updated 19 March 2019

2.3 additional exercises

1. Repeat the instructions for exercise 1 in the text for the system below.

$$x' = .1x - 3y$$
$$y' = 3x + .1y$$

For part (a), show that $(x, y) = (e^{.1t} \cos 3t, e^{.1t} \sin 3t)$ is a solution to the initial value problem $x(0) = 1, y(0) = 1$. Then do the same parts (b), (c), (d) as in exercise 1 in the text.

2.4 additional exercises

1. Solve the following system for $x = x(t), y = y(t)$, where a, b, c are constants and initial values are $x(0) = x_0$, $y(0) = y_0$. Notice that you will need to consider two cases: $a \neq c$ and $a = c$.

$$x' = ax$$
$$y' = bx + cy$$

2. Sketch the graph of $y = e^{at}(bt + c)$ for 8 possible cases: $a = \pm 1$, $b = \pm 1$, $c = 0, 1$.

3.1 and 3.2 additional exercises

1. Let A be an $n \times n$ matrix and suppose $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \neq 0$ and some scalar λ, that is, λ is an eigenvalue for A with a corresponding eigenvector $\vec{v}$. Let $\vec{w} = \frac{\vec{v}}{|\vec{v}|}$. Show that $|\vec{w}| = 1$ and that $A\vec{w} = \lambda\vec{w}$. This exercise justifies the common practice of assuming that eigenvectors have norm 1.

2. Let A be a 2×2 matrix with distinct real eigenvalues λ_1, λ_2 and corresponding eigenvectors $\vec{v}_1, \vec{v}_2$.

   (a) Show that $\vec{v}_1, \vec{v}_2$ are independent.

   (b) Let $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and let B be the 2×2 matrix whose first column is $\vec{v}_1$ and whose second column is $\vec{v}_2$. Show that

   $$A = B\Lambda B^{-1}.$$

   (c) Suppose that, in addition to being independent, the eigenvectors $\vec{v}_1, \vec{v}_2$ are orthogonal and have unit length. Show that

   $$A = \lambda_1 \vec{v}_1 \vec{v}_1^T + \lambda_2 \vec{v}_2 \vec{v}_2^T.$$

3.4 additional exercises

Consider the linear differential equation

$$\vec{x}' = A\vec{x} \tag{0.0.1}$$

where A be a 2×2 matrix with real entries, and suppose that A has a complex eigenvalue λ that is not real. Let $\vec{v}$ be a corresponding eigenvector.

1. Show that the eigenvalues λ_1, λ_2 of A have the form $\alpha + i\beta$ for some real numbers α, β, with $\beta \neq 0$.
2. Show that $\vec{x} = e^{\lambda t}\vec{v}$ is a solution to (0.0.1).
3. Show that $\vec{x} = e^{\overline{\lambda} t}\overline{\vec{v}}$ is also a solution (the horizontal bar symbol indicated complex conjugation).
4. Show that the real and imaginary parts of $e^{\lambda r}\vec{v}$ are both solutions, and argue why $\vec{v}, \overline{\vec{v}}$ are independent.
5. Conclude that the general solution to (0.0.1) is a superposition of the real and imaginary parts of $e^{\lambda t}\vec{v}$.

3.5 additional exercises

Consider the linear differential equation (0.0.1) above, but this time suppose that A has a repeated real eigenvalue λ with eigenvector an $\vec{v}$, and that there do *not* exist two independent eigenvectors.

1. Show that it is possible to find a vector $\vec{w}$ so that $A\vec{w} = \lambda\vec{w} + \vec{v}$, such that $\vec{v}, \vec{w}$ are independent.
2. Show that $e^{\lambda t}(\vec{w} + t\vec{v})$ is a solution to (0.0.1).
3. Conclude that the general solution to (0.0.1) is given by
$$c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{w} + t\vec{v}).$$
4. Find the general solution for (0.0.1) with $A = \frac{1}{2}\begin{pmatrix} 5 & 1 \\ -1 & 7 \end{pmatrix}$.