Theorem 1.7.2:

i. $\vdash \sim false \Leftrightarrow true$
   $\vdash \sim true \Leftrightarrow false$
ii. $\vdash (\alpha \Leftrightarrow \beta) \Leftrightarrow (\beta \Leftrightarrow \alpha)$
iii. $\vdash (\alpha \Leftrightarrow \beta) \Leftrightarrow (\sim \alpha \Leftrightarrow \sim \beta)$
iv. $\vdash (\alpha \Leftrightarrow \beta) \wedge (\beta \Leftrightarrow \gamma) \Rightarrow (\alpha \Leftrightarrow \gamma)$
v. $\vdash (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Leftrightarrow (\alpha \wedge \beta \Rightarrow \gamma)$

Sub$_=$ (*Substitutivity of* $=$)
$\frac{s=t,\ \alpha(s)}{\alpha(t)}$ $(\alpha(s)\ is\ free\ for\ t)$

Set Theory

Principle of Extension: $A = B\ iff\ (\forall x)(x \in A \Leftrightarrow x \in B)$

I3$_\wedge$: $\alpha \wedge \beta,\ \alpha \Rightarrow \gamma \vdash \gamma \wedge \beta$
I3$_\vee$: $\alpha \vee \beta,\ \alpha \Rightarrow \gamma \vdash \gamma \vee \beta$

Lemma 3.2.1: Let A, B, C be sets

i. $A = A$
ii. $if\ A = B,\ then\ B = A$
iii. $if\ A = B\ and\ B = C\ then\ A = C$

Lemma 3.2.2: $A \neq B\ iff\ (\exists x)(x \in A \wedge x \notin B) \vee (\exists x)(x \notin A \wedge x \in B)$

Corollary 3.2.2.1: Let A and B be sets
$(\exists x)(x \in A \wedge x \notin B) \Rightarrow A \neq B$

SUBSET: Let A and B be two sets
$A \subseteq B\ iff\ (\forall x)(x \in A \Rightarrow x \in B)$

PROPER SUBSET:
$A \subset B\ if\ A \subseteq B\ and\ A \neq B$

Lemma 3.2.3: Let A, B, C be sets

i. $A \subseteq A$
ii. $if\ A \subseteq B\ and\ B \subseteq A,\ then\ A = B$
iii. $if\ A \subseteq B\ and\ B \subseteq C,\ then\ A \subseteq C$

Lemma 3.2.4: $if\ A = B,\ then\ (A \subseteq B) \wedge (B \subseteq A)$
Lemma 3.2.5: $A \not\subseteq B \Leftrightarrow (\exists x)(x \in A \wedge x \notin B)$
Lemma 3.2.6: $if\ A \subset B,\ then\ (\exists x)(x \in B \wedge x \notin A)$

FE7$_\in$: $\sim (\forall x \in X) S(x) \equiv (\exists x \in X) \sim S(x)$
FE8$_\in$: $\sim (\exists x \in X) S(x) \equiv (\forall x \in X) \sim S(x)$

Principle of Specification: For every set A and every formula *S(x)*, there exists a set B whose elements are those of A for which *S(x)* is true.

$$\{x \mid x \in A \wedge S(x)\}\ or\ \{x \in A \mid S(x)\}$$

Lemma 3.5.7: $x \in \emptyset \Leftrightarrow false$
Lemma 3.5.8: $A \neq \emptyset \Leftrightarrow (\exists x) x \in A$
Corollary 3.5.8.1: $\sim (\exists x) x \in \emptyset$
Theorem 3.5.9: $\emptyset \subseteq A,\ for\ any\ set\ A$

POWER SET: Let A be a set. P(A) is the set $\{X | X \subseteq A\}$
UNION: Let A, B be two sets. $A \cup B = \{x \mid x \in A \vee x \in B\}$

Theorem 4.1.1: Let A, B, C be sets

i. $A \cup \emptyset = A$
ii. $A \cup A = A$

iii. $A \cup B = B \cup A$
iv. $(A \cup B) \cup C = A \cup (B \cup C)$
v. $A \subseteq B \; iff \; A \cup B = B$

