Discrete Mathematics
MH1812

Topic 10.3 - Graph Theory III
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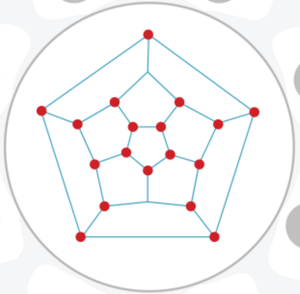
Topic Overview

What’s in store…

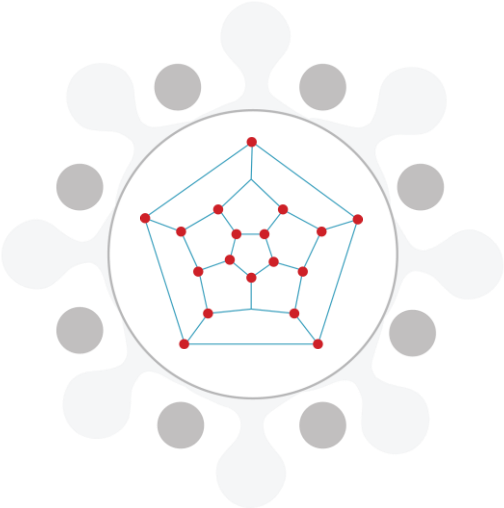
Hamiltonian Circuit

Graph Isomorphism



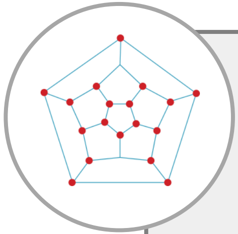
By the end of this lesson, you should be able to...

- Explain the concepts of the Hamiltonian circuit.
- Explain what is graph isomorphism.



Hamiltonian Circuit

Hamiltonian Circuit: Definition



A **Hamiltonian path** of a graph G is a walk such that every vertex is visited exactly once.

A **Hamiltonian circuit** of a graph G is a closed walk such that every vertex is visited exactly once (except the same start/end vertex).

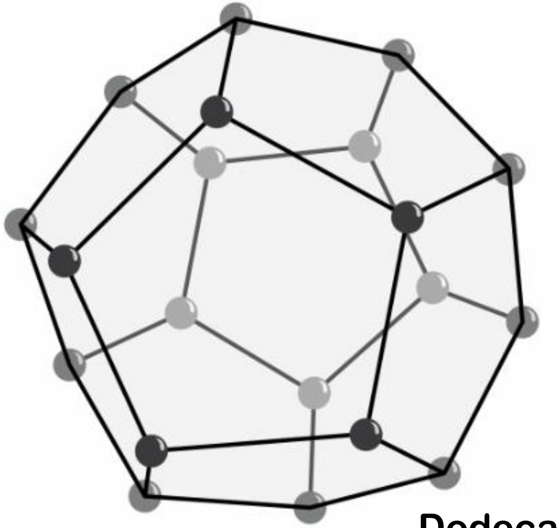
William Rowan Hamilton
1805 - 1865

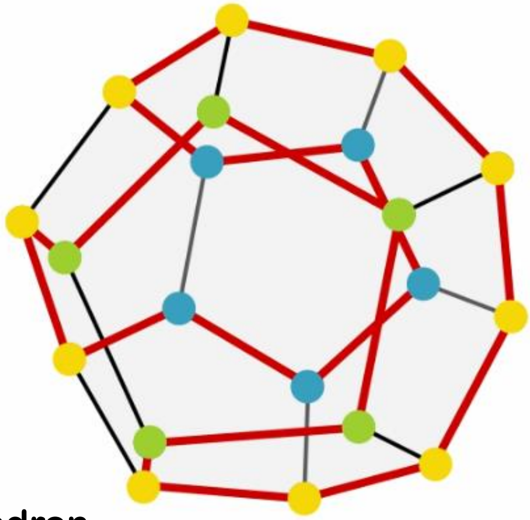
William Rowan Hamilton under WikiCommons (PD-US)

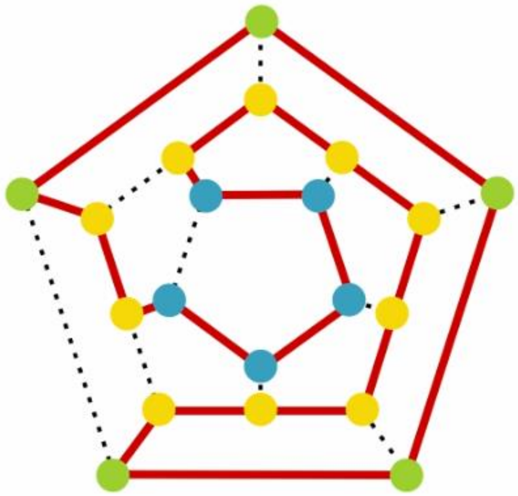
Hamiltonian Circuit: The Icosian Game (1857)

- Along the edges of a dodecahedron, find a path such that every vertex is visited a single time, and the ending point is the same as the starting point.

