Introduction to Modern Algebra and Geometry

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Abstract blurb goes here.

Preface

This is a short sentence. And some more. A few words about the purpose, aim, and scope of these notes.

$$x^2 + e^3 = 7$$

Contents

Chapter 0

Basics

This chapter is about some basics.

0.1 Complex Numbers

The set of complex numbers, denoted $\mathbf{C}$, is in one-to-one correspondence with the 2-dimensional real plane $\mathbf{R}^2$. We will write $z \leftrightarrow (x, y)$ to denote that the complex number z corresponds to the ordered pair (x, y) of real numbers.

0.1.1 Real and imaginary parts

Given a complex number z corresponding to the point (x, y) in $\mathbf{R}^2$, we say that x is the **real part** of z and that y is the **imaginary part** of z, denoted $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. The complex numbers contain the real numbers $\mathbf{R}$ as a subset. The real number x, which is also the complex number x, corresponds to the ordered pair $(x, 0)$, The complex number i corresponds to the ordered pair $(0, 1)$. Here is a summary so far.

$$\begin{aligned} z &\leftrightarrow (\mathrm{Re}(z), \mathrm{Im}(z)) \\ x \in \mathbf{R} &\leftrightarrow (x, 0) \\ i &\leftrightarrow (0, 1) \end{aligned}$$

0.1.2 Modulus and argument

Given a complex number $z \leftrightarrow (x, y)$, let (r, θ) be polar coordinates for the point (x, y) such that $r \geq 0$ and θ is measured in radians. The **modulus** or **norm** of z, denoted $|z|$, is defined to be the polar coordinate $r = \sqrt{x^2 + y^2}$ and the **argument** of z, denoted $\arg z$, is the polar coordinate θ, that is, the oriented angle made by the real vector (x, y) with the positive real axis. In other words, $(|z|, \arg z)$ are polar coordinates for the point (x, y). Here is a summary.

Norm and argument.

$$z \leftrightarrow (x, y) = (|z| \cos(\arg z), |z| \sin(\arg z)) \tag{0.1.1}$$

0.1.3 Addition and multiplication of complex numbers

Given complex numbers $z \leftrightarrow (x, y)$ and $z' \leftrightarrow (x', y')$, the sum $z + z'$ is defined by the following.

Complex addition.

$$z + z' \leftrightarrow (x + x', y + y') \tag{0.1.2}$$

In other words, complex addition corresponds to real vector addition. See Figure Figure 0.1.1.

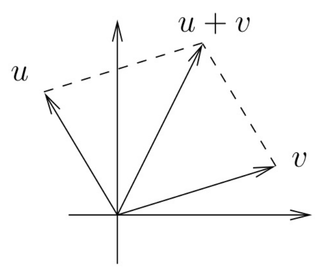

Figure 0.1.1: Complex addition is vector addition

The product zz' is defined as follows.

Complex multiplication.

$$|zz'| = |z||z'| \tag{0.1.3}$$
$$\arg(zz') = \arg z + \arg z' \tag{0.1.4}$$

The interaction between complex addition and multiplication is expressed by the following relationship, which holds for any complex z, u, v. See Figure Figure 0.1.2.

Distributive law.

$$z(u + v) = zu + zv \tag{0.1.5}$$

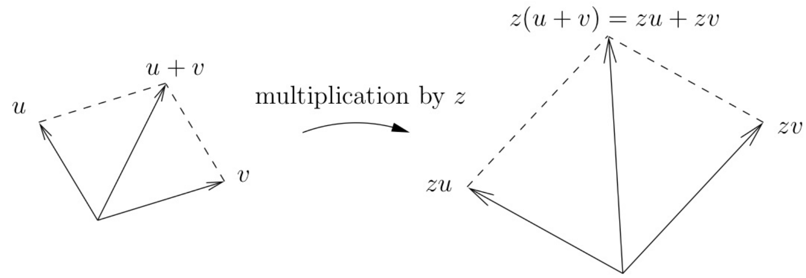

Figure 0.1.2: The distributive law: multiplication by z rotates the vectors u, v, and $u+v$ by angle $\arg z$ and scales by the factor $|z|$. The sum of the rotated vectors u, v is the rotated sum vector.

