Discrete Mathematics

MH1812

Topic 8.1 - Relations I

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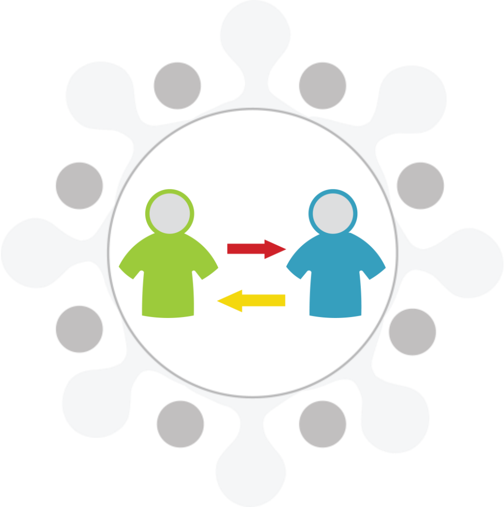
Topic Overview

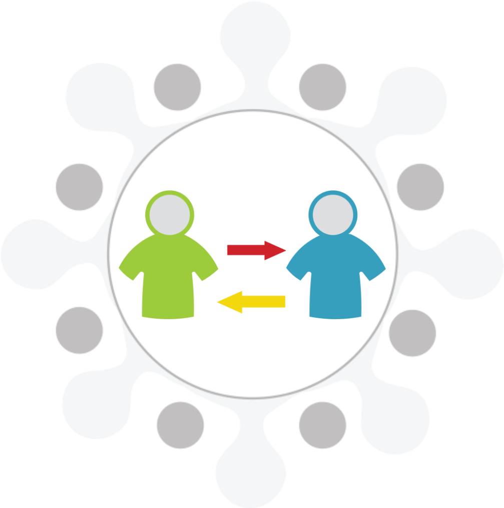
What's in store…

- **B**inary Relations
- **R**eflexivity
- **S**ymmetry
- **T**ransitivity



By the end of this lesson, you should be able to...

- Explain the different types of binary relations.
- Explain the concept of reflexivity.
- Explain the concept of symmetry.
- Explain the concept of transitivity.



Binary Relations

Binary Relations: Between Two Sets

Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$. Given (x,y) in $A \times B$, x is related to y by R $(xRy) \leftrightarrow (x,y) \in R$.

Example

$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \leftrightarrow (x-y)$ **is even**

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

$(1,1) \in R$, $(1,3) \in R$, $(2,2) \in R$

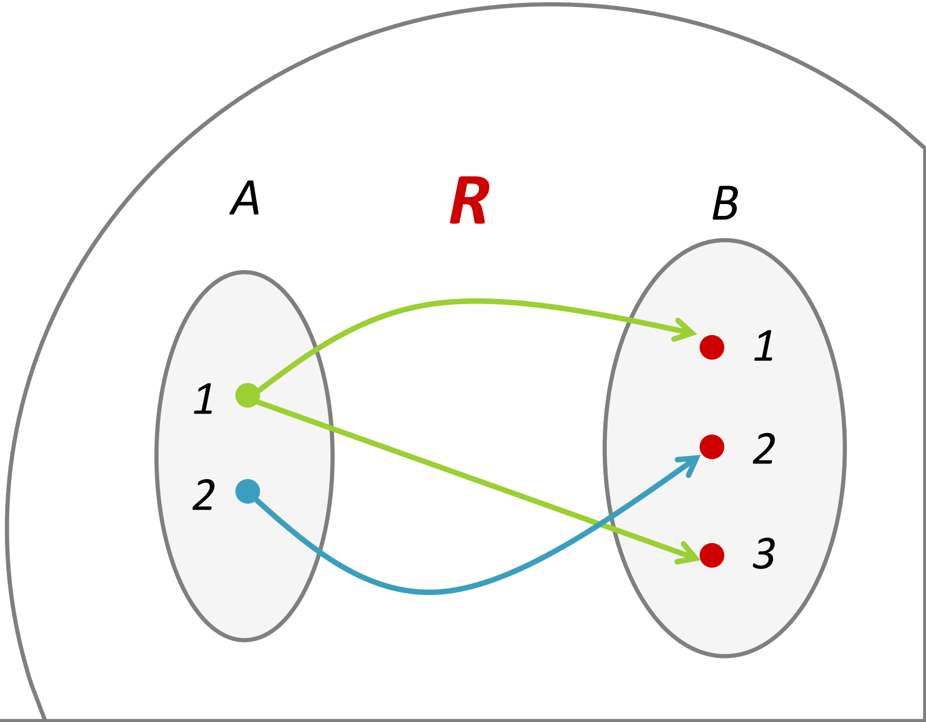
$x > y$, x **owes** y, x **divides** y

Binary Relations: Between Two Sets (Graphically)