- Hamilton sold it to a London game dealer in 1859 for 25 pounds.







Dodecahedron

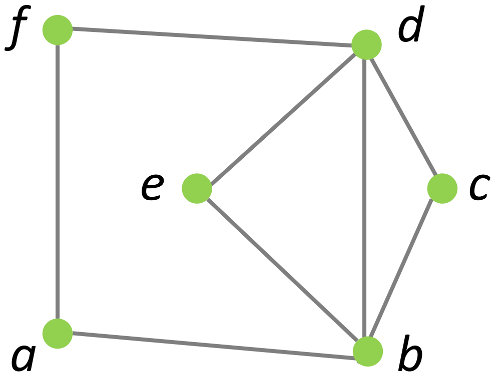
Hamiltonian Circuit: Hamiltonian vs. Eulerian

- Path (or trail) vs. circuit (or cycle):
  - For circuits, the walk starts and finishes at the same vertex.
  - But for a path, the starting vertex is different from the ending one.
- Eulerian: walk through every edge exactly once.
- Hamiltonian: walk through every vertex exactly once.

William Rowan Hamilton
1805 - 1865

Leonhard Euler
1707 - 1783

Hamiltonian Circuit: Examples

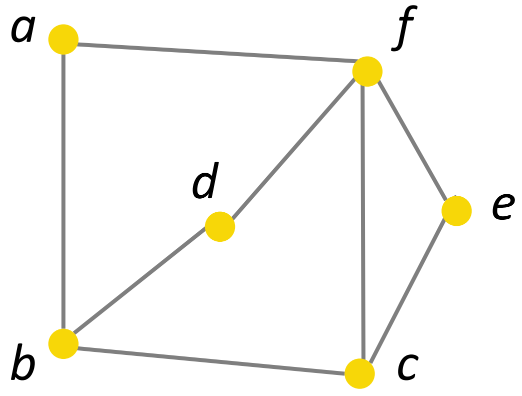


- **Euler circuit**
- **Hamiltonian path**
- **No Hamiltonian circuit**

$$a \rightarrow b \rightarrow e \rightarrow d \rightarrow c \rightarrow b \rightarrow d \rightarrow f \rightarrow a$$

$$f \rightarrow a \rightarrow b \rightarrow e \rightarrow d \rightarrow c$$

Hamiltonian Circuit: Examples



- **No Euler circuit, but Euler path**
- **Hamiltonian path**
- **No Hamiltonian circuit**

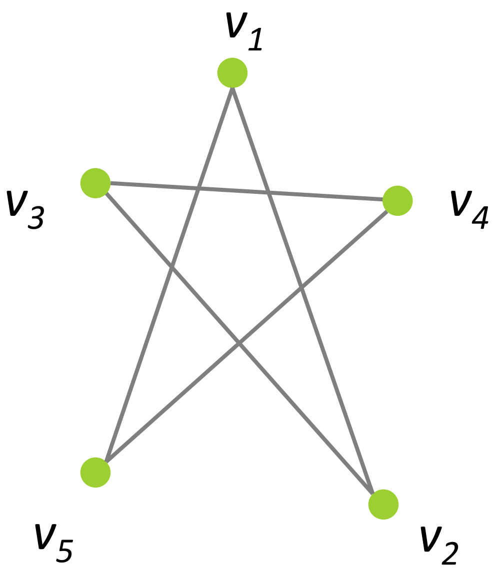
$$c \rightarrow e \rightarrow f \rightarrow c \rightarrow b \rightarrow d \rightarrow f \rightarrow a \rightarrow b$$

