Discrete Mathematics

MH1812

Topic 2.2 - Propositional Logic II

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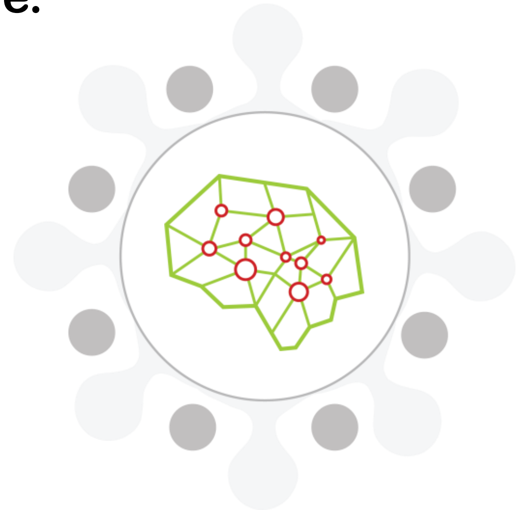
Topic Overview

What's in store…

- **L**ogical Equivalence Laws
- **C**onditional Operator
- **B**iconditional Operator
- **O**perator Precedence

By the end of this lesson, you should be able to…

- Prove equivalence using logical equivalence laws.
- Use the conditional operator to combine propositions.
- Use the biconditional operator to combine propositions.
- Evaluate logical expressions using operator precedence.



Logical Equivalence Laws

Logical Equivalence Laws: Already Seen

- Useful laws to **transform** one logical expression to an equivalent one

Axioms

T ≡ Tautology

C ≡ Contradiction

$$\neg\, T \equiv F$$

$$\neg\, F \equiv T$$

$$\neg\, T \equiv C \equiv F$$

$$\neg\, C \equiv T \equiv T$$

De Morgan

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Commutativity

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

Logical Equivalence Laws: More Laws

Double Negation

$$\neg(\neg p) \equiv p$$

Absorption

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Idempotent

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

Logical Equivalence Laws: Distributive Law

Distributivity

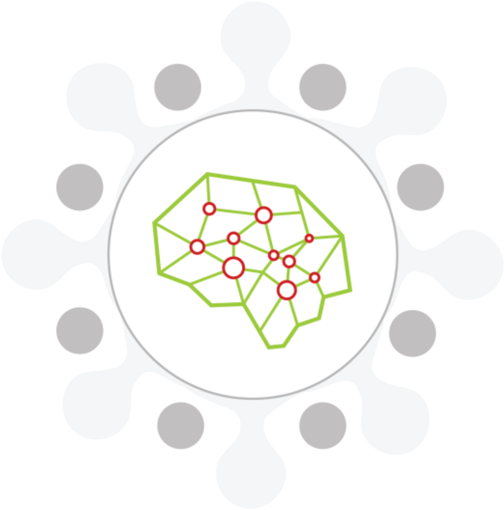
$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

Conditional Operator

Conditional Operator: Known Operators

- $\wedge$ Conjunction (and)
- $\vee$ Disjunction (or)
- $\neg$ Negation (not, alternatively $\sim$)



Conditional Operator: If Then

If p then q: $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

By definition, when p is false, $p \rightarrow q$ is true.

This is called **vacuously true** or **true by default**.

```
gap>
gap> a:=10;; if (a>5) then Print("yes"); fi;
yes
gap> a:=1;; if (a>5) then Print("yes"); fi;
gap>
gap>
```

Not really the same!

