

# An Elementary Introduction to the Hopf Fibration

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The Hopf fibration, named after Heinz Hopf who studied it in a 1931 paper [9], is an important object in mathematics and physics. It was a landmark discovery in topology and is a fundamental object in the theory of Lie groups. The Hopf fibration has a wide variety of physical applications including magnetic monopoles [14], rigid body mechanics [11], and quantum information theory [13].

Unfortunately, the Hopf fibration is little known in the undergraduate curriculum, in part because presentations usually assume background in abstract algebra or manifolds. However, this is not a necessary restriction. We present in this article an introduction to the Hopf fibration that requires only linear algebra and analytic geometry. In particular, no vector calculus, abstract algebra, or topology is needed. Our approach uses the algebra of quaternions and illustrates some of the algebraic and geometric properties of the Hopf fibration. We explain the intimate connection of the Hopf fibration with rotations of 3-space that is the basis for its natural applications to physics.

We deliberately leave some of the development as exercises, called “Investigations,” for the reader. The Investigations contain key ideas and are meant to be fun to think about. The reader may also take them as statements of facts that we wish to assume without interrupting the narrative.

## Hopf’s mapping

The *standard unit  $n$ -sphere*  $S^n$  is the set of points  $(x_0, x_1, \dots, x_n)$  in  $\mathbb{R}^{n+1}$  that satisfy the equation

$$x_0^2 + x_1^2 + \dots + x_n^2 = 1.$$

Geometrically,  $S^n$  is the set of points in  $\mathbb{R}^{n+1}$  whose distance from the origin is 1. Thus the 1-sphere  $S^1$  is the familiar unit circle in the plane, and the 2-sphere  $S^2$  is the surface of the solid unit ball in 3-space. The thoughtful reader may wonder what higher dimensional spheres look like. We address this issue at the end of this article, where we explain how *stereographic projection* is used to “see”  $S^3$ .

The Hopf fibration is the mapping  $h: S^3 \rightarrow S^2$  defined by

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)). \quad (1)$$

To be historically precise, Hopf’s original formula differs from that given here by a reordering of coordinates. We use this altered version to be consistent with the quaternion approach explained later in this article. It is easy to check that the squares of the three coordinates on the right-hand side sum to  $(a^2 + b^2 + c^2 + d^2)^2 = 1$ , so that the image of  $h$  is indeed contained in  $S^2$ .

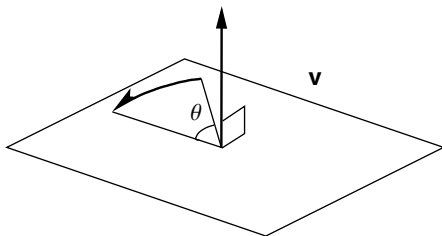
What problem was Hopf trying to solve when he invented this map? And how can one see any connection with physical rotations, as we have claimed?

Hopf's paper [9] represented an early achievement in the modern subject of *homotopy theory*. In loose terms, homotopy theory seeks to determine those properties of a space that are not altered by continuous deformations. One way to discover the properties of an unfamiliar space  $X$  is to compare  $X$  with a familiar one  $Y$  via the set of all continuous maps  $Y \rightarrow X$ . Two maps whose images can be continuously deformed from one to the other are called *homotopically equivalent*. Knowing something about  $Y$  and also about the set of homotopically equivalent maps from  $Y$  to  $X$  helps us understand  $X$ . This seemingly indirect method provides a powerful way to analyze spaces.

Ironically, one of the most intractable problems in homotopy theory is to determine the homotopy equivalence classes of maps  $Y \rightarrow X$  when  $X$  and  $Y$  are both spheres and the dimension of  $X$  is smaller than the dimension of  $Y$ . Many individual cases for particular pairs of dimensions of  $X$  and  $Y$  are understood, but there remain interesting unsolved problems. Hopf's map  $h: S^3 \rightarrow S^2$  was a spectacular breakthrough in this area. We cannot give the full story of this discovery here, but we can explain the Hopf fibration in a geometric way that indicates its connection to rotations.

## Rotations and quaternions

First, notice that a rotation about the origin in  $\mathbb{R}^3$  can be specified by giving a vector for the axis of rotation and an angle of rotation about that axis. We make the convention that the rotation will be counterclockwise for positive angles, when viewed from the tip of the vector (as in FIGURE 1), and clockwise for negative angles.