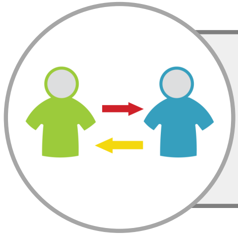
$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \leftrightarrow (x - y)$ **is even**

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

$(1,1) \in R$, $(1,3) \in R$, $(2,2) \in R$



Binary Relations: Inverse of a Binary Relation



Let R be a relation from A to B. The **inverse relation** R^{-1} from B to A is defined as: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

Binary Relations: Inverse of a Binary Relation (Example)

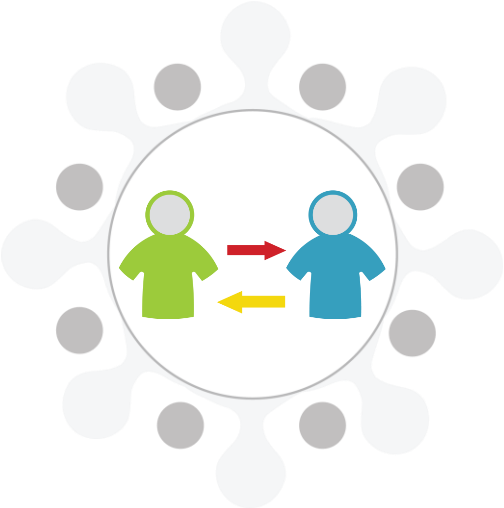
$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x, y) \in R \leftrightarrow x$ **divides** y

$A \times B = \{(2,2), (2,6), (2,8), (3,2), (3,6), (3,8), (4,2), (4,6), (4,8)\}$

$(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

$(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$

$(y, x) \in R^{-1} \leftrightarrow y$ **is a multiple of** x

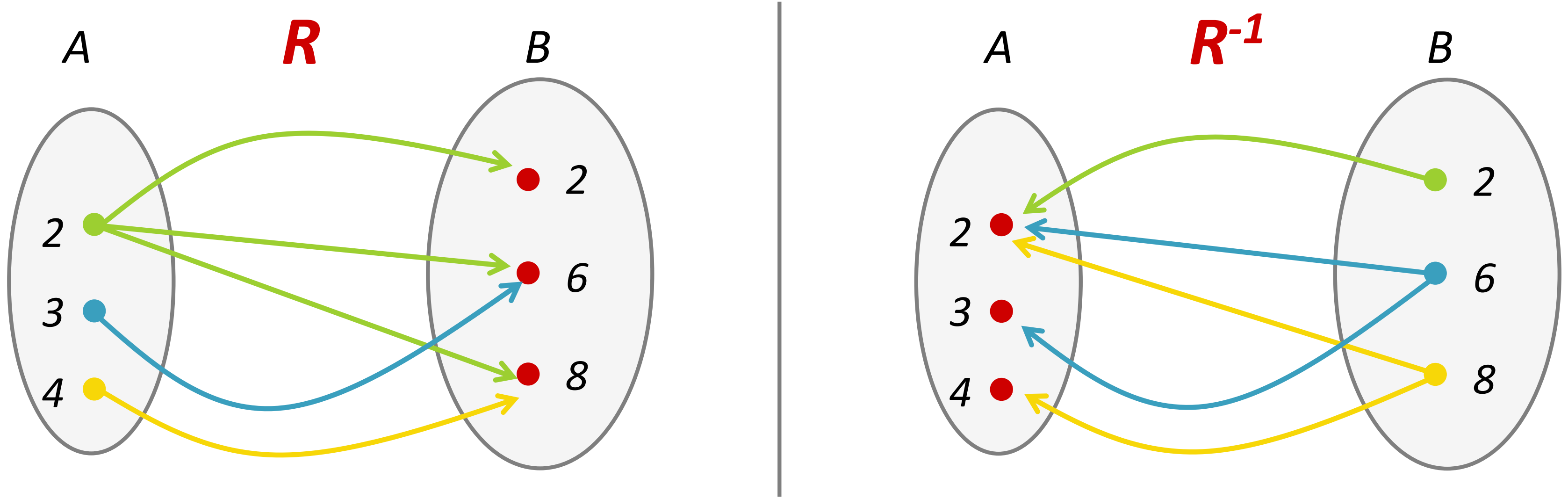


Binary Relations: Inverse of a Binary Relation (Graphically)

