Mathematical Reasoning II Course Notes
Spring 2006

David Lyons
Mathematical Sciences
Lebanon Valley College

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Contents

6 Nature of the Real Numbers

The line is an elegant, simple-looking model for the real numbers. Despite its apparent simplicity, the real line entails subtleties that are both interesting and instructive to contemplate. In this section we present a few ideas that require us to stretch our intuition.

6.1 Some Greek philosophy

In the philosophy of the Pythagorean school (named after Pythagoras, ca. 540 B.C.), the universe could be explained by relationships among whole numbers. The Pythagoreans attempted to use ratio and proportion of whole numbers to describe and explain everything from the planets and the stars to musical scales. The Pythagoreans viewed natural science and mathematics as parts of one great study.

Here is an explanation of the intuition behind this philosophy. Quantities can be represented by points on a line. The whole numbers lie at regularly spaced points on the line like markings on a ruler. Suppose we have a point p somewhere between the whole numbers 0 and 1, and we wish to associate a number with p. Just as rulers are marked with finer and finer subdivisions to obtain ever more precise measurements, we can subdivide the interval from 0 to 1 into equally spaced subintervals. It is intuitively reasonable to expect that the point p will coincide with one of these subdivision markings, although we admit that the number of subdivisions may need to be very large. If we divide the interval from 0 to 1 into n equal sized subintervals, and if p coincides with one of the markings obtained this way, then p is represented by the rational number m/n, where m is some whole number. Likewise, any other point q must be represented by some rational number, say s/t. Then the proportion of $p\colon q$ is $(m/n)\colon(s/t) = mt\colon ns$. The quantities p and q are in whole number proportion to one another.

The thinking outlined in the previous paragraph seemed so reasonable as to be obvious. Even more, the perfection of whole numbers and their ratios was central to the Greek philosophical world view. It was therefore a shocking discovery to find that there existed numbers on the real line which corresponded to *no* rational number, and therefore that there were quantities whose proportions could not be expressed in terms of whole numbers. The first evidence of irrational numbers was not easily accepted because it challenged established doctrine. A famous story tells that the Pythagorean member who discovered the first irrational number was murdered to suppress the dangerous new idea.

6.2 The existence of irrational numbers

It is likely that the first known irrational number is $\sqrt{2}$. How do we know that $\sqrt{2}$ is not a fraction of whole numbers? Here is the traditional argument which uses proof by contradiction.

(6.2.1) **Theorem.** The number $\sqrt{2}$ is irrational.

Proof: Suppose, on the contrary, that there are two positive integers m, n such that $\sqrt{2} = m/n$. We may assume, by reducing the fraction if necessary, that the numbers m, n share no common factors other than 1. Multiplying both sides by n and squaring, we obtain $2n^2 = m^2$. It follows that m^2, being twice the whole number n^2, is even. Since an odd number times an odd number is odd, the fact that m^2 is even implies that m must also be even. Since m is even, we can write $m = 2k$ for some whole number k. Then $2n^2 = m^2 = (2k)^2 = 4k^2$, so $n^2 = 2k^2$. From this it follows that n^2, and hence n, is even. But the fact that m and n are both even contradicts the fact that m, n have no common factors. We conclude that $\sqrt{2}$ can not be equal to a fraction of whole numbers.

A similar argument can be used to prove that the square root of any whole number which is not the square of another whole number is irrational. It is impressive that $\sqrt{2}$ has been known to be irrational for 2500 years, while some other famous irrational numbers such as π and e were not proved irrational until the 18th century.

6.3 Commensurable lengths

The Greek idea that any two numbers should compare in whole number proportion has the following consequence. Suppose that the ratio of quantities a and b is the fraction m/n of whole numbers m, n. Let $c = a/m = b/n$, so that $mc = a$ and $nc = b$. Think of c as the length of a measuring stick C, and think of a and b as lengths of line segments A and B. The measuring stick C can be laid end-to-end exactly m times to measure A, and exactly n times to measure B. Before the discovery of irrationals, it was assumed that any two segments could be measured by whole number multiples of some common measuring stick. The existence of irrational numbers implies that this is not always possible. Here is the formal vocabulary used to discuss these ideas.

Lengths A and B are said to be *commensurable* if there is a third line segment C such that the lengths of both A and B are whole number multiples of the length of C. Otherwise, A and B are called *incommensurable*.

For example, suppose the length of A is $2/5$ and the length of B is $7/3$. Both segments can be subdivided into segments of length $1/15$. Let C be a segment with a length of $1/15$. Segment A is exactly 6 times as long as C and segment B is exactly 35 times as long as C, so A and B are commensurable.

For another example, suppose A is a leg of an isosceles right triangle and B is the hypotenuse. Let a and b be the lengths of A and B, respectively. By the Pythagorean theorem, $a^2 + a^2 = b^2$, so $b = a\sqrt{2}$. Now suppose A and B are commensurable and that both a and b are whole number multiples of some length c. Then we have $a = mc$ and $b = nc$ for some whole numbers m, n. Solving $a = mc$ for c, we have $c = a/m$. Substituting this expression for c into $b = nc$, we get $b = a(n/m)$. Substituting $a\sqrt{2}$ for b, we obtain $a\sqrt{2} = a(n/m)$. Cancelling a, we have $\sqrt{2} = n/m$. But this is not possible because the square root of two is not rational. Having arrived at a contradiction, we conclude that A and B are *not* commensurable.

6.4 Decimal representation of real numbers

Where on the number line is $0.\overline{9} = 0.999\ldots$? It is certainly no bigger than 1, but is it less than 1? Let's call this number $x = 0.\overline{9}$. The first digit to the right of the decimal point tells us that x is no more than $1/10$ away from 1. The second digit to the right of the decimal tells us that x is no more than $1/100$

away from 1. Continuing in this manner, we conclude that x is no more than $1/10^n$ away from 1, for every positive integer n. If we let d be the difference $1 - x$ between x and 1, we have that $d < 1/10^n$ for every integer n. Taking reciprocals of both sides, $1/d > 10^n$ for every integer n. But there is no number that is larger than *every* power of 10. We conclude that d must be zero, and hence that $x = 1$.

