Writing Assignment 1, due 9/6. Let $\phi\colon \mathbf{R}^2 \to \mathbf{C}$ be given by $\phi(x,y) = x + iy$ and let $f\colon \mathbf{C} \to \mathbf{C}$ be given by $f(z) = ze^{i\theta_0}$, where θ_0 is a fixed real constant. Find a nice way to write $\hat{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$, where $\hat{f} = \phi^{-1} \circ f \circ \phi$. The goal of this writing is not just to write down a correct expression for $\hat{f}$, but also to explain how you derive it and why it is natural.

Writing Assignment 2, due 9/13. Chapter 3, Exercise 12.

Writing Assignment 3, due 9/20. Chapter 4, Exercises 8 and 14.

Writing Assignment 4, due 9/27. Prove the following proposition.

> Let z_1, z_2, z_3 be distinct points in the extended complex plane $\mathbf{C}^+$. There exists a unique cline that contains z_1, z_2, z_3.

Watch out what you assume! You may use all facts in the text up to and including the second Theorem on p.59. Do *not* use the theorem on p.60. In the proof of that Theorem, the author already assumes, without justification, that the above proposition holds. Hint: Look carefully at the proof of the second theorem on p.59. Interpret carefully what the author actually proves.

Writing Assignment 5, due 10/18. Describe an explicit method that generates a list

$$(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), \ldots$$

of *all* possible Pythagorean triples, that is, 3-tuples (a, b, c) that satisfy $a^2 + b^2 = c^2$. Your method should involve stereographic projection $s\colon S^1 \to \mathbf{R}$ given by $s(x,y) = \frac{x}{1-y}$ where $S^1 = \{(x,y)\colon x^2 + y^2 = 1\}$ is the unit circle in $\mathbf{R}^2$. Some hints: there's a connection between Pythagorean triples and points in the first quadrant on S^1 with rational coordinates, and stereographic projection sends points on S^1 with rational coordinates to rational numbers on the real line. More hints are that the inverse map for stereographic projection will be useful, and that somewhere in your past you have learned how to enumerate the positive rational numbers. Once you have your method for generating Pythagorean triples, what is the index for $(3, 4, 5)$? (That is, what is n such that $a_n = 3$, $b_n = 4$, and $c_n = 5$?) How about for $(5, 12, 13)$?

Writing Assignment 6, due 10/25. Chapter 7, Exercises 1 and 3, and Chapter 8, Exercise 4.

Writing Assignment 7, due 11/8. Chapter 9 Ex. 14, and Ch 10 Ex.5.

Writing Assignment 8, due 11/15. Chapter 11 Ex. 16 and 17.

Writing Assignment 9, due 12/4. This writing assignment is **OPTIONAL**. You amy resubmit any previous writing assignment except #1. The score on your resubmission will replace the original score.

Alternative for Writing Assignments 8 or 9 (or both). Work out the details and do a nice write-up for either part 1 or part 2 of the Theorem below. You may follow the suggested outlines, or come up with your own methods. Improvements are welcome!

Theorem. The set of rotations of the sphere are in one-to-one correspondence with Möbius transformations in the group $\mathbf{S}$ for elliptic geometry. Specifically, every rotation of the sphere is the lift, via sterographic projection, of an element of $\mathbf{S}$, and conversely, every element of $\mathbf{S}$ lifts to a rotation of the sphere.

We use the notation $R_{\vec{v},\theta}$ to denote a rotation of S^2 by θ radians about the axis determined by the normalized vector $\vec{v} \in \mathbf{R}^3$. By abuse of notation, we'll write X, Y, Z for the vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively. We will write $s\colon S^2 \to \mathbf{C}^+$ for stereographic projection.

Outline of the proof.

1. **Propositions: every rotation of S^2 lifts a transformation in S.**

   (a) $R_{X,\pi/2}$ lifts $z \to \frac{z+i}{iz+1}$. To prove this, check $s \circ R = T \circ s$.

   (b) $R_{Y,\theta} = R^{-1}_{X,\pi/2}\, R_{Z,\theta}\, R_{X,\pi/2}$, so rotations by arbitrary angles are lifts of transformations in $\mathbf{S}$

   (c) Let $R = R_{\vec{v},\theta}$ be an arbitrary rotation. We can decompose R^{-1} by "putting the north pole back into place, then putting the prime meridian back into place". Let $N_1 = R(N)$, where N is the north pole $N = (0,0,1)$. Suppose N_1 has polar angle θ and azimuthal angle ϕ. The rotation $R_{Z,-\phi}$ takes N_1 to a point N_2 in the X, Z-plane. Now $R_{Y,-\theta}$ takes N_2 to N. One more Z rotation by some angle $-\theta'$ brings the "prime meridian" (the great circle in the X, Z-plane) back into alignment. Now we have decomposed R^{-1}. Inverting gives us R. The last line shows how the basic building blocks are arbitrary Z rotations and quarter turn X rotations.

   $$\begin{aligned} R &= R_{Z,\phi}\, R_{Y,\theta}\, R_{Z,\theta'} \\ &= R_{Z,\phi}\, R^{-1}_{X,\pi/2}\, R_{Z,\theta}\, R_{X,\pi/2}\, R_{Z,\theta'} \end{aligned}$$

   Conclude that R is a lift of an element of $\mathbf{S}$.

2. **Propositions: every transformation in S lifts to a rotation of S^2.** (This is Henle's Ch 11, Exercise 2.)

   (a) Let $Uz = \frac{z-p}{\overline{p}z+1}$ with $|p| \le 1$. The rotation about the axis perpendicular to the great circle that contains N and $s^{-1}(p)$ that takes $s^{-1}(p)$ to the south pole (this works out to be $R_{N\times s^{-1}(p),2\arctan(|p|)}$) is the lift of some element W of $\mathbf{S}$ by part 2. W and U agree on the three points $p, -1/\overline{p}, 0$, so $W = U$, i.e., U lifts to a rotation.

   (b) Let $T \in \mathbf{S}$ be given. T has some fixed point. It is easy to show that the opposite end of its diameter is another fixed point (use $z_1\overline{z}_2 = -1$ implies $Tz_1\overline{(Tz_2)} = -1$). Let p be whichever of these is inside (or on the boundary of) the unit disk. Let U be defined as in the previous step. Use the usual normal form analysis to show that UTU^{-1} must be multiplication by a constant λ, and that this λ has norm 1. Let $Vz = \lambda z$ so that $T = U^{-1}VU$. Conclude that T lifts to a rotation.