$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x, y) \in R \leftrightarrow x$ **divides** y

$(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

$(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$



Binary Relations: Matrix Representation

$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3, b_{4,}),$

$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

(i, j)th **entry is** T **if** $a_i R b_j$ **:**

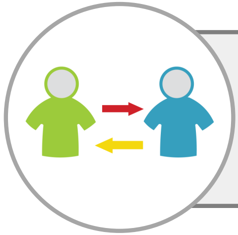
$$\begin{array}{c c} & \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix} \end{array}$$

Example

$A = \{2,3,4\},\ B = \{2,6,8\}, (x, y) \in R \leftrightarrow x$ **divides** y.

$A \backslash B$	2	6	8
2	T	T	T
3	F	T	F
4	F	F	T

Binary Relations: Matrix Representation



R **relation from** *A* **to** *B*: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

$$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3, b_{4,})$$

$$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$$

$$R^{-1} = \{(b_2, a_1), (b_1, a_2), (b_1, a_3), (b_4, a_3)\}$$

The matrix of R^{-1} **is the transpose of the matrix of** R**.**

$$a_i R b_j: \quad \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix} \end{array}$$

$$b_i R^{-1} a_j: \quad \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{bmatrix} F & T & T \\ T & F & F \\ F & F & F \\ F & F & T \end{bmatrix} \end{array}$$

Binary Relations: Composition of Relations

Given R in $A \times B$, and S in $B \times C$, the composition of R and S is a relation on $A \times C$ defined by
$S \circ R = \{(a, c) \in A \times C \mid \exists\, b \in B, aRb \text{ and } bSc\}$.

Example

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

$R = \{(a_1, b_1), (a_1, b_2)\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

What is $S \circ R$?

$S \circ R = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$

Binary Relations: Composition of Relations (Graphically)

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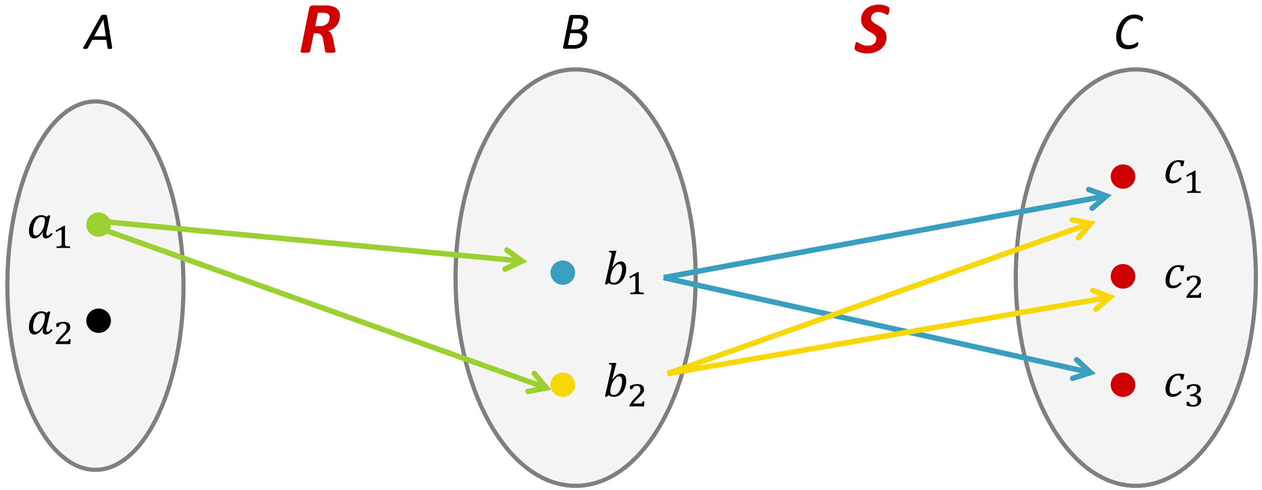