**Figure 1** A rotation in  $\mathbb{R}^3$  is specified by an angle  $\theta$  and a vector  $\mathbf{v}$  giving the axis

The specification of a rotation by an axis vector and an angle is far from unique. The rotation determined by the vector  $\mathbf{v}$  and the angle  $\theta$  is the same as the rotation determined by the pair  $(k\mathbf{v}, \theta + 2n\pi)$ , where  $k$  is any positive scalar and  $n$  is any integer. The pair  $(-\mathbf{v}, -\theta)$  also determines the same rotation. Nonetheless, we see that four real numbers are sufficient to specify a rotation: three coordinates for a vector and one real number to give the angle. This is far fewer than the *nine* entries of a  $3 \times 3$  orthogonal matrix we learn to use in linear algebra. In fact, we can cut the number of parameters needed to specify a rotation from four to three, for example, by giving an axis vector whose length determines the angle of rotation. However, we shall not pursue that here; it is the 4-tuple approach that turns out to be practical. Is there an efficient way to work with 4-tuples of real numbers to do practical calculations with rotations? Here are some questions that we recommend you ponder long enough to realize that they are cumbersome to answer by matrix methods. Revisit this topic after doing Investigation E below.

**INVESTIGATION A.** Show that the composition of two rotations is another rotation. (The composition of two rotations is the motion obtained by performing first one

rotation, then the other. Show by example that order counts.) Given geometric data (axes and angles) for two rotations, how do you determine the axis and angle for their composition?

The problem of finding a convenient algebraic method for computing with rotations led William Rowan Hamilton to invent the *quaternions* in the mid-19th century. The discovery of quaternions, and Hamilton's life in general, is a fascinating bit of history. For further reading, see biographies by Hankins [7] and O'Donnell [15]. Kuipers [10, § 6.2 ff] gives an exposition of the rotation problem in Investigation A and its solution, beyond what appears in this section.

Hamilton was inspired by the solution to the analogous problem in two dimensions: rotations of the plane about the origin can be encoded by unit length *complex numbers*. The angle of a rotation is the same as the angle made by its corresponding complex number, thought of as a vector in  $\mathbb{R}^2$ , with the positive real axis. The composition of rotations corresponds to the multiplication of the corresponding complex numbers. Hamilton tried for years to make an algebra of rotations in  $\mathbb{R}^3$  using ordered triples of real numbers. One day he realized he could achieve his goal using 4-tuples.

Here is Hamilton's invention: As a set (and as a vector space) the set of quaternions is identical to  $\mathbb{R}^4$ . The three distinguished coordinate vectors  $(0, 1, 0, 0)$ ,  $(0, 0, 1, 0)$ , and  $(0, 0, 0, 1)$  are given the names  $i$ ,  $j$ , and  $k$ , respectively. The vector  $(a, b, c, d)$  is written  $a + bi + cj + dk$  when thought of as a quaternion. The number  $a$  is referred to as the *real part* and  $b$ ,  $c$ , and  $d$  are called the *i*, *j*, and *k parts*, respectively. Like real and complex numbers, quaternions can be *multiplied*. The multiplication rules are encapsulated by the following relations.

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = k \quad jk = i \quad ki &= j \end{aligned}$$

The elements  $i$ ,  $j$ , and  $k$  do *not* commute. Reversing the left-right order changes the sign of the product.

$$ji = -k \quad kj = -i \quad ik = -j$$

Here is a sample multiplication.

$$\begin{aligned} (3 + 2j)(1 - 4i + k) &= 3 - 12i + 3k + 2j - 8ji + 2jk && \text{(distributing)} \\ &= 2 - 12i + 3k + 2i + 8k + 2i && \text{(applying relations)} \\ &= 3 - 10i + 2j + 11k && \text{(combining terms)} \end{aligned}$$

The *conjugate* of a quaternion  $r = a + bi + cj + dk$ , denoted  $\bar{r}$ , is defined to be  $\bar{r} = a - bi - cj - dk$ , which resembles the complex conjugate. The *length* or *norm* of a quaternion  $r$ , denoted  $\|r\|$ , is its length as a vector in  $\mathbb{R}^4$ ,  $\sqrt{a^2 + b^2 + c^2 + d^2}$ . (The term *norm*, when applied to quaternions, is sometimes used in other treatments to denote the *square* of the Euclidean norm defined here.)