INTERSECTION: $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Theorem 4.2.2: Let A, B, C be sets

i. $A \cap \emptyset = \emptyset$
ii. $A \cap A = A$
iii. $A \cap B = B \cap A$
iv. $(A \cap B) \cap C = A \cap (B \cap C)$
v. $A \subseteq B \; iff \; A \cap B = A$

Corollary 4.2.2.1: $A \cap B \subseteq A \subseteq A \cup B$
Theorem 4.2.3: Let A, B, C bet any three sets

i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DISJOINT: Two sets A and B are disjoint if $A \cap B = \emptyset$
RELATIVE COMPLEMENT: RC of B in A is the set $A - B = \{x \mid x \in A \wedge x \notin B\}$

Theorem 4.3.4: Let A, B, C be sets

i. $A \subseteq B \; iff \; A - B = \emptyset$
ii. $A - (A - B) = A \cap B$
iii. $A \cap (B - C) = (A \cap B) - (A \cap C)$

Lemma 4.3.5: Let $A \subseteq U. \; Then, \; x \in \overline{A} \Leftrightarrow x \notin A$
Lemma 4.3.6: DeMorgan's Theorem. Let A, B be any two subsets of U

i. $\overline{A \cup B} = \overline{A} \cap \overline{B}$
ii. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Lemma 4.3.7: Let A, B be any two subsets of U

i. $\overline{\overline{A}} = A$
ii. $\overline{\emptyset} = U \; and \; \overline{U} = \emptyset$
iii. $A \cap \overline{A} = \emptyset \; and \; A \cup \overline{A} = U$
iv. $A \subseteq B \; iff \; \overline{B} \subseteq \overline{A}$
v. $A - B = A \cap \overline{B}$

ORDERED PAIR: $(a, b) = \{\{a\}, \{a, b\}\} = \{x \mid x = \{a\} \vee x = \{a, b\}\}$
Lemma 4.4.8: $(a, b) = (x, y) \; implies \; that \; a = x \; and \; b = y$
Lemma 4.4.9: $a = c \wedge b = d \Rightarrow (a, b) = (c, d)$

CARTESIAN PRODUCT: $A \times B = \{x \mid (\exists a)(\exists b)(a \in A \wedge b \in B \wedge x = (a, b))\} = \{(a, b) \mid a \in A \wedge b \in B\}$
Lemma 4.5.10: Let A, B, X, Y be sets

i. $(A \cup B) \times X = (A \times X) \cup (B \times X)$
ii. $(A \cap B) \times X = (A \times X) \cap (B \times X)$
iii. $(A - B) \times X = (A \times X) - (B \times X)$
iv. $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$
v. $(A = \emptyset \vee b = \emptyset) \Leftrightarrow A \times B = \emptyset$

COLLECTION (union of): Let C be a collection of sets.
$\bigcup_{X \in C} X = \{x \mid (\exists X)(X \in C \wedge x \in X)\} = \{x \mid x \in X \; for \; some \; X \in C\}$
INTERSECTION of C: Let C be a collection of sets such that $C \neq \emptyset$
$\bigcap_{X \in C} X = \{x \mid (\forall X)(X \in C \Rightarrow x \in X)\} = \{x \mid x \in X \; for \; all \; X \in C\}$

Relations

RELATION: A set of ordered pairs. R is a relation if $(\forall x)(x \in R \Rightarrow (\exists a)(\exists b)x = (a,b))$
DOMAIN: domain of R is the set $DOM(R) = \{x \mid (\exists y)(x,y) \in R\}$
RANGE: range of R is the set $RAN(R) = \{y \mid (\exists x)(x,y) \in R\}$
Relation from X to Y: $R \subseteq X \times Y$
Relation in X: $R \subseteq X \times X$
Lemma 5.1.1: Let $R \subseteq X \times Y$. Then $DOM(R) \subseteq X$ *and* $RAN(R) \subseteq Y$

Properties of Relation

Let R be a relation in a set X:

