forest – an ensemble of randomized decision trees.

Question: What advantages does the random forests method have?

- Both training and prediction are very fast, because of the simplicity of the underlying decision trees. In addition, both tasks can be straightforwardly parallelized, because the individual trees are entirely independent entities.
- The multiple trees allow for a probabilistic classification: a majority vote among estimators gives an estimate of the probability (accessed in Scikit-Learn with the `predict_proba()` method).
- The nonparametric model is extremely flexible, and can thus perform well on tasks that are under-fit by other estimators.

Question: What disadvantages does random forests method have?

A primary disadvantage of random forests is that the results are not easily interpretable: that is, if you would like to draw conclusions about the *meaning* of the classification model, random forests may not be the best choice.

Boosting

Question: How does boosting work?

Boosting is a more complex version of ensemble learning. Unlike bagging or random forests, the *base learners* are **not** independent. In fact, on each boosting iteration the error from previous iterations is used to reweight examples before the current iteration begins.

As a very simple example showing how boosting works, we use as a base classifier a linear model that splits one of two dimensions *x* or *y* at some value. That is, it constructs a vertical or horizontal line on one of the two axes and classifies all points **greater than** the value as positive $+1$, otherwise as negative -1.

Then learning is just selecting which line makes the fewest errors. Since there is only a finite set of possible values to define the position of the line, this classifier is a little like a the regression random forest above. Alternatively, you could think of it as a one-level decision tree classifier.

To try to see what is going on, a two-dimensional dataset was created with data in the top right-hand corner being in one class, and the rest in another. A few of the datapoints were randomly mislabelled to simulate noise. Clearly, this dataset cannot be separated by a single horizontal or vertical decision boundary.

Question: What can we see after each time we run the boosting algorithm?

We can see that each time we run the boosting algorithm on such randomly generated datasets the output of the classifier on an independent test set is pretty good, since the algorithm typically gets only a small number of datapoints wrong, usually those that are close to one of the 'noisy' datapoints in the training data.

We generate plots showing the training data, the error curve on both the training and testing sets, and the boosting iterations of the classifier, each of which can only put in one horizontal or linear classification line (these alternate between iterations).

