Chapter 11 of Number Theory with Computer Applications

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Definition 11.1.3 A *continued fraction* has the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}}$$

where the a_i are integers.

It is called a *simple continued fraction* if it has the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

and if $a_1, a_2, \dots \geq 0$. A simple continued fraction of the above type is given the notation

$$[a_0, a_1, a_2, \dots].$$

A simple continued fraction is finite if it can be expressed as

$$[a_0, a_1, a_2, \dots, a_k] \text{ for some } k.$$

Algorithm to Convert a Rational Number into a Simple Continued Fraction

1. Apply the Euclidean algorithm to the numerator and the denominator.
2. Write the equations of the Euclidean algorithm as fractions.
3. Use the equations to convert to a simple continued fraction.

Example Convert 326/41 to a simple continued fraction.

SOLUTION:

$$
\begin{aligned}
326 &= 41(7) + 39 && \Rightarrow && \frac{326}{41} = 7 + \frac{39}{41} \\
41 &= 39(1) + 2 && \Rightarrow && \frac{41}{39} = 1 + \frac{2}{39} \\
39 &= 2(19) + 1 && \Rightarrow && \frac{39}{2} = 19 + \frac{1}{2} \\
2 &= 1(2)
\end{aligned}
$$

Thus $\frac{326}{41} = 7 + \frac{39}{41} = 7 + \cfrac{1}{\frac{41}{39}} = 7 + \cfrac{1}{1 + \frac{2}{39}} = 7 + \cfrac{1}{1 + \cfrac{1}{\frac{39}{2}}} = 7 + \cfrac{1}{1 + \cfrac{1}{19 + \frac{1}{2}}}$.

So $326/41 = [7, 1, 19, 2]$.

Example Convert the continued fraction $[5, 4, 3, 2]$ into a rational number in standard form.

SOLUTION: $[5, 4, 3, 2] = 5 + \cfrac{1}{4 + \cfrac{1}{3 + \frac{1}{2}}} = 5 + \cfrac{1}{4 + \frac{2}{7}} = 5 + \frac{7}{30} = \frac{157}{30}$.

Proposition 11.1.4 A real number α can be expressed as a finite simple continued fraction iff α is rational.

Proof. If α is rational, then the application of the Euclidean algorithm ends in a finite number of steps, and so gives a finite continued fraction.

If α is a finite continued fraction, then it can be turned into the form m/n by starting with the lower section and simplifying sequentially. ■

Definition: The infinite continued fraction $[a_0, a_1, a_2, \dots] = \lim_{n\to\infty}[a_0, a_1, \dots, a_n]$.

How to Convert an Irrational Number into a Simple Continued Fraction

Let $\lfloor\alpha\rfloor$ be the greatest integer (floor) function for α.

$\alpha = \lfloor\alpha\rfloor + r_0 = a_0 + r_0 = a_0 + \frac{1}{x_1}$.

Here $0 \le r_0 < 1$ so $x_1 > 1$. We repeat the procedure for x_1.

$x_1 = a_1 + \frac{1}{x_2}$. $a_1 = \lfloor x_1\rfloor$ and $\frac{1}{x_2} = [[x_1]]$ the fractional part of x_1.

$x_2 = a_2 + \dfrac{1}{x_3}$. $a_2 = \lfloor x_2 \rfloor$ and $\dfrac{1}{x_3} = [[x_2]]$ the fractional part of x_2.
Continuing gives the infinite continued fraction $[a_0, a_1, \dots]$.

Property: An irrational real number α has a unique simple continued fraction expression.

Example:

Find the continued fraction expansion for $\dfrac{1+\sqrt{5}}{2}$.

$\left\lfloor \dfrac{1+\sqrt{5}}{2} \right\rfloor = 1$.

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}-1}{2} = 1 + \cfrac{1}{\cfrac{1}{\cfrac{\sqrt{5}-1}{2}}} = 1 + \cfrac{1}{\cfrac{1+\sqrt{5}}{2}} = 1 + \cfrac{1}{1 + \cfrac{\sqrt{5}-1}{2}} =$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$$

$\frac{1+\sqrt{5}}{2} = [\overline{1}]$

Example:

Find the continued fraction expansion for $1 + 3\sqrt{2}$.

