Discrete Mathematics
MH1812

Topic 7.1 - Set Theory I
Dr. Guo Jian

Topic Overview

What’s in store…

Introduction to Set

Venn Diagram

Cartesian Product

Partition

By the end of this lesson, you should be able to...

- Explain the concepts of sets.
- Use Venn diagrams to show the relationship between sets.
- Explain what is cartesian product.
- Explain what is a partition of a set.

Introduction to Set

Introduction to Set: Definition

A **set** is a collection of abstract objects (e.g., prime numbers, domain in predicate logic).

- Determined by (distinct) elements/members:
  - E.g., $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} = \{1, 1, 1, 2, 3, 3, 3\}$
- Two common ways to specify a set:
  - **Explicit**: enumerate the members
    - E.g., $A = \{2, 3\}$
  - **Implicit**: description using predicates $\{x \mid P(x)\}$
    - E.g., $A = \{x \mid x \text{ is a prime number}\}$

Introduction to Set: Membership

We write $x \in S$ **iff** x **is an element (member) of** S.

- **E.g.,** $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} = \{1, 1, 1, 2, 3, 3, 3\}$

- **E.g.,** $A = \{x \mid x$ **is a prime number**$\}$ **then** $A = \{2, 3, 5, 7, \ldots\}$
  $2 \in A,\ 3 \in A,\ 5 \in A, \ldots, 1 \notin A,\ 4 \notin A,\ 6 \notin A, \ldots$

Introduction to Set: Subset

A set A **is a subset of the set** B, denoted by $A \subseteq B$ iff every element of A is also an element of B.

I.e.,:

- $A \subseteq B \triangleq \forall x(x \in A \rightarrow x \in B)$
- $A \not\subset B \triangleq \neg (A \subseteq B)$

$$\equiv \neg \forall x(x \in A \rightarrow x \in B)$$

$$\equiv \exists x(x \in A \wedge x \notin B)$$

- E.g., $B = \{1, 2, 3\}$, $A = \{1, 2\} \subseteq B$

Introduction to Set: Empty Set

The set that contains no element is called the **empty set** or **null set**.

- The empty set is denoted by $\varnothing$ or by $\{\ \}$.
- Note: $\varnothing \neq \{\varnothing\}$

Introduction to Set: Set Equality

$$A = B \triangleq \forall x (x \in A \leftrightarrow x \in B)$$

- Two sets A and B are equal iff they have the same elements.

$$A \neq B \triangleq \neg \forall x (x \in A \leftrightarrow x \in B)$$
$$\equiv \exists x \, [(x \in A \land x \notin B) \lor (x \in B \land x \notin A)]$$

- Two sets are not equal if they do not have identical members, i.e., there is at least one element in one of the sets which is absent in the other.
  – E.g., $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 3, 2\} = \{1, 1, 1, 2, 3, 3, 3\}$

Introduction to Set: Cardinality

The **cardinality** $|S|$ of S is the number of elements in S.
(E.g., for $S = \{1, 3\}$, $|S| = 2$)

- If $|S|$ is finite, S is a finite set; otherwise S is infinite.
  - The set of **positive** integers is an infinite set.
  - The set of **prime** numbers is an infinite set.
  - The set of **even prime** numbers is a finite set.

- Note: $|\varnothing| = 0$

Introduction to Set: Power Set

The **power set** $P(S)$ of a given set S is the set of all subsets of S: $P(S) = \{A \mid A \subseteq S\}$.

- E.g., for $S = \{1,2,3\}$

$$P(S) = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$$

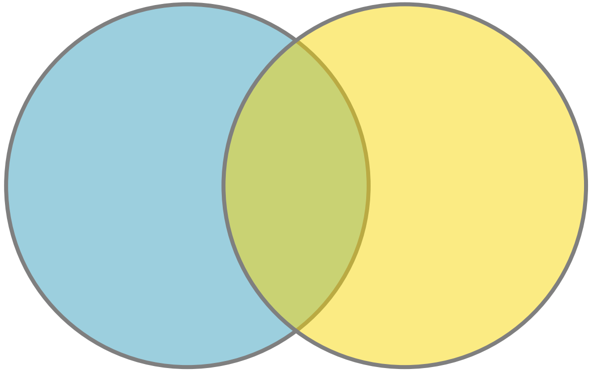
- If a set S has n elements, then $P(S)$ has 2^n elements.
  – Hint: Try to leverage the Binomial theorem.

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n,$$

Venn Diagram

Venn Diagram: Definition

A Venn diagram is used to show/visualise the possible relations among a collection of sets.

