# Discrete Mathematics

## MH1812

Topic 9.2 - Functions II

Dr. Wang Huaxiong

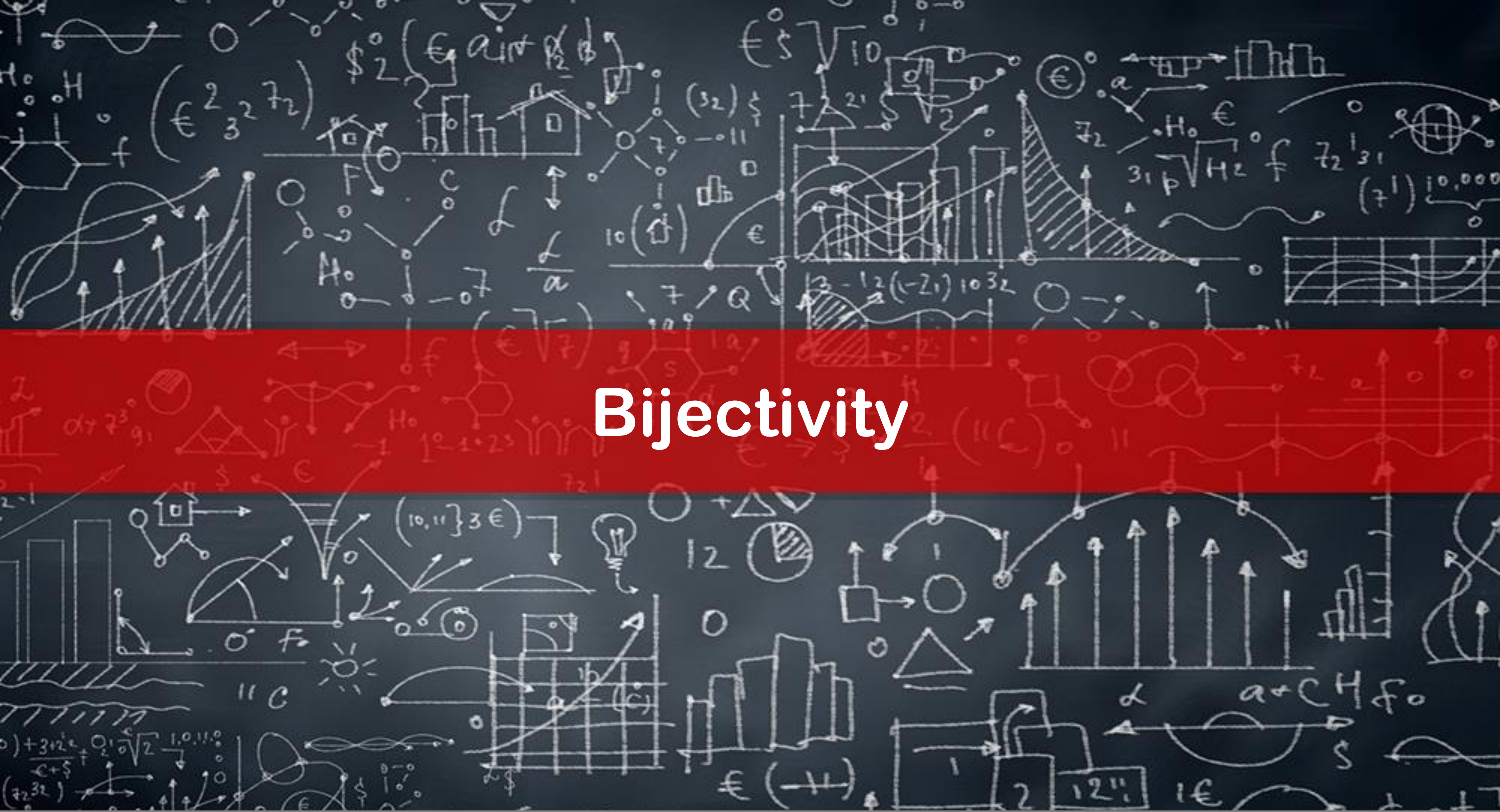
# Topic Overview

# What's in store…

- **B**ijectivity
- **I**dentity and Inverse
- **C**omposition and Properties

# By the end of this lesson, you should be able to…

- Explain the concepts of bijective functions.
- Explain the concepts of identity and inverse functions.
- Explain the composition of functions.