SOLUTION: Since $\lfloor 1 + 3\sqrt{2} \rfloor = 5$,

$$1 + 3\sqrt{2} = 5 + (1 + 3\sqrt{2} - 5) = 5 + (3\sqrt{2} - 4) = 5 + \cfrac{1}{\cfrac{1}{3\sqrt{2}-4}}.$$

Now $\dfrac{1}{3\sqrt{2}-4} = \dfrac{3\sqrt{2}+4}{(3\sqrt{2}+4)(3\sqrt{2}-4)} = \dfrac{3\sqrt{2}+4}{18-16} = \dfrac{3}{2}\sqrt{2} + 2 = 4 + (\dfrac{3\sqrt{2}-4}{2})$.
Thus

$$1 + 3\sqrt{2} = 5 + \cfrac{1}{\cfrac{1}{3\sqrt{2}-4}} = 5 + \cfrac{1}{4 + (\cfrac{3\sqrt{2}-4}{2})} = 5 + \cfrac{1}{4 + \cfrac{1}{(\cfrac{2}{3\sqrt{2}-4})}}.$$

Now $\dfrac{2}{3\sqrt{2}-4} = 8 + (3\sqrt{2} - 4)$. Thus

$$1 + 3\sqrt{2} = 5 + \cfrac{1}{4 + \cfrac{1}{8 + (3\sqrt{2}-4)}} = 5 + (3\sqrt{2} - 4)$$

so $3\sqrt{2} - 4 = \cfrac{1}{4 + \cfrac{1}{8 + (3\sqrt{2}-4)}}$

so $1+3\sqrt{2} = 5 + \cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{4+\cfrac{1}{8+\ldots}}}}$.

Hence $1+3\sqrt{2} = [5, \overline{4,8}]$

Some Classics

$\sqrt{2} = [1, \overline{2}]$.

$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots]$

$e^{1/n} = [1, n-1, 1, 1, 3n-1, 1, 1, 5n-1, 1, 1, 7n-1, 1, 1, \ldots]$

$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, \ldots]$.

$tan(1/n) = [0, n-1, 1, 3n-2, 1, 5n-2, 1, 7n-2, 1, 9n-2, 1, \ldots]$ for $n \geq 2$

Motivation:

- initial segments of continued fraction give rational approximations to the number
- continued fraction representation is finite if and only if number is rational whereas decimal representation of a rational number may be finite or infinite
- every rational number has essentially unique continued fraction representation: can be represented in exactly two ways: $[a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_{n-1}, (a_n - 1), 1]$ choose shorter as canonical representation

Convert an Infinite Continued Fraction to an irrational number

The following method SOMETIMES works (especially for square root expressions).

Find a closed expression for the continued fraction $[4, \overline{1,2}]$.

SOLUTION: Let $x = [4, \overline{1,2}] = 4 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\ldots}}}} = 4 + y$, where

$$y = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ldots}}} = \cfrac{1}{1+\cfrac{1}{2+y}} = \cfrac{1}{\left(\cfrac{3+y}{2+y}\right)} = \frac{2+y}{3+y}.$$

Thus $y^2 + 3y = 2 + y$ so $y^2 + 2y - 2 = 0$.

So $y = \dfrac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$. Discard the negative root. Thus $x = 4 + y =$

$4-1+\sqrt{3}=3+\sqrt{3}$.

Note You may want to try the sites

`http://www.numbertheory.org/php/nipell_pqa0.html`

`or`

`http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfCALC.html`

`to convert quadratics into continued fractions. \\`

`Also see`

`http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfCALC.html`

Exercises: p. 248 #1,3,4,5,6,8.

Some Solutions

Exercise p. 248 1a.) Determine the simple continued fraction expansion of $\frac{81}{19}$.

81=4(19)+5

19=3(5)+4

5=1(4)+1

4=4(1) Thus the continued fraction expansion is $[4,3,1,4]$.

3. Is the simple continued fraction expansion of a rational number unique? For 1a, the two possible expansions are $[4,3,1,4]$ and $[4,3,1,3,1]$

4. Note that $[a_0, a_1, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = a_0 + \cfrac{1}{[a_1, a_2, \dots]}$

11.2 Convergents

Definition 11.2.1 For $k \leq n$, the k th convergent C_k of the continued fraction $[a_0, a_1, \dots, a_n]$ or $[a_0, a_1, \dots]$ is $C_k = [a_0, a_1, \dots, a_k]$.

