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Fundamentals of Waveforms

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Table of contents

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苟日新，日日新，又日新。

——《礼记・大学》

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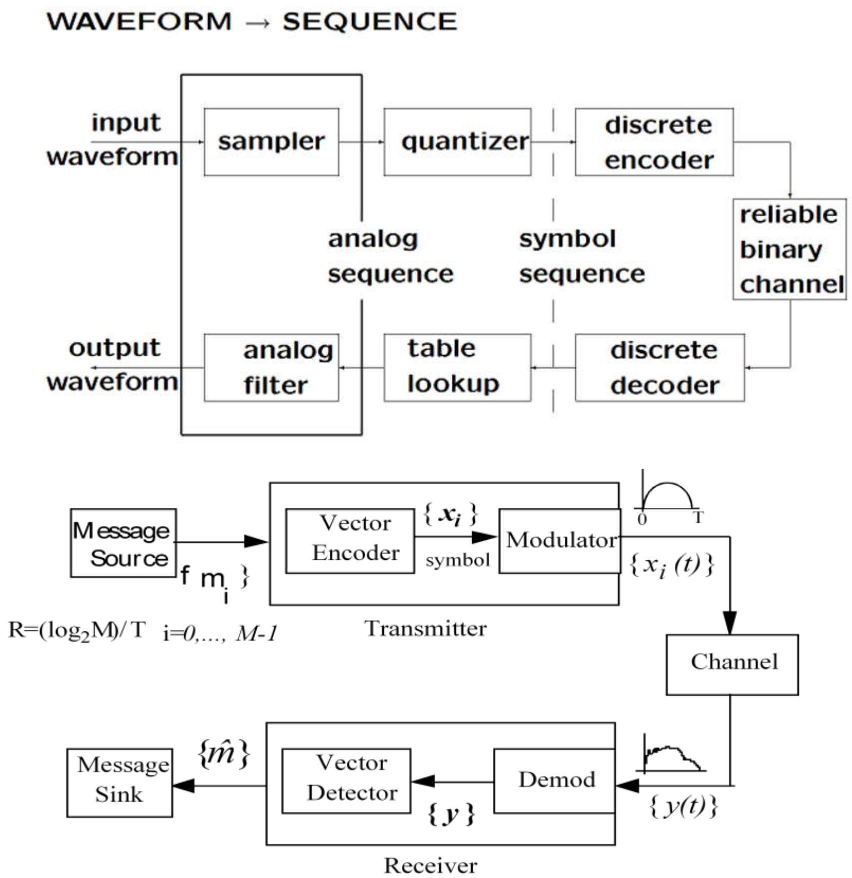
Introduction

- Successive transmission of discrete data messages is known as ***digital communication***.
- Binary logic familiar to most electrical engineers transmits some positive voltage level for a 1 and another voltage level for a 0 inside integrated circuits. Clearly such 1/0 transmission would not pass through a ***linear time-invariant channel***(that has the **Fourier transform indicated**). Instead the two modulated signals $x0(t)=+\cos(2\pi t)$ and $x1(t)=-\cos(2\pi t)$ (***B***inary ***P***hase-***S***hift ***K***eying) will easily pass through this channel and be readily distinguishable at the channel output.
- For the antenna example, a real waveform at the input in the appropriate frequency band is converted by the input antenna into electromagnetic radiation, part of which is received at the receiving antenna and converted back to a waveform.


WAVEFORM → SEQUENCE
input waveform
sampler
quantizer
discrete encoder
reliable binary channel
analog sequence
symbol sequence
output waveform
analog filter
table lookup
discrete decoder
Message Source
Vector Encoder
{ x_i }
symbol
Modulator
0
T
{ $x_i(t)$ }
{ m_i }
R=($\log_2$M)/ T
i=0,..., M-1
Transmitter
Channel
Message Sink
{ $\hat{m}$ }
Vector Detector
{ y }
Demod
{ y(t) }
Receiver

- The function of a channel encoder, i.e., a modulator, is to convert the incoming sequence of binary digits into a waveform in such a way that the noise corrupted waveform at the receiver can, with high probability, be converted back into the original binary digits.
- This is typically done by first converting the binary sequence into a sequence of analog signals, which are then converted to a waveform.
- ***Waveforms*** denoted as arbitrarily varying real or complex valued functions of time.
- An individual waveform from an analog source should be viewed as a ***sample waveform*** from a random process. These waveforms are a ***priori unknown***, so much mathematical precision is necessary here.
- Here the focus is on ways to map ***deterministic waveforms to sequences*** and vice versa.

Finite energy waveforms

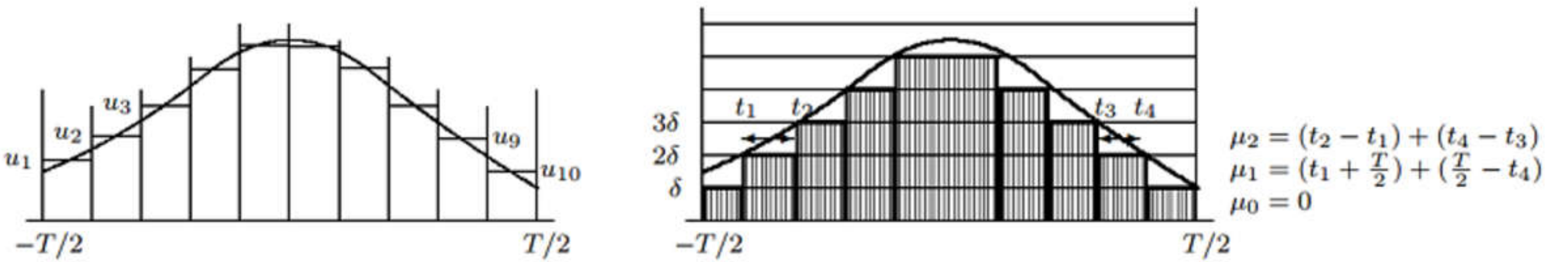
The energy in a real or complex waveform $u(t)$ is defined to be $\int_{-\infty}^{\infty} |u(t)|^2 \, dt$.

- The energy used over any finite interval T is limited both by regulatory agencies and by physical constraints on transmitters and antennas.
- Finite-energy waveforms have *measurability* properties, These finite-energy measurable functions are called L^2 functions. When time-constrained, they always have Fourier series, and without a time constraint, they always have Fourier transforms.
- Perhaps the most important property, however, is that L^2 functions can be treated essentially as conventional vectors.
- A major reason for restricting attention to finite-energy waveforms is that as their energy gets used up in different degrees of freedom (i.e., expansion coefficients), there is less energy available for other degrees of freedom, so that some sort of convergence must result.
- Unit impulses and constant functions are not physical waveforms, they are useful models of physical waveforms where energy is not important. However, that such waveforms can safely be limited to the finite-energy class.

