updated 14 October 2019

2.1 #14

(a) Let $p(x, y)$ be given by

$$p(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$

We have $p(0, 0) = 1$, but $\lim\limits_{(x,y)\to(0,0)} p(x, y)$ does not exist because approaching $(0, 0)$ along different paths produces different limits. Specifically, approching $(0, 0)$ along the postive x-axis, versus along the negative x-axis, leads to the two conflicting limits below.

$$\lim_{x\to 0^+} p(x,y) = 1$$
$$\lim_{x\to 0^-} p(x,y) = 0$$

(b) Let $q(x, y)$ be given by

$$q(x,y) = \begin{cases} \frac{x^2}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}.$$

Along any line $y = mx$ with $m \neq 0$, we have

$$\lim_{(x,y)\to(0,0)} q(x,y) = \lim_{x\to 0} q(x, mx) = \lim_{x\to 0} \frac{x^2}{mx} = 0,$$

along the line $y = 0$ we have

$$\lim_{(x,y)\to(0,0)} q(x,y) = \lim_{x\to 0} q(x, 0) = \lim_{x\to 0} 0 = 0,$$

but along the parabola $y = x^2$, we have

$$\lim_{(x,y)\to(0,0)} q(x,y) = \lim_{x\to 0} q(x, x^2) = \lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

Because these limits do not all agree, we conclude that $\lim_{(x,y)\to(0,0)} q(x, y)$ does not exist, and we also see that the limits agree along all lines $y = mx$.

(c) Such a function cannon exist because the definition of continuity requires that the limit in question exists.

(d) The limit $\lim_{(x,y)\to(0,0)}$ cannot exist because the other two given limits do not agree.

(e) Let $t(x, y)$ be given by $t(x, y) = 0$ on the domain that consists of the entire plane $\mathbf{R}^2$ minus the single point $(1, 1)$. It is clear that $\lim_{(x,y)\to(1,1)} t(x, y) = \lim_{(x,y)\to(1,1)} 0 = 0$, but $t(1, 1)$ is not defined.

2.1 #15

(a) and (b). The functions in the numerators and denominators of both f and g are continuous, so by the properties of continuous functions given in the text, f and g are continuous wherever the denominator is not zero. That means g is continuous everywhere, and f is continuous except for points along the line $x = y$, where f is not even defined.

(c) Approaching along the line $y = 0$ we get

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{x\to 0} \frac{0}{x^2} = 0,$$

but approaching along the line $y = x$ we get

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{x\to 0} \frac{x^2}{2x^2} = 1/2.$$

This is impossible, so we conclude that $\lim_{(x,y)\to(0,0)} h(x,y)$ does not exist.

(d) This function is continuous everywhere. We only need to show that

$$\lim_{(x,y)\to(0,0)} k(x,y) = 0.$$

Use polar coordinates to write $x = r\cos t, y = r\sin t$. Then for $r \neq 0$, we have

$$k(r,t) = \frac{r^6 \cos t \sin t}{r^2} = r^4 \cos t \sin t.$$

Since $|r^4 \cos t \sin t| \leq |r^4|$ and $r^4 \to 0$ as $r \to 0$. Since we must have $r \to 0$ along any path for which $(x,y) \to (0,0)$, we conclude that $\lim_{(x,y)\to(0,0)} k(x,y) = 0$, as desired.

2.2 #16

Given $f(x, y) = 8 - x^2 - 3y^2$.

(a) $\frac{\partial f}{\partial x} = -2x$, $\frac{\partial f}{\partial} = -6y$

(b)