updated 29 October 2019

2.1 #14

(a) Let $p(x,y)$ be given by

$$p(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} .$$

We have $p(0,0) = 1$, but $\displaystyle\lim_{(x,y)\to(0,0)} p(x,y)$ does not exist because approaching $(0,0)$ along different paths produces different limits. Specifically, approching $(0,0)$ along the postive x-axis, versus along the negative x-axis, leads to the two conflicting limits below.

$$\lim_{x\to 0^+} p(x,y) = 1$$
$$\lim_{x\to 0^-} p(x,y) = 0$$

(b) Let $q(x,y)$ be given by

$$q(x,y) = \begin{cases} \frac{x^2}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases} .$$

Along any line $y = mx$ with $m \neq 0$, we have

$$\lim_{(x,y)\to(0,0)} q(x,y) = \lim_{x\to 0} q(x, mx) = \lim_{x\to 0} \frac{x^2}{mx} = 0,$$

along the line $y = 0$ we have

$$\lim_{(x,y)\to(0,0)} q(x,y) = \lim_{x\to 0} q(x,0) = \lim_{x\to 0} 0 = 0,$$

but along the parabola $y = x^2$, we have

$$\lim_{(x,y)\to(0,0)} q(x,y) = \lim_{x\to 0} q(x,x^2) = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

Because these limits do not all agree, we conclude that $\lim_{(x,y)\to(0,0)} q(x,y)$ does not exist, and we also see that the limits agree along all lines $y = mx$.

(c) Such a function cannon exist because the definition of continuity requires that the limit in question exists.

(d) The limit $\lim_{(x,y)\to(0,0)}$ cannot exist because the other two given limits do not agree.

(e) Let $t(x,y)$ be given by $t(x,y) = 0$ on the domain that consists of the entire plane $\mathbf{R}^2$ minus the single point $(1,1)$. It is clear that $\lim_{(x,y)\to(1,1)} t(x,y) = \lim_{(x,y)\to(1,1)} 0 = 0$, but $t(1,1)$ is not defined.

2.1 #15

(a) and (b). The functions in the numerators and denominators of both f and g are continuous, so by the properties of continuous functions given in the text, f and g are continuous wherever the denominator is not zero. That means g is continuous everywhere, and f is continuous except for points along the line $x = y$, where f is not even defined.

(c) Approaching along the line $y = 0$ we get

$$\lim_{(x,y)\to(0,0)} h(x, y) = \lim_{x\to 0} \frac{0}{x^2} = 0,$$

but approaching along the line $y = x$ we get

$$\lim_{(x,y)\to(0,0)} h(x, y) = \lim_{x\to 0} \frac{x^2}{2x^2} = 1/2.$$

This is impossible, so we conclude that $\lim_{(x,y)\to(0,0)} h(x, y)$ does not exist.

(d) This function is continuous everywhere. We only need to show that

$$\lim_{(x,y)\to(0,0)} k(x, y) = 0.$$

Use polar coordinates to write $x = r\cos t, y = r\sin t$. Then for $r \neq 0$, we have

$$k(r, t) = \frac{r^6 \cos t \sin t}{r^2} = r^4 \cos t \sin t.$$

Since $|r^4 \cos t \sin t| \leq |r^4|$ and $r^4 \to 0$ as $r \to 0$. Since we must have $r \to 0$ along any path for which $(x, y) \to (0, 0)$, we conclude that $\lim_{(x,y)\to(0,0)} k(x, y) = 0$, as desired.

2.2 #16

Given $f(x, y) = 8 - x^2 - 3y^2$.

