An Elementary Introduction to the Hopf Fibration

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Introduction

The Hopf fibration, named after Heinz Hopf who studied it in a 1931 paper [8], is an important object in mathematics and physics. It was a landmark discovery in topology and is a fundamental object in the theory of Lie groups. The Hopf fibration has a wide variety of physical applications including magnetic monopoles [13], rigid body mechanics [10] and quantum information theory [12].

Unfortunately, the Hopf fibration is little known in the undergraduate curriculum, in part because presentations usually assume background in abstract algebra or manifolds. However, this is not a necessary restriction. We present in this article an introduction to the Hopf fibration that requires only linear algebra and analytic geometry. In particular, no vector calculus, abstract algebra or topology is needed. Our approach uses the algebra of quaternions and illustrates some of the algebraic and geometric properties of the Hopf fibration. We explain the intimate connection of the Hopf fibration with rotations of 3-space that is the basis for its natural applications to physics.

We deliberately leave some of the development as exercises, called "Investigations," for the reader. The Investigations contain key ideas and are meant to be fun to think about. The reader may also take them as statements of facts that we wish to assume without interrupting the narrative.

Hopf's mapping

The *standard unit n-sphere* S^n is the set of points $(x_0, x_1, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ that satisfy the equation
$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1.$$

Geometrically stated, S^n is the set of points in $\mathbb{R}^{n+1}$ whose distance from the origin is 1. Thus the 1-sphere S^1 is the familiar unit circle in the plane, and the 2-sphere S^2 is the surface of the solid unit ball in 3-space. The thoughtful reader may wonder what higher dimensional spheres look like. We address this issue at the end of this article, where we explain how *stereographic projection* is used to "see" inside S^3.

The Hopf fibration is the mapping $h \colon S^3 \to S^2$ defined by

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)). \tag{1}$$

To be historically precise, Hopf's original formula differs from that given here by a reordering of coordinates. We use this altered version to be consistent with the quaternion approach explained later in this article. It is easy to check that the squares of the three coordinates on the right hand side sum to $(a^2 + b^2 + c^2 + d^2)^2 = 1$, so that the image of h is indeed contained in S^2.

What problem was Hopf trying to solve when he invented this map? And how can one see any connection with physical rotations, as we have claimed?

The work in Hopf's paper [8] was an early achievement in the modern subject of *homotopy theory*. In loose terms, homotopy seeks to understand those properties of a space that are not altered by continuous deformations. One way to detect these properties in an unknown space X is to compare X with a well understood space Y via the set of all continuous maps $Y \to X$. Two maps whose images can be continuously deformed from one to the other are considered equivalent. Knowing something about Y and also about the set of homotopically equivalent maps from Y to X helps us understand X. This seemingly indirect method provides a powerful way to analyze spaces.

Ironically, one of the most intractable problems in homotopy theory is to determine the homotopy equivalence classes of maps $Y \to X$ when X and Y are both spheres and the dimension of X is smaller than the dimension of Y. Many individual cases for particular pairs of dimensions of X and Y are understood, but there remain interesting unsolved problems. Hopf's map $h \colon S^3 \to S^2$ was a spectacular breakthrough in this area. We cannot give the full story of this discovery here, but we can explain the Hopf fibration in a geometric way that indicates its connection to rotations.

Rotations and quaternions

First, notice that a rotation about the origin in $\mathbb{R}^3$ can be specified by giving a vector for the axis of rotation and an angle of rotation about the axis. We make the convention that the rotation will be counterclockwise for positive angles, and clockwise for negative angles, when viewed from the tip of the vector (see Figure 1).
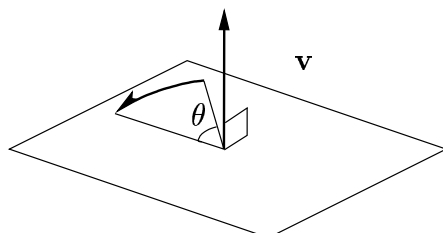
Figure 1: A rotation in $\mathbb{R}^3$ is specified by an angle θ and a vector $\mathbf{v}$ giving the axis.

The specification of a rotation by an axis vector and an angle is far from unique. The rotation determined by the vector $\mathbf{v}$ and the angle θ is the same as the rotation determined by the pair $(k\mathbf{v}, \theta + 2n\pi)$, where k is any positive scalar and n is any integer. The pair $(-\mathbf{v}, -\theta)$ also determines the same rotation. Nonetheless, we see that four real numbers are sufficient to specify a rotation: three coordinates for a vector and one real number to give the angle. This is far fewer than the *nine* entries of a 3×3 orthogonal matrix we learn to use in linear algebra. In fact, we can cut the number of parameters needed to specify a rotation from four to three, for example, by giving an axis vector whose length determines the angle of rotation. However, we shall not pursue that here; it is the 4-tuple approach that turns out to be practical. Is there an efficient way to work with 4-tuples of real numbers to do practical calculations with rotations? Here is a sample basic question.

