INTEGRAL WEYL INVARIANTS AND THE BOREL HOMOMORPHISM FOR Spin(n)

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Abstract

We examine the Weyl invariant subring $H^*(BT)^{\mathcal{W}}$ of the integral singular cohomology ring $H^*(BT)$ of the classifying space BT of the maximal torus T in a connected compact Lie group K. We show that $H^*(BT)^{\mathcal{W}}$ is *not* a polynomial algebra for $K = \mathrm{Spin}(n)$ when $n \geq 10$. We describe the connection of this result with the Borel homomorphism $H^*(BT) \to H^*(K/T)$.

1 Introduction

Given a connected compact Lie group K with maximal torus T, there is a fibration (described in §3.1 below)

$$K/T \to BT \to BK$$

where BT and BK are the classifying spaces for T and K, respectively. A key object in the study of the algebraic topology of K is the Borel homomorphism

$$\psi \colon H^*(BT) \to H^*(K/T)$$

induced on the integral cohomology rings by the inclusion $K/T \to BT$. The Weyl group $\mathcal{W}$ acts on T, BT, and K/T, and it is easy to see (§3.3) that the invariants $H^*(BT)^{\mathcal{W}}$ are contained in the kernel I of ψ. Analyzing the kernel I raises questions about $H^*(BT)^{\mathcal{W}}$; in particular, we wish to determine if $H^*(BT)^{\mathcal{W}}$ is a polynomial subring of $H^*(BT)$. In §2 of this paper, we prove our main result.

Theorem (2.1.1). *The subring of Weyl invariants $H^*(BT)^{\mathcal{W}}$ is not a polynomial ring for $K = \mathrm{Spin}(n)$, $n \geq 10$.*

Our proof utilizes a list of generators for $H^*(BT)^{\mathcal{W}}$ given by D. Benson and J. Wood in [1]. Theorem (2.1.1) was conjectured by Shrawan Kumar, and our proof generalizes a proof of his for the case $n = 11$.

In §3, we give the details of how this relates to the Borel homomorphism, and conclude with the following corollary.

Corollary (3.3.6). *For $K = \mathrm{Spin}(n)$, $n \geq 10$, the kernel I of the Borel homomorphism $\psi\colon H^*(BT) \to H^*(K/T)$ properly contains the ideal $\langle (H^*(BT)^{\mathcal{W}})_+ \rangle$ generated by Weyl invariants of positive degree.*

This paper presents some of the results from the author's doctoral dissertation, completed in December, 1996, at the University of North Carolina at Chapel Hill, under the direction of Shrawan Kumar.

1.1 Notation

We denote by $\mathbf{Z}$ and $\mathbf{Q}$ the integers and the rational numbers, respectively.

Let K be a compact connected Lie group with maximal torus $T \subseteq K$ of dimension $r = \mathrm{rank}(K)$. Denote by $M \subseteq (LT)^*$ the weight lattice of K, where LT is the Lie algebra of T. Let $R \subseteq M$ be the set of roots, $R_+ \subseteq R$ a choice of positive roots, and $\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq R_+$ the set of simple roots. Let $S(M)$ be the (graded) symmetric algebra (over $\mathbf{Z}$) of M.

Let $\mathcal{W} = N(T)/T$ denote the Weyl group, where $N(T)$ is the normalizer of T in K. Let $r_i \in \mathcal{W}$ be the reflection corresponding to $\alpha_i \in \Sigma$. For $w \in \mathcal{W}$, denote by $l(w)$ the length of the shortest expression for w as a product of the r_i. Let $w_0 \in \mathcal{W}$ denote the (unique) longest element, and let $e \in \mathcal{W}$ denote the identity element. There is a natural $\mathcal{W}$ action on LT, its dual $(LT)^*$, the weight lattice $M \subseteq (LT)^*$, and $S(M)$.

We denote by BG and EG the classifying space for G and the total space of the universal G-bundle, respectively, for a topological group G. Throughout this paper, we denote by $H^*(X)$ the singular cohomology ring, with coefficients in the integers $\mathbf{Z}$, of a topological space X.

We shall write A_+ to denote the positive part of a $\mathbf{Z}$-graded ring A. We denote by $\langle B \rangle$ the ideal generated by a subset B of a ring A.

1.2 Some algebra

We record in this section several results from commutative algebra to be used in §2 and §3. Let $\mathbf{F}$ be a field.

(1.2.1) **Proposition.** *Suppose $A \subseteq \mathbf{F}[z_1, \ldots, z_n]$ is a graded subalgebra of the graded polynomial algebra $\mathbf{F}[z_1, \ldots, z_n]$. If $\mathbf{F}[z_1, \ldots, z_n]$ is free as an A-module, then A itself is a polynomial algebra.*

For a proof, see ([9], Corollary 6.4.4, p. 155).

We have the following lemma of M. Demazure ([5], Lemme 6, p. 297).

(1.2.2) **Lemma.** *Let R be a principal ideal domain, A a $\mathbf{Z}$-graded (commutative, associative, with unit) R-algebra such that each graded piece A^n of A is R-free. Suppose that for any ring homomorphism from R to any field $\mathbf{F}$, the graded $\mathbf{F}$-algebra $A \otimes_R \mathbf{F}$ is a graded polynomial $\mathbf{F}$-algebra. Then A is a polynomial R-algebra.*

We shall make use of the following consequence of the above two results.

(1.2.3) **Corollary.** *Suppose $A \subseteq S = \mathbf{Z}[z_1, \ldots, z_n]$ is a graded subalgebra such that the $\mathbf{Z}$-module S/A is torsion free. If S is free as an A-module, then A is a polynomial ring.*

PROOF: For any field $\mathbf{F}$, there is only one homomorphism $\mathbf{Z} \to \mathbf{F}$. Since S is a free A-module, we have $S \otimes_{\mathbf{Z}} \mathbf{F}$ is a free $A \otimes_{\mathbf{Z}} \mathbf{F}$-module. Moreover, $A \otimes_{\mathbf{Z}} \mathbf{F}$ is a subalgebra of $S \otimes_{\mathbf{Z}} \mathbf{F}$ since S/A is torsion free by assumption. By Proposition (1.2.1), $A \otimes_{\mathbf{Z}} \mathbf{F}$ is a polynomial $\mathbf{F}$-algebra. By Lemma (1.2.2), we conclude that A is a polynomial $\mathbf{Z}$-algebra. □

We shall also make use of the following.

