Quantum Entanglement
Math Background Notes

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1 Matrix Groups and Their Lie Algebras

Unitary matrices play a central role in quantum mechanics. They are an example of a more general object called a *Lie group*, named after Sophus Lie and pronounced "Lee." The theory of Lie groups is useful in the study of quantum information theory. These notes are a basic introduction to concrete examples of Lie groups called matrix groups and some of their basic theory.

(1.1) **Definition.** A nonempty set of $n \times n$ matrices with real or complex entries is a ***matrix group*** if it is closed under matrix multiplication and inversion.

Key examples of matrix groups are the following.

- the ***trivial group*** $\{\mathrm{Id}\}$
- the ***orthogonal group*** $O(n) = \{g \mid gg^T = \mathrm{Id}\}$ (real entries)
- the ***special orthogonal group*** $SO(n) = \{g \mid g \in O(n), \det(g) = 1\}$
- the ***unitary group*** $U(n) = \{g \mid gg^\dagger = \mathrm{Id}\}$ (complex entries)
- the ***special unitary group*** $SU(n) = \{g \mid g \in U(n), \det(g) = 1\}$
- the ***general linear group*** $GL(n, \mathbf{R})$ or $GL(n, \mathbf{C})$ of all invertible $n \times n$ matrices (real or complex entries)

Each of these examples turns out to be a surface in $\mathbf{R}^{n^2}$ or $\mathbf{C}^{n^2} = \mathbf{R}^{2n^2}$. The trivial group is a single point and the general linear group is an open subset of $\mathbf{R}^{n^2}$ or $\mathbf{C}^{n^2}$. The unitary group $U(1)$ is the unit circle S^1 in $\mathbf{C} = \mathbf{R}^2$, and the special orthogonal group $SO(2)$ turns out to be the same thing even though it lives in $\mathbf{R}^4$. The special unitary group $SU(2)$ is the three dimensional sphere S^3 (even though it lives in $\mathbf{C}^4 = \mathbf{R}^8$).

Of particular interest in quantum information is the group $SU(2)$ of 2×2 unitary matrices with determinant 1.

(1.2) **Exercise.** Show that the following definitions of unitary matrix are equivalent.

1. An $n \times n$ complex matrix g is ***unitary*** if $gg^\dagger = \mathrm{Id}$.
2. An $n \times n$ complex matrix g is ***unitary*** if its columns form an orthonormal basis of $\mathbf{C}^n$.
3. An $n \times n$ complex matrix g is ***unitary*** if $\langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in \mathbf{C}^n$, where $\langle v, w \rangle$ denotes the inner product $v^\dagger w$ of v with w.

Comment on this exercise: the same facts hold for the orthogonal group if you interpret the Hermitian transpose † as ordinary transpose. Since inner product gives rise to length of vectors, and therefore to distances between vectors and angles between vectors, we say that unitary (or orthogonal) transformations are ***rigid***, ***distance preserving***, or are called ***isometries***.

(1.3) **Exercise.** Show that a 2×2 matrix g with complex entries is in $SU(2)$ if and only if

$$g = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix}$$

for some $a, b \in \mathbf{C}$ with $|a|^2 + |b|^2 = 1$. This affords a one-to-one correspondence $SU(2) \leftrightarrow S^3 \subset \mathbf{R}^4$ given by

$$\begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \leftrightarrow (x, y, z, w)$$

where $a = x + iy, b = z + iw$.

Here is a definition from multivariable calculus.

(1.4) **Definition.** Given a surface $\Sigma \subset \mathbf{R}^N$ and a point $p \in \Sigma$, the ***tangent space to*** Σ ***at*** p, denoted $T_p\Sigma$, is the set of all velocity vectors of curves at p.

$$T_p\Sigma = \{\alpha'(0) \mid \alpha \text{ a curve in } \Sigma, \alpha(0) = p\}$$

Familiar examples are tangent lines to curves in the plane and tangent planes to 2-dimensional surfaces in $\mathbf{R}^3$.

