Discrete Mathematics
MH1812

Topic 9.3 - Functions III
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Topic Overview

What's in store…

Ceiling and Floor

Pigeonhole Principle

Countable Sets

By the end of this lesson, you should be able to…

- Explain what is a ceiling function and floor function.
- Use the pigeonhole principle.
- Explain the difference between a countable set and an uncountable set.

Ceiling and Floor

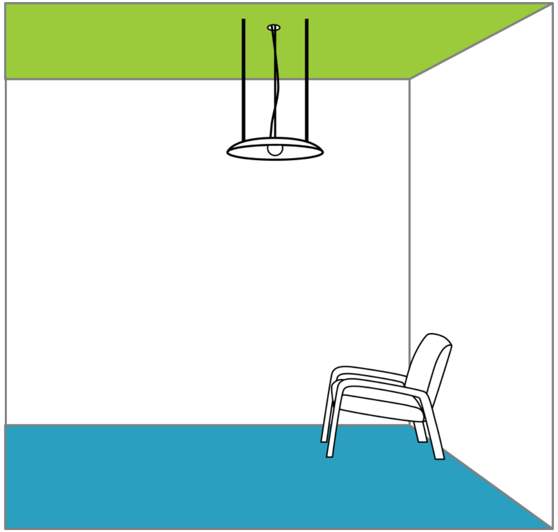
Ceiling and Floor: Definition

The **floor function** assigns to the real number x, the largest integer $\lfloor x \rfloor$ that is less than or equal to x. The **ceiling function** assigns to the real number x, the smallest integer $\lceil x \rceil$ that is greater than or equal to x.

Example

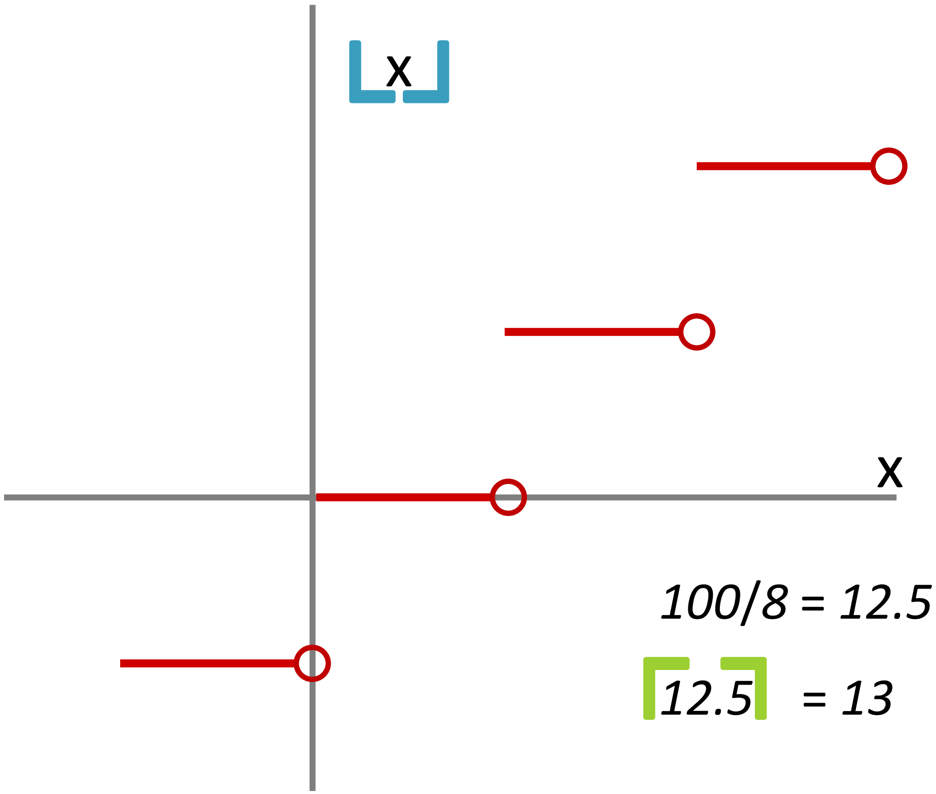
$\lfloor ½ \rfloor = 0$ $\lceil ½ \rceil = 1$

