NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

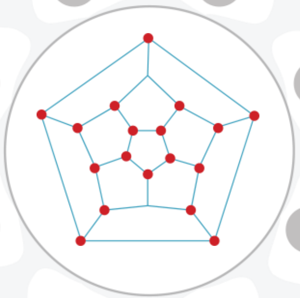
Topic 10.1 - Graph Theory I

Dr. Wang Huaxiong

Topic Overview

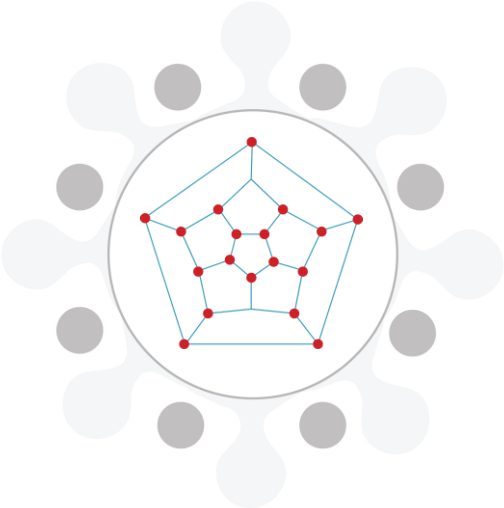
What's in store…

- **I**ntroduction to Graphs
- **T**ypes of Graphs
- **E**uler Theorem



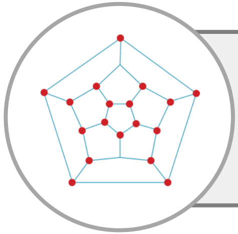
By the end of this lesson, you should be able to...

- Explain what is a graph.
- Explain the difference between the simple graph, multigraph and directed (multi) graph.
- Explain the concepts of the Euler path and circuit.
- Use the Euler theorem in graph theory.



Introduction to Graphs

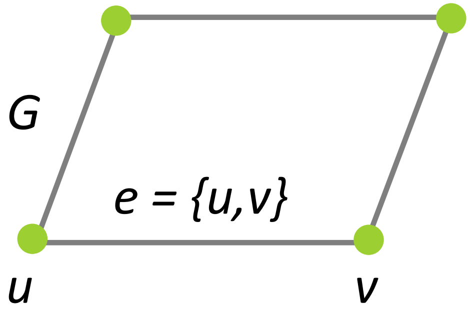
Introduction to Graphs: Definition



A **graph** $G = (V,E)$ is a structure consisting of a set V of vertices (nodes) and a set E of edges (lines joining vertices).

- Two vertices u and v are **adjacent** in G if $\{u,v\}$ is an edge of G.

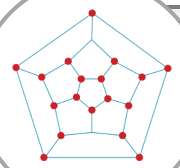
- If $e = \{u,v\}$, the edge e is called **incident** with the vertices u and v.



Graphs are useful to represent data.

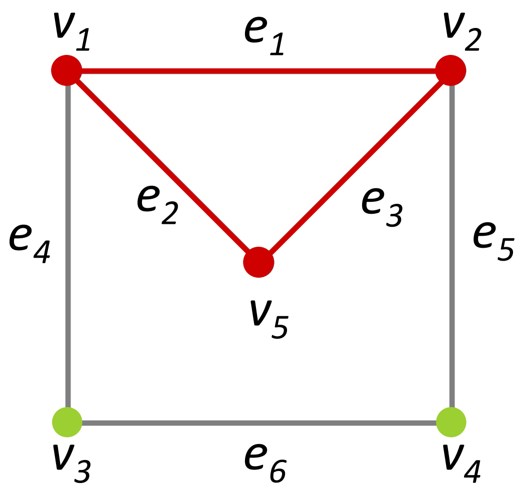
Types of Graphs

Types of Graphs: Subgraphs

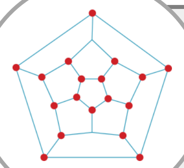


A graph $H = (V_H, E_H)$ is a **subgraph** of $G = (V_G, E_G)$ if V_H is a subset of V_G and E_H is a subset of E_G.