The fact that $0.\overline{9} = 1$ points out a feature of decimals that most people do not think about, namely, that decimal representations are not unique. It is possible for two *different* decimals to represent the *same* real number.

You are probably aware that the decimal representation for $1/3$ is $0.\overline{3} = 0.333\ldots$. Likely less familiar, but true nonetheless, is that $1/7 = 0.\overline{142857}$. What both of these fractions have in common is that their decimal representations *repeat*, that is, have a block of digits which repeat forever past a certain point. Of course, some decimals, such as $1/4 = 0.25$ come to an end, or *terminate*. It is a fact (see exercise 7c below) that every rational number has a decimal expansion that either terminates or repeats. Conversely, it is true (see exercise 7b) that any terminating or repeating decimal represents a rational number. It follows from these two statements that the decimal representation of any irrational number must never terminate or repeat, and that any non-terminating, non-repeating decimal must represent an irrational number.

6.5 Some vocabulary for whole numbers

The concept of commensurability entails the notion of whole number multiples. This is a basic and useful idea that can be expressed using several different phrases and has its own special notation. For two whole numbers a and b, the following phrases are equivalent.

- b is a whole number multiple of a
- $b = ka$ for some whole number k
- a *goes evenly into* b
- a is a *divisor* of b
- a *divides* b
- $a \mid b$

A positive integer $p \geq 2$ whose only positive divisors are 1 and p is called a *prime* number. Primes play the role of building blocks from which other positive integers are formed. Any positive integer n can be written as a product of primes $n = p_1p_2\ldots p_k$. Furthermore, this product is essentially *unique*. This means that if $n = q_1q_2\ldots q_l$ for some primes q_j, then $k = l$ and the list $p_1, p_2, \ldots, p_k$ is identical (except possibly listed in a different order) to the list $q_1, q_2, \ldots, q_l$. The fact that positive integers have unique prime factorizations is called the *fundamental theorem of arithmetic* and will be useful for exercises 1–3 below.

6.6 Exercises

1. Prove that $\sqrt{3}$ is irrational.

2. Let p be a prime number. Prove that $\sqrt{p}$ is irrational.

3. Let N be a positive whole number which is not the square of another positive whole number. Prove that $\sqrt{N}$ is irrational.

4. Are the radius and circumference of a circle commensurable? Explain.

5. Is it possible for a right triangle to have three commensurable sides? Explain.

6. Decide which of the following pairs A,B of line segments are commensurable. For those pairs which are commensurable, find a third segment C so that the lengths of A and B are whole number multiples of the length of C, and say what those multiples are.

	length of A	length of B
(i)	$2/5$	$2/7$
(ii)	$\sqrt{2}$	$\sqrt{8}$
(iii)	$\sqrt{3}$	$\sqrt{9}$
(iv)	5	7

7. (Characterization of rational and irrational numbers by decimal representation)

    (a) Let $x = 0.\overline{103}$ denote the repeating decimal $0.103103103\ldots$. Write x as a fraction m/n. (Hint: notice that $1000x - x$ is an integer. Solve for x to get a fraction m/n.)

    (b) Prove that any terminating or repeating decimal represents a rational number.

    (c) Prove that any rational number can be represented by a terminating or repeating decimal. Hint: think about what remainders are possible during the long division process.

    (d) Prove that if d, d' are two different decimals that represent the same number, then one of d, d' terminates and the other ends in an infinite string of nines.

    (e) Explain why any interval of the real line, no matter how small, contains infinitely many rational and irrational numbers.

8. The *integral lattice points* in the x, y coordinate plane $\mathbf{R}^2$ are all those points which have whole numbers for both their x and y coordinates (see the Figure 1). Is it possible to draw a straight line in the plane that passes through the origin $(0, 0)$ and *no other* lattice point? If so, give an equation for such a line and explain why your line contains only the single lattice point $(0, 0)$. If, on the other hand, all lines through the origin must contain some other lattice point, explain why. Hint: If a line hits a lattice point (p, q), what is its slope? Can you think of a way to pick a slope that makes the line miss *all* lattice points?

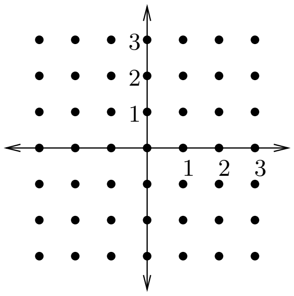


Figure 1
The integral lattice

7 Comparing sizes of two sets

Two sets are said to be *the same size* if there is a one-to-one correspondence between them. While this is a reasonable and harmless sounding definition, there are surprising results for infinite sets. For example, the set of integers and the set of even integers are the same size! The one-to-one correspondence matches the integer n with the even integer $2n$. This seems to go against good sense, for this shows that a set (the integers) can be the same size as one of its subsets (the even integers) that seems to be only "half a big." This points out one way in which infinite sets are very interesting.

7.1 Countability

Here are some basic definitions regarding sizes of sets. A set is *finite* if it has a finite number of members, otherwise it is called *infinite*. If a set X is finite, or if there is a one-to-one correspondence between X and the natural numbers, then X is called *countable*. Otherwise, X is called *uncountable*.

In the remainder of this subsection we discuss countability for three basic sets: the integers, the rationals, and the real numbers. We shall make use of the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers, also called the *counting numbers* or *natural numbers*[1].

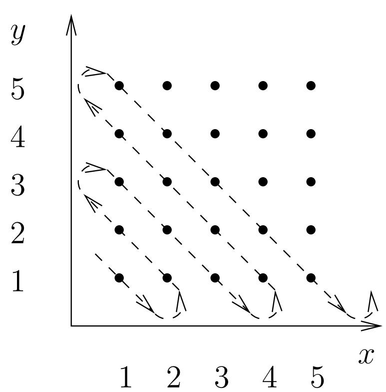


Figure 2
Counting $\mathbf{N} \times \mathbf{N}$

(7.1.1) **The set of integers is countable.** Consider the list

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

that continues in the obvious pattern. This establishes a one-to-one correspondence between the natural numbers and the integers by sending 1 to the first integer in the list, 2 to the second, 3 to the third, and so on.

