Intermediate Statistics Probability Problems

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Exercises

1. In a family of four children, what is more likely: two boys and two girls, or three of one gender and one of the other?

2. A box contains five letters A, B, C, D, and E, all equally likely to be drawn at random. Two draws are made at random with replacement. (Drawing "with replacement" means that after the first draw, the item drawn in replaced and the box is reshuffled, so that all the letters have the same chance of being drawn on the second draw as they did on the first.)

   (a) Find the probability that the two letters drawn are different.

   (b) Find the probability that at least one of the letters drawn is a vowel.

   (c) Work the same probability problems as in parts (a) and (b) where the two draws are taken *without* replacement. ("Without replacement" means that the first item is *not* placed back in the box after the first draw.)

3. Four cards are dealt from a well-shuffled, standard 52 card deck.

   (a) Find the probability that none of the cards are diamonds.

   (b) Find the probability that all four cards are diamonds.

   (c) Find the probability that all four cards are the same suit.

4. A box contains 1 red and 5 green marbles; each marble is equally likely to be selected on a random draw from the box. You draw four marbles from the box at random.

   (a) Find the probability that a red appears exactly three times if the draws are made with replacement.

   (b) Find the probability that a red appears exactly three times if the draws are made without replacement.

5. Let S be the sample space of all possible outcomes for the experiment which is making 5 random draws, with replacement, from a jar containing 2 red marbles, 3 green marbles and 4 blue marbles. For each individual draw, all marbles have the same chance of being selected. Outcomes in S are strings of 5 letters using the letters R, G and B, where the order counts. For example, the outcome RGGBR indicates draws of red, green, green, blue, and red marbles, in that order.

   (a) How many outcomes are in S?

   (b) Let E be the event "draw exactly 2 reds in 5 draws." How many outcomes are in E?

   (c) Is $P(E)$ equal to the number you found in (b) divided by the number you found in (a)? Why or why not? And if not, find $P(E)$. Give $P(E)$ as a decimal approximation correct to 3 significant digits.

(d) Let F be the event "draw exactly 2 blues in 5 draws." Are E and F independent? Show your calculations to support your answer. Give $P(F)$ and $P(E \cap F)$ as decimal approximations correct to 3 significant digits.

6. A fair coin is flipped 100 times.

   (a) Suppose you know that at least 99 of the flips are tails. What are the chances that all 100 are tails?

   (b) Suppose you know that the first 99 tosses are tails. What are the chances that all 100 are tails?

7. You play a lottery where you win if you correctly predict the six numbers that will be randomly chosen (without replacement) from the numbers 1–36. What are your chances of winning?

8. Four cards are dealt face down from a well-shuffled, standard 52 card deck.

   (a) Find the probability that the first two cards are red and the second two cards black.

   (b) Find the probability that exactly two of the cards are red.

9. 100 marbles are in a box; 54 are red, 46 are blue. Each marble has the same chance of being selected on a random draw from the box. Let E be the event "get exactly 3 reds in 5 random draws with replacement," and let F be the event "get a red on the first and last of 5 random draws with replacement."

   (a) Find $P(E)$.

   (b) Find $P(F)$.

   (c) Are E and F dependent or independent?

10. A fair die is rolled 5 times. Give the probabilities for each of the following events as a decimal approximation correct to 3 significant digits.

    (a) All rolls have an even number of spots.

    (b) It is not the case that all rolls have an even number of spots.

    (c) All rolls have an odd number of spots.

    (d) Exactly 3 of the rolls show an even number of spots.

11. Given $\Omega = \{a, b, c, d\}$, with $p(a) = 1/2$ and $p(b) = 1/4$. Find $p(c)$ and $p(d)$ if events $E = \{a, c\}$ and $F = \{c, d\}$ are independent.

12. Two cards are dealt from a well-shuffled standard 52-card deck.

    (a) What is the probability that either both cards are black or both cards are hearts?

(b) What is the probability that either both cards are black or both cards are aces?

13. A card is drawn at random from a well-shuffled standard 52 card deck, replaced, then the deck is reshuffled. Imagine this procedure is repeated N times. What is the smallest value of N for which there is a 70% or better chance that a king was drawn at least once?

14. A sample space S consists of 3 outcomes a, b, and c. Events U and V in S are given by $U = \{a, b\}$ and $V = \{b, c\}$.