- $V_H = \{v_1, v_2, v_5\}$ is a subset of V_G
- $E_H = \{e_1, e_2, e_3\}$ is a subset of E_G



Types of Graphs: Simple Graphs



A simple graph is a graph that has no loop (= edge $\{u,v\}$ with $u = v$) and no parallel edges between any pair of vertices.

From\To	Hong Kong	Singapore	Beijing	Taipei	London
Hong Kong		4 Flights			
Singapore	2 Flights		3 Flights	1 Flight	1 Flight
Beijing	1 Flight	2 Flights			
Taipei					
London		1 Flight	1 Flight		1 Flight

Draw a graph to see whether there are direct flights between any two cities (in either direction).

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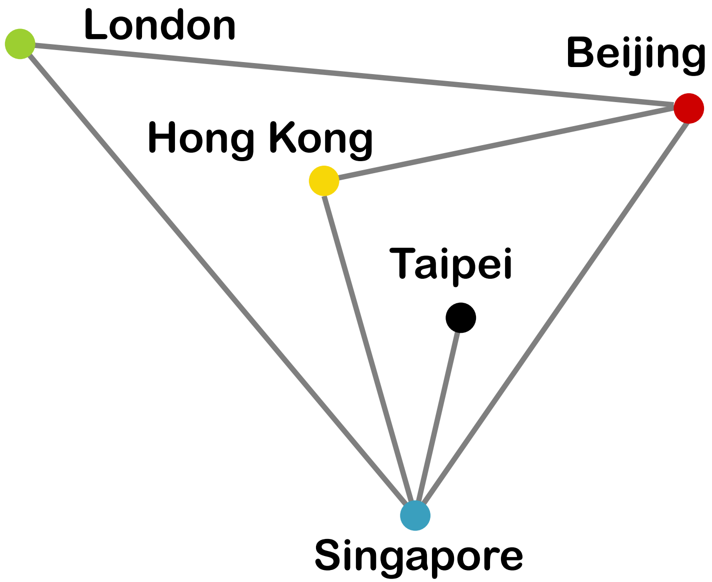
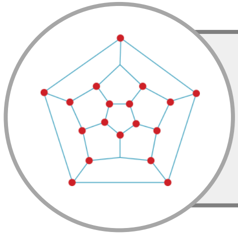


Types of Graphs: Multigraphs



A **multigraph** is a graph that has no loop and at least *2* parallel edges between some pair of vertices.

From\To	Hong Kong	Singapore	Beijing	Taipei	London
Hong Kong		4 Flights			
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Draw a graph with an edge for each flight that operates between two cities (in either direction).

Types of Graphs: Multigraphs

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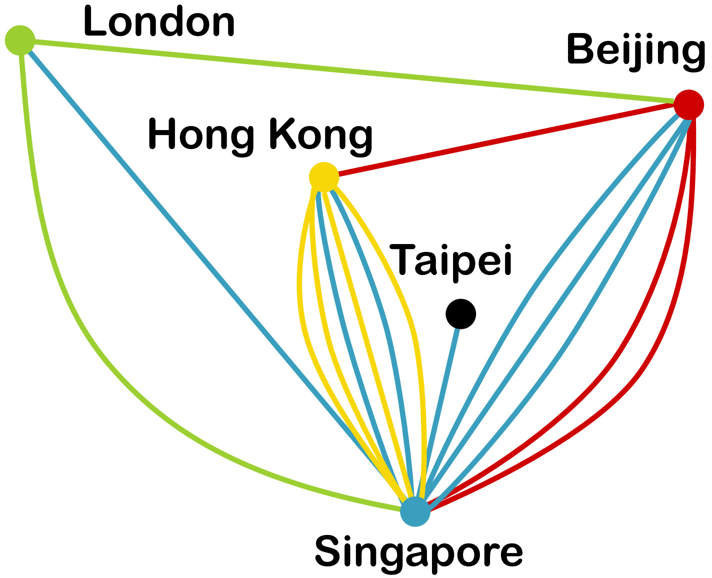
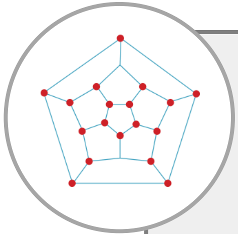


Types of Graphs: Directed (Multi) Graphs



A **directed** graph is a graph where edges $\{u,v\}$ are ordered, that is, edges have a direction. Parallel edges are allowed in **directed multigraphs**. Loops are allowed for both.

