NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics
MH1812

Topic 10.2 - Graph Theory II
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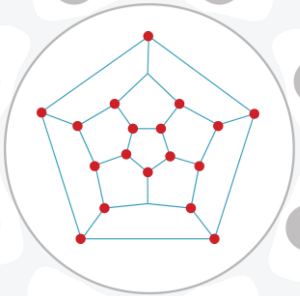
Topic Overview

What's in store…

Graphs

More on Node Degree

Adjacency Matrix

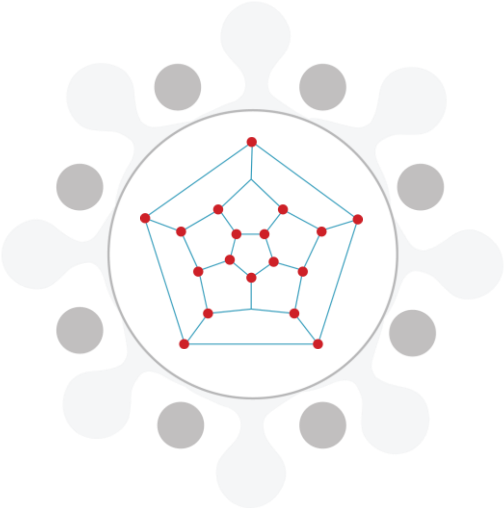


By the end of this lesson, you should be able to…

- Explain the difference between complete graph and bipartite graph.

- Explain how to find the total degree of an undirected graph.

- Explain how a graph can be represented by a matrix.



Graphs

Graphs: Wolf, Goat and Cabbage

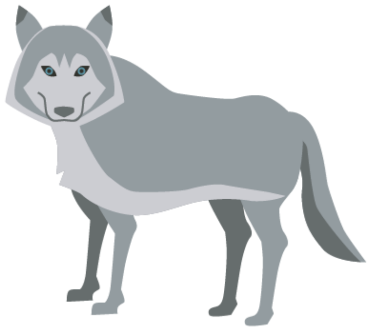
A classical puzzle that involves graphs.

From the left bank of the river, the ferryman has to transport the wolf, the goat and the cabbage to the right bank.

The boat is only big enough to transport one object/animal at a time, other than himself.

The wolf cannot be left alone with the goat, and the goat cannot be left alone with the cabbage.

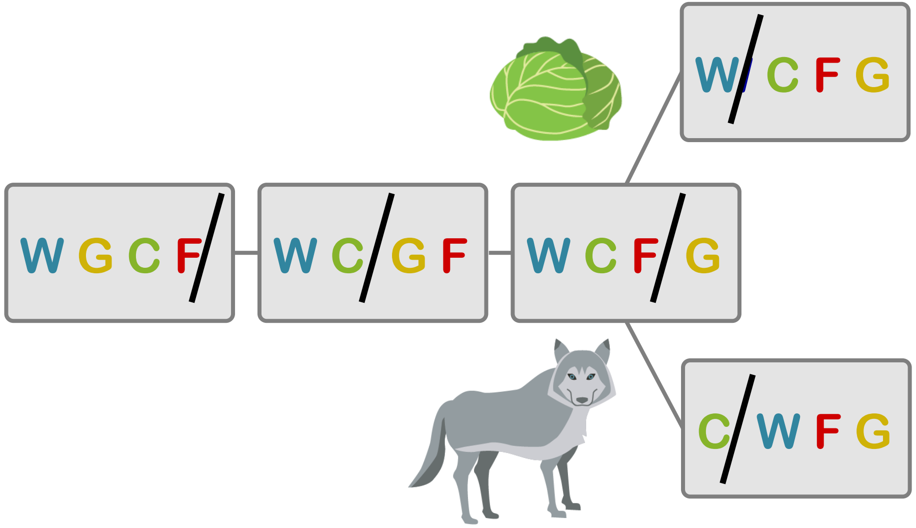
How should the ferryman proceed?



