Discrete Mathematics

MH1812

Topic 2.3 - Propositional Logic III

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Topic Overview

What's in store…

Arguments

Inference Rules

By the end of this lesson, you should be able to…

- **Determine whether or not an argument is valid.**
- **Apply basic inference rules.**

Arguments

Arguments: Valid Argument

An **argument** is a sequence of statements. The last statement is called the **conclusion**. All the previous statements are called **premises** (or **assumptions**/**hypotheses**).

A **valid argument** is an argument where the conclusion is true if the premises are all true.

Arguments: Valid Argument

A **valid argument** is an argument where the conclusion is true if the premises are all true.

Example

- "If you pay up in full then I will deliver it"
- "You pay up in full"

Premises

- "I will deliver it"

Conclusion

Arguments: Valid Argument

A **valid argument** is an argument where the conclusion is true if the premises are all true.

A series of statements form a valid argument if and only if "the conjunction of premises implying the conclusion" is a tautology.

$$((\text{Premise}) \land (\text{Premise})) \rightarrow \text{Conclusion}$$

Arguments: A Valid Argument Template

$$\left.\begin{array}{r} p \rightarrow q; \\ p; \end{array}\right\} \text{Premises}$$

$$\left.\begin{array}{r} \therefore\ q \end{array}\right\} \text{Conclusion}$$

- By definition, a valid argument satisfies: “If the premises are true, then the conclusion is true”.
- To check if the above argument is valid, we need to check that $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.

Arguments: Valid Argument Template

$$\begin{aligned} p \to q&; \\ p&; \\ \therefore\ q& \end{aligned}$$

Premises

Conclusion

- Critical rows are rows with all premises true.
- If in all critical rows the conclusion is true, then the argument is valid (otherwise it is invalid).

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

No need to calculate

Arguments: Counterexample

If in all critical rows the conclusion is true, then the argument is valid (otherwise it is invalid).

A critical row with a false conclusion is a counterexample.

A counterexample:

- Invalidates the argument (i.e., makes the argument not valid)
- Indicates a situation where the conclusion does not follow from the premises

Arguments: Invalid Argument Example

- "If it is falling and it is directly above me then I'll run"
- "It is falling"
- "It is not directly above me"

Premises

- "I will not run"

Conclusion

$$S = (f \land a \rightarrow r);$$
$$f;$$
$$\neg a;$$
$$\therefore \neg r$$

Premises

Conclusion

Arguments: Invalid Argument Example

$S = (f \wedge a \rightarrow r)$;
f;
$\neg a$;

$\therefore \neg r$

Premises

Conclusion

Counterexample

Critical rows

a r	f	$\neg a$	$f \wedge a$	S	$\neg r$
T T	T	F	T	T	F
T T	F	F	F	T	F
T F	T	F	T	F	T
T F	F	F	F	T	T
F T	T	T	F	T	F
F T	F	T	F	T	F
F F	T	T	F	T	T
F F	F	T	F	T	T

Arguments: Fallacy

A **fallacy** is an error in reasoning that results in an invalid argument.

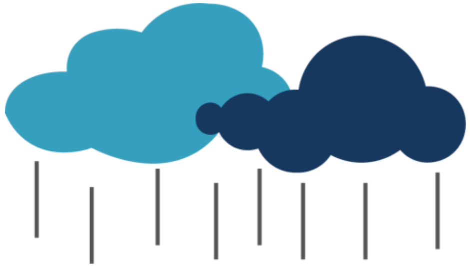
Arguments: Fallacy 1 (Converse Error)

Example

$$p \to q;$$
$$q;$$
$$\therefore\ p$$

- If it is Christmas then it is a holiday.
- It is a holiday. Therefore, it is Christmas!

Arguments: Fallacy 2 (Inverse Error)



Example

- If it is raining then I will stay at home.
- It is not raining. Therefore, I will not stay at home!

$$p \rightarrow q;$$
$$\neg p;$$
$$\therefore \neg q$$

Arguments: Invalid Argument, Correct Conclusion

An argument may be invalid, but it may still draw a correct conclusion (e.g., by coincidence).

Example

- If New York is a big city then New York has tall buildings.
- New York has tall buildings.
  - So, New York is a big city.

So, what happened?

- We have just made an invalid argument, i.e., converse error!
- But the conclusion is true (a fact true by itself).