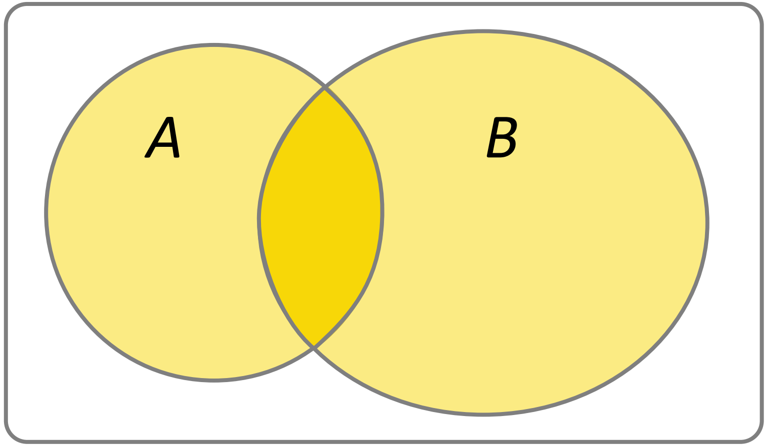


John Venn
(1834 - 1923)

Venn Diagram: Union and Intersection

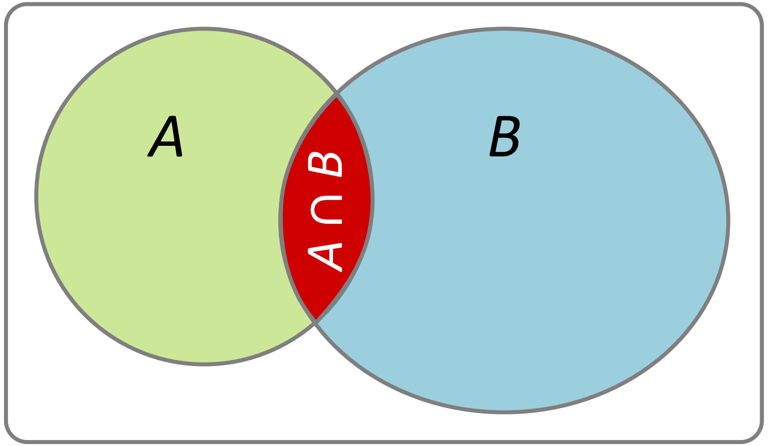
The union of sets A and B is the set of those elements that are either in A, in B, or both.

$$A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$$



The intersection of the sets A and B is the set of all elements that are in both A and B.

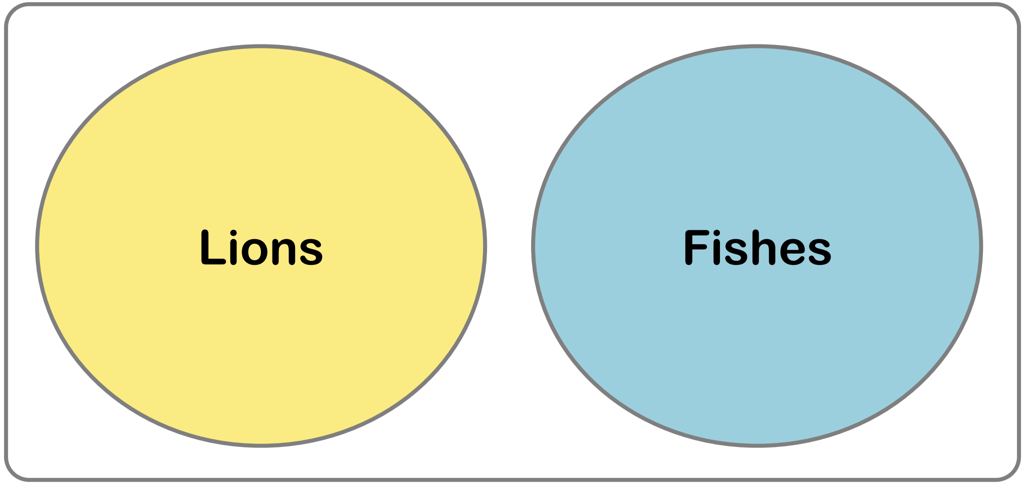
$$A \cap B \triangleq \{x \mid x \in A \wedge x \in B\}$$