(1.5) **Definition.** The ***Lie algebra*** of a matrix group G, denoted LG, is defined to be its tangent space at the identity matrix.

$$LG = T_{\mathrm{Id}}G = \{\alpha'(0) \mid \alpha \text{ a curve in } G, \alpha(0) = \mathrm{Id}\}$$

Since points in G live in $\mathbf{R}^{n^2}$ (or $\mathbf{C}^{n^2}$), the velocity vectors in LG also live in $\mathbf{R}^{n^2}$ (or $\mathbf{C}^{n^2}$), and are therefore naturally identified with $n \times n$ matrices. We usually think of LG as a set of $n \times n$ matrices. A concrete example is $U(1)$, which is the unit circle in the complex plane. The tangent space at identity is the vertical line through 1, which is the set of 1-dimensional complex vectors in the imaginary direction. Thus $L(U(1))$ is the set $\{it \mid t \in \mathbf{R}\}$.

It is customary to denote the Lie algebra of a matrix group by the same name as the group but using lower case letters. For example, $L(GL(n)) = gl(n)$, and $L(SU(n)) = su(n)$.

The important fact about LG is that it is a *vector space*, that is, it is closed under matrix addition and scalar multiplication (this is *not* true of G). While LG is not closed under matrix multiplication, it will turn out that it is closed under the ***Lie bracket*** or ***commutator*** defined by

$$[X, Y] = XY - YX$$

where X, Y are matrices in LG. Philosophically and practically, difficult questions about a matrix group are transformed into linear algebra questions about its Lie algebra. Multiplication at the group level is replaced by Lie bracket at the Lie algebra level. The bridge between Lie algebra and matrix group is the ***exponential map***, given by

$$\exp(A) = e^A = \mathrm{Id} + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots. \tag{1.6}$$

It is a fact that the infinite series converges for all square matrices A, and that $\exp(A)$ always lies in the Lie group for which A is an element of the Lie algebra. We shall prove particular cases of some of these statements in subsequent exercises. Here is an important fundamental fact.

(1.7) **Exercise.** Let A be an $n \times n$ matrix and consider the curve $\alpha(t) = \exp(tA)$ in $\mathbf{R}^{n^2}$ (or $\mathbf{C}^{n^2}$) . Show that

$$\alpha'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp(tA) = A.$$

Assume that the series defining $\exp(tA)$ converges and that term-by-term differentiation is valid.

Here is a series of exercises for the specific matrix groups $U(2)$ and $SU(2)$. The exercises will show that $L(U(2)) = u(2)$ is the set of 2×2 skew-Hermitian matrices, and that $L(SU(2)) = su(2)$ is the subset of skew-Hermitian matrices with trace zero.

(1.8) **Exercise.** Throughout this exercise, let X denote a 2×2 matrix with complex entries.

1. Show that X is skew-Hermitian (definition: X is skew-Hermitian means $X = -X^\dagger$) if and only if iX is Hermitian.
2. Show X is skew-Hermitian if and only if
$$X = \begin{bmatrix} it & -\overline{a} \\ a & is \end{bmatrix}$$
for some $t, s \in \mathbf{R}$, $a \in \mathbf{C}$.
3. Show that if X is skew-Hermitian, then e^X is unitary. Hint: Since the curve $e^{t(X+X^\dagger)}$ is stationary, we have
$$\mathrm{Id} = e^0 = e^{t(X+X^\dagger)} = e^{tX}e^{tX^\dagger} = e^{tX}\left(e^{tX}\right)^\dagger$$
for all t, and in particular for $t = 1$.
4. Use (1.7) to conclude that skew-Hermitian matrices are a subset of $L(U(2))$.
5. Prove this product rule for matrix-valued functions. Let $\alpha(t), \beta(t)$ be curves in the set of 2x2 unitary matrices. Show that
$$\frac{d}{dt}\left(\alpha(t)\beta(t)\right) = \alpha'(t)\beta(t) + \alpha(t)\beta'(t).$$
6. Let $X \in L(U(2))$ (do not assume X is skew-Hermitian—that is what you are trying to prove!). Let $\alpha(t)$ be a curve in $U(2)$ such that $\alpha(0) = \mathrm{Id}$ and $\alpha'(0) = X$. Apply the previous part of the exercise to the equation
$$\alpha(t)(\alpha(t))^\dagger = \mathrm{Id}$$

to get $X = -X^\dagger$, i.e., X is skew-Hermitian.

