Summer 2006 Results

ongoing work in quantum entanglement

David W. Lyons
lyons@lvc.edu
Mathematical Sciences
Lebanon Valley College

Scott N. Walck
walck@lvc.edu
Department of Physics
Lebanon Valley College

June 17, 2006

Abstract. In a series of previous papers we have proved *blah blah blah*. In this paper we prove *blah blah blah* amazing results about irreducible entanglement.

1 Introduction

context and implications

2 Notation

Let $H = (\mathbb{C}^2)^{\otimes n}$ denote the Hilbert space for a system of n qubits and let D denote the set of n-qubit density matrices (pure and mixed), that is, the set of positive semidefinite operators on H with trace 1. We define two local unitary groups $\widetilde{G} = SU(2)^n \times U(1)$ which acts on normalized vectors in H and $G = SU(2)^n$ which acts on D. We denote their respective Lie algebras by $L\widetilde{G} = su(2)^n \times u(1)$ and $LG = su(2)^n$.

actions of the two groups and their algebras

notation for partial trace over one qubit

Let A be the 1-qubit subsystem of H consisting of the k-th qubit, and let B the complementary subsystem of the remaining qubits. For a density matrix ρ, we shall write $\operatorname{tr}_k \rho$ or $\rho^{(k)}$ to denote $\rho^B = \operatorname{tr}_A \rho$. Similarly, we shall write $\operatorname{tr}_k X$ or $X^{(k)}$ to denote the partial trace of $X \in LG$ over the k-th qubit.

Observation: $X^{(k)} = 2P_{(k)}(X)$, where $P_{(k)}(X) = (X_1, X_2, \ldots, \hat{X}_k, \ldots X_n)$ is projection onto the local unitary Lie algebra for subsystem B.

3 A result on irreducible entanglement

Definition 3.1. We shall say that a density matrix ρ has *irreducible entanglement* if ρ is the unique density matrix that has $(\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)})$ as its reduced density matrices (*cite Linden and Wootters*).

Lemma 3.2. *Let $X \in LG$, $\rho \in D$. For $1 \leq k \leq n$, we have*

$$[X, \rho]^{(k)} = \frac{1}{2}\left[X^{(k)}, \rho^{(k)}\right].$$

PROOF. Here is the proof for $n = k = 2$. Let $X = (X_1, X_2)$ and write ρ as a sum $\rho = \sum_j \alpha_j \otimes \beta_j$. For a summand $\alpha \otimes \beta$ in ρ, we have

$$\begin{aligned}
\operatorname{tr}_2 [X, \alpha \otimes \beta] &= \operatorname{tr}_2 [X_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes X_2, \alpha \otimes \beta] \\
&= \operatorname{tr}_2 [X_1 \otimes \mathrm{Id}, \alpha \otimes \beta] + \operatorname{tr}_2 [\mathrm{Id} \otimes X_2, \alpha \otimes \beta] \\
&= \operatorname{tr}_2 ([X_1, \alpha] \otimes \beta) + \operatorname{tr}_2 (\alpha \otimes [X_2, \beta]) \\
&= \operatorname{tr} \beta \, [X_1, \alpha] + \operatorname{tr} [X_2, \beta] \, \alpha \\
&= \operatorname{tr} \beta \, [X_1, \alpha]
\end{aligned}$$

since the trace of a bracket is zero.

On the other side of the equation, we have

$$\begin{aligned}
[\operatorname{tr}_2 X, \operatorname{tr}_2 \alpha \otimes \beta] &= [\operatorname{tr}_2 (X_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes X_2), \operatorname{tr}_2 \alpha \otimes \beta] \\
&= [2X_1, (\operatorname{tr} \beta)\alpha] \\
&= 2 \operatorname{tr} \beta \, [X_1, \alpha] \, .
\end{aligned}$$

QED. □

Theorem 3.3. *Let ρ be a pure or mixed n-qubit density matrix. The following are equivalent.*

(i) There is an $X \in LG$ such that $X \notin K_\rho$ but $X^{(k)} \in K_{\rho^{(k)}}$ for $1 \le k \le n$.

(ii) There is a density matrix τ that is LU-equivalent but not equal to ρ such that $\tau^{(k)} = \rho^{(k)}$ for $1 \le k \le n$.

Definitions 3.4. If condition (i) holds, we shall say that ρ has the *enlarged kernel condition.* If (ii) holds, we shall say ρ has *LU irreducible entanglement*

Question: if ρ has irreducible entanglement, must it have LU irreducible entanglement?

PROOF. First we prove that (i) implies (ii). Suppose the enlarged kernel condition holds. Consider the curve

$$\rho(t) = e^{tX}(\rho) = e^{tX} \rho \left(e^{tX}\right)^{\dagger} .$$

There is some t_0 such that $\rho(t_0) \neq \rho$ (otherwise $0 = \rho'(0) = [X, \rho]$ so X would be in K_ρ). Let $\tau = \rho(t_0)$ and let $Y = t_0 X$, so

$$\tau = e^{Y} \rho \left(e^{Y}\right)^{\dagger} = \Phi(e^{Y})(\rho) = e^{\phi Y}(\rho)$$

where $\Phi \colon G \to gl(2^n, \mathbb{C})$ is conjugation action on $2^n \times 2^n$ matrices (in which D sits) and $\phi \colon LG \to gl(2^n, \mathbb{C})$ is its derivative at identity, given by $\phi(X)(A) = [X, A]$. Thus we have

$$\tau = e^{\phi Y}(\rho) = \rho + [Y, \rho] + (1/2) \, [Y, [Y, \rho]] + (\text{ higher order brackets}).$$

Taking partial trace over the k-th qubit, we obtain

$$\tau^{(k)} = \rho^{(k)} + [Y, \rho]^{(k)} + \operatorname{tr}_k(\text{ higher order brackets}).$$

Applying Lemma (3.2), all the brackets vanish since $Y^{(k)} \in K_{\rho^{(k)}}$. This is true for all k, so (i) implies (ii) is proved.

Now suppose that condition (ii) holds. Since $\exp\colon LG \to G$ is onto, there is an $X \in LG$ such that $\tau = \Phi(e^X)(\rho)$. Let $\rho(t) = \Phi(e^{tX})(\rho)$ and let $\alpha_k(t) = \Phi(e^{tX^{(k)}})(\rho^{(k)}) = e^{\phi t X^{(k)}}(\rho^{(k)})$ (where we abuse notation to let Φ denote the action of G on D and also the action of $SU(2)^{n-1}$ on D_{n-1}). We shall prove that X satisfies condition (i). Taking tr_k of

$$\rho(t) = \rho + t[X, \rho] + (\text{ higher order brackets})$$

a factor of 1/2 comes out of each bracket (by Lemma (3.2)), so we get

$$\begin{aligned}
\alpha_k(t) &= (\rho(t))^{(k)} \\
&= \rho^{(k)} + \frac{t}{2}[X^{(k)}, \rho^{(k)}] + (\text{ higher order brackets}) \\
&= \alpha_k(t/2).
\end{aligned}$$

Since this is true for all t, we must have

$$0 = \alpha_k'(t) = [X^{(k)}, \rho^{(k)}].$$

Since this holds for all k, condition (i) is satisfied by X. This concludes the proof. $\square$

4 Conclusion

summary of results and implications, brilliant conjectures

References

[1] David W. Lyons and Scott N. Walck. *Minimum orbit dimension for local unitary action on n-qubit pure states*, under review. e-print quant-ph/0503052

[2] David W. Lyons and Scott N. Walck. *Classification of n-qubit states with minimum orbit dimension* e-print quant-ph/0506241