Lebesgue Integration and L2 functions

- Every piecewise continuous function is *Riemann integrable*. Lebesgue integral can handle a very large class of functions, including all the Riemann integrable functions, but also even very discontinuous functions. In fact, it's safe to say that it can integrate any function that one actually needs in mathematics in real-life applications.
- The passage from Riemann's theory of integration to that of Lebesque is a process of completion. It is of the same fundamental importance in analysis as is the construction of the real number system from the rationals.
- Whenever the Riemann integral exists (i.e., the limit exists), the Lebesgue integral also exists and has the same value. The familiar rules for calculating Riemann integrals also apply for Lebesgue integrals.
- For some very weird functions, the Lebesgue integral exists, but the Riemann integral does not. There are also exceptionally weird functions for which not even the Lebesgue integral exists.

> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral.
>
> ——Lebesgue summarized his approach to integration in a letter to Paul Montel



$\int_{-T/2}^{T/2} u(t)\,dt \approx \sum_{i=1}^{i_0} u_i/i_0$

(a): Riemann

$\int_{-T/2}^{T/2} u(t)\,dt \approx \sum_m m\delta\,\mu_m$

(b): Lebesgue

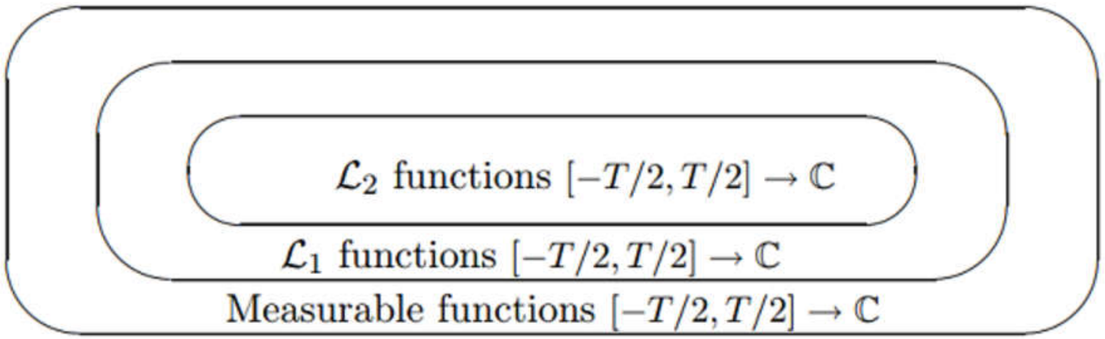
L^2 Functions over $[-T/2, T/2]$

L^1 functions

A function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is said to be L^1, or in the class L^1, if $u(t)$ is measurable and the Lebesgue integral of $|u(t)|$ is finite.
L^1 functions are sometimes called integrable functions.

L^2 functions

A function $\{u(t) : [-T/2, T/2] \to \mathbb{R}\}$ or $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is said to be an L^2 function, or a *finite-energy* function, if $u(t)$ is measurable and the Lebesgue integral of $|u(t)|^2$ is finite.
If $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is L^2, then it is also L^1.



Linear Time-Invariant System

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If $O\{x(t)\} = y(t)$,
Linearity(Additivity and Homogeneity) :

$$O\{a_1x_1(t) + a_2x_2(t)\} = a_1O\{x_1(t)\} + a_2O\{x_2(t)\} = a_1y_1(t) + a_2y_2(t)$$

Time-invariance :

$$O\{x(t-\tau)\} = O\{y(t-\tau)\}$$

Eigenfunctions

If the output signal $O\{x(t)\}$ is a scalar λ multiple of the input signal $x(t)$, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.

$$O\{x(t)\} = \lambda x(t)$$

If $x(t) = e^{st}$ and $h(t)$ is the impulse response of LTI then $O\{e^{st}\} = (h * x)(t) = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-st}d\tau = H(s)e^{st}$.
Furthermore, the eigenvalue associated with e^{st} is $H(s)$.

Why play a fundamental role

- Many communication channels possess LTI property.
- We hope the channels can be approximated as LTI or extended from LTI in a range of applications.

More explanation about the sencond item:

- Unit-sample response $h[n]$ or unit-impulse response $h(t)$ is a complete description of an LTI system.
- Many systems are naturally described by their frequency response; LTI systems can be characterized by responses to eternal sinusoids; frequency response is easy to calculate from the system function.
- Sinusoidal and complex exponential signals are used to describe the characteristics of many physical processes, in particular physical systems in which energy is conserved. Periodic complex exponentials serve as extremely useful building blocks for many other signals(harmonic). If the input to an LTI system is expressed as a linear combination of periodic complex exponentials or sinusoids, the output can also be expressed in this form, with coefficients that are related in a straightforward way to those of the input.
- For many physical channels, they introduce distortions in their passbands, such a channel can be modeled by an LTI filter followed by AWGN noise, the approach to remove ISI(inter-symbol interference) is usually known as equalization.
- Linear operations preserve Gaussianity.
- A large class of interesting functions(L^2) could be represented by linear combinations of complex exponentials(Fourier Theorem).

Fourier series

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Given in most engineering texts

The *Fourier series* for function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ is given by

$$u(t) = \begin{cases} \sum\limits_{-\infty}^{\infty} \hat{u}_k e^{2\pi ikt/T} & T/2 \leq t \leq T/2 \\ 0 & elsewhere \end{cases}$$

Electrical engineers formerly reserved the symbol i for electrical current and thus often use j to denote $\sqrt{-1}$. The Fourier series of a time-limited function maps function to a sequence of complex coefficients $\hat{u}_k$ satisfy

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi ikt/T} dt, \ -\infty < k < \infty \tag{1}$$

For any integer n, the functions $cos(2\pi nx), sin(2\pi nx), e^{2\pi inx}$ are all $\mathbb{Z}$-*periodic*(1-periodic). So in some math book the $\hat{u}_k$ often denoted as:

$$\hat{u}_k = \int_0^1 u(t) e^{-2\pi ikt} dt, -\infty < k < \infty$$

$u(t)$ can be expressed as a linear combination of truncated complex sinusoids by the standard rectangular function as follows:

$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi ikt/T} rect(t/T) = \sum_{k\in(Z)} \hat{u}_k \theta_k(t) \tag{2}$$

where

$$rect(t) = \begin{cases} 1 & -1/2 \leq t \leq 1/2 \\ 0 & elsewhere \end{cases} \quad , \quad and\ \theta_k(t) = e^{2\pi ikt/T} rect(t/T)$$

Fourier series energy equation

$$\begin{aligned} \int_{-\infty}^{\infty} |u(t)|^2 d(t) &= \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \hat{u}_k e^{2\pi ikt/T} \sum_{\ell=-\infty}^{\infty} \hat{u}_\ell^* e^{2\pi i\ell t/T} dt \\ &= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \hat{u}_k \hat{u}_\ell^* \int_{-T/2}^{T/2} e^{2\pi i(k-\ell)t/T} dt \\ &= T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \hat{u}_k \hat{u}_\ell^* \delta[k-\ell] \\ &= T \sum_{k=-\infty}^{\infty} |\hat{u}_k|^2 \end{aligned}$$