- R is ***reflexive*** if $(\forall x \in X)(x,x) \in R$
- R is ***irreflexive*** if $(\forall x \in X)(x,x) \notin R$
- R is ***symmetric*** if $(x,y) \in R \Rightarrow (y,x) \in R$
- R is ***antisymmetric*** if $(x,y) \in R \wedge (y,x) \in R \Rightarrow x = y$
- R is ***asymmetric*** if $\forall x, y \in X, \sim ((x,y) \in R \wedge (y,x) \notin R$, or equivalently, $\forall x, y \in X, (x,y) \in R \Rightarrow (y,x) \notin R$
- R is ***transitive*** if $(x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R$

Lemma 5.2.1: Let R be a relation in X. If $(\exists x \in X)(x,x) \in R$, then R is not asymmetric
EQUIVALENCE RELATION: A relation R in a set X is an ER if R is reflexive, symmetric, and transitive
EQUIVALENCE CLASS OF x wrt R: Let R be an equivalence relation in a set X. For each $x \in X$, the **equiv class of x wrt R** is the set $[x]/R = \{y \mid y \in X \wedge (x,y) \in R\} = \{y \in X \mid (x,y) \in R\}$
COLLECTION OF EQUIV CLASSES wrt R: $[X]/R = \{S \mid (\exists x)(x \in X \wedge S = [x]/R)\} = \{[x]/R \mid x \in X\}$

PARTITION: Let X be a set. A collection of sets C is a ***partition*** of X if

i. $\bigcup_{S \in C} S = X$, and
ii. $\forall S \in C, S \neq \emptyset$, and
iii. $\forall S, S' \in C, S \neq S' \Rightarrow S \cap S' = \emptyset$

CP 5.3a: Let R be a reflexive relation in X and $a \in X$, then $(a,a) \in R$
CP 5.3b: Let R be a symmetric relation in X and $(a,b) \in R$, then $(b,a) \in R$
CP 5.3c: Let R be a antisymmetric relation in X, and $(a,b) \in R \wedge (b,a) \in R$, then $a = b$
CP 5.3d: Let R be a transitive relation in X, and $(a,b) \in R \wedge (b,c) \in R$, then $(a,c) \in R$

Lemma 5.3.1: If R is an equivalence relation in X, then the collection of equivalence classes $[X]/R$ is a partition of X
INDUCED RELATION: Let C be a partition of X. The relation induced by C, denoted by X/C, is a relation in X, such that: $X/C = \{(x,y) \mid (\exists S \in C)(x \in S \wedge y \in S)\}$
Lemma 5.3.2: Let R be an equivalence relation in X. The relation induced by the collection of equivalence classes $[X]/R$ is identical to R, or, $X/([X]/R) = R$
Lemma 5.3.3: Let C be a partition of X. The induced relation X/C is an equivalence relation in X.
Lemma 5.3.4: Let C be a partition of X. Then $A \in C \wedge a \in A \Rightarrow A = [a]/(X/C)$
Lemma 5.3.5: Let C be a partition of X. Then $[X]/(X/C) = C$
Theorem 5.3.6:

i. If R is an equivalence relation in X, then the set of equivalence classes $[X]/R$ is a partition of X that induces the relation R, and
ii. If C is a partition of X, then the induced relation X/C is an equivalence relation in X whose set of equivalence classes is identical to C

PARTIAL ORDER: A relation R in a set X is a *partial order* if R is reflexive, antisymmetric, and transitive
PARTIALLY ORDERED SET: an ordered pair $(X, \precsim)$ in which X in a set and $\precsim$ is a partial order in X. If $\forall x, y \in X, (x \precsim y) \vee (y \precsim x)$ then $\precsim$ is called a ***total order*** and $(X, \precsim)$ is a ***totally ordered set*** or a ***chain***
STRICT ORDER: a relation R in a set X is a ***strict order*** if R is asymmetric and transitive

Lemma 5.4.1: let $\precsim$ be a partial order in X, and $X_=$ be the relation of equality in X. The relation $\precsim -X_=$ is a strict order in X

INVERSE RELATION: Let R be a relation in a set X. The *inverse* of R is the relation $R^{-1} = \{(y,x) \mid (x,y) \in R\}$

COMPOSITION: Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$. The *composition* of R_1 and R_2 is the relation $R_1 \circ R_2 = \{(x,z) \in A \times C \mid (\exists y \in B)((x,y) \in R_1 \wedge (y,z) \in R_2)\}$