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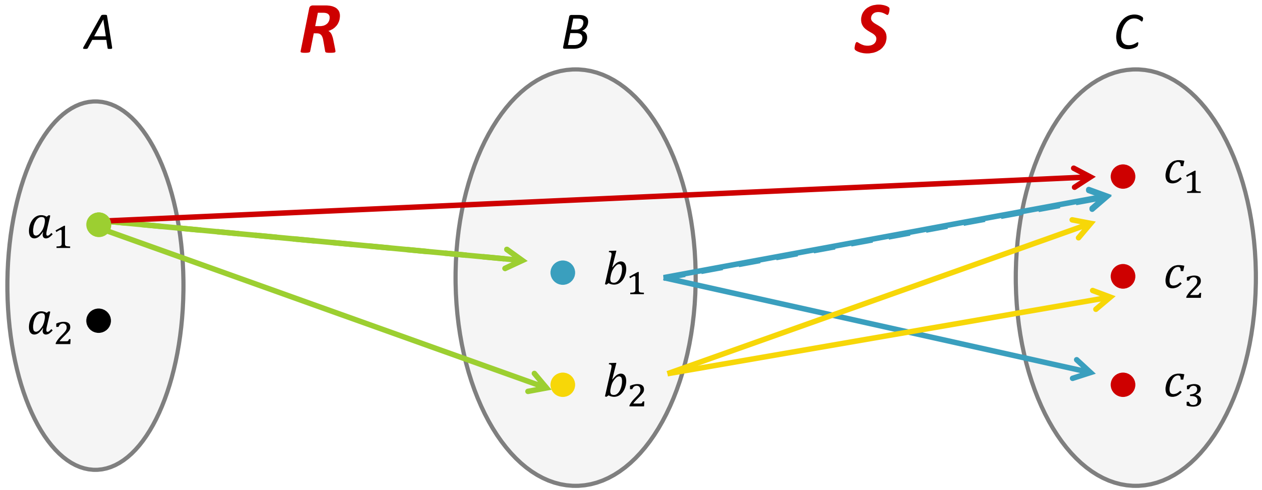
$S \circ R = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$



Binary Relations: Composition of Relations (Graphically)

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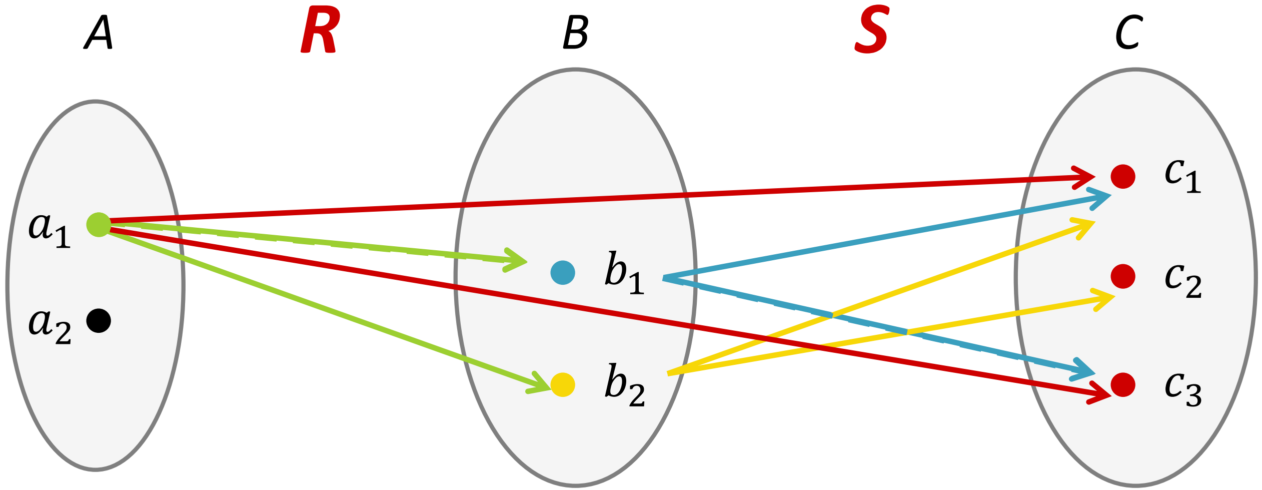