(1.2.4) **Observation.** *Let G be a group, and let S be a free $\mathbf{Z}$-module on which G acts $\mathbf{Z}$-linearly. Then S/S^G is torsion free.*

PROOF: Let $n \in \mathbf{Z}$ and $x \in S$. If $nx \in S^G$, then $nx = g \cdot nx = ng \cdot x$ for all $g \in G$. Since S is free, we have $x = g \cdot x$ for all $g \in G$, i.e., $x \in S^G$. □

2 Weyl invariants for $\mathrm{Spin}(n)$

In this section we examine the Weyl invariants $H^*(BT)^{\mathcal{W}}$ for $K = \mathrm{Spin}(n)$. We utilize calculations of D. Benson and J. Wood of generators for $H^*(BT)^{\mathcal{W}}$ in their paper [1]. We will adopt their notation in this section, with the following exception. Given elements $f_1, f_2, \ldots, f_k$ in a polynomial ring $R[x_1, x_2, \ldots, x_n]$ over R, we write $R\{f_1, f_2, \ldots, f_k\}$ to denote the subring of $R[x_1, x_2, \ldots, x_n]$ generated by 1 and the f_i. (Benson and Wood use $R[f_1, f_2, \ldots, f_k]$, which might cause confusion with the notation for a polynomial ring.)

2.1 The Main Theorem

As in ([4], IV, sec. 3), let $T(n) = \mathrm{SO}(2)^r$ be the maximal torus of $\mathrm{SO}(n)$, where $r = [n/2] = \mathrm{rank}(\mathrm{SO}(n)) = \mathrm{rank}(\mathrm{Spin}(n))$. Let $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the double covering and let $\tilde{T}(n) = \pi^{-1}(T)$ be the maximal torus of $\mathrm{Spin}(n)$. We have $H^*(BT(n)) = \mathbf{Z}[x_1, x_2, \ldots, x_r]$ and $H^*(B\tilde{T}(n)) = \mathbf{Z}\{A, x_1, x_2, \ldots, x_r\}$, where $A = \frac{1}{2}\sum x_i$, and $\pi^*\colon H^*(BT(n)) \to H^*(B\tilde{T}(n))$ takes x_i to x_i. The ring $H^*(B\tilde{T}(n))$ is polynomial in the variables A and any $r-1$ of the x_i.

(2.1.1) **Theorem.** $H^*(B\tilde{T}(n))^{\mathcal{W}} \subseteq H^*(B\tilde{T}(n))$ *is not a polynomial ring for any* $n \geq 10$.

2.2 Outline of the proof of (2.1.1)

To prove Theorem (2.1.1), we tensor all the rings over $\mathbf{Z}$ with $\overline{\mathbf{F}_2}$, the algebraic closure of $\mathbf{Z}/2\mathbf{Z}$. This simplifies expressions for generators of the invariant subring and allows us to use the following geometric argument.

Since $H^*(B\tilde{T}(n))/H^*(B\tilde{T}(n))^{\mathcal{W}}$ is torsion free (by Observation (1.2.4)), the ring $H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ is a subalgebra of $H^*(B\tilde{T}(n)) \otimes \overline{\mathbf{F}_2} = \overline{\mathbf{F}_2}[\xi_1, \ldots, \xi_r]$, and hence is an integral domain. We will exhibit $H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ (for $n \geq 10$) as a quotient $\overline{\mathbf{F}_2}[u_1, \ldots, u_N]/J$, where $0 \neq J \subseteq \langle u_1, u_2, \ldots, u_N\rangle^2$. Thus $H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ can be thought of as the affine coordinate ring of an affine variety V over $\overline{\mathbf{F}_2}$, where V is defined by the ideal J and embedded in affine space $\mathbf{A}^N$. Since $J \subseteq \langle u_1, u_2, \ldots, u_N\rangle^2$, the Jacobian matrix $(\partial\varphi_i/\partial u_j)$ vanishes at 0 for any choice of generators $\varphi_1, \varphi_2, \ldots, \varphi_k$ for J, so we have that $0 \in V$ is a singular point. Therefore $H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ (and hence $H^*(B\tilde{T}(n))^{\mathcal{W}}$) cannot be a polynomial ring.

We denote by $\overline{x}$ the element $x \otimes 1 \in H^*(B\tilde{T}(n)) \otimes \overline{\mathbf{F}_2}$.

2.3 Computational Lemmas

We begin by defining polynomials c_i, p_i, and f_i in $H^*(B\tilde{T}(n))$. Let c_i, p_i be defined by

$$
\begin{aligned}
c_i &= \sum_{1 \leq j_1 < j_2 < \cdots j_i \leq r} x_{j_1} x_{j_2} \cdots x_{j_i} \\
p_i &= \sum_{1 \leq j_1 < j_2 < \cdots j_i \leq r} x_{j_1}^2 x_{j_2}^2 \cdots x_{j_i}^2
\end{aligned}
$$

for $1 \leq i \leq r$, $c_0 = p_0 = 1$, and $c_j = p_j = 0$ for $j > r$. Following Benson and Wood, we take the degree of each x_i to be 2, so then $\deg c_i = 2i$ and $\deg p_i = 4i$. The polynomials f_i are homogeneous of degree 2^{i+1} in the x_j. Furthermore, f_i lies in the subring $\mathbf{Z}\{c_2, \ldots, c_{2^i}\} = \mathbf{Z}[c_2, \ldots, c_{2^i}]$ and we will write $f_i(c) = f_i(c_2, \ldots, c_{2^i})$ to indicate f_i as a polynomial in the c_j. The f_i are defined inductively ([1], p. 16) beginning with $f_1 = c_2$. Assume now that $f_1, \ldots, f_i$ have been defined. Since