Example: Find the first four convergents of [3,7,15,1,292].

Solution: They are

$C_0 = 3$, $C_1 = 3 + 1/7 = 22/7$, $C_2 = 3 + \cfrac{1}{7 + \cfrac{1}{15}} = 333/106$,

$C_3 = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = 355/113$.

Property 11.2.3: Let $[a_0, a_1, \dots]$ be a continued fraction. Let $C_i = \dfrac{p_i}{q_i}$ for $i = 0, 1, \dots$. Then $C_0 = a_0 = \dfrac{a_0}{1} = \dfrac{p_0}{q_0}$ and $C_1 = a_0 + \dfrac{1}{a_1} = \dfrac{a_0 a_1 + 1}{a_1}$. Then the numerator p_k and denominator q_k of the k^{th} convergent satisfy the recurrence relation

$$p_k = a_k p_{k-1} + p_{k-2} \tag{1}$$

$$q_k = a_k q_{k-1} + q_{k-2} \tag{2}$$

with initial values $p_0 = a_0$, $p_1 = a_0 a_1 + 1$.

$q_0 = 1$, $q_1 = a_1$.

PROOF: see text.

Example $[a_0, a_1, a_2, a_3, a_4] = [3, 7, 15, 1, 292]$. Here $p_0 = 3$, $p_1 = 22$, $q_0 = 1$, $q_1 = 7$. So $C_1 = 22/7 \approx 3.142857$.

Thus $p_2 = a_2 p_1 + p_0 = 15(22) + 3 = 333$; $q_2 = a_2 q_1 + q_0 = 15(7) + 1 = 106$ so $C_2 = \dfrac{p_2}{q_2} = 333/106 \approx 3.141509$.

Next $p_3 = a_3 p_2 + p_1 = 1(333) + 22 = 355$; $q_3 = a_3 q_2 + q_1 = 1(106) + 7 = 113$ so $C_3 = \dfrac{p_3}{q_3} = 355/113 \approx 3.1415929204$.

Next $p_4 = a_4 p_3 + p_2 = 292(355) + 333 = 103993$; $q_4 = a_4 q_3 + q_2 = 292(113) + 106 = 33102$ so $C_4 = \dfrac{p_4}{q_4} = 103993/33102 \approx 3.141592653$.

The real value of π is $3.14159265458979\dots$.

Aside: Fibonacci Numbers:

Define $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n = 2, 3, \dots$. Then the Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

From $F_n = F_{n-1} + F_{n-2}$ we get $\dfrac{F_n}{F_{n-1}} = 1 + \dfrac{F_{n-2}}{F_{n-1}}$. Assume $\lim_{n\to\infty} \dfrac{F_n}{F_{n-1}}$ exists and call it γ. Then

$$\gamma = 1 + \frac{1}{\gamma}.$$

Thus $\gamma^2 = \gamma + 1$ or $\gamma^2 - \gamma - 1 = 0$. This is a quadratic and from the quadratic formula, we get $\gamma = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$. We discard the negative root to get

$$\gamma = \frac{1+\sqrt{5}}{2} \text{ (the golden ratio).}$$

Back to Continued Fractions. Example:

For the continued fraction $[\overline{1}]$, find the convergents.

SOLUTION: $p_k = p_{k-1} + p_{k-2}$ for $k \geq 2$ with $p_0 = a_0 = 1$, $p_1 = a_0 a_1 + 1 = 2$.

$q_k = q_{k-1} + q_{k-2}$ for $k \geq 2$ with $q_0 = 1$, $q_1 = a_1 = 1$.

Therefore $p_k = F_{k+1}$ the $k+1^{\text{st}}$ Fibonacci number.

$q_k = F_k$ the k^{th} Fibonacci number.

$C_k = F_{k+1}/F_k$ and $\lim_{k\to\infty} C_k = \dfrac{1+\sqrt{5}}{2}$.

$$[\overline{1}] = \gamma = \frac{1+\sqrt{5}}{2}.$$

Note: the larger the a_k is in the continued fraction the closer the corresponding convergent is to the number being approximated. Thus $\dfrac{1+\sqrt{5}}{2}$ is the hardest number to approximate rationally since the a_k are 1 everywhere.