(2) **Investigation.** Given geometric data (axes and angles) for two rotations, how do you determine the axis and angle for their composition? (The composition of two rotations is the motion obtained by performing first one rotation, then the other. Order counts!) Suggestion: think about this investigation long enough to realize that it is difficult, or at least tedious, if you restrict yourself to matrix methods; then revisit after you do Investigation (7) below.

The problem of finding a convenient algebraic method for computing with rotations led William Rowan Hamilton to invent the *quaternions* in the mid-19th

century. The discovery of quaternions, and Hamilton's life in general, is a fascinating bit of history. For further reading, see [6] or [14]. For an exposition of the rotation problem in Investigation 2 and its solution, beyond what appears in this section, see [9], § 6.2 ff.

Hamilton was inspired by the solution to the analogous problem in two dimensions: rotations of the plane about the origin can be encoded by unit length *complex numbers*. The angle of a rotation is the same as the angle made by its corresponding complex number, thought of as a vector in $\mathbb{R}^2$, with the positive real axis. The composition of rotations corresponds to the multiplication of the corresponding complex numbers. Hamilton tried for years to make an algebra of rotations in $\mathbb{R}^3$ using ordered triples of real numbers. One day he realized he could achieve his goal using 4-tuples. Here is his invention.

As a set (and as a vector space) the set of quaternions is identical to $\mathbb{R}^4$. The three distinguished coordinate vectors $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are given the names i, j and k, respectively. The vector (a, b, c, d) is written $a + bi + cj + dk$ when thought of as a quaternion. The number a is referred to as the *real part* and b, c and d are called the *i, j and k parts*, respectively. Like real and complex numbers, quaternions can be *multiplied*. The multiplication rules are encapsulated by the following relations.

$$i^2 = j^2 = k^2 = -1$$

$$ij = k \qquad jk = i \qquad ki = j$$

The elements i, j and k do *not* commute. Reversing the left-right order changes the sign of the product.

$$ji = -k \qquad kj = -i \qquad ik = -j$$

Here is a sample multiplication.

$$
\begin{aligned}
(3 + 2j)(1 - 4i + k) &= 3 - 12i + 3k + 2j - 8ji + 2jk \quad \text{(distributing)} \\
&= 3 - 10i + 2j + 11k \quad \text{(applying relations and combining terms)}
\end{aligned}
$$

Similar to the complex numbers, the *conjugate* of a quaternion $r = a + bi + cj + dk$, denoted $\bar{r}$, is defined to be $\bar{r} = a - bi - cj - dk$. The *length* or *norm* of a quaternion r, denoted $||r||$, is its length as a vector in $\mathbb{R}^4$. (The term *norm*, when applied to quaternions, is also commonly used to refer to the *square* of the Euclidean norm defined here.) The formula for the norm of $r = a + bi + cj + dk$ is $||r|| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

(3) **Investigation.** Quaternion multiplication is associative but noncommutative. Associativity means that

$$p(qr) = (pq)r$$

for all quaternions p, q and r. Another formula for the norm of $r = a + bi + cj + dk$ is $||r|| = \sqrt{r\overline{r}}$. Norm has the property $||rs|| = ||r|| \, ||s||$ for all quaternions r and s. Because of this, multiplying two unit length quaternions yields another unit length quaternion. The set of unit length quaternions, viewed as points in $\mathbb{R}^4$, is the 3-sphere S^3. Each nonzero quaternion r has a *multiplicative inverse*, denoted r^{-1}, given by

$$r^{-1} = \frac{\overline{r}}{||r||^2}.$$

When r is a unit quaternion, r^{-1} is the same as $\overline{r}$. (For these and other details about quaternion algebra, see [9] Ch. 5.)

Here is how a quaternion r determines a linear mapping $R_r\colon \mathbb{R}^3 \to \mathbb{R}^3$. To a point $p = (x, y, z)$ in 3-space, we associate a quaternion $xi + yj + zk$ which we will also call p (a quaternion whose real part is zero is called *pure*). The quaternion product rpr^{-1} is also pure, that is, of the form $x'i + y'j + z'k$, and hence can be thought of as a point (x', y', z') in 3-space. We define the mapping R_r by
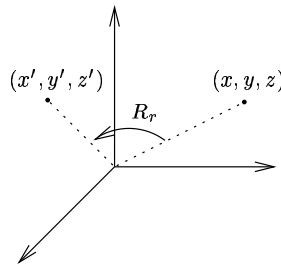
$$R_r(x, y, z) = (x', y', z'). \tag{4}$$

Figure 2: A nonzero quaternion r gives rise to a rotation R_r in $\mathbb{R}^3$.

(5) **Investigation.** Verify that the mapping R_r described in the previous paragraph is indeed a linear map. Moreover, show that the map determined by any nonzero real scalar multiple of r is equal to R_r, that is, show that $R_{kr} = R_r$ for any quaternion r and any nonzero real number k. Show that when $r \neq 0$, R_r is invertible with inverse $(R_r)^{-1} = R_{(r^{-1})}$.