Functions

FUNCTION: A function from sets X to Y, denoted by $f : X \to Y$, is a relation from X to Y with:

i) $Dom(f) = X$
ii) $(x,y) \in f \wedge (x,z) \in f \Rightarrow y = z$

MAP x ONTO y: Let $f : X \to Y$ and $(x,y) \in f$, or equivalently, $y = f(x)$. Then f is said to map x onto y, or y is the *image* of x under f

Lemma 6.1.1: Let $f : X \to Y$ and $g : X \to Y$. Then, $f = g$ iff $\forall x \in X, f(x) = g(x)$

IMAGE OF A SET UNDER f: Let $f : X \to Y$ and $A \subseteq X$. The image of A under f is the set:

$$f(A) = \{y \in Y \mid (\exists x \in A)(x,y) \in f\} = \{y \in Y \mid (\exists x)(x \in A \wedge y = f(x))\} = \{f(x) \in Y \mid x \in A\}$$

RESTRICTION/EXTENSION: Let f, g be two functions such that $g \subseteq f$. Then g is a *restriction* of f and f is an *extension* of f. Let $f : X \to Y$ and $Z \subseteq X$ such that $Dom(g) = Z$. Then g is *restriction* of $f \to Z$ and f is an *extension* of g to X. The function g is denoted by $f|Z$, and $f|Z = \{(x,y) \in f \mid x \in Z\}$

Y^X: The set of all function from $X \to Y$ is the set $Y^X = \{f \mid f : X \to Y\}$. Note $Y^X \subseteq P(X \times Y)$

ONTO: A function $f : X \to Y$ is *onto* or *surjective*, denoted by $f : X \xrightarrow{\text{onto}} Y$, iff $(\forall y \in Y)(\exists x \in X)(x,y) \in f$, or equivalently, $(\forall y \in Y)(\exists x \in X) y = f(x)$, or, $(\forall y)(y \in Y \Rightarrow ((\exists x) x \in X \wedge y = f(x)))$

1-1: A function $f : X \to Y$ is *one-to-one* or *injective*, denoted by $f : X \xrightarrow{\text{1-1}} Y$, iff $(x,z) \in f \wedge (y,z) \in f \Rightarrow x = y$, or equivalently, $f(x) = f(y) \Rightarrow x = y$

1-1 CORRESPONDANCE: A function $f : X \to Y$ is a *one-to-one correspondance* or *bijective*, denoted by $f : X \xrightarrow[\text{onto}]{\text{1-1}} Y$, iff it is both 1-1 and onto

INVERSE: Let $f : A \to B$. The *inverse* of f is the relation f^{-1} from $B \to A$ such that $f^{-1} = \{(y,x) \mid (x,y) \in f\}$. f^{-1} may not be a function

IMAGE OF A SET UNDER R: Let $R \subseteq X \times Y$ be a relation from X to Y and $A \subseteq X$. The *image* of A under R is the set $R(A) = \{y \in Y \mid (\exists x \in A)(x,y) \in R\}$$

INVERSE IMAGE: Let $f : X \to Y$. Then $f^{-1} \subseteq Y \times X$ is a relation from Y to X. Let $B \subseteq Y$. The *inverse image* of B under f is the image of B under f^{-1} which is the set:

$$f^{-1}(B) = \{x \in X \mid (\exists y \in B)(y,x) \in f^{-1}\} = \{x \in X \mid (\exists y \in B)(x,y) \in f\} = \{x \in X \mid f(x) \in B\}$$

Lemma 6.3.1: Let $f : A \to B$. Then f^{-1} is a function from B to A iff f is 1-1 and onto

Theorem 6.3.2: $f : A \xrightarrow[\text{onto}]{\text{1-1}} B$ iff $f^{-1} : B \xrightarrow[\text{onto}]{\text{1-1}} A$

COMPOSITION: Let $f : A \to B$ and $g : B \to C$. The *composition* of f and g is the function $f \circ g : A \to C$

