PARTITION THEORY (notes by M. Hlynka and W.L. Yee, University of Windsor)

Definition: A partition of a positive integer n is an expression of n as a sum of positive integers. Partitions are considered the same if the summands differ only by order. Let $p(n)$ be the number of partitions of n. By convention, take $p(0) = 1$.

Example:

$$
\begin{aligned}
5 &= 5 & & \\
&= 4+1 & &= 3+2 \\
&= 3+1+1 & &= 2+2+1 \\
&= 2+1+1+1 & &= 1+1+1+1+1
\end{aligned}
$$

Thus $p(5) = 7$.

Exercise 1: Show that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$ by listing all partitions for each n.

Exercise 2: Find p(8).

$$
\begin{aligned}
8 &= 8 & &= 7+1 \\
&= 6+2 & &= 6+1+1 \\
&= 5+3 & &= 5+2+1 \\
&= 5+1+1+1 & &= 4+4 \\
&= 4+3+1 & &= 4+2+2 \\
&= 4+2+1+1 & &= 4+1+1+1+1 \\
&= 3+3+2 & &= 3+3+1+1 \\
&= 3+2+2+1 & &= 3+2+1+1+1 \\
&= 3+1+1+1+1+1 & &= 2+2+2+2 \\
&= 2+2+2+1+1 & &= 2+2+1+1+1+1 \\
&= 2+1+1+1+1+1+1 & &= 1+1+1+1+1+1+1+1
\end{aligned}
$$

Therefore $p(8) = 22$.

(Hardy-Ramanujan asymptotic formula), Rademacher:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left[\frac{sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right]$$

where $A_k(n) = \sum_{0\leq m<k,(m,k)=1} \omega_{h,k}e^{-\frac{2\pi i h n}{k}}$ and $\omega_{h,k}$ are certain $24k^{\text{th}}$ roots of unity.

Notation: Let

$p(n)$ = the number of partitions of n,

$p_m(n)$ = the number of partitions of n into at most m summands,

$\pi_m(n)$ = the number of partitions of n into summands of size at most m,

$p^o(n)$ = the number of partitions of n into odd summands

$p^e(n)$ = the number of partitions of n into even summands

$q(n)$ = the number of partitions of n into distinct summands

$q^e(n)$ = the number of partitions of n into an even number of distinct summands

$q^o(n)$ = the number of partitions of n into an odd number of distinct summands.

Property 1: $\pi_m(n) = \pi_{m-1}(n) + \pi_m(n-m)$ for $n \geq m > 1$.

Proof. The partitions counted in $\pi_{m-1}(n)$ are the partitions of n in $\pi_m(n)$ that do not have a part of size m. Any other partition of n counted in $\pi_m(n)$ must have m as one of its summands. Other summands are $\leq m$. So we can find the remaining partitions counted in $\pi_m(n)$ by taking all partitions counted in $\pi_m(n-m)$ and adding the summand m to them. The result follows. ■

Note: (a) $\pi_n(n) = p(n)$ since all partitions of n have summands at most n.

(b) $\pi_1(n) = 1$ since $n = 1 + \cdots + 1$.

Thus we can use the Property 1 to recursively find $p(n)$.

Example: Find $p(7)$.

$$\begin{aligned}
p(7) &= \pi_7(7) = \pi_6(7) + \pi_7(7-7) = \pi_6(7) + 1 = \pi_5(7) + \pi_6(7-6) + 1 \\
&= \pi_5(7) + \pi_6(1) + 1 = \pi_5(7) + 1 + 1 = \pi_4(7) + \pi_5(2) + 2 \\
&= \pi_4(7) + 4 = \pi_3(7) + \pi_4(3) + 4 = \pi_3(7) + 3 + 4 \\
&= \pi_2(7) + \pi_3(4) + 7 = \pi_1(7) + \pi_2(5) + \pi_2(4) + \pi_3(1) + 7 \\
&= 1 + \pi_1(5) + \pi_2(3) + \pi_1(4) + \pi_2(2) + 1 + 7 \\
&= 1 + 1 + 2 + 1 + 2 + 1 + 7 = 15
\end{aligned}$$

Exercise 3 Find $p(8)$ using the recursive property.

Graphical Representation

A partition can be written with a graphical representation, known as a Ferrers graph, named after the British mathematician Norman Macleod Ferrers. See
http://www.gap-system.org/∼history/Biographies/Ferrers.html
For example $3 + 2$ can be written as

```
* * *
* *
```

As another example, the graphical representation

```
* * * *
* * *
*
```

represents the partition 4+3+1.

Conjugate Graph

The conjugate graph is obtained from a graph by writing the rows as columns.
The conjugate graphs for the two previous graphs are

```
* *              * * *
* *              * *
*                * *
        and      *
```

The two corresponding partitions are $2 + 2 + 1$ and $3 + 2 + 2 + 1$.