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Hong Kong		4 Flights			
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Draw a graph to see whether there are direct flights between any two cities (direction matters).

Types of Graphs: Directed (Multi) Graphs

From\To	Hong Kong	Singapore	Beijing	Taipei	London
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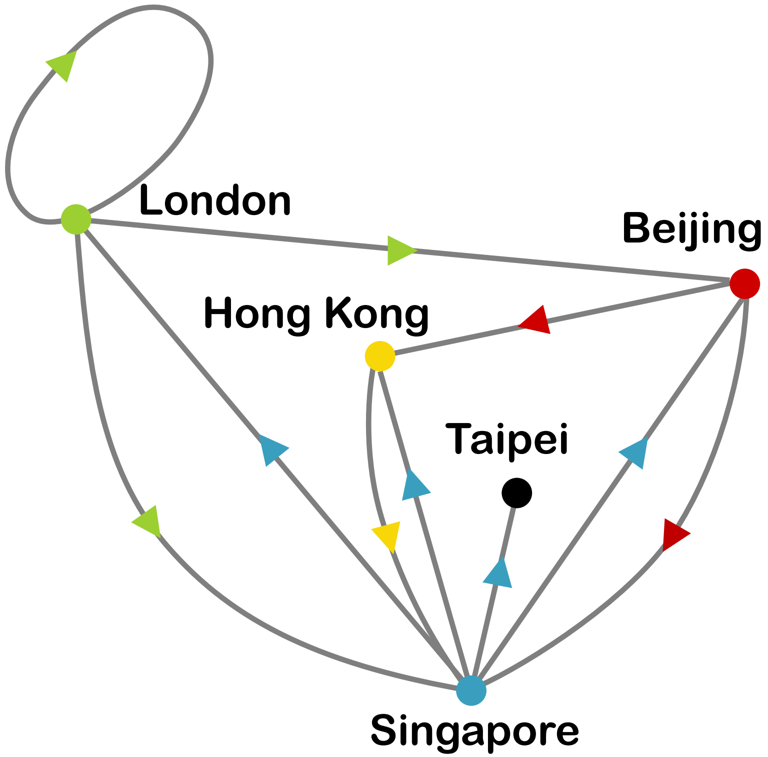
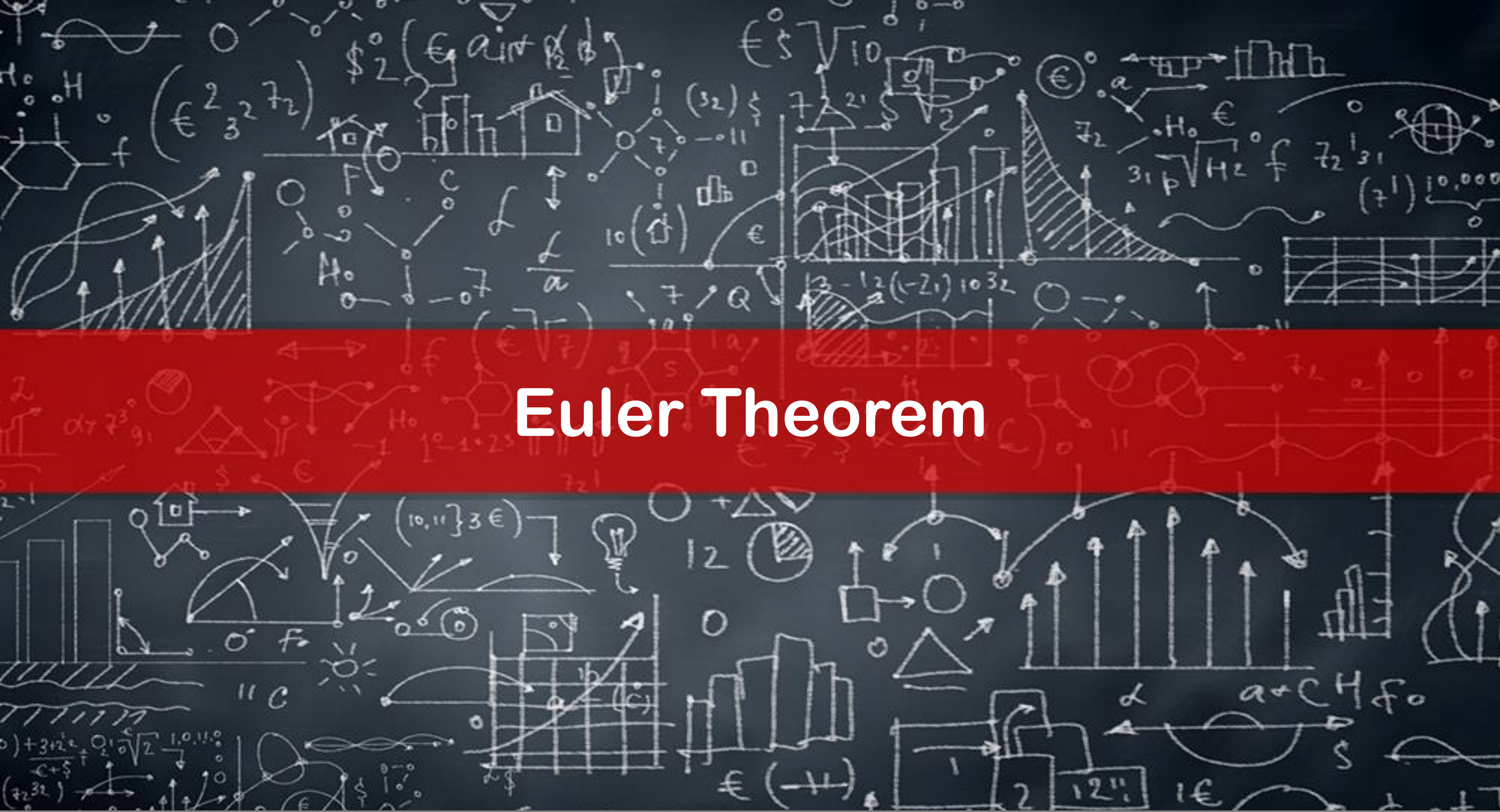
Draw a graph to see whether there are direct flights between any two cities (direction matters).