INVESTIGATION B. What algebraic properties do the quaternions share with the real or complex numbers? How are they different? In particular, verify the following things: Show that quaternion multiplication is associative but noncommutative. (Associativity means that  $p(qr) = (pq)r$  for all quaternions  $p$ ,  $q$  and  $r$ .) The norm of  $r = a + bi + cj + dk$  can also be written as  $\|r\| = \sqrt{r\bar{r}}$ . The norm has the property  $\|rs\| = \|r\| \|s\|$  for all quaternions  $r$  and  $s$ . (Because of this, multiplying two unit length quaternions yields another unit length quaternion.) The set of unit length

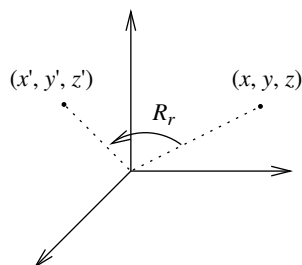
quaternions, viewed as points in  $\mathbb{R}^4$ , is the 3-sphere  $S^3$ . Each nonzero quaternion  $r$  has a *multiplicative inverse*, denoted  $r^{-1}$ , given by

$$r^{-1} = \frac{\bar{r}}{\|r\|^2}.$$

When  $r$  is a unit quaternion,  $r^{-1}$  is the same as  $\bar{r}$ . (Kuipers [10, Ch. 5] is a good source for other details about quaternion algebra.)

Here is how a quaternion  $r$  determines a linear mapping  $R_r: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . To a point  $p = (x, y, z)$  in 3-space, we associate a quaternion  $xi + yj + zk$ . By slight abuse of notation, we will also call this  $p$ . Since the real part of  $p$  is zero we call it a *pure* quaternion. The quaternion product  $rpr^{-1}$  can be shown to be pure, and hence can be thought of as a point  $x'i + y'j + z'k = (x', y', z')$  in 3-space. We define the mapping  $R_r$  by

$$R_r(x, y, z) = rpr^{-1} = (x', y', z'). \quad (2)$$



**Figure 2** A nonzero quaternion  $r$  gives rise to a rotation  $R_r$  in  $\mathbb{R}^3$

**INVESTIGATION C.** Is the mapping  $R_r$  described in the previous paragraph indeed a linear map? Verify that this is the case. Moreover, show that the map determined by any nonzero real scalar multiple of  $r$  is equal to  $R_r$ , that is, show that  $R_{kr} = R_r$  for any quaternion  $r$  and any nonzero real number  $k$ . Show that when  $r \neq 0$ ,  $R_r$  is invertible with inverse  $(R_r)^{-1} = R_{(r^{-1})}$ .

From the “moreover” statement in this Investigation, whenever  $r \neq 0$ , we are free to choose  $r$  to have norm 1 when working with the map  $R_r$ , and we shall do so since this makes the analysis simpler; we may restrict our consideration to points on the 3-sphere  $S^3$  in order to work with rotations given by quaternions.

For  $r \neq 0$ , it turns out that  $R_r$  is a rotation of  $\mathbb{R}^3$ . The axis and angle of the rotation  $R_r$  are elegantly encoded in the four coordinates  $(a, b, c, d)$  in the following way, when  $r$  is a unit quaternion. If  $r = \pm 1$ , it is easy to see that  $R_r$  is the identity mapping on  $\mathbb{R}^3$ . Otherwise,  $R_r$  is a rotation about the axis determined by the vector  $(b, c, d)$ , with angle of rotation  $\theta = 2 \cos^{-1}(a) = 2 \sin^{-1}(\sqrt{b^2 + c^2 + d^2})$ . To appreciate how nice this is, have a friend write down a  $3 \times 3$  orthogonal matrix, say, with no zero entries; now find the axis and angle of rotation. You will quickly appreciate the elegance that quaternions bring to this problem, as compared with matrix methods.

The facts stated in the preceding paragraph are not supposed to be obvious. The next investigation gives a sequence of exercises that outline the proof. For a detailed discussion, see Kuipers [10, § 5.15].

INVESTIGATION D. How does a unit quaternion encode geometric information about its corresponding rotation? Let  $r = a + bi + cj + dk$  be a unit quaternion. Verify that if  $r = \pm 1$ , then  $R_r$  defined above is the identity mapping. Otherwise, show that  $R_r$  is the rotation about the axis vector  $(b, c, d)$  by the angle  $\theta = 2 \cos^{-1}(a) = 2 \sin^{-1}(\sqrt{b^2 + c^2 + d^2})$ , as follows.

1. Show that  $R_r$  preserves norm, that is, that  $\|R_r(p)\| = \|p\|$  for any pure quaternion  $p = xi + yj + zk$ . (This follows from the fact that the norm of a quaternion product equals the product of the norms.)
2. Show that the linear map  $R_r$  has eigenvector  $(b, c, d)$  with eigenvalue 1.
3. Here is a strategy to compute the angle of rotation. Choose a vector  $\mathbf{w}$  perpendicular to the eigenvector  $(b, c, d)$ . This can be broken down into two cases: if at least one of  $b$  and  $c$  is nonzero, we may use  $\mathbf{w} = ci - bj$ . If  $b = c = 0$ , we may use  $\mathbf{w} = i$ . Now compute the angle of rotation by finding the angle between the vectors  $\mathbf{w}$  and  $R_r \mathbf{w}$  using the following formula from analytic geometry, where the multiplication in the numerator on the right-hand side is the dot product in  $\mathbb{R}^3$ .

$$\cos \theta = \frac{\mathbf{w} \cdot R_r \mathbf{w}}{\|\mathbf{w}\|^2}$$

In all cases the right-hand side equals  $a^2 - b^2 - c^2 - d^2 = 2a^2 - 1$ . Now apply a half-angle identity to get  $a = \cos(\theta/2)$ .

