Discrete Mathematics

MH1812

Topic 8.3 - Relations III

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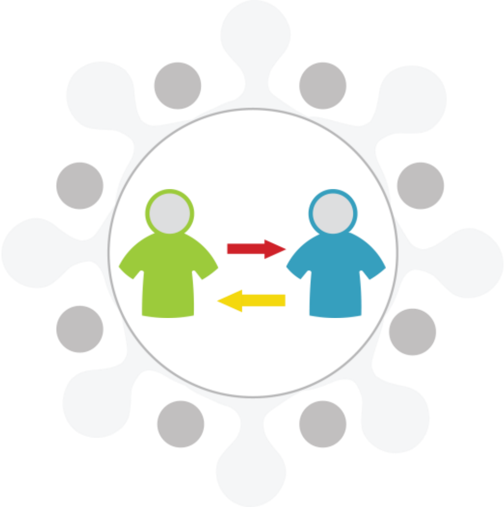
Topic Overview

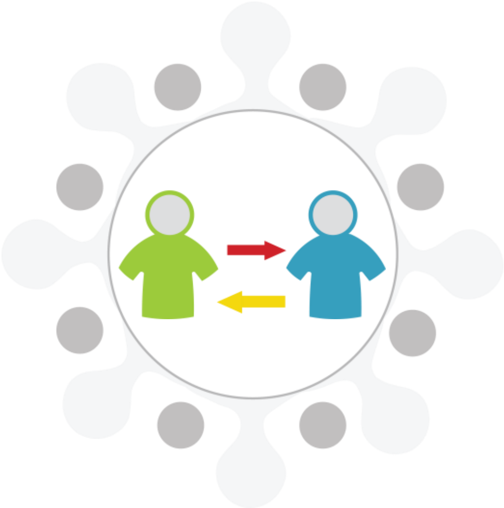
What’s in store…

- **P**artial Order
- **T**ransitive Closure
- **B**eyond Binary Relations
- **O**perations on Relations



By the end of this lesson, you should be able to…

- Explain the concept of partial order.
- Explain the three properties of transitive closure.
- Explain the concept of non-binary relations.
- Explain the different operations on relations.



Partial Order

Partial Order: Definition

R is **a partial order** on A if R is reflexive, antisymmetric and transitive.

Example

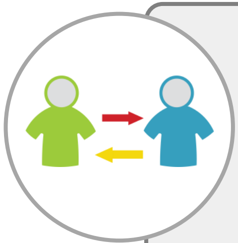
$A = \mathbb{Z}$, $xRy \leftrightarrow x \leq y$

Notion of partial order is useful for scheduling problems across possibly different domains.

Transitive Closure

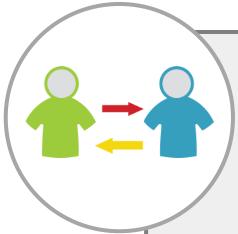
Transitive Closure: What is Closure?

Let A be a set and R a binary relation on A.



The **closure of a relation** $R \subseteq A \times A$ with respect to a property P (P being reflexive, symmetric, or transitive) is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

Transitive Closure: Definition



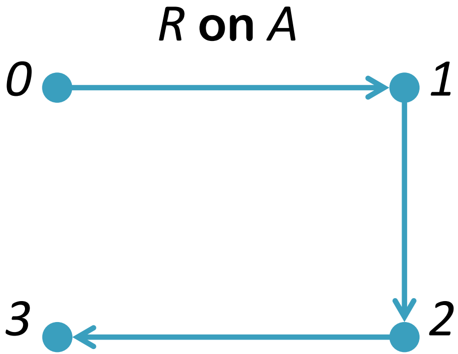
Let A be a set and R a binary relation on A. The **transitive closure** of R is the binary relation R^t on A that satisfies the following three properties:

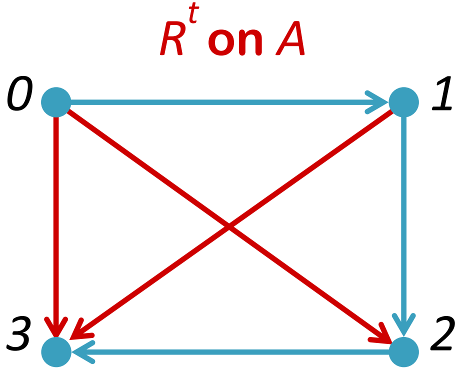
1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R then $R^t \subseteq S$

