Discrete Mathematics

MH1812

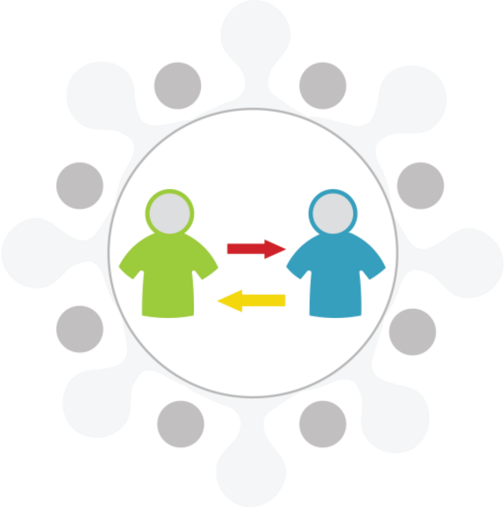
Topic 8.2 - Relations II

Dr. Guo Jian

Topic Overview

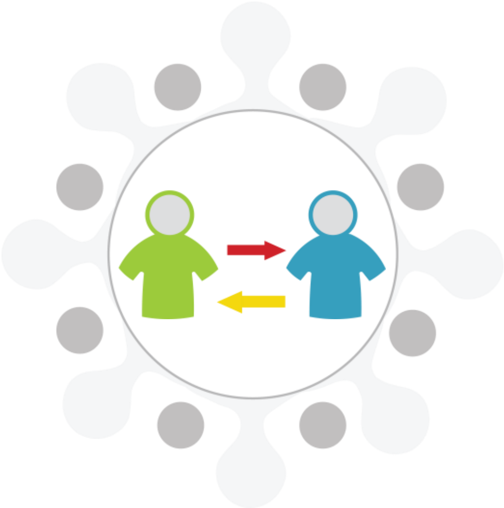
What’s in store…

- **E**quivalence Relations
- **A**ntisymmetry



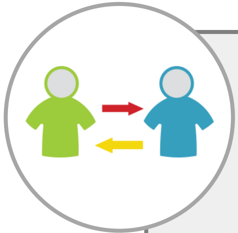
By the end of this lesson, you should be able to...

- Explain the conditions for an equivalence relation.
- Explain the concept of antisymmetry.



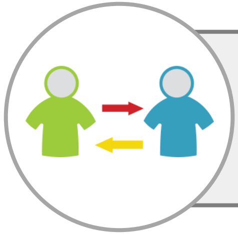
Equivalence Relations

Equivalence Relations: Definition



A relation R on a set A is **an equivalence relation** if:

1. R is reflexive: $\forall x \in A, xRx$
2. R is symmetric: $\forall x \forall y\ xRy \rightarrow yRx$
3. R is transitive: $\forall x \forall y \forall z\ xRy \wedge yRz \rightarrow xRz$



Equivalence class of a in A: $[a] = \{x \in A |\ aRx\}$ for R an equivalence relation.

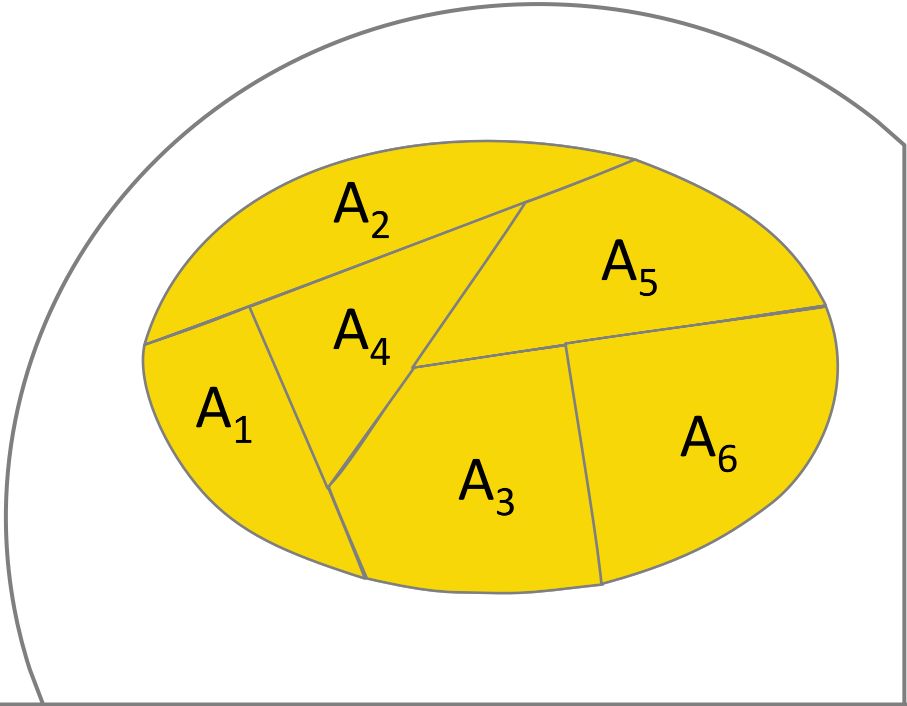
Equivalence Relations: Equivalence Classes

Partition of a set *A*:

$A_i \cap A_j = \varphi$ **whenever** $i \neq j$

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = A$

Equivalence classes of *A* form a partition of *A*.