Venn Diagram: Disjoint Sets

Sets A and B are disjoint iff $A \cap B = \varnothing$

$|A \cap B| = 0$

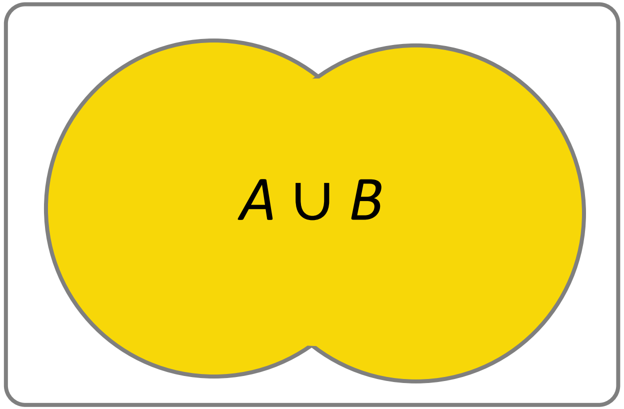


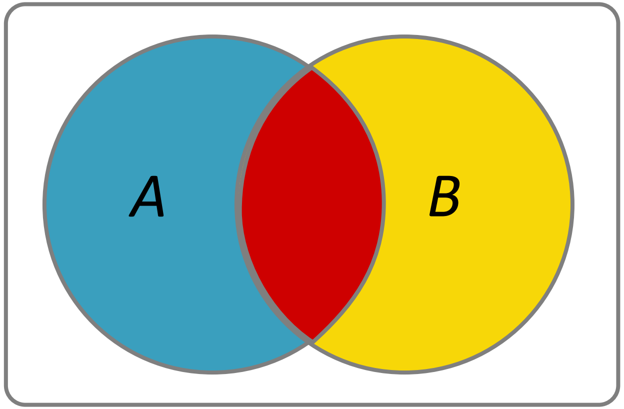
$\text{Lions} \cap \text{Fishes} = \varnothing$

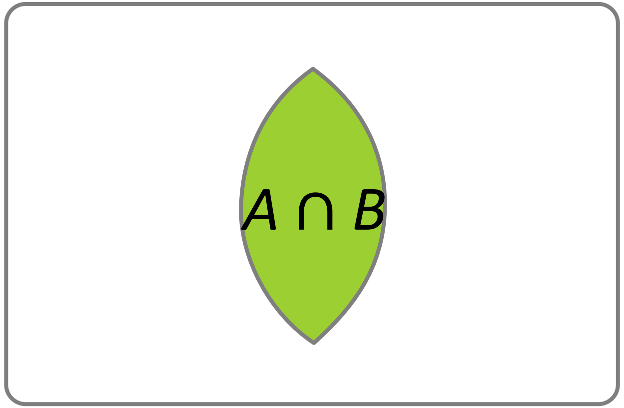
What about the Merlion?

Venn Diagram: Cardinality of Union

$$|A \cup B| = |A| + |B| - |A \cap B|$$



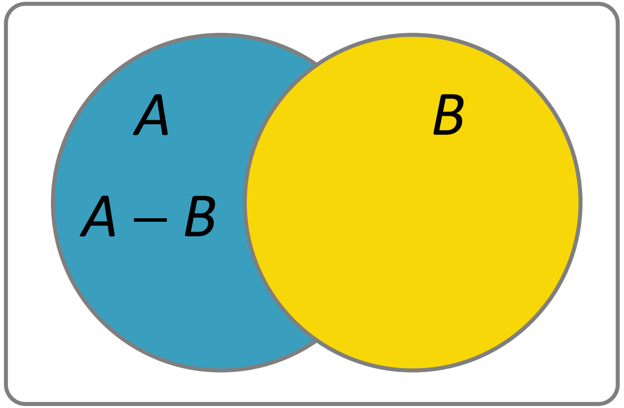




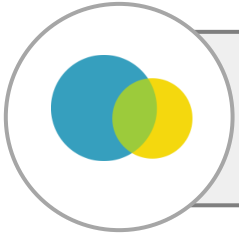
Venn Diagram: Set Difference and Complement

The **difference of A and B** (or **complement of B with respect to A**) is the set containing those elements that are in A but not in B.

$$A - B \triangleq \{x \mid x \in A \wedge x \notin B\}$$

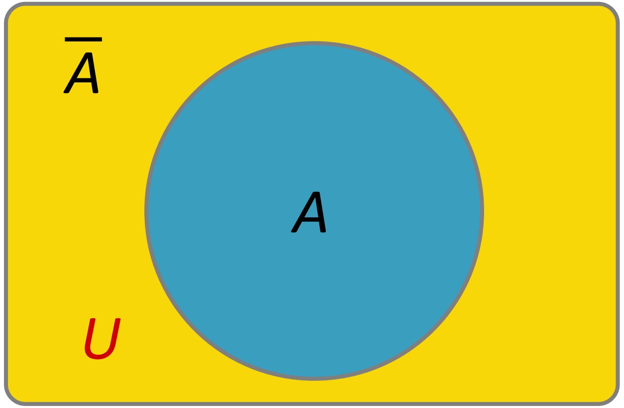


Venn Diagram: Set Difference and Complement



The **complement of A** is the complement of A with respect to U.