Conclude that the Lie algebra of $U(2)$ is contained in the set of skew-Hermitian matrices.

Conclude that the Lie algebra of $U(2)$ is equal to the set of skew-Hermitian matrices.

7. Show that the condition $\det(g) = 1$ that defines the subset $SU(2)$ of $U(2)$ gives the condition $\mathrm{trace}(X) = 0$ that defines the subset $su(2)$ of $u(2)$. Thus the special unitary Lie algebra is the set of traceless skew-Hermitian matrices.

   Solution outline: First verify this lemma: For any normal matrix A, we have $\det e^A = e^{\mathrm{trace} A}$. Hint for the lemma: use spectral decomposition for A. Now let $X \in L(SU(2))$. By the facts stated in the sentence after (1.6), we know e^{tX} lies in $SU(2)$ for all t. Consider the constant function $t \mapsto \det e^{tX} = 1$. Take the derivative of both sides at $t = 0$, and use the lemma to get $\mathrm{trace}(X) = 0$. This shows that $L(SU(2))$ is contained in the set of traceless skew-Hermitian matrices. Conversely, if $\mathrm{trace}(X) = 0$, use the lemma again to show that e^{tX} lies in $SU(2)$. Since X is the velocity vector at time zero of the curve $t \mapsto e^{tX}$, we have $X \in L(SU(2))$.

2 Matrix Group and Lie Algebra Actions

A matrix group G is said to ***act on*** a set S if there is a map

$$\Phi \colon G \times S \to S$$

that satisfies

(i) $\Phi(\mathrm{Id}, s) = s$ for all $s \in S$

(ii) $\Phi(g, \Phi(h, s)) = \Phi(gh, s)$ for all $g, h \in G$, $s \in S$.

If S is a vector space, then we will also demand that

(iii) the map $S \to S$ given by $s \mapsto \Phi(g, s)$ is linear for all $g \in G$.

It is customary to write gs for $\Phi(g, s)$. In this shorter notation, condition (ii) says $g(hs) = (gh)s$.

The fundamental example of a group action is called the ***natural action***, in which G is a matrix group of $n \times n$ real or complex matrices and S is $\mathbf{R}^n$ or $\mathbf{C}^n$, respectively, with the action defined by

$$\Phi(g, s) = \text{ the matrix } g \text{ times the column vector } s.$$

(2.1) **Exercise.** Verify that the natural action satisfies conditions (i)–(iii) in the definition of action above.

A nontrivial and important example is the action of the unitary group on the set of density matrices (positive semidefinite matrices with trace 1), given by

$$\Phi(U, \rho) = U \rho U^\dagger.$$

(2.2) **Exercise.** Verify that the action of the unitary group on density matrices satisfies conditions (i) and (ii) in the definition of action. In particular, show that the image of Φ is indeed contained in the set of density matrices. Does it make sense to check condition (iii)?

Another important action is the ***adjoint action*** of a matrix group G on its Lie algebra LG given by

$$\Phi(g, X) = gXg^{-1}.$$

(2.3) **Exercise.** Verify that the Adjoint action satisfies conditions (i)–(iii) in the definition of action above.

Associated to every matrix group action on a vector space there is a corresponding Lie algebra action. Given an action $\Phi\colon G \times V \to V$, we define a map $\phi\colon LG \times V \to V$ by

$$\phi(X, v) = \frac{d}{dt}\bigg|_{t=0} (e^{tX})v = \lim_{h\to 0} \frac{e^{hX}v - v}{h}.$$

As with the group action notation, we usually shorten $\phi(X, v)$ by writing Xv.