Equivalence Relations: Integers mod n

$$a \equiv b \pmod{n} \leftrightarrow a = qn + b$$

$\equiv \pmod{n}$ **is an equivalence relation**:

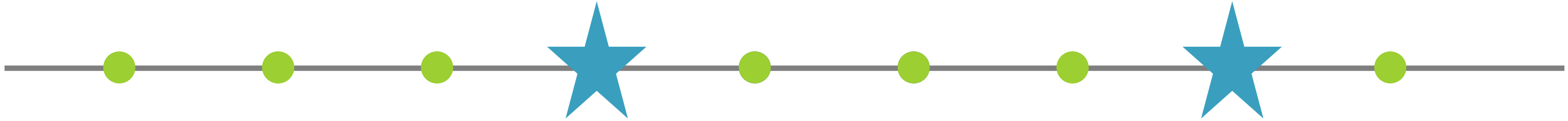
1. $\equiv \pmod{n}$ **is reflexive:** $\forall x \in A, x \equiv x \bmod n$
2. $\equiv \pmod{n}$ **is symmetric:** $\forall x \forall y\ x \equiv y \pmod{n} \rightarrow y \equiv x \pmod{n}$
3. $\equiv \pmod{n}$ **is transitive:** $\forall x \forall y \forall z\ x \equiv y \pmod{n} \land y \equiv z \pmod{n} \rightarrow x \equiv z \pmod{n}$

Equivalence Relations: Integers mod n

Equivalence class of $[0] = \{0,n,2n,3n,\ldots,-n,-2n,-3n\ldots\}$

Equivalence class of $[1] = \{1,n+1,2n+1,3n+1,\ldots,-n+1,-2n+1\ldots\}$

Example: Integers mod 4



Integers mod n **can be represented as elements between** 0 **and** $n-1$:
$\{0,1,2,\ldots,n-1\}$

Antisymmetry

Antisymmetry: Definition

A relation R on a set A is **antisymmetric** if $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$: $\forall x\ \forall y\ xRy \land yRx \rightarrow x = y$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: antisymmetric

$A = \mathbb{Z}$, $xRy \leftrightarrow x \geq y$: antisymmetric

$BRC \leftrightarrow B \subseteq C$: antisymmetric

Antisymmetry: Graphically

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

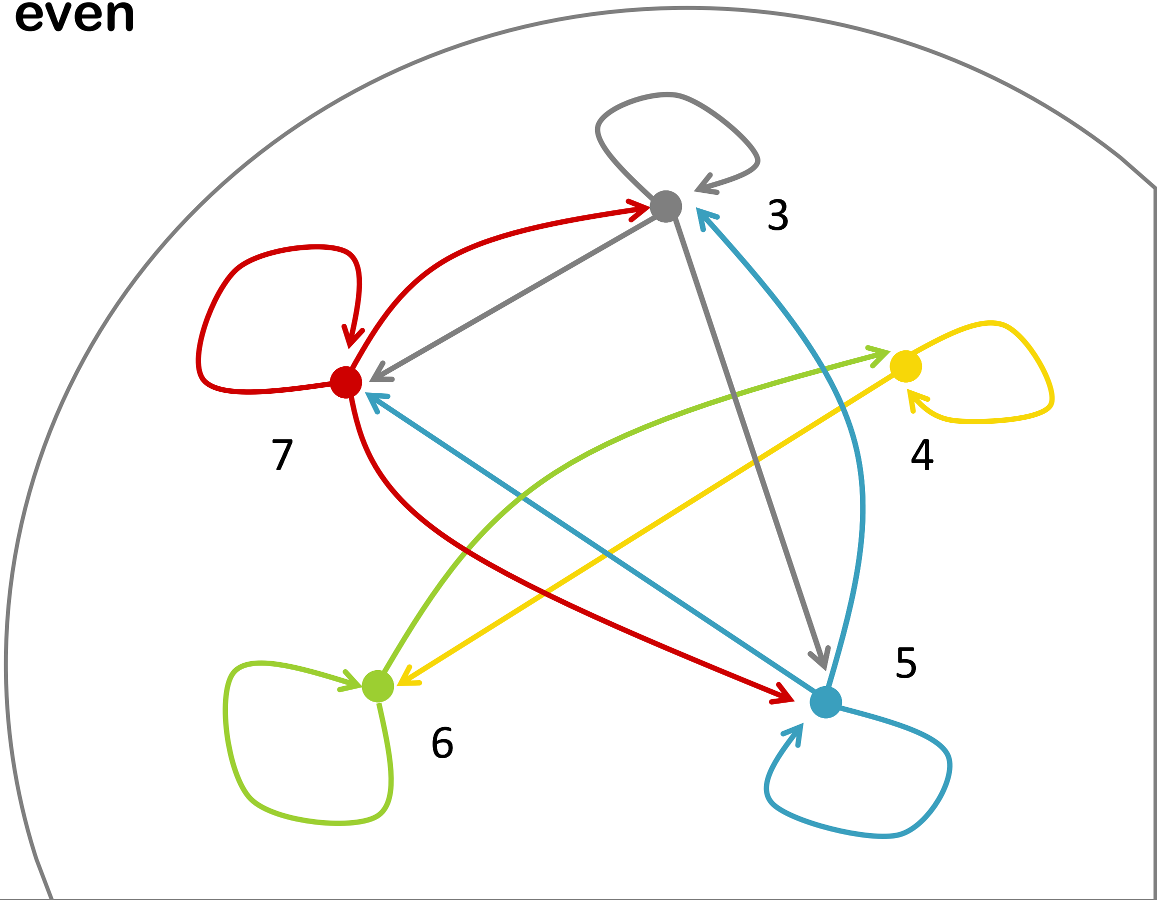
$[3] = \{3,5,7\}$, $[4] = \{4,6\}$

R **reflexive**

R **symmetric**

R **transitive**

R **is not antisymmetric**



Topic Summary

Let's recap…

- Equivalence relations: equivalence class

- Partial order: antisymmetry