Lemma 6.3.3: Let $f : A \xrightarrow{\text{1-1}} B$ and $g : B \xrightarrow{\text{1-1}} C$. Then $f \circ g : A \xrightarrow{\text{1-1}} C$

Lemma 6.3.4: Let $f : A \xrightarrow{\text{onto}} B$ and $g : B \xrightarrow{\text{onto}} C$. Then $f \circ g : A \xrightarrow{\text{onto}} C$

Theorem 6.3.5: Let $f : A \xrightarrow[\text{onto}]{\text{1-1}} B$ and $g : B \xrightarrow[\text{onto}]{\text{1-1}} C$. Then $f \circ g : A \xrightarrow[\text{onto}]{\text{1-1}} C$

CHARACTERISTIC FUNCTION: Let U be the universal set, and $A \subseteq U$. The *characteristic function* of A is the function $\chi_A : U \to \{0, 1\}$ such that $\chi_A(x) = 0$ if $x \notin A$ or 1 if $x \in A$

Finite and Infinite Sets

EQUINUMEROUS: two sets are equinumerous, denoted by $X \sim Y$, if there exists a function $F : X \xrightarrow[\text{onto}]{\text{1-1}} Y$. Informally, two sets are equinumerous if they have the same number of elements

Theorem 7.1.1: Let C be a collection of sets. The relation $\sim$ is an equivalence relation in C

INITIAL SEGMENT: Let N be the set of all positive integers and $k \in N$. The *initial segment* of k is the set $N_k = \{x \in N \mid x \leq k\}$

FINITE/INFINITE: A set A is a *finite* set if $A = \emptyset$ or $(\exists k \in N) N_k \sim A$. A set is *inifinite* if it is not finite. The *size* of A, denote by $|A|$, is 0 if $A = \emptyset$ and is k if $N_k \sim A$ for some $k \in N$

Theorem 7.2.2: Let A, B be two finite sets. $|A| = |B|$ iff $A \sim B$
Lemma 7.2.3: Let A and B be two non-empty such that $A \cap B = \emptyset$. Then $|A \cup B| = |A| + |B|$
Theorem 7.2.4: Let A and B be two non-empty finite sets. $|A \cup B| = |A| + |B| - |A \cap B|$
Theorem 7.2.5: Let A and B be two non-empty finite sets. If $|A| > k \cdot |B|$, where $k \in N$, then $(\forall f) f : A \to B, (\exists b \in B) |f^{-1}(\{b\})| \geq k + 1$

Pigeonhole Principle: To put m pigeons into n pigeonholes such that $m > n$, there will be one pidgeonhole containing at least $\lceil \frac{m}{n} \rceil$ pigeons

Theorem 7.3.6: A set A is an infinite set iff there exists Y such that $Y \subset A$ and $A \sim Y$

COUNTABLE: Let N be the set of all positive integers. A set X is *denumerable* or *countably infinite* if $N \sim X$. A set is *countable* if it is finite or denumerable and is *uncountable* otherwise

Lemma 7.4.7: Let A be a denumerable set. Any infinite subset of A is denumerable

Family: A *family* is a function $y : I \to Y$ whose domain I is called an *index set.* Every element in I is called an *index* and the range of y is called an *indexed set.* A *family of sets* is a family whose indexed set is a collection of sets.

Graph Theory

GRAPH: a triple $G = (V, E, \psi)$ where

- V and E are two disjoint ($V \cap E = \emptyset$) finite sets such that $V \neq \emptyset$
  - each element in V is called a *vertex* of G. V is the *vertex set* of G
  - each element in E is called an *edge* of G. E is the *edge set* of G
- ψ (*incidence function* of G) is a function such that
  - $\psi : E \to \{\{u, v\} \mid u, v \in V\}$ (if G is an undirected graph)
  - $\psi : E \to V \times V$ (if G is a directed graph)

ORDER: *order* of G is the number of vertices in G (i.e. $|V|$)
SIZE of G: *size* of G is the number of edges in G (i.e. $|E|$)