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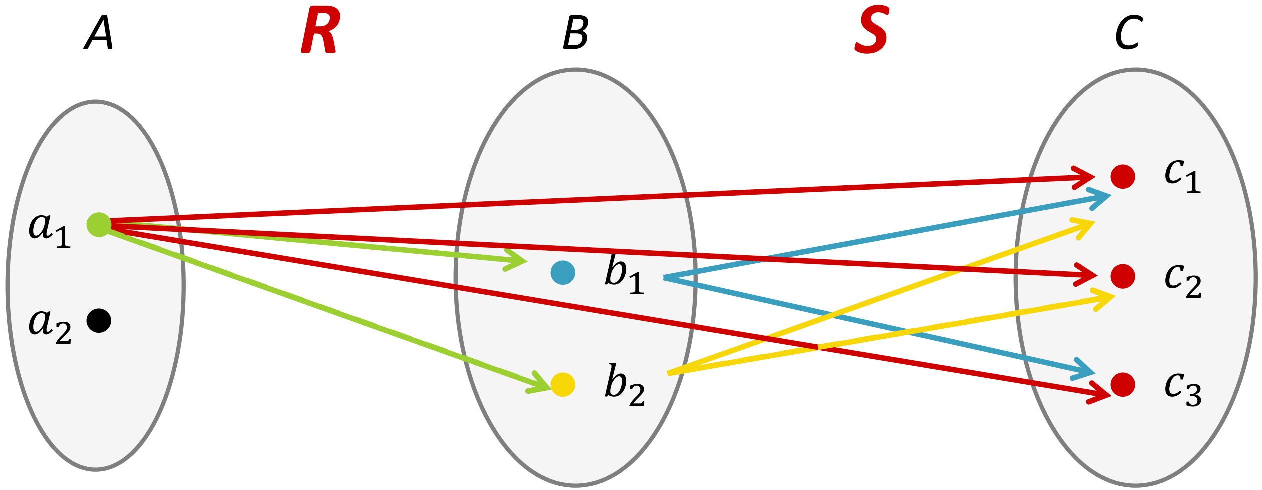


Reflexivity

Reflexivity: Definition

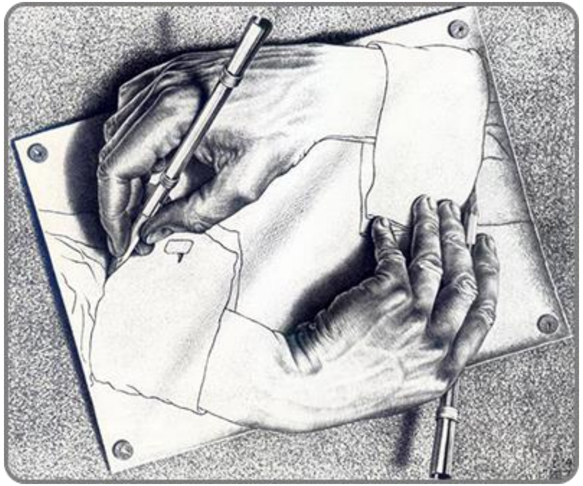
A relation R on a set A is **reflexive** if every element of A is related to itself: $\forall x \in A, xRx$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: reflexive

$A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: not reflexive

What is the reflexivity on the matrix representing R?



