MH1812 Tutorial
Chapter 7: Set Theory

Q1: Deduce that the cardinality of the power set $P(S)$ of a finite set S with n element is 2^n.

Solution: The number of subsets of i elements from set S is: $\binom{n}{i}$, hence the total number of subsets is $\sum_{i=0}^{n} \binom{n}{i} = (1+1)^n = 2^n$. $\square$

Q2: Consider the set $A = \{1, 2, 3\}$, $P(A)$ = power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

Solution:

- $2^3 = 8$.
- The subsets are $\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$, hence in total 8.

$\square$

Q3: Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B), P(A), P(A \cup B), P(A \times B).$$

Solution: $A \cap B = \{2\}$, hence $P(A \cap B) = P(\{2\}) = \{\{\ \}, \{2\}\}$.
$P(A) = P(\{1, 2\}) = \{\{\ \}, \{1\}, \{2\}, \{1, 2\}\}$.
$A \cup B = \{1, 2, 3\}$, hence $P(A \cup B) = P(\{1, 2, 3\}) = \{\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$, hence
$P(A \times B) = \{$
$\{\ \}$,
$\{(1, 2)\}, \{(1, 3)\}, \{(2, 2)\}, \{(2, 3)\}$,
$\{(1, 2), (1, 3)\}, \{(1, 2), (2, 2)\}, \{(1, 2), (2, 3)\}, \{(1, 3), (2, 2)\}, \{(1, 3), (2, 3)\}, \{(2, 2), (2, 3)\}$,
$\{(1, 2), (1, 3), (2, 2)\}, \{(1, 2), (1, 3), (2, 3)\}, \{(1, 2), (2, 2), (2, 3)\}, \{(1, 3), (2, 2), (2, 3)\}$,
$\{(1, 2), (1, 3), (2, 2), (2, 3)\}\}$. $\square$

Q4: Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \emptyset.$$

Solution: Prove by contradiction: assume $(A - B) \cap (B - A) \neq \emptyset$. Then there exists some element $x \in (A-B)\cap(B-A)$. This means x belongs to both $(A-B)$ and $(B-A)$. Then $x \in A$ (since $x \in A-B$), and $x \notin A$ (since $x \in B-A$), hence contradiction. $\square$

Q5: Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B$$

Solution: To prove $P(A) = P(B) \iff A = B$, we need to prove both $P(A) = P(B) \to A = B$ and $P(A) = P(B) \leftarrow A = B$:

- When $P(A) = P(B)$, the subsets of single element should be the same for both, which means A and B are the same.
- When $A = B$, it implies $P(A) = P(B)$ due to the definition.

$\square$

Q6: Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Solution: To prove $P(A) \subseteq P(B) \iff A \subseteq B$, we need to prove both $P(A) \subseteq P(B) \to A \subseteq B$ and $P(A) \subseteq P(B) \leftarrow A \subseteq B$:

- $P(A) \subseteq P(B)$, so the subsets of single element in $P(A)$ are also in $P(B)$, hence all elements in A are also in B, i.e., $A \subseteq B$.
- For any $X \in P(A)$, $X \subseteq A \subseteq B$, i.e., $X \subseteq B$, hence $X \in P(B)$.

$\square$

Q7: Show that the empty set is a subset of all non-null sets.

Solution: Recall the definition of subset: $Y \subseteq X$ means by definition that $\forall x, (x \in Y \to x \in X)$. Now take Y to be the empty set $\emptyset$. Since $x \in Y$ is necessarily false (one cannot take any x from the empty set), then the conditional statement is vacuously true. $\square$

Q8: Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Solution:

$$\begin{aligned}
A \neq B &= \neg\forall x(x \in A \leftrightarrow x \in B) \\
&\equiv \exists x \neg(x \in A \leftrightarrow x \in B) \text{ (negation of universal quantifier)} \\
&\equiv \exists x \neg[(x \in A \to x \in B) \wedge (x \in B \to x \in A)] \text{ (definition)} \\
&\equiv \exists x[(\neg(x \in A \to x \in B)) \vee (\neg(x \in B \to x \in A))] \text{ (DeMorgan)} \\
&\equiv \exists x[(\neg(x \notin A \vee x \in B)) \vee (\neg(x \notin B \vee x \in A))] \text{ (Conversion)} \\
&\equiv \exists x[(x \in A \wedge x \notin B)) \vee (x \in B \wedge x \notin A))] \text{ (DeMorgan)}
\end{aligned}$$

$\square$

Q9: Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold ?

