NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 3.2 - Predicate Logic II

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Topic Overview

What's in store…

Negation of Quantification

Determining Truth Values

Conditional Quantification

$\forall\ \exists$

By the end of this lesson, you should be able to…

- Express the negation of a quantified statement.
- Find the truth value of a quantified statement.
- Manipulate quantified conditional statements.

Negation of Quantification

Negation of Quantification: Truth vs. False

Statement	When True	When False
$\forall x \in D, P(x)$	$P(x)$ is **true** for every x in D.	There is one x for which $P(x)$ is **false**.
$\exists x \in D, P(x)$	There is one x in D for which $P(x)$ is **true**.	$P(x)$ is **false** for every x in D.

Assume that D consists of $x_1, x_2, \dots, x_n$

$$\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$$

$$\exists x \in D, P(x) \equiv P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)$$

Negation of Quantification: Example 1

"**Not all** SCSE students study hard."

=

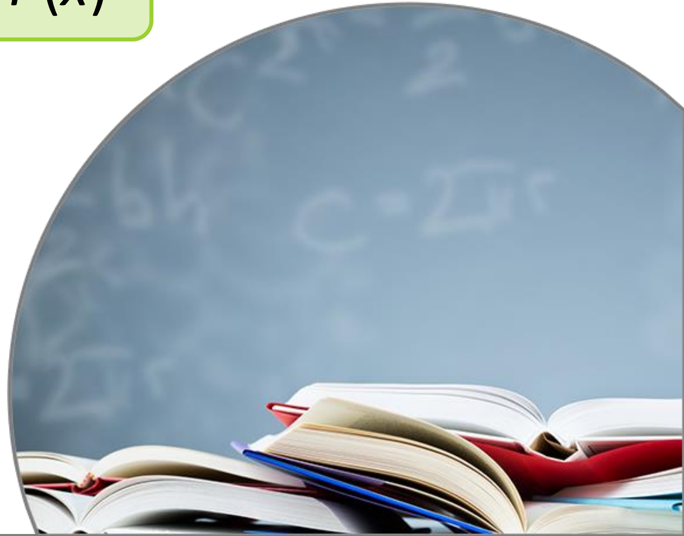
"**There is at least one** SCSE student who does not study hard."

$\neg (\forall x \in D, P(x))$ $\equiv$ $\exists x \in D, \neg P(x)$

D = {SCSE students}

$P(x)$ = "x studies hard"

Negation of a universal quantification becomes an existential quantification.



Negation of Quantification: Example 2

"It is not the case that some students in this class are from NUS."

≡

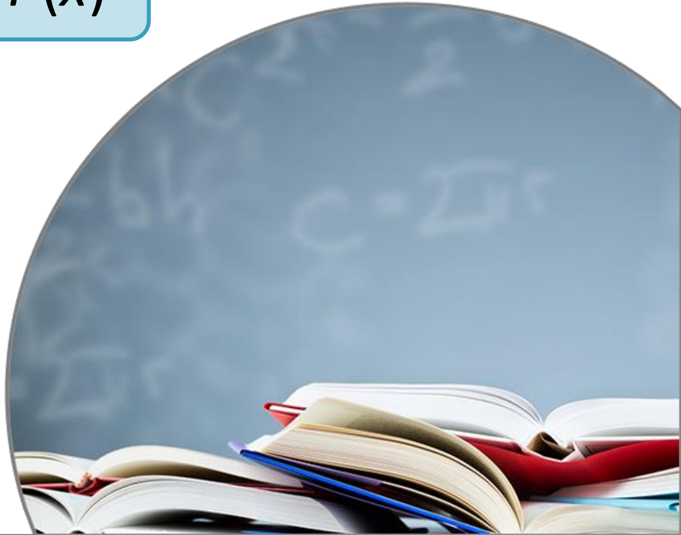
"All students in this class are not from NUS."

$$\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

D = {Students}

$P(x)$ = "x is from NUS"

Negation of an existential quantification becomes an universal quantification.



Negation of Quantification: Example 3

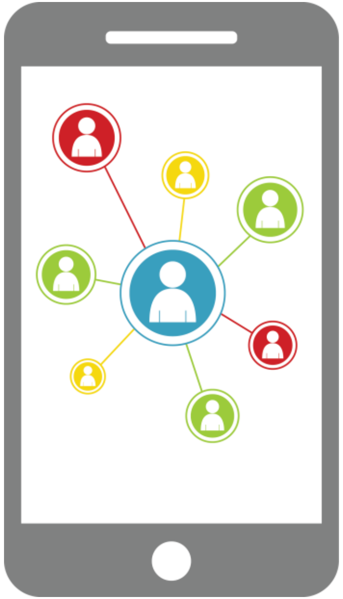
$$\neg\,(\forall x \in D,\ P(x) \wedge Q(x))$$

$$\equiv\ \exists x \in D,\ \neg\,(P(x) \wedge Q(x))$$

Negation of Quantification

$$\equiv\ \exists x \in D,\ (\neg\, P(x) \vee \neg\, Q(x))$$

De Morgan



- Not all students in this class are using Facebook and (also) Google+.
  - There is some (at least one) student in this class who is not using Facebook or not using Google+ (or may be using neither).