$\lfloor -½ \rfloor = -1$ $\lceil -½ \rceil = 0$



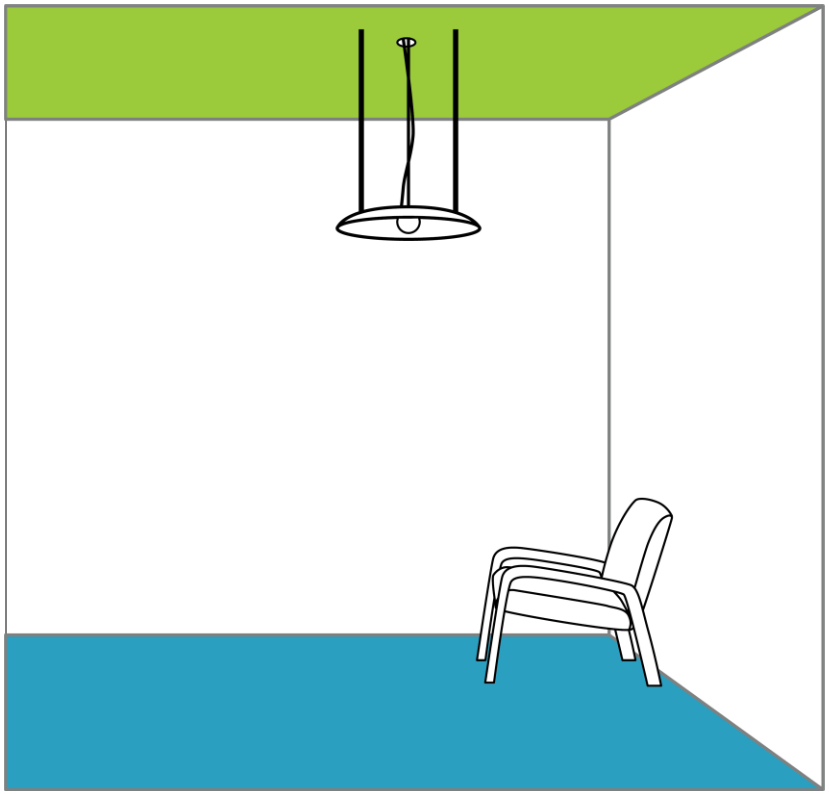
Ceiling and Floor: Example

How many bytes are required to encode *100* bits of data?



$$100/8 = 12.5$$

$$\lceil 12.5 \rceil = 13$$



Pigeonhole Principle

Pigeonhole Principle: Definition

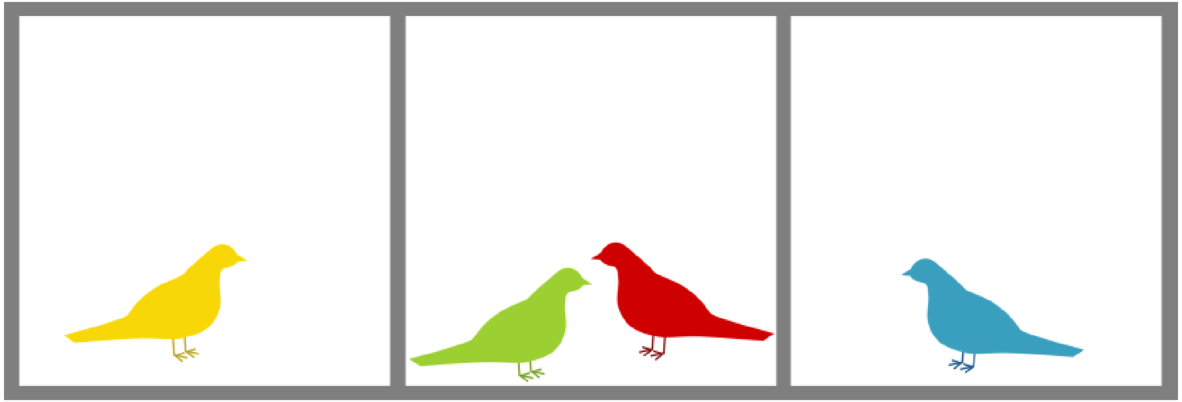
- *k* pigeonholes, *n* pigeons, $n > k$
- At least one pigeonhole contains at least two pigeons

Peter Gustav Lejeune Dirichlet (1805 - 1859)

Peter Gustav Lejeune Dirichlet under WikiCommons (PD-US)