Here is the fact that illustrates how Hamilton accomplished his goal to make an algebra of rotations.

INVESTIGATION E. Let  $r$  and  $s$  be unit quaternions. Verify that

$$R_r \circ R_s = R_{rs}.$$

In words rather than symbols: the *composition* of rotations can be accomplished by the *multiplication* of quaternions. Now go back and try Investigation A.

The next investigation is appropriate for a student who has some experience with groups, or could be a motivating problem for an independent study in the basics of group theory. (Armstrong [2] gives an excellent introduction to group theory with a geometric point of view.)

INVESTIGATION F. The set of unit quaternions,  $S^3$ , with the operation of quaternion multiplication satisfies the axioms of a *group*. The set of rotations in 3-space, with the operation of composition, is also a group, called  $SO(3)$ . The map  $\varphi: S^3 \rightarrow SO(3)$  given by  $r \mapsto R_r$  is a *group homomorphism*. Each rotation  $R$  in  $SO(3)$  can be written in the form  $R = R_r$  for some  $r \in S^3$  (that is, the map  $\varphi$  is *surjective*), and each rotation  $R_r$  has precisely two preimages in  $S^3$ , namely  $r$  and  $-r$ . The *kernel* of  $\varphi$  is the subgroup  $\{1, -1\}$ , and we have an *isomorphism* of groups

$$S^3 / \{1, -1\} \approx SO(3).$$

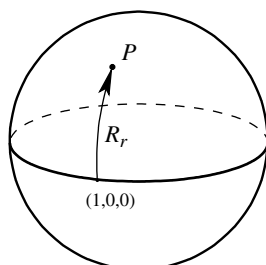
## The 3-sphere, rotations, and the Hopf fibration

We now reformulate the Hopf map in terms of quaternions. First, fix a distinguished point, say,  $P_0 = (1, 0, 0) = i$ , on  $S^2$ . (Any other point would work as well, but this one makes the formulas turn out particularly nicely.) Given a point  $(a, b, c, d)$  on  $S^3$ , let  $r = a + bi + cj + dk$  be the corresponding unit quaternion. The quaternion  $r$  then

defines a rotation  $R_r$  of 3-space given by (2) above. The Hopf fibration maps this quaternion to the image of the distinguished point under the rotation; in formulas, this is

$$r \mapsto R_r(P_0) = rir^{-1} = ri\bar{r}. \quad (3)$$

INVESTIGATION G. Verify that the two formulas (1) and (3) for the Hopf fibration are equivalent.



**Figure 3** The unit quaternion  $r$  moves  $(1, 0, 0)$  to  $P$  via  $R_r$ . The Hopf map takes  $r$  to  $P$ .

Consider the point  $(1, 0, 0)$  in  $S^2$ . One can easily check that the set of points

$$C = \{(\cos t, \sin t, 0, 0) \mid t \in \mathbb{R}\}$$

in  $S^3$  all map to  $(1, 0, 0)$  via the Hopf map  $h$ . In fact, this set  $C$  is the *entire* set of points that map to  $(1, 0, 0)$  via  $h$ . In other words,  $C$  is the preimage set  $h^{-1}((1, 0, 0))$ . You may recognize that  $C$  is the unit circle in a plane in  $\mathbb{R}^4$ . As we shall see, this is typical: for *any* point  $P$  in  $S^2$ , the preimage set  $h^{-1}(P)$  is a circle in  $S^3$ . We will also refer to the preimage set  $h^{-1}(P)$  as the *fiber* of the Hopf map over  $P$ .

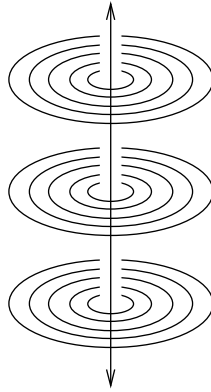
We devote the remainder of this article to study one aspect of the geometry of the Hopf fibration, namely, the configuration of its fibers in  $S^3$ . Using stereographic projection (to be explained below) we get a particularly elegant decomposition of 3-space into a union of disjoint circles and a single straight line. Because this arrangement is fun to think about, we cast it first in the form of a puzzle.