From the "moreover" statement in the previous Investigation, whenever $r \neq 0$, we are free to choose r to have norm 1 when working with the map R_r, and we shall do so since this makes the analysis simpler.

For $r \neq 0$, it turns out that R_r is a rotation of $\mathbb{R}^3$. The axis and angle of the rotation R_r are elegantly encoded in the four coordinates (a, b, c, d) in the following way, when r is a unit quaternion. If $r = \pm 1$, it is easy to see that R_r is the identity mapping on $\mathbb{R}^3$. Otherwise, R_r is a rotation about the axis determined by the vector (b, c, d), with angle of rotation $\theta = 2 \cos^{-1}(a) = 2 \sin^{-1}(\sqrt{b^2 + c^2 + d^2})$. To appreciate how nice this is, have a friend write down a 3×3 orthogonal matrix, say, with no zero entries; now quickly find the axis and angle of rotation!

The facts stated in the preceding paragraph are not supposed to be obvious. Here is a sequence of exercises that outline the proof. For a detailed discussion see [9] § 5.15.

(6) **Investigation.** Let $r = a + bi + cj + dk$ be a unit quaternion. Verify that if $r = \pm 1$, then R_r defined above is the identity mapping. Otherwise, show that R_r is the rotation about the axis vector (b, c, d) by the angle $\theta = 2 \cos^{-1}(a) = 2 \sin^{-1}(\sqrt{b^2 + c^2 + d^2})$, as follows.

1. Show that R_r preserves norm, i.e., that $||R_r(p)|| = ||p||$ for any pure quaternion $p = xi + yj + zk$. (This follows from the fact that the norm of a quaternion product equals the product of the norms.)

2. Show that the linear map R_r has eigenvector (b, c, d) with eigenvalue 1.

3. Here is a strategy to compute the angle of rotation. Choose a vector $\mathbf{w}$ perpendicular to the eigenvector (b, c, d). This can be broken down into two cases: if at least one of b, c is nonzero, we may use $\mathbf{w} = ci - bj$. If $b = c = 0$, we may use $\mathbf{w} = i$. Now compute the angle of rotation by finding the angle between the vectors $\mathbf{w}$ and $R_r\mathbf{w}$ using the following formula from analytic geometry, where the multiplication in the numerator on the right hand side is the dot product in $\mathbb{R}^3$.

$$\cos \theta = \frac{\mathbf{w} \cdot R_r\mathbf{w}}{||\mathbf{w}||^2}$$

In all cases the right hand side equals $a^2 - b^2 - c^2 - d^2 = 2a^2 - 1$. Now apply a half-angle identity to get $a = \cos(\theta/2)$.

Here is the fact that illustrates how Hamilton accomplished his goal to make an algebra of rotations.

(7) **Investigation.** Let r and s be unit quaternions. Then

$$R_r \circ R_s = R_{rs}.$$

In words, the *composition* of rotations can be accomplished by the *multiplication* of quaternions. Now go back and try Investigation (2).

The next Investigation is appropriate for a student who has some experience with groups, or could be a motivating problem for an independent study in the basics of group theory. (For an excellent introduction to group theory with a geometric point of view, see [2].)

(8) **Investigation.** The set S^3 with the operation of quaternion multiplication satisfies the axioms of a *group*. The set of rotations in 3-space, with the operation of composition, is also a group, called SO(3). The map $\varphi\colon S^3 \to$ SO(3) given by $r \mapsto R_r$ is a *group homomorphism*. Each rotation R in SO(3) can be written in the form $R = R_r$ for some $r \in S^3$ (i.e., the map φ is *surjective*), and each rotation R_r has precisely two preimages in S^3, namely r and $-r$. The *kernel* of φ is the subgroup $\{1, -1\}$, and we have an *isomorphism* of groups

$$S^3/\{1, -1\} \approx SO(3).$$

The 3-sphere, Rotations, and the Hopf Fibration

We now give a reformulation of the Hopf map in terms of quaternions. First, fix a distinguished point, say, $P_0 = (1, 0, 0)$, on S^2. Given a point (a, b, c, d) on S^3, let $r = a + bi + cj + dk$ be the corresponding unit quaternion. The quaternion r then defines a rotation R_r of 3-space given by (4) above. Then the Hopf fibration is defined by

$$r \mapsto R_r(P_0) = ri\overline{r}. \tag{9}$$

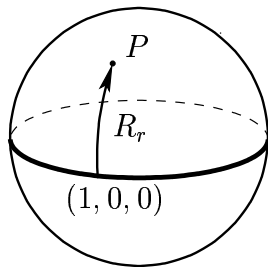(10) **Investigation.** The two formulas (1) and (9) for the Hopf fibration are equivalent.

Figure 3: The unit quaternion r moves $(1, 0, 0)$ to P via R_r. The Hopf map takes r to P.