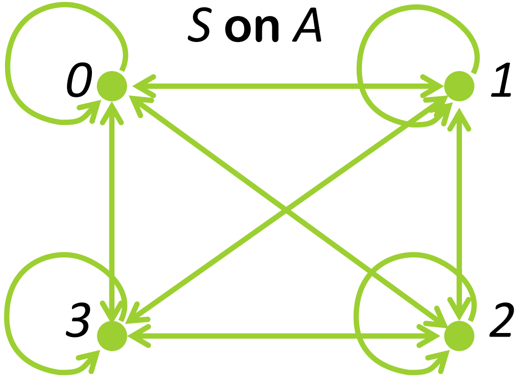
Transitive Closure: Example

Let $A = \{0,1,2,3\}$

Consider a relation $R = \{(0,1),(1,2),(2,3)\}$ on A







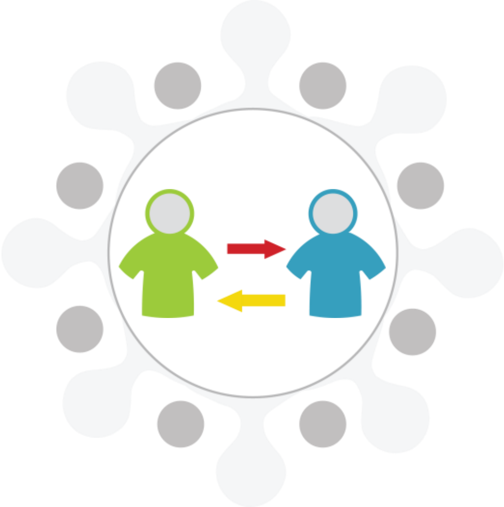
S is transitive and $R \subseteq S$

Thus $R^t \subseteq S$

$R^t = \{(0,1),(1,2),(2,3),(0,2),(0,3),(1,3)\}$

Transitive Closure: Construction

- Let A be a set and R a binary relation on A.
- Start with R, and do the following: $\forall x, y, z \in A$, if $(xRy \land yRz \land x\not{R}z)$ then add (x,z).
- Repeat until the obtained relation is transitive (will stop if $|A|$ is finite).
- The ordering in which the edges are added does not matter.

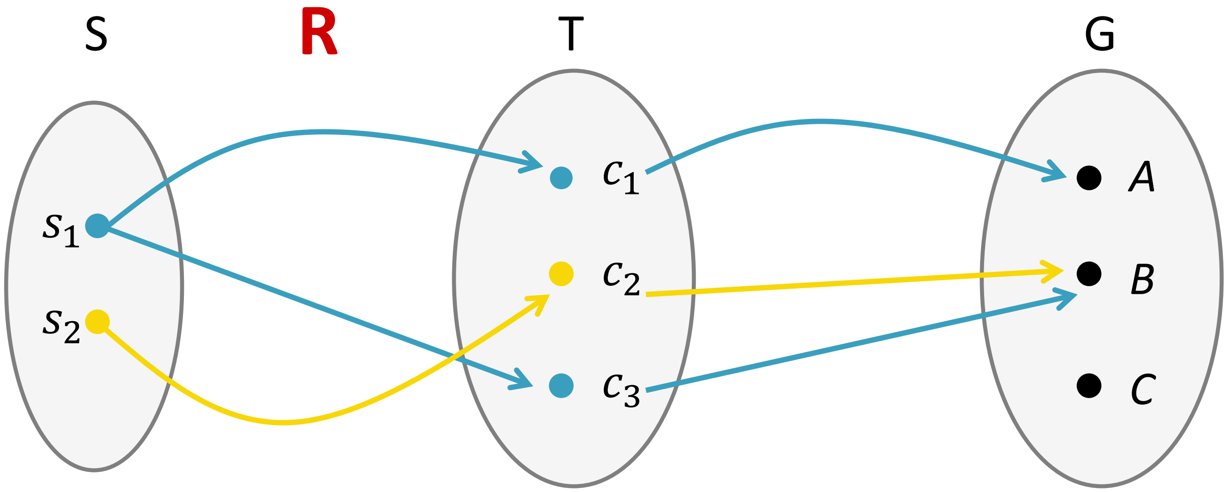


Beyond Binary Relations

Beyond Binary Relations: Non-binary Relations (Example)

$S = \{s_1, s_2\}$ **students,** $T = \{c_1, c_2, c_3\}$ **courses**

$G = \{A, B, C\}$ **grades,** (s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)



Beyond Binary Relations: *n-ary* Relations

Let $A_1, \ldots, A_n$ be sets. A *n-ary* **relation** R is a subset of $A_1 \times \cdots \times A_n$. $a_1, \ldots, a_n$ are related if $(a_1, \ldots, a_n) \in R$.