INVESTIGATION H. (LINKED CIRCLES PUZZLE) Using disjoint circles and a single straight line, can you fill up 3-space in such a way that each pair of circles is linked, and the line passes through the interior of each circle?

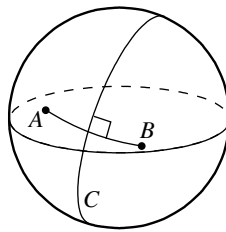
It is the linked nature of the circles that makes this puzzle interesting. If the circles are not required to be linked, there are easy solutions. For example, just take stacks of concentric circles whose centers lie on the given line (see FIGURE 4). We will show that the Hopf fibers themselves give rise to a solution to this puzzle, but see if you can think of your own solution first.

We begin with an observation, presented in the form of an Investigation, on how to find rotations that take a given point  $A$  to a given point  $B$ .

INVESTIGATION I. Given two points  $A$  and  $B$  on  $S^2$  that are not antipodal, how can we describe the set of *all* possible rotations that move  $A$  to  $B$ ? First, choose an arc of a great circle joining  $A$  to  $B$  and call this arc  $\overline{AB}$ ; note that the choice of arc is not unique, although the great circle is. Convince yourself that if  $R$  is a rotation taking  $A$  to  $B$ , then the axis of  $R$  must lie somewhere along the great circle bisecting  $\overline{AB}$  (see FIGURE 5). Along this great circle there are two axes of rotation for which the angle of rotation is easy to compute.

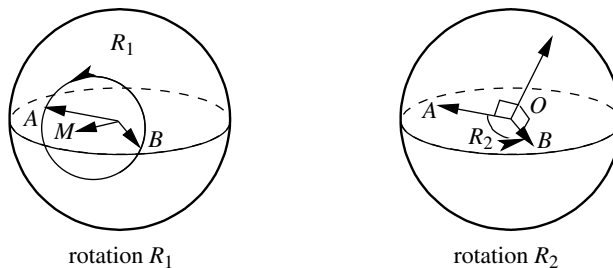


**Figure 4** One way to fill  $\mathbb{R}^3$  with disjoint circles and a line. Now try to arrange for every pair of circles to be linked!



**Figure 5** The axis of any rotation taking  $A$  to  $B$  must pass through the great circle  $C$  that bisects  $\overline{AB}$

1. When the axis of rotation passes through the midpoint  $M$  of  $\overline{AB}$ , the angle of rotation  $\theta$  is  $\pi$  radians or 180 degrees. Let us call this rotation  $R_1$  (see the drawing on the left in FIGURE 6).
2. When the axis of rotation is perpendicular to the vectors  $\mathbf{v} = \vec{OA}$  and  $\mathbf{w} = \vec{OB}$ , the angle of rotation is (plus or minus) the angle between  $\mathbf{v}$  and  $\mathbf{w}$  and is given by  $\cos(\theta) = \mathbf{v} \cdot \mathbf{w}$ . We will call this rotation  $R_2$  (see the drawing on the right in FIGURE 6).



**Figure 6** Two rotations taking  $A$  to  $B$

If a point  $r$  in  $S^3$  is sent by the Hopf map to the point  $P$  in  $S^2$ , then by Investigation G we know that the rotation  $R_r$  moves the point  $(1, 0, 0)$  to  $P$ . We can use Investigation I to find the axis and angle of rotation for two rotations that map  $(1, 0, 0)$  to  $P$ .

Once we have axes and angles of rotation for the rotations  $R_1$  and  $R_2$  of Investigation I, we can use Investigation D to find the quaternions  $r_1$  and  $r_2$  that map to  $R_1$  and  $R_2$  under the map  $\varphi$ , that is,  $R_1 = R_{r_1}$  and  $R_2 = R_{r_2}$ .

INVESTIGATION J. What are explicit formulas for the quaternions  $r_1$  and  $r_2$  described above? For the point  $P = (p_1, p_2, p_3)$  on  $S^2$ , verify that the quaternions  $r_1$  and  $r_2$  are given by

$$r_1 = \frac{1}{\sqrt{2(1+p_1)}} ((1+p_1)i + p_2j + p_3k),$$

$$r_2 = \sqrt{\frac{1+p_1}{2}} \left( 1 + \frac{-p_3j}{1+p_1} + \frac{p_2k}{1+p_1} \right).$$

Let us write  $e^{it}$  for  $\cos t + i \sin t$ . The fiber  $h^{-1}(P)$  is given as a parametrically defined circle in  $\mathbb{R}^4$  by either of the following.

$$h^{-1}(P) = \{r_1 e^{it}\}_{0 \leq t \leq 2\pi}$$

$$h^{-1}(P) = \{r_2 e^{it}\}_{0 \leq t \leq 2\pi}$$

The point  $P = (-1, 0, 0)$  is a special case, and  $h^{-1}((-1, 0, 0))$  is given by

$$h^{-1}((-1, 0, 0)) = \{ke^{it}\}_{0 \leq t \leq 2\pi}.$$

## Seeing the Hopf fibration

Next we demonstrate a method that allows us to *see* a little of what is going on with the Hopf fibration. Our aim is to show pictures of fibers. We do this by means of *stereographic projection*, which may be familiar to readers from an article by Delman and Galperin [6] in the previous issue of the MAGAZINE.