In [28]:
```
import matplotlib.pyplot as pl
import numpy as np

def train(data,classes,weights,whichdim):

    error = np.zeros(10)
    for value in range(0,10,1):
        val = float(value)/10                                   ### tr
        classn = np.where(data[whichdim,:]<val,-1,1)            ### al
        ind = np.where(classes!=classn)                         ### al
        error[value] = np.sum(weights[ind])                     ### su
    return whichdim,float(np.argmin(error))/10,1-whichdim       ### re

def classify(data,classes,dim,value):
    dim = int(dim)# modify here
    # modify here
    classn = np.where(data[dim,:]<value,-1,1)
    ind = np.where(classes!=classn,1,0)
    return classn, ind

def boost(data,classes,testdata,testclasses):
    T = 20
    N = np.shape(data)[1]
    ndim = np.shape(data)[0]
    classifiers = np.zeros((2,T))          ### a 2 x 20 array; a cla
    whichdim = 0                           ### initial choice of x o

    w = np.ones((N,T+1),dtype=float)/N     ### initialise each of N
    index = np.ones((N,T+1))               ### same shape, used for
    e = np.zeros(T)                        ### error store for each
    alpha = np.zeros(T+1)                  ### for the classifier we

    err = np.zeros((2,T+1))                ### for classification er

    poutput = np.zeros((T+1,N))
    ptoutput = np.zeros((T+1,N))
    po = np.zeros(T+1)
    pto = np.zeros(T+1)
    f = pl.figure(figsize=(14,8))
    for t in range(T):
        classifiers[0,t],classifiers[1,t],whichdim = train(data,cla
        outputs,errors = classify(data,classes,classifiers[0,t],cla
        toutputs,terrors = classify(testdata,testclasses,classifier
        which = np.where(outputs<=0)
        which2 = np.where(outputs>0)

        ax=f.add_subplot(5,4,t+1)
        ax.plot(data[0,which],data[1,which],'bo',ms=3)
        ax.plot(data[0,which2],data[1,which2],'r^',ms=3)
        dummy1=np.linspace(0., 1., num=11)
        dummy0=np.zeros(11)
        dummy2=np.add(dummy0,classifiers[1,t])

        if  classifiers[0,t] == 0:
            ax.plot(dummy2,dummy1,'c-')
        else:
            ax.plot(dummy1,dummy2,'c-')
        cur_title = 'iteration ' + str(t+1)
        ax.set_title(cur_title)