Consider the point $(1, 0, 0)$ in S^2. One can easily check that the set of points

$$C = \{(\cos t, \sin t, 0, 0) \mid t \in \mathbb{R}\}$$

in S^3 all map to $(1, 0, 0)$ via the Hopf map h. In fact, this set C is the *entire* set of points that map to $(1, 0, 0)$ via h. In other words, C is the preimage set $h^{-1}((1, 0, 0))$. You may recognize that C is the unit circle in a plane in $\mathbb{R}^4$. As we shall see, this is typical: for *any* point P in S^2, the preimage set $h^{-1}(P)$ is a circle in S^3. We will also refer to the preimage set $h^{-1}(P)$ as the *fiber* of the Hopf map over P.

We devote the remainder of this article to study one aspect of the geometry of the Hopf fibration, namely, the configuration of its fibers in S^3. Using stereographic projection (to be explained below) we get a particularly elegant decomposition of 3-space into a union of disjoint circles and a single straight line. Because this arrangement is fun to think about, we cast it first in the form of a puzzle.

(11) **Linked Circles Puzzle.** Using disjoint circles and a single straight line, can you fill up 3-space in such a way that each pair of circles is linked, and the line passes through the interior of each circle?

It is the linked-ness of the circles that makes this puzzle interesting. If the circles are not required to be linked, there are easy solutions. For example, just take stacks of concentric circles whose centers lie on the given line (see Figure 4). We will show that the Hopf fibers themselves give rise to a solution to this puzzle, but see if you can think of your own solution first!

We begin with an observation, presented in the form of an Investigation, on how to find rotations that take a given point A to a given point B.
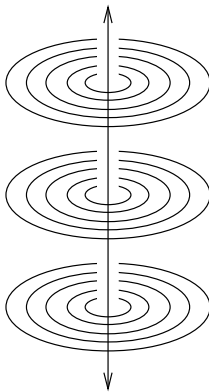
Figure 4: One way to fill $\mathbb{R}^3$ with disjoint circles and a line. Now try to arrange for every pair of circles to be linked!

(12) **Investigation.** Given two points A and B on S^2, choose an arc of a great circle joining A to B and call this arc $\overline{AB}$; note that this choice is not unique. If R is a rotation taking A to B, then the axis of R must lie somewhere along the great circle bisecting $\overline{AB}$ (see Figure 5).

Along this great circle there are two axes of rotation for which the angle of rotation is easy to compute.

1. When the axis of rotation passes through the midpoint M of $\overline{AB}$, the angle of rotation θ is π radians or 180 degrees. Let us call this rotation R_1 (see the drawing on the left in Figure 6).

2. When the axis of rotation is perpendicular to the vectors $\mathbf{v} = \vec{OA}$ and $\mathbf{w} = \vec{OB}$, the angle of rotation is (plus or minus) the angle between $\mathbf{v}$ and $\mathbf{w}$ and is given by $\cos(\theta) = \mathbf{v} \cdot \mathbf{w}$. We will call this rotation R_2 (see the drawing on the right in Figure 6).
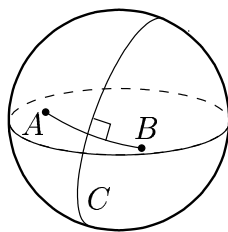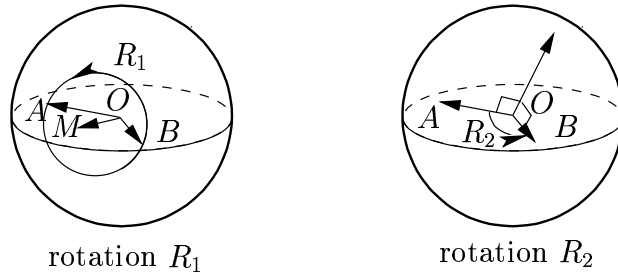

Figure 5: The axis of any rotation taking A to B must pass through the great circle C that bisects $\overline{AB}$.

If a point r in S^3 is sent by the Hopf map to the point P in S^2, then by (10) we know that the rotation R_r moves the point $(1, 0, 0)$ to P. We can use (12) to find

Figure 6: Two rotations taking A to B.

the axis and angle of rotation for two rotations that map $(1, 0, 0)$ to P. Let us call R_1 and R_2 the rotations described in (12) parts 1 and 2, respectively.

Once we have axes and angles of rotation for R_1 and R_2, we can use (6) to find the quaternions r_1 and r_2 that map to R_1 and R_2 under the map φ, i.e., $R_1 = R_{r_1}$ and $R_2 = R_{r_2}$.