We begin by describing the stereographic projection of the 2-sphere to the  $x, y$ -plane. Imagine a light source placed at the “north pole”  $(0, 0, 1)$ . Stereographic projection sends a point  $P$  on  $S^2$  to the intersection of the light ray through  $P$  with the plane as in FIGURE 7.

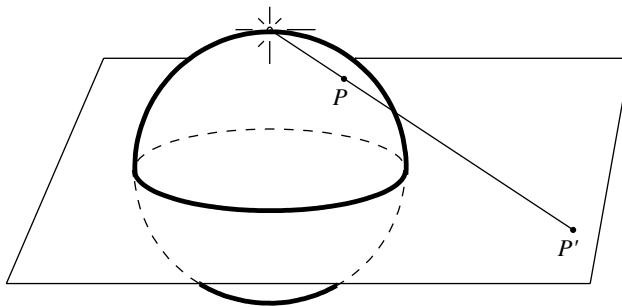


Figure 7 Stereographic projection

The alert reader will notice that the point  $(0, 0, 1)$  has no sensible image under this projection. Therefore we restrict the stereographic projection to  $S^2 \setminus (0, 0, 1)$ .



INVESTIGATION K. Verify that the stereographic projection described above is given by

$$(x, y, z) \mapsto \left( \frac{x}{1-z}, \frac{y}{1-z} \right).$$

Write out the *inverse* map  $\mathbb{R}^2 \rightarrow S^2 \setminus (0, 0, 1)$ . That is, given a point  $(a, b)$  in the plane, what are the  $(x, y, z)$  coordinates of the point on  $S^2$  sent to  $(a, b)$  by the stereographic projection? Show that a circle on  $S^2$  that contains  $(0, 0, 1)$  is mapped to a straight line in the plane. Prove that a circle on  $S^2$  that does not pass through the point of projection  $(0, 0, 1)$  is mapped by the stereographic projection to a circle in the plane. (Ahlfors [1, Ch. 1 § 2.4] gives a proof that stereographic projection preserves circles based on elementary geometry of complex numbers.)

Like the definition of the sphere, stereographic projection generalizes to all dimensions, and in particular, it provides a projection map  $S^3 \setminus (1, 0, 0, 0) \rightarrow \mathbb{R}^3$  given by

$$(w, x, y, z) \mapsto \left( \frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w} \right). \quad (4)$$

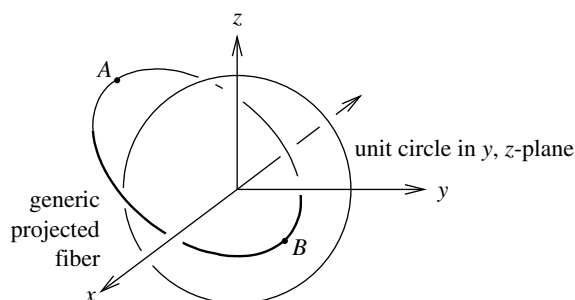
The point  $(1, 0, 0, 0)$  on  $S^3$  from which we project is an arbitrary choice, but it does make the formulas simple.

The real power of stereographic projection is this: it allows us to see all of the 3-sphere (except one point) in familiar 3-space. This is remarkable because  $S^3$  is a curved object that resides in 4-space.

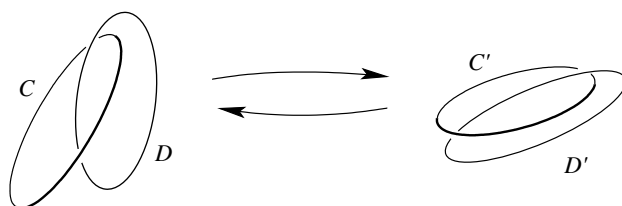
The last property in Investigation K above—that stereographic projection preserves circles—holds in all dimensions [4, Chapter 18]. We know from the previous section that fibers of the Hopf map are circles in  $S^3$ . It follows that stereographic projection sends them to circles (or a line, if the fiber contains the point  $(1, 0, 0, 0)$ ) in  $\mathbb{R}^3$ . We conclude with two Investigations that show how the stereographic images of the Hopf fibers solve the linked circles puzzle.