Complex Exponentials

A *complex-valued function* of the real variable x may be written as $u(x) + iv(x)$ (u,v real valued), its derivative and integral with respect to x are defined to be

$$\frac{d}{dx}(u + iv) = \frac{du}{dx} + i\frac{dv}{dx}\ , \quad and \int (u + iv)dx = \int udx + i\int udx$$

From this it follows easily that

$$\frac{d}{dx}(e^{(a+ib)x}) = (a + ib)e^{(a+ib)x}\ , \quad and \int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x}$$

The truncated complex sinusoids are orthogonal for $k \neq m \in \mathbb{Z}$

$$\int_{-\infty}^{\infty} \theta_k(t)\theta_m^*(t)d(t) = \int_{-T/2}^{T/2} e^{2\pi i(k-m)t/T} = \frac{T}{2\pi i(k-m)}e^{2\pi i(k-m)t/T}\Big|_{-T/2}^{T/2} = T\frac{sin\pi(k-m)}{\pi(k-m)}$$

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Fourier Theorem

More precision and interpretation

Let $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ be an L^2 function. Then for each $k \in \mathbb{Z}$, the Lebesgue integral

$$\hat{u}_k = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-2\pi ikt/T}dt, \ -\infty < k < \infty$$

exists and satisfies $|\hat{u}_k| \leq \frac{1}{T}\int |u(t)|dt < \infty$. Furthermore,

$$\lim_{\ell\to\infty}\int_{-T/2}^{T/2}\left|u(t) - \sum_{k=-\ell}^{\ell}\hat{u}_k e^{2\pi ikt/T}\right|^2 dt = 0$$

where the limit is monotonic in ℓ. Also, the Fourier energy equation (3) is satisfied.

Conversely, if $\{\hat{u}_k; k \in \mathbb{Z}\}$ is a two-sided sequence of complex numbers satisfying $\sum_{k=-\infty}^{\infty}|\hat{u}_k|^2 \leq \infty$, then an L^2 function $\{u(t) : [-T/2, T/2] \to \mathbb{C}\}$ exists such that (3) and (4) are satisfied.

There is an important theorem due to Carleson, stating that if $u(t)$ is L^2, then $\sum_k \hat{u}_k e^{2\pi ikt/T} rect(t/T)$ converges almost everywhere(convergence with probability 1) on $[-T/2, T/2]$.

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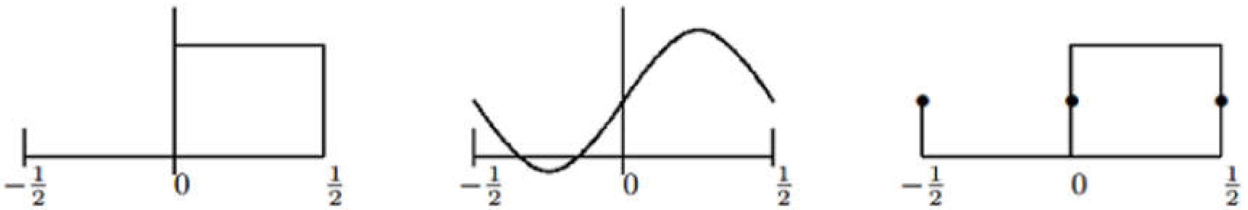
L^2 converge

A series is defined to converge in L^2 if (4) holds. The notation $l.i.m.$ (limit in mean-square)is used to denote L^2 convergence, so (4) is often abbreviated by

$$u(t) = l.i.m. \sum_k \hat{u}_k e^{2\pi ikt/T} rect(t/T)$$

The following example illustrate there is isolated discontinuity at $t = -1/2, 0, 1/2$, the middle figure depicts a partial expansion with k = −1,0,1.



$v(t) = \mathrm{rect}(2t - \frac{1}{4})$ $\quad$ $\frac{1}{2} + \frac{2}{\pi}\sin(2\pi t)$ $\quad$ $\sum_{k=-\infty}^{\infty} v_k e^{2\pi ikt}\mathrm{rect}(t)$

Fourier Transform and L2 Waveforms

The Fourier transform and its inverse are defined by

$$\hat{u}(f) = \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} dt \quad u(t) = \int_{-\infty}^{\infty} \hat{u}(f) e^{2\pi i f t} df$$

The first integral exists for all f, second exists for all t.
If we use $\omega = 2\pi f$, these integral become

$$\hat{u}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) e^{i\omega t} d\omega$$

Some book denote as $\hat{u}(j\omega)$, that's in the view of systems, for set $s = j\omega$ then Laplace transform becomes Fourier transform; also frequency response lives on the $j\omega$ axis of the Laplace transform.

The Fourier transform maps a function of time $\{u(t) : R \to C\}$ into a function of frequency $\{\hat{u}(f) : R \to C\}$. The Fourier transform does not exist for all functions, and when the Fourier transform does exist, there is not necessarily an inverse Fourier transform. L^2 functions always have Fourier transforms, but only in the sense of L^2 equivalence.

A Few Standard Fourier Transform Pair

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Two useful special cases of any Fourier transform pair are:

$$u(0) = \int_{-\infty}^{\infty} \hat{u}(f)df \quad \hat{u}(0) = \int_{-\infty}^{\infty} u(t)dt$$

Parseval's theorem:

$$\int_{-\infty}^{\infty} u(t)v^*(t)dt = \int_{-\infty}^{\infty} \hat{u}(f)\hat{v}^*(f)df$$

Energy equation(replacing $v(t)$ by $u(t)$):

$$\int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{u}(f)|^2 df$$

$|\hat{u}(f)|^2$ is called the **spectral density** of u(t).

$$
\begin{array}{rcll}
au(t)+bv(t) & \leftrightarrow & a\hat{u}(f)+b\hat{v}(f) & \text{linearity} \\
u^*(-t) & \leftrightarrow & \hat{u}^*(f) & \text{conjugation} \\
\hat{u}(t) & \leftrightarrow & u(-f) & \text{time/frequency duality} \\
u(t-\tau) & \leftrightarrow & e^{-2\pi i f\tau}\hat{u}(f) & \text{time shift} \\
u(t)\,e^{2\pi i f_0 t} & \leftrightarrow & \hat{u}(f-f_0) & \text{frequency shift} \\
u(t/T) & \leftrightarrow & T\,\hat{u}(fT) & \text{scaling (for } T>0) \\
du(t)/dt & \leftrightarrow & 2\pi i f\hat{u}(f) & \text{differentiation} \\
\int_{-\infty}^{\infty} u(\tau)v(t-\tau)\,d\tau & \leftrightarrow & \hat{u}(f)\hat{v}(f) & \text{convolution} \\
\int_{-\infty}^{\infty} u(\tau)v^*(\tau-t)\,d\tau & \leftrightarrow & \hat{u}(f)\hat{v}^*(f) & \text{correlation} \\
\operatorname{sinc}(t)=\dfrac{\sin(\pi t)}{\pi t} & \leftrightarrow & \operatorname{rect}(f)=\begin{cases} 1 & \text{for} \quad |f|\le 1/2 \\ 0 & \text{for} \quad |f|>1/2 \end{cases} & \\
e^{-\pi t^2} & \leftrightarrow & e^{-\pi f^2} & \\
e^{-at};\ t\ge 0 & \leftrightarrow & \dfrac{1}{a+2\pi i f} \quad \text{for } a>0 & \\
e^{-a|t|} & \leftrightarrow & \dfrac{2a}{a^2+(2\pi i f)^2} \quad \text{for } a>0 &
\end{array}
$$

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Fourier transforms of L^1 functions

L^1 functions always have well-defined Fourier transforms, but the inverse transform does not always have very nice properties.