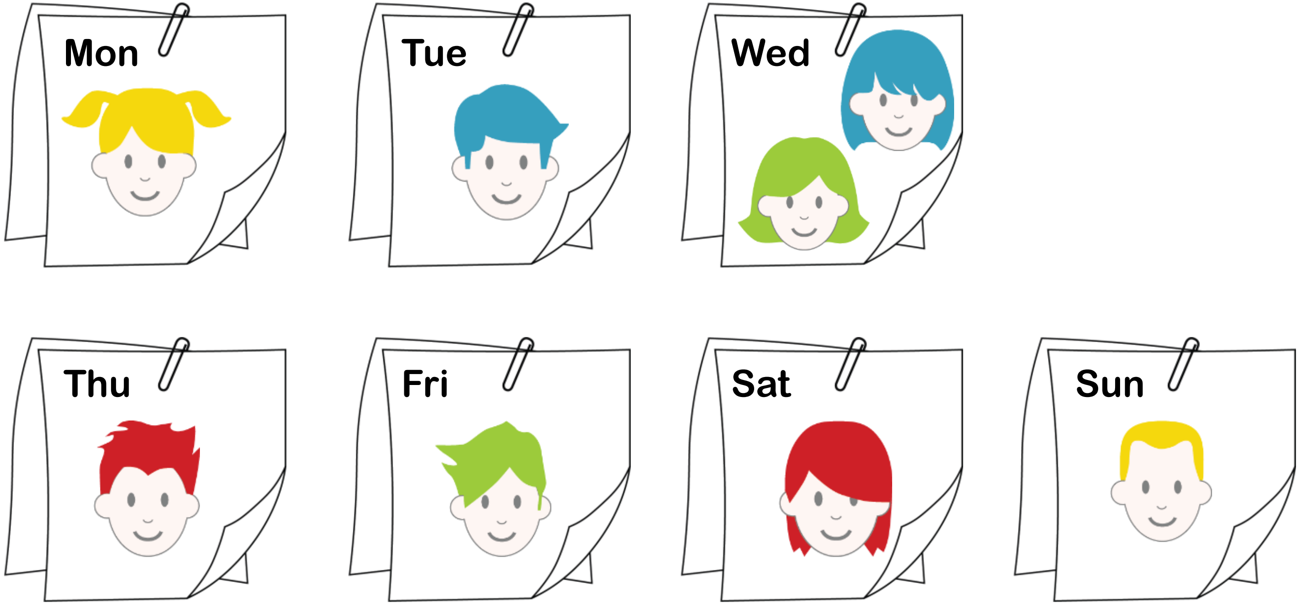
Pigeonhole Principle

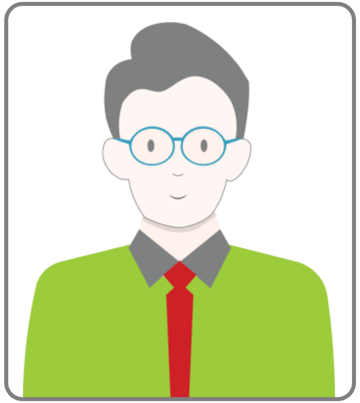
A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements in the domain that have the same image in the codomain.



Pigeonhole Principle: Scenario 1

Consider Bob and his *8* children. At least two of his children were born on the same day of the week.

