The Borel-Weil Theorem

Seminar Presentation by David Lyons, October 1994

The purpose of this talk is to illuminate the subject of Lie theory through the exposition of one of its beautiful and important theorems. Lie theory is about Lie groups, Lie algebras, and their representations. The theorem I will present is called the Borel-Weil theorem; I will sketch a proof due to my advisor, Shrawan Kumar.

I will begin with a broad outline of Lie theory to set the stage and introduce objects which play a role in the Borel-Weil theorem. If the subject is new to you, I hope to provide a tour of its landscape, giving some idea of the basic notions and problems of representation theory.

1 Some Elements of Lie Theory (Outline)

(1.1) **Three Geometric Categories.**
Smooth Manifolds $\supseteq$ Complex Manifolds $\supseteq$ Smooth Algebraic Varieties

- Examples: S^1, surfaces in $\mathbf{R}^3$
- Tangent spaces and tangent maps

(1.2) **Definitions and Examples of Lie Groups, Lie Algebras, and Representations.**

- Definition: A Lie Group is a group which is also a geometric space in one of the above categories, where the group operations are morphisms in the appropriate category.
- Examples: $GL(n), SL(n)$
- Definition: The Lie algebra $L(G)$ or $\mathfrak{g}$ of a Lie group G is its tangent space at identity. Bracket structure coming from vector fields. Abstract Lie Algebra.
- Examples: $gl(n), sl(n)$
- Group homomorphisms, derivatives at identity, exponential map, and commutative diagram

- some correspondences: Groups $\longleftrightarrow$ Algebras, normal subgroup $\longleftrightarrow$ ideal, abelian $\longleftrightarrow$ abelian
- Definition of representation, realization by matrices
- Examples: $SL(n)$ acts on $\mathbf{C}^n$, $SL(2)$ acts on homogeneous polynomials of degree n in two variables
- Correspondence: G acts by identity $\longleftrightarrow$ $L(G)$ acts by 0
- Definition of matrix coefficients $\mathcal{M}(G) = \{d^V_{\alpha,v}\}_{V,\alpha,v}$ where

  $$(1.3) \quad d^V_{\alpha,v}(g) = \alpha(g.v) \quad \text{for } g \in G, v \in V, \alpha \in V^*, V \text{ a repn. of } G$$
- Example: the Peter-Weyl theorem
- Submodules, complete reducibility, picture in matrices
- The semisimple condition for a group

(1.4) **Some Problems in Lie Theory, and One Application.**

- Given G, does it admit a faithful representation?
- Given repn. V, what are its submodules? Is V a direct sum of its submodules?
- Given G, describe all of its irreps.
- Describe the structure of infinite dimensional representations.
- Describe the topology of Lie Groups.
- Classify Lie Groups.
- Answers to those problems
- The Hard Lefschetz Theorem: for M compact Kähler, $sl(2)$ acts on $H^*(M) \rightsquigarrow$ isomorphism $H^{n-i}(M) \xrightarrow{\sim} H^{n+i}(m)$ where $[\alpha] \mapsto [\alpha \wedge \omega^i]$, where ω is the Kähler form on M. For details, see [3].

(1.5) **General Analysis of semisimple $G, \mathfrak{g}, G \to \mathbf{Aut}(V)$.**

- The Adjoint repn.
- Cartan subgroup $H \subseteq G$

- Decompose $\mathfrak{g}$ with respect to $H \rightsquigarrow$ roots and the Weyl group $\rightsquigarrow$ classification
- (Integral) weights, highest weight, every repn has a highest weight, dominant weight

(1.6) **More notions needed for Borel-Weil.**

- Action of G on maps between G-modules: Given G-modules V, W and $f : V \to W$

$$(g.f)(v) = g.(f(g^{-1}.v)) \qquad \text{for } g \in G, v \in V \tag{1.7}$$

- There are two standard left G actions, L and R, of G on functions $\{f : G \to \mathbf{C}\}$, defined as follows.

$$(L_g f)(h) = f(g^{-1}h) \tag{1.8}$$
$$(R_g f)(h) = f(hg) \tag{1.9}$$

for all $g, h \in G, f : G \to \mathbf{C}$

- Line bundles, how they generalize functions
- B, U, Borel is semidirect product of Cartan and Unipotent, how U acts on highest weight vectors, how $e^{\lambda} : H \to \mathbf{C}^*$ extends to B
- Homogeneous spaces, G/B in particular