Example

$S = \{s_1, s_2\}$ students, $T = \{c_1, c_2, c_3\}$ courses

$G = \{A, B, C\}$ grades, (s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)

Operations of Relations: Complement of a Relation

Let $R \subseteq A_1 \times \cdots \times A_n$ be a relation.
$\overline{R} = (A_1 \times \cdots \times A_n - R)$ is the relational complement of R,
i.e., $(a_1, a_2, a_3,\ldots, a_n) \in \overline{R} \Leftrightarrow (a_1, a_2, a_3,\ldots, a_n) \notin R$.

Example

$A = \{1,2\}$, $B = \{3,5\}$ **and** $R = \{(1,3), (2,5)\}$

Then $\overline{R} = A \times B - R = \{(1,5), (2,3)\}$

Operations on Relations

Operations of Relations: Union of Relations

Let $R, S \subseteq A_1 \times \cdots \times A_n$ **be two relations.** $R \cup S$ **is the relation such that** $(a_1, a_2, a_3, \ldots, a_n) \in R \cup S \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \in R \vee (a_1, a_2, a_3, \ldots, a_n) \in S$.

Example

$A = \{1,2,3\}$, $B = \{1,2,3,4\}$ **and** $R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Then $R \cup S = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

Operations of Relations: Intersection of Relations

Let $R, S \subseteq A_1 \times \cdots \times A_n$ **be two relations.** $R \cap S$ **is the relation such that** $(a_1, a_2, a_3, \ldots, a_n) \in R \cap S \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \in R \wedge (a_1, a_2, a_3, \ldots, a_n) \in S$.

Example

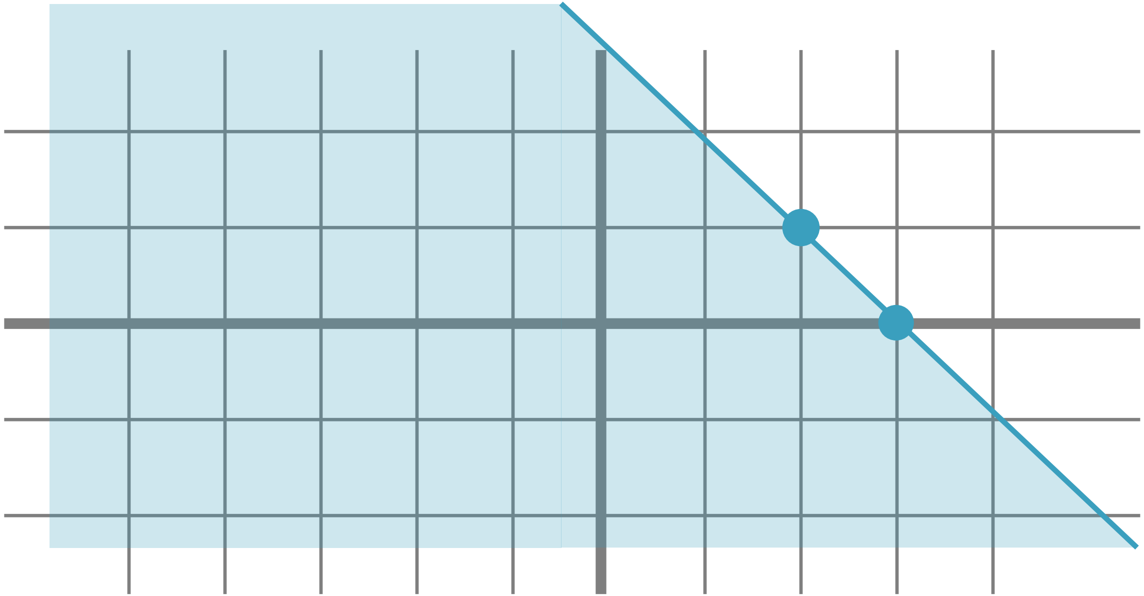
$A = \{1,2,3\}$, $B = \{1,2,3,4\}$ **and** $R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Then $R \cap S = \{(1,1)\}$