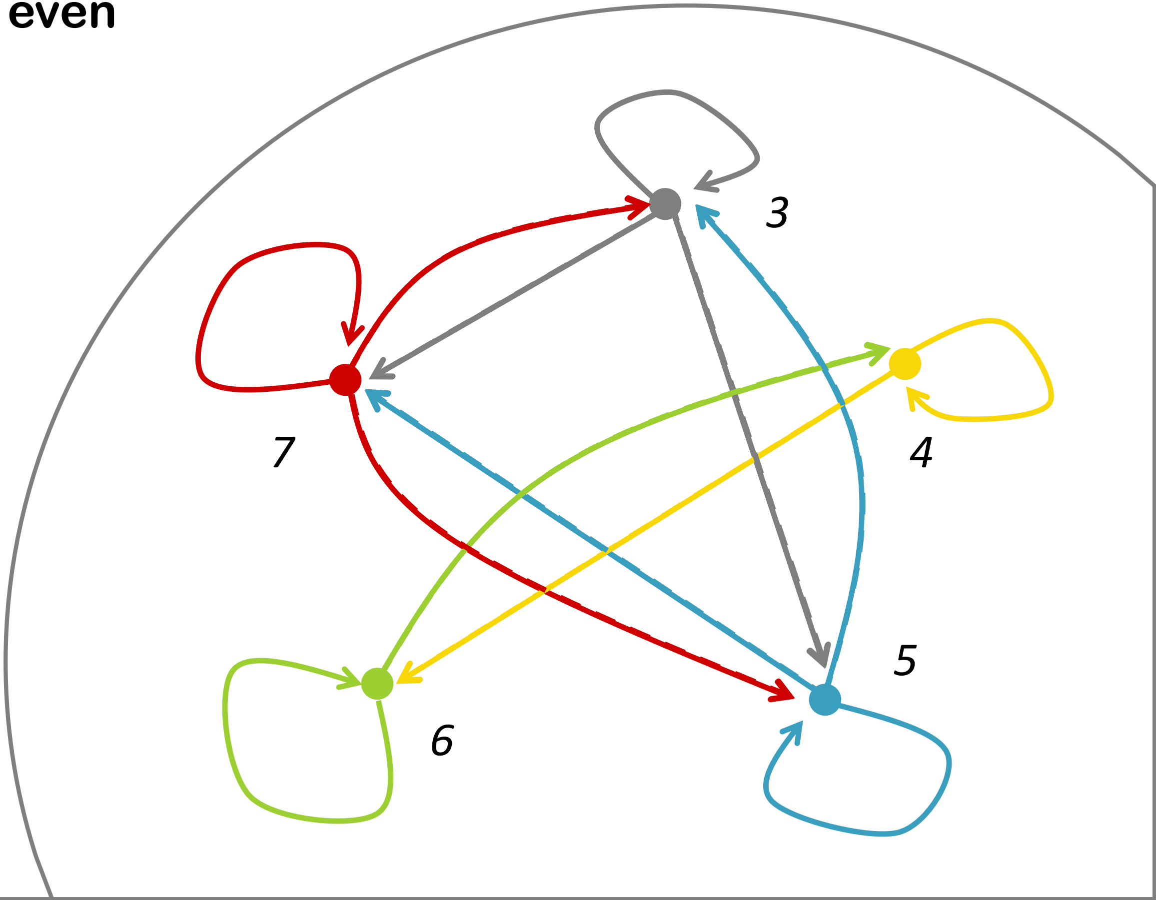
Drawing Hands (M.C. Escher)

Retrieved Mar 10, 2018 from https://www.wikiart.org/en/m-c-escher/drawing-hands

Reflexivity: Graphically

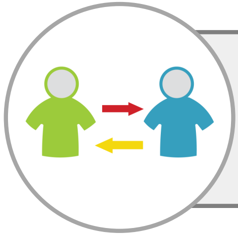
$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ is even

R reflexive

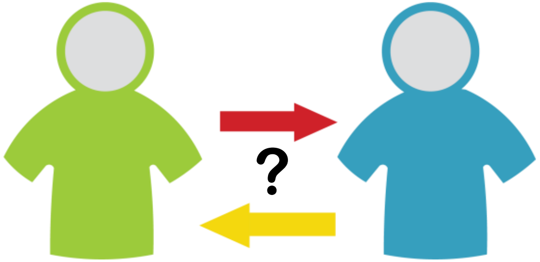


Symmetry

Symmetry: Definition

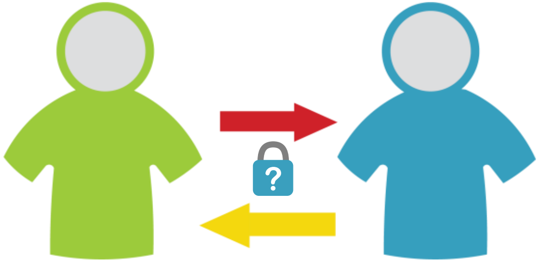


A relation R on a set A is **symmetric** if $(x,y) \in R$ implies $(y,x) \in R$: $\forall x \in A \ \forall y \in A, xRy \rightarrow yRx$.



