Voting Methods

MAS 100 Concepts of Mathematics
Sample Paper
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September 2013

One of the most fundamental social tasks is group decision making: given a set of possible choices, a group of people must choose one of them. The decision making process should be *fair* in the sense that all participants should be able to agree, before the process has been carried out, they they will accept the outcome. The main point of this essay is that there is no simple solution to this problem. To illustrate, we will present several methods for group decision making, called *voting methods*, and will analyze their strengths and weaknesses.

But isn't there already a simple solution? Isn't majority rule a well accepted method that is guaranteed to make the maximum number of people happy? The shortcoming of majority rule is that it only works when there are exactly two alternatives to choose from. If there are three alternatives, say A, B, and C, then there may be no majority winner. For example, there could be 40% in favor of A and 30% each in favor of B and C. In voting terminology, the term *majority* means *more than half*. The highest percentage of votes, like A's 40% in this example, is called a *plurality*.

Okay, so how about plurality? Just declare the winner to be the candidate with the highest number of votes. Isn't this a fair system that makes the greatest possible number of people happy? Surprisingly, no! Consider this example: 40% of voters rank three didates A, B, C in order ABC, that is, A first, B second, and C third. 30% rank their preferences BCA, and the remaining 30% rank their preferences CBA. In a plurality vote, A wins with 40%. But 60% of the voters prefer B to A! And 70% of voters prefer B to C. Shouldn't B be the winner? This illustrates a serious flaw in the plurality method: a candidate who is preferred over every other candidate, in each case by a majority, can fail to be a plurality winner. This is not just a theoretical shortcoming. It happened in the 2000 US presidential election, where Al Gore was prefered by a majority over George Bush and Ralph Nader, but Bush won the plurality vote [1].

The above example suggests a method that might be preferable to plurality. This method, called the *Condorcet method*[1], will elect as winner a candidate (B in the example) who is prefered by a majority when compared to any other candidate, one at a time. In a Condorcet election, each voter submits a list that ranks all the candidates in order, from most to least prefered. After preference list ballots are submitted, votes are counted for a one-versus-one majority election for each pair of candidates. In the contest between candidates X and Y, a preference list that ranks X above Y counts as a vote for X. If there is one candidate who beats all other candidates in one-versus-one majority votes, then that candidate is declared the Condorcet winner. For example, Al Gore would have been the Condorcet winner of the 2000 US presidential election. This method has such a strong sense of fairness, one could ask why we don't use it. The problem is that the Condorcet method sometimes does not pick a winner. In a vote with 3 preference lists ABC, BCA, CAB, there is no Condorcet winner. In one-on-one contests, A beats B, B beats C, and C beats A. Because of this shortcoming, Condorcet's method is not practical.

It seems reasonable that some tweak or variation on Condorcet's method

[1] Condorcet's method is named after the 18th-century French mathematician and philosopher Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet, 1743–1794.

might guarantee the election of a winner, and retain the desirable property that any candidate who beats all others in one-versus-one contests will be the winner. A natural idea is to add ranking scores to the preference lists. The *Borda count*[2] gives a score of 0 to the last candidate on a preference list, a score of 1 to the next-to-last, and so on up the list. For example, for the preference list $ABCD$, A gets 3 points, B gets 2, C gets 1, and D gets 0. To run a Borda election, each voter submits a ranked preference list of candidates. After preference list ballots are collected, each candidate receives a total score that is the sum of that candidate's scores on each of the preference lists. The highest score is the winner. Here's an example. 4 voters submit the list ABC, 3 voters submit the list BCA, and 3 voters CBA. Candidate A gets a Borda score of $2 \cdot 4 = 8$ (two points from each of the four voters in the first group, and no points from any other voters), candidate B gets $1 \cdot 4 + 2 \cdot 3 + 1 \cdot 3 = 13$ points, and C gets $1 \cdot 3 + 2 \cdot 3 = 7$ points. Thus B is the Borda winner. The Borda count does more than just pick a winner with the highest number of points. It produces a final ranking of all the candidates by total points. In this example, the ranking is BAC. In fact, the Borda method is used to rank sports teams by taking votes, or ranking lists, from coaches and sports writers [3].

It might appear that the Borda method is a solution to the search for a fair voting method. But alas, there are shortcomings here, too. First, the Borda count does not break the tie in the example we gave above where the Condorcet method fails to pick a winner. With preference lists ABC, BCA, CAB, all three candidates get the same Borda score 3, so we have a three-way tie. Even more spectacularly, it is possible for there to be a *majority* winner who does not win the Borda count. For example, a 10-voter election with with 6 preference lists $ABCD$ and 4 preference lists $BCDA$, A wins the majorty vote with 60%, but A's Borda score is $3 \cdot 6 = 18$ points, while B's score is $2 \cdot 6 + 3 \cdot 4 = 24$ points. C has 14 and D has 4 points. Thus the Borda winner is not the same as the majority winner!

So what method *does* elect candidates who win in one-on-one contests against all other candidates (like the Condorcet method), but does not suffer the breakdown of Condorcet's method by failing to pick a winner in some cases, and does not suffer from paradoxes of plurality and the Borda count? The sad fact is that there is no such method. A famous result of Kenneth Arrow in 1950 proves that there exists no voting method that meets a list of basic requirements of fairness. There are many voting methods, all with their own good properties, but also with their own flaws. Arrow's theorem says the flaws are unavoidable.

What does this mean for the problem of fair selection among alternatives for a society? Because every voting method must have some drawback, we have to be educated about the strengths and weaknesses of each method so that we can choose the most appropriate method for a particular context. Among other things, Arrow's theorem guarantees that the need for mathematics in social science will never go away.

[2]The Borda count is named after Jean-Charles de Borda, who devised the method in 1770.

References

[1] COMAP, *For All Practical Purposes*

> This is a widely used textbook for liberal arts mathematics courses. It introduces a variety of applications of mathematics at an elementary level. In particular, it has two chapters on voting methods in a larger section of the book on "Social Choice".

[2] Wikipedia pages on Condorcet's Method, the Borda Count, and Arrow's Theorem
http://en.wikipedia.org/wiki/Condorcet_method
http://en.wikipedia.org/wiki/Borda_count
http://en.wikipedia.org/wiki/Arrow's_impossibility_theorem

> These wikipedia pages give a bit more history, detail, and examples than the textbook source above.

[3] Electorama web site
http://wiki.electorama.com/wiki/Borda_count

> The Electorama site is devoted to voting methods of all kinds.