NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 9.1 - Functions I

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Topic Overview

What's in store…

- **I**ntroduction to Functions
- **I**njectivity
- **S**urjectivity

By the end of this lesson, you should be able to…

- Explain the concepts of functions.
- Explain the concepts of injective functions.
- Explain the concepts of surjective functions.

Introduction to Functions

Introduction to Functions: Definition

Let X and Y be sets. A **function** f from X to Y is a rule that assigns every element x of X to a unique y in Y. We write $f: X \rightarrow Y$ and $f(x) = y$.

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

X =	Domain
Y =	Codomain
y =	**Image** of x under f
x =	**Preimage** of y under f
Range =	Subset of Y with preimages

Introduction to Functions: Example 1

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

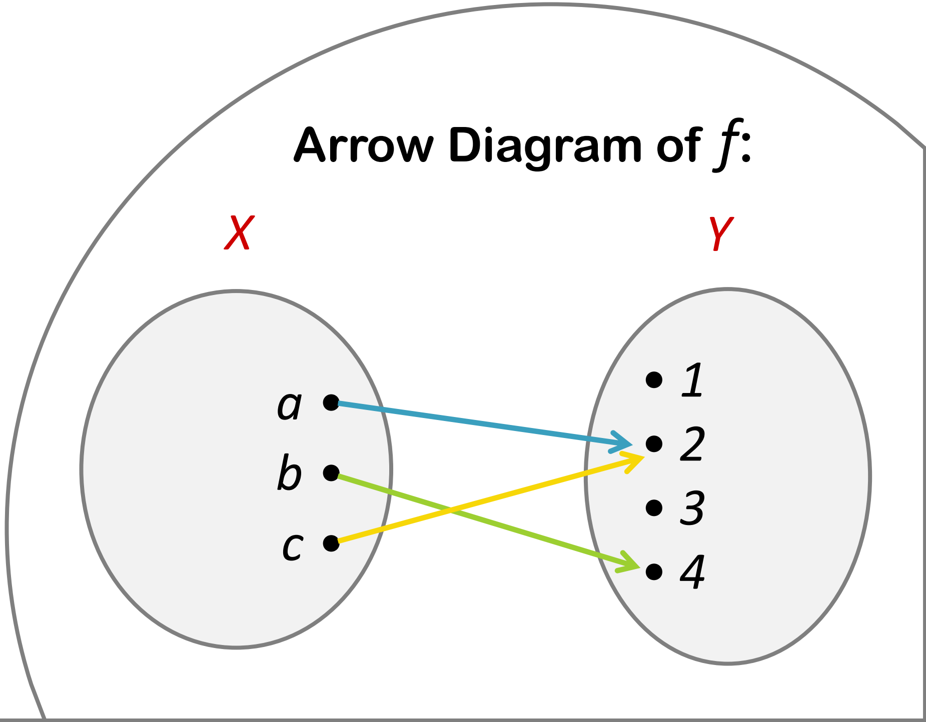
Domain $X = \{a,b,c\}$

Codomain $Y = \{1,2,3,4\}$

$f = \{(a,2), (b,4), (c,2)\}$

Preimage of 2 is $\{a,c\}$

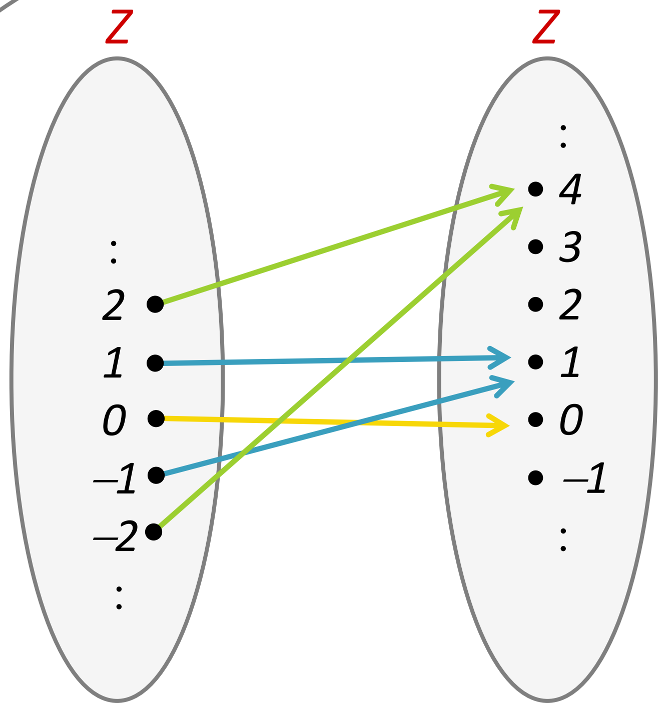
Range = $\{2,4\}$



Introduction to Functions: Example 2

Let f be the function from Z to Z that assigns the square of an integer to this integer.