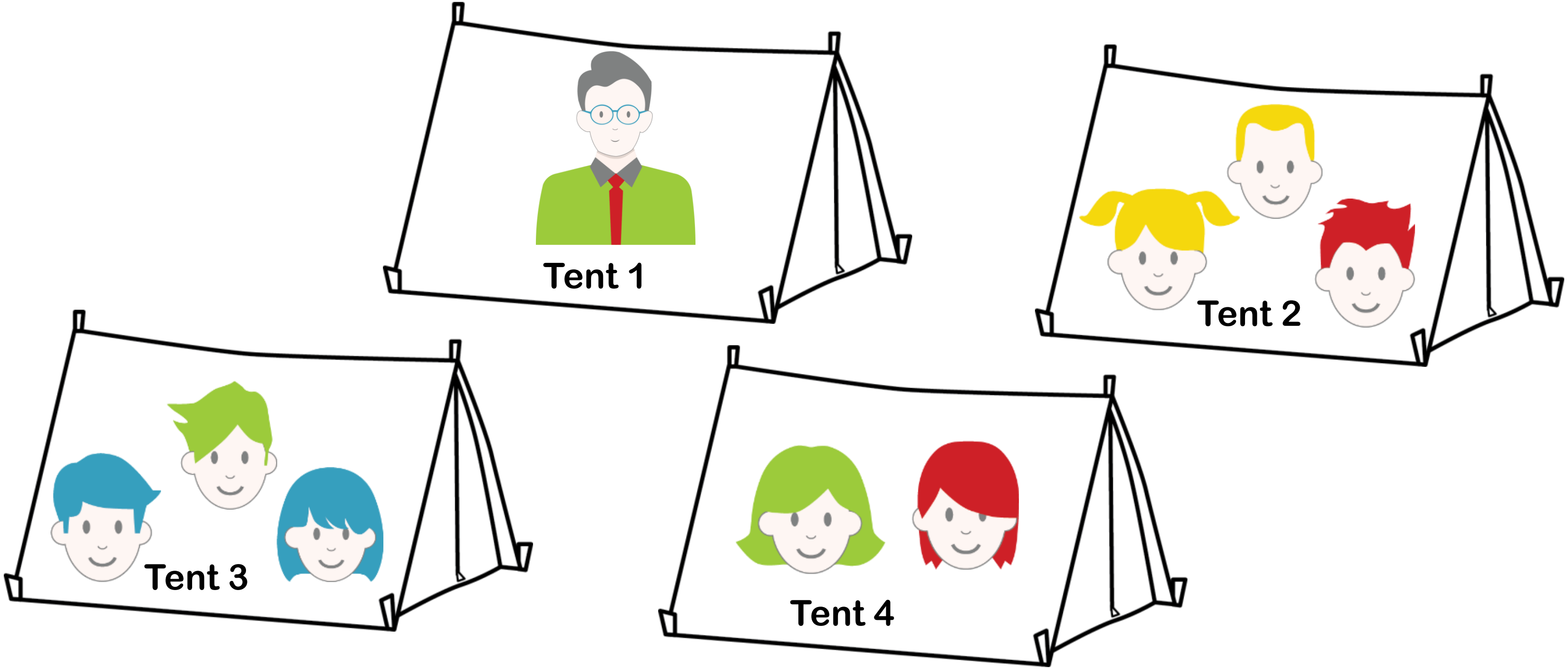




Bob

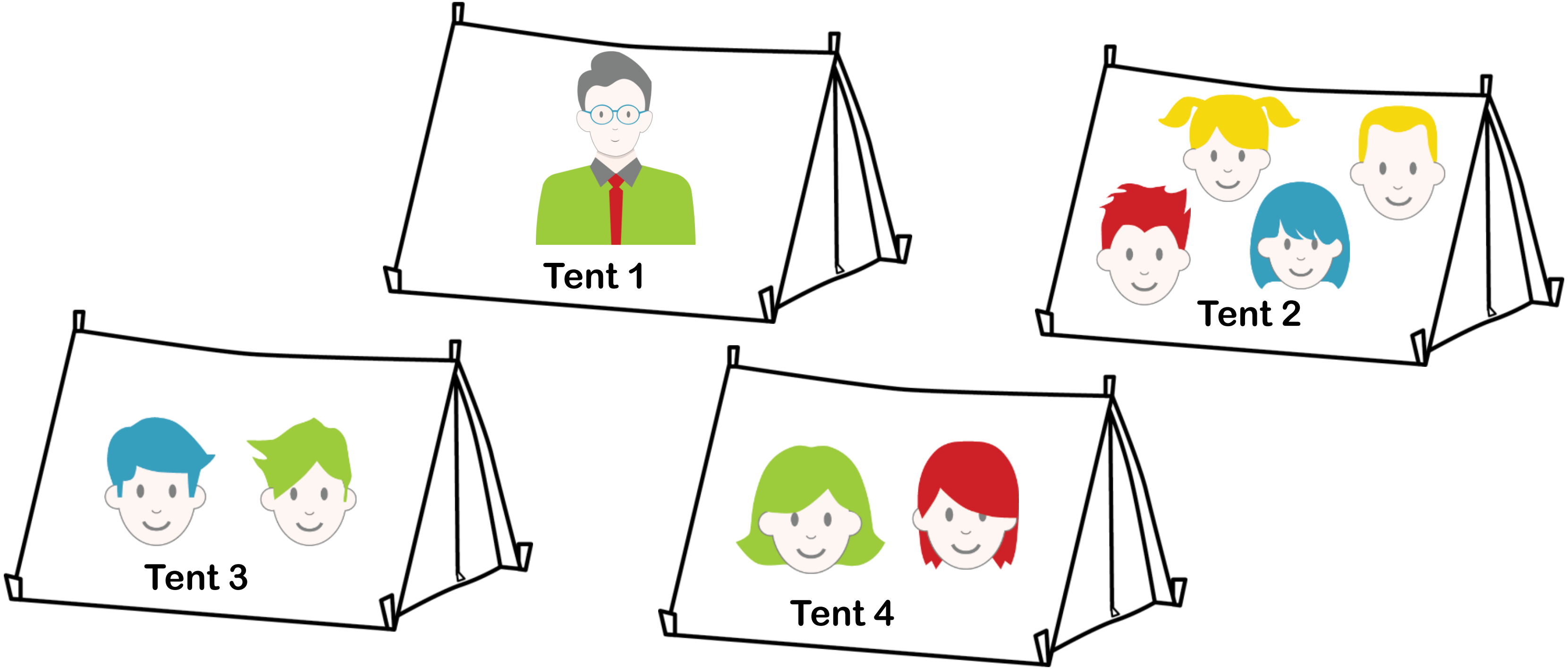
Pigeonhole Principle: Scenario 2

They go camping at the lake. Bob gets a tent of his own, but the others get to share 3 tents. Then, there are at least 3 children sleeping in at least one of them.



Pigeonhole Principle: Scenario 3