(7.1.2) **The set $\mathbf{N} \times \mathbf{N}$ is countable.** Figure 2 shows the set $\mathbf{N} \times \mathbf{N}$ (the set of ordered pairs of positive whole numbers) as a subset of points in the x, y-plane. The path indicated by dashed lines with direction arrows shows how to weave along diagonal lines through the set $\mathbf{N} \times \mathbf{N}$. This counts the points in $\mathbf{N} \times \mathbf{N}$ by producing an ordered list.

$$(1, 1), (2, 1), (1, 2), (1, 3), (2, 2), (3, 1), (4, 1), \ldots$$

(7.1.3) **The set of rationals is countable.** Label each point (a, b) in the set $\mathbf{N} \times \mathbf{N}$ with the rational number a/b. Beginning with the first ordered pair $(1, 1)$ from the list in (7.1.2) above, we make a list of rational numbers by writing down the corresponding label. Here is the start of the list of rationals.

$$1, 2, 1/2, 1/3$$

When we get to $(2, 2)$, the corresponding rational number is $2/2 = 1$ which already appears on our list, so we skip over it and proceed to $(3, 1)$. We continue in this manner, skipping over any rational that has already been listed. This

[1] In some contexts, the set natural numbers includes zero. Unfortunately, there is not a standard meaning for "natural numbers," and authors must always specify whether the natural numbers include zero or not.

produces a list of all positive rational numbers, each appearing exactly one time. Once the positive rationals have been listed

$$q_1, q_2, q_3, \ldots$$

we can list *all* rational numbers using the same "positive-negative alternation trick" that we used for the integers in (7.1.1) above.

$$0, q_1, -q_1, q_2, -q_2, q_3, -q_3, \ldots$$

In this manner, we have produced a one-to-one correspondence of the natural numbers with the rational numbers.

(7.1.4) **The set $[0,1]$ is uncountable.** Here is a proof by contradiction, known as *Cantor's diagonal argument* in honor of Georg Cantor (1845–1918), the inventor of set theory.

Suppose there were a one-to-one correspondence between the natural numbers and the closed interval $[0,1]$ of real numbers. Then we would have a list

$$r_1, r_2, r_3, \ldots$$

of all the real numbers in $[0,1]$. First, choose a decimal representation for each real number on the list. Let d_j denote the digit j places to the right of the decimal in the decimal representation of r_j (for example, if $r_5 = 0.2659132$, then $d_5 = 1$ because 1 is the digit in the fifth place to the right of the decimal).

Next we construct a list of digits.

$$x_1, x_2, x_3, \ldots$$

Beginning with x_1, let $x_1 = 1$, unless $d_1 = 1$, in which case let $x_1 = 2$. Now let $x_2 = 1$, unless $d_2 = 1$, in which case let $x_2 = 2$. Continuing, let $x_3 = 1$, unless $d_3 = 1$, in which case let $x_3 = 2$. And so on.

Now let x be the real number whose decimal representation is

$$0.x_1x_2x_3\ldots.$$

$$\begin{aligned} r_1 &= 0.\,d_1\mathrm{XXX}\ldots \\ r_2 &= 0.\,\mathrm{X}d_2\mathrm{XX}\ldots \\ r_3 &= 0.\,\mathrm{XX}d_3\mathrm{X}\ldots \\ r_4 &= 0.\,\mathrm{XXX}d_4\ldots \end{aligned}$$

Figure 3
Idea for
Cantor's diagonal argument

Clearly, x is a number in the interval $[0,1]$, and yet x cannot be equal to any of the real numbers in the list $r_1, r_2, r_3, \ldots$ because the decimal representation for x is different in at least one decimal place from each of the decimal representations for each r_j (for example, x cannot equal r_{100} because the one hundredth digit to the right of the decimal place is different in the decimal representations for x and r_{100}; this is guaranteed by the way we constructed the digit x_{100})[2].

But this is a contradiction! (Why?) We conclude that it is not possible to make a one-to-one correspondence between the natural numbers and the set $[0,1]$.

The idea behind this proof, and the reason for the name *diagonal argument*, is indicated in Figure 3. The X's represent digits in the decimal representations. The construction of x is made by going down the diagonal line through $d_1, d_2, d_3, \ldots$.

[2]There is a fine point here. There are numbers that have two distinct decimal representations. For example, $1.\overline{0} = 0.\overline{9}$. In light of this example, it is conceivable that x could possibly equal some r_j even though x and r_j have different decimal representations. We leave it as an exercise for the reader to see that the only way two different decimal representations can equal the same real number is when one of the decimals terminates and the other has repeating 9's after some decimal place. Since the decimal representation for x does not terminate or have any 9's, we know then that there is no other possible decimal representation for x.

Having established that the interval $[0, 1]$ is uncountable, intuition suggests that the set of real numbers must also be uncountable because it contains $[0, 1]$ as a subset. More generally, the following facts are true (the proof can be found in texts on elementary set theory).

(7.1.5) **A subset of a countable set is countable. A set that contains an uncountable set is uncountable.**

Putting (7.1.4) and (7.1.5) together, we conclude that the real numbers are uncountable.

(7.1.6) **The set of real numbers is uncountable.**

7.2 Exercises

1. Explain why it is impossible to find a one-to-one correspondence between the set of rational numbers and the points in the interval $[0, 1]$ of the real line.

2. Explain why every real number has either: (a) a unique decimal representation, or; (b) exactly two decimal representations.

3. Here is a flawed argument to prove that the set of rational numbers is uncountable. First, choose a decimal representation for each rational number. Suppose the rational numbers are countable, and make a list $q_1, q_2, q_3, \ldots$ of all the rationals. Now use Cantor's diagonal argument to construct a decimal string that does not already appear in the list. This violates the assumption that the list was complete, so the rationals are not countable. What is wrong with this argument?

4. Consider the following situation.

   > There are four sets called A, B, C and D. Set A is a subset of B and set C is a subset of D. There are members of B that are not members of A and there are members of D that are not members of C. There is a one-to-one correspondence between A and D, and there is a one-to-one correspondence between B and C.

   Is this possible? If so, describe four sets and two one-to-one correspondences that fit the above description. If not, explain (using complete sentences) why not.

8 Symmetry

Symmetry is an ancient and rich field that has contributed to every natural science and every branch of mathematics. We introduce the mathematical language and some of the ideas used to study symmetry.