Let $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ be L^1. Then $\hat{u}(f) = \int_{-\infty}^{\infty} u(t)e^{-2\pi ift}dt$ both *exists and satisfies* $|\hat{u}(f)| \leq \int |u(t)|dt$ for each $f \in \mathbb{R}$. Furthermore, $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ is a *continuous* function of f.

Not enough functions are L^1 to provide suitable models for communication systems. For example, $sinc(t)$ is not L^1.

Also, functions with discontinuities cannot be Fourier transforms of L^1 functions.

Finally, L^1 functions might have infinite energy. L^2 functions turn out to be the “right” class.

Fourier transforms of L^2 functions

For any L^2 function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ and any positive number A, define $\hat{u}_A(f)$ as the Fourier transform of the truncation of $u(t)$ to $[-A, A]$,

$$\hat{u}_A(f) = \int_{-A}^{A} u(t)e^{-2\pi ift}dt$$

Plancherel part 1

For any L^2 function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$, an L^2 function $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ exists satisfying both

$$\lim_{A \to \infty} \int_{-\infty}^{\infty} |\hat{u}(f) - \hat{u}_A(f)|^2 df = 0$$

and the energy function.

For any L^2 function $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ and any positive number B, define the inverse transform

$$u_B(t) = \int_{-B}^{B} \hat{u}(f) e^{2\pi i f t} df$$

Plancherel part 2

For any L^2 function $\{u(t) : \mathbb{R} \to \mathbb{C}\}$, let $\{\hat{u}(f) : \mathbb{R} \to \mathbb{C}\}$ be the Fourier transform of Plancherel part 1. Then

$$\lim_{B \to \infty} \int_{-\infty}^{\infty} |u(t) - u_B(t)|^2 dt = 0$$

All L^2 functions have Fourier transforms in the sense of limit in mean-square equivalent (L^2 equivalent).

$$\hat{u}(f) = \underset{A \to \infty}{\mathrm{l.i.m.}} \int_{-A}^{A} u(t) e^{-2\pi i f t} dt; \qquad u(t) = \underset{B \to \infty}{\mathrm{l.i.m.}} \int_{-B}^{B} \hat{u}(f) e^{2\pi i f t} df$$

All the Fourier transform relations in the above picture except differentiation hold for all L^2 functions.

The DTFT and the Sampling Theorem

The discrete-time Fourier transform (**DTFT**) is the time/frequency dual of the Fourier series. In the sense of L^2 convergence, Fourier series uses the sequence of coefficients to represent the function, **DTFT** uses the frequency function to represent the sequence. **DTFT** is similar to modulation from discrete to waveform.

Familiar DTFT expression

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega n})e^{j\omega n}d\omega \qquad \text{(synthesis)}$$

$$X(\Omega) = X(e^{j\omega n}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad \text{(analysis)}$$

DTFT

Assume $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ is L^2 (and thus also L^1). The **DTFT** of $\hat{u}(f)$ over $[-W, W]$ is then defined by

$$\hat{u}(f) = l.i.m. \sum_k u_k e^{-2\pi i k f/(2W)} rect(f/2W)$$

where the **DTFT** coefficients $\{u_k; k \in \mathbb{Z}\}$ are given by

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k f/(2W)} df \qquad (1)$$

We also write this as

$$\hat{u}(f) = l.i.m. \sum_k u_k \hat{\phi}_k(f) \qquad (2)$$

where,

$$\hat{\phi}_k(f) = e^{-2\pi i k f/(2W)} rect(f/2W)$$

DTFT theorem

Let $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ be an L^2. Then for each $k \in \mathbb{Z}$, the Lebesgue integral (1) exists and satisfies $|u_k| \leq \frac{1}{2W} \int |\hat{u}(f)| df < \infty$. Furthermore,

$$\lim_{\ell \to \infty} \int_{-W}^{W} |\hat{u}(f) - \sum_{k=-\ell}^{\ell} u_k e^{-2\pi i k f/(2W)}|^2 df = 0 \tag{3}$$

$$\int_{-W}^{W} |\hat{u}(f)| df = 2W \sum_{-\infty}^{\infty} |u_k|^2 \tag{4}$$

Finally, if $\{u_k; k \in \mathbb{Z}\}$ is a sequence of complex numbers satisfying $\sum_k |u_k|^2 < \infty$, then an L^2 function $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ exists satisfying (3) and (4).

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The sampling theorem

$\hat{u}(f)$ for an L2 baseband waveform is both L^1 and L^2. Thus, at every t

$$u(t) = \int_{-W}^{W} \hat{u}(f) e^{2\pi i f t} df$$

and $u(t)$ is continuous. Since the transform pair

$$rect(\frac{f}{2W}) \leftrightarrow 2W sinc(2Wt) \qquad \text{(scaling relation)}$$

the inverse transform of $\hat{\phi}_k(f)$ is

$$\phi_k(t) = 2W sinc(2Wt - k) \leftrightarrow \hat{\phi}_k(f) = e^{-2\pi i k f/(2W)} rect(f/2W) \qquad \text{(time-shift relation)}$$

Then

$$u(t) = \sum_k u_k \phi_k(t) = \sum_k 2W u_k sinc(2Wt - k) = \sum_{k=-\infty}^{\infty} u(k/2W) sinc(2Wt - k) \qquad (5)$$