(13) **Investigation.** For the point $P = (p_1, p_2, p_3)$ on S^2, the quaternions r_1 and r_2 described above are given by

$$
\begin{aligned}
r_1 &= \frac{1}{\sqrt{2(1 + p_1)}} \left((1 + p_1)i + p_2 j + p_3 k \right), \\
r_2 &= \sqrt{\frac{1 + p_1}{2}} \left(1 + \frac{-p_3 j}{1 + p_1} + \frac{p_2 k}{1 + p_1} \right).
\end{aligned}
$$

Let us write e^{it} for $\cos t + i \sin t$. The fiber $h^{-1}(P)$ is given as a parametrically defined circle in $\mathbb{R}^4$ by either of the following.

$$
\begin{aligned}
h^{-1}(P) &= \{ r_1 e^{it} \}_{0 \leq t \leq 2\pi} \\
h^{-1}(P) &= \{ r_2 e^{it} \}_{0 \leq t \leq 2\pi}
\end{aligned}
$$

The point $P = (-1, 0, 0)$ is a special case, and $h^{-1}((-1, 0, 0))$ is given by

$$
h^{-1}((-1, 0, 0)) = \{ k e^{it} \}_{0 \leq t \leq 2\pi}.
$$

Seeing the Hopf fibration

Next we demonstrate a method that allows us to *see* a little of what is going on with the Hopf fibration. Our aim is to show pictures of fibers. We do this by means of *stereographic projection*.

We begin by describing the stereographic projection of the 2-sphere to the x, y-plane (see Figure 7). Imagine a light source placed at the "north pole" $(0, 0, 1)$. Stereographic projection sends a point P on S^2 to the intersection of the light ray through P with the plane.
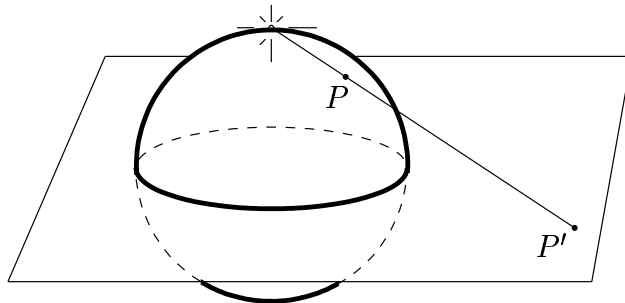

Figure 7: Stereographic projection.

The alert reader will notice that the point $(0, 0, 1)$ has no sensible image under this projection. Therefore we restrict the stereographic projection to $S^2 \setminus (0, 0, 1)$.

(14) **Investigation.** Verify that the stereographic projection described above is given by
$$(x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right).$$
Write out the *inverse* map $\mathbb{R}^2 \to S^2 \setminus (0, 0, 1)$. That is, given a point (a, b) in the plane, what are the (x, y, z) coordinates of the point on S^2 sent to (a, b) by the stereographic projection? Show that a circle on S^2 that contains $(0, 0, 1)$ is mapped to a straight line in the plane. Prove that a circle on S^2 that does not pass through the point of projection $(0, 0, 1)$ is mapped by the stereographic projection to a circle in the plane. (For a proof of the circle preservation property using elementary geometry of complex numbers, see [1] Ch. 1 § 2.4.)

Like the definition of the sphere, stereographic projection generalizes to all dimensions, and in particular, it provides a projection map $S^3 \setminus (1, 0, 0, 0) \to \mathbb{R}^3$ given by
$$(w, x, y, z) \mapsto \left(\frac{x}{1 - w}, \frac{y}{1 - w}, \frac{z}{1 - w} \right). \tag{15}$$
Note that the point $(1, 0, 0, 0)$ on S^3 from which we project is an arbitrary choice.

The real power of stereographic projection is this: it allows us to see all of the 3-sphere (except one point) in familiar 3-space. This is remarkable because S^3 is a

curved object that resides in 4-space.

The last property in (14) above—that stereographic projection preserves circles—holds in all dimensions (see [4], Chapter 18). We know from the previous section that fibers of the Hopf map are circles in S^3. It follows that stereographic projection sends them to circles (or a line, if the fiber contains the point $(1,0,0,0)$) in $\mathbb{R}^3$. We conclude with two Investigations that show how the stereographic images of the Hopf fibers solve the linked circles puzzle (11).

(16) **Investigation.** Let us denote by s the stereographic projection $S^3 \setminus (1,0,0,0) \to \mathbb{R}^3$ given in (15). Then $s \circ h^{-1}((1,0,0))$ is the x-axis, $s \circ h^{-1}((-1,0,0))$ is the unit circle in the y,z-plane, and for any other point $P = (p_1, p_2, p_3)$ on S^2 not equal to $(1,0,0)$ or $(-1,0,0)$, $s \circ h^{-1}(P)$ is a circle in $\mathbb{R}^3$ that intersects the y,z-plane in exactly two points A and B, one inside and one outside the unit circle in the y,z-plane. This establishes that $s \circ h^{-1}(P)$ is linked with the unit circle in the y,z-plane. The points A and B lie on a line through the origin containing the vector $(0, p_3, -p_2)$. The plane of the circle $s \circ h^{-1}(P)$ cannot contain the x-axis (if it did, $s \circ h^{-1}(P)$ would intersect $s \circ h^{-1}((1,0,0))$, but fibers are disjoint). From these observations we can conclude that the x-axis passes through the interior of the circle $s \circ h^{-1}(P)$. See Figure 8.
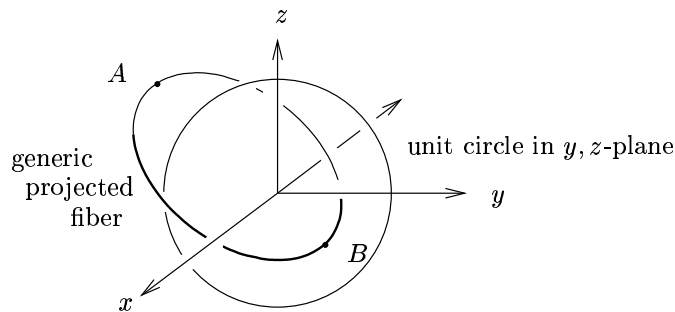

Figure 8: A generic projected Hopf fiber. A and B mark the intersections of the fiber with the y,z-plane.