8.1 Mathematical Meaning of Symmetry

In everyday language, we say an object is *symmetrical* if it is identical to its own mirror reflection. This most common use of the term *symmetrical* refers to *bilateral symmetry*. We say an object has *rotational symmetry* if a specific placement of the object is indistinguishable from some replacement of the object after a rotation. See Figure 4.

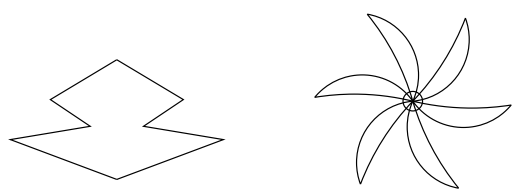


Figure 4 Symmetrical objects

In mathematics, we think of symmetries as motions. This is natural for the case of a rotation, but it is not obvious how a reflection is represented by a motion. Here is how we do it. A mirror is represented by a plane. Given a plane Π and a point p not in Π, *the reflection of p across* Π is the point p' that is the same distance as p from Π, but on the other side of Π, in such a way that the line through p and p' is perpendicular to Π. If q is a point on the mirror Π, the reflection q' of q is q itself. See Figure 5. The reflection through the mirror plane Π can be thought of as the motion or transformation of space that sends each point p to its reflection p'.

Viewing a reflection as a motion sending each point p to its reflection p' means that a reflection is a *function* from 3-dimensional Euclidean space $\mathbf{R}^3$ to itself. Similarly, we shall think of a rotation as a function.

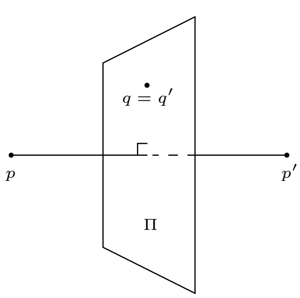


Figure 5
Reflections of p and q
through mirror plane Π

Now we can define bilateral and rotational symmetry in mathematical terms. An object X (a set points in space $\mathbf{R}^3$) has *bilateral (or mirror) symmetry* if there is a mirror reflection that takes every point in X to some other point in X. Similarly, we say X has *rotational symmetry* is there is a rotation of space that takes every point in X to some other point in X.

Once we begin to analyze symmetry, it is natural to ask more questions. Are there other types of symmetry besides reflection and rotation? What should be the definition of symmetry? We address the second question first.

A feature shared by reflection and rotation is that the image of an object is left undistorted. There is no squashing or stretching or tearing. One way to express this property is to say that distances between points in an object are left unchanged by reflections and rotations. This is the property we shall use to define symmetry.

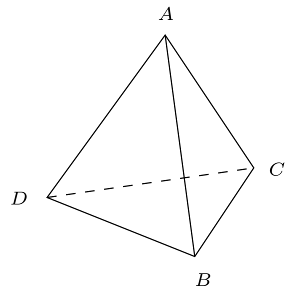


Figure 6
Tetrahedron

(8.1.1) **Definition of Symmetry.** Let X be a set of points in $\mathbf{R}^3$. A *symmetry of* X is a function $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ with the following properties:

(i) for every point x in X, the image point $f(x)$ also lies in X

(ii) for every pair of points x and y in X, the distance from $f(x)$ to $f(y)$ is the same as the distance from x to y

Note that under this definition, the identity function $\mathrm{Id}\colon \mathbf{R}^3 \to \mathbf{R}^3$ that leaves all points unmoved qualifies as a symmetry.

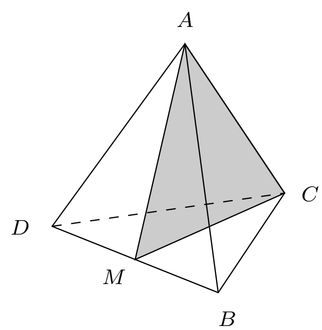


Figure 7
Plane of symmetry
in the tetrahedron

Now we address the question of whether there may be other kinds of symmetry besides reflection and rotation. Intuition may suggest that the answer is no, but this is not obvious. The confirmation that intuition is essentially correct is given by the following fact, which we state without proof.

(8.1.2) **Possible Types of Symmetry.** Let $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ be a symmetry of an object $X \subseteq \mathbf{R}^3$ according to Definition (8.1.1). Then f is either

(i) a reflection,

(ii) a rotation, or

(iii) a composition of a reflection with a rotation

(where "composition" means composition of functions).

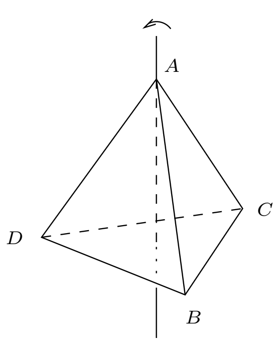


Figure 8
Axis of rotational symmetry

(8.1.3) **Example: Symmetries of the tetrahedron.** The tetrahedron is the regular solid whose four faces are equilateral triangles. See Figure 6. Since any symmetry of the tetrahedron must permute the vertices, there can be no more than $4! = 24$ (the number of permutations of four things) possible symmetries. We claim that there are in fact exactly 24 symmetries. The tetrahedron is divided symmetrically in half by a plane that contains the edge AC and the midpoint M of the opposite edge BD (see Figure 7). Since the tetrahedron has six edges, there are six planes of mirror symmetry of this type. The axis of rotation that passes through vertex A and the midpoint of the opposite triangular face $\triangle BCD$ yields two rotational symmetries by one-third and two-thirds of a rotation. See Figure 8. Since there are four vertices, there are 8 rotational symmetries of this type. The axis of rotation that passes through midpoints M and N of opposite edges BD and AC yields a rotational symmetry by one-half rotation. See Figure 9. Since there are three pairs of opposite edges, there are three symmetries of this type. So far we have a total of 6 mirror reflection symmetries, 8 rotations about the type of axis shown in Figure 8, and 3 rotations about the type of axis shown in Figure 9. Together with the identity, this makes a total of 18 symmetries. What are the remaining $24-18 = 6$ symmetries? We claim that the remaining 6 permutations of the vertices can be realized by a composition of one of the 6 reflections with one of the 11 rotations. We encourage the reader to verify the statements made in this example with her own model of a tetrahedron.

