Finite Mathematics Supplementary Notes

Fall 2018

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1.1 Puzzle

1. Item X costs \$50. Every year, the price goes up \$7. Draw a graph of price as a function of time. When will the price reach \$100? When will the price reach \$200?

2. Item X costs \$50. Every year, the price goes up 10%. Draw a graph of price as a function of time. When will the price reach \$100? When will the price reach \$200?

1.2 Linear and Exponential Functions

Definition. A quantity $y = y(t)$ that varies depending on t according to an equation of the form

$$y = a + dt$$

is called a ***linear function*** of t.

Definition. A quantity $y = y(t)$ that varies depending on t according to an equation of the form

$$y = a \cdot b^t$$

where $a \neq 0$, b is positive and $b \neq 1$, is called an ***exponential function*** of t. We also write $\exp_b(t)$ for b^t.

Definition. The function $u = g(v)$ is called the ***inverse*** of the function f if $g(f(u)) = u$ and $f(g(v)) = v$ for every input value u for f and every input value v for g. This is the same as saying that the point (u, v) is on the graph of f if and only if the point (v, u) is on the graph of g.

Definition. The inverse of the function exp_b is written $\log_b$ and is called the **logarithm base** b function. This means that $b^u = v$ if and only if $u = \log_b v$.

Vocabulary: standard bases for exp **and** log**.**

The most-used bases (the values of b for exponential and logarithm functions are $b = 10$, $b = e \approx 2.718$, and $b = 2$. The function $y = \log_{10}(x)$ is called the *common logarithm function* and is often written simply $y = \log x$. The function $y = \log_e(x)$ is called the *natural logarithm function* and is written $y = \ln x$. The natural logarithm is the most widely used logarithm in science and in this course.

Key fact: any base can be rewritten using base e**.** Using the fact that $\exp_e$ and $\log_e = \ln$ are inverses, we have

$$b = e^{\ln b} \tag{1.2.1}$$

for any nonnegative b. We can use (1.2.1) to rewrite b^t with base e.

$$\begin{aligned}
b^t &= \left(e^{\ln b}\right)^t \quad \text{(using (1.2.1))} && (1.2.2)\\
&= e^{((\ln b)t)} \quad \text{(using } (a^b)^c = a^{bc}\text{)} && (1.2.3)\\
&= e^{t \ln b} && (1.2.4)
\end{aligned}$$

Application of key fact to solve for t. Here is how to solve $y = ab^t$ for t.

$$y = ab^t \tag{1.2.5}$$
$$\frac{y}{a} = b^t \tag{1.2.6}$$
$$\frac{y}{a} = e^{t \ln b} \tag{1.2.7}$$
$$\ln\left(\frac{y}{a}\right) = t \ln b \tag{1.2.8}$$
$$\frac{\ln\left(\frac{y}{a}\right)}{\ln b} = t \tag{1.2.9}$$

Exercises.

1. Values for three of the four quantities y, a, d and t in the equation

$$y = a + dt$$

are given in each row of the table below. For each row, fill in the value for the fourth quantity.

y	a	d	t
	1	2	3
1		2	3
1	2		3
1	2	3	

2. Values for three of the four quantities y, a, b and t in the equation

$$y = ab^t$$

are given in each row of the table below. For each row, fill in the value for the fourth quantity.

y	a	b	t
	1	2	3
1		2	3
1	2		3
1	2	3	

1.3 Vocabulary of Growth

Definition of Growth Factor. For an exponential function $y = ab^t$ and a specific time value $t = u$, the value of y at time $t = u + 1$ is

$$y(u + 1) = ab^{u+1} = ab^u b^1 = y(u) \cdot b.$$

In words, this says that the quantity y grows by a factor b during any time interval of length one time unit. For this reason, b is called the ***growth factor (per unit time)*** of the quantity y. For example, if y grows by a factor of $b = 1.1$ in one year, we say that 1.1 is the "annual growth factor".

