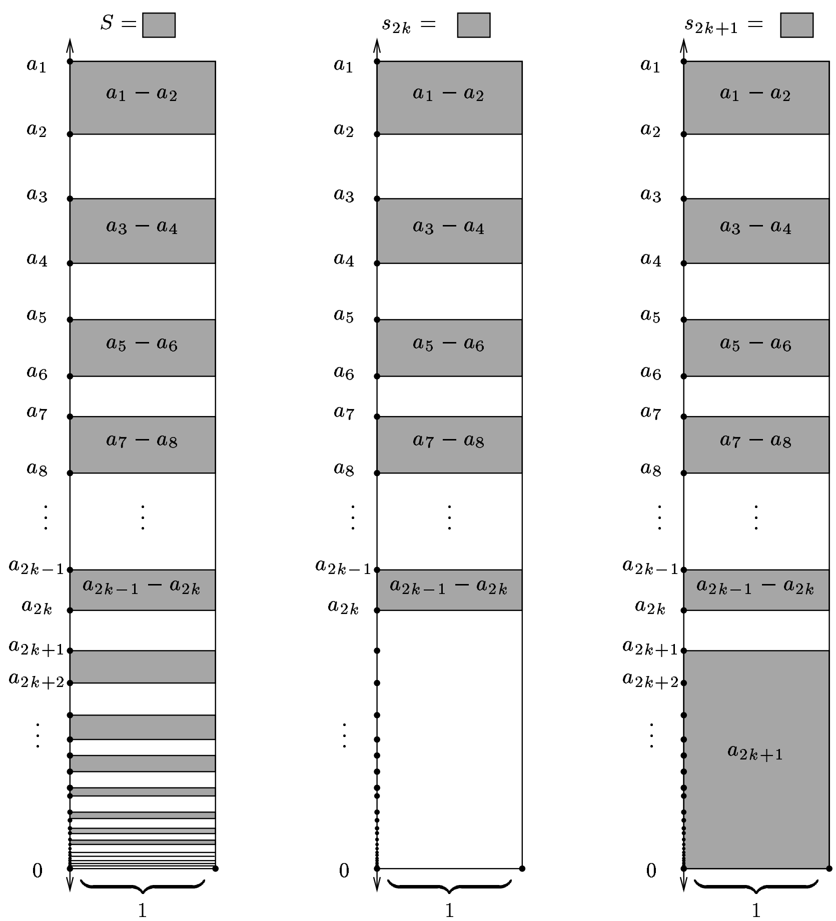
Theorem (Alternating Series Test). *An alternating series* $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + \cdots$ *converges to a sum* S *if* $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq 0$ *and* $a_n \to 0$. Moreover, for every k, $s_{2k} < S < s_{2k+1}$ and for every n, $|S - s_n| < a_{n+1}$.

Proof We use the hypotheses of the theorem to draw the figure below, which shows the terms $a_1, a_2, a_3, \ldots$ of the alternating series as a decreasing sequence converging to zero along a vertical axis. Horizontal line segments one unit wide establish a strip of rectangles whose areas are the same as their vertical heights.

Let S be the area of the shaded rectangles in the left-most tower. This tower is followed by towers for even and odd partial sums s_{2k} and s_{2k+1}. Comparison of these figures immediately establishes $s_{2k} < S < s_{2k+1}$.



Moreover, it is clear from the picture that the dark rectangles in the left-most tower below a_{2k+1} represent the difference $S - s_{2k}$, and hence $|S - s_{2k}| < a_{2k+1}$. Also the white rectangles below a_{2k+2} represent the difference $s_{2k+1} - S$, hence $|S - s_{2k+1}| < a_{2k+2}$. Combining these inequalities gives $|S - s_n| < a_{n+1}$ for all n, as desired. As $a_n \to 0$, it further follows that $s_n \to S$, so the series indeed converges to S. QED