MH1812 Tutorial
Chapter 9: Functions

Q1: Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and intersection $\cap$ function: $P(A) \times P(A) \to P(A)$, i.e., for any $x, y \in P(A)$, $f(x, y) = x \cap y$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

Solution:
$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
Domain: $P(A) \times P(A)$
Co-Domain: $P(A)$
Range: $P(A)$
9 $\square$

Q2: Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

Solution: $\sin(0) = \sin(\pi) = 0$ $\square$

Q3: Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not onto, but $\sin : \mathbb{R} \to [-1, 1]$ is.

Solution: Since $-1 \leq \sin \leq 1$, there exists no $x \in \mathbb{R}$ such that $\sin(x) = 2$.
There exists $x \in [-\pi/2, \pi/2]$ such that for any $y \in [-1, 1]$, $\sin(x) = y$. $\square$

Q4: Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

Solution: No. Since for $y = 1 \in \mathbb{Z}$, there exists no n such that $4n-1 = 1$ ($n = 1/2 \notin \mathbb{Z}$) $\square$

Q5: Is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, a bijection (one-to-one correspondence)?

Solution: YES.
Surjective: for any $y \in \mathbb{R}$ there exists $x = \sqrt[3]{y}$ such that $x^3 = y$.
One-to-One: for any $x_1, x_2 \in \mathbb{R}$, if $x_1^3 = x_2^3$, $(x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0 \Rightarrow x_1 = x_2$, hence one-to-one. $\square$

Q6: Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 5$. What is $g \circ f$? What is $f \circ g$?

Solution: $g \circ f(x) = x^2 + 5$, $f \circ g(x) = (x + 5)^2$. $\square$

Q7: Consider $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. What is $g \circ f$? What is $f \circ g$?

Solution: $g \circ f(n) = (n + 1)^2$, $f \circ g(n) = n^2 + 1$. □

Q8: Given two functions $f : X \to Y$, $g : Y \to Z$. If $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.

Solution:f must be one-to-one, but necessarily for g.
Counter-Example: $X = \mathbb{Z}^+, Y = Z = \mathbb{Z}$, $f(x) = x$, and $g(x) = x^2$.
So $g \circ f : \mathbb{Z}^+ \to \mathbb{Z}$ and $g \circ f(x) = x^2$, this is one-to-one.
However, $g : \mathbb{Z} \to \mathbb{Z}$ and $g(x) = x^2$ is NOT one-to-one, e.g., $g(1) = g(-1) = 1$. □

Q9: Show that if $f : X \to Y$ is invertible with inverse function $f^{-1} : Y \to X$, then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.

Solution: For any $x \in X$, let $y = f(x)$, and by the definition of f^{-1}, $f^{-1}(y) = x$, i.e., $f^{-1} \circ f(x) = f^{-1}(y) = x$, hence $f^{-1} \circ f = i_X$.
For any $y \in Y$, since f^{-1} exists, and if $x \in X$ such that $f(x) = y$, then by the definition of f^{-1}, $f^{-1}(y) = x$, then $f \circ f^{-1}(y) = f(x) = y$, hence $f \circ f^{-1} = i_Y$. □

Q10: Prove or disprove $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$, for x, y two real numbers.

Solution:Counter-Example: $x = y = 0.4$, $\lceil x + y \rceil = \lceil 0.8 \rceil = 1$ while $\lceil x \rceil + \lceil y \rceil = \lceil 0.4 \rceil + \lceil 0.4 \rceil = 1 + 1 = 2$. □

Q11: If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.

Solution: 4 pigeonholes with 5 pigeons, at leat two will fall in the same pigeonhole. □

Q12: If you have 10 black socks and 10 white socks, and you are picking socks randomly, you will only need to pick three to find a matching pair.

Solution: 2 pigeonholes with 3 pigeons, at least two will fall in the same pigeonhole. □