# Bijectivity

# Bijectivity: One-to-one Correspondence

$f(x)$

A function $f$ is a **one-to-one correspondence** (or **bijection**), if and only if it is both one-to-one and onto.

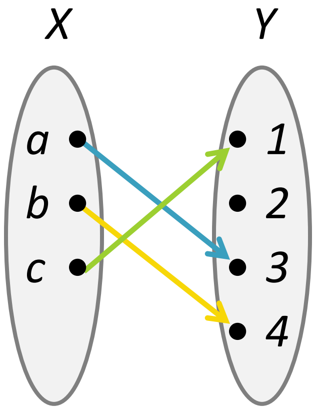
## In words…

"No element in the codomain of $f$ has two (or more) preimages" (one-to-one)

and

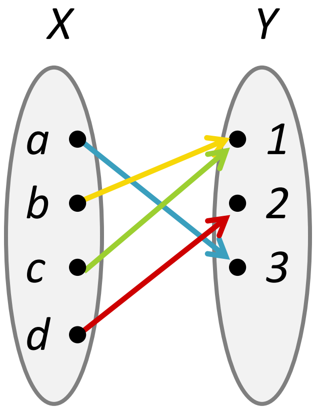
"Each element in the codomain of $f$ has a preimage" (onto)

# Bijectivity: Example (Bijection)



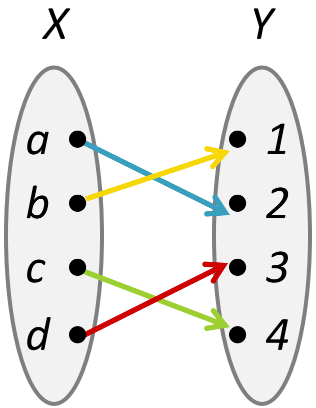

**No!**

**(Not onto as *2* has no preimage)**



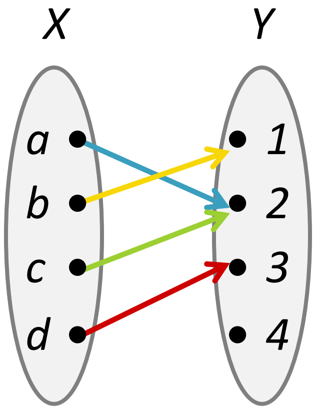

**No!**

**(Not one-to-one as *1* has two preimages)**



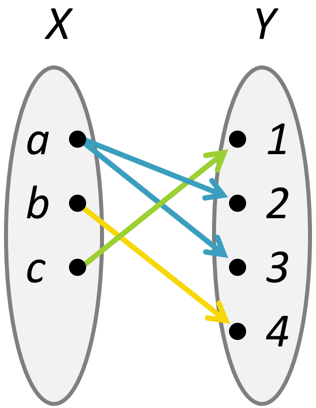

**Yes!**

**(Each element has exactly one preimage)**




**No!**

**(Neither one-to-one nor onto)**




**No!**

**(Not a function as *a* has two images)**

# Identity and Inverse

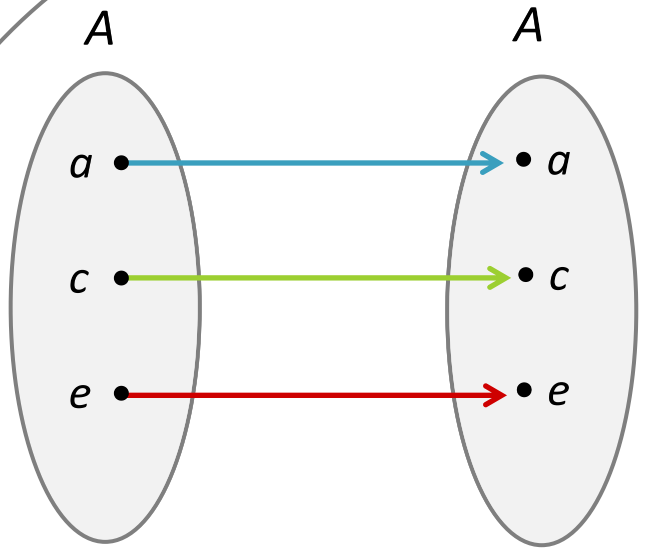
# Identity and Inverse: Identity Function

The **identity function** on a set $A$ is defined as:

$\boldsymbol{i_A}: A \rightarrow A,\ i_A(x) = x.$

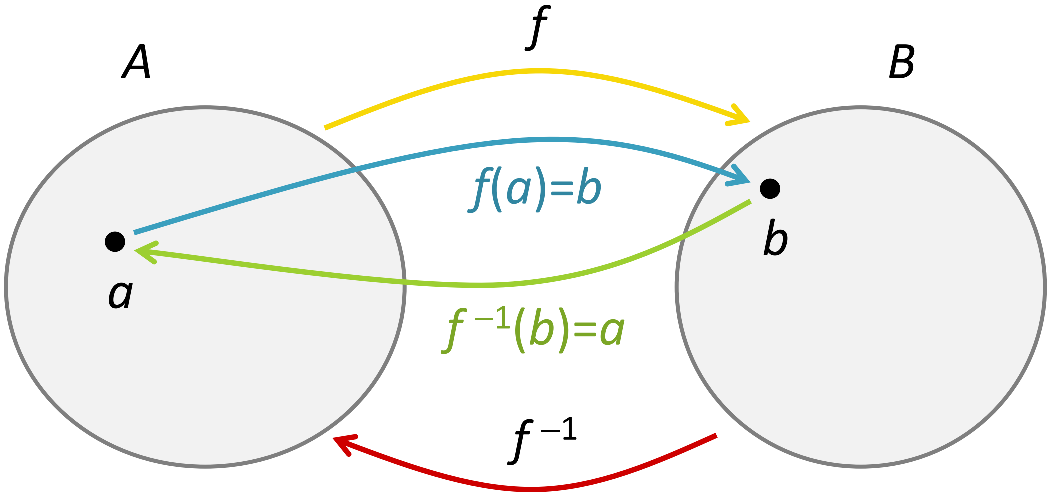
**Example**

**All identity functions are bijections (e.g., for $A = \{a, c, e\}$).**



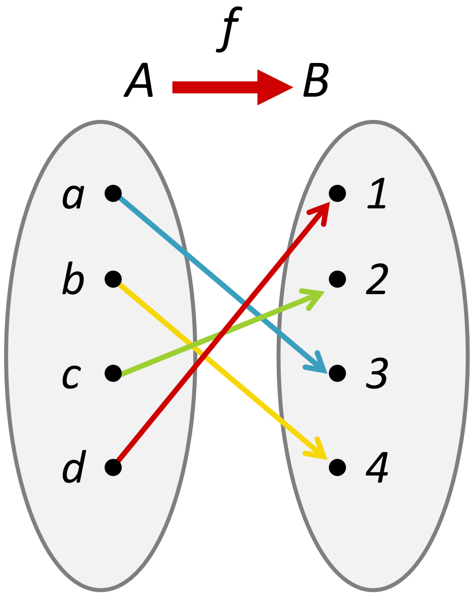
# Identity and Inverse: Inverse Function

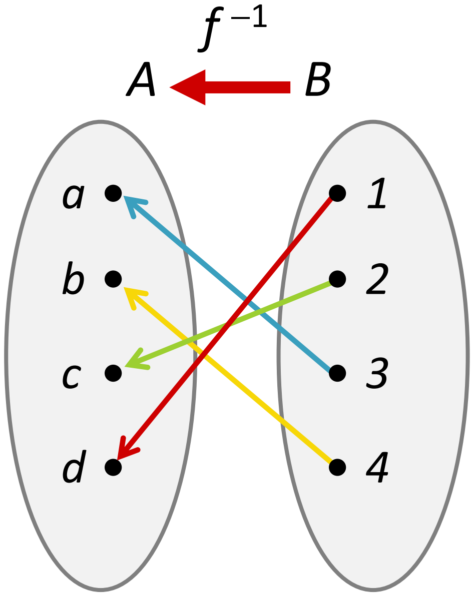
Let $f: A \to B$ be a one-to-one correspondence (bijection). Then the **inverse function of $f$**, $f^{-1}: B \to A$, is defined by: $f^{-1}(b)$ = that unique element $a \in A$ such that $f(a) = b$. We say that $f$ is **invertible**.



# Identity and Inverse: Example 1

**Find the inverse function of the following function:**



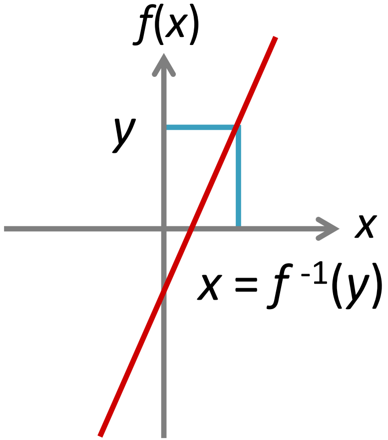




**Let** $f: A \to B$ **be a one-to-one correspondence and** $f^{-1}: B \to A$ **its inverse.**
**Then** $\forall b \in B\ \forall a \in A\ (f^{-1}(b) = a \iff b = f(a))$.