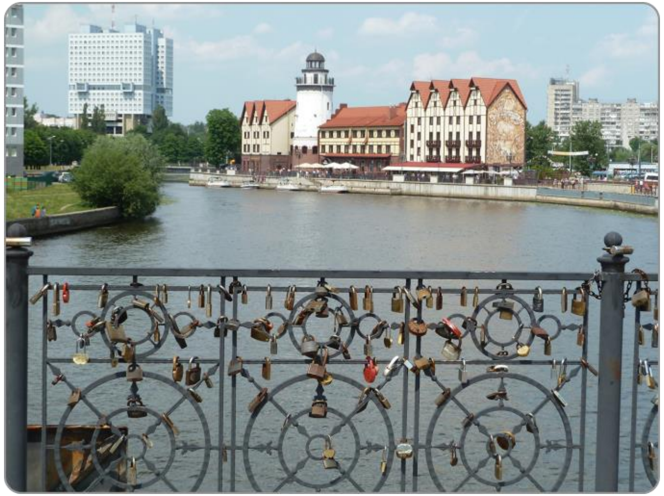


Euler Theorem

Euler Theorem: The Mathematician

Leonhard Euler introduced graphs in 1736 to solve the Königsberg Bridge problem.

What is the "Königsberg Bridge problem"?