Determining Truth Values

Determining Truth Values: Three Methods

Systematic Approaches

Method of:

- Exhaustion
- Case
- Logical derivation

Determining Truth Values: Method of Exhaustion

Let $D = \{5,6,7,8,9\}$

Is $\exists x \in D, x^2 = x$ **true or false?**

x	x^2	$x^2 = x$
5	$5^2 = 25$	**False**
6	$6^2 = 36$	**False**
7	$7^2 = 49$	**False**
8	$8^2 = 64$	**False**
9	$9^2 = 81$	**False**

Limitation?

- **Domain may be too large to try out all options, e.g., all integers.**

Determining Truth Values: Method of Case

Positive Example to Prove Existential Quantification

Let $\mathbb{Z}$ denote all integers.

Is $\exists x \in \mathbb{Z}, x^2 = x$ true or false?

Take $x = 0$ or 1 and we have it.

True!

Counterexample to Disprove Universal Quantification

Let $\mathbb{R}$ denote all reals.

Is $\forall x \in \mathbb{R}, x^2 > x$ true or false?

Take $x = 0.3$ as a counterexample.

False!

Determining Truth Values: Method of Case

Positive Example

It is not a proof of universal quantification.

Negative Example

It is not disproof of existential quantification.

Note that it may be hard to find suitable "cases" even if such cases do exist!

Determining Truth Values: Method of Logical Derivation

Consider an (arbitrary) domain X with n members.

Is $\exists x \in X, (P(x) \vee Q(x)) \equiv (\exists x \in X, P(x)) \vee (\exists x \in X, Q(x))$?

$$\exists x \in X, (P(x) \vee Q(x))$$

$$\equiv [P(x_1) \vee Q(x_1)] \vee \ldots \vee [P(x_n) \vee Q(x_n)]$$

$$\equiv [P(x_1) \vee \ldots \vee P(x_n)] \vee [Q(x_1) \vee \ldots \vee Q(x_n)]$$

$$\equiv (\exists x \in X, P(x)) \vee (\exists x \in X, Q(x))$$

Conditional Quantification

Conditional Quantification: Example 1

For any real number x, if $x > 1$ then $x^2 > 1$ (i.e., any real number greater than 1 has a square larger than 1).

- **Let $P(x)$ denote "$x > 1$".**
- **Let $Q(x)$ denote "$x^2 > 1$".**
- **Recall: $\mathbb{R}$ is the collection of all real numbers.**

In Symbolic Form: $\forall x \in \mathbb{R}, (P(x) \rightarrow Q(x))$

Conditional Quantification: Example 2

Many statements can be restated as conditional statements. Consider the statement "lions are fierce animals".

- **Let A denote the collection of all animals.**
- **Let $P(x)$ denote "x is a lion".**
- **Let $Q(x)$ denote "x is fierce".**
- **The statement can be rephrased as: "If an animal x is a lion then x is fierce".**

In Symbolic Form: $\forall x \in A, (P(x) \rightarrow Q(x))$

Conditional Quantification: Definitions

Given a conditional quantification such as…

$$\forall x \in A\ (P(x) \rightarrow Q(x))$$

Then, we define…

Contrapositive	$\forall x \in A,\ \neg Q(x) \rightarrow \neg P(x)$
Converse	$\forall x \in A,\ Q(x) \rightarrow P(x)$
Inverse	$\forall x \in A,\ \neg P(x) \rightarrow \neg Q(x)$

Note: a conditional proposition is logically equivalent to its contrapositive.

Conditional Quantification: Negation

What is $\neg\,(\forall x \in X,\ P(x) \rightarrow Q(x))$**?**

$\neg\,(\forall x \in X,\ P(x) \rightarrow Q(x))$

$\equiv \exists x \in X, \neg\,(P(x) \rightarrow Q(x))$ **Negation of Quantified Statements**

$\equiv \exists x \in X, \neg\,(\neg\,P(x) \vee Q(x))$ **Conversion of Conditionals**

$\equiv \exists x \in X, P(x) \wedge \neg\,Q(x)$ **De Morgan**

Topic Summary

Let's recap…

- Negation of quantification
- Determining truth value of a quantification:
  - Methods for proving quantified statements
- Conditional quantification