(17) **Investigation.** To show that *any two* projected fiber circles C and D are linked, we exhibit a continuous one-to-one map $\psi \colon \mathbb{R}^3 \to \mathbb{R}^3$ that takes C to the unit circle in the y, z-plane, and takes D to some other projected fiber circle E. Since E is linked with the unit circle in the y, z-plane, C and D must also be linked. See Figure 9. (Students who have never studied topology may accept the intuitively reasonable statement that linked-ness of circles cannot be altered by a continuous bijective map. Students with experience in topology may enjoy trying to prove this.)

Here is how to construct the map ψ. Let P be any point on the circle C, and let $r = s^{-1}(P)$. Define $f \colon \mathbb{R}^4 \to \mathbb{R}^4$ by $f(x) = kr^{-1}x$ (quaternion multiplication). The map ψ is the composition $s \circ f \circ s^{-1}$.
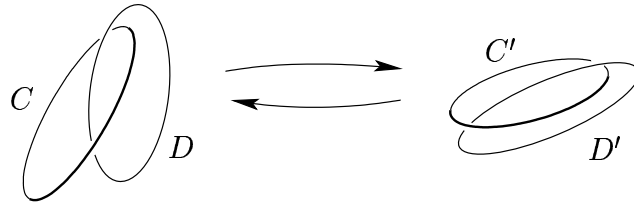

Figure 9: If the continuous bijective images C', D' of circles C, D are linked, then C and D must also be linked.
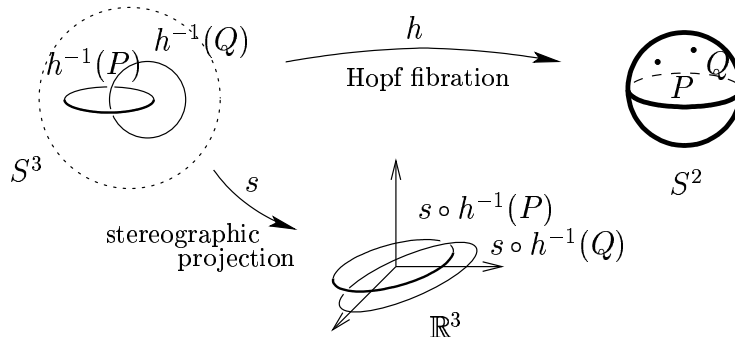

Figure 10: Stereographic projections of Hopf fibers. Any two projected fibers are linked circles, except $s \circ h^{-1}(1, 0, 0)$ is a line.

Conclusion

We have explained how the Hopf fibration can be understood in terms of quaternions. In the process of doing so, we showed how the algebra of rotations in 3-space is built into the workings of the Hopf map.

Topics raised in the Investigations suggest many lines of inquiry for independent student research. For example, making computer animations of linked Hopf fibers has been an independent study research project for two of our undergraduate students. Figure 11 shows an image from the software written by Nick Hamblet.[1] The image shows marked points on the codomain S^2 in the left panel and their corresponding fibers, projected to $\mathbb{R}^3$ via stereographic projection, in the right panel. An ongoing project is to build a web tutorial site featuring the animations. The reader who finds topics in this article appealing will enjoy a related article [17]. For general inspiration, and more on the geometry of $\mathbb{R}^3$ and rotations, see Hermann Weyl's lovely book *Symmetry* [15].
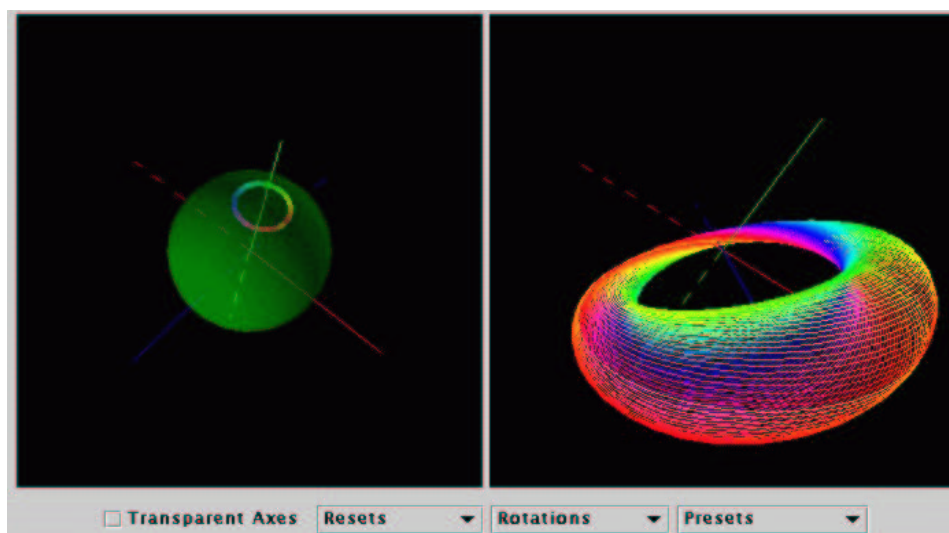