Definitions about Growth Rates. For an exponential function $y = ab^t$ with growth rate b, let $r = b - 1$ and let $k = \ln b$. With these definitions, we have $b = 1 + r = e^k$. Using the fact that $(e^k)^t = e^{kt}$, we have three useful ways to write the exponential function.

$$\begin{aligned} y &= ab^t \\ y &= a(1 + r)^t \\ y &= ae^{kt} \end{aligned}$$

The number $r = b - 1$ is called the ***growth rate (per unit time)*** and the number $k = \ln b$ is called the ***continuous growth rate (per unit time)*** of the function. For example, an annual growth factor of $b = 1.1$ has an annual growth rate of $0.1 = 10\%$, and has a continuous annual growth rate of approximately $0.0953 = 9.53\%$.

Some useful words to symbols translations.

- p percent of a quantity x is $\frac{p}{100}x$
- "quantity x grows by p percent" means $x \to x + \frac{p}{100}x$
- when quantity x grows by p percent, the *growth rate* is $r = \frac{p}{100}$
- "quantity x grows by growth rate r" means $x \to x + rx = x(1 + r)$
- "quantity x grows by a factor b" means $x \to xb$
- "growth by p percent" is the same as "growth by the factor $1 + \frac{p}{100}$"

Exercises.

1. The value for one of the three quantities b, r, and k for the function

$$y = b^t = (1 + r)^t = e^{kt}$$

is given in each row of the table below. For each row, fill in the values for the other two quantities.

b	r	k
1.07		
	0.0354	
		0.0967

1.4 Geometric Sequences

Definitions. A ***sequence*** is a list of numbers

$$a_0, a_1, a_2, a_3, \ldots$$

where the subscripts $0, 1, 2, 3, \ldots$ keep track of location in the sequence. For an exponential function $y = ab^t$, the sequence

$$y(0), y(1), y(2), \ldots = a, ab, ab^2, \ldots$$

is called an ***exponential*** or ***geometric sequence***.

Observation/Motivation. Many applications of mathematics, including important financial formulas, involve the sum of terms of a geometric sequence. Given a positive number n, notice that the first n terms of a geometric sequence have this form.

$$y(0), y(1), y(2), \ldots, y(n-1) = a, ab, ab^2, \ldots, ab^{n-1}$$

Fact: Sum of Terms in a Geometric Series. For a sum

$$s = a + ab + ab^2 + \cdots + ab^{n-1}$$

of the first n terms of a geometric series, we have

$$bs = ab + ab^2 + \cdots + ab^{n-1} + ab^n.$$

Subtracting, many terms cancel and we have

$$bs - s = ab^n - a.$$

Solving for s and simplifying, we have

$$s = a\left(\frac{b^n - 1}{b - 1}\right).$$

Exercises.

1. Find the value of the sum
$$1 + 1.01 + 1.01^2 + 1.01^3 + \cdots + 1.01^{49}.$$

2. Fill in the missing terms of the following geometric sequences.
   (a) $5, 2, 0.8, \_\_, \_\_, \_\_, \ldots$
   (b) $\_\_, 2, \_\_, 4, \_\_, 8, \ldots$

3. Find the given sums of terms of geometric sequences.
   (a) $2 + 6 + 18 + 54 + \cdots + 2(3^{100})$
   (b) $2 + 6 + 18 + 54 + \cdots + 9565938$

1.5 More Exercises

1. Consider the following true life situation.

   > "I want to rent this room for the month of July," he said. The clerk wheezed. He peered through the narrow slits of his bloodshot eyes, glaring through the murk of the humid dusty darkness of the fleabag lobby, and said, "for you—a deal." "How much?" said the big guy, sweat trickling down his face, staining the collar of his dingy shirt which didn't appear to have been washed in weeks. Noticing the telltale bulge of a revolver under the stranger's dirt stained jacket, the clerk replied, "First day—one cent. Second day—two cents. Third day—four. Every day it doubles." The stranger's face drew into a knot as he scrutinized the greasy poker faced clerk. He said, "That's nothin'. What's the hitch?"

   (a) How much would the stranger pay on July 31st?

   (b) What would the bill be for the month of July?

2. You get a letter in the mail that says, "Send a dollar to each of the five people on this list. Add your name to the bottom, take the top name off, and send a copy of the new list plus these instructions to five new people. P.S. If you break the chain you will have to sit in a math lecture every day for the rest of your life."