$$
\begin{aligned}
p_j &= c_j^2 + 2 \sum_{1 \leq k \leq j} (-1)^k c_{j-k} c_{j+k} \qquad \text{([1], p. 15)} \\
&= c_j^2 + 2 \sum_{1 \leq k \leq j, k \neq j-1} (-1)^k c_{j-k} c_{j+k} + 2(-1)^{j-1} c_1 c_{2j-1},
\end{aligned}
$$

we may write

$$
f_i(p) - f_i(c)^2 = a_i + c_1 b_i,
$$

where $f_i(p) = f_i(p_2, \ldots, p_{2^i})$ is obtained by replacing c_j by p_j in $f_i(c)$ and a_i and b_i are polynomials in the variables $c_2, \ldots, c_{2^{i+1}}$. It is easy to see that a_i and b_i are divisible by 2. Benson and Wood then define g_i to be $\frac{b_i}{2}$, and set

$$
f_{i+1}(c) = \frac{1}{2}\left[f_i(p) - f_i(c)^2 - 4Ag_i\right] = \frac{a_i}{2}.
$$

In order to prove some technical lemmas about the f_i, it is convenient to use a "c_1 free" description, i.e., we eliminate the need for the term $-4Ag_i$. We set

$$
s_j = p_j + 2(-1)^j c_1 c_{2j-1}
$$

$$= c_j^2 + 2 \sum_{1\le k\le j, k\ne j-1} (-1)^k c_{j-k}c_{j+k}.$$

Now it is easy to see that $f_i(s) - f_i(c)^2 = a_i$, so that f_{i+1} is given by

$$f_{i+1}(c) = \frac{1}{2}\left[f_i(s) - f_i(c)^2\right].$$

The first three f_i, which are easy to check by hand, are

$$\begin{aligned} f_1 &= c_2, \\ f_2 &= c_4, \\ \text{and } f_3 &= -c_3c_5 + c_2c_6 + c_8. \end{aligned} \tag{2.3.1}$$

(2.3.2) **Lemma.** *For $i \ge 3$, $f_i = c_{2^i} + c_2c_{2^i-2} + ($a polynomial in $c_2, \ldots, c_{2^i-3})$.*

PROOF: We have $f_3 = -c_3c_5 + c_2c_6 + c_8$. Assume that the lemma holds for f_i. Since (for $i \ge 2$)

$$s_{2^i} = 2c_{2^{i+1}} + 2c_2c_{2^{i+1}-2} + (\text{ a polynomial in } c_3, \ldots, c_{2^{i+1}-3})$$

and since s_j is a polynomial in $c_2, \ldots, c_{2^{i+1}-4}$ for $2 \le j \le 2^i - 2$, we have

$$\begin{aligned} f_{i+1} &= \frac{1}{2}\left[s_{2^i} + s_2s_{2^i-2} + (\text{ a polynomial in } s_2, \ldots, s_{2^i-3}) - f_i(c)^2\right] \\ &= c_{2^{i+1}} + c_2c_{2^{i+1}-2} + (\text{ a polynomial in } c_2, \ldots, c_{2^{i+1}-3}). \end{aligned}$$

This completes the induction and establishes the lemma. □

In fact, we have the following sharper lemma.

(2.3.3) **Lemma.** *For $i \ge 4$, $f_i = c_{2^i} + c_2c_{2^i-2} - c_2^2c_{2^{i-1}-3}c_{2^{i-1}-1} + e_i$, where e_i is a polynomial in the c_j $(2 \le j \le 2^i - 3)$ which does not involve the monomial $c_2^2c_{2^{i-1}-3}c_{2^{i-1}-1}$.*

PROOF: By the proof of Lemma (2.3.2), for $i \ge 3$ we have

$$f_{i+1} = \frac{1}{2}\left[s_{2^i} + s_2s_{2^i-2} + d - f_i(c)^2\right],$$

where d is a polynomial in $s_2, \ldots, s_{2^i-3}$. We have

$$\begin{aligned} s_2 &= c_2^2 + 2c_4, \\ s_{2^i-2} &= c_{2^i-2}^2 - 2c_{2^i-3}c_{2^i-1} + 2 \sum_{2 \leq k \leq 2^i-2, k \neq 2^i-3} (-1)^k c_{2^i-2-k}c_{2^i-2+k}, \end{aligned}$$

whence

$$s_2 s_{2^i-2} = -2c_2^2 c_{2^i-3}c_{2^i-1} + e$$

where e is a polynomial in the c_j with no other multiples of $c_2^2 c_{2^i-3}c_{2^i-1}$. Any monomials in the c_j occurring in d have the form $\alpha \prod c_{j-k}c_{j+k}$, where $\alpha \in \mathbf{Z}$ and the indices j, k run over a subset of $2 \leq j \leq 2^i - 3$ and $0 \leq k \leq j, j-k \neq 1$. Thus d cannot contain a multiple of $c_2^2 c_{2^i-3}c_{2^i-1}$. Any monomials in $f_i(c)^2$ must be a product of two monomials of homogeneous degree 2^{i+1} in the x_j, so none of these can be a multiple of $c_2^2 c_{2^i-3}c_{2^i-1}$. This establishes that the coefficient of $c_2^2 c_{2^i-3}c_{2^i-1}$ in f_{i+1} is -1, which together with Lemma (2.3.2), proves Lemma (2.3.3). □

Benson and Wood construct certain invariants q_i ([1], section 3), η_i (for odd n) and μ_i (for even n) ([1], section 4). The precise definitions of these elements do not concern us; we collect here the properties we shall need. The q_i are defined in terms of the f_i mentioned above, and certain h_i, by $q_i = 2Ah_i - f_i$ for $i \geq 1$. Thus we have

(2.3.4) $$\overline{q_i} = \overline{f_i}$$

(the h_i no longer concern us mod 2). We also have

(2.3.5) $$\overline{p_i} = \overline{c_i}^2.$$

The only property of the η_i and μ_i which we need are their degrees.

$$\begin{aligned} \deg \eta_i &= 2^{i+1} \quad \text{for } 0 \leq i \leq r, \\ \deg \mu_i &= 2^i \quad \text{for } 1 \leq i \leq r. \end{aligned}$$

We record here Benson and Wood's list of generators for $H^*(B\tilde{T}(n))^{\mathcal{W}}$ ([1], Theorem 7.1, p. 21).