$$\overline{A} = U - A \triangleq \{x \mid x \notin A\}$$



Cartesian Product

Cartesian Product: Definition

The **Cartesian product** $A \times B$ of the sets A and B is the set of all **ordered pairs** (a,b) where $a \in A$ and $b \in B$.

$$A \times B \triangleq \{(a,b) \mid a \in A \wedge b \in B\}$$

René Descartes
(1596 - 1650)

Portrait of René Descartes by André Hatala under WikiCommons (PD-US)

Cartesian Product: Example

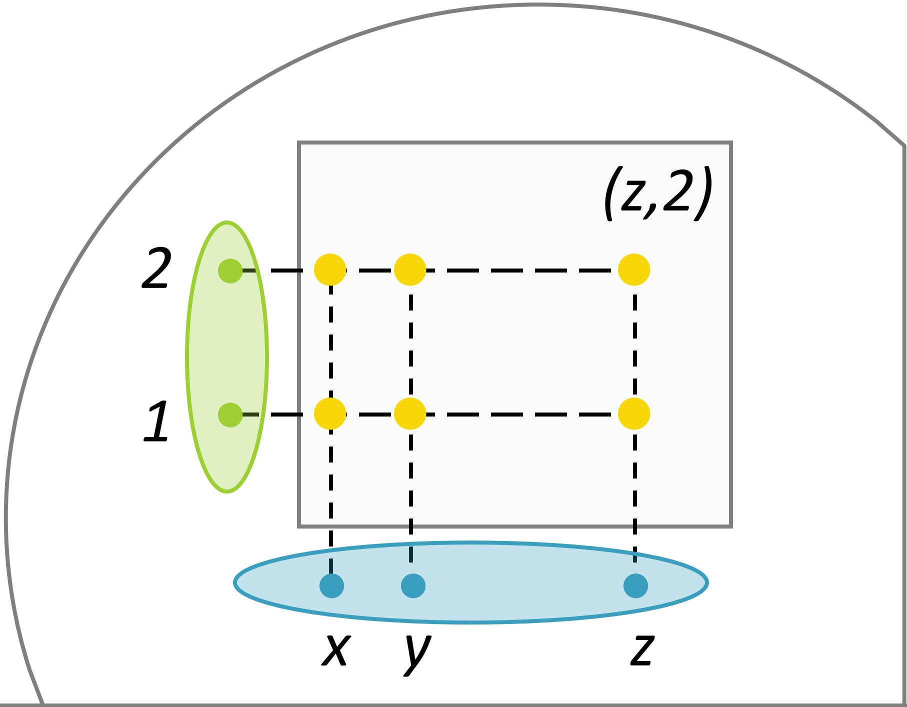
$A = \{1,2\}$, $B = \{x,y,z\}$

$A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$

$B \times A = \{(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)\}$

In general: $A_1 \times A_2 \times \ldots \times A_n \triangleq \{(a_1,a_2, \ldots, a_n) \mid a_i \in A_i$ **for** $i = 1,2, \ldots, n\}$

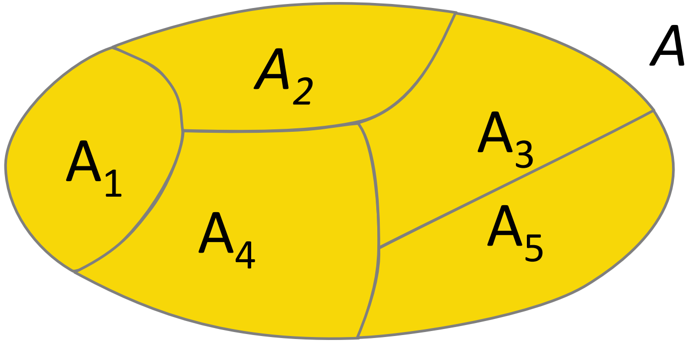
$| A_1 \times A_2 \times \ldots \times A_n | = |A_1| \, |A_2| \ldots |A_n|$



Partition

Partition: Definition

A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A, iff $A = A_1 \cup A_2 \cup \ldots A_n$ and $A_1, A_2, \ldots, A_n$ are **mutually disjoint**, i.e., $A_i \cap A_j = \varnothing$ for all $i, j = 1, 2, \ldots, n$, and $i \neq j$.



Topic Summary

Let’s recap…

- Sets:
  - Membership
  - Subset
  - Null set
  - Equality

- Venn diagram

Let’s recap…

- Set operations:
  - Union
  - Intersection
  - Complement
  - Difference

- Cartesian Product

- Partition