(2.4) **Exercise.** Verify that the Lie algebra action corresponding to the natural action of G on $\mathbf{R}^n$ or $\mathbf{C}^n$ is given by

$$\phi(X, v) = \text{ the matrix } X \text{ times the column vector } v.$$

Hint: Write out the Taylor series for e^{hX} and evaluate the limit on the right hand side.

(2.5) **Exercise.** Verify that the Lie algebra action corresponding to the action of the unitary group on density matrices is given by

$$\phi(X, \rho) = X\rho - \rho X.$$

Same hint as the last exercise.

Comment on this exercise: The alert reader may have noticed that we defined Lie algebra action for the case when G acts on a vector space, but not in general for G acting on a set (why?). But the set of density matrices is not a vector space, and $\phi(X, \rho)$ is not in general a density matrix. In this case we think of ϕ as taking values in the codomain $gl(n)$ of all $n \times n$ matrices. Then the definition works.

(2.6) **Exercise.** Verify that the Lie algebra action corresponding to the adjoint action of G on LG is given by

$$\phi(X, Y) = XY - YX.$$

This action is called the ***adjoint*** action of LG on itself (so "adjoint" refers to both the group and Lie algebra actions). The expression $XY - YX$ is denoted $[X, Y]$, and the binary operation given by the adjoint action is called the ***Lie bracket***. Verify that the Lie bracket is bilinear, that is, show that

1. $[cX + Z, Y] = c[X, Y] + [Z, Y]$ and
2. $[X, cY + Z] = c[X, Y] + [X, Z]$

for all $X, Y, Z \in LG$ and $c \in \mathbf{C}$ (or $\mathbf{R}$).

(2.7) **Exercise.**

1. Verify by direct checking that $u(2)$ and $su(2)$ are closed under bracket.
2. Show that $su(2)$ with its Lie bracket is ***isomorphic*** to $\mathbf{R}^3$ with cross product. That is, show there is an invertible linear map $\alpha\colon su(2) \to \mathbf{R}^3$ such that $\alpha([X,Y]) = \alpha(X) \times \alpha(Y)$ for all $X, Y \in su(2)$. Here is an outline. Let $E_1 = \begin{bmatrix} i/2 & 0 \\ 0 & -i/2 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 0 & i/2 \\ i/2 & 0 \end{bmatrix}$. Show that E_1, E_2, E_3 form a basis for the real vector space $su(2)$. Define α by $\alpha(E_i) = e_i$ for $1 \le i \le 3$, where e_i is the ith standard basis vector for $\mathbf{R}^3$. Show that α has the desired property.

Given an action of a matrix group G on a set S and an element $s \in S$, we define the ***stabilizer*** or ***isotropy subgroup for*** s to the the subset of G given by

$$\mathrm{Stab}_s = \{g \in G\colon gs = s\}.$$

(2.8) **Exercise.** Verify that Stab_s is indeed a ***subgroup*** of G, that is, a subset of G that satisfies the definition of matrix group.

For an example of an isotropy subgroup, let G be $U(n)$ acting naturally on $S = \mathbf{C}^n$, and let $s = (1, 0, \ldots, 0)$. The isotropy subgroup for s is the set of all matrices that fix s, that is, the set of all unitary matrices whose first column (and therefore also first row), is $(1, 0, \ldots, 0)$. The remaining $(n-1) \times (n-1)$ lower right-hand corner submatrix can contain any $U(n-1)$ matrix, so Stab_s is in natural one-to-one correspondence with the matrix group $U(n-1)$.

The Lie algebra $L(\mathrm{Stab}_s)$ of an isotropy subgroup is called the ***stabilizer*** or ***isotropy subalgebra*** for s and plays an important role in our research.