Graphs: Wolf, Goat and Cabbage

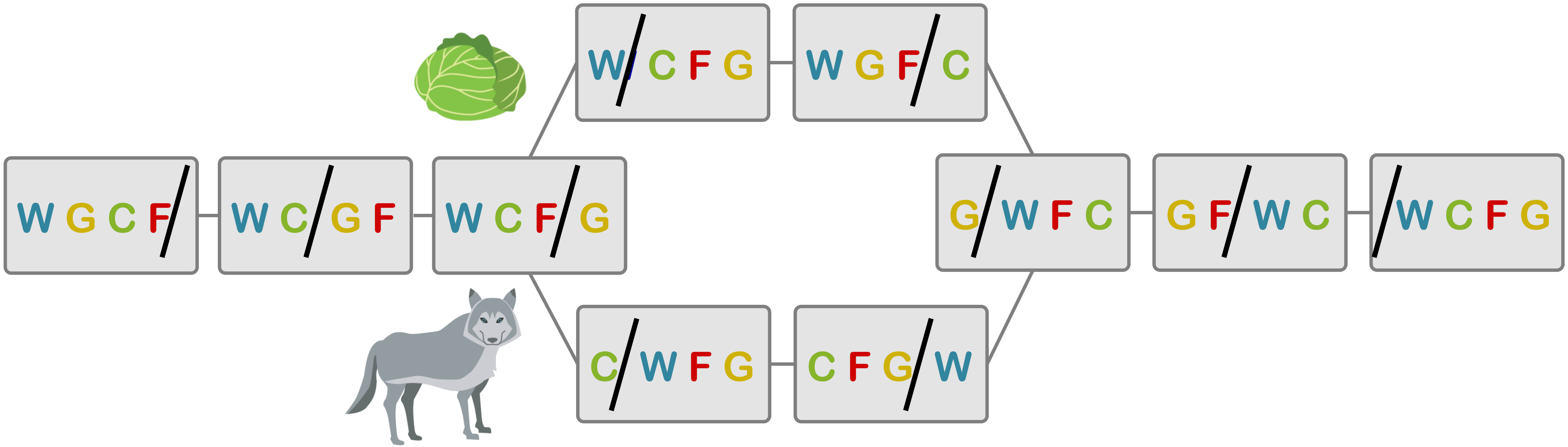
1. The ferryman takes the goat (no other choice)
2. The ferryman returns
3. Either he takes the cabbage or the wolf

F = ferryman
G = goat
W = wolf
C = cabbage



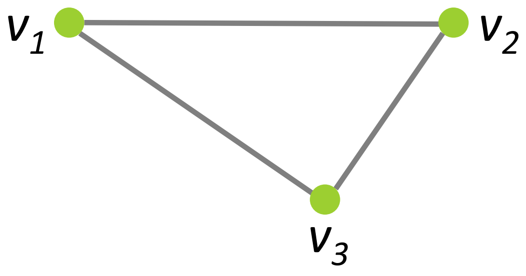
Graphs: Wolf, Goat and Cabbage

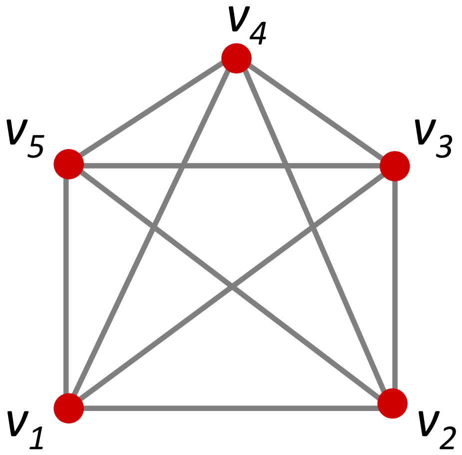
4. Either he takes:
   a. The cabbage, brings back the goat, leaves the goat and takes the wolf across, returns, and takes the goat across.
   b. The wolf, brings back the goat, leaves the goat and takes the cabbage across, returns, and takes the goat across.