Kaliningrad (Königsberg) in Russia

Leonhard Euler
1707 - 1783

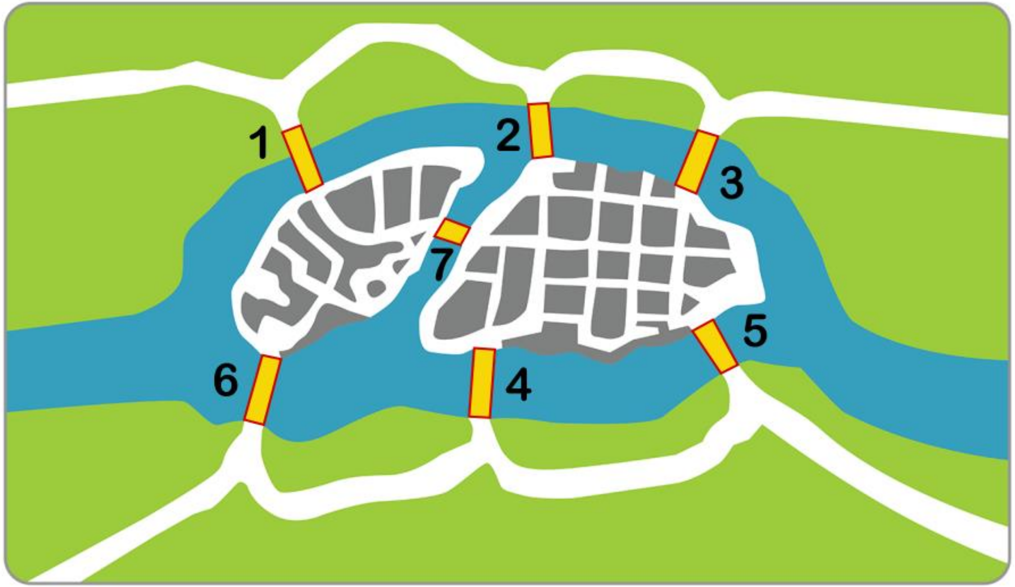
Portrait of Leonhard Euler by Jakob Emanuel Handmann under WikiCommons (PD-US)

Euler Theorem: Origin (Bridges of Königsberg)

- Königsberg (now known as Kaliningrad in Russia) has *7* bridges.

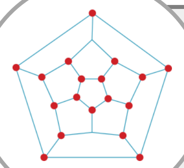
- People tried (without success) to find a way to walk all *7* bridges without crossing a bridge twice.

- Leonhard Euler proved that it was impossible to walk all seven bridges without crossing a bridge twice.



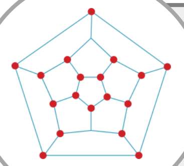
Seven Bridges of Königsberg

Euler Circuit: Definitions



A **Euler path** (Eulerian trail) is a walk on the edges of a graph which uses each edge in the original graph exactly once.

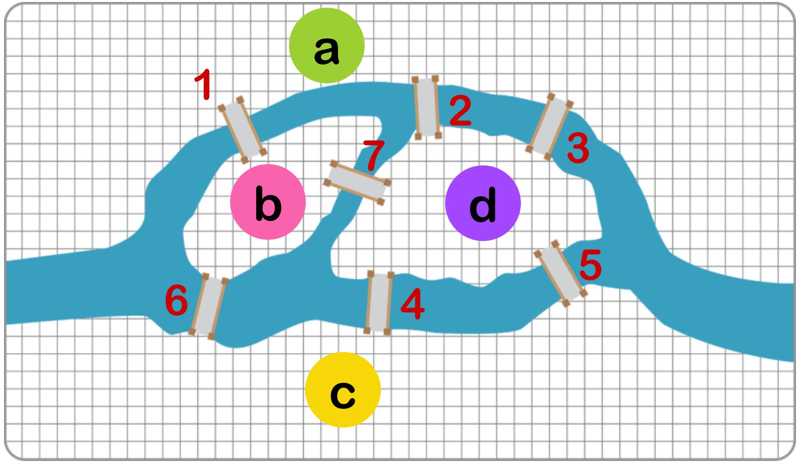
The beginning and end of the walk may or may not be the same vertex.