Self Conjugate Partitions

A partition is said to be self conjugate if its graph and the conjugate of its graph are the same.

Example The partition 4+3+3+1 is self conjugate because the graph is

```
* * * *                                  * * * *
* * *                                    * * *
* * *                                    * * *
*            with conjugate              *
```

Property 2: The number of partitions of n into self conjugate parts equals the number of partitions of n into distinct odd parts.

Proof. Every self conjugate partition maps to a partition with distinct odd parts by reading off the nested Γ shaped sections. Conversely, any odd part can be folded into a Γ shape and by nesting these, we get a self conjugate partition. ∎

Example: The self conjugate partition 4+3+3+1 (of 11) has graph

```
* * * *
* * *
* * *
*
```

The partition 7+3+1 (of 11) into distinct odd parts can be converted to Γ shapes and nested to give

```
x x x x
x y y
x y z
x
```

The corresponding self conjugate partition is 4+3+3+1.

Example

There are 3 partitions of 12 into distinct odd parts and 3 partitions of 12 into self conjugate parts. The correspondence is

$11 + 1 \leftrightarrow 6 + 2 + 1 + 1 + 1 + 1$

$9 + 3 \leftrightarrow 5 + 3 + 2 + 1 + 1$

$7 + 5 \leftrightarrow 4 + 4 + 2 + 2$

Exercise 4 Find all partitions of 13 into distinct odd parts and find the corresponding self conjugate partition for each.

Odd parts:

$13 \leftrightarrow 7 + 1 + 1 + 1 + 1 + 1 + 1$

$$9 + 3 + 1 \leftrightarrow \begin{array}{ccccc} * & * & * & * & * \\ * & * & * & & \\ * & * & * & & \\ * & & & & \\ * & & & & \end{array} \leftrightarrow 5 + 3 + 3 + 1 + 1$$

$$7 + 5 + 1 \leftrightarrow \begin{array}{cccc} * & * & * & * \\ * & * & * & * \\ * & * & * & \\ * & * & & \end{array} \leftrightarrow 4 + 4 + 3 + 2$$

Exercise 5 Find all self conjugate partitions of 14. Find the corresponding partition of each with distinct odd parts.

Exercise 6 How can one recognize whether or not 6+5+3+2+2+1 is a self conjugate partition without drawing a graph? State a general rule.

Property 3:

The number of partitions of n into at most m parts is the number of partitions into parts whose largest part is at most m. i.e. $p_m(n) = \pi_m(n)$.

Proof: **Exercise 7:** Give a proof based on Ferrers Diagram. (Homework)

Exercise 8:

The number of partitions of n into exactly k parts is the number of partitions into parts such that ______________________________.

Generating Functions for Partitions

How may ways can we make 72 cents using pennies, nickles and dimes?

Solution: Use generating functions.

Pennies: $p_1(x) = 1 + x^1 + x^2 + x^3 + \ldots$

Nickles: $p_2(x) = 1 + x^5 + x^{10} + x^{15} + \ldots$

Dimes: $p_3(x) = 1 + x^{10} + x^{20} + \ldots$

Product: $p_1(x)p_2(x)p_3(x) = (1 + x + x^2 + x^3 + x^4 + 2x^5 + \cdots + 2x^9 + 4x^{10} + \ldots$

Using Maple:

```
p1:=1; for i from 1 to 100 do p1:=p1+x^i end do;
p2:=1; for i from 1 to 20 do p2:=p2+x^(5*i) end do;
p3:=1; for i from 1 to 10 do p3:=p3+x^(10*i) end do;
expand(p1*p2*p3);
```

$1 + x + x^2 + x^3 + x^4 + 2 * x^5 + 2 * x^6 + 2 * x^7 + 2 * x^8 + 2 * x^9 + 4 * x^{10} + \cdots + 64x^{72} + \cdots$

So the answer is 64 ways.

The generating function

$f_1(x) = (1 + x + x^{1+1} + x^{1+1+1} + \dots) = (1 + x + x^2 + x^3 + \dots)$ gives the number of partitions of i into parts all of size 1 as the coefficient of x^i.

The generating function $(1 + x^2 + x^{2+2} + x^{2+2+2} = 1 + x^2 + x^4 + x^6$ gives the number of partitions of i of size 2 as the coefficient of x^i.

Then the generating function

$f_2(x) = (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots) = 1 + x^1 + 2x^2 + 2x^3 + 3x^4 + \dots$ gives the number of partitions of i into parts of size 1 or 2 as the coefficient of x^i.

Here the coefficient of x^4 is 3 which means there are 3 partitions of 4 into parts consisting of 1's and 2's. They are 2+2, 2+1+1, and 1+1+1+1. One part of the product comes from the first polynomial and the second part comes from the second polynomial. These came from the terms $1x^4, x^2x^2, x^4 1$ or $x^{2+2}, x^{1+1}x^2, x^{1+1+1+1}$.

