NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 7.2 - Set Theory II

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Topic Overview

What's in store…

Set Identities

Proving Set Identities

By the end of this lesson, you should be able to...

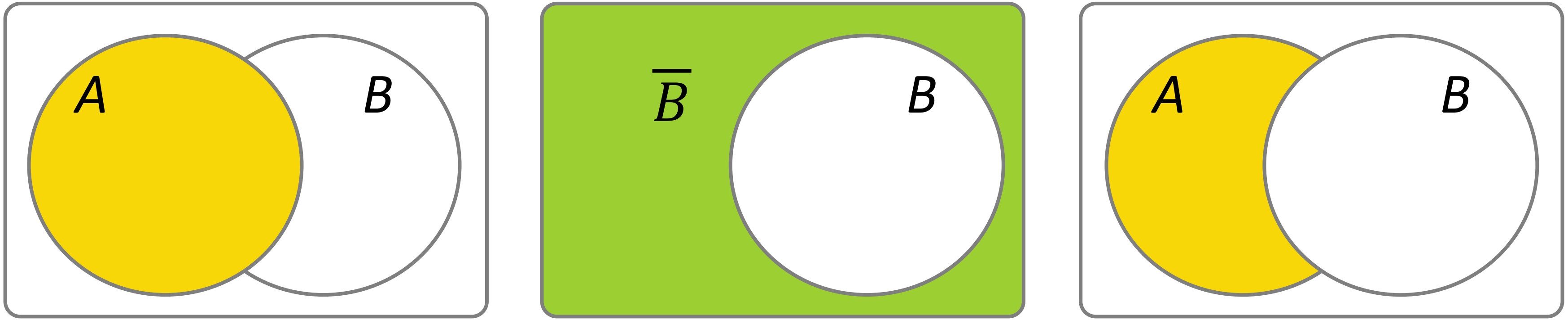
- Explain the different types of set identities.

- Apply the three methods to prove set identities.

Set Identities

Set Identities: Set Difference

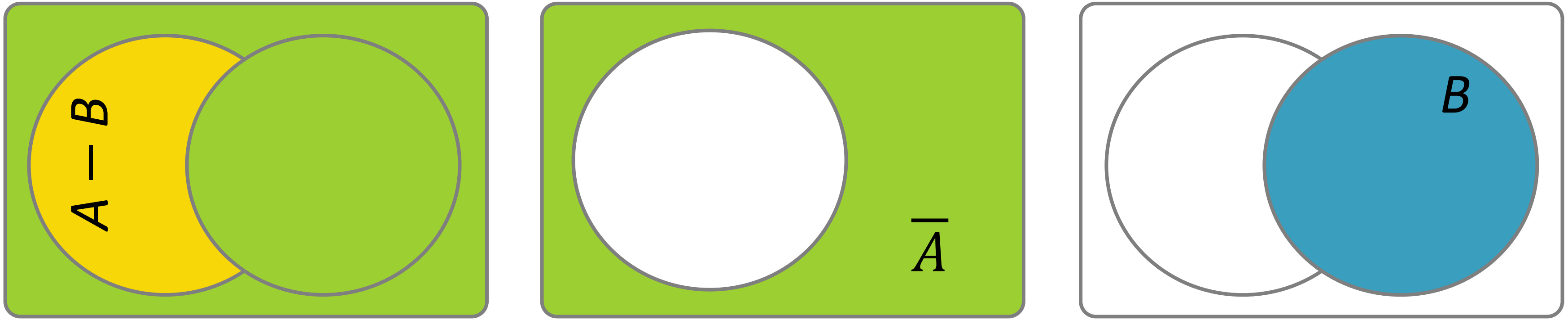
$$A \cap \overline{B} = A - B$$



Compare $A \cap \overline{B}$ **with** $A - B = \{x \mid x \in A \wedge x \notin B\}$

Set Identities: Set Difference

$$\overline{A \cap \overline{B}} = \overline{A} \cup B$$



- Consider $\overline{A - B} = \overline{A \cap \overline{B}}$
- This is De Morgan's Law $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$ with $X = A$ and $Y = \overline{B}$