Solution: For any $x \in (A\times B)\cup(C\times D)$, $x \in (A\times B)$ or $x \in (C\times D)$. When $x \in (A\times B)$, $x = (x_1, x_2) \in (A \times B)$, by definition $x_1 \in A \subseteq A \cup C$, and $x_2 \in B \subseteq (B \cup D)$, hence $x \in (A \cup C) \times (B \cup D)$. We can prove the same when $x \in (C \times D)$.
The equality does NOT necessarily hold: take $A = [-1, 0]$, $B = [-1, 0]$, $C = [0, 1]$, $D = [0, 1]$ (all the sets are intervals, that is $[a, b]$ means the interval from a to b). Then

$$[-1, 0] \times [-1, 0] \neq [-1, 1] \times [-1, 1].$$

$\square$

Q10: Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold ?

Solution: It does NOT hold. For example, take $A_1 = \{0\}, A_2 = \{1\}, B_1 = \{0\}, B_2 = \{1\}$, then

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

while

$$(A_1 \times B_1) \cup (A_2 \times B_2) = \{(0, 0), (1, 1)\}$$

$\square$

Q11: How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

Solution:

$$\sum_{i=0}^{2i\leq n} \binom{n}{i}$$

$\square$

Q12: Prove the following set equality:

$$\{12a + 25b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Solution: First of all, it is easy to see, all elements in the LHS are integers, hence LHS $\subseteq \mathbb{Z}$. Then we have to prove every integer is also in LHS. Note for $12 \times (-2) + 25 = 1$, so any integer x can be expressed as $x \times (12 \times (-2) + 25) = 12 \times (-2x) + 25x$, i.e., with $(a = -2x, b = x)$. $\square$

Q13: Let A, B, C be sets. Prove or disprove the following set equality:

$$A - (B \cup C) = (A - B) \cap (A - C).$$

Solution: LHS $= A - (B \cup C) = A \cap \overline{B \cup C} = A \cap \overline{B} \cap \overline{C}$
RHS $= (A \cap \overline{B}) \cap (A \cap \overline{C}) = (A \cap A) \cap \overline{B} \cap \overline{C} = A \cap \overline{B} \cap \overline{C} =$ LHS. $\square$

Q14: For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \overline{A} \cup B.$$

using set identities.

Solution: LHS $= \overline{(A \cap \overline{B}) \cap \overline{\overline{B \cap \overline{C}}}} = \overline{(A \cap \overline{B}) \cap (\overline{B} \cup C)} = \overline{A \cap (\overline{B} \cap (\overline{B} \cup C))} = \overline{A \cap \overline{B}}. = \overline{A} \cup B =$ RHS. $\square$

Q15: This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Solution: To prove LHS $\subseteq$ RHS:

$$\begin{aligned} x \in (A \cup B) \cap \overline{A \cap B} &\equiv x \in (A \cup B) \wedge x \in \overline{A \cap B} \\ &\equiv (x \in A \wedge x \notin A \cap B) \vee (x \in B \wedge x \notin A \cap B) \\ &\equiv (x \in A - B) \vee (x \in B - A) \\ &\equiv x \in (A - B) \cup (B - A) \end{aligned}$$

$\square$

Q16: The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.
2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

Solution:

- By definition: $A \oplus B = (A - B) \cup (B - A)$, hence:

$$\begin{aligned} (A \oplus B) \oplus B &\equiv (((A - B) \cup (B - A)) - B) \cup (B - ((A - B) \cup (B - A))) \\ &\equiv ((A - B - B) \cup (B - A - B)) \cup ((B - (A - B)) \wedge (B - (B - A))) \\ &\equiv ((A - B) \cup \emptyset) \cup (B \wedge (B \cap A)) \\ &\equiv (A - B) \cup (B - A) \end{aligned}$$

A	B	$A \oplus B$	$(A \oplus B) \oplus B$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

$\square$

Q17: In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Solution: Lets denote the set of students who ate durians, mangoes, litchis as D, M, L, respectively. From the question we have

$$|D| = 88, |M| = 73, |L| = 46, |D \cap M| = 34, |D \cap L| = 16, |M \cap L| = 12, |D \cap M \cap| = 5.$$

Now, we want to find out $|\overline{D \cup M \cup L}|$, note $|D \cup M \cup L| = |D| + |M| + |L| - |D \cap M| - |D \cap L| - |M \cap L| + |D \cap M \cap L| = 88 + 73 + 46 - 34 - 16 - 12 + 5 = 150$, hence $|\overline{D \cup M \cup L}| = 200 - 150 = 50$. $\square$

Q18: Let A, B, C be sets. Prove or disprove the following set equality:

$$A \times (B - C) = (A \times B) - (A \times C).$$

Solution: For any $x = (x_1, x_2) \in A \times (B - C)$, by definition it means $x_1 \in A$ and $x_2 \in (B - C)$ and $x_2 \in (B - C) \iff x \in B \wedge x \notin C \iff (x_1, x_2) \in A \times B \wedge (x_1, x_2) \notin A \times C \iff (x_1, x_2) \in (A \times B) - (A \times C)$. $\square$