Figure 11: Screenshot of Hopf fiber software.

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[1] Lebanon Valley College, class of 2004. Nick's work continues a similar project at Wake Forest University with Paul Hemler, Prof. of Computer Science, and Keely Chorn, class of 1999.

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Referees' Appendix

Formula for Hopf fibration

Many versions of formula (1) for the Hopf fibration are given in different texts. Each can be obtained from any other by a permutation of the argument variables and the output variables. I have chosen to use this version because it matches the standard quaternion version (9) of the Hopf fibration. Using this Hopf fibration, fiber circles are *left* cosets of the canonical torus $T = \{e^{it} \mid t \in \mathbb{R}\}$ which is the complex numbers embedded in the quaternions. This is also consistent with the identification of S^2, the base of the Hopf fibration, as the homogeneous space $SU(2)/T$ by the standard *right* action of T on $SU(2)$. One commonly finds these left and right actions reversed, which accounts for different versions of the Hopf formula. While I do not discuss tori, matrix groups or homogeneous spaces in this article, I still like to give a version of the Hopf fibration that generalizes well for students who may have more advanced background.

Solution to Investigation (2)

Note that I intend that the reader work out this solution *after* reading about quaternions and rotations further in the paper.

Let R be the rotation with axis (b, c, d) and angle θ, and let S be the rotation with axis (b', c', d') and angle θ'. Let $a = \cos(\theta/2)$, $a' = \cos(\theta'/2)$. Let $r = a + bi + cj + dk$ and $s = a' + b'i + c'j + d'k$. Normalize r and s so that each has norm 1. Let us assume that has been done. The composition $R \circ S$ has the corresponding quaternion

$$\begin{aligned}
rs \quad = \quad & aa' - bb' - cc' - dd' \\
+ \quad & (ab' + ba' + cd' - dc')i \\
+ \quad & (ac' + ca' - bd' + db')j \\
+ \quad & (ad' + da' + bc' - cb')k.
\end{aligned}$$

We can now read the geometric data for the rotation $R \circ S$ from the coordinates of the quaternion rs. The axis is

$$(ab' + ba' + cd' - dc', ac' + ca' - bd' + db', ad' + da' + bc' - cb')$$

and the angle is $2\cos^{-1}(aa' - bb' - cc' - dd')$.

Verification, Investigation (6), part 3

For $\mathbf{w} = ci - bj$, we have

$$
\begin{aligned}
\mathbf{w} \cdot R_r \mathbf{w} &= (ci - bj) \cdot [r(ci - bj)\overline{r}] \\
&= (ci - bj) \cdot [(a + bi + cj + dk)(ci - bj)(a - bi - cj - dk)] \\
&= (ci - bj) \cdot [\{(ac + bd)i + (-ab + cd)j + (-b^2 - c^2)j\}(a - bi - cj - dk)] \\
&= (ci - bj) \cdot [(a^2c + 2abd - cd^2 - b^2c - c^3)i + (2acd + bd^2 - a^2b + b^3 + bc^2)j - 2a(b^2 + c^2)k] \\
&= (b^2 + c^2)(a^2 - b^2 - c^2 - d^2) \\
&= (b^2 + c^2)(2a^2 - 1)
\end{aligned}
$$

from which the formula for θ follows.

For $\mathbf{w} = i$ (when $b = c = 0$), we have

$$
\begin{aligned}
\mathbf{w} \cdot R_r \mathbf{w} &= i \cdot ri\overline{r} \\
&= i \cdot [(a + dk)(i)(a - dk)] \\
&= i \cdot [(ai + dj)(a - dk)] \\
&= i \cdot [(a^2 - d^2)i + 2adj] \\
&= a^2 - d^2 \\
&= 2a^2 - 1.
\end{aligned}
$$

Verification, Investigation (10)

$$
\begin{aligned}
R_r(P_0) &= ri\overline{r} \\
&= (a + bi + cj + dk)(i)(a - bi - cj - dk) \\
&= (-b + ai + dj - ck)(a - bi - cj - dk) \\
&= (a^2 + b^2 - c^2 - d^2)i + 2(ad + bc)j + 2(bd - ac)k \\
&= h(a, b, c, d)
\end{aligned}
$$

Verification, Investigation (13)

Let $A = (1, 0, 0)$ and $B = P = (p_1, p_2, p_3)$ as in Investigation (12). As long as $P \neq (-1, 0, 0)$, the (non-unique) midpoint M of an arc connecting A to B can be determined by normalizing the average $(A + B)/2$. Thus M is collinear with $(1 + p_1, p_2, p_3)$. Since $\theta = \pi$ for R_1, the real part of r_1 is $\cos(\pi/2) = 0$. Thus r_1

is $(1 + p_1, p_2, p_3)$ divided by its norm. The norm is $[1 + 2p_1 + p_1^2 + p_2^2 + p_3^2]^{1/2} = \sqrt{2(1 + p_1)}$. This verifies the equation for r_1.