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Sampling theorem

Let $\{\hat{u}(f) : [-W, W] \to \mathbb{C}\}$ be L^2 (and thus also L^1). For $u(t)$ in (5), let $T = 1/(2W)$. Then $u(t)$ is continuous, L^2, and bounded by $u(t) \leq \int_{-W}^{W} |\hat{u}(f)| df$. Also, for all $t \in \mathbb{R}$,

$$u(t) = \sum_{-\infty}^{\infty} u(kT) sinc(\frac{t - kT}{T})$$

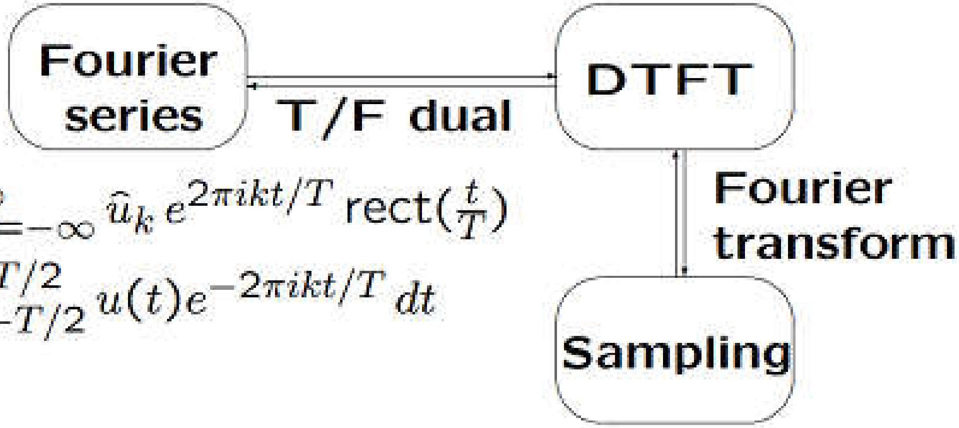
This says that a baseband-limited function is specified by its *samples intervals* $T = 1/(2W)$.
The sinc function is nonzero over all noninteger times. Recreating the waveform at the receiver from a set of samples thus requires infinite delay(band-limited functions are not time-limited).
Practically sinc functions can be truncated, but the sinc waveform decays to zero as $1/t$, which is impractically slow.

Baseband-limited

An L^2 function is **baseband-limited** to W if it is the pointwise inverse transform of an L^2 function $\hat{u}(f)$ that is 0 for $|f| > W$.
Equivalently, it is baseband-limited to W if it is **continuous** and its Fourier transform is 0 for $|f| > 0$.
There are other bandlimited functions, limited to $[-W, W]$, which are not continuous. The sampliing theorem does not hold for these functions.

$$\hat{u}(f) = \sum_k u_k e^{-2\pi i k \frac{f}{2W}} \mathrm{rect}\left(\frac{f}{2W}\right)$$

$$u_k = \frac{1}{2W} \int_{-W}^{W} \hat{u}(f) e^{2\pi i k \frac{f}{2W}} \, df$$



$$u(t) = \sum_{k=-\infty}^{\infty} \hat{u}_k \, e^{2\pi i k t/T} \, \mathrm{rect}\left(\frac{t}{T}\right)$$

$$\hat{u}_k = \frac{1}{T} \int_{-T/2}^{T/2} u(t) e^{-2\pi i k t/T} \, dt$$

$$u(t) = \sum_{k=-\infty}^{\infty} 2W u_k \, \mathrm{sinc}(2Wt - k)$$

$$u_k = \frac{1}{2W} u\left(\frac{k}{2W}\right)$$

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The sampling theorem for $[\Delta - W, \Delta + W]$

Consider an L^2 frequency function $\{\hat{v}(f) : [\Delta - W, \Delta + W] \to \mathbb{C}\}$. The *shifted DTFT* for $\hat{v}(f)$ is then

$$\hat{v}(f) = l.i.m. \sum_k v_k e^{-2\pi i k f/(2W)} rect(\frac{f-\Delta}{2W}) = l.i.m. \sum_k v_k \hat{\theta}_k(f)$$

where

$$v_k = \frac{1}{2W} \int_{\Delta - W}^{\Delta + W} \hat{v}(f) e^{2\pi i k f/(2W)} df$$

The inverse Fourier transform of $\hat{\theta}_k(f)$ can be calculated by shifting and scaling to be

$$\theta_k(t) = 2W sinc(2Wt - k) e^{2\pi i \Delta (t - \frac{k}{2W})} \leftrightarrow \hat{\theta}_k(f) = e^{-2\pi i k f/(2W)} rect(\frac{f-\Delta}{2W})$$

This generalizes the sampling equation to the frequency band $[\Delta - W, \Delta + W]$, let $T = 1/2W$,

$$v(t) = \sum_k v_k \theta_k(t) = \sum_k v(\frac{k}{2W}) sinc(2Wt - k) e^{2\pi i \Delta (t - \frac{k}{2W})} = \sum_k v(kT) sinc(\frac{t}{T} - k) e^{2\pi i \Delta (t - kT)}$$

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Aliasing and Degrees of Freedom

An important rule of thumb used by communication engineers is that the class of real functions that are approximately baseband-limited to W_0 and approximately time-limited to $[-T_0/2, T_0/2]$ have about $2T_0W_0$ real degrees of freedom if $T_0W_0 >> 1$. This means that any function within that class can be specified approximately by specifying about $2T_0W_0$ real numbers as coefficients in an orthogonal expansion.

Degrees of freedom

Degrees of freedom is somewhat difficult to state precisely, since time-limited functions cannot be frequency-limited and vice-versa.

Applying the sampling theorem, real (complex) functions $u(t)$ strictly baseband-limited to W_0 are specified by its real (complex) samples at rate $2W_0$. If the samples are nonzero only within the interval $[-T_0/2, T_0/2]$, then there are about $2T_0W_0$ nonzero samples, and these specify $u(t)$ within this class. Here a precise class of functions have been specified, but *functions* that are zero outside of an interval have been *replaced with functions whose samples are zero* outside of the interval.

Consider a large time interval T_0 and a baseband limited band W_0. There are $T0/T$ segments of duration T and $W_0/2W = W_0T$ positive frequency segments. Counting negative frequencies also, there are $2T_0W_0$ time/frequency blocks and $2T_0W_0$ coefficients.

If one ignores coefficients outside of $T0$, $W0$, then the function is specified by $2T_0W_0$ complex numbers.
For real functions, it is T_0W_0 complex numbers.