(2.9) **Exercise.** When G is a matrix group of $n \times n$ matrices acting naturally on $\mathbf{C}^n$, verify that $L(\mathrm{Stab}_s)$ is given by

$$L(\mathrm{Stab}_s) = \{X \in LG\colon Xs = 0\}.$$

Show that $L(\mathrm{Stab}_s)$ is closed under addition, scalar multiplication, and Lie bracket.

To complete the basic vocabulary list for group actions, we define the ***orbit*** of $s \in S$ to be the subset of S given by

$$\mathcal{O}_s = \{gs\colon g \in G\}.$$

In the example above of the unitary group acting on $\mathbf{C}^n$ with $s = (1, 0, \ldots, 0)$, the orbit of s is the set of unit length vectors in $\mathbf{C}^n$. The orbits of a group action of G on S partition S into disjoint sets whose union is all of S. This is equivalent to saying that the relation

$$s \sim s' \text{ if there is some } g \text{ such that } gs = s'$$

is an equivalence relation.

(2.10) **Exercise.** Show that there is a one-to-one correspondence

$$\mathcal{O}_s \leftrightarrow G/\mathrm{Stab}_s$$

given by $gs \leftrightarrow [g]$, where G/Stab_s denotes the set of equivalence classes of the equivalence relation on G given by $g \sim h$ if and only if $h^{-1}g \in \mathrm{Stab}_s$.

3 Actions on Tensor Products

If matrix group G acts naturally on V and matrix group H acts naturally on W (where V, W are not necessarily the same dimension, but are either both real or both complex vector spaces) then $G \times H$ acts on the tensor product $V \otimes W$ where the action is given by

$$(g,h)(v \otimes w) = gv \otimes hw = (g \otimes h)(v \otimes w). \tag{3.1}$$

The expression $g \otimes h$ denotes the Kronecker product of g times h as matrices. The Lie Algebra $L(G \times H) = LG \times LH$ also acts on $V \otimes W$ by

$$(X,Y)(v \otimes w) = Xv \otimes w + v \otimes Yw = (X \otimes \mathrm{Id} + \mathrm{Id} \otimes Y)(v \otimes w). \tag{3.2}$$

(3.3) **Exercises.**

1. Derive (3.2). The definition of Lie algebra action says that

$$(X,Y)(v \otimes w) = \left.\frac{d}{dt}\right|_{t=0} \left[(e^{tX}, e^{tY})(v \otimes w)\right].$$

Write out the series for the exponential function and take the derivative at $t = 0$. It may be helpful to prove this lemma: Let $v(t), w(t)$ be curves in V, W, respectively. Then

$$\frac{d}{dt}(v(t) \otimes w(t)) = v'(t) \otimes w(t) + v(t) \otimes w'(t).$$

2. Let $G = U(2) \times U(2)$ and $V = \mathbf{C}^2 \otimes \mathbf{C}^2$. Let $v = |01\rangle + |10\rangle$.
   (a) Calculate gv and write the result in the standard basis for
   $$g = \left(\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$
   (b) Calculate gv and write the result in the standard basis for
   $$g = \left(\begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix}, \begin{bmatrix} c & -\overline{d} \\ d & \overline{c} \end{bmatrix} \right).$$
   (c) Calculate Xv and write the result in the standard basis for $X = (i\sigma_x, i\sigma_y)$.
   (d) Calculate Xv and write the result in the standard basis for
   $$X = \left(\begin{bmatrix} it & -\overline{a} \\ a & -it \end{bmatrix}, \begin{bmatrix} is & -\overline{b} \\ b & -is \end{bmatrix} \right).$$

4 Partial Trace

Given operators $g: V \to V$ and $h: W \to W$ on vector spaces V, W, we have an operator $g \otimes h: V \otimes W \to V \otimes W$ as defined in the previous section. We define the ***partial trace*** over the subsystem V of $g \otimes h$ to be the operator

$$\mathrm{tr}_V(g \otimes h) = (g \otimes h)^W = \mathrm{tr}(g)\, h: W \to W.$$

Similar, we define the partial trace over W to be

$$\operatorname{tr}_W(g \otimes h) = (g \otimes h)^V = \operatorname{tr}(h)\, g\colon V \to V.$$

