MH1812 Tutorial
Chapter 4: Proof Techniques

Q1: Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Solution: We will prove by contrapositive: "if $\sqrt{q}$ is NOT irrational, then q is NOT irrational", which is equivalent with "if $\sqrt{q}$ is rational, then q is rational". If $\sqrt{q}$ is rational, it can be re-written as $\frac{a}{b}$ with $a, b \in \mathbb{Z}$, then $q = (\sqrt{q})^2 = \frac{a^2}{b^b}$, which is also rational by definition since both a^2 and b^2 are integers. $\square$

Q2: Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Solution: First of all, the first n odd positive integers refer to the set $1, 3, 5, \cdots, 2n-1$, so the sum is $S = \sum_{i=1}^{n}(2i-1)$.
Prove by mathematical induction:
Basis Step: when $n = 1$, $S = \sum_{i=1}^{n=1}(2i-1) = 2 \times 1 - 1 = 1$, and $(n = 1)^2 = 1$. The two numbers are equal, so the basis step is verified.
Inductive Step: assume for $n = k$, the statement is true, i.e., $\sum_{i=1}^{n=k}(2i-1) = (n = k)^2 = k^2$. Then for $n = k+1$, $S = \sum_{i=1}^{n=k+1}(2i-1) = \sum_{i=1}^{n=k}(2i-1) + \sum_{i=k+1}^{k+1}(2i-1) = k^2 + (2(k+1)-1) = k^2 + 2k + 1 = (k+1)^2$, verified.
Hence, by mathematical induction, the equation holds for all positive integer n. $\square$

Q3: Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Solution: Basis Step: the smallest positive integer is 1, so we check the basis case $n = 1$. When $n = 1$, $n^3 - n = 1^3 - 1 = 0$, which is multiple of 3 (actually 0 is multiple of any non-zero integer).
Inductive Step: assume for $n = k$, $n^3 - n = k^3 - k = 3x$ for some $x \in \mathbb{Z}$. Then, for $n = k+1$, $n^3 - n = (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1) = k^3 + 3k^2 + 2k = 3x + k + 3k^2 + 2k = 3(x + k^2 + k)$, which is also a multiple of 3.
Hence, proved by mathematical introduction. $\square$

Q4: Prove by mathematical induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

Solution: Here n is positive integer, and the base case is $n = 1$.
Basis Step: for $n = 1$, LHS $= 1^2 = 1$, and RHS $= \frac{1}{6} \times 1 \times (1+1) \times (2 \cdot 1 + 1) = 1$, verified.
Inductive Step: assume for $n = k$, the statement is true, i.e., $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$. For $n = k+1$, $LHS = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 = \frac{1}{6}(k+1)(k(2k+1) + 6(k+1)) = \frac{1}{6}(k+1)(2k^2 + 7k + 6) = \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$, verified for $n = k+1$.
Hence, proved by mathematical introduction. $\square$