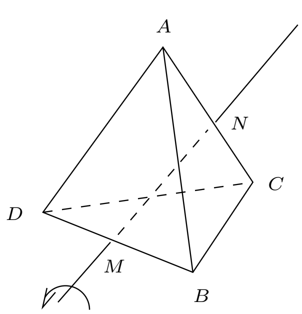


Figure 9
Another axis of
rotational symmetry

8.2 Symmetry Groups

The set of symmetries of an object X is called the *symmetry group of* X.

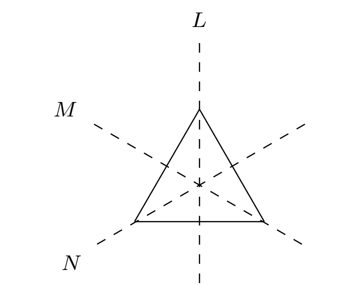


Figure 10
Lines of bilateral symmetry
for an equilateral triangle

(8.2.1) **Example: The symmetry group of the equilateral triangle.** The equilateral triangle has three lines of bilateral symmetry and rotational symmetry about its center point. Let us denote the three lines with the letters L, M and N (see Figure 10). Let us denote by f_L, f_M and f_N the reflections across the three lines L, M and N, respectively, and denote by $r_{1/3}$ and $r_{2/3}$ rotations of 1/3 and 2/3 of a full counterclockwise rotation about the center of the triangle. Together with the identity, this makes six symmetries. We know there cannot be any others since the symmetry group must permute the set of vertices, and there are only $3! = 6$ such permutations. Thus, the symmetry group of the equilateral triangle is the following set.

$$\{f_L, f_M, f_N, r_{1/3}, r_{2/3}, \mathrm{Id}\}$$

Some objects, such as the equilateral triangle, have finite symmetry groups. Others, such as the circle or the sphere, have infinite symmetry groups.

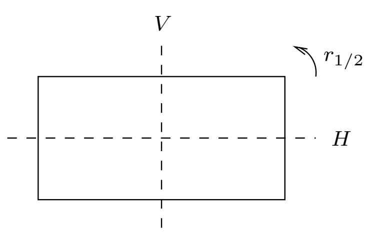


Figure 11
Rectangle

Symmetries groups possess a natural structure given by the compositions of pairs of symmetries. For example, in the symmetry group of the equilateral triangle, we have $r_{1/3} \circ f_L = f_M$ and $f_N \circ f_M = r_{1/3}$.

The *multiplication table* (also called the *composition table* or *Cayley table*) for a symmetry group is a rectangular array of rows and columns that records all possible compositions of pairs of symmetries in the group. Rows and columns of the table are labeled by the symmetries in the group, one row and one column for each symmetry. The table entry in the row labeled by the symmetry f and the column labeled by the symmetry g is the composition $f \circ g$, which is usually written fg.

(8.2.2) **Example: Cayley table for the symmetry group of an equilateral triangle.** Let the symmetries of the equilateral triangle be denoted as in the example (8.2.1) above. Here is the Cayley table for the symmetry group.

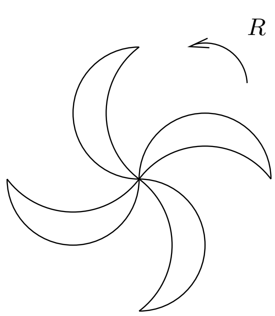


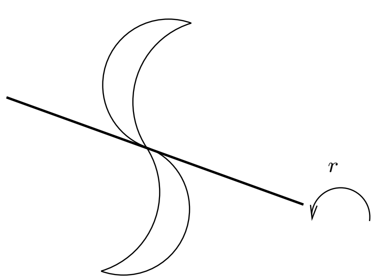
Figure 12
Four-bladed fan

	f_L	f_M	f_N	$r_{1/3}$	$r_{2/3}$	Id
f_L	Id	$r_{2/3}$	$r_{1/3}$	f_N	f_M	f_L
f_M	$r_{1/3}$	Id	$r_{2/3}$	f_L	f_N	f_M
f_N	$r_{2/3}$	$r_{1/3}$	Id	f_M	f_L	f_N
$r_{1/3}$	f_M	f_N	f_L	$r_{2/3}$	Id	$r_{1/3}$
$r_{2/3}$	f_N	f_L	f_M	Id	$r_{1/3}$	$r_{2/3}$
Id	f_L	f_M	f_N	$r_{1/3}$	$r_{2/3}$	Id

8.3 Isomorphism

Two symmetry groups are called *isomorphic* if their multiplication tables are the same. The term “isomorphic” comes from the Greek roots meaning “same structure.”

By “the same,” we mean that there is a one-to-one correspondence between the two symmetry groups that consistently matches every entry in the two multiplication tables. Such a one-to-one correspondence is called an *isomorphism*.



To illustrate this concept we introduce three objects and their symmetry groups. The first object is the flat (non-square) rectangle (see Figure 11). It has two lines of symmetry: the vertical line V, and the horizontal line H. Let F_V and F_H denote the reflections across these lines. It also has a symmetry $r_{1/2}$ by a rotation of 180 degrees. Here is the Cayley table for the group of symmetries of the rectangle.

	F_V	F_H	$r_{1/2}$	Id
F_V	Id	$r_{1/2}$	F_H	F_V
F_H	$r_{1/2}$	Id	F_V	F_H
$r_{1/2}$	F_H	F_V	Id	$r_{1/2}$
Id	F_V	F_H	$r_{1/2}$	Id

The second object is the four-bladed fan (see Figure 12). Let R denote the symmetry which is counter-clockwise rotation by one-quarter turn. Here is the Cayley table for the symmetry group of the fan.

	R	R^2	R^3	Id
R	R^2	R^3	Id	R
R^2	R^3	Id	R	R^2
R^3	Id	R	R^2	R^3
Id	R	R^2	R^3	Id

The third object is called "S on a stick" (see Figure 13). This is a 3-dimensional object consisting of a letter "S" fixed on an axle perpendicular to the plane of the letter. The object has mirror symmetry through the plane of the letter "S" and has rotational symmetry about the axle. Let f denote the reflection through the plane of the letter "S" and let r denote rotation about the axle by 180 degrees. Here is the Cayley table for the symmetry group of "S on a stick."

	f	r	fr	Id
f	Id	fr	r	f
r	fr	Id	f	r
fr	r	f	Id	fr
Id	f	r	fr	Id

Consider the following one-to-one correspondence between the symmetry group of the rectangle and the symmetry group of the "S on a stick."

