MH1812 Tutorial
Chapter 2: Propositional Logic

Q1: Decide whether the following statements are propositions. Justify your answer.

1. 2+2 = 5.
   Solution: Yes, because this statement always takes the truth value "false". □
2. 2+2 = 4.
   Solution: Yes, because this statement always takes the truth value "true". □
3. x = 3.
   Solution: No, because this statement can be "true" when x is 3 and "false" when x is not 3. □
4. Every week has a Sunday.
   Solution: Yes, because this statement always takes the truth value "true". □
5. Have you read "Catch 22" ?
   Solution: No, because the truth value depends on who is answering the question. □

Q2: Show that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

This is the second law of De Morgan.

Solution: We show the equivalence using truth tables:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Since both truth tables are the same, the two logical expressions are equivalent. □

Q3: Show that second absorption law $p \wedge (p \vee q) \equiv p$ holds.

Solution: We show the equivalence using a truth table:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Since the columns of p and $p \wedge (p \vee q)$ are identical, so these two logical expressions are equivalent. □

Q4: These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
   Solution:

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \wedge (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

□

2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.
   Solution:

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$p \wedge r$	$q \wedge r$	$(p \wedge r) \vee (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

□

Q5: Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
  Solution:

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p \wedge \neg q$	$\neg(p \vee \neg q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

□

- developing a series of logical equivalences.

  Solution:

$$
\begin{aligned}
\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q) \text{ DeMorgan} \\
&\equiv \neg p \wedge (q \vee \neg q) \text{ Distributivity} \\
&\equiv \neg p \wedge T \text{ since } (q \vee \neg q) \equiv T \\
&\equiv \neg p.
\end{aligned}
$$

□

Q6: Using a truth table, show that:

$$\neg q \to \neg p \equiv p \to q.$$

Solution:

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

□

Q7: Show that $p \vee q \to r \equiv (p \to r) \wedge (p \to r)$.

Solution:

$$
\begin{aligned}
p \vee q \to r &\equiv (p \vee q) \to r \text{ precedence} \\
&\equiv \neg(p \vee q) \vee r \text{ conversion theorem} \\
&\equiv (\neg p \wedge \neg q) \vee r \text{ De Morgan} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) \text{ Distributivity} \\
&\equiv (p \to r) \wedge (p \to r) \text{ conversion theorem}
\end{aligned}
$$

$\square$

Q8: Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements ?

Solution: They are not equivalent. Here is a proof using truth table:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \vee (q \to r)$	$p \to r$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

We can see that the second row are giving different truth values, for example. This can be done using equivalences as well:

$$\begin{aligned}(p \to q) \vee (q \to r) &\equiv (\neg p \vee q) \vee (\neg q \vee r) \text{ conversion theorem}\\ &\equiv \neg p \vee r \vee T \text{ since } \neg q \vee q \equiv T\\ &\equiv T\end{aligned}$$

Since $p \to r$ is not equivalent to T, both statements cannot be equivalent. $\square$

Q9: Show that this argument is valid:

$$\neg p \to F; \therefore p.$$

Solution: The premise is $\neg p \to F \equiv p \vee F$, which is true only when p is true. $\square$

Q10: Show that this argument is valid, where C denotes a contradiction.

$$\neg p \to C; \therefore p.$$

Solution: The premise is $\neg p \to C \equiv p \vee C$, which is true only when p is true. $\square$

Q11: Determine whether the following argument is valid:

$$\begin{aligned}&\neg p \to r \wedge \neg s\\ &t \to s\\ &u \to \neg p\\ &\neg w\\ &u \vee w\\ &\therefore t \to w.\end{aligned}$$

Solution: We start by noticing that we have

$$u \vee w; \neg w; \therefore u.$$

Indeed, if $u \vee w$ and $\neg w$ are both true, then w is false, and u must be true (case elimination). Next

$$u \to \neg p; u; \therefore \neg p.$$

Indeed, if $u \to \neg p$ is true, either u is true and $\neg p$ is true, or u is false. But u is true, thus $\neg p$ is true (Modus Ponens). Then

$$\neg p \to r \wedge \neg s; \neg p; \therefore r \wedge \neg s,$$

this is again Modus Ponens. Then

$$r \wedge \neg s; \therefore \neg s.$$

Indeed, for $r \wedge \neg s$ to be true, it must be that $\neg s$ is true. Finally,

$$t \to s; \neg s; \therefore \neg t$$

Since for $t \to s$ to be true, we need either t to be false, or t and s to be true, but since s is false, t must be false (Modus Tollens), and

$$\neg t \therefore \neg t \vee w$$

or equivalently

$$\neg t \vee w \equiv t \to w$$

using the Conversion theorem, which shows that the argument is valid. □

Q12: Determine whether the following argument is valid:

$$\begin{aligned}
&p \\
&p \vee q \\
&q \to (r \to s) \\
&t \to r \\
&\therefore \neg s \to \neg t.
\end{aligned}$$

Solution: For this question, there is no obvious way to combine the known statements with inference rules. The only 2 related statements are p and $p \vee q$, and assuming that both are true, all can be deduced is that q is either true or false (this gives no information about q at all). Now if q is false, $q \to (r \to s)$ is always true, while if q is true, $q \to (r \to s)$ is true only if $(r \to s)$ is true, which excludes the possibility

$r = T$ and $s = F$. Now we look at the last premise $t \to r$. For it to be true, we need t false, or t true and r true. If s is true, then $\neg s$ is always false, and the conclusion is always true. We thus focus on s is false, and $\neg t$ is false, that is t is true. So we have a counter-example (which makes all premises true and conclusion false):

$$q = F, r = T, s = F, t = T.$$

One can also draw the truth table, and find the counter-example from the critical rows.
$\square$