Writing Assignment 1, due 8/31. State and prove Supplementary Notes Proposition (1.1.4).

Writing Assignment 2, due 9/7. Using only the definition of probability measure (1.2.1), prove all four parts of Proposition (1.2.4). Here is a more properly complete statement of the Proposition than the sketch in the notes.

Proposition (1.2.4) Let A, B be events in a probability space Ω with probability measure P. The following hold.

1. $P(\emptyset) = 0$
2. $A \subseteq B \Rightarrow P(A) \leq P(B)$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. $P(A^c) = 1 - P(A)$

Writing Assignment 3, due 9/14. Let $\Omega = \{1, 2, 3, \ldots\}$. Let $p\colon \Omega \to \mathbf{R}$ be given by $p(n) = 1/2^{n+1}$. Let $\mathcal{A} \subset \mathcal{P}(\Omega)$ be the collection of all finite and cofinite subsets of Ω (recall that a cofinite set is a set whose complement is finite). Let $P\colon \mathcal{A} \to \mathbf{R}$ be given by

$$P(A) = \begin{cases} \displaystyle\sum_{n \in A} p(n) & \text{if } A \text{ is finite} \\ 1 - P(A^c) & \text{if } A \text{ is cofinite} \end{cases}$$

Show the following.

1. $\mathcal{A}$ is an algebra of subsets of Ω.
2. $\mathcal{A}$ is *not* a σ-algebra of subsets of Ω.
3. $0 \leq P(A) \leq 1$ for all A in $\mathcal{A}$ (be careful! you have to prove something for the "less than or equal to 1" part)
4. P satisfies $P(A \cup B) = P(A) + P(B)$ for every disjoint pair of elements A, B in $\mathcal{A}$.
5. There exists a countable collection $A_1, A_2, A_2, \ldots$ of elements of $\mathcal{A}$ such that
$$P\left(\bigcup_{n \in \mathbf{N}} A_n\right) \neq \sum_{n \in \mathbf{N}} P(A_n).$$

Note added 9/12/2018: Facts about sets in Writing Assignment 3. For writing assignment 3, you may use facts about set algebra without proof. As an alternative to stating such facts within your proofs of parts 1 through 5, you may find it convenient state your set facts as separate lemmas. For example, let's say that, in two different places in your proofs, you want to use the facts that $U \subset V$ implies $V^c \subset U^c$, and that $U \cap V = \emptyset$ implies $U \subset V^c$. You could write something like this.

Lemma. Let U, V be subsets of a universal set S. The following hold.

1. If $U \subset V$ then $V^c \subset U^c$.
2. If $U \cap V = \emptyset$ then $U \subset V^c$.

Then when you use those facts in your proofs in the writing exercise, say something like "by part 1 of the Lemma stated above" or "by part 2 of the Lemma above".

Writing Assignment 4, due 10/5. Let a, d, r be positive real numbers with $r \neq 1$.

1. Derive a closed-form expression for the truncated arithmetic series

$$a + (a + d) + (a + 2d) + \cdots + (a + nd).$$

   Explain your methods.

2. Derive a closed-form expression for the truncated geometric series

$$a + ar + ar^2 + \cdots + ar^n.$$

   Explain your methods.

3. Use your result from part 2 to explain why

$$a + ar + ar^2 + \cdots = a\left(\frac{1}{1 - r}\right)$$

   for $|r| < 1$.

4. Let $s_0, s_1, s_2, \ldots$ be given by the following recursive rule.

$$\begin{aligned} s_0 &= a \\ s_k &= rs_{k-1} + d \qquad \text{for } k \geq 1 \end{aligned}$$

   Use mathematical induction to prove that

$$s_n = ar^n + d\left(\frac{r^n - 1}{r - 1}\right)$$

   for $n \geq 0$.

Writing Assignment 5, due 10/19. Suppose that $\mathcal{A}$ is a σ-algebra of subsets of the set of real numbers and that $\mathcal{A}$ contains all sets of the form $\{x \colon x \leq C\}$ for all real C. Let a, b be real numbers with $a < b$, and let S be the interval $S = \{x \colon a < x < b\}$. Show that S is an element of $\mathcal{A}$.

Writing Assignment 6, due 11/9. Let Ω be the set of permutations of the set $\{1, 2, \ldots, n\}$ with a uniform probability function. Let $Y \colon \Omega \to \mathbf{N}$ be the random variable on Ω defined by $Y(\alpha) = |\{k \colon \alpha(k) = k\}|$, that is, $Y(\alpha)$ is the number of fixed points of the permutation α. Find

1. $P(Y = 0)$, and
2. $P(Y = k)$ for $k = 0, 1, 2, \ldots, n$

Write your work as an exposition with rationale. For example, it makes sense to begin with the problem of counting derangements. You may cite the inclusion-exclusion formula and use it without having to prove it.

Here is one version of inclusion-exclusion.

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then we have the following.

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| \\
&+ \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| \\
&+ \cdots \\
&+ (-1)^{r+1} \sum_{1\le i_1<i_2<\cdots<i_r\le n} \left|\bigcap_{t=1}^{r} A_{i_t}\right| \\
&+ \cdots \\
&+ (-1)^{n+1} \left|\bigcap_{i=1}^{n} A_i\right|
\end{aligned}$$

Writing Assignment 7, due 11/16.

1. Find a function $S\colon (0,1] \to \mathbf{R}$ that meets these conditions.
   - S is differentiable
   - $S(1) = 0$
   - $S(p) \to \infty$ as $p \to 0^+$
   - $S(pq) = S(p) + S(q)$
2. Show that if S_1, S_2 both satisfy the conditions above, then $S_1 = kS_2$ for some positive constant k.
3. Show that $\lim_{x\to 0^+} x\ \ln x = 0$, and use this to explain why

   $$H(p_1, p_2, \ldots, p_m) = \sum_{i=1}^{n} p_i S(p_i)$$

   is defined for any finite discrete probability distribution $p_1, p_2, \ldots, p_n$.
4. Find the probability distribution p_1, p_2 that maximizes H for $n = 2$ (by part (2), it does not matter what constant multiple of S you choose).
5. Find the probability distribution $p_1, p_2, \ldots, p_n$ that maximizes H for arbitrary n. Hint: use Lagrange multipliers.

Writing Assignment 8, due 12/5. This writing assignment is **OPTIONAL**. You may resubmit any previous writing assignment. If you are resubmitting Writing #3, submit *part 3 only*. The score on your resubmission will replace the original score.