INVESTIGATION L. Let us denote by  $s$  the stereographic projection  $s : S^3 \setminus (1, 0, 0, 0) \rightarrow \mathbb{R}^3$  given in (4). Then  $s \circ h^{-1}((1, 0, 0))$  is the  $x$ -axis,  $s \circ h^{-1}((-1, 0, 0))$  is the unit circle in the  $y, z$ -plane, and for any other point  $P = (p_1, p_2, p_3)$  on  $S^2$  not equal to  $(1, 0, 0)$  or  $(-1, 0, 0)$ ,  $s \circ h^{-1}(P)$  is a circle in  $\mathbb{R}^3$  that intersects the  $y, z$ -plane in exactly two points  $A$  and  $B$ , one inside and one outside the unit circle in the  $y, z$ -plane. This establishes that  $s \circ h^{-1}(P)$  is linked with the unit circle in the  $y, z$ -plane. The points  $A$  and  $B$  lie on a line through the origin containing the vector  $(0, p_3, -p_2)$ . The plane of the circle  $s \circ h^{-1}(P)$  cannot contain the  $x$ -axis (if it did,  $s \circ h^{-1}(P)$  would intersect  $s \circ h^{-1}((1, 0, 0))$ , but fibers are disjoint). From these observations we can conclude that the  $x$ -axis passes through the interior of the circle  $s \circ h^{-1}(P)$ . See FIGURE 8.

INVESTIGATION M. To show the linked nature of *any two* circles  $C$  and  $D$  that are projections of fibers, we exhibit a continuous one-to-one map  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  that takes  $C$  to the unit circle in the  $y, z$ -plane, and takes  $D$  to some other projected fiber circle. Since the image of  $D$  is linked with the unit circle in the  $y, z$ -plane, as in FIGURE 8,  $C$  and  $D$  must also be linked. (Students who have never studied topology may accept the intuitively reasonable statement that the linked nature of circles cannot be altered by a continuous bijective map, aided by FIGURE 9. Students with experience in topology may enjoy trying to prove this.)

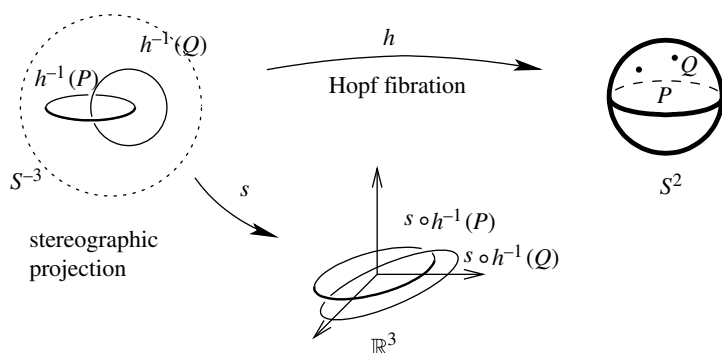


**Figure 8** A generic projected Hopf fiber.  $A$  and  $B$  mark the intersections of the fiber with the  $y, z$ -plane.



**Figure 9** If the continuous bijective images  $C', D'$  of circles  $C, D$  are linked, then  $C$  and  $D$  must also be linked.

Here is how to construct the map  $\psi$ . Let  $P$  be any point on the circle  $C$ , and let  $r = s^{-1}(P)$ . Define  $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$  by  $f(x) = kr^{-1}x$  (quaternion multiplication). The map  $\psi$  is the composition  $s \circ f \circ s^{-1}$ .



**Figure 10** Stereographic projections of Hopf fibers. Any two projected fibers are linked circles, except  $s \circ h^{-1}(1, 0, 0)$ , which is a line.

## Conclusion

We have explained how to understand the Hopf fibration in terms of quaternions. In the process, we showed how the algebra of rotations in 3-space is built into the workings of the Hopf map.

Topics raised in the Investigations suggest many lines of inquiry for independent student research. For example, making computer animations of linked Hopf fibers has been an independent study research project for two of our undergraduate students. FIGURE 11 shows an image from the software written by Nick Hamblet (see Acknowledgment below). The left panel shows a set of points lying on a circle in the codomain  $S^2$  of the Hopf fibration. The right panel shows, via stereographic projection, the fibers corresponding to those points. An ongoing project is to build a web tutorial site featuring the animations. The reader who finds topics in this article appealing will enjoy a related article [18]. For general inspiration, and more on the geometry of  $\mathbb{R}^3$  and rotations, see Hermann Weyl's lovely book *Symmetry* [16].