   (a) Suppose the 3 outcomes are equally likely. Are U and V dependent or independent? Explain.

   (b) If $p(a) = 1/2$, find values for $p(b)$ and $p(c)$ so that U and V are independent, or explain why this cannot be done.

   (c) If possible, give a complete list of events in S.

15. Independent events E and F in a sample space Ω have probabilities $P(E) = 2/3$ and $P(F) = 1/3$. Find the probability of the event $E \cup F$, or say why there is not enough information.

16. A fair coin is flipped ten times.

   (a) Which is more likely: (i) getting at least one head in the first five flips or (ii) getting at least two heads in all ten flips? Show your work.

   (b) Find the probability of getting exactly two heads on the first five flips and exactly four heads on the last five flips. Give a decimal approximation correct to three significant digits.

17. Five cards are dealt face down from a well shuffled standard 52 card deck.

   (a) Find the probability that all the cards are red.

   (b) Find the probability that the cards are, in order, the ace, king, queen, jack and ten of diamonds.

   (c) Find the probability that the five cards are the ace, king, queen, jack and ten of diamonds, in any order.

18. A fair coin is tossed 10 times. Find the probability that exactly 3 of the first five tosses are heads, while at least 1 of the last five tosses is a head. Calculate and give the probability as a decimal approximation correct to 2 decimal places.

Solutions to Exercises

Note: Most of the "solutions" posted here are not solutions at all, but are merely final answer keys, although some are complete. These are provided so that you can check your work; reading the answer keys is not a substitute for working the problems yourself.

Probability Solutions

1. $P(\text{even gender split}) = \binom{4}{2}/16 = 6/16 = 3/8$
   $P(\text{3-1 gender split}) = 2\binom{4}{3}/16 = 8/16 = 1/2$, the greater of the two

2. (a)
   $$
   \begin{aligned}
   & P(2 \text{ draws different}) \\
   = \ & P(\text{1st draw is a letter AND 2nd draw is something else}) \\
   = \ & P(\text{1st draw is a letter})P(\text{2nd is something else|1st is something}) \\
   = \ & 1 \cdot 4/5 = 4/5
   \end{aligned}
   $$

   (b)
   $$
   \begin{aligned}
   & P(\text{at least one vowel}) \\
   = \ & 1 - P(\text{no vowels}) \\
   = \ & 1 - P(2 \text{ consonants}) \\
   = \ & 1 - P(\text{1st draw consonant AND 2nd draw consonant}) \\
   = \ & 1 - 3/5 \cdot 3/5 = 16/25
   \end{aligned}
   $$

   (c) $P(2 \text{ draws different}) = 1,$
   $$
   \begin{aligned}
   & P(\text{at least one vowel}) \\
   = \ & 1 - P(\text{no vowels}) \\
   = \ & 1 - P(2 \text{ consonants}) \\
   = \ & 1 - P(\text{1st draw consonant AND 2nd draw consonant}) \\
   = \ & 1 - P(\text{1st draw consonant})P(\text{2nd draw consonant|1st draw consonant}) \\
   = \ & 1 - 3/5 \cdot 2/4 = 7/10
   \end{aligned}
   $$

3. (a) $P(\text{no diamonds}) = 39/52 \cdot 38/51 \cdot 37/50 \cdot 36/49$
   (b) $P(\text{all diamonds}) = 13/52 \cdot 12/51 \cdot 11/50 \cdot 10/49$
   (c) $P(\text{all same suit}) = 4 \cdot (\text{above})$

4. (a) (with replacement) $P(\text{exactly 3 R in 4 draws}) = \binom{4}{3} \cdot (1/6)^3 \cdot 5/6$
   (b) (without replacement) $P(\text{exactly 3 R in 4 draws}) = 0$

5. (a) $3^5 = 243$

   (b) $\binom{5}{2} \cdot 2^3 = 80$

   (c) $P(E) = \binom{5}{2} \cdot (2/9)^2 \cdot (7/9)^3 \approx .232$, this does not equal $80/243 \approx .329$