Then $f: Z \rightarrow Z, f(x) = x^2$



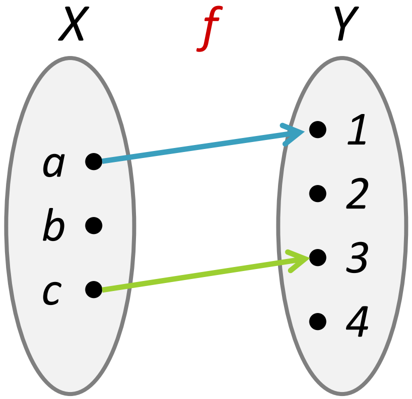
Domain and codomain of f: Z

Range $(f) = \{0, 1, 4, 9, 16, 25, \ldots.\}$

Introduction to Functions: Functions vs. Non-functions

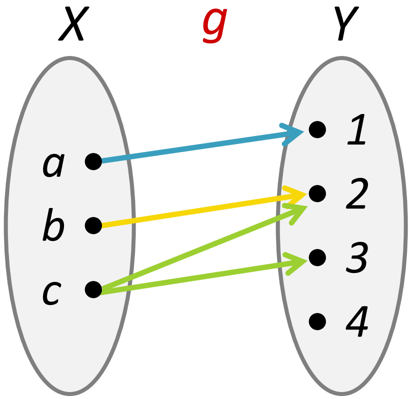
$$(\forall x \in X\ \exists y \in Y, y = f(x)) \land (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

X = {a,b,c} **to** Y = {1,2,3,4}



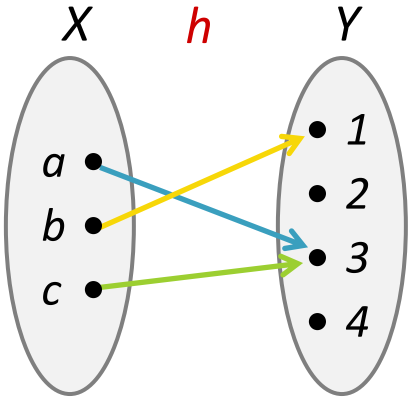
No!

(*b* has no image)



No!

(*c* has two images)

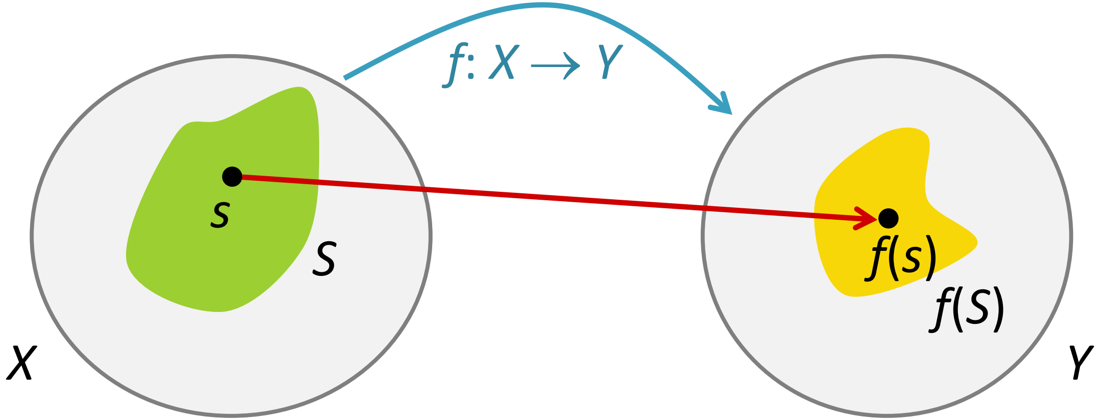


Yes!