# Identity and Inverse: Example 2

**What is the inverse of**

$f{:}R \to R$ , $f(x) = 4x{-}1$**?**



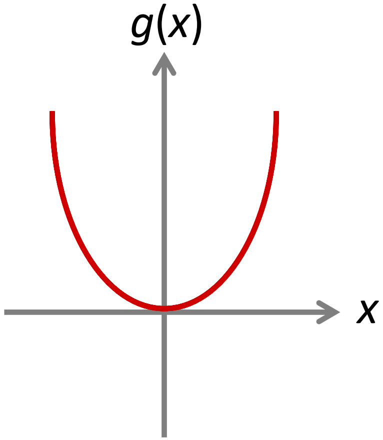

**Let** $y \in R$**.**

**Calculate** $x$ **with** $f(x) = y$: $y = 4x{-}1 \Leftrightarrow (y{+}1)/4 = x$.

**Hence** $f^{-1}(y) = (y{+}1)/4$.

**What is the inverse of**

$g{:}R \to R$, $g(x) = x^2$ **?**



# Identity and Inverse: One-to-one Correspondence

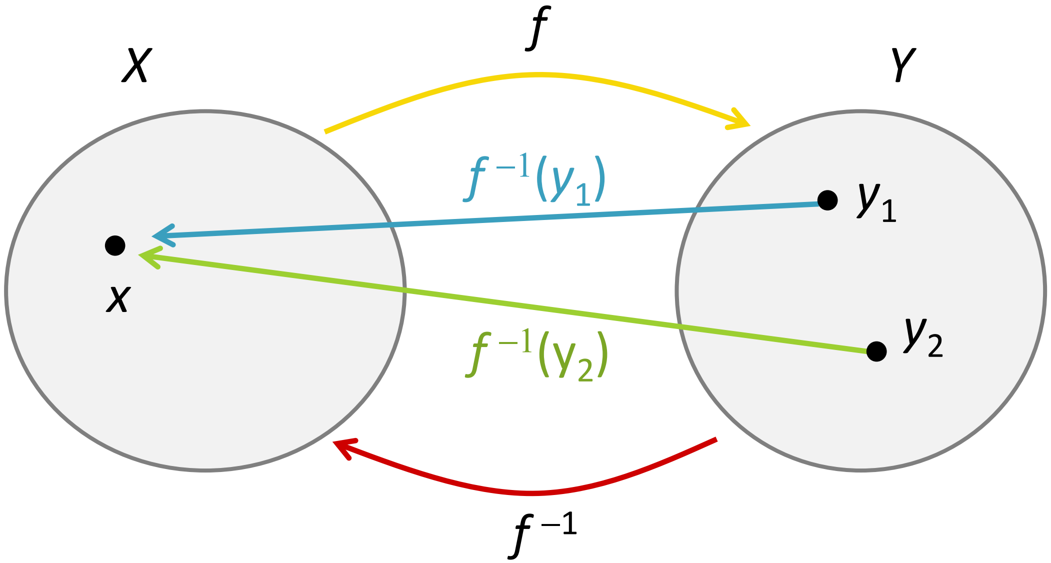
$f(x)$

**Theorem 1**: If $f: X \to Y$ is a one-to-one correspondence, then $f^{-1}: Y \to X$ is a one-to-one correspondence.

**Proof: $f^{-1}$ is one-to-one**

Take $y_1, y_2 \in Y$ such that $f^{-1}(y_1) = f^{-1}(y_2) = x$.

Then $f(x) = y_1$ and $f(x) = y_2$, thus $y_1 = y_2$.



