NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 3.1 - Predicate Logic I

Dr. Gary Greaves

Limitation of Propositional Logic

- Every SCSE student must study discrete mathematics.
- Jackson is an SCSE student.
  - So, Jackson must study discrete mathematics.

This argument can't be expressed with propositional logic.

What propositional logic allows to express:

- If Jackson is an SCSE student, then he must study discrete mathematics.
- Jackson is an SCSE student.
  - So, Jackson must study discrete mathematics.

Topic Overview

What's in store…

Predicates

Quantification

By the end of this lesson, you should be able to…

- Identify a statement containing a predicate.
- Use quantifiers to express a property about “all” and “some”.

Predicates

Predicates: Definition

Is the statement "x^2 is greater than x" a proposition?

- Define $P(x)$ = "x^2 is greater than x"
  - Is $P(1)$ a proposition?
  - $P(1)$ = "1^2 is greater than 1"

Predicates: Definition

A **predicate** is a statement that contains variables (**predicate variables**) and that is either true or false depending on the values of these variables.

- $P(x)$ = "x^2 is greater than x"
- $P(1)$ = "1^2 is greater than 1"
- $P(x)$ is a predicate

```
frederique@frederique-desktop:~$ ./bool
Is 10 equal to 3 ? 0
Is 10 different from 3? 1
frederique@frederique-desktop:~$
```

```c
#include <stdio.h>

void main()
{
    int a,b;
    a=10;
    b=3;
    printf("Is %d equal to %d ? %d\n",a,b,a==b);
    printf("Is %d different from %d? %d\n",a,b,a!=b);
```

Predicates: Predicate Instantiated/Domain

A predicate is a statement that contains variables (predicate variables) and that is either true or false depending on the values of these variables.

A predicate instantiated (where variables are substituted for specific values) is a proposition.

- $P(x)$ = "x^2 is greater than x"
- $P(1)$ = "1^2 is greater than 1"

Predicates: Predicate Instantiated/Domain

The **domain** of a predicate variable is the collection of all possible values that the variable may take.

- E.g., the domain of x in $P(x)$: the integers
- A predicate may have more than one variable
- Different variables may have different domains

Predicates: Example

Let $P(x, y)$ = "$x > y$"

Domain: integers (i.e., both x and y are integers)

***P*(4, 3)**

This means "$4 > 3$", so $P(4, 3)$ is **TRUE**.

***P*(1, 2)**

This means "$1 > 2$", so $P(1, 2)$ is **FALSE**.

***P*(3, 4)**

This means "$3 > 4$", so $P(3, 4)$ is **FALSE**.

In general, $P(x, y)$ and $P(y, x)$ are not equal.

Quantification

Quantification: Statements Like…

Some birds are angry.

The square of any real number is non-negative.

Not all SCSE students study hard.

Quantification: Universal Quantification

A **universal quantification** is a quantifier (something that tells the amount or quantity) meaning "**given any**" or "**for all**".

E.g., "$\forall x \in D$, $P(x)$ is true" iff "$P(x)$ is true **for every** x in D"

$\forall$

Symbol

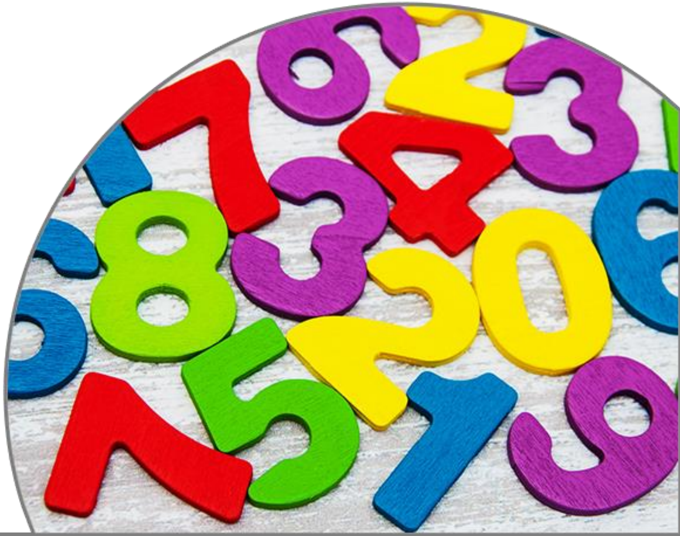
$\forall$	Universal quantifier, "for all", "for every"
$\in$	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