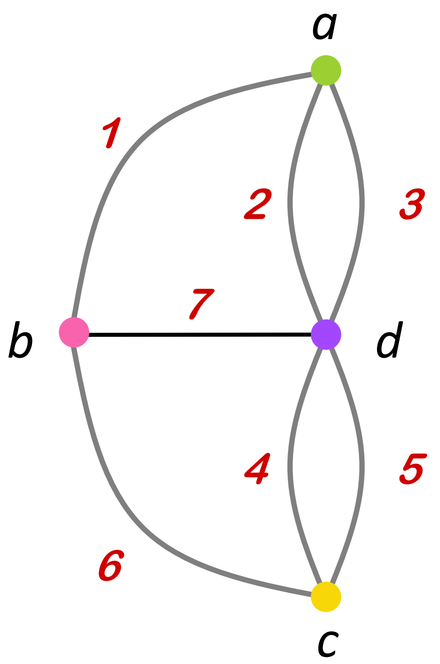


A **Euler circuit** (Eulerian cycle) is a walk on the edges of a graph which starts and ends at the same vertex, and uses each edge in the original graph exactly once.

Euler Circuit

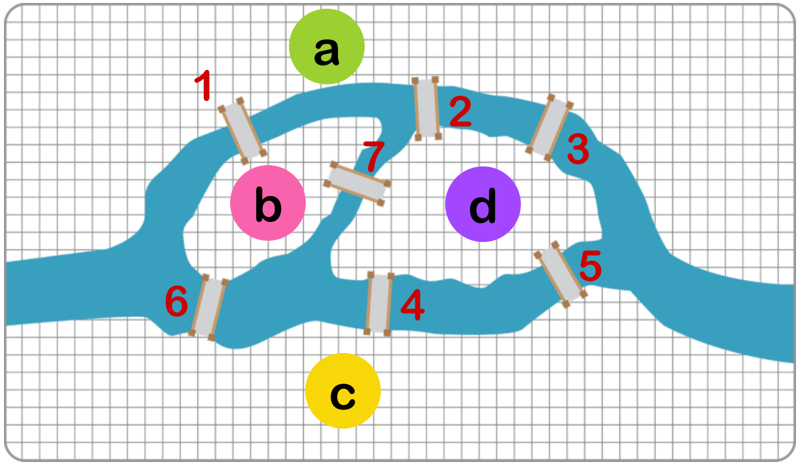
- Suppose the beginning and end are the same node u.
- The graph must be connected.
- At every vertex $v \neq u$, we reach v along one edge and go out along another, thus the number of edges incident at v (called the degree of v) is even.

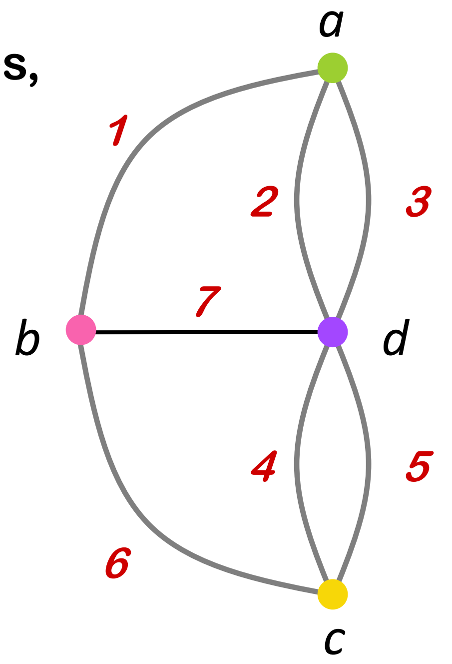




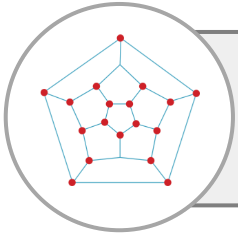
Euler Circuit

- The node u is visited once the first time we leave, and once the last time we arrive, and possibly in between (back and forth), thus the degree of u is even.
- Since the Königsberg Bridges graph has odd degrees, it has no solution!

