MAS 372 Mathematical Statistics
Ch 8 #16a Sample Solution
January 2018

First, note that we must assume that $\sigma > 0$ in order for f to be a legitimate density function. Next, assuming that $E(X^2)$ exists, we have the following (the fact that we obtain a value at the end of the calculation establishes the convergence of the doubly infinite improper integral in (1) and shows that $E(X^2)$ does indeed exist).

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x|\sigma)\, dx && (1) \\ &= 2\int_{0}^{\infty} x^2 f(x|\sigma)\, dx && (2) \\ &= \frac{1}{\sigma}\int_{0}^{\infty} x^2 e^{-x/\sigma}\, dx && (3) \end{aligned}$$

We will complete the calculation in the following steps.

Step 1. Use integration by parts twice to evaluate the necessary antiderivative for (3). We will show the following.

$$\int x^2 e^{-x/\sigma}\, dx = -e^{-x/\sigma}\left(2\sigma^3 + \sigma x^2 + 2\sigma^2 x\right). \tag{4}$$

Step 2. Evaluate the necessary limit to compute (3). We will show the following.

$$E(X^2) = \frac{1}{\sigma}\lim_{b\to\infty}\left[-e^{-b/\sigma}\left(2\sigma^3 + \sigma b^2 + 2\sigma^2 b\right)\right]_0^b = 2\sigma^2. \tag{5}$$

For Step 1, we use integration by parts twice. In (6) below, use

$$\begin{aligned} u &= x^2 \\ du &= 2x\, dx \\ v &= -\sigma e^{-x/\sigma} \\ dv &= e^{-x/\sigma}\, dx \end{aligned}$$

and in (7), use

$$\begin{aligned} u &= x \\ du &= dx \\ v &= -\sigma e^{-x/\sigma} \\ dv &= e^{-x/\sigma}\, dx. \end{aligned}$$

Using these substitutions, we derive (4).

$$\begin{aligned} \int x^2 e^{-x/\sigma}\, dx &= -\sigma x^2 e^{-x/\sigma} + 2\sigma \int x e^{-x/\sigma}\, dx && (6) \\ &= -\sigma x^2 e^{-x/\sigma} + 2\sigma\left[-\sigma x e^{-x/\sigma} + \sigma\int e^{-x/\sigma}\, dx\right] && (7) \\ &= -\sigma x^2 e^{-x/\sigma} + 2\sigma\left[-\sigma x e^{-x/\sigma} - \sigma^2 e^{-x/\sigma}\right] && (8) \\ &= -\sigma e^{-x/\sigma}\left(x^2 + 2\sigma x + 2\sigma^2\right) && (9) \end{aligned}$$

For Step 2, use the fact that $\lim_{x\to\infty} p(x)e^{-ax} = 0$ for any $a > 0$ (by repeated application of L'Hopital's rule, say) to obtain the limit (5).

$$E(X^2) = \mu_2 = 2\sigma^2$$

This leads us to the method of moments estimate

$$\hat{\sigma} = \sqrt{\frac{\hat{\mu}_2}{2}} = \frac{1}{\sqrt{2}}\sqrt{\frac{\sum_i X_i^2}{n}}.$$