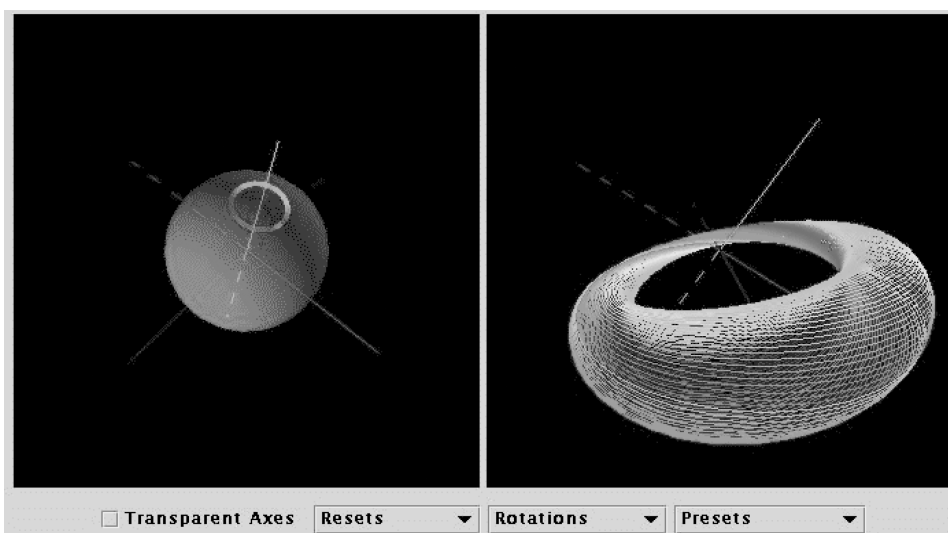


Figure 11 Screenshot of Hopf fiber software

**Acknowledgment.** We are grateful to Lebanon Valley College for summer support for Nick Hamblet's software development project. Nick Hamblet is a student at Lebanon Valley College, class of 2004. His work builds on joint work of the author with Paul Hemler, Professor of Computer Science, and Keely Chorn, class of 1999, both at Wake Forest University.

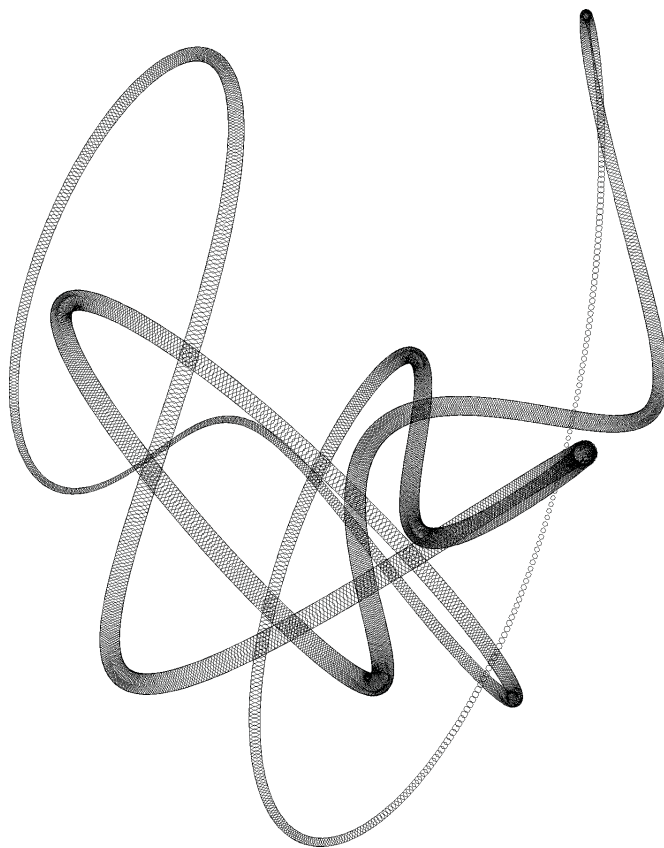
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## Mathematics Awareness Month: Mathematics and Art

Bob Brill makes art by composing simple computer algorithms that generate imagery. There are worlds of order and beauty lying dormant in our various mathematical systems, waiting to be made visible by these algorithmic processes. This is the beauty of pattern, rhythm, symmetry, asymmetry, balance, and movement. These are the worlds he explores in his art. “Mathematics,” Brill says, “more than any other human activity, seems to offer connections to the underlying order of the world. This is a great inspiration for an artist and a great challenge.”



“Beyond Lissajous,” by Bob Brill. Visit the artist’s website: see <http://users.migate.net/~bobbrill>.

For more about Math Awareness Month, see page 118.