(2.3.6) **Theorem (Benson and Wood).**

1. *If $n = 2r+1 \geq 7$ is odd, then*

$$H^*(B\tilde{T}(n))^{\mathcal{W}} = \mathbf{Z}\{p_1, \ldots, p_r, q_1, \ldots, q_{r-2}, \eta_{r-1}\}.$$

2. *If $n = 2r \geq 6$ is even, then*

$$H^*(B\tilde{T}(n))^{\mathcal{W}} = \begin{cases} \mathbf{Z}\{p_1, \ldots, p_{r-1}, c_r, q_1, \ldots, q_{r-2}, \mu_r\}, & r \text{ odd} \\ \mathbf{Z}\{p_1, \ldots, p_{r-1}, c_r, q_1, \ldots, q_{r-3}, \mu_{r-1}\}, & r \text{ even} . \end{cases}$$

Thus we immediately have a list of generators for $H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$. Observe that $\overline{p_1} = 0$ since $\overline{p_1} = \overline{c_1}^2 = 4\overline{A}^2$.

(2.3.7) **List of generators mod 2.**

1. *If $n = 2r+1 \geq 7$ is odd, then*

$$H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2} = \overline{\mathbf{F}_2}\{\overline{p_2}, \ldots, \overline{p_r}, \overline{q_1}, \ldots, \overline{q_{r-2}}, \overline{\eta_{r-1}}\}.$$

2. *If $n = 2r \geq 6$ is even, then*

$$H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2} = \begin{cases} \overline{\mathbf{F}_2}\{\overline{p_2}, \ldots, \overline{p_{r-1}}, \overline{c_r}, \overline{q_1}, \ldots, \overline{q_{r-2}}, \overline{\mu_r}\}, & r \text{ odd} \\ \overline{\mathbf{F}_2}\{\overline{p_2}, \ldots, \overline{p_{r-1}}, \overline{c_r}, \overline{q_1}, \ldots, \overline{q_{r-3}}, \overline{\mu_{r-1}}\}, & r \text{ even} . \end{cases}$$

For each rank $r \geq 1$, let m be the unique integer satisfying $2^m \leq r < 2^{m+1}$. We shall prove that $\overline{q_{m+1}}$ must be included in any minimal sublist of generators from the above lists, for $r \geq 5$. To do so, we shall make use of lemmas (2.3.1) and (2.3.3) to make statements about the form of the $\overline{q_{m+1}} = \overline{f_{m+1}}$ as a polynomial in the $\overline{c_j}$. Though there are no relations among the c_j over $\mathbf{Z}$, it is not *a priori* clear that this holds mod 2. It is an elementary matter, however, settled in the following lemma.

(2.3.8) **Lemma.** $\overline{\mathbf{F}_2}\{\overline{c_2},\ldots,\overline{c_r}\} \subseteq H^*(B\tilde{T}(n)) \otimes \overline{\mathbf{F}_2}$ *is a polynomial ring.*

PROOF: First note that there are no relations among the c_j in $\mathbf{Z}[c_2,\ldots,c_r] \subseteq H^*(B\tilde{T}(n))$ since $\mathbf{Q}[c_2,\ldots,c_r] \subseteq \mathbf{Q}[c_1,c_2,\ldots,c_r] = \mathbf{Q}[x_1,\ldots,x_r]^{\Sigma_r}$ is a polynomial ring (Σ_r denotes the symmetric group).

The kernel K of the map

$$\mathbf{Z}[c_2,\ldots,c_r] \otimes \mathbf{Z}/2\mathbf{Z} \to H^*(B\tilde{T}(n)) \otimes \mathbf{Z}/2\mathbf{Z} \tag{2.3.9}$$

is all those $\xi \otimes 1$ such that $\xi = 2\zeta$ for some $\zeta \in H^*(B\tilde{T}(n))$. Let $\xi \otimes 1 \in K$. Since ξ is a symmetric invariant, so is ζ, since $2\zeta = \sigma(2\zeta) = 2\sigma\zeta$ for all $\sigma \in \Sigma_r$. Benson and Wood prove ([1], Lemma 3.1, p. 15) that $\mathbf{Z}[A,c_2,\ldots,c_r] = H^*(B\tilde{T}(n))^{\Sigma_r}$. Therefore $\zeta \in \mathbf{Z}[A,c_2,\ldots,c_r]$, and hence $\zeta \in \mathbf{Z}[c_2,\ldots,c_r]$ since ξ does not involve c_1. Thus $K = 0$.

Since $\overline{\mathbf{F}_2}$ is a free $\mathbf{Z}/2\mathbf{Z}$-module, tensoring (2.3.9) with $\overline{\mathbf{F}_2}$ over $\mathbf{Z}/2\mathbf{Z}$ acquires no kernel, and we have

$$\mathbf{Z}[c_2,\ldots,c_r] \otimes \overline{\mathbf{F}_2} = \overline{\mathbf{F}_2}[\overline{c_2},\ldots,\overline{c_r}] \approx \overline{\mathbf{F}_2}\{\overline{c_2},\ldots,\overline{c_r}\} \subseteq H^*(B\tilde{T}(n)) \otimes \overline{\mathbf{F}_2}$$

is a polynomial ring, as claimed. □

Next we state and prove our main lemma.