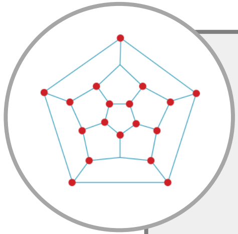




Euler Theorem



The **degree** of a vertex is the number of edges incident with it.



Theorem: consider a connected graph G.

1. If G contains an Euler path that starts and ends at the same node, then all nodes of G have an even degree.
2. If G contains an Euler path, then exactly two nodes of G have an odd degree.

Euler Theorem

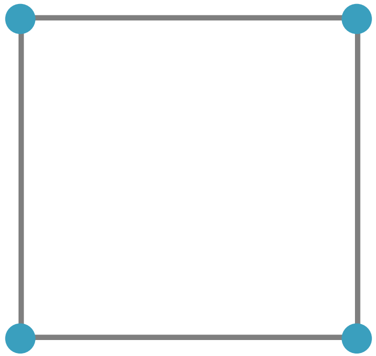
- Suppose G has an Euler path, which starts at v and finishes at w.

- Add the edge $\{v,w\}$.

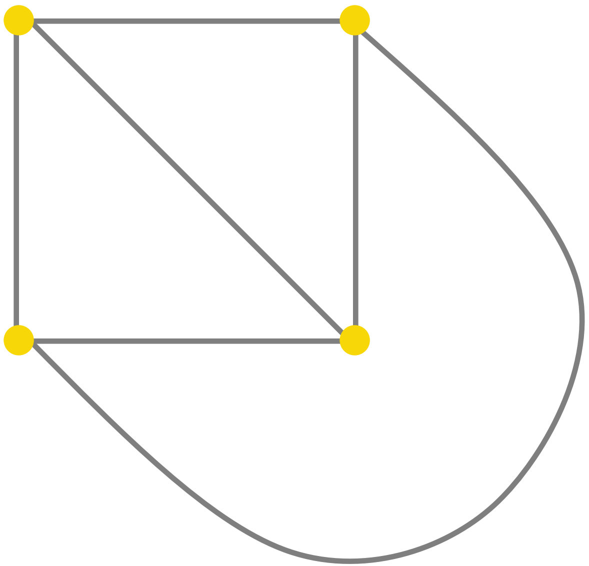
- Then by the first part of the theorem, all nodes have even degrees, except for v and w which have odd degrees.

Euler Theorem: Examples

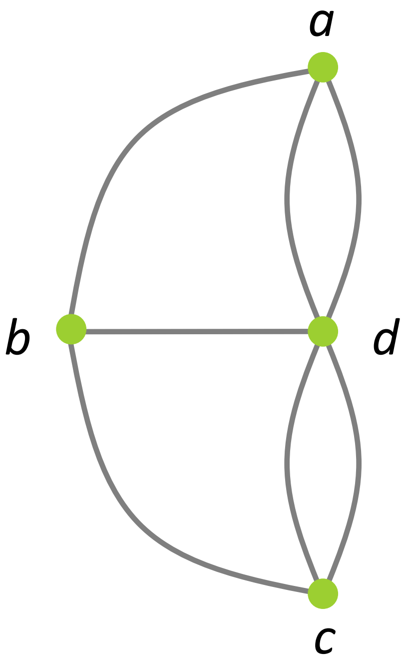
Note: Euler Theorem actually states an "if and only if" statement.



Euler Circuit



No Euler Path



No Euler Path

Topic Summary

Let’s recap…

- Graph theory has numerous applications (e.g., networks, distributed systems, coding theory)

- Parts of a graph:
  - Vertex
  - Edge
  - Adjacent
  - Incident

Let's recap…

- Types of graphs:
  - Simple graph
  - Multigraph
  - Directed (multi) graph

- Euler path, Euler circuit and Euler theorem