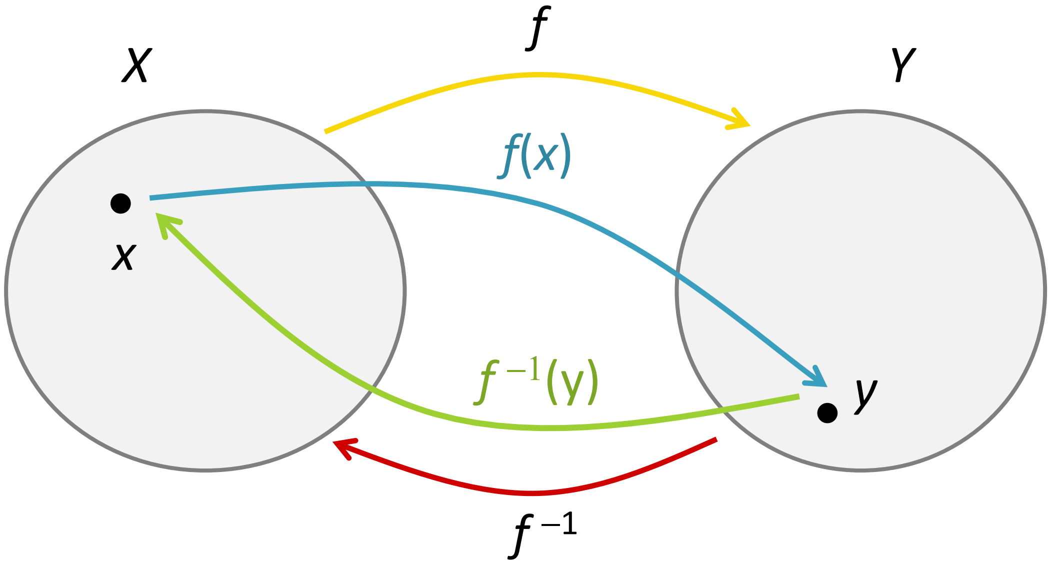
# Identity and Inverse: One-to-one Correspondence

$f(x)$ **Theorem 1**: If $f: X \to Y$ is a one-to-one correspondence, then $f^{-1}: Y \to X$ is a one-to-one correspondence.

**Proof: $f^{-1}$ is onto**

Take some $x \in X$, and let $y = f(x)$.

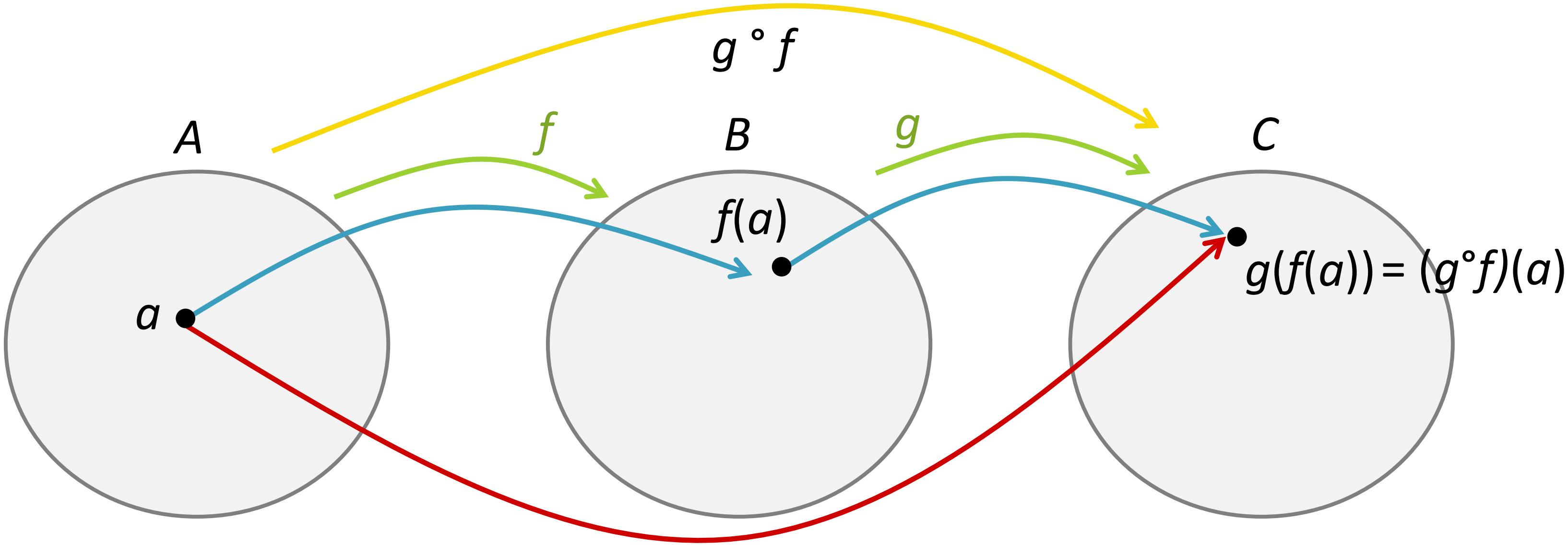
Then $f^{-1}(y) = x$.