(1.10) **Passing Between Categories.**

- Maximal compact $K \subseteq G$
- $L(G) = L(K) \otimes \mathbf{C}$
- Irreps(G)=Irreps($L(G)$)=Irreps($L(K)$)=Irreps(K) (for simply connected G)
- "Peter-Weyl" for algebraic groups

$$A(G) = \mathcal{M}(G) \xrightarrow{\sim} \mathcal{M}(K) \tag{1.11}$$

2 The Borel-Weil Theorem

For the remainder of this discussion G shall be a simply connected semisimple algebraic group. Fix a choice of Cartan subgroup $H \subseteq G$ and Borel subgroup $B \subseteq G$. Recall that G/B is a smooth projective variety. We will construct, for any integral weight $\lambda \in L(H)^*$, a line bundle $\mathcal{L}_\lambda$ on G/B. G will act on G/B and $\mathcal{L}_\lambda$, and hence on sections $\Gamma(G/B, \mathcal{L}_\lambda)$ by the standard action (1.7) of G on maps between G-spaces. The Borel-Weil Theorem asserts that when λ is dominant, $\Gamma(G/B, \mathcal{L}_\lambda)$ is (the dual of) a finite dimensional irreducible G-module of highest weight λ. This achieves our goal of constructing a geometric realization of all the irreducible representations of G.

We begin by constructing the line bundle $\mathcal{L}_\lambda$ on G/B. Let $e^\lambda : H \to \mathbf{C}^*$ denote the representation of H whose derivative is λ. The representation e^λ extends to $e^\lambda : B \to \mathbf{C}^*$. Let $\mathbf{C}_\lambda$ denote the one dimensional B-module with action given by e^λ, i.e., $b.v = e^\lambda(b)v$ for all $b \in B, v \in \mathbf{C}_\lambda$. B acts on G by $b.g = gb^{-1}$. Finally, B acts on $G \times \mathbf{C}_\lambda$ by combining the two actions given above, i.e., $b.(g,v) = (gb^{-1}, e^\lambda(b)v)$. Let $G \times_B \mathbf{C}_\lambda$ denote the quotient of $G \times \mathbf{C}_\lambda$ by this B action. Then $G \times_B \mathbf{C}_\lambda$ is a line bundle on G/B with projection $[g,v] \mapsto [g]$, where square brackets denote equivalence classes in the appropriate spaces. We define $\mathcal{L}_\lambda$ to be $G \times_B \mathbf{C}_{-\lambda}$ (note the minus sign!). Now we can state the Borel-Weil theorem.

(2.1) **Theorem (Borel-Weil).** *Let λ be a dominant integral weight for G, with $\mathcal{L}_\lambda$ as above. Then the vector space $\Gamma(G/B, \mathcal{L}_\lambda)^*$ is a finite dimensional irreducible G-module of highest weight λ.*

Before giving the proof, we point out why this statement is significant. It is not *a priori* clear that $\Gamma(G/B, \mathcal{L}_\lambda)$ should even be finite dimensional, much less irreducible. Secondly, G/B is a "nice" space; this space of sections is particularly well-behaved, and hence useful.

The proof consists of establishing the following G-module maps:

$$\Gamma(G/B, \mathcal{L}_\lambda) \overset{(1)}{\hookrightarrow} A(G) \overset{(2)}{=} \mathcal{M}(G) \overset{(3)}{\longleftarrow} \bigoplus_{\mu \text{ dominant}} (V_\mu^* \otimes V_\mu)$$

where V_μ is the irreducible representation for G of highest weight μ. We will explicitly identify the image M of $\Gamma(G/B, \mathcal{L}_\lambda)$ under (1) in A(G). Then we show that the preimage of M under (3) is contained in the single summand $V_\lambda^* \otimes V_\lambda$, and that $M \approx \Gamma(G/B, \mathcal{L}_\lambda)$ is indeed irreducible.

We proceed with (1). The G actions on $G/B, \mathcal{L}_\lambda$, and $\Gamma(G/B, \mathcal{L}_\lambda)$ are, respectively,

$$\begin{aligned} h.[g] &= [hg] \\ h.[g,v] &= [hg, v] \\ (h.\sigma)([g]) &= h.\sigma([h^{-1}g]) \end{aligned}$$

for all $g, h \in G, v \in \mathbf{C}_{-\lambda}$ and $\sigma \in \Gamma(G/B, \mathcal{L}_\lambda)$. We define a map $\Gamma(G/B, \mathcal{L}_\lambda) \to A(G)$ by $\sigma \mapsto [g \mapsto v]$, where v is given by $\sigma[g] = [g, v]$. One checks that this is indeed a G-map, where G acts on $A(G)$ by the action L given in (1.8). Let $M \subseteq A(G)$ be defined by $M = \{f : f(gb) = e^\lambda(b) f(g)\}$. We claim that M is is the image of $\Gamma(G/B, \mathcal{L}_\lambda)$ in $A(G)$. It is straightforward to check that the image of $\Gamma(G/B, \mathcal{L}_\lambda)$ does indeed lie in M. Map (1) has an inverse on M given by $f \mapsto [[g] \mapsto [g, f(g)]]$. Thus we have $\Gamma(G/B, \mathcal{L}_\lambda) \xrightarrow{\sim} M \subseteq A(G)$, which establishes (1).