(a) $f_x(x, y) = -2x,\ f_y(x, y) = -6y$

(b) $f(2, 1) = 1,\ f_x(2, 1) = -4$, we we have trace $y = 1$ tangent given by

$$x = t, y = 1, z = 1 - 4t$$

(c) $f(2, 1) = 1,\ f_y(2, 1) = -6$, we we have trace $x = 2$ tangent given by

$$x = 2, y = 1, z = 1 - 6t$$

(d) direction vectors are $(1, 0, -4)$, $(0, 1, -6)$

(e) a normal vector for the plane is the cross product of the two vectors in part (d), that is $(-f_x, -f_y, 1) = (4, 6, 1)$, so the tangent plane equation is

$$4(x - 2) + 6(y - 1) + (z - 1) = 0$$

2.3 #13ab

(a)

$$\begin{aligned} C(x, y) &= 25e^{-(x-1)^2-(y-1)^3} \\ C_x(x, y) &= -2(x - 1)C(x, y) \\ C_{xx}(x, y) &= -2C(x, y) + 4(x - 1)^2C(x, y) \\ &= 2(2(x - 1)^2 - 1)C(x, y) \\ C_y(x, y) &= -3(y - 1)^2C(x, y) \\ C_{yy}(x, y) &= -6(y - 1)C(x, y) + 9(y - 1)^4C(x, y) \\ &= 3(y - 1)(3(y - 1)^3 - 2)C(x, y) \end{aligned}$$

(b) $C_{xx}(1.1, 1.2) = 2(\frac{2}{100} - 1)25e^{-\frac{1}{100}-\frac{8}{1000}} = -1.96 \cdot 25e^{-\frac{18}{1000}} \approx -48.13$

The rate of change of the rate of change of the ant's temperature is about -48 degrees Celsius per inch per inch.

2.4 #13ac

(a)

$$\begin{aligned}
f(x,y) &= (\cos x)(2e^{2y} + e^{-2y}) \\
f_x(x,y) &= -(\sin x)(2e^{2y} + e^{-2y}) \\
f_y(x,y) &= (\cos x)(4e^{2y} - 2e^{-2y}) \\
f(0,0) &= 3 \\
f_x(0,0) &= 0 \\
f_y(0,0) &= 2 \\
L(x,y) &= 3 + 2y \\
f(0.1, 0.2) &\approx L(0.1, 0.2) = 3.4
\end{aligned}$$

This compares okay to $f(0.1, 0.2) = (\cos 0.1)(2e^{0.4} + e^{-.4}) \approx 3.308$.

(c)

$$\begin{aligned}
h(x,y,z) &= e^{2x}(y + z^2) \\
h_x(x,y,z) &= 2h(x,y,z) \\
h_y(x,y,z) &= e^{2x} \\
h_z(x,y,z) &= 2ze^{2x} \\
h(0,1,-2) &= 5 \\
h_x(0,1,-2) &= 10 \\
h_y(0,1,-2) &= 1 \\
h_z(0,1,-2) &= -4 \\
L(x,y,z) &= 5 + 10x + (y-1) - 4(z+2) \\
h(-0.1, 0.9, -1.8) &\approx L(-0.1, 0.9, -1.8) = 5 - 1 - 0.1 - 4(0.2) = 3.1
\end{aligned}$$

This compares okay to $h(-0.1, 0.9, -1.8) = e^{-0.2}(0.9 + 1.8^2) \approx 3.39$.

2.4 #15d

(i) Definition 2.4.14 in the textbook says that $z = f(x,y)$ is differentiable at (x_0, y_0) if there are constants a, b (the textbook uses m, n, but we're changing to a, b to avoid a clash with the use of m, n in Supp. Notes (5.1.1)) such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - ah - bk}{\sqrt{h^2+k^2}} = 0.$$

If we set $\mathbf{x} = (x,y)$, $\mathbf{x}_0 = (x_0, y_0)$, $\mathbf{h} = (h,k)$, and $L(h,k) = (a,b)\cdot(h,k)$ (here we ignore the use of L in the textbook), the above limit is a perfect match for Supp. Notes (5.1.1) for the case $n = 2$, $m = 1$.

(ii) For the case $f(x,y) = |x| + |y|$, $(x_0, y_0) = (0,0)$, the limit above becomes

$$\lim_{(h,k)\to(0,0)} \frac{|h| + |k| - ah - bk}{\sqrt{h^2+k^2}}.$$

Along the line $k = 0$, the limit becomes

$$\lim_{h\to 0} \frac{|h| - ah}{|h|}.$$

It is easy to see that this limit cannot be zero, because the one-sided limits

$$\lim_{h\to 0^+} \frac{|h| - ah}{|h|} = \lim_{h\to 0^+} \frac{h - ah}{h} = 1 - a$$

$$\lim_{h\to 0^-} \frac{|h| - ah}{|h|} = \lim_{h\to 0^+} \frac{-h - ah}{-h} = 1 + a$$

cannot both be zero. We conclude the limit that defines differentiability cannot be zero, so the function in question is not differentiable at $(0,0)$.