Below are a number of relationships arising from the definitions of complex addition and multiplication. Let a, b, c, d be real numbers and let z, u, v be complex numbers. The following relationships hold. [1]

$$a + ib \leftrightarrow (a, b) \tag{0.1.6}$$
$$a + b \ \text{ (complex sum)} = a + b \ \text{ (real sum)} \tag{0.1.7}$$
$$ab \ \text{ (complex product)} = ab \ \text{ (real product)} \tag{0.1.8}$$
$$|a| \ \text{ (complex norm)} = |a| \ \text{ (real absolute value)} \tag{0.1.9}$$
$$1z = z1 = z \tag{0.1.10}$$
$$i^2 = -1 \tag{0.1.11}$$
$$(a + ib) + (c + id) = (a + c) + i(b + d) \tag{0.1.12}$$
$$(a + ib)(c + id) = (ac - bd) + i(ad + bc) \tag{0.1.13}$$

0.1.4 The complex exponential function

The Taylor series for the real function $y = e^x$ is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots .$$

Convergence for sequences and series of complex numbers can be defined in a way that naturally extends the definitions for real numbers. It turns out that the complex power series

$$1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

converges for every *complex* number z, so we define the *complex* exponential function by

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots .$$

The complex exponential obeys familiar laws of the real exponential. For z, w in $\mathbf{C}$, we have

$$e^z e^w = e^{z+w}$$

[1] Equations (0.1.7), (0.1.8), and (0.1.9) say that complex addition, multiplication, and norm *extend* the corresponding operations on the reals. Equation (0.1.10) is expressed by saying 1 is the *multiplicative identity*.

$$e^0 = 1$$

A key property of the complex exponential is the following, called **Euler's formula**.

Euler's formula.

$$e^{it} = \cos t + i \sin t \ \text{ (for real)} \tag{0.1.14}$$

For z with $r = |z|$ and $t = \arg(z)$, the expression $z = re^{it}$ is called the **polar form** for z. By contrast, we call $z = x + iy$ the **rectangular form** (or the **Cartesian form**) for z. Figure Figure 0.1.3 shows a summary of the geometric content of the rectangular and polar forms for a complex number z.

Here is how complex multiplication looks in polar form. For $z = re^{i\theta}, w = se^{i\phi}$, we have

$$zw = (re^{i\theta})(se^{i\phi}) = rse^{i(\theta+\phi)}. \tag{0.1.15}$$

From this it is easy to see that for $r \neq 0$, we have

$$\left(re^{i\theta}\right)\left(\frac{1}{r}e^{-i\theta}\right) = 1.$$

For $z = re^{i\theta}$ with $r \neq 0$, we call $\frac{1}{r}e^{-i\theta}$ the **multiplicative inverse** of z, denoted $1/z$ or z^{-1}.

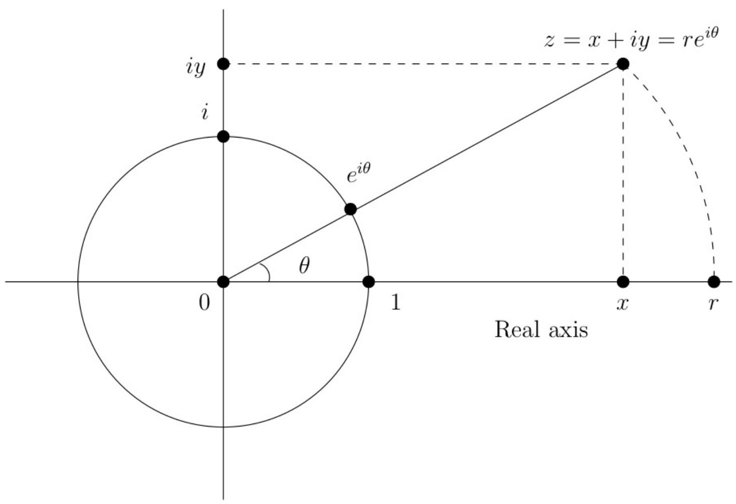

Figure 0.1.3: Rectangular and polar forms for a complex number z

0.1.5 Conjugation

The **conjugate** of the complex number $z = x + iy = re^{i\theta}$, denoted $\overline{z}$ or z^*, is defined to be $z^* = x - iy = re^{-i\theta}$. Geometrically, z^* is the reflection of z across the real axis (the x-axis) in $\mathbf{R}^2$. Here are some relations involving conjugates.