(2.3.10) **Main Lemma.** *For $r \geq 5$, we have*

(1) $\overline{q_{m+1}}^2 \in \overline{\mathbf{F}_2}\{\overline{p_2},\ldots,\overline{p_r}\} \subseteq H^*(B\tilde{T}(n)) \otimes \overline{\mathbf{F}_2}$, *and*

(2) $\overline{q_{m+1}} \notin \overline{\mathbf{F}_2}\{\overline{p_2},\ldots,\overline{p_{r-1}},\overline{c_r},\overline{q_1},\ldots,\overline{q_m}\} \subseteq H^*(B\tilde{T}(n)) \otimes \overline{\mathbf{F}_2}$,

where m is given by $2^m \leq r < 2^{m+1}$.

PROOF:

(1) Since $\overline{q_i} = \overline{f_i}$ is a polynomial in the $\overline{c_j}$, we have that $\overline{q_i}^2$ is a polynomial in the $\overline{c_j}^2 = \overline{p_j}$. This proves (1).

(2) By Lemma (2.3.8) it suffices to exhibit a monomial in the $\overline{c_j}$ occurring in $\overline{q_{m+1}}$ which could not be a product of monomials in the $\overline{c_j}$ occurring in $\overline{p_2},\ldots,\overline{p_{r-1}},\overline{c_r}$, and $\overline{q_1},\ldots,\overline{q_m}$. We examine four cases.

(Case $r = 5$) When $r = 5$, we have $m + 1 = 3$. By (2.3.1), we have $\overline{q_3} = \overline{f_3} = \overline{c_3}\,\overline{c_5}$ since $\overline{c_6} = \overline{c_8} = 0$. This can clearly not be written in terms of $\overline{p_2}, \ldots, \overline{p_4}, \overline{c_5}, \overline{q_1} = \overline{c_2}$ and $\overline{q_2} = \overline{c_4}$.

(Case $r = 6$) When $r = 6$, we have $m + 1 = 3$. By (2.3.1), we have $\overline{q_3} = \overline{f_3} = \overline{c_3}\,\overline{c_5} + \overline{c_2}\,\overline{c_6}$ since $\overline{c_8} = 0$. This can clearly not be written in terms of $\overline{p_2}, \ldots, \overline{p_5}, \overline{c_6}, \overline{q_1} = \overline{c_2}$ and $\overline{q_2} = \overline{c_4}$.

(Case $r = 7$) When $r = 7$, we have $m + 1 = 3$. By (2.3.1), we have $\overline{q_3} = \overline{f_3} = \overline{c_3}\,\overline{c_5} + \overline{c_2}\,\overline{c_6}$ since $\overline{c_8} = 0$. This can clearly not be written in terms of $\overline{p_2}, \ldots, \overline{p_6}, \overline{c_7}, \overline{q_1} = \overline{c_2}$ and $\overline{q_2} = \overline{c_4}$.

(Case $r \geq 8$) When $r \geq 8$, we have $m + 1 \geq 4$, so by Lemma (2.3.3), $\overline{q_{m+1}}$ contains the monomial $\overline{c_2}^2\overline{c_{2^m-3}}\,\overline{c_{2^m-1}}$ when written as a polynomial in the $\overline{c_j}$. (This term is non zero in $H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ since $2^m \leq r$.) Since $\overline{q_i}$ is a polynomial in $\overline{c_2}, \ldots, \overline{c_{2^i}}$, we see that $\overline{c_{2^m-1}}$ can appear in $\overline{q_i}$ only when $i \geq m$. But, from degree considerations, $\overline{c_{2^m-1}}$ can appear in $\overline{q_m}$ only in a monomial of the form $\overline{c_1}\,\overline{c_{2^m-1}}$, which is not the case ($\overline{c_1} = 0$). It follows that $\overline{q_{m+1}}$ cannot be written in terms of the $\overline{p_k} = \overline{c_k}^2$, $2 \leq k \leq r-1$, $\overline{c_r}$ and $\overline{q_j}$ for $j < m+1$, since none of these could supply $\overline{c_{2^m-1}}$ in the monomial $\overline{c_2}^2\overline{c_{2^m-3}}\,\overline{c_{2^m-1}}$ in $\overline{q_{m+1}}$.

□

(2.3.11) **Proposition.** *For $r \geq 5$, the element $\overline{q_{m+1}}$ must be included in any sublist of generators from the above lists (2.3.7).*

Proof: By part (2) of the main lemma (2.3.10), we need only show that $\deg(q_{m+1})$ is strictly less than the degree of the remaining generator μ or η.

When $r = 5$, we have $m + 1 = 3$. Now $\deg(q_3) = 16 < 32 = \deg(\mu_5)$ (for the case $n = 10$) $= \deg(\eta_4)$ (for the case $n = 11$). For $r \geq 6$, we have

$$\deg(q_{m+1}) = 2^{m+2} \leq 4r < 2^{r-1}$$

which is less than or equal to the degree of μ_{r-1}, μ_r, and η_{r-1}. □

2.4 Proof of Theorem (2.1.1)

For a given n and rank $r \geq 5$, now choose a minimal sublist of generators from the appropriate list in (2.3.7) and label them $a_1, a_2, \ldots, a_N$. By the above proposition, $\overline{q_{m+1}}$ is one of the a_j, say $\overline{q_{m+1}} = a_1$. Let $Q = \overline{\mathbf{F}_2}[u_1, \ldots, u_N]$ and define a map $F\colon Q \to H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ by $u_i \mapsto a_i$. Let $J = \ker F$.