They go camping at the lake. Bob gets a tent of his own, but the others get to share 3 tents. Then, there are at least 3 children sleeping in at least one of them.



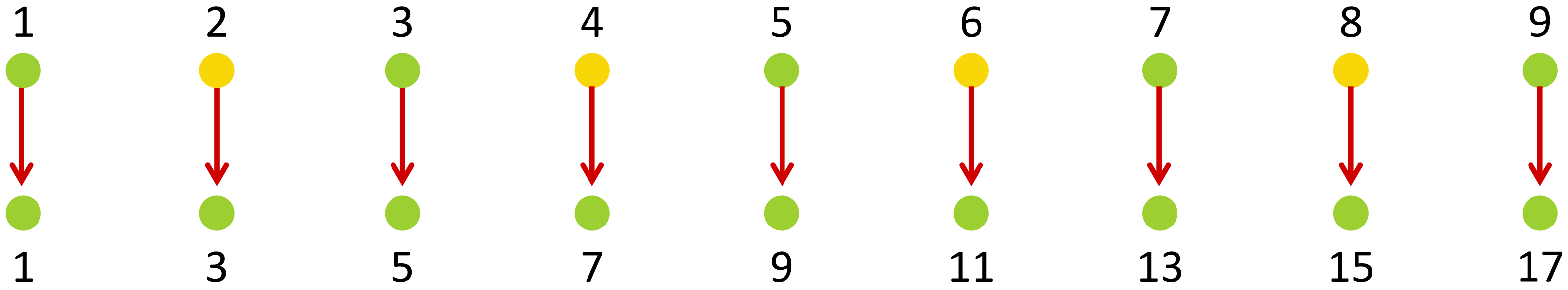
Countable Sets

Countable Sets: Definition

A set that is either finite, or has the same cardinality as the set of positive integers is called **countable**.
A set that is not countable is called **uncountable**.