        index[:,t] = errors
        e[t] = np.sum(w[:,t]*index[:,t])/np.sum(w[:,t])     ### tot

        if t>0 and (e[t]==0 or e[t]>=0.5):                  ### Hal
            T=t
            alpha = alpha[:t]
            index = index[:,:t]
            w = w[:,:t]
            break

        alpha[t] = np.log((1-e[t])/e[t])                   ### vers
        w[:,t+1] = w[:,t]* np.exp(alpha[t]*index[:,t])     ### rewe
        w[:,t+1] = w[:,t+1]/np.sum(w[:,t+1])               ### norm

        outputs = np.zeros((N,t))
        toutputs = np.zeros((N,t))
        for i in range(t):
            outputs[:,i],errors  = classify(data,classes,classifier
            toutputs[:,i],terrors  = classify(testdata,testclasses,

        for n in range(N):
            poutput[t,n] = np.sum(alpha[:t]*outputs[n,:])/sum(alpha
            ptoutput[t,n] = np.sum(alpha[:t]*toutputs[n,:])/sum(alp
        poutput[t,:] = np.where(poutput[t,:]>0,1,-1)
        ptoutput[t,:] = np.where(ptoutput[t,:]>0,1,-1)
        po[t] = np.shape(np.where(poutput[t,:]!=classes))[1]
        pto[t] = np.shape(np.where(ptoutput[t,:]!=testclasses))[1]
    pl.tight_layout()

    outputs = np.zeros((N,np.shape(w)[1]))
    for t in range(T):
        outputs[:,t],errors  = classify(data,classes,classifiers[0,

    output = np.zeros(N)
    for n in range(N):
        output[n] = np.sum(alpha*outputs[n,:])/np.sum(alpha)
    pl.figure()
    f = pl.figure(figsize=(10,6))
    ax=f.add_subplot(2,2,1)
    classes = np.where(((data[0,:]>0.4) & (data[1,:]>0.4)),1,-1)
    which = np.where(classes==-1)
    which2 = np.where(classes==1)
    ax.plot(data[0,which],data[1,which],'bo',ms=5)
    ax.plot(data[0,which2],data[1,which2],'r^',ms=5)
    ax.set_title('Training Data')

    ax=f.add_subplot(2,2,2)
    which = np.where(output<=0)
    which2 = np.where(output>0)
    ax.plot(data[0,which],data[1,which],'bo',ms=5)
    ax.plot(data[0,which2],data[1,which2],'r^',ms=5)
    ax.set_title('Output on training data')

    outputs = np.zeros((N,np.shape(w)[1]))
    for t in range(T):
        outputs[:,t],errors  = classify(testdata,testclasses,classi

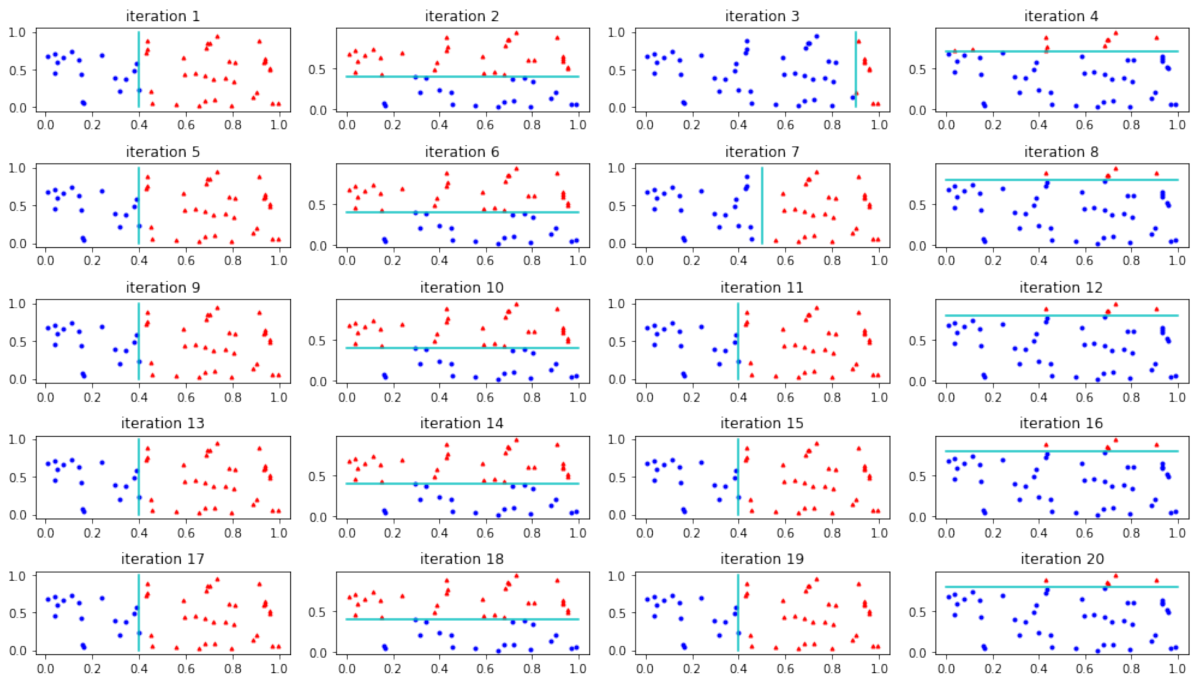
    output = np.zeros(N)
    for n in range(N):
        output[n] = np.sum(alpha*outputs[n,:])/np.sum(alpha)
