A Simple Way To Teach Logarithms

by Richard Hammack and David Lyons, published in Mathematics Teacher, May 1995

We have taught logarithms in high school and college algebra and introductory calculus courses. Before we began using the method of teaching logs described in this article, we found that many students had difficulties mastering the concept, more so than with other functions. Other teachers we talked to reported a similar experience.

The conceptual way to understand the function $y = \log_a x$ is to view it as the inverse of $y = a^x$. When students are beginning to learn functions, they can get lost in the details of working with inverses. Our approach is nothing more than a simple change of notation; we replace $\log_a$ by $a^\square$ (read "a-box"). We have used this device in our classes and have found a great improvement in student comprehension of logarithms.

Beginning with examples rather than general definitions, we write on the board:

$$2^\square(8) = \underline{\quad\quad}?$$

We say, "Two-box of eight equals blank. What number goes in the box so that 2 raised to that power is 8?" Since $2^3 = 8$, we fill in the blank:

$$2^\square(8) = \underline{3}\,.$$

We ask the class to supply the answers to the next few examples.

$$\begin{aligned} 2^\square(16) &= 4 \\ 2^\square(2) &= 1 \\ 3^\square(27) &= 3 \\ 3^\square(9) &= 2 \end{aligned}$$

By the fifth example, everyone in the room is chiming in, "Three-box of nine is two." Next we introduce some examples which require a little more thought.

$$2^\square(\frac{1}{2}) = \underline{\quad\quad}?$$

Give the class a few moments to consider the answer, then explain that since $2^{-1} = \dfrac{1}{2}$, we have

$$2^\square(\frac{1}{2}) = \underline{-1}\,.$$

We do some more examples with negative and fractional answers, carefully explaining how to use the appropriate properties of exponents.

$$\begin{aligned} 2^\square(\sqrt{2}) &= \frac{1}{2} \\ 3^\square(\frac{1}{9}) &= -2 \\ 8^\square(2) &= \frac{1}{3} \\ 2^\square(\frac{1}{16}) &= -4 \end{aligned}$$

Everyone in the class usually understands the pattern after a few examples. At this point we begin to point out some general properties of the function $y = a^{\Box}(x)$. Do these examples one at a time, asking the students to reason them out before giving the answer.

$$2^{\Box}(1) = 0$$
$$3^{\Box}(1) = 0$$

The observation here is that no matter what "a" is, we have $a^{\Box}(1) = 0$ since $a^0 = 1$. Then we ask,

$$2^{\Box}(-2) = \underline{\quad\quad}?$$

Since two raised to *any* exponent is positive (we like to reinforce this by pointing to the graph of $y = 2^x$), we see that $2^{\Box}(-2)$ cannot exist. Indeed for *any* $a > 0$, we have $a^{\Box}(x)$ does not exist for $x \leq 0$. At this point it is appropriate to draw the graphs of a couple of examples of this new function, say $y = 2^{\Box}(x)$ and $y = 3^{\Box}(x)$, pointing out their common features: the x-intercept is (1,0); the y axis is a vertical asymptote; and their domain excludes the negative numbers and zero. We also establish the essential connection with the exponential functions by demonstrating that $y = 2^{\Box}(x)$ is the inverse of $y = 2^x$.

Finally we announce that another way to write $2^{\Box}(x)$ is $\log_2(x)$ and in general, $a^{\Box} = \log_a$. Again a few examples are in order.

$$\begin{aligned} \log_3(81) &= 3^{\Box}(81) = 4 \\ \log_4(2) &= 4^{\Box}(2) = \frac{1}{2} \\ \log_2(-1) &= 2^{\Box}(-1) \text{ does not exist} \end{aligned}$$

Since "two-box" isn't a scary function, neither is this strange new word "logarithm." In short, the a-box notation is a bridge to the standard $\log_a$ notation. After some practice, a-box can be dropped entirely.

The nice feature about a-box notation is that it makes evaluation of the log function transparent. It can be used to establish the basic properties of logs. For example, $a^{\Box}(a^x) = x$. Translating into standard notation we have the familiar rule $\log_a(a^x) = x$.

This presentation can be embellished and adapted to fit your students and your teaching style. In fact, the box idea can be used for any inverse function. For example, you can explain arcsin by using $\sin \Box$.

We have had tremendous success using a-box to teach logarithms. It is our hope that others will try it and that their students will benefit from it. We taught a college algebra course where an identical quiz on logs was given to three sections of the same course. In one of the sections a-box was used, and the other two sections had logarithms presented using only the standard notation. The students in the a-box section evaluated the logs on the quiz with one hundred percent accuracy. Most of them translated from $\log_a$ to a-box to find the result. Students in the other two sections showed the usual wide range of confusions we commonly see with logs. Please send us reports of your successes or criticisms if you try this method in your class. Our address is Math Dept., CB #3250, UNC-CH, Chapel Hill, N.C. 27599. We can be reached by email at hammack@math.unc.edu and dwl@math.unc.edu.