Writing Assignment 3, part 4, sample solution. Let A, B be disjoint elements in $\mathcal{A}$. We consider two cases: (a) A, B both finite; and (b) one of A, B finite, the other cofinite. It is not necessary to consider that case when A, B are both cofinite because there does not exist any pair of disjoint cofinite sets.

Case (a). Suppose A, B are both finite. Then $A \cup B$ is also finite, so we have

$$\begin{aligned}
P(A \cup B) &= \sum_{n \in A \cup B} p(n) \quad \text{(definition of } P\text{)} \\
&= \sum_{n \in A} p(n) + \sum_{n \in B} p(n) \quad \text{(rearranging terms)} \\
&= P(A) + P(B) \quad \text{(definition of } P\text{)}.
\end{aligned}$$

Case (b). Suppose one of A, B is finite, and the other is cofinite. Without loss of generality, we may assume A is finite and B is cofinite. It is convenient to give a name to the finite complement of B, so let $C = B^c$. In what follows, we will make use of a decomposition of C into a disjoint union of finite sets. Using the fact that the two-set collection A, A^c forms a partition of Ω, we may write

$$C = (A \cap C) \cup (A^c \cap C). \tag{1}$$

Because A, B are disjoint, we know $A \subset B^c = C$, so $A \cap C = A$. Using De Morgan's Law, we have $A^c \cap C = (A \cup B)^c$. Using these observations, equation (1) becomes

$$C = A \cup (A \cup B)^c. \tag{2}$$

By part 1 of this writing exercise, we know that $A \cup B$ is cofinite, so $(A \cup B)^c$ is finite. Thus we may apply Case (a) to (2).

$$P(C) = P(A) + P((A \cup B)^c) \tag{3}$$

Now we can establish the equation to be shown, starting with the right-hand side.

$$\begin{aligned}
P(A) + P(B) &= P(A) + 1 - P(B^c) \quad \text{(definition of } P\text{)} \\
&= P(A) + 1 - P(C) \\
&= P(A) + 1 - (P(A) + P((A \cup B)^c)) \quad \text{(applying (3))} \\
&= 1 - P((A \cup B)^c) \quad \text{(simplifying)} \\
&= P(A \cup B) \quad \text{(definition of } P\text{)}
\end{aligned}$$

This concludes the proof of Case 2, and the proof of part 4 is complete.