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T-spaced sinc-weighted sinusoid expansion

Let $u(t) \leftrightarrow \hat{u}(f)$ be an arbitrary L^2 transform pair, and segment $\hat{u}(f)$ into intervals of width $2W$.

$$\hat{u}(f) = l.i.m. \sum_k \hat{v}_m(f); \quad \hat{v}_m(f) = \hat{u}(f) rect(\frac{f}{2W} - m)$$

$\hat{v}_m(f)$ is non-zero over $[2Wm - W, 2Wm + W]$, so from the sampling theorem for $[\Delta - W, \Delta + W]$,

$$v_m(t) = \sum_k v_m(kT) sinc(\frac{t}{T} - k) e^{2\pi i(m/T)(t-kT)} = \sum_k v_m(kT) sinc(\frac{t}{T} - k) e^{2\pi imt/T}$$

Combining all of these *frequency segments*,

$$u(t) = l.i.m. \sum_m v_m(t) = l.i.m. \sum_{m,k} v_m(kT) \psi_{m,k}(t)$$

The doubly indexed set of orthogonal functions are the time and frequency shifts of the basic function $\psi_{0,0}(t) = sinc(t/T)$. The time shifts are in multiples of T and the frequency shifts are in multiples of $1/T$.

$$\psi_{m,k}(t) = sinc(\frac{t}{T} - k) e^{2\pi imt/T}; \quad m, k \in \mathbb{Z}$$

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T-spaced truncated sinusoid expansion

Suppose that an L^2 waveform $\{u(t) : R \to C\}$ is segmented into segments $u_m(t)$ of duration T. Expressing $u(t)$ as the sum of these segments

$$u(t) = l.i.m. \sum_m u_m(t); \qquad u_m(t) = u(t) rect(\frac{t}{T} - m)$$

For a function $\{v(t) : [\Delta - T/2, \Delta + T/2] \to \mathbb{C}\}$ centered around some arbitrary time Δ, the *shifted Fourier series* over that interval is

$$v(t) = l.i.m. \sum_k \hat{v}_k e^{2\pi ikt/T} rect(\frac{t-\Delta}{T})$$

$$\hat{v}_k = \frac{1}{T} \int_{\Delta-T/2}^{\Delta+T/2} v(t) e^{-2\pi ikt/T} dt, -\infty < k < \infty$$

Expanding each segment $u_m(t)$ by the shifted Fourier series

$$u_m(t) = l.i.m. \sum_k \hat{u}_{k,m} e^{2\pi ikt/T} rect(\frac{t}{T} - m)$$

$$\hat{u}_{k,m} = \frac{1}{T} \int_{mT-T/2}^{mT+T/2} u_m(t) e^{-2\pi ikt/T} dt = \frac{1}{T} \int_{-\infty}^{\infty} u(t) e^{-2\pi ikt/T} rect(\frac{t}{T} - m) dt$$

This expands $u(t)$ as a weighted sum of doubly indexed functions

$$u(t) = l.i.m. \sum_{m,k} \hat{u}_{k,m} \theta_{m,k}(t), \qquad \text{where}$$

$$\theta_{m,k}(t) = e^{2\pi ikt/T} rect(\frac{t}{T} - m)$$

ALIASING

The samples from different frequency slices get summed together in the samples of $u(t)$. This phenomenon is called *aliasing*.

A time domain approach

Suppose we approximate a function u(t) that is not quite baseband limited by the sampling expansion $s(t) \approx u(t)$.

$$s(t) = \sum_k u(kT) sinc(\frac{t}{T} - k)$$

$$u(t) = l.i.m. \sum_{m,k} v_m(kT) sinc(\frac{t}{T} - k) e^{2\pi imt/T}$$

$$s(kT) = u(kT) = \sum_m v_m(kT) (Aliasing)$$

$$s(t) = \sum_k \sum_m v_m(kT) sinc(\frac{t}{T} - k)$$

$$u(t) - s(t) = \sum_m \sum_{m \neq 0} v_m(kT) [e^{2\pi imt/T} - 1] sinc(\frac{t}{T} - k)$$

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A frequency domain approach

$s(t)$ can be separated into the contribution from each frequency band as

$$s(t) = \sum_m s_m(t), \quad s_m(t) = \sum_k v_m(kT) sinc(\frac{t}{T} - k)$$

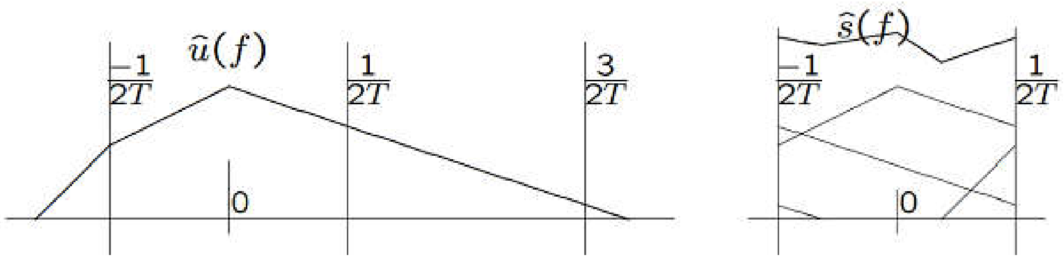
Comparing $s_m(t)$ to $v_m(t)$, it is seen that

$$v_m(t) = s_m(t)e^{2\pi imt/T}, \quad \hat{s}_m(f) = \hat{v}_m(f + \frac{m}{T})$$

Since $\hat{v}_m(f) = \hat{u}(f)rect(fT - m)$, one sees that $\hat{v}_m(f + \frac{m}{T}) = \hat{u}(f + \frac{m}{T})rect(fT)$. Thus,

$$\hat{s}(f) = \sum_m \hat{u}(f + \frac{m}{T})rect[fT]$$

Each frequency slice $\hat{v}_m(f)$ is shifted down to baseband in this equation, and then all these shifted frequency slices are summed together.



Aliasing theorem

Let $\hat{u}(f)$ be L^2, and let $\hat{u}(f)$ satisfy the condition

$$\lim_{|f|\to\infty} \hat{u}(f)|f|^{1+\epsilon}$$

Then $\hat{u}(f)$ is L^1, and the inverse Fourier transform $u(t) = \int \hat{u}(f)e^{2\pi i f t}df$ converges pointwise to a continuous bounded function. For any given $T > 0$, the sampling approximation $\sum_k u(kT)sinc(\frac{t}{T} - k)$ converges pointwise to a continuous bounded L^2 function $s(t)$. The Fourier transform of $s(t)$ satisfies

$$\hat{s}(f) = l.i.m. \sum_m \hat{u}(f + \frac{m}{T})rect[fT]$$

The condition that $\lim \hat{u}(f)f^{1+\epsilon} = 0$ implies that $\hat{u}(f)$ goes to 0 with increasing f at a faster rate than $1/f$.

Without the mathematical convergence details, what the aliasing theorem says is that, corresponding to a Fourier transform pair $u(t) \leftrightarrow \hat{u}(f)$, there is another Fourier transform pair $s(t)$ and $\hat{s}(f)$; $s(t)$ is a baseband sampling expansion using the T-spaced samples of $u(t)$ and $\hat{s}(f)$ is the result of folding the transform $\hat{u}(f)$ into the band $[-W, W]$ with $W = 1/(2T)$.

Vector space

Public

Linear algebra is the study of linear maps on finite-dimensional vector spaces. Use of vectors to represent a countably infinite sequence is a small conceptual extension. Viewing waveforms as vectors is a larger conceptual extension. We have to view vectors as abstract objects rather than as n-tuples. Orthogonal expansions are best viewed in vector space terms. L2 waveforms can be viewed as vectors in the inner product space, such inner product space known as **signal space**.

A *vector space* $\mathcal{V}$ is a set of elements, $\vec{v} \in \mathcal{V}$, called vectors, along with a set of rules for operating on both these vectors and a set of ancillary elements called scalars.