Symmetries of the rectangle	$\leftrightarrow$	Symmetries of "S on a stick"
Id	$\mapsto$	Id
F_V	$\mapsto$	f
F_H	$\mapsto$	r
$r_{1/2}$	$\mapsto$	fr

This is an example of of an isomorphism. If you substitute the entry in the right-hand column for the entry in the left-hand column everywhere in the Cayley table for the symmetries of the rectangle, you get the Cayley table for the symmetries of the "S on a stick."

Here is another one-to-one correspondence between the same two groups. This is also an isomorphism.

$$\begin{array}{rcl}
\text{Symmetries of the rectangle} & \leftrightarrow & \text{Symmetries of “S on a stick”} \\
\text{Id} & \mapsto & \text{Id} \\
F_V & \mapsto & f \\
F_H & \mapsto & fr \\
r_{1/2} & \mapsto & r
\end{array}$$

Here is a one-to-one correspondence that is *not* an isomorphism.

$$\begin{array}{rcl}
\text{Symmetries of the rectangle} & \leftrightarrow & \text{Symmetries of “S on a stick”} \\
\text{Id} & \mapsto & r \\
F_V & \mapsto & f \\
F_H & \mapsto & fr \\
r_{1/2} & \mapsto & \text{Id}
\end{array}$$

Why is this not an isomorphism? In the symmetry group of the rectangle, we have the equation $F_V F_V = \text{Id}$. If we substitute the entries in the right-hand column for the entries in the left-hand column everywhere in the Cayley table for the symmetries of the rectangle, we get the equation $ff = r$. Since this is not true in the symmetry group of “S on a stick,” this correspondence is *not* an isomorphism.

Note that we have exhibited three one-to-one correspondences between the two symmetry groups; two of them are isomorphisms and one of them is not. Because there exists at least one isomorphism, we call the two symmetry groups isomorphic, and think of them as the same.

Finally, we remark that while there are many one-to-one correspondences between the symmetry groups of the four-bladed fan and the rectangle, *none of them* are isomorphisms. How do we know this? Consider the element R in the symmetry group of the four-bladed fan. The element R has the property that its successive powers $R, R^2, R^3, R^4 = \text{Id}$ constitute the entire group. There is no such element in the symmetry group of the rectangle. This rules out the possibility of an isomorphism, as follows. Suppose, on the contrary, that there is a one-to-one correspondence that is an isomorphism. Let x be the symmetry of the rectangle that corresponds to the symmetry R of the four-bladed fan. Then x, x^2, x^3, x^4 would have to be the whole symmetry group for the rectangle. But this is not possible. We conclude that there cannot be any isomorphism, and we say that the two symmetry groups are not isomorphic.

Here is another way to see that the symmetry groups of the four-bladed fan and the rectangle are not isomorphic. For any element x in the symmetry group of the rectangle, we have $x^2 = \text{Id}$. That is to say, the square any element is the identity. If there were an isomorphism, the squares of all the elements in the symmetry group of the four-bladed fan would have to be equal, but this is not the case.

8.4 Exercises

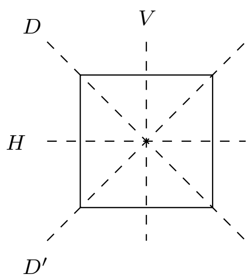


Figure 14
Square

1. Let V, H, D and D' denote the vertical, horizontal, main diagonal (sloping downwards from left to right), and secondary diagonal (sloping upwards from left to right) lines of bilateral symmetry of the square (see Figure 14). Let f_V, f_H, f_D and $f_{D'}$ denote the reflection symmetries across these line. Let r_{90}, r_{180} and r_{270} denote counterclockwise rotational symmetries by 90, 180 and 270 degrees. Let Id denote the identity symmetry. Complete the following Cayley table for the symmetry group of the square.

	f_V	f_H	f_D	$f_{D'}$	r_{90}	r_{180}	r_{270}	Id
f_V								
f_H								
f_D								
$f_{D'}$								
r_{90}								
r_{180}								
r_{270}								
Id								

2. (a) How many *rotational* symmetries does the cube have? Draw diagrams illustrating them.

   (b) How many planes of bilateral symmetry does the cube have? Draw diagrams illustrating them.

3. Let X be the equilateral triangle, oriented so that it has a vertical line of symmetry. Let f denote the reflection or flip across this line of symmetry, and let r denote a rotational symmetry of one third revolution counterclockwise. Explain why the list

$$\{\text{Id}, r, r^2, f, fr, fr^2\}$$

   gives *all* the symmetries of X, and write the Cayley table for the symmetry group (label the rows and columns of the table in the same order as the above list).

4. Let X be the equilateral triangle, oriented so that it has a vertical line of symmetry. Let f denote the reflection or flip across this line of symmetry, and let r denote a rotational symmetry of one third revolution counterclockwise. Explain why the list

$$\{\text{Id}, r, f, fr, rf, frf\}$$

   gives *all* the symmetries of X, and write the Cayley table for the symmetry group (label the rows and columns of the table in the same order as the above list).

5. Your friend presents you with a multiplication table for a symmetry group. You notice that one of the rows does not contain the identity. Explain (using complete sentences) how you know that your friend made a mistake.

6. The symmetry group G of an object X consists of three symmetries f, g, and h. A friend gives you the following Cayley table for G.

	f	g	h
f	g	h	f
g	f	g	h
h	g	f	f

Explain (using complete sentences) how you know that your friend made a mistake.

7. Here is a part of the Cayley table for the symmetry group G of an object X.

	f	g	fg	gf
f	f^2	fg	f^2g	h
g	gf	g^2	k	g^2f
fg		fg^2	m	n
gf	gf^2		q	m

Fill in the missing entries, and explain why hg must equal gh.