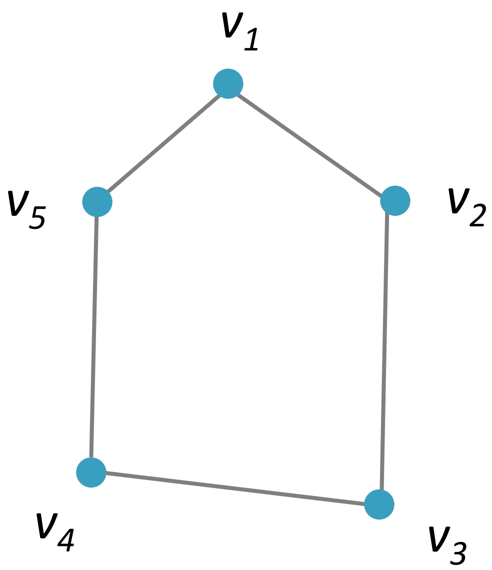
$$e \rightarrow c \rightarrow f \rightarrow d \rightarrow b \rightarrow a$$

Graph Isomorphism

Graph Isomorphism: Pictorial Representations

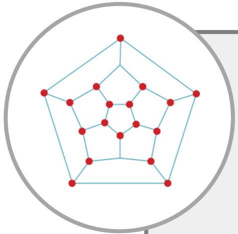
A graph can have many pictorial representations.





$$\begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Graph Isomorphism: Definition

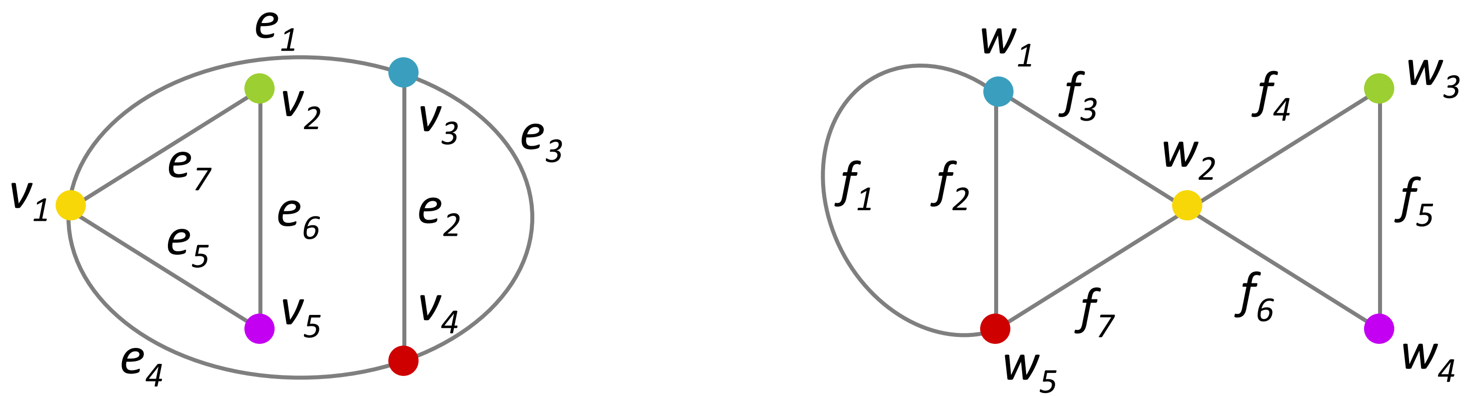


A graph $G = (V_G,E_G)$ is isomorphic to a graph $H = (V_H,E_H)$ if and only if there exists two bijections mapping the vertex sets and edge sets, respectively:

$$g: V_G \to V_H,\ h: E_G \to E_H$$

such that an edge $e \in E_G$ is incident on $v,w \in V_G \Leftrightarrow$ the edge $h(e) \in E_H$ is incident on $g(v), g(w) \in V_H$.

Graph Isomorphism: Example



Vertex and edge bijections:

$g = \{(v_1,w_2), (v_2,w_3), (v_3,w_1), (v_4,w_5), (v_5,w_4)\}$

$h = \{(e_1, f_3), (e_2, f_2), (e_3, f_1), (e_4, f_7), (e_5, f_6), (e_6, f_5), (e_7, f_4)\}$

Topic Summary

Let’s recap…

- Basic definitions: graph, vertex (node), edge, loop
- Node degree, graph degree, handshaking theorem
- Types of graphs: simple, multigraph, (un)directed, complete, bipartite
- Euler path and circuit
- Hamiltonian path and circuit
- Adjacency matrix and graph isomorphism

Your Learning Roadmap