Context: A set of p people (a "nation") is partitioned into subgroups ("states") of sizes $p_1, p_2, \ldots, p_n$ ("state populations"). A positive whole number h of items ("house seats") is to be allocated among them. The quantity $q_i = \frac{p_i}{p} h$, called the ith state's *quota*, is the ith state's ideal fair share of house seats h by population proportion. An *apportionment* is a collection of numbers $h_1, \ldots, h_n$ so that the h_i's are nonnegative whole numbers that sum to h. Intuitively, apportionment methods seek a good approximation to the quota, that is, $\frac{h_i}{h} \approx q_i$. We would like to find a fair method for rounding each q_i up or down to produce h_i.

An apportionment method that always produces apportionments with

$$\lfloor q_i \rfloor \leq \frac{h_i}{h} \leq \lceil q_i \rceil$$

is said to have the *quota condition*.

To define the apportionment methods used for US Congressional House seats, we introduce the notion of a *divisor*. The *standard divisor* is

$$s = \frac{p}{h}.$$

The standard divisor is the ideal number of citizens per representative. We can write

$$q_i = \frac{p_i}{p} h = \frac{p_i}{s}.$$

Intuitively, we want to determine h_i by applying some kind of rounding procedure (rounding up or down to the nearest integer, by some rule) to q_i. This turns out to need tweaking to avoid paradoxes and biases. So-called *divisor* methods address the paradoxes and biases by perturbing the quotas. This is accomplished by perturbing the standard divisor s. We call $d(t) = s + t$ (for some $|t|$ small) a *trial divisor*. Divisor methods then apply a rounding procedure to the new trial quotas $q_i(t) = \frac{p_i}{d(t)}$ to produce apportionments. The idea is that by adjusting t (a process of guess and check), we will arrive at apportionments h_i that sum to h. In theory, this can fail when two tentative apportionments $h_1(t), h_2(t)$ make integer jumps at the same value of t, but this does not happen in practice when the p_i are large.

Here is a summary of the quantities involved. Let X be a finite set partitioned by subsets X_i, and let h be a positive integer.

$$p = |X| \qquad \text{(total population)}$$
$$p_i = |X_i| \qquad \text{(state population)}$$
$$h \qquad \text{(number of house seats)}$$
$$s = \frac{p}{h} \qquad \text{(standard divisor)}$$
$$q_i = \frac{p_i}{p}h = \frac{p_i}{s} \qquad \text{(ith quota)}$$
$$R \qquad \text{(rounding function)}$$
$$R(q_i) \qquad \text{(tentative apportionment)}$$
$$\lfloor q_i \rfloor \qquad \text{(lower quota)}$$
$$\lceil q_i \rceil \qquad \text{(upper quota)}$$
$$q_i - \lfloor q_i \rfloor \qquad \text{(fractional part of q_i)}$$
$$d(t) = s + t, |t| \text{ small} \qquad \text{(trial divisor)}$$
$$q_i(t) = \frac{p_i}{d(t)} \qquad \text{(trial quota)}$$
$$R(q_i(t)) \qquad \text{(trial apportionment)}$$

Vocabulary note: COMAP uses the term "apportionment quotient" for what I call the "trial quota".

Divisor method note: if $\sum_i R(q_i)$ (the total tentative apportionments) is not equal to h, a way to pick a starting value for the perturbation t is to make the fractional change in the divisor equal and opposite to the fractional error in the total tentative apportionment. This produces the following values of t, d.

$$t = s\left(\frac{\text{(total tentative apportionment)} - h}{h}\right)$$
$$d(t) = s\left(1 + \frac{\text{(total tentative apportionment)} - h}{h}\right)$$

Method	Hamilton (Hare)	Jefferson (d'Hondt)	Webster (Saint-Laguë)	Hill-Huntington
Rounding function R	$R(q) = \text{floor}(q) = \lfloor q \rfloor$	$R(q) = \text{floor}(q) = \lfloor q \rfloor$	$R(q) = \text{round}(q) =$ $\lceil q \rceil$ if $q - \lfloor q \rfloor \geq .5$, $\lfloor q \rfloor$ if $q - \lfloor q \rfloor < .5$	$R(q) =$ $\text{fancyround}(q) =$ $\lceil q \rceil$ if $q \geq \sqrt{\lfloor q \rfloor \lceil q \rceil}$, $\lfloor q \rfloor$ if $q < \sqrt{\lfloor q \rfloor \lceil q \rceil}$
Initial perturbation t	$t = 0$	t small and negative	$t = 0$	$t = 0$
Step 1	Allocate trial apportionments $R(q_i(t))$. If the tentative apportionments sum to h, we are done. Otherwise go to Step 2.			
Step 2, if house seats remain after step 1.	Give a seat to the state with the maximum $q_i - \lfloor q_i \rfloor$. Repeat until all seats are used up.	Decrease the value of t slightly. If the last t value was positive, make the new t value also positive. Repeat Step 1.		
Step 2, if too many house seats were apportioned in step 1.	This never happens for Hamilton	Increase the value of t slightly. If the last t value was negative, make the new t value also negative. Repeat Step 1.		
Comments	Suffers from the Alabama Paradox, the Population Paradox, and the New States Paradox	Is biased in favor of most populous states. Does not satisfy the quota condition.	Optimizes apportionment equity based on absolute differences in representative share. Has no population bias.	Optimizes apportionment equity based on relative differences in representative share. Produces no zero apportionments. This is the method currently in use.