# Composition and Properties

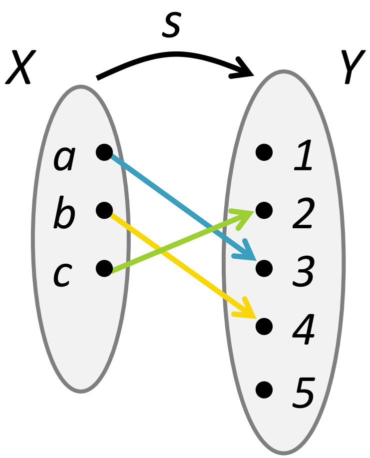
# Composition and Properties: Composition of Functions

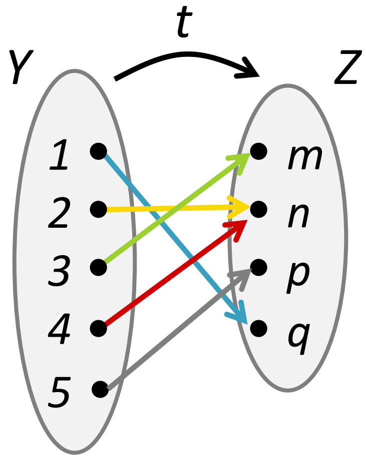
Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. The **composition** of the functions $f$ and $g$, denoted as $g \circ f$, is defined by: $g \circ f: A \rightarrow C$, $(g \circ f)(a) = g(f(a))$.

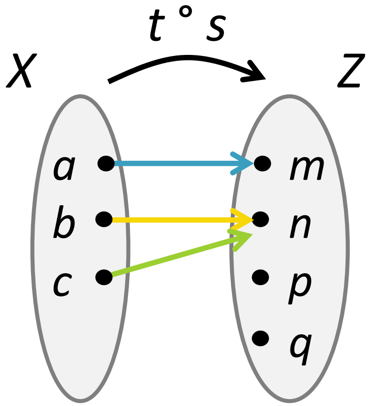


# Composition and Properties: Example

**Given functions** $s: X \rightarrow Y$ **and** $t: Y \rightarrow Z$. **Find** $t \circ s$ **and** $s \circ t$.



# Composition and Properties: Example

$f: Z \rightarrow Z,\ f(n) = 2n + 3,\ g: Z \rightarrow Z,\ g(n) = 3n + 2$

**What is $g \circ f$ and $f \circ g$?**

$(f \circ g)(n) = f(g(n)) = f(3n + 2) = 2(3n + 2) + 3 = 6n + 7$

$(g \circ f)(n) = g(f(n)) = g(2n + 3) = 3(2n + 3) + 2 = 6n + 11$

$f \circ g \neq g \circ f$ **(No commutativity for the composition of functions!)**

# Composition and Properties: One-to-one Propagation

$f(x)$

**Theorem 2**: Let $f: X \to Y$ and $g: Y \to Z$ be both one-to-one functions. Then $g \circ f$ is also one-to-one.

**Proof:** $\forall x_1, x_2 \in X\ ((g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2)$

Suppose $x_1, x_2 \in X$ with $(g \circ f)(x_1) = (g \circ f)(x_2)$.

Then $g(f(x_1)) = g(f(x_2))$.

Since $g$ is one-to-one, it follows $f(x_1) = f(x_2)$.

Since $f$ is one-to-one, it follows $x_1 = x_2$.

# Composition and Properties: Onto Propagation

$f(x)$

**Theorem 3**: Let $f: X \to Y$ and $g: Y \to Z$ be both onto functions. Then $g \circ f$ is also onto.

**Proof: $\forall z \in Z\ \exists x \in X$ such that $(g \circ f)(x) = z$**

Let $z \in Z$.

Since $g$ is onto, $\exists y \in Y$ with $g(y) = z$.

Since $f$ is onto, $\exists x \in X$ with $f(x) = y$.

Hence, with $(g \circ f)(x) = g(f(x)) = g(y) = z$.

# Topic Summary

# Let's recap…

- Bijective functions
- Identify and inverse functions
- Composition of functions and their properties