Activity 8.4.7 solutions

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Note: In the solutions that follow we assume the following as known facts that do not require explanation.

(1) (for $a \neq 0$) $\sum_{n=0}^{\infty} ar^n = \begin{cases} \text{converges to } \frac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$

(2) $\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } 0 < p \leq 1 \end{cases}$

(a) Method 1: Integral test

We have $\int \frac{1}{\sqrt{x-2}}\,dx = \int (x-2)^{-1/2}\,dx = 2(x-2)^{1/2}$.

So $\int_3^{\infty} \frac{2}{\sqrt{x-2}}\,dx = \lim_{b\to\infty} \int_3^b \frac{2}{\sqrt{x-2}}\,dx = \lim_{b\to\infty} \left(4(b-2)^{1/2} - 4\cdot 1\right) = \infty.$

By the integral test, we conclude that $\boxed{\sum_{k=3}^{\infty} \frac{2}{(k-2)^{1/2}} \text{ diverges.}}$

Method 2: Limit Comparison test.

Let $a_k = \frac{2}{\sqrt{k-2}} = \frac{1}{\sqrt{k}} \cdot \frac{2}{\sqrt{1-\frac{2}{k}}}$, and let

$b_k = \frac{1}{\sqrt{k}}$, so $\lim_{k\to\infty} \frac{a_k}{b_k} = 2$.

By the limit comparison test, $\sum a_k$ and $\sum b_k$ either both converge or both diverge. We know by fact (2) above that $\sum b_k$ diverges $(p = 1/2 < 1)$ so we conclude that $\boxed{\sum a_k \text{ also diverges.}}$

(b) Since $\frac{k}{1+2k} \to \frac{1}{2}$ as $k\to\infty$, we conclude that $\boxed{\sum_{k=1}^{\infty} \frac{k}{1+2k} \text{ diverges}}$ by the Divergence Test.

(c) $\frac{2k^2+1}{k^3+k+1} \sim \frac{2k^2}{k^3} \sim 2\cdot\frac{1}{k}$, $\sum \frac{1}{k}$ diverges $(p=1)$

so $\sum \frac{2k^2+1}{k^3+k+1}$ also diverges

Quick Analysis in your brain

Careful reasoning:

Let $a_k = \frac{2k^2+1}{k^3+k+1} = \frac{k^2}{k^3} \frac{\left(2+\frac{1}{k^2}\right)}{\left(1+\frac{1}{k^2}+\frac{1}{k^3}\right)} = \frac{1}{k}\cdot\begin{pmatrix}\text{expression} \\ \text{that} \to 2 \\ \text{as} \\ k\to\infty\end{pmatrix}$.

and let $b_k = \frac{1}{k}$, so $\lim_{k\to\infty} \frac{a_k}{b_k} = 2$. By the Limit Comparison Test, $\sum a_k$ and $\sum b_k$ either both converge or both diverge. Since $\sum b_k$ diverges $(p=1)$, we conclude that $\boxed{\sum a_k \text{ also diverges.}}$

(d) Analysis in your brain! You can't integrate factorial. This is no p-series. So try ratio test!

Let $a_n = \frac{100^n}{n!}$.

We have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{100^{n+1}}{(n+1)!}\right)}{\left(\frac{100^n}{n!}\right)} = \lim_{n\to\infty} \frac{100^{n+1}}{100^n}\cdot\frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} 100\cdot\frac{1}{n+1}$

$= 0 < 1.$

By the Ratio Test, we conclude that $\boxed{\sum a_n \text{ converges.}}$

(e) This is geometric with $r = \frac{2}{5} < 1$, so the series converges.

(f) $\frac{k^3-1}{k^5+1} \approx \frac{k^3}{k^5} = \frac{1}{k^2}$, $\sum \frac{1}{k^2}$ converges $(p=2)$

so $\sum \frac{k^3-1}{k^5+1}$ must converge

Let $a_k = \frac{k^3-1}{k^5+1} = \frac{k^3}{k^5} \frac{\left(1-\frac{1}{k^3}\right)}{\left(1+\frac{1}{k^5}\right)} = \frac{1}{k^2}\begin{pmatrix}\text{expression} \\ \text{that} \to 1 \\ \text{as } k\to\infty\end{pmatrix}$.

Let $b_k = \frac{1}{k^2}$, so $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$. By the Limit Comparison Test, $\boxed{\sum a_k \text{ converges}}$ because $\sum b_k$ converges $(p=2)$.

(g) This series is geometric with $r = \frac{3}{7} < 1$, so the series converges.

(h) Method 1: Ratio Test

Let $a_k = \frac{1}{k^k}$. Then $\frac{a_{k+1}}{a_k} = \frac{k^k}{(k+1)^{k+1}} = \underbrace{\left(\frac{k}{k+1}\right)^k}_{A} \cdot \underbrace{\frac{1}{k+1}}_{B}$.

First we'll do $\lim_{k\to\infty} A$. The trick is to take $\ln\left(\lim_{k\to\infty} A\right) = k \lim_{k\to\infty} \ln\frac{k}{k+1} = 0.$

So $\lim_{k\to\infty} A = e^0 = 1$. So $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} A \cdot \lim_{k\to\infty} B$

$= 1\cdot 0 = 0 < 1.$

By the Ratio Test, $\boxed{\sum a_k \text{ converges.}}$

Method 2: Root Test.

We have $\lim_{k\to\infty} \sqrt[k]{\frac{1}{k^k}} = \lim_{k\to\infty} \frac{1}{k} = 0 < 1.$

So $\boxed{\sum a_k \text{ converges}}$ by the Root Test.

(i) This series alternates, $\frac{1}{\sqrt{k+1}}$ is decreasing and $\frac{1}{\sqrt{k+1}} \to 0$ as $k\to\infty$, so $\boxed{\text{the series converges}}$ by the Alternating Series Test.

(j) We have $\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln u = \ln\ln x + C.$

$u = \ln x$

$du = \frac{1}{x}\,dx$

Thus we have $\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \left[\ln\ln b - \ln\ln 2\right] = \infty.$

By the Integral Test, we conclude that $\boxed{\sum \frac{1}{k\ln k} \text{ diverges.}}$

(k) Using the formula $|S - S_n| < a_{n+1}$,

we solve $\frac{1}{\ln(n+1)} < .001$ to get $n+1 > e^{1000}$,

So we can pick any whole number n greater than $e^{1000} - 1$.