(Each element of *X* has exactly one image)

Introduction to Functions: Image of a Set

Let f be a function from X to Y and $S \subseteq X$. The **image of S** is the subset of Y that consists of the images of the elements of S: $f(S) = \{f(s) \mid s \in S\}$.



Injectivity

Injectivity: One-to-one Function

A function f is **one-to-one** (or **injective**), if and only if $f(x) = f(y)$ implies $x = y$ for all x and y in the domain of f.

In words…

"All elements in the domain of f have different images".

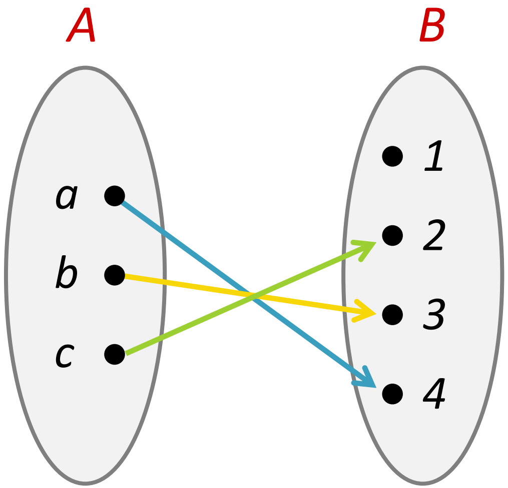
Mathematical Description

$$f\colon A \to B \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \Longrightarrow x_1 = x_2)$$

or

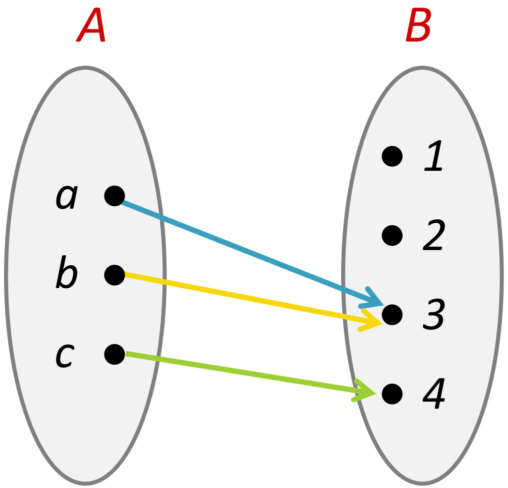
$$f\colon A \to B \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in A\ (x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2))$$

Injectivity: One-to-one Example



One-to-one

(All elements in A have a different image)

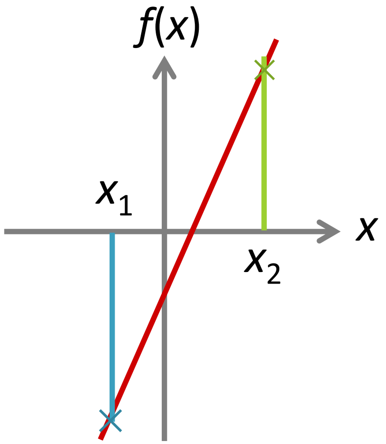


Not one-to-one

(a and b have the same image)

Injectivity: One-to-one Example

$f: R \to R, f(x) = 4x - 1$



Does each element in R have a different image?

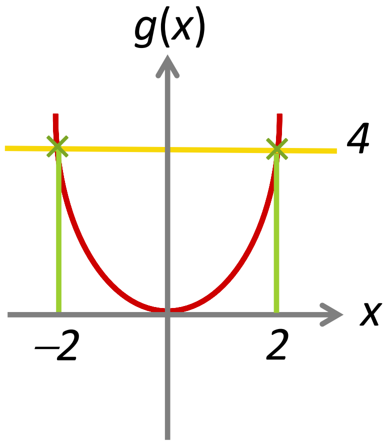
Yes!

To show $\forall x_1, x_2 \in R\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$,

take some $x_1, x_2 \in R$ **with** $f(x_1) = f(x_2)$.

Then $4x_1 - 1 = 4x_2 - 1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$.

$g: R \to R, g(x) = x^2$



No!

Take $x_1 = 2$ **and** $x_2 = -2$.

Then $g(x_1) = 2^2 = 4 = g(x_2)$ **and** $x_1 \neq x_2$.