   (a) Assuming nobody broke the chain, and every letter was passed on in one day, how much money would you have after 10 days? 20 days? One hundred days?

   (b) Assuming no person ever received the letter twice, and each letter was passed on in one day (and nobody broke the chain) how long would it take for everyone on the planet to get a letter?

3. Gumby and Pokey decide to go on a diet together. Gumby and Pokey both weigh 10 ounces. Starting their diets on the same day, Gumby loses 1/10 of an ounce each day, while Pokey loses half his body weight each day.

   (a) Who wins the race to the body weight of 1 ounce?

   (b) Explain how you know, without calculating, that Gumby will win the race to zero body weight.

   (c) How much of Pokey is left when Gumby vanishes?

1.6 Answer Key for Exercises and Puzzles

1.1 Answer Key for Puzzles

1. The graph is a straight line that passes through $(0, 50)$ and has slope 7 dollars per year. The horizontal axis is labeled by the variable t in years, and the vertical axis is the variable y in dollars. The formula for the function is $y = 50 + 7t$, where t is the number of years since the start, and y is the price in dollars. Solving for t when $y = 100$, we get $t = 7\frac{1}{7}$ years. Since the price only goes up once each year, the answer to the question "when will the price reach \$100?" is 8 years. Solving for t when $y = 200$, we get $t = 21\frac{3}{7}$ years, so it takes 22 years for the price to reach \$200.

2. The graph is an upward-bending exponential curve that passes through $(0, 50)$, and $(1, 55)$. The horizontal axis is labeled by the variable t in years, and the vertical axis is the variable y in dollars. The formula for the function is $y = 50(1.1)^t$, where t is the number of years since the start, and y is the price in dollars. Solving for t when $y = 100$, we get $t \approx 7.27$, so it takes 8 years for the price to reach \$100. Solving for t when $y = 200$, we get $t \approx 14.55$, so it takes 15 years for the price to reach \$200.

1.2 Answer Key for Exercises

1. $y = 7$, $a = -5$, $d = -1/3$, $t = -1/3$

2. $y = 8$, $a = 1/8$, $b = (1/2)^{1/3} \approx 0.794$, $t = \frac{\ln(1/2)}{\ln 3} \approx -0.631$

1.3 Answer Key for Exercises

1. row 1: $r = .07 = 7.0\%$, $k = \ln 1.07 \approx 0.0677 = 6.77\%$, row 2: $b = 1.0354$, $k = \ln 1.0354 \approx .0348 = 3.48\%$, row 3: $b = e^{.0967} \approx 1.102$, $r = b - 1 \approx .102 = 10.2\%$

1.4 Answer Key for Exercises

1. $(1.01^{50} - 1)/(.01) \approx 64.46$

2. (a) $5(2/5)^3, 5(2/5)^4, 5(2/5)^5$, $a = 5$, $b = 2/5$
   (b) $2^{1/2}, 2^{3/2}, 2^{5/2}$, $a = 2^{1/2}$, $b = 2^{1/2}$

3. (a) $3^{101} - 1 \approx 1.55 \times 10^{48}$
   (b) $3^{15} - 1 = 14,348,906$

1.5 Answer Key for Exercises

1. (a) 2^{30} cents

   (b) $2^{31} - 1$ cents

2. (a) $-5 + 5 + 5^2 + 5^3 + 5^4 + 5^5 = \$3900$

   (b) Solve $\left(\dfrac{5^{n+1} - 1}{5 - 1}\right) > 7 \times 10^9$ (or pick your favorite world population) to get $n = 14$ days (there is ambiguity about what we are counting: days from when you receive the chain letter, or days that start after you send your first mail? so you might get a different answer by a day)

3. (a) Body weight functions are $G(t) = 10 - t/10$ and $P(t) = 10(1/2)^t$. Solve $G(t) = 1$ and $P(t) = 1$ to see that Pokey wins (about 3.3 days versus 90 days).

   (b) There is no t for which $P(t) = 0$.

   (c) It takes 100 days for $G(t)$ to hit zero. At that time, Pokey weights $10(1/2)^{100} \approx 7.9 \times 10^{-30}$ ounces.