8. Are the symmetry groups for the equilateral triangle and the six-bladed fan (pictured on the right in Figure 4) isomorphic? Explain why or why not.

9. Let G be the symmetry group of the rectangle and let H be the symmetry group of the "S on a stick" described in 8.3.

   (a) Suppose that $f\colon G \to H$ is a one-to-one correspondence that is an isomorphism. Explain why $f(\mathrm{Id}) = \mathrm{Id}$.

   (b) Find *all* possible isomorphisms between G and H. How many are there?

Solutions to Exercises

Note: Most of the "solutions" posted here are not solutions at all, but are merely final answer keys, although some are complete. These are posted so that you can check your work; reading the answer keys is not a substitute for working the problems yourself. For homework, quizzes and exams, you need to show the steps of whatever procedure you are using—not just the final result. Sometimes you will be asked to explain your thinking in complete sentences.

6.5 Solutions

1. (student writing project)
2. Let p be prime, and suppose that $\sqrt{p}$ is rational, say $\sqrt{p} = m/n$, where m, n are positive integers with no common factors. Squaring and simplifying, we have $pn^2 = m^2$, so p must be one of the prime factors of m, say $m = kp$ for some positive integer k. Then we have $pn^2 = (kp)^2$, so simplifying yields $n^2 = pk^2$, so p must be a prime factor of n. But this contradicts the assumption that m and n have no common factors. We conclude that $\sqrt{p}$ must not be rational.
3. Let N be a positive whole number which is not a perfect square and suppose that $\sqrt{N}$ is rational, say $\sqrt{N} = m/n$, where m, n are positive integers with no common factors. Squaring and simplifying, we have $Nn^2 = m^2$. Now consider the prime factorizations of the left and right hand sides of this last equation. Every power of every prime in the factorization of m^2 must be even. This is also true of for n^2. Therefore the same must be true for N. But this is a contradiction since N is not a perfect square. We conclude that $\sqrt{N}$ is irrational.
4. (student writing project)
5. Yes. For example, the three sides of a 3-4-5 right triangle can be measured by a common yardstick of length 1.
6. Let a, b, c denote the lengths of segments A, B, C, respectively.
   (a) These are commensurable with $c = 1/35$. We have $a = 14c$ and $b = 10c$.
   (b) Since $\sqrt{8} = 2\sqrt{2}$, these are commensurable with $c = \sqrt{2}$. We have $a = c$ and $b = 2c$.
   (c) These are not commensurable since a is irrational and b is rational.
   (d) These are commensurable with $c = 1$, so $a = 5c$ and $b = 7c$.
7. (a) $x = 103/999$
   (b) (student writing project)
   (c) (student writing project)
   (d) Let d, d' be two different decimals that represent the same real number x. Let d_i, d'_i denote the digit i places to the right of the decimal point, so we have

$$x = \sum_{i=-\infty}^{\infty} d_i 10^{-i} = \sum_{i=-\infty}^{\infty} d'_i 10^{-i}.$$

Let N be the leftmost decimal place where d and d' disagree. By renaming if necessary, we may assume $d_N < d'_N$. We claim that $d'_N = d_N + 1$, that $d_i = 9$ for $i > N$, and $d'_i = 0$ for $i > N$. Let

$$y = \sum_{i=-\infty}^{N-1} d_i 10^{-i} = \sum_{i=-\infty}^{N-1} d'_i 10^{-i}.$$

Then we have

$$x = y + d_N 10^{-N} + \sum_{i=N+1}^{\infty} d_i 10^{-i} = y + d'_N 10^{-N} + \sum_{i=N+1}^{\infty} d'_i 10^{-i}.$$

Since the maximum possible value of $\sum_{i=N+1}^{\infty} d_i 10^{-i}$ is $\sum_{i=N+1}^{\infty} 9 \cdot 10^{-i} = 10^{-N}$ (sum of a geometric series), and the minimum possible value of $\sum_{i=N+1}^{\infty} d'_i 10^{-i}$ is zero, the claim follows. We conclude that if two different decimal expansions represent the same real number, then one of them terminates and the other ends in an infinite string of nines.

(e) Let J be an interval of numbers, let x be a number in J, and choose N large enough that the open interval $(x-10^{-N}, x+10^{-N})$ is contained in J. If x is rational, let $y = x$. If x is irrational, let y be the number whose decimal representation is the same as that of x up to the $(N+1)$st decimal place, and zeros thereafter. Observe that J contains the infinite set of rational numbers $\{y+10^{-i} \colon i > N+2\}$ and also contains the infinite set of irrational numbers $\{y + 10^{-i}/\sqrt{2} \colon i > N+2\}$.

8. Yes, it is possible. Any irrational slope will do. For example, consider the line $y = \sqrt{2}x$. If this line passes through the lattice point (p, q), then the slope of the line is $\sqrt{2} = q/p$. But this is impossible since $\sqrt{2}$ is irrational.

7.2 Solutions

1. Suppose that there were a one-to-one correspondence between the rational numbers and the points in the interval $[0, 1]$, say $f \colon \mathbf{Q} \to [0, 1]$. Let $g \colon \mathbf{N} \to \mathbf{Q}$ denote the one-to-one correspondece between the natural numbers and the rational numbers described in (3.1.3). Since the composition of two one-to-one correspondences is another one-to-one correspondence, we have $f \circ g \colon \mathbf{N} \to [0, 1]$ is a one-to-one correspondence. But this is impossible, by the argument in (3.1.4). We conclude that the one-to-one correspondence f cannot exist.

2. (Outline of a geometric argument) Let x be a point on the real number line to the right of zero. Here is a recipe to make a decimal expansion for x.

   Starting at zero, travel to the rightmost possible whole number that is either (a) less than x, or (b) less than *or equal to* x (you may choose (a) or (b) arbitrarily). This is the whole number part of the decimal expansion of

x. Next, go to the rightmost tenths subdivision that is either (a) less than x, or (b) less than *or equal to* x (again, you may choose arbitrarily). This gives the tenths place digit for the decimal expansion for x. Continue in this manner, choosing either (a) or (b) for each decimal place, to generate all the digits for the decimal expansion of x. Note that for every decimal place, (a) and (b) are the only possible choices, so this recipe produces all possible decimal representations for x.