END-VERTICES: if $\psi(e) = \{a, b\}$, then a and b are *end-vertices* of e
HEAD & TAIL: if $\psi(e) = (a, b)$, then a is the *tail* of e and b is the *head* of e
INCIDENT, ADJACENT: from the previous two definitions, we say edge e *joins* vertices a and b. Edge e is *incident upon* vertices a and b. Vertex a is *adjacent* to vertex b, and vertex b is *adjacent* to vertex a

SELF-LOOP: an edge whos two end-vertices are identical (i.e. $\psi(e) = \{u, u\}$)
PARALLEL EDGES: two edges are *parallel edges* if they have the same end-vertices. In a directed graph, they must also have the same tail and head
SIMPLE GRAPH: a graph having no self-loops and no parallel edges
NULL GRAPH: a graph without any edges
TRIVIAL GRAPH: a graph with exactly 1 vertex
LOOP GRAPH: a graph consisting of 1 vertex and 1 self-loop

DEGREE: the number of edges incident upon a vertex v in G, with self-loops counted twice, is the *degree* of v in G, denoted by $deg_G(v)$. Formally,

$$deg_G(v) = |\{e \in E \mid v \in \psi(e) \wedge \psi(e) \neq \{v, v\}\}| + 2|\{e \in E \mid \psi(e) = \{v, v\}\}|$$

DEGREE SEQUENCE: degree sequence of $G = (V, E, \psi)$ is the sequnce $deg(v_1), deg(v_2), ..., deg(v_p)$ such that $deg(v_i) \geq deg(v_{i+1}), 1 \leq i < p$ (i.e. degree sequence of vertices of G is listed in decreasing order)
GRAPHICAL SEQUENCE: a decreasing sequence of integers that is a degree sequence of a simple undirected graph
REGULAR GRAPH: a graph in which all the vertices are of the same degree
k-REGULAR GRAPH: a regular graph in which every vertex is of degree k
CUBIC GRAPH: a 3-regular graph
ISOLATED VERTEX & PENDANT: a vertex v where $deg(v) = 0$ is *isolated.* v is a *pendant* if $deg(v) = 1$

THEOREM 8.4.1: Let $G = (V, E, \psi)$ be an undirected graph. Then, $\sum_{v \in V} deg(v) = 2|E|$

THEOREM 8.4.2: The number of vertices of odd degree in an undirected graph is even

Corollary 8.4.3: Let $G = (V, E)$ be a simple undirected graph. Then, $|E| \leq \frac{|V|(|V|-1)}{2}$

IDENTICAL GRAPH: Let $G = (V, E, \psi)$ and $G = (V', E', \psi')$ ne two graphs. G is identical to G', denoted by $G = G'$ if $V = V'$, $E = E'$, and $\psi = \psi'$.

SUBGRAPH & SUPERGRAPH: G' is a *subgraph* of G and G is a *supergraph* of G', denoted by $G' \sqsubseteq G$ if $V' \subseteq V$, $E' \subseteq E$, and $\psi' \subseteq \psi$

PROPER SUBGRAPH: G' is a *proper subgraph* of G, denoted by $G' \sqsubset G$, if $G' \sqsubseteq G$ and $G \neq G'$

SPANNING SUBGRAPH: G' is a *spanning subgraph* of G if $V = V'$ and G' is a subgraph of G

UNDERLYING SIMPLE GRAPH: of G, is a simple graph resulting from G after all the self-loops are removed, and for every pair of adjacent vertices having parallel edges joining them, all but one of the edges joining them are removed.

LEMMA 8.5.4: The relation $\sqsubseteq$ is a partial order
LEMMA 8.5.5: The relation $\sqsubset$ is a strict order

INDUCED SUBGRAPH: Let $G = (V, E, \psi)$ be a graph. Let $U \subseteq V$ such that $U \neq \emptyset$. The *subgraph of G induced by* U, denoted by $\langle U \rangle_G$, is the largest subgraph of G with vertex set U. i.e. $\langle U \rangle_G \in \zeta_U \wedge (\forall H \in \zeta_U)(H \sqsubseteq \langle U \rangle_G)$, where ζ_U is the set of all subgraphs of G with vertex set U

COMPLETE GRAPH: a *complete* graph is a simple undirected graph $G = (V, E)$ in which $\{u, v\} \in E, \forall u, v \in V, u \neq v$