Propostion 11.2.4: Let $C_k = \dfrac{p_k}{q_k}$ be the kth convergent of $[a_0, \ldots, a_k]$. Then

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \tag{3}$$

$$p_k q_{k-2} - q_k p_{k-2} = (-1)^k a_k. \tag{4}$$

Proof. (of (3) only) From $p_k = a_k p_{k-1} + p_{k-2}$ (1) and $q_k = a_k q_{k-1} + q_{k-2}$ (2), we have

$\begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix} = \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1} & q_{k-1} \\ p_{k-2} & q_{k-2} \end{bmatrix}$ so taking determinants of both sides and using $det(AB) = det(A)det(B)$ gives

$$\begin{aligned} p_kq_{k-1} - p_{k-1}q_k &= (-1)^1(p_{k-1}q_{k-2} - p_{k-2}q_{k-1}) = (-1)^2(p_{k-2}q_{k-3} - p_{k-3}q_{k-2}) \\ &= (-1)^3(p_{k-3}q_{k-4} - p_{k-4}q_{k-3}) = \cdots = (-1)^{k-1}(p_1q_0 - p_0q_1) \\ &= (-1)^{k-1}((a_0a_1+1)(1) - (a_0)(a_1)) = (-1)^{k-1} \end{aligned}$$

■

Corollary: p_k and q_k are relatively prime.

Proof. Since $p_kq_{k-1} - q_kp_{k-1} = (-1)^{k-1}$, then any divisor of both p_k and q_k also divides 1. ■

Example 11.2.5 Solve $257x - 97y = 1$ using continued fractions.

SOLUTION: Note that (257,97)=1 so by the Euclidean algorithm, there is a solution.

257=2(97)+63

97=1(63)+34

63=1(34)+29

34=1(29)+5

29=5(5)+4

5=1(4)+1

4=4(1)

so $\dfrac{257}{97} = [2, 1, 1, 1, 5, 1, 4]$ so the convergents are found from

$p_0 = a_0 = 2$, $p_1 = a_0a_1 + 1 = 3$, $p_2 = 1(3) + 2 = 5$, $p_3 = 1(5) + 3 = 8$, $p_4 = 5(8) + 5 = 45$, $p_5 = 1(45) + 8 = 53$, $p_6 = 4(53) + 45 = 257$;

$q_0 = 1$, $q_1 = a_1 = 1$, $q_2 = 1(1) + 1 = 2$, $q_3 = 1(2) + 1 = 3$, $q_4 = 5(3) + 2 = 17$, $q_5 = 1(17) + 3 = 20$, $q_6 = 4(20) + 17 = 97$.

Thus $C_0 = p_0/q_0 = 2$, $C_1 = 3$, $C_2 = 5/2$, $C_3 = 8/3$, $C_4 = 45/17$, $C_5 = 53/20$, $C_6 = 257/97$. Using $k = 6$ in Proposition 11.2.4, we get $p_6q_5 - q_6p_5 = (-1)^5 \Rightarrow$ $257(20) - 97(53) = -1$ so $257(-20) - 97(-53) = 1$ so $x = -20$, $y = -53$.

If we use the continued fraction $[2, 1, 1, 1, 5, 1, 3, 1]$ instead, $C_0, \ldots, C_5$ are the same, but $C_6 = 204/77$ and $C_7 = 257/97$. Again, using $k = 7$ in Proposition 11.2.4, we get $257(77) - 97(204) = 1$.

Corollary 11.2.6: Let $C_i = p_i/q_i$ be the ith convergent of $[a_0, a_1, \ldots]$. Then for $k = 0, \ldots, n$,

$$C_k - C_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}} \tag{6}$$

$$C_k - C_{k-2} = \frac{(-1)^k a_k}{q_k q_{k-2}} \tag{7}$$

Proof. Divide equation (3) by $q_k q_{k-1}$ to get (6). Divide equation (4) by $q_k q_{k-2}$ to get (7). ■

Property 11.2.8: The odd convergents form a decreasing sequence and the even convergents from an increasing sequence.

$C_0 \leq C_2 \leq C_4 \leq \ldots \alpha \cdots \leq C_5 \leq C_3 \leq C_1$.

Proof. $C_2 - C_0 = \dfrac{a_2(-1)^2}{q_2 q_0} \geq 0$, $C_4 - C_2 = \dfrac{a_4(-1)^4}{q_4 q_2} \geq 0$, etc. So $C_0 \leq C_2 \leq C_4 \leq \ldots$.