Not Symmetric Relationship

E.g., $A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: not symmetric



Symmetric Relationship

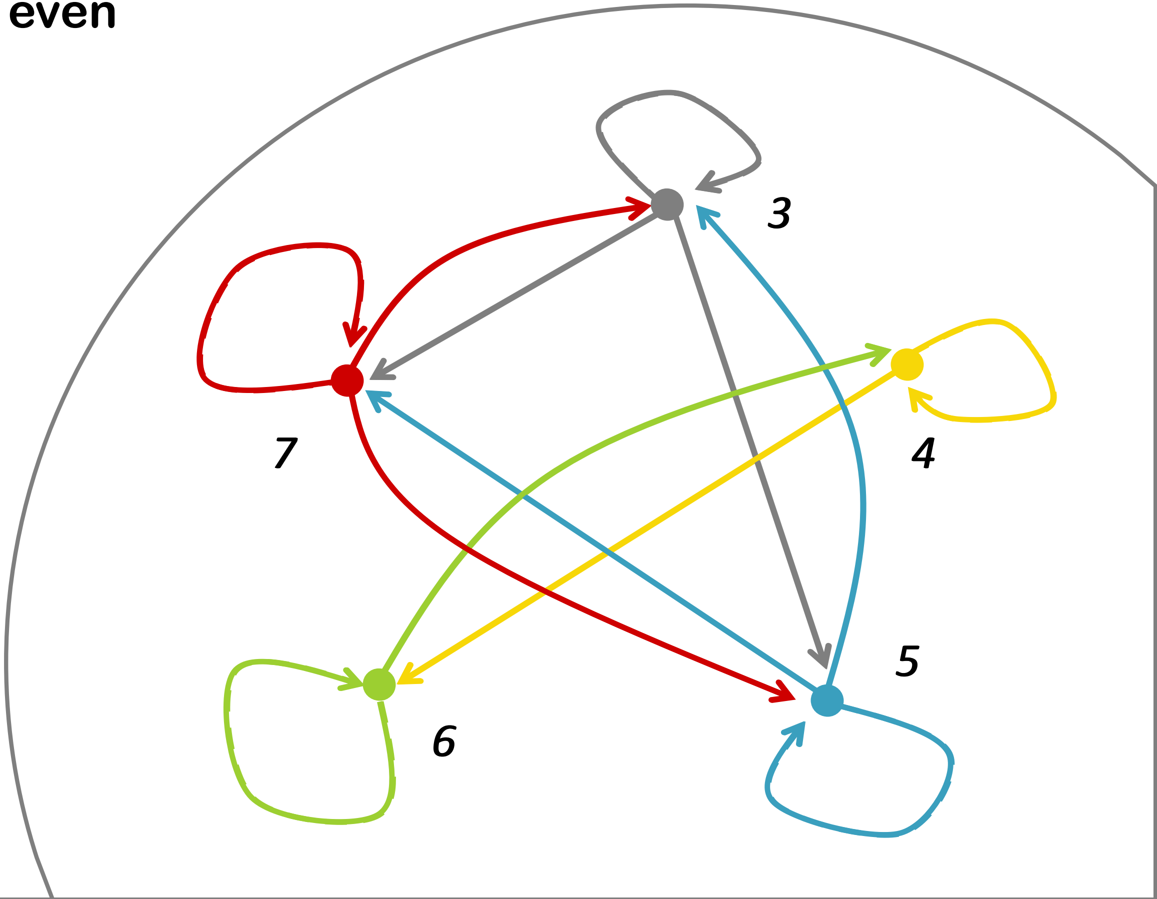
E.g., $A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: symmetric

Symmetry: Graphically

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ is even

R reflexive

R symmetric



Transitivity

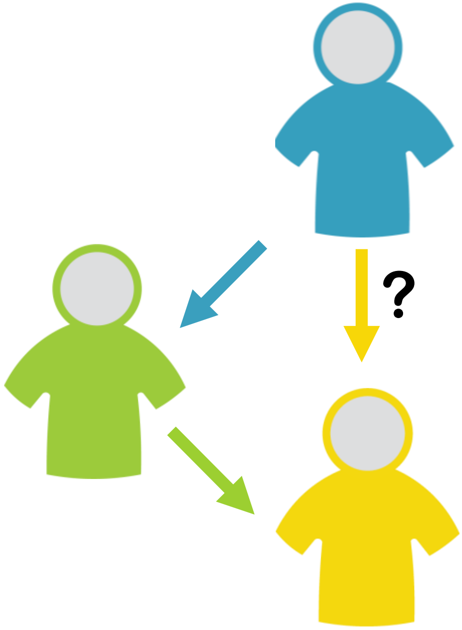
Transitivity: Definition

A relation R on a set A is **transitive** if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in$ R: $\forall x\ \forall y\ \forall z\ xRy \land yRz \rightarrow xRz$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: transitive

