MH1812 Tutorial
Chapter 6: Linear Recurrence

Q1: Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3, a_0 = 0$.

1. Solve it using the backtracking method.
2. Solve it using the characteristic equation.

Solution:

- We have $a_n = 2a_{n-1} - a_{n-2}$, thus $a_{n-1} = 2a_{n-2} - a_{n-3}$, $a_{n-2} = 2a_{n-3} - a_{n-4}$, $a_{n-3} = 2a_{n-4} - a_{n-5}$, etc therefore

$$\begin{aligned}
a_n &= 2a_{n-1} - a_{n-2} \\
&= 2(2a_{n-2} - a_{n-3}) - a_{n-2} = 3a_{n-2} - 2a_{n-3} \\
&= 3(2a_{n-3} - a_{n-3}) - 2a_{n-2} = 4a_{n-2} - 3a_{n-3} \\
&= 4(2a_{n-4} - a_{n-3}) - 3a_{n-2} = 5a_{n-2} - 4a_{n-3} \\
&= \cdots
\end{aligned}$$

  We see that a general term is $(i+1)a_{n-i} - ia_{n-(i+1)}$. Therefore the last term is when $n - i - 1 = 0$ that is $i = n - 1$, for which we have $na_1 - (n-1)a_0$, therefore with initial condition $a_0 = 0$ and $a_1 = 3$, we get

$$a_n = 3n.$$

- The characteristic equation is $x^2 - 2x + 1$ that is

$$(x-1)^2 = 0$$

  We get two same solution $x = 1$, therefore

$$a_n = u \times 1^n + v \times n \times 1^n = u + vn.$$

  The initial conditions are:

$$a_0 = u = 0, a_1 = u + v = 3.$$

  Solving above two equations, we obtain:

$$u = 0, v = 3.$$

  Hence, $a_n = 3n$.

$\square$

Q2: What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of this recurrence is:

$$x^2 = x + 2 \Longleftrightarrow x^2 - x - 2 = (x-2)(x+1) = 0$$

We obtain two distinct solutions $x = 2$, and $x = -1$. Therefore $a_n = u \times 2^n + v \times (-1)^n$. The initial conditions are:

$$a_0 = u + v = 2, a_1 = 2u - v = 7.$$

Solving above results in $u = 3$, and $v = -1$. Therefore

$$a_n = 3 \times 2^n - (-1)^n.$$

$\square$

Q3: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n,a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Solution:

$$\begin{aligned} a_n + a'_n &= (c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) + (c_1 a'_{n-1} + c_2 a'_{n-2} + \ldots + c_k a'_{n-k}) \\ &= c_1(a_{n-1} + a'_{n-1}) + c_2(a_{n-2} + a'_{n-2}) + \ldots + c_k(a_{n-k} + a'_{n-k}) \end{aligned}$$

Thus $a_n + a'_n$ is a solution of the recurrence. Similarly,

$$\begin{aligned} \alpha a_n &= \alpha(c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) \\ &= c_1 \alpha a_{n-1} + c_2 \alpha a_{n-2} + \ldots + c_k \alpha a_{n-k} \end{aligned}$$

Therefore αa_n is a solution of the recurrence. $\square$

Q4: Solve the following two recurrence relations:

$$a_n = 3a_{n-1}, a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2}, b_1 = 0, b_2 = 12.$$

Solution:

- $a_n = 3a_{n-1} = 3 \times 3 \times a_{n-2} = \cdots = 3^{n-1}a_1 = 4 \times 3^{n-1}$.
- The characteristic equation is $x^2 = 4x - 3 \Longleftrightarrow x^2 - 4x + 3 = (x-3)(x-1) = 0$. Solving it results in two distinct solutions $x = 3$, and $x = 1$. Therefore $b_n = u3^n + v1^n = u3^n + v$. The initial conditions are:

$$b_1 = 3u + v = 0, b_2 = 9u + v = 12$$

  Solving above results in $u = 2$, and $v = -6$, Hence $b_n = 2 \times 3^n - 6$.

□

Q5: Solve the following linear recurrence relation:

$$b_n = 4b_{n-1} - 4b_{n-2}, b_0 = 2, b_1 = 4.$$

Solution: The characteristic equation is $x^2 = 4x - 4 \Longleftrightarrow x^2 - 4x + 4 = (x-2)^2 = 0$. Solving it results in two same solutions $x = 2$. Therefore $b_n = u2^n + vn2^n$. The initial conditions are:

$$b_0 = u = 2, b_1 = 2u + 2v = 4$$

Solving above results in $u = 2$, and $v = 0$, Hence $b_n = 2 \times 2^n = 2^{n+1}$. □