Quantification: Universal Quantification

$\forall$	**Universal quantifier, "for all", "for every"**
$\in$	**"Is a member (or) element of", "belonging to"**
D	**Domain of predicate variable**

The square of any real number is non-negative.

$$\forall x \in \mathbb{R},\ x^2 \geq 0$$



Quantification: Existential Quantification

An **existential quantification** is a quantifier (something that tells the amount or quantity) meaning "**there exists**", "**there is at least one**" or "**for some**".

E.g., "$\exists\, x \in D$, $P(x)$ is true" iff "$P(x)$ is true for **at least one** x in D"

$\exists$

Symbol

$\exists$	Existential quantifier, "there exists"
$\in$	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

Quantification: Existential Quantification

$\exists$

$\exists$	**Existential quantifier, "there exists"**
$\in$	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

Some birds are angry.

$\exists\, x \in$ {birds}, x is angry

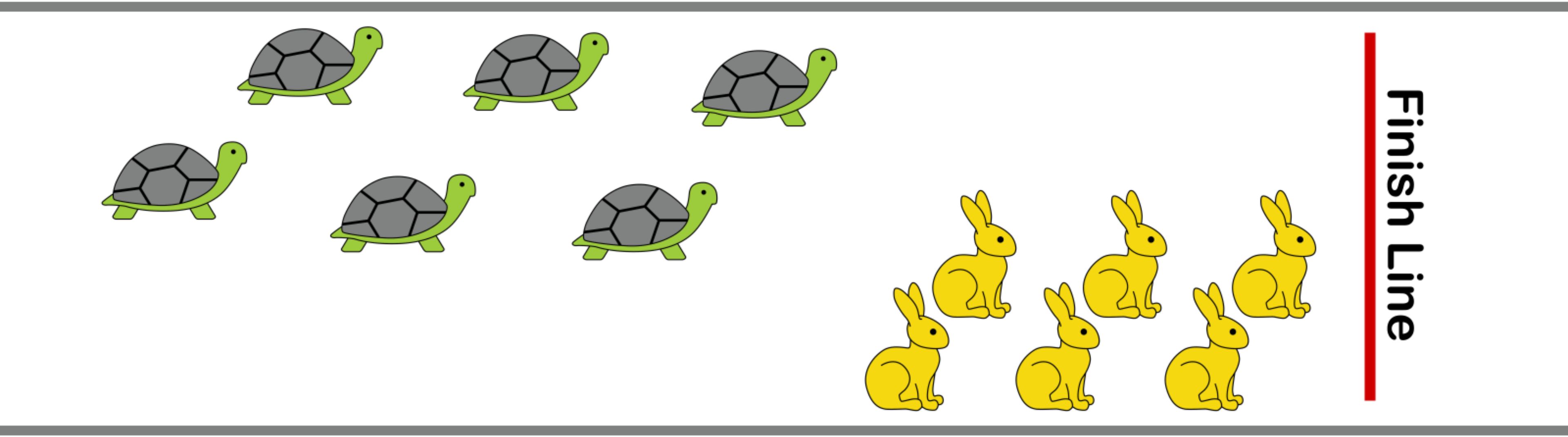
Quantification: Nested Quantification (I)

- A proposition may contain multiple quantifiers:
  - "All rabbits are faster than all tortoises."
  - Domains: R = {rabbits}, T = {tortoises}
  - Predicate $C(x,y)$: Rabbit x is faster than tortoise y

In Symbols	$\forall x \in R, (\forall y \in T, C(x, y))$ or $\forall x \in R, \forall y \in T, C(x, y)$
In Words	For any rabbit x, and for any tortoise y, x is faster than y.