Any $\overline{p_i}$ is expressible as a polynomial in terms of the a_j, so for each i, $2 \leq i \leq r$ let us write

$$\overline{p_i} = P_i(a_2, \ldots, a_N) + a_1 Q_i(a_2, \ldots, a_N) \tag{2.4.1}$$

for some polynomials P_i and Q_i of $a_2, \ldots, a_N$. To see why no a_1^2 is needed in the above expression, recall that m satisfies $r < 2^{m+1}$, so that

$$\deg \overline{p_i} \leq 4r < 2^{m+3} < 2^{2m+4} = \deg(a_1^2) = \deg(\overline{q_{m+1}}^2) \tag{2.4.2}$$

for $2 \leq i \leq r$.

(2.4.3) **Claim.** *Any polynomial Q_i in (2.4.1) cannot be a nonzero constant.*

Proof: There are three cases to consider:

(1) n is odd,

(2) n is even and $2 \leq i \leq r-1$, and

(3) n is even and $i = r$.

In cases (1) and (2), if some Q_{i_0} were a nonzero constant, then

$$\overline{p_{i_0}} = P_{i_0}(a_2, \ldots, a_N) + a_1 Q_{i_0} \tag{2.4.4}$$

would be a counterexample to Proposition (2.3.11) since p_{i_0} is in the list of generators (2.3.7). In case (3), simply replace $\overline{p_r}$ by $\overline{c_r}^2$ in (2.4.4), and again we have a counterexample to Proposition (2.3.11). □

Let $\overline{q_{m+1}}^2 = P(\overline{p_2}, \ldots, \overline{p_r})$ be the relation guaranteed by part (1) of the main lemma (2.3.10). By (2.4.2), every monomial in the $\overline{p_i}$ in P is divisible by at least two of the $\overline{p_i}$. Consider any monomial $\alpha\overline{p_{i_1}}\,\overline{p_{i_2}}\cdots\overline{p_{i_k}}$ ($\alpha \in \overline{\mathbf{F}_2}, 2 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq r$ and $k \geq 2$) appearing in P. By (2.4.3) and degree considerations, at most one $Q_{i_j} \neq 0$. Hence we get

$$a_1^2 = R(a_2, \ldots, a_N) + a_1 S(a_2, \ldots, a_N)$$

for some polynomials R and S in $a_2, \ldots, a_N$. Therefore we have

$$0 \neq u_1^2 - R(u_2, \ldots, u_N) - u_1 S(u_2, \ldots, u_N) \in J.$$

Hence J is a non zero ideal. Since the list of generators is minimal and all the generators are of positive degree, we clearly have $J \subseteq \langle u_1, u_2, \ldots, u_N \rangle^2$.

Now $Q/J = H^*(B\tilde{T}(n))^{\mathcal{W}} \otimes \overline{\mathbf{F}_2}$ is the affine coordinate ring of a singular variety, as explained in §2.2 above, and is therefore not a polynomial ring. This completes the proof of Theorem (2.1.1). $\square$

3 Application to Topology

In this section we describe the topological problem which provides a context for the results of §2. A reference for facts given in this section is [2].

3.1 The Borel Homomorphism

(We follow the notation of §1.1.)

The torus T acts on the total space EK, and we have a fibration

$$EK/T \to EK/K$$

with fiber K/T. Since EK is contractible, EK/T is homotopy equivalent to BT. Hence the above fibration is nothing but

$$K/T \to BT \to BK.$$

The map $\psi\colon H^*(BT) \to H^*(K/T)$ induced in integral cohomology by the inclusion $K/T \to BT$ is called the *Borel homomorphism* for K. Of course we may consider $\psi_R\colon H^*(BT;R) \to H^*(K/T;R)$ for any coefficient ring R. The Weyl group $\mathcal{W}$ of K acts on T, BT, and K/T, and hence on cohomology rings of these spaces. The Borel homomorphism is equivariant with respect to these $\mathcal{W}$-actions. We will denote by I and I_R the kernels of ψ and ψ_R, respectively.

It is a fact that the *rational* Borel homomorphism

$$\psi_{\mathbf{Q}}\colon H^*(BT;\mathbf{Q}) \to H^*(K/T;\mathbf{Q})$$

is surjective, and that the kernel $I_{\mathbf{Q}}$ of $\psi_{\mathbf{Q}}$ is equal to the ideal $\langle (H^*(BT;\mathbf{Q})^{\mathcal{W}})_+\rangle$ generated by positive degree Weyl invariants. In general, the *integral* Borel homomorphism ψ is *not* surjective . The main result of this section is the following.

Corollary (3.3.6). *For $K = \mathrm{Spin}(n)$, $n \geq 10$, the kernel I of the Borel homomorphism $\psi\colon H^*(BT) \to H^*(K/T)$ properly contains the ideal $\langle (H^*(BT)^{\mathcal{W}})_+\rangle$ generated by Weyl invariants of positive degree.*

To analyze the Borel homomorphism ψ and the relationship between its kernel I and the Weyl invariants $H^*(BT)^{\mathcal{W}}$, it is convenient to use an identification of $H^*(BT)$ with the symmetric algebra (over the integers) $S(M)$ of the weight lattice M of K, as follows.

First we describe $\mathcal{W}$-equivariant isomorphisms

$$M \xrightarrow{\Phi} H^1(T) \xrightarrow{\Psi} H^2(BT).$$

Let $\check{M} = \ker[\exp\colon LT \to T]$ denote the coroot lattice for K. Since LT is the universal cover for T, we may identify $\check{M}$ with $\pi_1(T) = H_1(T)$. Hence, taking duals, $M \approx \mathrm{Hom}(H_1(T),\mathbf{Z})$, which by "universal coefficients" is $H^1(T)$. Denote by Φ this identification $M \xrightarrow{\approx} H^1(T)$. Next, Ψ is defined to be the differential (in the Leray-Serre spectral sequence) $d_2(\xi)\colon H^1(T) \to H^2(BT)$, where

$$\xi: T \to ET \to BT$$

is the universal T-bundle. Since ET is contractible (in particular, $H^1(ET) = H^2(ET) = 0$), we have that Ψ is an isomorphism.