   (d) $P(F) = \binom{5}{2} \cdot (4/9)^2 \cdot (5/9)^3 \approx .339$,
   $|E \cap F| = \binom{5}{2} \cdot \binom{3}{2} = 30$,
   $P(E \cap F) = 30 \cdot (2/9)^2 \cdot (4/9)^2 \cdot 3/9 \approx .098$,
   $P(E \cap F)$ does not equal $P(E)P(F)$, so E, F are dependent

6. (a) $P(100 \text{ T}|\text{at least 99 T}) = 1/101$

   (b) $P(100 \text{ T}|\text{first 99 T}) = 1/2$

7. $P(\text{win lottery}) = 1/\binom{36}{6}$

8. (a) $P(\text{RRBB}) = 26/52 \cdot 25/51 \cdot 26/50 \cdot 25/49$

   (b) $P(\text{exactly 2R}) = \binom{4}{2} \cdot (\text{above})$

9. (a) $P(E) = \binom{5}{3}(.54)^3(.46)^2$

   (b) $P(F) = (.54)^2$

   (c) Event $E \cap F$ consists of the 3 outcomes RRBBR, RBRBR, and RBBRR. Each of these outcomes has probability $(.54)^3(.46)^2$, so $P(E \cap F) = 3(.54)^3(.46)^2$, which does not equal $P(E)P(F)$, so we conclude that E and F are dependent.

10. (a) $P(\text{all rolls even}) = (1/2)^5 = 1/32 \approx .0313$

    (b) $P(\text{not all rolls even}) = 1 - 1/32 \approx .969$

    (c) $P(\text{all rolls odd}) = 1/32 \approx .0313$

    (d) $P(\text{exactly 3 even}) = \binom{5}{3}(1/2)^3(1/2)^2 \approx .313$

11. $P(c) = 1/6$, $P(d) = 1/12$

12. (a)

$$
\begin{aligned}
&P(\text{both black OR both hearts}) \\
= \quad &P(\text{both black}) + P(\text{both hearts}) \\
= \quad &26/52 \cdot 25/51 + 13/52 \cdot 12/51
\end{aligned}
$$

   (b)

$$
\begin{aligned}
&P(\text{both black OR both aces}) \\
= \quad &P(\text{both black}) + P(\text{both aces}) - P(\text{both black aces}) \\
= \quad &26/52 \cdot 25/51 + 4/52 \cdot 3/51 - 2/52 \cdot 1/51
\end{aligned}
$$

13.

$$P(\text{draw at least one K in N draws}) = 1 - P(\text{draw no K in N draws})$$
$$= 1 - (12/13)^N$$

So we want to solve $1 - (12/13)^N \geq .7$, or equivalently $.3 \leq (12/13)^N$. You can find N by trial and error, or solve as follows.

$$\log(.3) \leq N\log(12/13) \quad (\text{take log both sides})$$
$$\log(.3)/\log(12/13) \geq N \quad (\text{divide both sides by } \log(12/13))$$

Get N slightly less than 16, so the smallest whole number solution is $N = 16$.

14. (a) Dependent. $P(U \cap V) = P(b) = 1/3$. This does not equal $P(U)P(V) = 2/3 \cdot 2/3 = 4/9$.

(b) $P(b) = 1/2$ and $P(c) = 0$ makes U and V independent.

(c) $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$

15.

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$= 2/3 + 1/3 - 2/3 \cdot 1/3 = 7/9$$

16. (a) $P(i) = 1 - P(\text{no heads}) = 1 - (1/2)^5 = 31/32$
$P(ii) = 1 - P(\text{zero H or one H}) = 1 - (1/2)^10 - \binom{10}{1}(1/2)^10 = 1013/1024$
$P(ii)$ is greater

(b) $P(\text{exactly 2 H in first 5 flips AND exactly 4 H in last 5 flips})$
$= \binom{5}{2}(1/2)^5 \binom{5}{4}(1/2)^5 = 50/1024 \approx .0488$

17. (a) $P(\text{all red}) = 26/52 \cdot 25/51 \cdot 24/50 \cdot 23/49 \cdot 22/48$

(b) $P(\text{AKQJten diamonds in order}) = 1/(52 \cdot 51 \cdot 50 \cdot 49 \cdot 48)$

(c) $P(\text{AKQJten diamonds in any order}) = 5! \cdot (\text{above})$

18. $\binom{5}{3}(1/2)^5(1 - (1/2)^5) = 310/(32^2) \approx .303$