Writing Assignment 1. Let $\phi\colon \mathbf{R}^2 \to \mathbf{C}$ be given by $\phi(x, y) = x + iy$ and let $f\colon \mathbf{C} \to \mathbf{C}$ be given by $f(z) = ze^{i\theta_0}$, where θ_0 is a fixed real constant. Find a nice way to write $\hat{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$, where $\hat{f} = \phi^{-1} \circ f \circ \phi$. The goal of this writing is not just to write down a correct expression for $\hat{f}$, but also to explain how you derive it and why it is natural.

Sample solution. Let f, ϕ be as given above, and let $(x, y) \in \mathbf{R}^2$. The heart of the calculation is the evaluation of f in Cartesian form. Using Euler's formula to write $e^{i\theta_0} = \cos\theta_0 + i \sin\theta_0$, we have

$$\begin{aligned} f(x + iy) &= (x + iy)(\cos\theta_0 + i \sin\theta_0) \\ &= (x \cos\theta_0 - y \sin\theta_0) + i(x \sin\theta_0 + y \cos\theta_0). \end{aligned} \tag{1}$$

Now we evaluate $\hat{f}$.

$$\begin{aligned} \hat{f}(x, y) &= \phi^{-1}(f(\phi(x, y))) \\ &= \phi^{-1}(f(x + iy)) \quad \text{(evaluating } \phi) \\ &= \phi^{-1}((x \cos\theta_0 - y \sin\theta_0) + i(x \sin\theta_0 + y \cos\theta_0)) \quad \text{(substituting (1))} \\ &= (x \cos\theta_0 - y \sin\theta_0, x \sin\theta_0 + y \cos\theta_0) \quad \text{(evaluating } \phi^{-1}) \end{aligned}$$

Thus we have the desired formula for $\hat{f}$.

$$\hat{f}(x, y) = (x \cos\theta_0 - y \sin\theta_0, x \sin\theta_0 + y \cos\theta_0)$$

We know that the effect of f is rotation of the complex plane $\mathbf{C}$ about 0 by θ_0 radians. The function ϕ matches our usual picture of $\mathbf{R}^2$ with the usual picture of $\mathbf{C}$, so $\hat{f}$ is rotation of the real plane $\mathbf{R}^2$ about the origin by θ_0 radians.