Similarly, by (7),

$C_3 - C_1 = \dfrac{a_3(-1)^3}{q_3 q_1} \leq 0$, $C_5 - C_3 = \dfrac{a_5(-1)^5}{q_5 q_3} \leq 0$, etc. So $C_1 \geq C_3 \geq C_5 \geq \ldots$.

The rest follows from $\lim_{n\to\infty} C_n = \alpha$. ■

Exercises: p. 255 # 1, 3, 5, 6, 7,12, 15a

Problem Solutions:

1(b) Compute the convergents of $[1, 3, 6, 11]$.

SOLUTION:

$$\begin{aligned}
p_0 = a_0 &= 1 & p_1 &= a_0 a_1 + 1 = (1)(3) + 1 = 4 \\
p_2 = a_2 p_1 + p_0 = 6(4) + 1 &= 25 & p_3 &= a_3 p_2 + p_1 = 11(25) + 4 = 279 \\
q_0 &= 1 & q_1 &= a_1 = 3 \\
q_2 = a_2 q_1 + q_0 = 6(3) + 1 &= 19 & q_3 &= a_3 q_2 + q_1 = 11(19) + 3 = 212.
\end{aligned}$$

$C_0 = \frac{p_0}{q_0} = 1$; $C_1 = \frac{p_1}{q_1} = 4/3$; $C_2 = \frac{p_2}{q_2} = 25/19$; $C_3 = \frac{p_3}{q_3} = 279/212$.

3(a)If $c > d > 0$, show that $[a, c] < [a, d]$.
SOLUTION: $c > d > 0 \Rightarrow \frac{1}{c} < \frac{1}{d} \Rightarrow a + \frac{1}{c} < a + \frac{1}{d} \Rightarrow [a, c] < [a, d]$.

4. Let $a_1, a_2, \ldots, a_n, x$ be positive real numbers. Determine values of n for which $[a_0, a_1, a_2, \ldots, a_n] > [a_0, a_1, a_2, \ldots, a_n + x]$.
SOLUTION: From #3, this is true for $n = 1$ but false for $n = 2$. Since $[a_0, a_1, \ldots, a_{n-1}, a_n] = [a_0, [a_1, \ldots, a_n]]$ we get an alternating type of result. Thus $n = 1, 3, 5, \ldots$.

We need the following two exercises for proving there exist solutions to Pell's Equation in Chaper 14.

Eg. p. 262 # 3) Show that

$$\alpha - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(x_{k+1}q_k + q_{k-1})}.$$

Proof: Allowing non-integers in continued fractions, we write

$$\alpha = [a_0, a_1, a_2, \ldots, a_k, x_{k+1}].$$

By Property 11.2.3,

$$\begin{aligned} p_{k+1} &= a_{k+1}p_k + p_{k-1} = x_{k+1}p_k + p_{k-1} \\ q_{k+1} &= a_{k+1}q_k + q_{k-1} = x_{k+1}q_k + q_{k-1} \end{aligned}$$

so since $a_{k+1} = x_{k+1}$,

$$\alpha = \frac{p_{k+1}}{q_{k+1}} = \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}}.$$

Thus

$$\begin{aligned}
\alpha - \frac{p_k}{q_k} &= \frac{x_{k+1}p_k + p_{k-1}}{x_{k+1}q_k + q_{k-1}} - \frac{p_k}{q_k} \\
&= \frac{(x_{k+1}p_k + p_{k-1})q_k - (x_{k+1}q_k + q_{k-1})p_k}{q_k(x_{k+1}q_k + q_{k-1})} \\
&= \frac{p_{k-1}q_k - p_k q_{k-1}}{q_k(x_{k+1}q_k + q_{k-1})} \\
&= \frac{(-1)^k}{q_k(x_{k+1}q_k + q_{k-1})} \qquad \text{by Proposition 11.2.4}
\end{aligned}$$

Eg. p. 262 # 4) Prove that if $\frac{p}{q}$ is a convergent of the simple continued fraction of an irrational number α, show that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Proof: By # 3,

$$\left|\alpha - \frac{p_k}{q_k}\right| = \frac{1}{|q_k(x_{k+1}q_k + q_{k-1})|} < \frac{1}{q_k^2}$$

since $x_{k+1} > 1$ and $q_{k-1} \geq 0$ making the denominator $\geq q_k^2$,