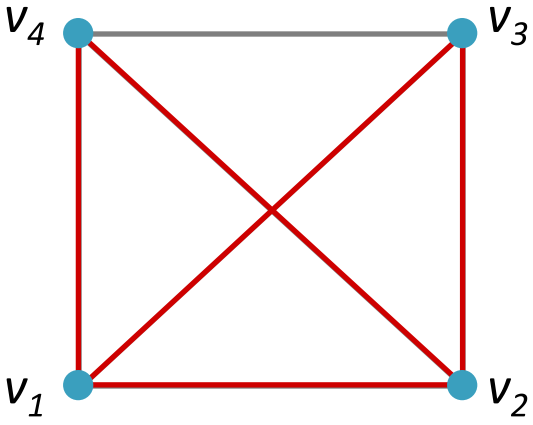


Graphs: Complete Graphs

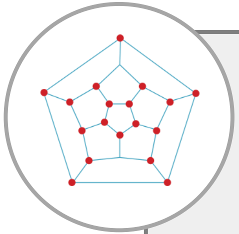
A **complete graph with *n* vertices** is a simple graph that has every vertex adjacent to every other distinct vertex.



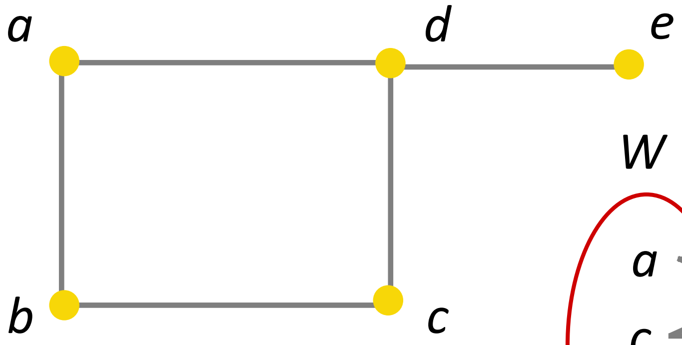




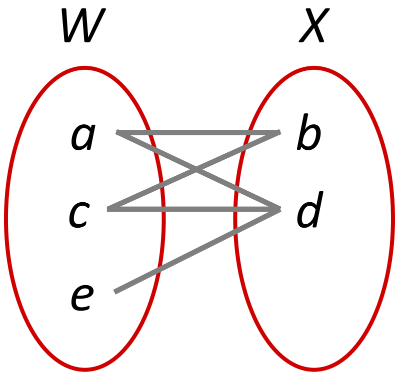
Graphs: Bipartite Graphs



A **bipartite graph** is a graph whose vertices can be partitioned into *2* (disjoint) subsets W and X such that each edge connects a $w \in W$ and a $x \in X$.



$W \cap X = \varnothing$



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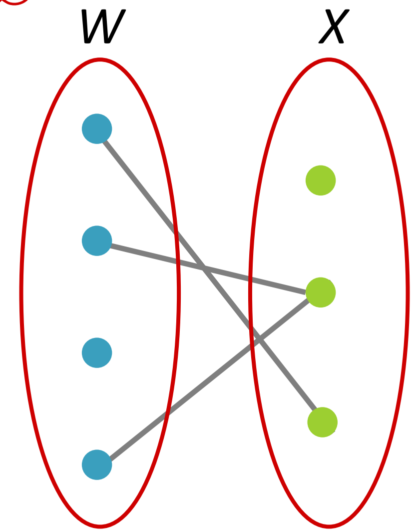
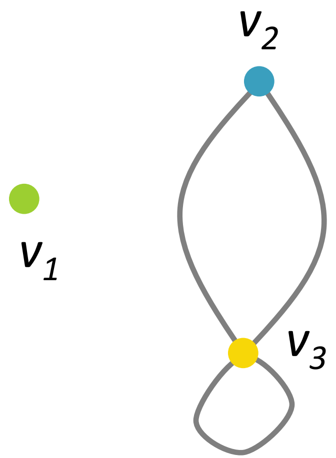


More on Node Degree

Mode on Node Degree: Definitions

The **degree deg(v)** of a vertex v in an undirected graph is the number of edges incident with it (a loop at a vertex contributes twice). In-degree and out-degree are distinguished for directed graphs.