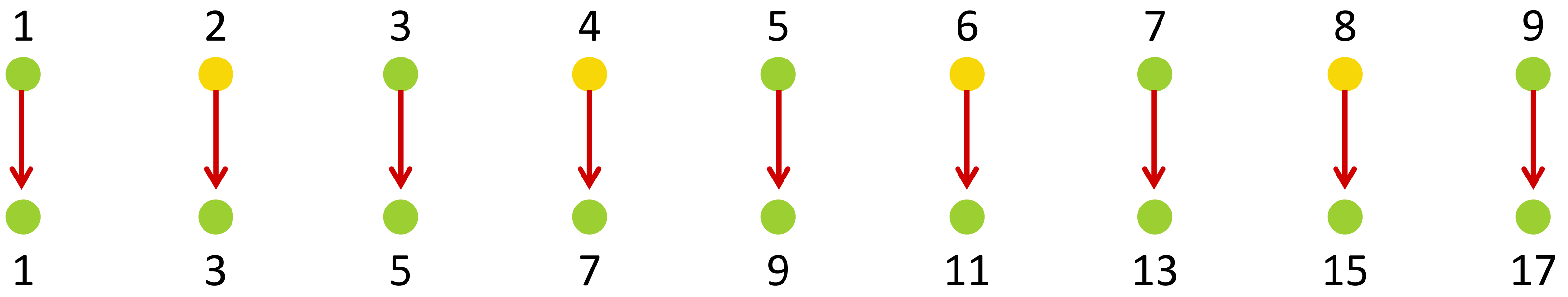
Countable Sets: Example

The set of odd positive integers is a countable set.



- To show that the set of positive odd integers is countable, find a one-to-one correspondence between this set and the set of positive integers.
- Consider the function $f(n) = 2n - 1$.
- $f(n)$ goes from the set of positive integers to the set of odd positive integers.

Countable Sets: Example

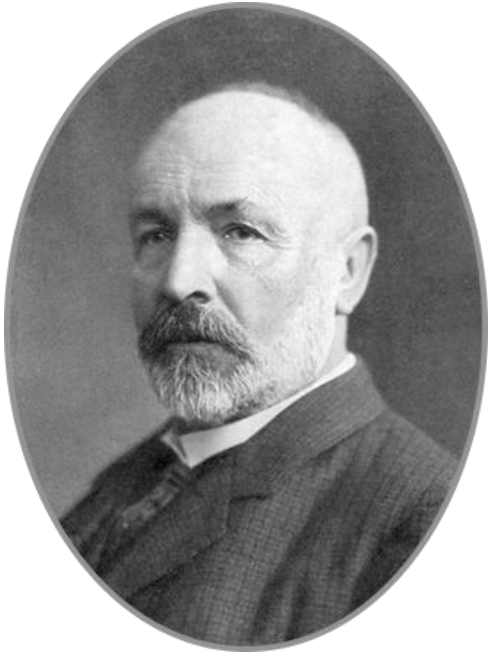


- $f(n)$ is one-to-one: suppose $f(n) = f(m)$, then $2n - 1 = 2m - 1$. Hence, $n = m$.
- $f(n)$ is onto: take m as an odd positive integer. Then m is less than an even integer $2k$ (k a natural number). Thus $m = 2k - 1 = f(k)$.

Countable Sets: An Uncountable Set?

What would be an example of an uncountable set?

- **Real numbers**
- **Proven in 1879 by Cantor**
- **Proof is called “Cantor diagonalisation argument”**
- **Proof method is widely used in the theory of computation**



Georg Ferdinand Ludwig Philipp Cantor 1845 - 1918

Countable Sets: Cantor Diagonalisation

- Suppose that the set of real numbers is countable.
- Then, we will get a contradiction.
- If the set of real numbers is countable, then the set of real numbers that falls between *0* and *1* is also countable.
- Since there is a one-to-one correspondence with positive integers, we can label all of them:

$$r_1, r_2, r_3, \ldots$$

Countable Sets: Cantor Diagonalisation

- **Write these numbers in decimal representation:**

$$r_1 = 0.\, d_{11}\ d_{12}\ d_{13} \ldots$$
$$r_2 = 0.\, d_{21}\ d_{22}\ d_{23} \ldots$$
$$r_3 = 0.\, d_{31}\ d_{32}\ d_{33} \ldots$$

- **Note that all d_{ij} belong to $\{0,1,2,\ldots 9\}$**

- **Form a new real number r with decimal expansion**

$$r = 0.\, d_1\ d_2\ d_3 \ldots$$

where d_i is 5 if $d_{ii} = 4$ and 4 otherwise

Countable Sets: Cantor Diagonalisation

- **The number r is different from all other real numbers listed in the interval $[0,1]$.**
- **This is because r differs from the decimal expansion of r_i in the *ith* place by construction.**
- **We thus found a contradiction to the fact that we are able to list all the real numbers in $[0,1]$, since r does not belong!**

Topic Summary

Let’s recap…

- Ceiling and floor functions
- Pigeonhole principle
- Countable and uncountable sets