We extend the partial trace linearly to functions

$$\begin{array}{rcl} \operatorname{tr}_V & : & L(V \otimes W) \to L(W) \\ \operatorname{tr}_W & : & L(V \otimes W) \to L(V) \end{array}$$

(where $L(\text{vector space})$ denotes the space of linear operators on that vector space). It is a fact that the partial trace functions are well-defined by these formulas.

(4.1) **Exercises.**

1. Let $e_1, \ldots, e_n$ be an orthonormal basis for V and $f_1, \ldots, f_m$ be an orthonormal basis for W so that $\{e_i \otimes f_j\}$ is a basis for $V \otimes W$. Given $\rho\colon V \otimes W \to V \otimes W$, let $\rho_{ij,k\ell}$ denote the matrix entry in ρ given by

$$\rho_{ij,k\ell} = \langle e_i \otimes f_j \mid \rho(e_k \otimes f_\ell)\rangle.$$

Then we have

$$(\rho_V)_{ik} = (\operatorname{tr}_W \rho)_{ik} = \sum_{\ell=1}^{m} \rho_{i\ell,k\ell}.$$

2. Consider the special case where $H = V \otimes W$ is n-qubit state space with W the subsystem of a single qubit, say the kth. We write $\rho^{(k)}$ to denote the partial trace $\rho^V = \operatorname{tr}_k \rho$ over the kth qubit for $\rho \in L(H)$. Verify the following formula for partial trace of a single qubit. In the standard computational basis with

$$\rho = \sum_{I,J} \rho_{I,J} E_{I,J} = \sum_{I,J} \rho_{I,J} \left|i_1\right\rangle \left\langle j_1\right| \otimes \cdots \left|i_n\right\rangle \left\langle j_n\right|$$

we have

$$\begin{aligned} \rho^{(k)} &= \operatorname{tr}_k \rho \\ &= \sum_{I,J: i_k = j_k} \rho_{I,J} \left|i_1\right\rangle \left\langle j_1\right| \otimes \cdots \widehat{\left|i_k\right\rangle \left\langle j_k\right|} \cdots \otimes \left|i_n\right\rangle \left\langle j_n\right| \\ &\quad \text{(since } \operatorname{tr} E_{ij} = \delta_{ij}) \\ &= \sum_{I',J'} (\rho_{I'_{k;0},J'_{k;0}} + \rho_{I'_{k;1},J'_{k;1}}) \left|i'_1\right\rangle \left\langle j'_1\right| \otimes \cdots \otimes \left|i'_{n-1}\right\rangle \left\langle j'_{n-1}\right| \end{aligned}$$

where the circumflex "hat" symbol denotes omission of a factor, the symbols I', J' are $(n-1)$-qubit multi-indices, and $I'_{k;r}$ denotes the n-qubit multi-index $i'_1 i'_2 \ldots i'_{k-1} r i_k \ldots i'_{n-1}$ formed by inserting r in the kth position of $I' = i'_1 i'_2 \ldots i'_{n-1}$.

3. (continues the previous problem) Let

$$\sigma_1 = \frac{1}{2}\sigma_x, \qquad \sigma_2 = \frac{1}{2}\sigma_y, \qquad \sigma_3 = \frac{1}{2}\sigma_z$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices, and let

$$\sigma_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \frac{1}{2}\text{Id}.$$

Let $A = \sum_I a_I \sigma_I$ be a matrix in $gl(2^n, \mathbf{C}) = L(H)$. Verify that the trace over the k-th qubit of A is

$$\mathrm{tr}_k A = \sum_{I:i_k=0} a_I \sigma_{i_1} \otimes \cdots \widehat{\sigma_{i_k}} \cdots \otimes \sigma_{i_n}.$$

5 Research Problems

This section is a brief and broad introduction to the areas of investigation of past and current interest to the LVC Mathematical Physics Research Group.