Set Identities: Laws

Identity	Name
$A \cup \varnothing = A$ $A \cap U = A$	**Identity laws**
$A \cup U = U$ $A \cap \varnothing = \varnothing$	**Domination laws**
$A \cup A = A$ $A \cap A = A$	**Idempotent laws**
$\overline{\overline{A}} = A$	**Double Complement laws**

Set Identities: Laws

Identity	Name
$A \cup B = B \cup A$ $A \cap B = B \cap A$	**Commutative laws**
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	**Associative laws**
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**Distributive laws**
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	**De Morgan's laws**

Set Identities: Laws

Identity	Name
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	**Absorption laws**
$A - B = A \cap \overline{B}$	**Alternate representation for set difference**

Proving Set Identities

Proving Set Identities: Three Methods

- Recall: two sets are **equal if and only if** they contain exactly the same elements, i.e., iff $A \subseteq B$ and $B \subseteq A$.

Three Methods to Prove Set Identities

- Show that each set is a subset of the other
- Apply set identity theorems
- Use membership table

Proving Set Identities: Each Others' Subset

Show that $(B - A) \cup (C - A) = (B \cup C) - A$

For any $x \in$ LHS, $x \in (B - A)$ **or** $x \in (C - A)$ **(or both)**

When $x \in B - A$

$\Rightarrow (x \in B) \land (x \notin A)$

$\Rightarrow (x \in B \cup C) \land (x \notin A)$

$\Rightarrow x \in (B \cup C) - A$

When $x \in C - A$

$\Rightarrow (x \in C) \land (x \notin A)$

$\Rightarrow (x \in B \cup C) \land (x \notin A)$

$\Rightarrow x \in (B \cup C) - A$

Therefore LHS $\subseteq$ RHS

Proving Set Identities: Each Others' Subset

Show that $(B - A) \cup (C - A) = (B \cup C) - A$

For any $x \in \text{RHS}$, $x \in (B \cup C)$ **and** $x \notin A$

When $x \in B$ **and** $x \notin A$	$(x \in B) \land (x \notin A)$	$\Rightarrow x \in B - A$
		$\Rightarrow x \in (B - A) \cup (C - A)$
When $x \in C$ **and** $x \notin A$	$(x \in C) \land (x \notin A)$	$\Rightarrow x \in C - A$
		$\Rightarrow x \in (B - A) \cup (C - A)$

Therefore RHS $\subseteq$ **LHS**

With LHS $\subseteq$ **RHS and RHS** $\subseteq$ **LHS, we can conclude that LHS = RHS.**

Proving Set Identities: Using Set Identity Theorems

Show that $(A - B) - (B - C) = A - B$

$$(A-B)-(B-C) = (A \cap \overline{B}) \cap (B \cap \overline{C}) \quad \textbf{(By alternate representation for set difference)}$$

$$= (A \cap \overline{B}) \cap (\overline{B} \cup C) \quad \textbf{(By De Morgan's laws)}$$

$$= [(A \cap \overline{B}) \cap \overline{B}] \cup [(A \cap \overline{B}) \cap C] \quad \textbf{(By Distributive laws)}$$

$$= [A \cap (\overline{B} \cap \overline{B})] \cup [A \cap (\overline{B} \cap C)] \quad \textbf{(By Associative laws)}$$

$$= (A \cap \overline{B}) \cup [A \cap (\overline{B} \cap C)] \quad \textbf{(By Idempotent laws)}$$

$$= A \cap [\overline{B} \cup (\overline{B} \cap C)] \quad \textbf{(By Distributive laws)}$$

$$= A \cap \overline{B} \quad \textbf{(By Absorption laws)}$$

$$= A - B \quad \textbf{(By alternate representation for set difference)}$$

Proving Set Identities: Using Membership Tables

Similar to truth table (in propositional logic):

- Columns for different set expressions
- Rows for all combinations of memberships in constituent sets
- "*1*" = membership, "*0*" = non-membership
- Two sets are equal iff they have identical columns

Proving Set Identities: Using Membership Tables

Prove that $(A \cup B) - B = A - B$

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

Topic Summary

Let’s recap…

- Set identities
- Prove set identities:
  - Each others’ subset
  - Set identity theorems
  - Membership table