$A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: transitive



Transitivity: Graphically

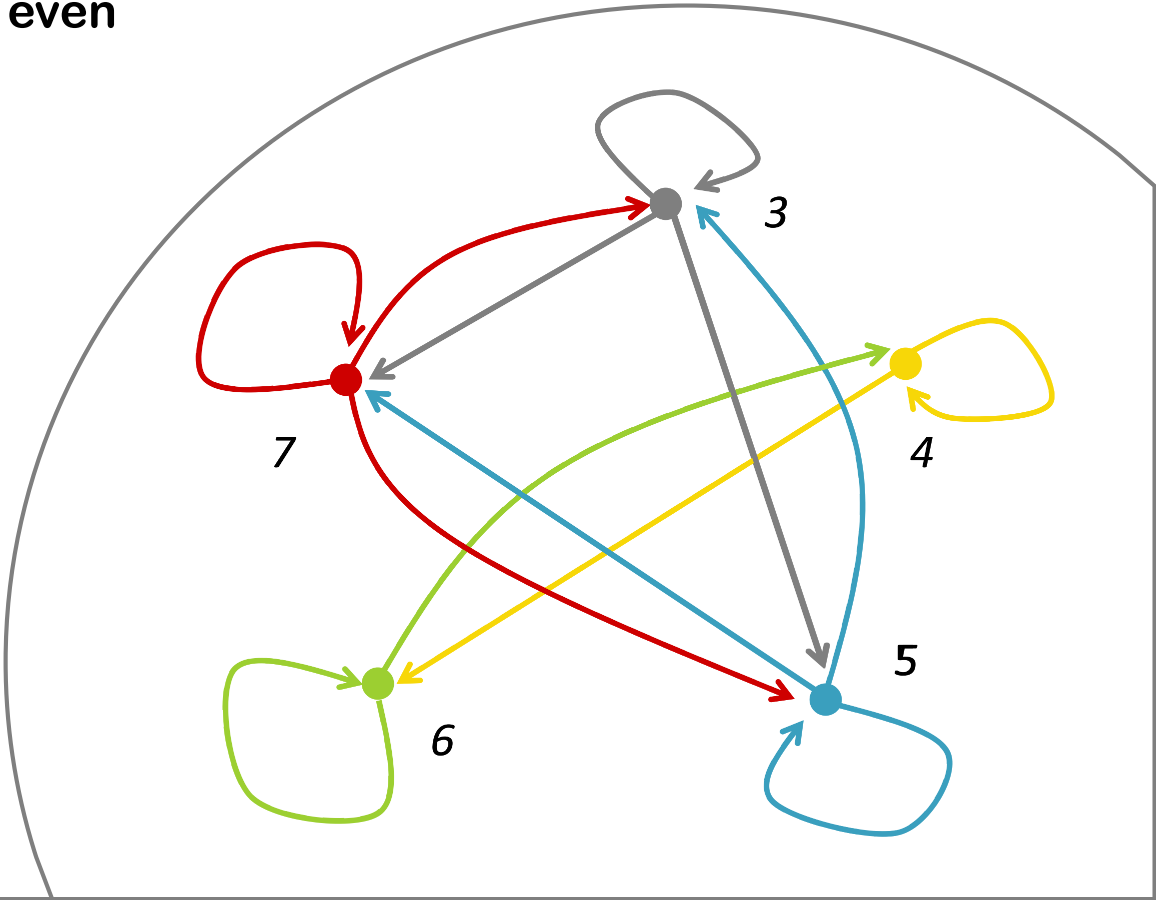
A = {3,4,5,6,7}, $xRy \leftrightarrow (x - y)$ **is even**

[3] = {3,5,7}, [4] = {4,6}

R **reflexive**

R **symmetric**

R **transitive**



Topic Summary

Let’s recap…

- Binary relations:
  - Inverse and composition
  - Graphical representation

- Properties:
  - Reflexivity
  - Symmetry
  - Transitivity