COMPLEMENT GRAPH: Let $G = (V, E)$ be a simple graph. The *complement* of G is the simple graph $\overline{G} = (V, \overline{E})$, such that $\forall u, v \in V$, $(u \neq v) \Rightarrow (\{u, v\} \notin E \Leftrightarrow \{u, v\} \in \overline{E})$

GRAPH UNION: Let $G = (V, E, \psi)$ and $G' = (V', E', \psi')$ be two simple graphs such that $\psi \cup \psi'$ is a function. The *union* of G and G' is the graph $G \cup G' = (V \cup V', E \cup E', \psi \cup \psi')$

LEMMA 8.6.6: Let $G = (V, E)$ be a simple graph and $\overline{G} = (V, \overline{E})$ be the complement of G

i. $E \cap \overline{E} = \emptyset$
ii. $G \cup \overline{G} = K_{|V|}$, where $K_{|V|} = (V, E_{K_{|V|}})$ is the complete graph with vertex set V

BIPARTITE GRAPH: a graph $G = (V, E, \phi)$ is a *bipartite graph* if there exists $\{X, Y\}$ such that $X \cup Y = V$ and $X \cap Y = \emptyset$, and every edge in the graph has one end-vertex in X and one end-vertex in Y.

Isomorphism

ISOMORPHIC: let $G_1 = (V_1, E_1, \psi_1)$ and $G_2 = (V_2, E_2, \psi_2)$. G_1 and G_2 are *isomorphic*, denoted by $G_1 \cong G_2$, if $\exists \Phi : V_1 \to V_2$ such that $\{a, b\} \in E_1$ iff $\{\Phi(a), \Phi(b)\} \in E_2$. Φ is called an *isomorphic function* from G_1 to G_2

LEMMA 9.1.1: the relation $\cong$ is an equivalence relation in the set of all graphs

THEOREM 9.1.2: let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs such that $G_1 \cong G_2$. Then, $|V_1| = |V_2|$ and $|E_1| = |E_2|$

THEOREM 9.1.3: let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two sumple graphs such that $G_1 \cong G_2$. Then, $deg(u) = deg(\Phi(u)), \forall u \in V_1$, where Φ is an isomorphic function from G_1 to G_2

THEOREM 9.1.4: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple undirected graphs. Then, $G_1 \cong G_2$ iff $\overline{G_1} \cong \overline{G_2}$

Connected Graphs

u-v WALK: alternating sequence of vertices and edges of G, $w_0, e_1, w_1, ..., w_{k-1}, e_k, w_k$, s.t. $w_0 = u$ and $w_k = v$ and $psi(e_i) = \{w_{i-1}, w_i\}, 1 \leq i \leq k$
internal vertex: $w_i, 1 \leq i \leq k-1$
terminating vertices: $w_0 = u$ and $w_k = v$ of a $u - v$ walk
CLOSED WALK: a $u - v$ walk where $u = v$

NULL WALK: $u - v$ walk where $k = 0$

u-v TRAIL: a $u - v$ walk with no repeated edges
u-v PATH: a $u - v$ walk with no repeated vertices
CIRCUIT: non-null $u - v$ trail where $u = v$
CYCLE: circuit with no repeated vertices except the first and last

LEMMA 10.1.1: Every $u - v$ walk contains a $u - v$ path
LEMMA 10.1.2: Every circuit contains a cycle

CONNECTED: two vertices u and v in a graph G are connected, denoted by $u \sim_G v$, if there exists a $u - v$ path in G.
CONNECTED GRAPH: a graph is connected if every two vertices in it are connected. $(\forall u, v \in V) u \sim_G v$
DISCONNECTED GRAPH: a graph that is not connected $(\exists u, v \in V) u \nsim_G v$

THEOREM 10.2.3: binary relation $\sim$ is an equivalence relation in V_G

CONNECTED COMPONENT: of G is a subgraph of G induced by an equivalence class wrt $\sim$. # of components in G is denoted by $\omega(G)$