Quantification: Nested Quantification (I)

In Symbols	$\forall x \in R, (\forall y \in T, C(x, y))$ or $\forall x \in R, \forall y \in T, C(x, y)$
In Words	For any rabbit x, and for any tortoise y, x is faster than y.



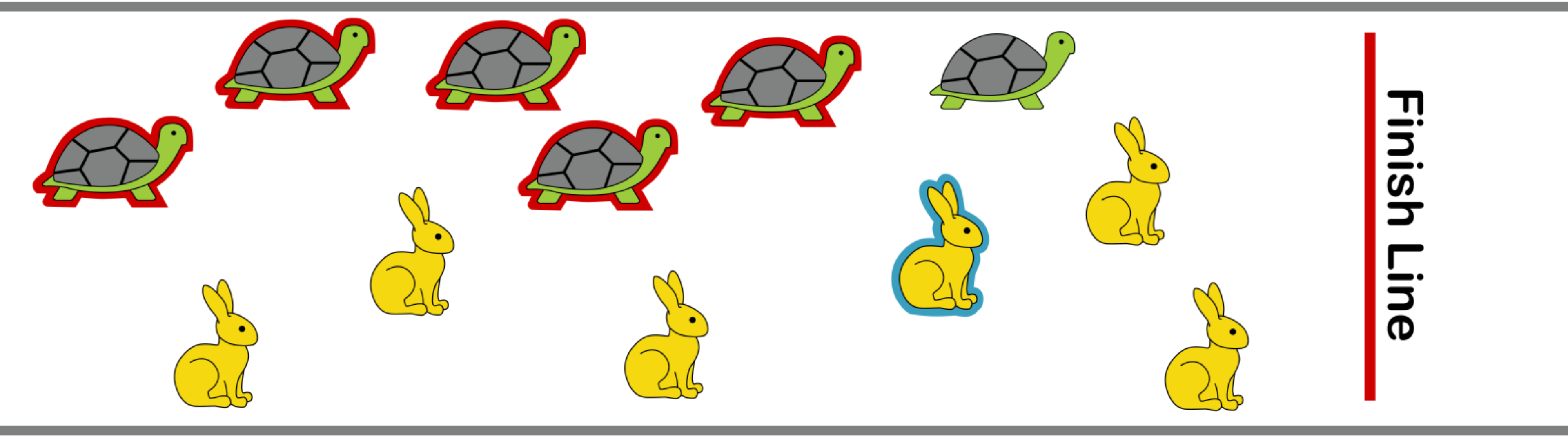
Quantification: Nested Quantification (II)

- Another example:
  - "Every rabbit is faster than some tortoise."
  - Domains: R = {rabbits}, T = {tortoises}
  - Predicate $C(x,y)$: Rabbit x is faster than tortoise y

In Symbols	$\forall x{\in}R, (\exists y{\in}T, C(x, y))$ or $\forall x{\in}R, \exists y{\in}T, C(x, y)$
In Words	For any rabbit x, there exists a (some) tortoise y, such that x is faster than y.

Quantification: Nested Quantification (II)

In Symbols	$\forall x \in R, (\exists y \in T, C(x, y))$ or $\forall x \in R, \exists y \in T, C(x, y)$
In Words	For any rabbit x, there exists a (some) tortoise y, such that x is faster than y.