Let H_A, H_B be the state spaces of quantum systems A, B, respectively, and let $H = H_A \otimes H_B$ the the state space of the composite system AB. A state $|\psi\rangle$ is a ***product state*** if $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$ for some $|\alpha\rangle$ in H_A, and $|\beta\rangle$ in H_B. If $|\psi\rangle$ is *not* a product, then $|\psi\rangle$ is said to be ***entangled*** with respect to subsystems A, B.

(5.1) **Exercise.** Show that the 2-qubit singlet state $|00\rangle + |11\rangle$ is entangled with respect to the two 1-qubit subsystems.

Entanglement is an important feature of quantum systems. It seems to be necessary for better-than-classical performance in quantum algorithms. It is easy to define, but not so easy to study.

Let $U(H_A), U(H_B)$ be the groups of unitary operators on H_A, H_B, respectively. The group $U(H_A) \times U(H_B)$ is called the ***local unitary group*** (or *LU* group) of operators on the composite system AB. The *LU* group sits inside of the full unitary group $U(H)$ by the embedding

$$g, h \mapsto g \otimes h$$

for $g \in U(H_A), h \in U(H_B)$.

It is natural to say that two states $|\psi\rangle, |\psi'\rangle$ in H ***have the same entanglement type*** if there is a local unitary operator g, h that transforms $|\psi\rangle$ into $|\psi'\rangle$, that is, if $(g \otimes h)|\psi\rangle = |\psi'\rangle$. Thus the study of entanglement leads to the following main problem, which is currently unsolved.

> **Local Unitary Equivalence Classification Problem.** Classify the equivalence classes of states under local unitary equivalence. That is, provide a description of all the possible mutually inequivalent states in a composite system. This could also be called the problem of **classification of entanglement types**.

This problem is solved for 2 and 3 qubit systems. Only partial results are known for higher numbers of qubits. For multi-qubit systems (sometimes called "many party systems") the local unitary group is $U(2) \times U(2) \times \cdots \times U(2)$, the product of n copies of $U(2)$, where n is the number of qubits. It is convenient to consolidate the overall phase into one factor and work with the *LU* group

$$U(1) \times SU(2) \times SU(2) \times \cdots \times SU(2).$$

The corresponding local unitary Lie algebra is

$$u(1) \times su(2) \times su(2) \times \cdots \times su(2).$$

Lyons and Walck and students have achieved a series of partial results towards the main problem. We have used a two-step method.

1. **Stabilizer Lie algebra classification.** Identify possible stabilizer subalgebras, possibly with certain characteristics.
2. **Classify states with a given stabilizer.** Given a stabilizer subalgebra type, find and classify all possible states that have that stabilizer, up to local unitary equivalence.

Thus we achieve a partial classification of local unitary equivalence classes.

A second area of ongoing investigation is to consider what information is retained or lost under the operation of partial trace. Let D_m denote the space of m-qubit density operators. Given an n-qubit density operator ρ and a subsystem V of m qubits, there is a map

$$P_V\colon D_n \to D_m$$

that takes ρ to ρ^V by taking partial trace over the compliment of V. One can ask how much of ρ can be recovered from a knowledge of $P_V(\rho)$ for various subsystems V.

> **Parts and whole problem.** Given various reduced density matrices (partial traces) of a possibly unknown density matrix ρ, how much information can be recovered about ρ?

(5.2) **Exercises.**

1. Let $|\psi\rangle = |01\rangle - |10\rangle$. Find Stab_ψ and $K_\psi := L(\mathrm{Stab}_\psi)$.
2. Let $|\psi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$. Find Stab_ψ and K_ψ when $\alpha\beta \neq 0$.
3. For $|\psi\rangle$ in the previous problem, find all the reduced density matrices. That is, find all possible partial traces over all possible subsystems of the 3-qubit system.