    which = np.where(output<=0)
    which2 = np.where(output>0)
    ax=f.add_subplot(2,2,4)
    ax.set_title('Output on test data')
    ax.plot(testdata[0,which],testdata[1,which],'bo',ms=5)
    ax.plot(testdata[0,which2],testdata[1,which2],'r^',ms=5)

    ax=f.add_subplot(2,2,3)
    ax.set_title('Test Data')
    which = np.where(testclasses==-1)
    which2 = np.where(testclasses==1)
    ax.plot(testdata[0,which],testdata[1,which],'bo',ms=5)
    ax.plot(testdata[0,which2],testdata[1,which2],'r^',ms=5)
    pl.tight_layout()

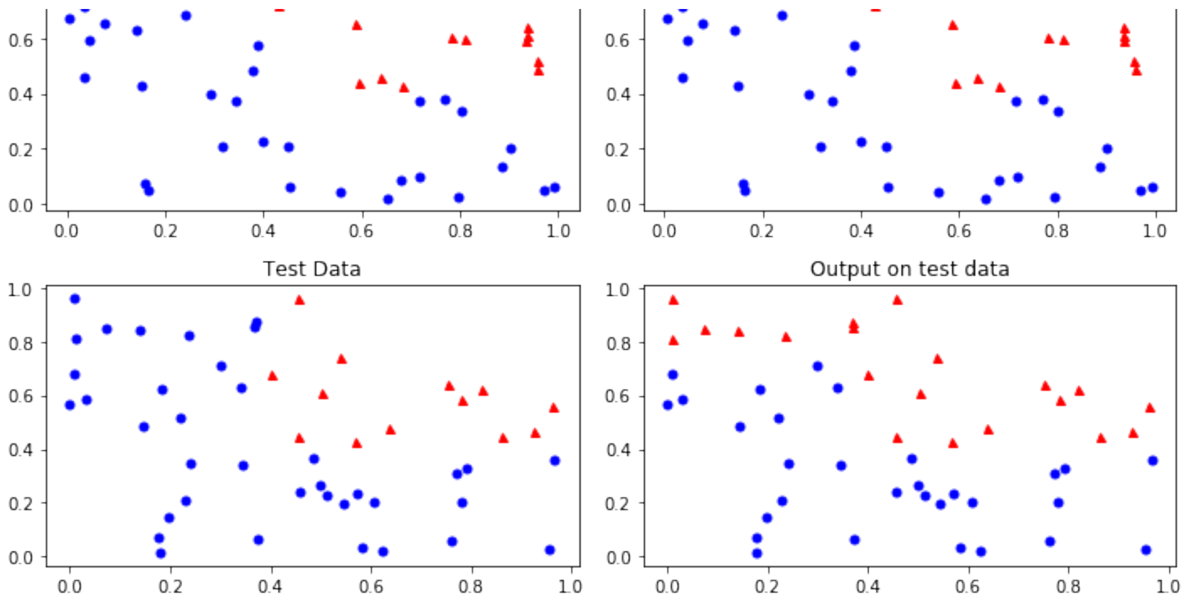
    pl.figure()
    pl.plot(np.arange(T),po[:T]/N,'k-',np.arange(T),pto[:T]/N,'k--'
    pl.legend(('Training error','Test error'))
    pl.xlabel('Iterations')
    pl.ylabel('Error')
    return output

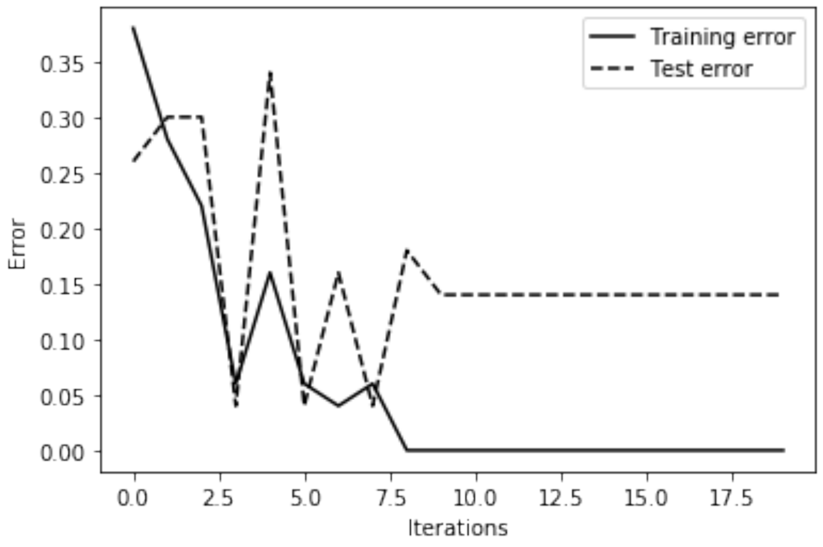
def test_boost():
    pl.ion()                               ### interactive plotting
    ndata = 50
    data = np.random.rand(2,ndata)         ### uniformly distribute
    classes = np.where(((data[0,:]>0.4) & (data[1,:]>0.4)),1,-1)
    testdata = np.random.rand(2,ndata)
    testclasses = np.where(((testdata[0,:]>0.4) & (testdata[1,:]>0.
    boost(data,classes,testdata,testclasses)


test_boost()
plt.show()
```



```
<Figure size 432x288 with 0 Axes>
```





Discussion of boosting output

In the plots above we can see the story of how boosting works on this simple binary classification problem. Clearly neither the training nor the test set are linearly separable. So the boosting algorithm incrementally constructs the ensemble by selecting on each iteration a linear classifier to partition the data. On some of these, such as iterations 1 and 8, the learner seems to be identifying a component of the target decision boundary. On other iterations, such as 3, 12 and 18, the classifier is focusing on quite a small subset of the data to separate from the rest of the points. However, after the 20 iterations are complete we see that boosting has in fact managed to learn an ensemble model which implements an accurate non-linear classifier (the learning curve actually shows that the best error is already achieved before 5 iterations of boosting).

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In [ ]:
```