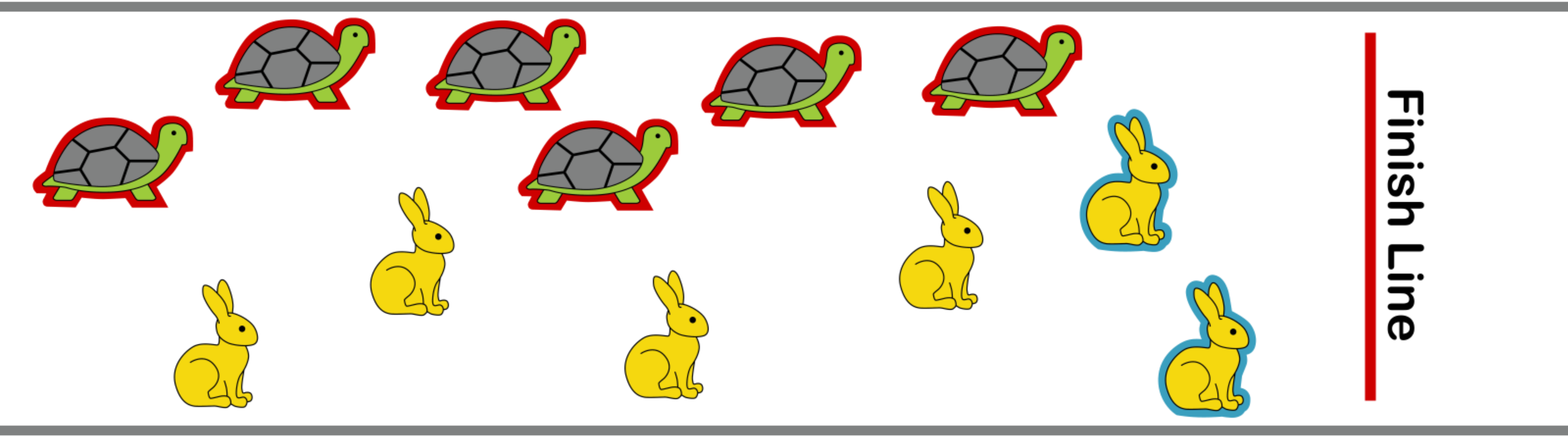
Quantification: Nested Quantification (III)

- Another example:
  - "There is a rabbit that is faster than all tortoises."
  - Domains: R = {rabbits}, T = {tortoises}
  - Predicate $C(x,y)$: Rabbit x is faster than tortoise y

In Symbols	$\exists x \in R, (\forall y \in T, C(x, y))$ or $\exists x \in R, \forall y \in T, C(x, y)$
In Words	There exists a rabbit x, such that for any tortoise y, this rabbit x is faster than y.

Quantification: Nested Quantification (III)

In Symbols	$\exists x \in R, (\forall y \in T, C(x, y))$ or $\exists x \in R, \forall y \in T, C(x, y)$
In Words	**There exists a rabbit** x**, such that for any tortoise** y**, this rabbit** x **is faster than** y**.**



Quantification: Order of Nesting Matters

Is $\forall x \in D, \exists y \in D, P(x,y) \equiv \exists y \in D, \forall x \in D, P(x,y)$ in general?