Conditional Operator: Conversion Theorem

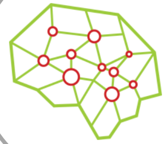
Theorem: $p \to q \equiv \neg p \vee q$

Proof

p q	$p \to q$	$\neg p$	$\neg p \vee q$
T T	T	F	T
T F	F	F	F
F T	T	T	T
F F	T	T	T

Conditional Operator: Converse, Inverse, Contrapositive

Statement	$p \rightarrow q$
Converse	$q \rightarrow p$
Inverse	$\neg p \rightarrow \neg q$
Contrapositive	$\neg q \rightarrow \neg p$



Theorem: $\neg q \rightarrow \neg p \equiv p \rightarrow q$

Proof

$\neg q \rightarrow \neg p$

$\equiv \neg(\neg q) \vee \neg p$

$\equiv q \vee \neg p$

$\equiv \neg p \vee q$

$\equiv p \rightarrow q$

Conditional Operator: Only If

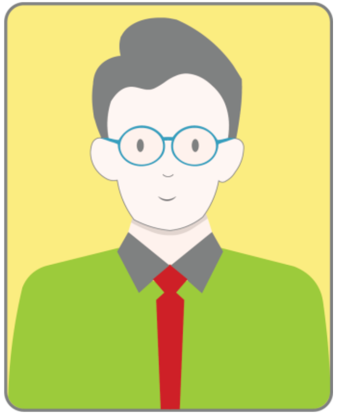
- p **only if** $q \triangleq \neg q \rightarrow \neg p$
- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
- (If not q then not p) $\equiv$ $(p \rightarrow q)$ (why?)

Example

"Bob pays taxes **only if** his income $\geq \$1000$"

$\triangleq$ "if Bob's income $< \$1000$ then he does not pay taxes"

$\equiv$ "if Bob pays tax then his income $\geq \$1000$"



Bob

Conditional Operator: Sufficient and Necessary Conditions

When $p \rightarrow q$, p is called a sufficient condition for q,
q is a necessary condition for p.

- Being an apple is a sufficient condition for being a fruit.

  ≡ If it is an apple then it must be a fruit.

- Being a fruit is a necessary condition for being an apple.

  ≡ If it is not a fruit then it cannot be an apple.

Conditional Operator: Example

Let f: “you fix my ceiling”, p: “I will pay my rent”

“You fix my ceiling **or** I won’t pay my rent!”

$$f \vee \neg p \equiv p \rightarrow f$$

Landlord

“**If** you do not fix my ceiling, **then** I won’t pay my rent.”

$$\neg f \rightarrow \neg p \equiv p \rightarrow f$$

Tenant

“I will pay my rent **only if** you fix my ceiling.”

$$\neg f \rightarrow \neg p \equiv p \rightarrow f$$

Biconditional Operator

Biconditional Operator: Known Operators

- $\neg$ Negation (not)
- $\wedge$ Conjunction (and)
- $\vee$ Disjunction (or)
- $\rightarrow$ Conditional (if then)

