Solution 10

Exercise96 Prove that if a connected graph G has exactly two vertices which have odd degree, then it contains an Euler path.

Solution Suppose that v and w to be the two vertices that have odd degree, while the other vertices have an even degree. Create a new graph G' with one more edge e which connects node v and w, then every vertices in G' has even degree, so there is an Euler cycle as

$$v,\, e_1,\, v_2,\, e_2,\, \cdots,\, w,\, e,\, v$$

then

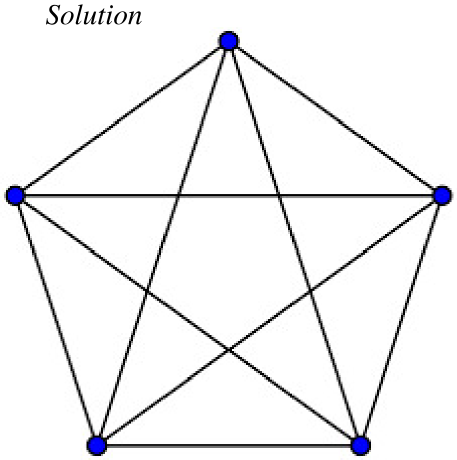
$$v,\, e_1,\, v_2,\, e_2,\, \cdots,\, w$$

is an Euler path.

To find the Euler circuit for any graph with even degrees for all vertices, you can refer to this note for more information: http://www.math.unm.edu/~loring/links/discrete_f05/euler.pdf

Exercise97 Draw a complete graph with 5 vertices.

Solution



Exercise98 Show that in every graph G, the number of vertices of odd degree is even.

Solution Proof by contradiction. In every graph $G = (V, E)$, write the set V of vertices as $V' \cup V''$ where V' is the set of vertices with odd degree and V'' is the set of vertices with even degree. Suppose that the number of vertices of odd degree is odd.

By Handshaking Theorem,

$$2e = \sum_{v \in V} deg(v) = \sum_{v \in V'} deg(v) + \sum_{v \in V''} deg(v)$$

where the first sum $\sum_{v \in V'} deg(v)$ is odd and the second sum $\sum_{v \in V''} deg(v)$ is always even, a contradiction.

Exercise99 Show that in very simple graph (with at least two vertices), there must be two vertices that have the same degree.

Solution Proof by contradiction. Suppose there is a graph G with n vertices. If all degrees are different, there must be n vertices with n different degrees $0, 1, \cdots, n-1$. There exist two vertices v and w such that $deg(v) = 0$ and $deg(w) = n-1$, which is impossible since $deg(w) = n-1$ means w is connected to all other vertices including v, but $deg(v) = 0$ means v is connected to no other vertices including w.

Exercise100 Decide whether the following graphs contain a Euler path/cycle.

Solution By Euler Theorem, consider a connected graph G,

- If G contains an Euler path that starts and ends at the same node, then all nodes of G have an even degree.
- If G contains an Euler path, then exactly two nodes of G have an odd degree.

For first graph, there are exactly two nodes of G have an odd degree, then it contains an Euler path.

For second graph, all nodes of G have an even degree, then it contains an Euler cycle.

For third graph, there are four nodes of G have an odd degree, then it does not contain an Euler path or cycle.