Surjectivity

Surjectivity: Onto Function

$f(x)$

A function f from X to Y is **onto** (or **surjective**), if and only if for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$.

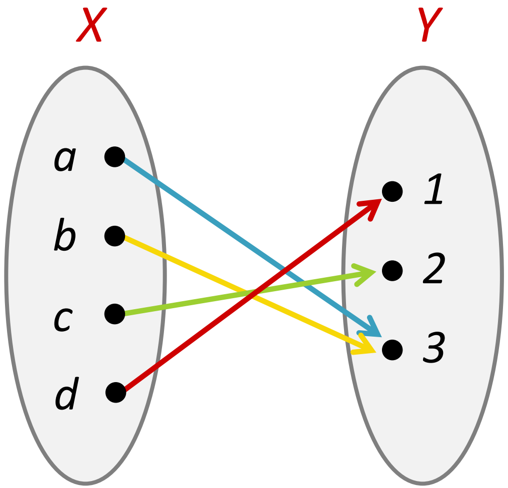
In words…

"Each element in the codomain of f has a preimage".

Mathematical Description

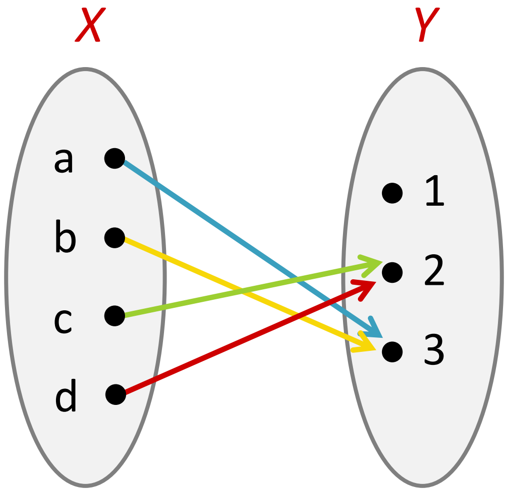
$f: X \rightarrow Y$ is onto $\Leftrightarrow \forall y \in Y \; \exists x \in X, \;\; f(x) = y$

Surjectivity: Onto Example



Onto

(All elements in *Y* have a preimage)



Not onto

(*1* has no preimage)

Surjectivity: Onto Example

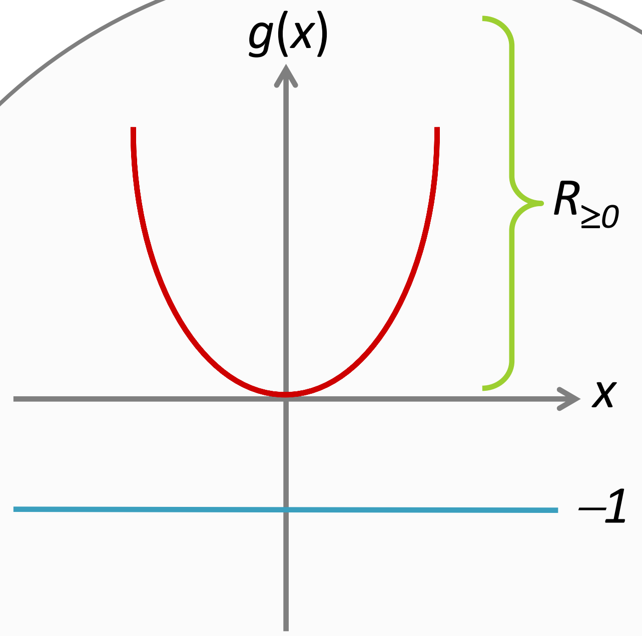
$g: R \to R,\ g(x) = x^2$

Does each element in *R* have a preimage?

No!

- **To show** $\exists y \in R$ **such that** $\forall x \in R\ g(x) \neq y$
- **Take** $y = -1$
- **Then any** $x \in R$ **holds** $g(x) = x^2 \neq -1 = y$

But $g{:}R \to R_{\geq 0}$, $g(x) = x^2$ **(where** $R_{\geq 0}$ **denotes the set of non-negative real numbers) is onto!**



Topic Summary

Let's recap…

- Functions:
  - Domain
  - Codomain
  - Image
  - Preimage
  - Range

- Injective functions (one-to-one)

- Surjective functions (onto)