A vector space with real scalars is called a *real vector space*, and one with complex scalars is called a *complex vector space*.

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Axioms of vector space

Addition

For each $\vec{v} \in \mathcal{V}$ and $\vec{u} \in \mathcal{V}$, there is a vector $\vec{v} + \vec{u} \in \mathcal{V}$ called the sum of $\vec{v}$ and $\vec{u}$ satisfying:

- Commutativity: $\vec{v} + \vec{u} = \vec{u} + \vec{v}$.
- Associativity: $\vec{v} + (\vec{u} + \vec{w}) = (\vec{v} + \vec{u}) + \vec{w}$, for each $\vec{v}, \vec{u}, \vec{w} \in \mathcal{V}$.
- Zero: there is a unique vector $\mathbf{0} \in \mathcal{V}$ such that $\vec{v} + \mathbf{0} = \vec{v}$ for all $\vec{v} \in \mathcal{V}$.
- Negation: for each $\vec{v} \in \mathcal{V}$, there is a unique vector $-\vec{v}$ such that $\vec{v} + (-\vec{v}) = \mathbf{0}$.

Scalar multiplication

For each scalar α and each $\vec{v} \in \mathcal{V}$ there is a unique vector $\alpha\vec{v} \in \mathcal{V}$ called the scalar product of α and $\vec{v}$ satisfying:

- Scalar associativity: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for all scalars α, β, and all $\vec{v} \in \mathcal{V}$.
- Unit multiplication: for the unit scalar 1, $1\vec{v} = \vec{v}$ for all $\vec{v} \in \mathcal{V}$.

Distributive laws

- For all scalars α and all $\vec{v}, \vec{u} \in \mathcal{V}$, $\alpha(\vec{v} + \vec{u}) = \alpha\vec{v} + \alpha\vec{u}$.
- For all scalars α, β and all $\vec{v} \in \mathcal{V}$, $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$.

Finite-dimensional vector spaces

The set of finite-energy complex waveforms can be viewed as a complex vector space. For space L^2 of finite energy complex functions, we can define $\vec{u} + \vec{v}$ as function $\vec{w}$ where $w(t) = u(t) + v(t)$ for each t. Define $\alpha\vec{v}$ as vector $\vec{u}$ for which $u(t) = \alpha v(t)$.

- **Span**: A set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n} \in \mathcal{V}$ spans $\mathcal{V}$ (and is called a *spanning set* of $\mathcal{V}$) if every vector $\vec{v} \in \mathcal{V}$ is a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$.
- **Finite-dimensional**: A vector space $\mathcal{V}$ is finite-dimensional if a finite set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$ exist that span $\mathcal{V}$.
- **Linearly dependent**: A set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n} \in \mathcal{V}$ is linearly dependent if $\sum_{j=1}^{n} \alpha_j \overrightarrow{v_j}$ for some set of scalars not all equal to 0. This implies that each vector $\overrightarrow{v_k}$ for which $\alpha_k \neq 0$ is a linear combination of the others.
- **Basis**: A set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n} \in \mathcal{V}$ is defined to be a basis for $\mathcal{V}$ if the set both spans $\mathcal{V}$ and is linearly independent. The **dimension** of a finite-dimensional vector space is defined as the number of vectors in a basis.

Inner product spaces

Vector space definition lacks distance and angles. Inner product adds these features. The inner product of $\vec{v}$ and $\vec{u}$ is denoted $\langle \vec{v}, \vec{u} \rangle$.

Axioms of inner product space

- Hermitian symmetry: $\langle \vec{v}, \vec{u} \rangle = \langle \vec{u}, \vec{v} \rangle^*$
- Hermitian bilinearity: $\langle \alpha\vec{v} + \beta\vec{u}, \vec{w} \rangle = \alpha\langle \vec{v}, \vec{w} \rangle + \beta\langle \vec{u}, \vec{w} \rangle$; $\langle \vec{v}, \alpha\vec{u} + \beta\vec{w} \rangle = \alpha^* \langle \vec{v}, \vec{u} \rangle + \beta^* \langle \vec{v}, \vec{w} \rangle$
- Strict positivity:

$$\langle \vec{v}, \vec{v} \rangle \geq 0, \text{equality iff } \vec{v} = \vec{0} \qquad (*)$$

For $\mathbb{C}^n$, we usually define $\langle \vec{v}, \vec{u} \rangle = \sum_i v_i u_i^*$.
If $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ are unit vectors in $\mathbb{C}^n$, then $\langle \vec{v}, \vec{e}_i \rangle = v_i, \langle \vec{e}_i, \vec{v} \rangle = v_i^*$.
$\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$ is **squared norm** of $\vec{v}$. $\|\vec{v}\|$ is **length** of $\vec{v}$.
$\vec{v}$ and $\vec{u}$ are are orthogonal if $\langle \vec{v}, \vec{u} \rangle = 0$. More generally $\vec{v}$ can be broken into a part $\vec{v}_{\perp\vec{u}}$ that is orthogonal to $\vec{u}$ and another part collinear $\vec{v}_{|\vec{u}}$ with $\vec{v}$.

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One-dimensional projection theorem

Let $\vec{v}$ and $\vec{u}$ be arbitrary vectors with $\vec{u} \neq 0$ in a real or complex inner product space. Then there is a unique scalar α for which $\langle \vec{v} - \alpha\vec{u}, \vec{u} \rangle = 0$. That α is given by $\alpha = \langle \vec{v}, \vec{u} \rangle / ||\vec{u}||^2$.

$$\vec{v}_{|\vec{u}} = \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u} = \langle \vec{v}, \frac{\vec{u}}{||\vec{u}||} \rangle \frac{\vec{u}}{||\vec{u}||}$$

$$cos(\angle(\vec{u}, \vec{v})) = \langle \frac{\vec{v}}{||\vec{v}||}, \frac{\vec{u}}{||\vec{u}||} \rangle \frac{\vec{u}}{||\vec{u}||}$$

Pythagorean theorem: If $\vec{v}$ and $\vec{u}$ are orthogonal, then $||\vec{v} + \vec{u}||^2 = ||\vec{v}||^2 + ||\vec{u}||^2$.
Schwarz inequality: Let $\vec{v}$ and $\vec{u}$ be vectors in a real or complex inner product space, then $|\langle \vec{v}, \vec{u} \rangle| \leq ||\vec{v}|| ||\vec{u}||$.

The inner product space of L^2 functions

L^2 becomes an inner product space if we define the inner product of L^2 as

$$\langle \vec{v}, \vec{u} \rangle = \int_{-\infty}^{\infty} v(t)u^*(t)dt$$

Strict positivity axiom (*) does not hold for finite-energy waveforms, so we must define equality as L^2 equivalence.
The vectors in this space are equivalence classes. Alternatively, view a vector as a set of coefficients in an orthogonal expansion.