For r_2, first we find an axis vector $A \times B = (0, -p_3, p_2)$. For this to work, we must make the restriction $p_1 \neq 1$. The cosine of the angle θ for R_2 is $A \cdot B = p_1$, so the real part of r_2 is $\cos(\theta/2) = \sqrt{(1 + p_1)/2}$. So far we have

$$r_2 = \sqrt{(1 + p_1)/2} + \alpha(-p_3 j + p_2 k)$$

for some scalar α. To find α, we set the norm of r_2 equal to 1 and solve.

$$(1 + p_1)/2 + \alpha^2(p_2^2 + p_3^2) = 1$$
$$(1 + p_1)/2 + \alpha^2(1 - p_1^2) = 1$$
$$\alpha = 1/\sqrt{2(1 + p_1)}$$

Factoring out, we get the desired expression for r_2. Observe that when $p_1 = 1$, the formula for r_2 still works, so we can remove the restriction $p_1 \neq 1$.

Finally, we examine the special case $P = (-1, 0, 0)$. The rotations R_1 and R_2 don't make good sense in this case. To find the fiber $h^{-1}(P)$, we just need some point in the fiber to start with. It is easy to check that $R_k(P_0) = P$, so k is in $h^{-1}(P)$.

Verification, Investigation (16)

The fiber $h^{-1}(1, 0, 0)$ is the set $\{\cos t, \sin t, 0, 0\}$. The point e^{it} maps under stereographic projection to $(\frac{\sin t}{1 - \cos t}, 0, 0)$. This sweeps out the x-axis as t varies. The fiber $h^{-1}(-1, 0, 0)$ is the set $\{ke^{it}\}$. The point $ke^{it} = (0, 0, \sin t, \cos t)$ maps under stereographic projection to $(0, \sin t, \cos t)$.

Now we analyze the generic point $P = (p_1, p_2, p_3)$ for $p_1 \neq \pm 1$. The fiber is the set $\{r_1 e^{it}\}$. The point

$$r_1 e^{it} = \left(-\sqrt{\frac{1 + p_1}{2}} \sin t, \sqrt{\frac{1 + p_1}{2}} \cos t, \right.$$

$$\left. \frac{1}{\sqrt{2(1 + p_1)}}(p_2 \cos t + p_3 \sin t), \frac{1}{\sqrt{2(1 + p_1)}}(-p_2 \sin t + p_3 \cos t) \right)$$

maps under stereographic projection to

$$\left(\frac{\sqrt{\frac{1+p_1}{2}} \cos t}{1 + \sqrt{\frac{1+p_1}{2}} \sin t}, \frac{\frac{1}{\sqrt{2(1+p_1)}}(p_2 \cos t + p_3 \sin t)}{1 + \sqrt{\frac{1+p_1}{2}} \sin t}, \frac{\frac{1}{\sqrt{2(1+p_1)}}(-p_2 \sin t + p_3 \cos t)}{1 + \sqrt{\frac{1+p_1}{2}} \sin t} \right).$$

By inspection, the x-coordinate of the image point is zero when $t = \pm\pi/2$. So the points of intersection of $s \circ h^{-1}(P)$ with the y, z-plane are

$$\left(0, \frac{\frac{1}{\sqrt{2(1+p_1)}}p_3}{1 + \sqrt{\frac{1+p_1}{2}}}, \frac{\frac{1}{\sqrt{2(1+p_1)}}(-p_2)}{1 + \sqrt{\frac{1+p_1}{2}}}\right)$$

and

$$\left(0, \frac{\frac{1}{\sqrt{2(1+p_1)}}(-p_3)}{1 - \sqrt{\frac{1+p_1}{2}}}, \frac{\frac{1}{\sqrt{2(1+p_1)}}p_2}{1 - \sqrt{\frac{1+p_1}{2}}}\right).$$

The product of the norms of these points is

$$\begin{aligned}
&(p_2^2 + p_3^2)\frac{1}{2(1 + p_1)}\frac{1}{1 - \frac{1+p_1}{2}} \\
= \ &(1 - p_1^2)\frac{1}{2(1 + p_1)}\frac{2}{1 - p_1} \\
= \ &\frac{1 - p_1^2}{(1 + p_1)(1 - p_1)} = 1.
\end{aligned}$$

This implies that one of A, B is inside the unit circle and the other is outside.