Extending $M \stackrel{\Psi \circ \Phi}{\to} H^2(BT)$ yields a $\mathcal{W}$-equivariant isomorphism $S(M) \approx H^*(BT)$. Through this identification, we will speak of the "Borel homomorphism"

$$\psi: S(M) \to H^*(K/T).$$

We shall write $S_{\mathbf{Q}}$ for $S(M) \otimes \mathbf{Q}$, and speak of the rational Borel homomorphism $\psi_{\mathbf{Q}}: S_{\mathbf{Q}} \to H^*(K/T; \mathbf{Q})$ with kernel $I_{\mathbf{Q}}$.

3.2 BGG operators

In order to prove facts about the kernel I of the Borel homomorphism ψ and the Weyl invariants $S(M)^{\mathcal{W}}$, we utilize operators

$$\Delta_w: S(M) \to S(M)$$

defined for each $w \in \mathcal{W}$ by $\Delta_{r_i}(P) = (P - r_i P)/\alpha_i$ for $1 \le i \le r$, $P \in S(M)$. Then Δ_w is defined to be $\Delta_{r_{i_1}} \Delta_{r_{i_2}} \cdots \Delta_{r_{i_j}}$, where $w = r_{i_1} r_{i_2} \cdots r_{i_j}$ is a reduced expression (Δ_w is indeed independent of the choice of reduced expression). Each Δ_w stabilizes I, and hence defines an operator A_w on $S(M)/I$. We will also write Δ_w, A_w for the induced operators on $S_{\mathbf{Q}}$ and $S_{\mathbf{Q}}/I_{\mathbf{Q}}$. Since $\psi_{\mathbf{Q}}$ is surjective, each A_w is defined on all of $H^*(K/T; \mathbf{Q})$. In fact, each A_w is a well-defined operator on $H^*(K/T; \mathbf{Z})$ (see the appendix in [6] for an alternate description of the A_w). In [2], the following formulas are proved.

(3.2.1)
$$A_{r_i} u_w = \begin{cases} 0 & \text{if } l(wr_i) = l(w) + 1 \\ u_{wr_i} & \text{if } l(wr_i) = l(w) - 1\ . \end{cases}$$

(3.2.2)
$$A_{w^{-1}w_0} u_{w_0} = u_w, \ \text{ for all } w \in \mathcal{W}\ .$$

Repeated application of (3.2.1) gives that $A_w u_w = u_e = 1$, and if $l(v) = l(w)$ and $v \neq w$, we have $A_v u_w = 0$. Thus we have

$$A_v u_w = \delta_{v,w} \text{ when } l(v) = l(w) \; . \tag{3.2.3}$$

We have the following (easily verified) properties of the operators Δ_w.

$$P \in S(M)^{\mathcal{W}} \Longleftrightarrow \Delta_{r_i} P = 0 \text{ for all } i, 1 \leq i \leq r, \tag{3.2.4}$$

$$\Delta_w P = 0 \text{ if } l(w) > \deg P \; , \tag{3.2.5}$$

$$\Delta_{r_i}(PQ) = (\Delta_{r_i} P) \cdot Q + r_i P \Delta_{r_i} Q \; , \tag{3.2.6}$$

for all $P, Q \in S(M)$, $1 \leq i \leq r$.

3.3 The kernel of the Borel homomorphism and Weyl invariants

In this section we establish some connections between the kernel I of the Borel homomorphism and $S(M)^{\mathcal{W}}$.

We begin with a fact alluded to in §1, namely, that the $\mathcal{W}$-equivariance of ψ implies that the ideal $\langle S(M)^{\mathcal{W}}{}_{+} \rangle$ is contained in I. Indeed, if $P \in S(M)^{\mathcal{W}}{}_{+}$, then $\psi(P) = \psi(r_i P) = r_i \psi(P)$ for $1 \leq i \leq r$, so $\psi(P) \in H^*(K/T)^{\mathcal{W}}$. Since $H^*(K/T)^{\mathcal{W}} = H^0(K/T)$ ([7]), we have that $\psi(P)$ must be zero.

Next we examine a consequence of the well known fact that $I_{\mathbf{Q}} = \langle (S_{\mathbf{Q}})^{\mathcal{W}}{}_{+} \rangle$. Let $U_{w_0} = \dfrac{\rho^N}{N!} \in S_{\mathbf{Q}}$, where $\rho = \frac{1}{2} \sum_{\gamma \in R_+} \gamma$ and $N = l(w_0)$. By ([2], Corollary 3.16) we have $\psi_{\mathbf{Q}}(U_{w_0}) = u_{w_0}$. Now set $U_w = \Delta_{w^{-1} w_0} U_{w_0}$ for $w \in \mathcal{W}$. Then

$$\psi_{\mathbf{Q}}(U_w) = \psi_{\mathbf{Q}}(\Delta_{w^{-1} w_0} U_{w_0}) = A_{w^{-1} w_0} u_{w_0} = u_w$$

for all $w \in \mathcal{W}$ by (3.2.2). Observe that when $l(v) = l(w)$ we have

$$\psi_{\mathbf{Q}}(\Delta_v U_w) = A_v u_w = \delta_{v,w} \tag{3.3.1}$$

by (3.2.3), where δ denotes the Kronecker delta. Since $\psi_{\mathbf{Q}}\colon S^0_{\mathbf{Q}} \to H^0(K/T;\mathbf{Q})$ is an isomorphism ($1 \mapsto 1$), we have

$$\Delta_v U_w = \delta_{v,w} \text{ when } l(v) = l(w). \tag{3.3.2}$$

The following result is known, but we include a proof since the same argument is used in the proof of the subsequent proposition (3.3.4).