Equality (2) is the fact mentioned earlier in (1.11). Map (3) simply takes $\alpha \otimes v \in V_\mu^* \otimes V_\mu$ to the matrix coefficient $d_{\alpha,v}^{V_\mu}$ defined in (1.3). We put a G action on $V_\mu^* \otimes V_\mu$ by defining $g.(\alpha \otimes v) = g\alpha \otimes v$. With the action L on $A(G)$ as above, it is easy to check that (3) is a G-module map. We now show that (3) is a bijection. Since G is semisimple, any finite dimensional representation is the sum of irreducible representations. From this one deduces that (3) is surjective. To show injectivity, first choose bases $\{\alpha_i^\mu\}_{i=1}^{\dim(V_\mu)}, \{v_i^\mu\}_{i=1}^{\dim(V_\mu)}$ for V_μ^* and V_μ, respectively, so that the collection $\{\alpha_i^\mu \otimes v_j^\mu\}_{\mu,i,j}$ forms a basis for $\bigoplus_\mu (V_\mu^* \otimes V_\mu)$. Now let K be a maximal compact subgroup of G. By (1.11), it suffices to establish injectivity of $\bigoplus_\mu (V_\mu^* \otimes V_\mu) \overset{(3)}{\hookrightarrow} \mathcal{M}(K)$. On $\mathcal{M}(K)$ we have an $\mathcal{L}^2$ inner product available; we can use orthogonality relations in $\mathcal{M}(K)$ to show that the image of $\{\alpha_i^\mu \otimes v_j^\mu\}_{\mu,i,j}$ under (3) is indeed a linearly independent set. For a reference, see [1], II,4.

Finally, we identify the preimage of M under (3). We begin by considering the action R of G on $A(G)$ given by (1.9). For $d_{\alpha,v}^V \in \mathcal{M}(G)$, this becomes $R_g d_{\alpha,v}^V = d_{\alpha,gv}^V$. Using the identification (3) the R action takes the form

$$R_g(\alpha \otimes v) = \alpha \otimes gv \qquad \forall g \in G, \alpha \otimes v. \tag{2.2}$$

The condition

$$f(gb) = e^\lambda(b) f(g) \qquad \forall g \in G, b \in B$$

which defines M, can now be written

$$R_b f = e^\lambda(b) f \qquad \forall b \in B. \tag{2.3}$$

We wish to explicitly identify the preimage of $f \in M$ under (3), which we shall call again f. Choose a basis $\{\alpha_i^\mu\}$ for each V_μ^*. Then there are uniquely determined vectors $w_i^\mu \in V_\mu$ such that $f = \sum_{\mu,i} \alpha_i^\mu \otimes w_i^\mu$. Applying (2.2) we have, for all $b \in B$,

$$R_b f = \sum \alpha_i^\mu \otimes b w_i^\mu.$$

On the other hand, applying (2.3) yields (again for all $b \in B$)

$$R_b f = e^\lambda(b) \sum \alpha_i^\mu \otimes w_i^\mu = \sum \alpha_i^\mu \otimes e^\lambda(b) w_i^\mu.$$

We conclude that w_i^μ is a highest weight vector of weight λ for every μ and i, and hence that f lies in $V_\lambda^* \otimes \langle v^\lambda \rangle$, where v^λ is a vector of weight λ inside V_λ. It is easy to verify that the image of any $\alpha \otimes v \in V_\lambda^* \otimes \langle v^\lambda \rangle$ under (3) satisfies (2.3) and hence lies in M. Thus we have

$$\Gamma(G/B, \mathcal{L}_\lambda) \approx M \approx V_\lambda^* \otimes \langle v^\lambda \rangle \approx V_\lambda^*.$$

Q.E.D.

References

[1] Theodor Bröcker and Tammo tom Dieck. *Representations of Compact Lie Groups.* Springer-Verlag, New York, 1985.

[2] William Fulton and Joe Harris. *Representation Theory, A First Course.* Springer-Verlag, New York, 1991.

[3] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry.* John Wiley and Sons, New York, 1978.

[4] James E. Humphries. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, New York, 1972.

[5] Frank Warner. *Foundations of Differentiable Manifolds and Lie Groups.* Springer-Verlag, New York, 1983.

David W. Lyons
University of North Carolina at Chapel Hill
E-mail address: `dwl@math.unc.edu`