MH1812 Tutorial
Chapter 8: Relations

Q1: Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \Leftrightarrow (x - y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Solution: $R = \{(1, 1), (1, 3), (2, 2)\}$, so $R^{-1} = \{(1, 1), (3, 1), (2, 2)\}$ and the matrix representation is:

$$R^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} T & F \\ F & T \\ T & F \end{bmatrix} \end{array}$$

□

Q2: Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \Leftrightarrow x|y$. Compute the matrix of the inverse relation R^{-1}.

Solution: $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$, so $R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$, and the matrix representation is:

$$R^{-1} = \begin{array}{c} \\ 2 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} T & F & F \\ T & T & F \\ T & F & T \end{bmatrix} \end{array}$$

□

Q3: Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x - y)$. Show that if n is odd, then n is related to 1.

Solution: For any odd n, it can be written as $n = 2k + 1$ for some $k \in \mathbb{Z}$, $1 - n = -2k$ a multiple of 2, i.e., $2|(1 - n)$, hence $(1, n) \in R$. □

Q4: This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

$$R = \{(a_1, b_1), (a_1, b_2)\}, \qquad S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

What is the matrix of $S \circ R$?

Solution: $S \circ R = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$, hence,

$$S \circ R = \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} T & T & T \\ F & F & F \end{bmatrix} \end{array}$$

□

2. In general, what is the matrix of $S \circ R$?

   Solution: A matrix M of $|A|$ rows and $|C|$ columns, and $M_{i,j}$ is T iff there exists $b \in B$ such that $(a_i, b) \in R$ and $(b, c_j) \in S$, otherwise F. □

Q5: Consider the relation R on $\mathbb{Z}$, given by $aRb \Leftrightarrow a - b$ divisible by n. Is it symmetric?

Solution: YES. For any $(a, b) \in R$, $n|(a - b)$, i.e., $a - b = nk$ for some $k \in \mathbb{Z}$, so $b - a = -nk$, i.e., $n|(b - a)$, hence $(b, a) \in R$. R is symmetric. □

Q6: Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Solution: For any $(x, y) \in R$, $(y, x) \in R^{-1}$ by definition, R is symmetric $\Leftrightarrow (x, y) \in R$ and $(y, x) \in R \Leftrightarrow R = R^{-1}$. □

Q7: Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

Solution: R is reflexive since $\{(a, a), (b, b), (c, c), (d, d)\} \subseteq R$.
R is symmetric since it appears in pairs:
(a, a)
$(a, b), (b, a)$
$(a, d), (d, a)$
(b, b)
(c, c)
(d, d)
R is NOT transitive, as $(b, a), (a, d) \in R$, but $(b, d) \notin R$. □

Q8: Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R anti-symmetric?

Solution: R is NOT anti-symmetric, since $(0, 2)$ and $(2, 0)$ are both in R and $0 \neq 2$. □

Q9: Are symmetry and antisymmetry mutually exclusive?

Solution:NO. For example, R is defined over set $A = \{a\}$ as $R = \{(a, a)\}$, R is both symmetric and anti-symmetric.
Additional remark on this topic: There are also examples for relation to be neither symmetric nor anti-symmetric, e.g., $A = \{a, b, c, d\}$ and $R = \{(a, b), (b, a), (c, d)\}$. □

Q10: Consider the relation R given by divisibility on positive integers, that is $xRy \Leftrightarrow x|y$. Is this relation reflexive? symmetric? antisymmetric? transitive? What if the relation R is now defined over non-zero integers instead?

Solution: R is reflexive, anti-symmetric, and transitive. When R is defined over non-zero integers, R is reflexive, transitive (it is not longer anti-symmetric). □

Q11: Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Show that the relation $xRy \Leftrightarrow 2|(x - y)$ is an equivalence relation.

Solution: R is reflexive, symmetric, and transitive, hence an equivalence relation, and the equivalence relations are $[0] = \{0, 2, 4, 6, 8\}$ and $[1] = \{1, 3, 5, 7\}$. □

Q12: Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Solution: Let the equivalence classes to be $A_1, A_2, \ldots, A_n$, prove R partition A, we need to prove:

1. $A_1 \cup A_2 \cup \cdots \cup A_n = A$
2. For any $x \in A_i$ and $y \in A_j$ with $i \neq j$, $(x, y) \notin R$

Since for any element in A, it has to belong to one of the equivalence class (if not, this element alone will form a new equivalence class), hence $A_1 \cup A_2 \cup \cdots \cup A_n = A$. To prove 2. we prove by contradiction, i.e., assume there exists $x \in A_i$ and $y \in A_j$ with $i \neq j$, $(x, y) \in R$, then for any element $z \in A_j$, $z \in A_i$ as well, since $(y, z) \in R$ (both y, z in A_j) and $(x, y) \in R$, hence $(x, z) \in R$ (transitive), hence $z \in A_i$ as well, so $A_j \subseteq A_i$, similarly we can prove $A_i \subseteq A_j$, so $A_i = A_j$ which contradicts with $i \neq j$. □

Q13: Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation $xRy \Leftrightarrow \exists c \in \mathbb{Z}, y = cx$. Is R an equivalence relation? is R a partial order?

Solution: Similar with Q11, R is reflexive, anti-symmetric, and transitive, hence a partial order. □