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \tag{0.1.16}$$

$$\mathrm{Im}(z) = \frac{z - z^*}{2i} \tag{0.1.17}$$

$$|z|^2 = zz^* \tag{0.1.18}$$

$$2\arg z = \frac{z}{z^*} \quad \text{(for)} \tag{0.1.19}$$

$$\frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2} \quad \text{(for)} \tag{0.1.20}$$

$$(zw)^* = z^* w^* \tag{0.1.21}$$

0.1.6 Exercises

1. What is the difference between polar coordinates and polar form? What is the difference between rectangular coordinates and rectangular form? Write formulas for converting from polar to rectangular coordinates and vice-versa.

Solution. Let z be a complex number, let $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $r = |z|$ and $\theta = \arg(z)$. The pair (r, θ) is called the polar coordinates for z, while the expression $re^{i\theta}$ is called the polar form for z. The pair (x, y) is called the rectangular coordinates for z, while the expression $x + iy$ is called the rectangular form for z.

To convert from polar to rectangular, use the equations $x = r\cos\theta, y = r\sin\theta$ (show sketches to explain these formulas). To convert from rectangular to polar, use $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. For the last equation, you must use judgment when $x = 0$ to decide whether θ should be $\pi/2$ or $-\pi/2$. You must also use judgment when calculating $\theta = \arctan(y/x)$. The standard codomain for arctan is the interval $(\pi/2, \pi/2)$, so you need to use $\theta = \arctan(y/x) + \pi$ for $x < 0$.

2. Express each of the following in rectangular and polar form.

(a) $3(2 - i) + 6(1 + i)$ **Answer**. $12 + 3i = \sqrt{153}\, e^{i \arctan(1/4)}$

(b) $\left(2e^{i\pi/6}\right)\left(3e^{-i\pi/3}\right)$ **Answer**. $6e^{-i\pi/6} = 3\sqrt{3} - 3i$

(c) $(2 + 3i)(4 - i)$ **Answer**. $11 + 10i = \sqrt{221}\, e^{i \arctan(10/11)}$

(d) $(1 + i)^3$ **Answer**. $-2 + 2i = 2\sqrt{2}\, e^{i3\pi/4}$

3. Prove the following property of norm.

The triangle inequality.

For any two complex numbers z, w, we have

$$|z + w| \leq |z| + |w|.$$

Solution. The simplest approach is geometric: Sketch the parallelogram for vector addition and use the fact that the length of any side of a triangle is less than the sum of the lengths of the other two sides.

4. Prove Equation (0.1.18).

Solution. Let $z = re^{i\theta}$. Then $\overline{z} = re^{-i\theta}$, and we have

$$\begin{aligned} z\overline{z} &= re^{i\theta} re^{-i\theta} \\ &= r^2 e^0 \\ &= r^2 = |z|^2. \end{aligned}$$

5. Let p and q be complex numbers. Prove that the distance (ordinary distance between points in the plane) between p and q is $|p - q|$.

Hint. Hint: Use rectangular form.

Solution. Let $p = a + ib$ and $q = c + id$. We have

$$\begin{aligned} |p - q| &= |(a + ib) - (c + id)| \\ &= |(a - c) + i(b - d)| \\ &= \sqrt{(a - c)^2 + (b - d)^2}. \end{aligned}$$

The latter expression is the distance from p to q, so we are done.

6. Express each of the following in rectangular and polar form.

(a) $\dfrac{2 + i}{3 - i}$ **Answer**. $\dfrac{1}{2} + \dfrac{1}{2}i = \dfrac{\sqrt{2}}{2}e^{i\pi/4}$

(b) $\dfrac{1 + 2i}{1 - 2i}$ **Answer**. $-\dfrac{3}{5} + \dfrac{4}{5}i = e^{i(\arctan(-4/3)+\pi)}$

(c) $\dfrac{2e^{i\pi/4}}{3e^{-i\pi/2}}$ **Answer**. $\dfrac{2}{3}e^{i3\pi/4} = -\dfrac{\sqrt{2}}{3} + \dfrac{\sqrt{2}}{3}i$

7. Verify the formulas (0.1.16) and (0.1.17).

Solution. Let $z = x + iy$. Then we have

- $\dfrac{z + \overline{z}}{2} = \dfrac{2x}{2} = x = \mathrm{Re}(z)$, and
- $\dfrac{z - \overline{z}}{2i} = \dfrac{2iy}{2i} = y = \mathrm{Im}(z)$.