Inference Rules

Inference Rules: Definition

A **rule of inference** is a logical construct which takes premises, analyses their syntax and returns a conclusion.

We already saw…

$$
\begin{array}{r}
p \rightarrow q; \\
p; \\
\therefore \quad q
\end{array}
$$

Modus Ponens
(Method of Affirming)

$$
\begin{array}{r}
p \rightarrow q; \\
\neg q; \\
\therefore \neg p
\end{array}
$$

Modus Tollens
(Method of Denying)

Inference Rules: More Inference Rules

Conjunctive Simplification (Particularising)

$p \wedge q;$
$\therefore p$

Disjunctive Syllogism (Case Elimination)

$p \vee q;$
$\neg p;$
$\therefore q$

Conjunctive Addition (Specialising)

$p;$
$q;$
$\therefore p \wedge q$

Rule of Contradiction

$\neg p \rightarrow C;$
$\therefore p$

Disjunctive Addition (Generalisation)

$p;$
$\therefore p \vee q$

Alternative Rule of Contradiction

$\neg p \rightarrow F;$
$\therefore p$

Inference Rules: Dilemma

Dilemma (case by case discussions)

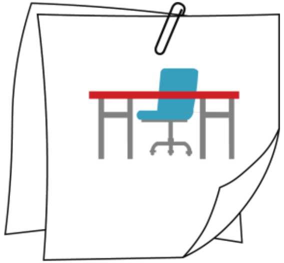
$p \lor q$;
$p \rightarrow r$;
$q \rightarrow r$;
$\therefore\ r$

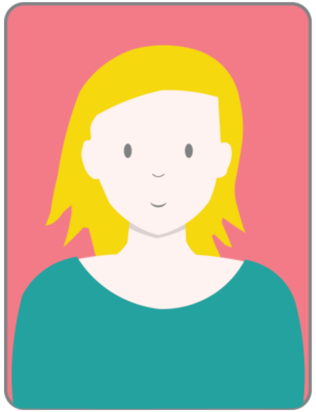
Inference Rules: Hypothetical Syllogism

Hypothetical Syllogism

$p \to q;$
$q \to r;$
$\therefore p \to r$

Example





Alice

- If I do not wake up, then I cannot go to work.
- If I cannot go to work, then I will not get paid.
- Therefore, if I do not wake up, then I will not get paid.

Inference Rules: Proof Hypothetical Syllogism

$(p \rightarrow q) \wedge (q \rightarrow r)$	(Hypotheses; Assumed True)
$\equiv (p \rightarrow q) \wedge (\neg q \vee r)$	(Conversion Theorem)
$\equiv [(p \rightarrow q) \wedge \neg q] \vee [(p \rightarrow q) \wedge r]$	(Distributive)
$\equiv [((p \rightarrow q) \wedge \neg q) \vee (p \rightarrow q)] \wedge [((p \rightarrow q) \wedge \neg q) \vee r]$	(Distributive)
$\equiv (p \rightarrow q) \wedge [((p \rightarrow q) \wedge \neg q) \vee r]$	(Recall absorption law: $a \vee (a \wedge b) \equiv a$, hence $[((p \rightarrow q) \wedge \neg q) \vee (p \rightarrow q)] \equiv p \rightarrow q$)
$\equiv (p \rightarrow q) \wedge [((\neg p \vee q) \wedge \neg q) \vee r]$	(Conversion)
$\equiv (p \rightarrow q) \wedge [((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee r]$	(Distributive)
$\equiv (p \rightarrow q) \wedge [((\neg p \wedge \neg q) \vee F) \vee r]$	(Contradiction)
$\equiv (p \rightarrow q) \wedge [(\neg p \wedge \neg q) \vee r]$	(Unity)
$\equiv (p \rightarrow q) \wedge [(\neg p \vee r) \wedge (\neg q \vee r)]$	(Distributive)
$\equiv [(p \rightarrow q) \wedge (\neg q \vee r)] \wedge (\neg p \vee r)$	(Commutative; Associative)
$\therefore (\neg p \vee r) \equiv p \rightarrow r$	(Conjunctive Simplification; Conversion)

Topic Summary

Let's recap…

- Arguments:
  - Valid arguments
  - Invalid arguments
  - Counterexample
  - Fallacy
- Inference rules:
  - Derive conclusions from a bunch of information
  - Some basic inference rules