2.5 #14abc

(a) Using the standard parametrization taught in the textbook, we get

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = (1,4) + t(5-1, 6-4)$$

which is the same as the pair of equations $x = 1 + 4t$, $y = 4 + 2t$. There are infinitely many correct alternative ways to parametrize this line, but they will all have the form $x = f(t)$, $y = \frac{1}{2}(f(t) + 7)$ for some function f.

(b) If you use the standard parameterization from part (a), you get $dx/dt = 4$, $dy/dt = 2$. But if you use an alternative parameterization, you get $dx/dt = f'(t)$, $dy/dt = \frac{1}{2}f'(t)$.

(c) If you use the standard parameterization from part (a), you get

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = -8x - 16y.$$

At the point $(3,5)$, we have $\frac{dT}{dt} = -104$ degrees Fahrenheit per time unit. If you use an alternative parameterization, you get

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = -16f'(t).$$

If t_0 is the time value such that $f(t_0) = 3$, we have $\left.\frac{dT}{dt}\right|_{t=t_0} = -16f'(t_0)$. Thus we see that we could engineer a final answer to have any value we want by changing the parameterization of the line.

2.6 #12,13

12. For $f(x,y) = ye^{-xy}$, we have $\nabla f = (-y^2e^{-xy}, (1-xy)e^{-xy})$, so $\nabla f(0,2) = (-4,1)$. Thus we have

$$1 = D_{\mathbf{u}}f(0,2) = \nabla f(0,2) \cdot (u_1, u_2) = -4u_1 + u_2$$

Substituting $u_2 = 1+4u_1$ into $u_1^2+u_2^2 = 1$, we get $u_1 = 0, -\frac{8}{17}$. Substituting back to get u_2 values, we get the two solutions $\mathbf{u} = (0,1), (-\frac{8}{17}, -\frac{15}{17})$.

13. From $\partial f/\partial x = \sin(yz)$ we get

$$f(x,y,z) = x\sin(yz) + h(y,z)$$

for some function h. From $\partial f/\partial y = xz\cos(yz) + 2y$ we get

$$f(x,y,z) = x\sin(yz) + y^2 + j(x,z)$$

for some function j. From $\partial f/\partial x = xy\cos(yz) + \frac{5}{z}$ we get

$$f(x,y,z) = x\sin(yz) + 5\ln z + k(x,y)$$

for some function k. The function

$$f(x,y,z) = x\sin(yz) + y^2 + 5\ln z + C$$

where C is any constant, is consistent with all three conditions.

2.7 #19

For $f(x,y) = 3xe^y - x^3 - e^{3y}$, we have

$$\nabla f = (3e^y - 3x^2, 3xe^y - 3e^{3y}).$$

Setting both components equal to zero, we see that the only critical point is where the graphs $y = 2\ln x$ and $y = \frac{1}{2}\ln x$ intersect, namely, at $(1,0)$.

To see that this critical point is a local maximum, take second derivatives and do the Second Derivative Test. We have

$$\begin{aligned} f_{xx} &= -6x \\ f_{yy} &= 3xe^y - 9e^{3y} \\ f_{xy} &= 3e^y \end{aligned}$$

At the critical point $(1,0)$, we have

$$\begin{aligned} f_{xx}(1,0) &= -6 < 0 \\ f_{yy}(1,0) &= -6 \\ f_{xy}(1,0) &= 3 \\ D(1,0) &= (-6)(-6) - 3^2 = 27 > 0 \end{aligned}$$

so we conclude that the critical point is a local maximum.

To see that f has no global maximum, observe that along $y = 0$, the value $f(x,0) = 3x - x^3 - 1$ has no maximum as $x \to -\infty$.