8. Given a nonzero complex number z, explain why z has exactly two square roots, and explain how to find them.

Solution. Since squaring a number squares the norm and doubles the argument, a square root can be found by taking the square root of the norm and dividing the argument by two. That is, for $z = re^{i\theta}$, a square root of z is $\sqrt{r}e^{i\theta/2}$. Another square root of z is the negative of that expression. Any other square root of z would have to have norm $\sqrt{|z|}$ and argument $\theta/2$ plus or minus an integer multiple of π, so these must be all the square roots of z.

9. Find all complex solutions of the following equations.

(a) $z^2 + 3z + 5 = 0$ **Answer**. $-\dfrac{3}{2} \pm i\dfrac{\sqrt{11}}{2}$

(b) $(z - i)(z + i) = 1$ **Answer**. 0

(c) $\dfrac{2z + i}{-z + 3i} = z$ **Answer**. $(1/2)[(-2 \pm 281^{1/4}\cos\varphi) + i(3 \pm 281^{1/4}\sin\varphi)]$, where $\varphi = (\arctan(16/5) + \pi)/2$

10. Use the fact that $e^{ia}e^{ib} = e^{i(a+b)}$ together with Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to derive the trigonometric angle sum formulas below.

$$\begin{aligned} \cos(a + b) &= \cos a \cos b - \sin a \sin b \\ \sin(a + b) &= \cos a \sin b + \sin a \cos b \end{aligned}$$

Solution. Using Euler's formula for the first equality below, and then using complex multiplication for the second equality, we have

$$e^{ia}e^{ib} = (\cos a + i \sin a)(\cos b + i \sin b)$$

$$= (\cos a \cos b - \sin a \sin b) + i(\cos a \sin b + \sin a \cos b). \qquad (\star)$$

On the other hand, Euler's formula also gives

$$e^{i(a+b)} = \cos(a+b) + i \sin(a+b). \qquad (\star\star)$$

Equating real and and imaginary parts of $(\star)$ and $(\star\star)$ gives the desired trigonometric identities.

0.2 Equivalence Relations

section about equivalence relations here

0.2.1 Definitions

an equivalence relation is

0.3 Modular Arithmetic

section about modular arithmetic here

0.3.1 Definitions

the integers are awesome

0.4 Linear Algebra Basics

section about basics of linear algebra

0.4.1 Definitions

a vector is a thing

Chapter 1

Algebra

This is algebra.

1.1 Examples of groups

hiya

1.1.1

Examples:

- permutation groups
- symmetries of planar figures, R_θ, F_L notation
- 2×2 matrix groups
- integers, integers mod n

1.2 Definitions and properties for groups

hiya

1.2.1

define group and tie formal definition to examples from previous section, use more vocabulary such as *subgroup*

1.3 Group homomorphisms

hiya

1.3.1

define vocabulary including homomorphism, isomorphism, kernel, image
develop properties of normal subgroup

1.4 Fundamental theorems of groups

hiya

1.4.1

group mod kernel is isomorphic to the image

Lagrange's theorem as a consequence for a sepcial case

1.5 Group actions

hiya

1.5.1

defintions: action, orbit, stabilizer

orbit-stabilizer theorem(s), applications

Chapter 2

Geometry

This is geometry. We will spend most of our time on planar geometries.

2.1 Euclidean geometry

hiya

2.1.1

Euclidean geometry in the plane via algebra of the complex numbers

2.2 Möbius geometry

hiya

2.2.1

Möbius geometry as a $GL(2, \mathbf{C})$ action on $\mathbf{C}$

2.3 Möbius geometry as projective geometry

hiya

2.3.1

Möbius geometry as projective geometry of the projective space of $\mathbf{C}^2$, introduce stereographic projection as a tool

2.4 Hyperbolic geometry

hiya

2.4.1

Hyperbolic geometry as a subgeometry of Möbius geometry

2.5 Elliptic geometry

hiya

2.5.1

Elliptic geometry as a subgeometry of Möbius geometry

2.6 Projective geometry

hiya

2.6.1

Projective (real and complex) geometry

2.7 The quaternions

hiya

2.7.1

algebra of rotations in 3-dimensional Euclidean space

Chapter 3

More Topics

This is more topics.

3.1 Further topics

3.1.1

more algebra topics: group classification problems, Sylow theorems, fundamental theorem of finitely generated Abelian groups, more structures like rings, fields, vector spaces and modules, some properties

more geometry topics: finite geometries, using stereographic projection to count Pythagorean triples