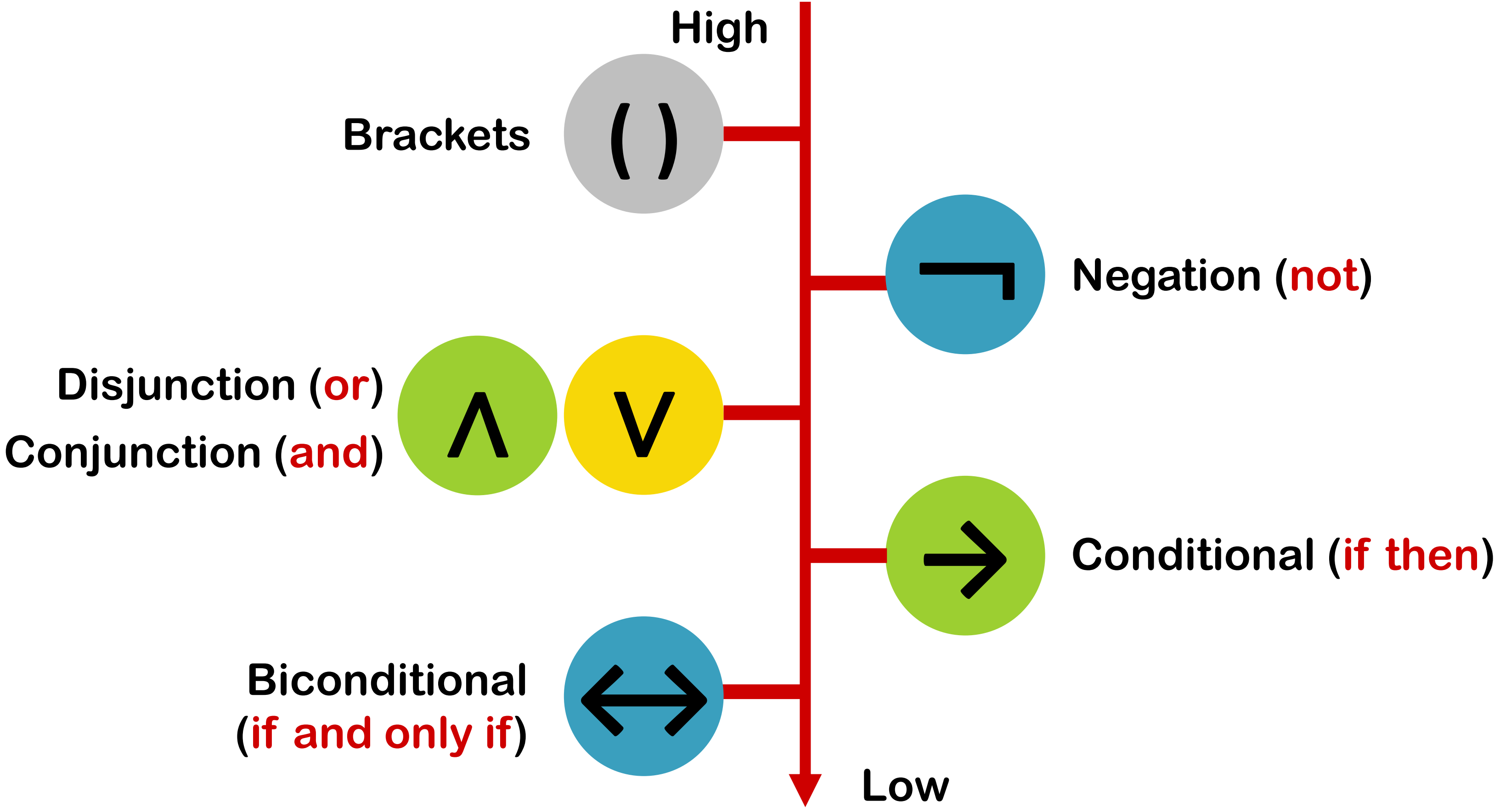
Biconditional Operator: If and Only If

- The **biconditional** of p and q: $p \leftrightarrow q \triangleq (p \rightarrow q) \wedge (q \rightarrow p)$
  - True only when p and q have identical truth value
- If and only if (**iff**)

p q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T T	T	T	T
T F	F	T	F
F T	T	F	F
F F	T	T	T

Operator Precedence

Operator Precedence: High to Low



Operator Precedence: Leftmost and Rightmost

Leftmost Precedence

When equal priority instances of binary connectives are not separated by (), the leftmost one has precedence.

E.g., $p \to q \to r \equiv (p \to q) \to r$

High → Low

Rightmost Precedence

When instances of $\neg$ are not separated by (), the rightmost one has precedence.

E.g., $\neg\neg\neg p \equiv \neg(\neg(\neg p))$

Low ← High

Operator Precedence: Example

Show that $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

$p \vee q \rightarrow r$

$\equiv (p \vee q) \rightarrow r$ **Operator precedence**

$\equiv \neg(p \vee q) \vee r$ **Why?**

$\equiv (\neg p \wedge \neg q) \vee r$ **De Morgan's**

$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ **Why?**

$\equiv (p \rightarrow r) \wedge (q \rightarrow r)$ **Why?**

Topic Summary

Let's recap…

- Useful logical equivalence laws:
  - Proving equivalence using these laws
- Conditional and biconditional operators:
  - Sufficient and necessary conditions
- Operator precedence