Total degree = $\deg(v_1) + \deg(v_2) + \deg(v_3) = 0 + 2 + 4 = 6$

The **total degree deg(G)** of an undirected graph G is the sum of the degrees of all the vertices of G: $\sum_{v \in V} \deg(v)$

Mode on Node Degree: The Handshaking Theorem

Let $G = (V,E)$ **be an undirected graph with** e **edges.**

Then

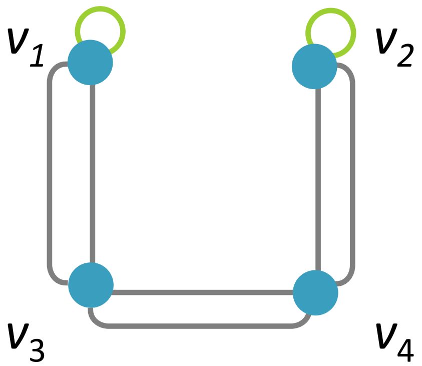
$$\boxed{2e = \sum_{v \in V} \deg(v)}$$

(Note that this even applies if multiple edges and loops are present.)

Mode on Node Degree: The Handshaking Theorem

Proof

Choose an $e \in E(G)$ with endpoints $v, w \in V$. e contributes 1 to $\deg(v)$ and 1 to $\deg(w)$. This is true even when $v = w$. Thus, each edge contributes 2 to the total degree.

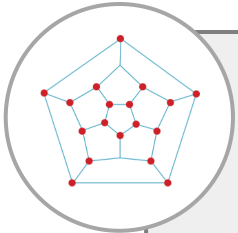


$$\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4) = 4$$

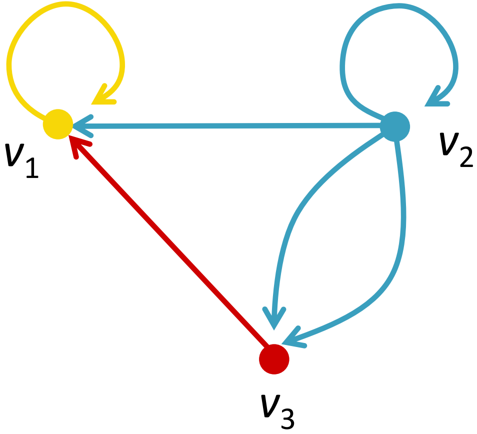
$$2e = \sum \deg(v) = 4 \times 4 = 16 \text{ and } e = 8$$

Adjacency Matrix

Adjacency Matrix: Definition



A graph can be represented by a matrix $A = (a_{ij})$ called adjacency matrix, with a_{ij} = the number of arrows from v_i to v_j.



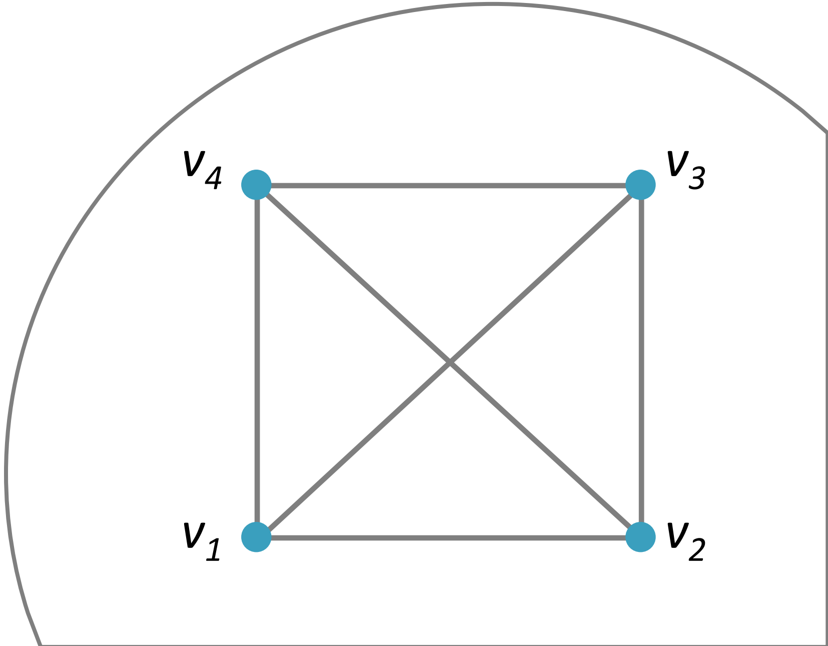
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

What is the adjacency matrix of a complete graph?

Adjacency Matrix: Example

What is the adjacency matrix of a complete graph?

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



Topic Summary

Let's recap…

- Types of graphs:
  - Complete graph
  - Bipartite graph
- Handshaking theorem
- A graph represented by a matrix