LEMMA 10.2.4: $\omega(G) \geq 1$, for every graph $G = (V, E)$
LEMMA 10.2.5: let $G_i = (V_i, E_i), 1 \leq i \leq \omega$, be the connected components of $G = (V, E)$. Then, $\bigcup_{i=1}^{\omega} V_i = V$ and $\bigcup_{i=1}^{\omega} E_i = E$
LEMMA 10.2.6: let H and H' be two distinct connected components of $G = (V, E)$. Then, $V_H \cap V_{H'} = \emptyset$ and $E_H \cap E_{H'} = \emptyset$
LEMMA 10.2.7: let H and H' be two distinct connected components of $G = (V, E)$. Then, $(\forall u \in V_H)(\forall v \in V_{H'}) u \nsim v$
THEOREM 10.2.8: A graph $G = (V, E)$ is disconnected iff $\omega(G) \geq 2$
LEMMA 10.2.9: let $G_i = (V_i, E_i), 1 \leq i \leq \omega$ be the connected components of $G = (V, E)$ s.t. the order and size of G_i are p_i and q_i, respectively. Then, $\sum_{i=1}^{\omega} p_i = |V|$ and $\sum_{i=1}^{\omega} q_i = |E|$

Length of a path: number of edges in the path
Distance: between two vertices u and v in graph G, $dist_G(u, v)$, is the length of the shortest $u - v$ path
Diameter: of a graph, denoted by $diameter(G)$, is the longest distance in that graph, i.e. $diameter(G) = max\{dist_G(u, v) \mid u, v \in V_G\}$
Girth: of G is the length of the shortest cycle in G

CYCLE GRAPH: a graph of order n consisting of *exactly one cycle*. A cycle of $order = 3$ is called a *triangle*
PATH GRAPH: a graph of order n consisting of *exactly one path*

Bridges and Cut Vertices

G-e: let G be a graph and v and e be a vertex and edge of G, respectively. Then, $G - e$ is the graph resulting from G after e is removed. $G - v$ is the graph resulting from G after v is removed

BRIDGE: an edge e is a *bridge* in G if $\omega(G - e) > \omega(G)$

CUT-VERTEX: a vertex v in a non-trivial graph G is a *cut-vertex* if $\omega(G - v) > \omega(G)$

LEMMA 11.1.1: let G be a connected graph. If e is a *bridge* in G, then $G - e$ is a disconnected graph. If e is a *cut-vertex* in G, then $G - v$ is a disconnected graph.

THEOREM 11.1.1: let $G = (V, E, \psi$ be a connected graph. An e of G is a bridge iff e does not lie on any cycle in G

Corollary 11.1.1.1: let $G = (V, E, \psi)$ be a graph and $\psi(e) = \{a, b\}$ be an edge in G. Then e is a bridge in G iff the path $a\ e\ b$ is the only path connecting vertices a and b

THEOREM 11.1.2: let $G = (V, E, \psi)$ be a non-trivial connected graph. A vertex v of G is a *cut-vertex* iff there exists two other vertices x, y in G s.t. every $x - y$ path in G passes through v

THEOREM 11.1.3: let $G = (V, E, \psi)$ be a connected graph with $|V| \geq 3$. If e is a bridge in G, then e is incident upon a cut-vertex G

Trees

TREE: a connected, undirected, simple graph without any circuits

FOREST: a graph whose connected components are trees

Lemma 12.1.1: A tree is a simple graph

Theorem 12.1.2: Let G=(V,E) be an undirected graph. The following six statements are equivalent:

- G is a tree
- G is circuit-free, but if any new edge is added to G, a circuit is formed
- G contains no self-loops and for every two vertices in G, there is a unique path connecting them
- G is connected but if any edge is removed from G, then the resulting graph is disconnected
- G is connected and $|E| = |V| - 1$
- G is circuit-free and $|E| = |V| - 1$

Theorem 12.1.3: In any tree G of order $p \geq 2$, there are at least two *pendants*

SPANNING TREE: of a connected graph G is a spanning subgraph of G which is a tree. Specifically, a graph $T = (V_T, E_T)$ is a *spanning tree* of a connected graph $G = (V, E)$ if $T \sqsubseteq G$ and $V_T = V$

Theorem 12.2.4: A graph is connected *iff* it has a spanning tree