(3.3.3) **Proposition.** $S_{\mathbf{Q}} = \bigoplus_{w\in\mathcal{W}} (S_{\mathbf{Q}})^{\mathcal{W}} U_w.$

PROOF: First we show that the sum $\sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$ is direct. Suppose $0 = \sum_w F_w U_w$, $F_w \in (S_{\mathbf{Q}})^{\mathcal{W}}$. Note that by (3.2.6) and (3.2.4) we have, for any i $1 \le i \le r$, and for any $w \in \mathcal{W}$, that

$$\Delta_{r_i}(F_w U_w) = \Delta_{r_i} F_w \cdot U_w + r_i F_w \Delta_{r_i} U_w = F_w \Delta_{r_i} U_w.$$

By repeated application of the above, we have

$$\Delta_v(F_w U_w) = F_w \Delta_v U_w \text{ for all } v, w \in \mathcal{W}.$$

This, together with the facts (3.2.5) and (3.3.2), allows us to conclude

$$0 = \Delta_{w_0}(\sum_w F_w U_w) = \sum_w F_w \Delta_{w_0} U_w = F_{w_0}.$$

Now assume inductively that $F_w = 0$ for $l(w) > d$ (induction downwards on the length d). Then for $v \in \mathcal{W}$ such that $l(v) = d$, we have $0 = \Delta_v(\sum_{l(w)\le d} F_w U_w) = \sum_{l(w)\le d} F_w \Delta_v U_w = F_v$, completing the induction. This establishes that the sum $\sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$ is direct.

To show that $\sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$ is all of $S_{\mathbf{Q}}$, suppose this is not the case and let $X \in S_{\mathbf{Q}} \setminus \sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$ be a homogeneous element of least degree, say d. Write $\psi_{\mathbf{Q}}(X) = \sum_w \alpha_w u_w$, $\alpha_w \in \mathbf{Q}$. Let $Y = X - \sum_w \alpha_w U_w$ so $Y \in I_{\mathbf{Q}} = \langle (S_{\mathbf{Q}})^{\mathcal{W}}{}_+ \rangle$. Thus we may write $Y = \sum_i F_i G_i$, for some homogeneous $F_i \in (S_{\mathbf{Q}})^{\mathcal{W}}{}_+$, $G_i \in S_{\mathbf{Q}}$. Since $\deg G_i < d$, each G_i lies in $\sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$, so Y

also lies in $\sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$. But then $X = Y + \sum_w \alpha_w U_w$ lies in $\sum_w (S_{\mathbf{Q}})^{\mathcal{W}} U_w$, which contradicts our assumption on X. $\square$

Next we prove the following, a certain integral version of the above proposition. Let $\{e_i\}$ be a homogeneous $\mathbf{Z}$-basis for $\psi(S(M)) \subseteq H^*(K/T)$, and let $\{E_i\}$ be a set of homogeneous preimages in $S(M)$.

(3.3.4) **Proposition.** *If $I = \langle S(M)^{\mathcal{W}}{}_+ \rangle$ then $S(M) = \bigoplus_i S(M)^{\mathcal{W}} E_i$.*

Proof: First we establish that $S_{\mathbf{Q}} = \bigoplus_i (S_{\mathbf{Q}})^{\mathcal{W}} E_i$. Since $\psi_{\mathbf{Q}}$ is surjective, the rank of $\psi(S(M))$ equals the dimension over $\mathbf{Q}$ of $\psi_{\mathbf{Q}}(S_{\mathbf{Q}}) = H^*(K/T;\mathbf{Q})$. Thus $\{e_i\}$ is a $\mathbf{Q}$-basis for $H^*(K/T;\mathbf{Q})$, and we may choose (non-uniquely) a bijection $f\colon \{1,2,\ldots,|\mathcal{W}|\} \to \mathcal{W}$ such that $\deg E_i = l(f(i)) = \deg U_{f(i)}$. The fact that $\sum_i (S_{\mathbf{Q}})^{\mathcal{W}} E_i$ spans all of $S_{\mathbf{Q}}$ is established by the same argument as in the proposition above, replacing U_w by E_i. To see that the sum $\sum_i (S_{\mathbf{Q}})^{\mathcal{W}} E_i$ is direct, consider the $(S_{\mathbf{Q}})^{\mathcal{W}}$-module map $F\colon S_{\mathbf{Q}} \to S_{\mathbf{Q}}$ defined by $U_{f(i)} \mapsto E_i$. F is a surjective $\mathbf{Q}$-linear map of vector spaces over $\mathbf{Q}$ of the same dimension in each degree, so F is an isomorphism. Thus we have $0 = \sum_i F_i E_i \Leftrightarrow 0 = \sum_i F_i U_{f(i)} \Leftrightarrow F_i = 0$ for all i, for $F_i \in (S_{\mathbf{Q}})^{\mathcal{W}}$.

Now we come to the proof that $S(M) = \bigoplus_i S(M)^{\mathcal{W}} E_i$. First of all, by the same argument as in the proof of (3.3.3), it follows that $S(M) = \sum_i S(M)^{\mathcal{W}} E_i$ (by making use of the hypothesis $I = \langle S(M)^{\mathcal{W}}{}_+ \rangle$). Next, the sum $\sum_i S(M)^{\mathcal{W}} E_i$ is clearly direct, since a relation $0 = \sum_i F_i E_i$ in $S(M)$ is a relation in $S_{\mathbf{Q}}$, so all the F_i are zero. $\square$

Applying Observation (1.2.4) to $S = S(M)$ and $G = \mathcal{W}$ and Proposition (3.3.4) shows that the hypotheses of Corollary (1.2.3) are satisfied with $S = S(M)$ and $A = S(M)^{\mathcal{W}}$, so that we have the following.

(3.3.5) **Corollary.** *If $I = \langle S(M)^{\mathcal{W}}{}_+ \rangle$, then $S(M)^{\mathcal{W}}$ is a polynomial ring.*

We conclude with a corollary of Theorem (2.1.1), which follows immediately from Corollary (3.3.5).

(3.3.6) **Corollary.** *For $K = \mathrm{Spin}(n)$, $n \geq 10$, the kernel of the Borel homomorphism properly contains the ideal generated by Weyl invariants of positive degree.*

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