Operations of Relations: Example

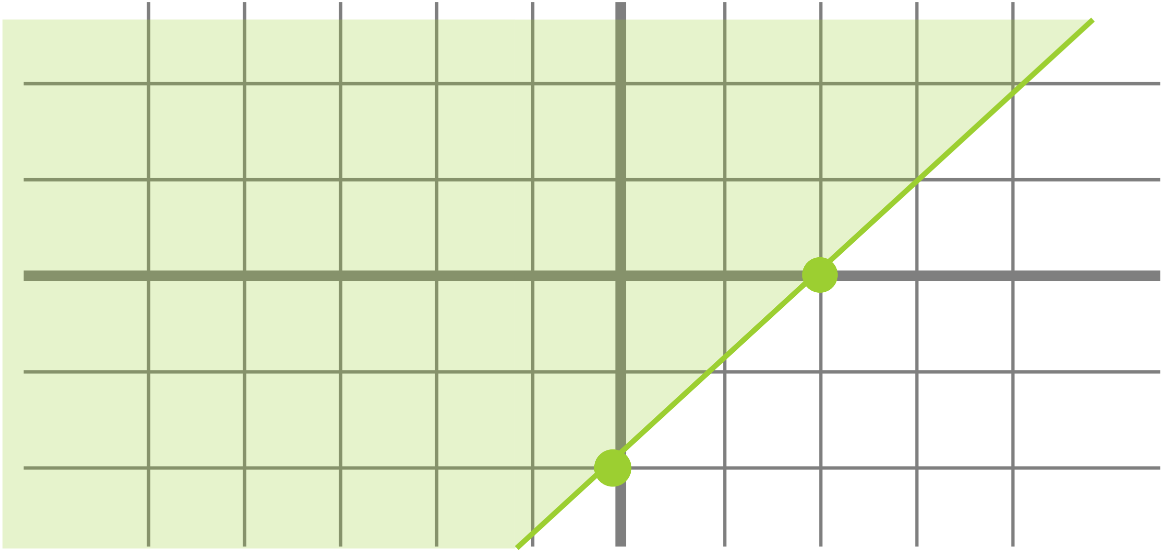
$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$



Operations of Relations: Example

$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$

$S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \leq 2 \}$

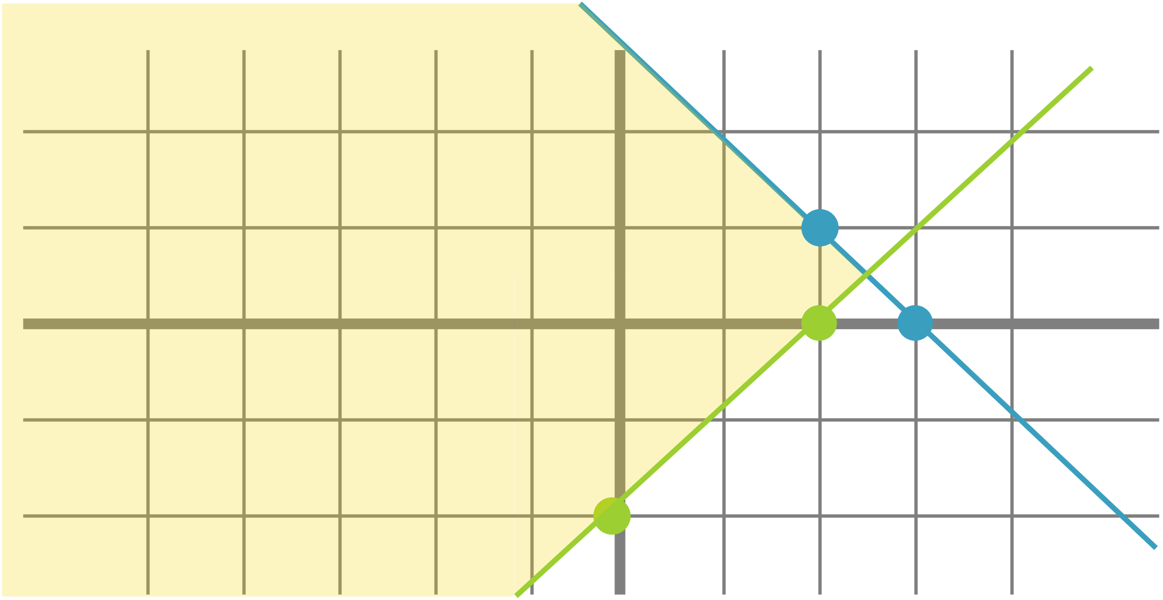


Operations of Relations: Example

$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$

$S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \leq 2 \}$

$T \cap S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid (x + y \leq 3) \wedge (x - y \leq 2) \}$



Topic Summary

Let’s recap…

- Partial Order
- Transitive Closure
- Beyond binary relations
- Operations on relations
  – Complement
  – Union
  – Intersection