The generating function

$$\begin{aligned} f_3(x) &= (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots \\ &= 1 + x^1 + 2x^2 + 3x^3 + 5x^4 + \dots = \sum p(i)x^i \end{aligned}$$

gives the number of partitions of i as the coefficient of x^i.

Exercise 9:

(a) What is the generating function for partitions into odd parts?

(b) What is the generating function for partitions into distinct parts?

(c) What is the generating function for partitions into odd parts which can be repeated at most three times?

Note Issues of convergence will be ignored. The x terms are simply place holders. However, we can manipulate the series as if they were polynomials with domain equal to $\Re$.

Property 4:

The number of partitions of n into odd parts equals the number of partitions of n into distinct parts. i.e. $p^o(n) = q(n)$.

PROOF.

The generating function for partitions into odd parts is

$g_1(x) = (1 + x + x^2 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^5 + x^{10} + \dots)\dots$

The generating function for partitions into distinct parts is

$$
\begin{aligned}
g_2(x) &= (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots = \prod_{i=1}^{\infty}(1+x^i) \\
&= \frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\dots \\
&= \frac{\prod_{i=1}^{\infty}(1-x^{2i})}{\prod_{i=1}^{\infty}(1-x^{2i})\prod_{i=1}^{\infty}(1-x^{2i-1})} = \frac{1}{\prod_{i=1}^{n}(1-x^{2i-1})} \\
&= (1 + x + x^2 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^5 + x^{10} + \dots)\dots
\end{aligned}
$$

The result follows. □

Exercise 10:

Use the method in the above property to show that every positive integer can be expressed uniquely as a sum of powers of 2, with each power appearing at most once (i.e. there is a unique representation in base 2). (Homework)

Example:

Use the method in the property to show that the number of partitions into parts that appear at most twice is equal to the number of partitions into parts not divisible by 3.
SOLUTION: Let $f(x)$ be the generating function for parts that appear at most twice. Then $f(x) = (1+x+x^2)(1+x^2+x^4)\cdots$. Let $g(x)$ be the generating function for parts not divisible by 3. Then

$$g(x) = (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^4+x^8+\ldots)$$
$$(1+x^5+x^{10}+\ldots)\cdots$$

Thus

$$\begin{aligned}
g(x) &= (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^4+x^8+\ldots)\cdots \\
&= \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)\cdots \\
&= \frac{1}{\prod_{i \not\equiv 0 \, mod 3}(1-x^i)} \\
&= \frac{\prod(1-x^{3i})}{\prod(1-x^{3i})\prod(1-x^{3i+1})\prod(1-x^{3i+2})} \\
&= \frac{\prod(1-x^{3i})}{\prod(1-x^i)} = \left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^6}{1-x^2}\right)\left(\frac{1-x^9}{1-x^3}\right)\cdots \\
&= (1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)\cdots = f(x)
\end{aligned}$$

Exercise 11:

What would the next result corresponding to the previous result? Prove it?

Using Inclusion/Exclusion

In probability, there is the rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Similarly, we have a count of partitions of n that include 1, namely $p(n-1)$ (we can add 1 to each partition counted in $p(n-1)$). We have a count of partitions of n that include 2, namely $p(n-2)$ (we can add 2 to each partition counted in $p(n-2)$). We have a count of partitions of n that include both 1 and 2, namely $p(n-3)$ (we can add 1+2 to each partition included in $p(n-3)$). We also have partitions all of whose parts are greater than 2. Thus

$$p(n) = p(n-1) + p(n-2) - p(n-3) + p_{\geq 3}(n).$$

Example We apply the above rule to compute $p(4), p(5), \ldots$ given $p(0) = 1, p(1) = 1, p(2) = 2, p(3) = 3$.
$p(4) = p(3)+p(2)-p(1)+p_{\geq 3}(4) = 3+2-1+\#\{4\} = 3+2-1+1 = 5$
$p(5) = p(4)+p(3)-p(2)+p_{\geq 3}(5) = 5+3-2+\#\{5\} = 5+3-2+1 = 7$
$p(6) = P(5) + p(4) - p(3) + p_{\geq 3}(6) = 7 + 5 - 3 + \#\{6, 3+3\} = 7 + 5 - 3 + 2 = 11$.

Exercise: Continue as above and compute $p(7), p(8), p(9)$.

Exercise: In probability, we can extend the level of inclusion and exclusion to get the rule
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.
(a) If we apply this to $p(n)$ using parts with parts of size 1,2,3, what

recursive formula do we get?

(b) Apply your formula to obtain $p(7), p(8), p(9), p(10)$.