LHS

$\forall x \in D, \exists y \in D, P(x,y)$

y can **vary** with x

RHS

$\exists y \in D, \forall x \in D, P(x,y)$

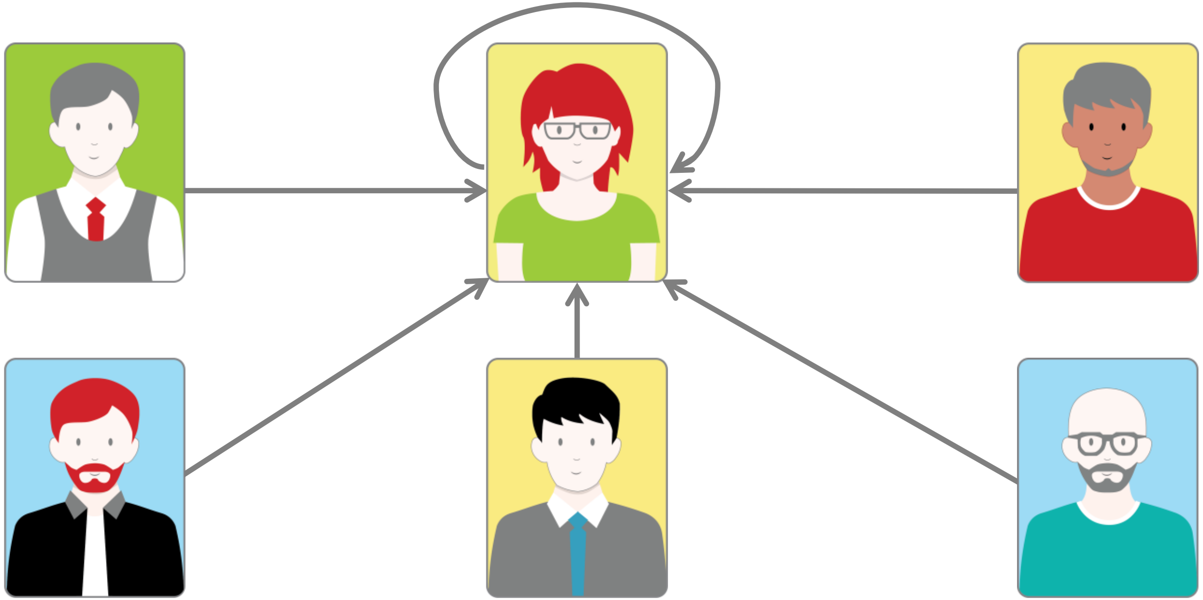
y is **fixed**, but x **varies**

Let $P(x,y)$ = "x admires y"

LHS: "Every person admires someone"

RHS: "Some people are admired by everyone"

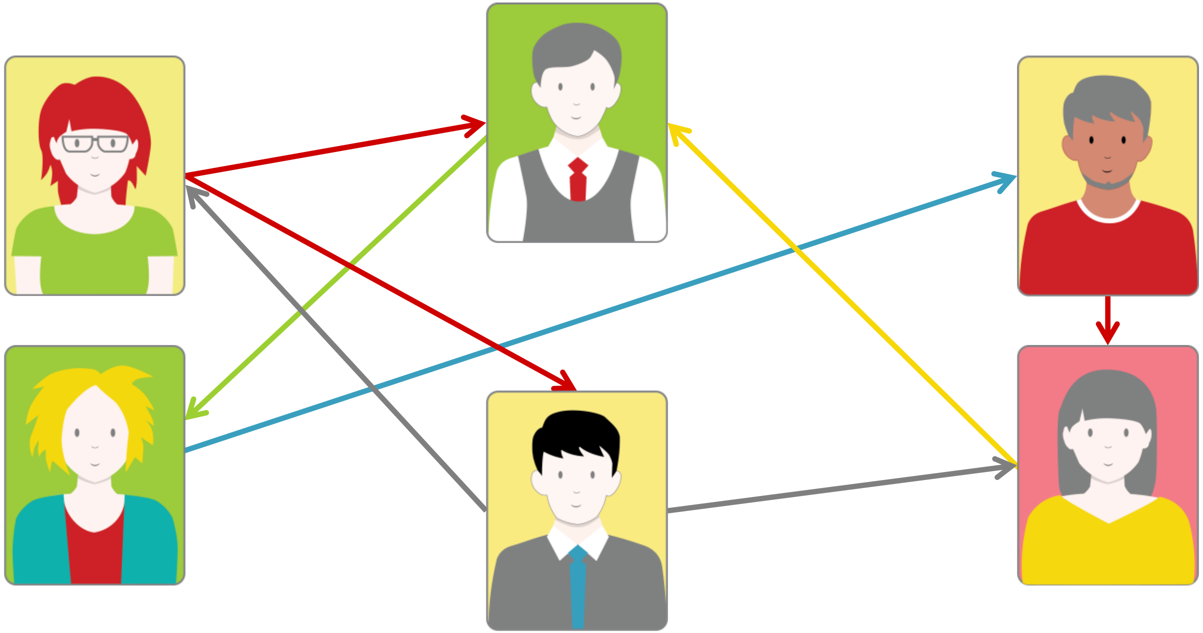
Quantification: Order of Nesting Matters



LHS $\forall x \in D, \exists y \in D, P(x,y)$ "Every person admires someone"

RHS $\exists y \in D, \forall x \in D, P(x,y)$ "Some people are admired by everyone"

Quantification: Order of Nesting Matters



LHS $\forall x \in D, \exists y \in D, P(x,y)$ "Every person admires someone"

RHS $\exists y \in D, \forall x \in D, P(x,y)$ "Some people are admired by everyone"

Topic Summary

Let’s recap…

- Predicates:
  - Statements with variables
- Quantifiers:
  - Use to express a property about “all” and “some”
  - Universal
  - Existential