Finite-dimensional vector spaces

The set of finite-energy complex waveforms can be viewed as a complex vector space. For space L^2 of finite energy complex functions, we can define $\vec{u} + \vec{v}$ as function $\vec{w}$ where $w(t) = u(t) + v(t)$ for each t. Define $\alpha\vec{v}$ as vector $\vec{u}$ for which $u(t) = \alpha v(t)$.

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- **Basis**: A set of vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n} \in \mathcal{V}$ is defined to be a basis for $\mathcal{V}$ if the set both spans $\mathcal{V}$ and is linearly independent. The **dimension** of a finite-dimensional vector space is defined as the number of vectors in a basis.

Orthonormal Bases and the Projection Theorem

The use of orthonormal bases simplifies almost everything concerning inner product spaces, and for infinite-dimensional expansions, orthonormal bases are even more useful. Gram-Schmidt procedure starting from an arbitrary basis $s_1, \ldots, s_n$ for an n-dimensional inner product subspace $\mathcal{S}$, generates an orthonormal basis for $\mathcal{S}$. The infinite dimensional projection theorem can provide simple and intuitive proofs and interpretations of limiting arguments and the approximations suggested by those limits.

Vector subspaces

A subspace of a vector space $\mathcal{V}$ is a subset $\mathcal{S}$ of $\mathcal{V}$ that forms a vector space in its own right.
Equivalent: For all $\vec{u}, \vec{v} \in \mathcal{V}, \alpha\vec{u} + \beta\vec{v} \in \mathcal{S}$.
The notion of linear combination (which is at the heart of both the use and theory of vector spaces) depends on what the scalars are.
So $\mathbb{R}^n$ is not a subspace of $\mathbb{C}^n$; real L^2 is not a subspace of complex L^2.
A subspace of an inner product space (using the same inner product) is an inner product space.

Finite-dimensional Projections

Assume $\mathcal{V}$ is an inner product space, A vector $\phi \in \mathcal{V}$ is normalized if $\|\phi\| = 1$.
The projection $v_{|\phi} = \langle v, \phi \rangle \phi$ for $\|\phi\| = 1$.
An orthonormal set $\{\phi_j\}$ is a set such that $\langle \phi_j, \phi_k \rangle = \delta_{jk}$.
If $\{v_j\}$ is orthogonal set, then $\{\phi_j\}$ is an orthonormal set where $\phi_j = v_j / \|v_j\|$.

Projection theorem

Assume that $\{\phi_1, \ldots, \phi_n\}$ is an orthonormal basis for an n-dimensional subspace $\mathcal{S} \subset \mathcal{V}$. For each $v \in V$, there is a unique $v_{|\mathcal{S}} \in \mathcal{S}$ such that $\langle v - v_{|\mathcal{S}}, s \rangle = 0$ for all $s \in \mathcal{S}$. Furthermore,

$$v_{|s} = \sum_j \langle v, \phi_j \rangle \phi_j$$

.

Proof outline: Let $v_{|\mathcal{S}} = \sum_i \alpha_i \phi_i$. Find the conditions on $\alpha_1, \ldots, \alpha_n$ such that $v - v_{|\mathcal{S}}$ is orthogonal to each ϕ_i.

$$0 = \langle v - \sum_i \alpha_i \phi_i, \phi_j \rangle = \langle v, \phi_j \rangle - \alpha_j$$

For $v \in \mathcal{S}, v = \sum_j \alpha_j \phi_j, \phi_j$ orthonormal basis of $\mathcal{S}$,

$$\|v\|^2 = \langle v, \sum_j \alpha_j \phi_j \rangle = \sum \alpha_j^* \langle v, \phi_j \rangle = \sum_j |\alpha_j|^2$$

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Bessel's inequality

Let $\mathcal{S} \subseteq \mathcal{V}$ be the subspace spanned by the set of orthonormal vectors $\phi_1, \ldots, \phi_n$. For any $v \in V$

$$0 \leq \sum_{j=1}^{n} |\langle v, \phi_j \rangle|^2 \leq \|v\|^2$$

is the key to understanding the convergence of orthonormal expansions.

LS error property

$v_{|s}$ is the choice for s that yields the Least Square Error(LS) or Minimum Square Error(MSE).
The projection $v_{|\mathcal{S}}$ is the unique closest vector in S to v; i.e., for all $s \in \mathcal{S}$,

$$\|v - v_{|\mathcal{S}}\|^2 \leq \|v - s\|^2$$

The individual basis functions themselves have a trivial vector representation; namely $\phi_n(t)$ is represented by $\phi_n = [00, \ldots, 1, \ldots, 0]^T$. In effect, $x_{in} = \langle x_i \phi_n \rangle$ is the projection of the ith modulated waveform on the n^{th} basis function.

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Gram-Schmidt orthonormalization

Given basis $s_1, \ldots, s_n$ for an inner product subspace, find an orthonormal basis.
$\phi_1 = s_1/\|s_1\|$ is an orthonormal basis for subspace $\mathcal{S}_1$ generated by s_1.
Given orthonormal basis $\phi_1, \ldots, \phi_k$ of subspace $\mathcal{S}_k$ generated by $s_1, \ldots, s_k$, project s_{k+1} onto $\mathcal{S}_k$.

$$\phi_{k+1} = \frac{(s_{k+1})_{\perp \mathcal{S}_k}}{\|(s_{k+1})_{\perp \mathcal{S}_k}\|}$$

The Gram-Schmidt algorithm gives a simple construction of the q's from the a's in factorization of A = QR (orthonormal columns times upper triangular).

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Orthonormal expansions in L^2

For L^2, the projection theorem can be extended to a countably infinite dimension.

Infinite-dimensional projection

Let $\{\phi_m, 1 \leq m < \infty\}$ be a set of orthonormal functions, and let v be any L^2 vector. Then there is a unique L^2 vector u such that $v - u$ is orthogonal to each ϕ_m and

$$\lim_{n\to\infty} \|u - \sum_{m=1}^{n} \langle v, \phi_m \rangle \phi_m \| = 0$$

Outline of proof: Let $\mathcal{S}_n$ be subspace spanned by $\phi_1, \ldots, \phi_n$.

$$v_{|\mathcal{S}_n} = \sum_{k=1}^{n} \alpha_k \phi_k = \sum_{k=1}^{n} \langle v, \phi_k \rangle \phi_k$$

$$\|v_{|\mathcal{S}_m} - v_{|\mathcal{S}_n}\|^2 = \sum_{k=n}^{m} |\alpha_k|^2 \to 0$$

$v_{|\mathcal{S}_n}$ forms a *Cauchy sequence*. By the *Riesz-Fischer theorem*(L^2 waveforms has an L^2 limit), $l.i.m.\, v_{|\mathcal{S}_n} = u$ exists. This shows that the fourier series converges in L^2.

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贤者以其昭昭使人昭昭，今以其昏昏使人昭昭。

——孟子·尽心下

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