If x is a whole number, then making choice (b) at the first step of the above recipe produces a terminating decimal representation with zero in all places to the right of the decimal point. Making choice (a) at the first step produces a decimal expansion ending in repeating nines.

If x is not a whole number, but can be represented by a terminating decimal, terminating, say, in the kth place to the right of the decimal, then making choice (b) at the kth step of the recipe produces the terminating expansion. Making choice (a) at the kth step produces a decimal expansion ending with repeating nines.

If x cannot be represented by a terminating decimal (for example, $x = 1/3$), then any two sequences of choices (a) and (b) result in the same expansion.

Thus for any positive number x, there is either a unique decimal representation (in the case that x cannot be represented by a terminating decimal) or there are precisely two decimal representations—one which terminates, and the other ending in repeating nines.

A similar argument works for negative numbers, generating digits by starting at zero and traveling to the left. Finally, there can be only one decimal expansion for zero (zeros in every decimal place).

3. The decimal string constructed by this argument cannot be rational because it cannot terminate or repeat.

4. Yes, this is possible. Let $A = C = \mathbf{N}$, and let $B = D = 2\mathbf{N} = \{2, 4, 6, \ldots\}$.

8.4 Solutions

1.

	f_V	f_H	f_D	$f_{D'}$	r_{90}	r_{180}	r_{270}	Id
f_V	Id	r_{180}	r_{270}	r_{90}	$f_{D'}$	f_H	f_D	f_V
f_H	r_{180}	Id	r_{90}	r_{270}	f_D	f_V	$f_{D'}$	f_H
f_D	r_{90}	r_{270}	Id	r_{180}	f_V	$f_{D'}$	f_H	f_D
$f_{D'}$	r_{270}	r_{90}	r_{180}	Id	f_H	f_D	f_V	$f_{D'}$
r_{90}	f_D	$f_{D'}$	f_H	f_V	r_{180}	r_{270}	Id	r_{90}
r_{180}	f_H	f_V	$f_{D'}$	f_D	r_{270}	Id	r_{90}	r_{180}
r_{270}	$f_{D'}$	f_D	f_V	f_H	Id	r_{90}	r_{180}	r_{270}
Id	f_V	f_H	f_D	$f_{D'}$	r_{90}	r_{180}	r_{270}	Id

2. (a) Type I: axis through centers of a pair of opposite faces. There are 1/4, 2/4 and 3/4 revolution symmetries about these axes. There are 6 faces, so there are three pairs of opposite faces, so 3 of these axes. So there are a total of $3 \cdot 3 = 9$ rotational symmetries of this type. (sketch)

Type II: axis through midpoints of a pair of opposite edges. There is a 1/2 revolution symmetry about each of these axes. There are 12 edges, so there are 6 pairs of opposite edges, so there are 6 of these axes. So there are a total of 6 rotational symmetries of this type. (sketch)

Type III: axis through a pair of opposite vertices. There are 1/3 and 2/3 revolution symmetries about these axes. There are 8 vertices, so there are 4 pairs of opposite vertices, so there are 4 of these axes. So there are a total of $4 \cdot 2 = 8$ rotational symmetries of this type. (sketch)

There are no further axes of rotational symmetry. Thus there are a total of 9 (Type I) plus 6 (Type II) plus 8 (Type III) plus 1 (the identity) equals 24 total rotational symmetries of the cube.

(b) Type I: plane midway between a pair of opposite faces. There are 3 of these (as counted in 2a Type I above).

Type II: plane midway between a pair of opposite edges. There are 6 of these (as counted in 2a Type II above).

There are no more planes of symmetry. So there are a total of 9 bilateral symmetries of the cube.

3. By direct checking, we see that the given symmetries are all different. Since there are only six symmetries of the equilateral triangle, we know this list is the entire symmetry group. Here is the Cayley table.

	1	r	r^2	f	fr	fr^2
1	1	r	r^2	f	fr	fr^2
r	r	r^2	1	fr^2	f	fr
r^2	r^2	1	r	fr	fr^2	f
f	f	fr	fr^2	1	r	r^2
fr	rf	fr^2	f	r^2	1	r
fr^2	fr^2	f	fr	r	r^2	1

4. (same reasoning as previous problem)

	1	r	f	fr	rf	frf
1	1	r	f	fr	rf	frf
r	r	frf	rf	f	fr	1
f	f	fr	1	r	frf	rf
fr	fr	rf	frf	1	r	f
rf	rf	f	r	frf	1	fr
frf	rfr	1	rf	rf	f	r

5. Let f be the symmetry whose row does not contain the identity. Since f is a symmetry, so is its inverse f^{-1}. The entry in the row labeled f and the column labeled f^{-1} must be $f(f^{-1}) = 1$, the identity. Thus we know there has been some mistake.

6. Reading the last two entries in the last row, we have the following compositions.

$$\begin{aligned} hg &= f \\ hh &= f \end{aligned}$$

Applying h^{-1} to the left on both sides, we have

$$\begin{aligned} g &= h^{-1}f \\ h &= h^{-1}f, \end{aligned}$$

so we see that $g = h$. Now we see our friend has made a mistake by claiming that f, g, and h were three distinct symmetries.

7. In the third row, first column, the missing entry is fgf. We see from the first row, fourth column that $fgf = h$, so the missing entry is h.

   In the fourth row, second column, the missing entry is gfg. We see from the second row, third column that $gfg = k$, so the missing entry is k.

   To see why hg must equal gh, we again use the first row, fourth column to see that $h = fgf$, so $hg = fgfg$. On the other hand, $gh = gfgf$. But both of these products equal m (third row third column and fourth row fourth column). We conclude that $hg = gh$.

8. No. (many possible explanations)

9. (a) Let g be any symmetry in G. Then we have

   $$\begin{aligned} f(g) &= f(\mathrm{Id}\ g) \\ &= f(\mathrm{Id})f(g) \quad \text{(since } f \text{ is an isomorphism)} \end{aligned}$$

   Multiply both sides of the equation

   $$f(g) = f(\mathrm{Id})f(g)$$

   on the right by $(f(g))^{-1}$ to obtain $f(\mathrm{Id}) = \mathrm{Id}$.

   (b) There are 6. (describe them)