Quantum Information Background Notes

Summer 2012

David Lyons
Mathematical Sciences
Lebanon Valley College

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Contents

1 Complex Numbers

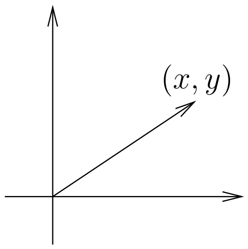

Figure 1
The vector (x, y)

1.1 Motivation

The Euclidean line is the set used to measure a great many "real world" phenomena such as time, distance, mass, temperature, and so on. It is for this reason that we call the line *real.* Analysis of mathematical models involving functions on the real line is made rich and powerful by the algebraic structure of the real numbers. Key features of this algebra are the operations of addition and multiplication, together with the distributive law which governs their interaction.

The Euclidean plane is the set used to measure any kind of data that consists of *pairs* of real numbers. Examples include graphs of functions on the real line and two-dimensional geometric figures. A natural question is whether the algebra of the line extends in any useful ways to the plane. The answer is yes. The extension of the algebra of the reals to the plane forms the *complex* numbers.

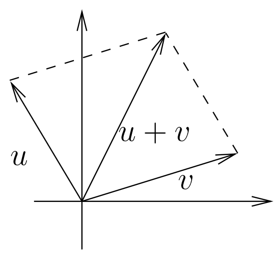

Figure 2
The parallelogram law
of vector addition

1.2 Vectors and vector operations

The complex numbers, denoted $\mathbf{C}$, is defined to be the set of points in the Euclidean plane, that is, the set

$$\mathbf{R}^2 = \mathbf{R} \times \mathbf{R} = \{(x, y) \mid x, y \in \mathbf{R}\}$$

where $\mathbf{R}$ denotes the set of real numbers. We can think of a point (x, y) as a *vector*, or directed line segment, which begins at the origin and ends at (x, y). In drawings, we represent vectors by arrows as in Figure 1. This comes from physics where vectors model physical quantities such as displacement, velocity or force.

The way in which physical vector quantities combine leads to the operation of *vector addition.* The sum of two vectors (a, b), (c, d) is defined to be

$$(a, b) + (c, d) = (a + c, b + d). \tag{1.2.1}$$

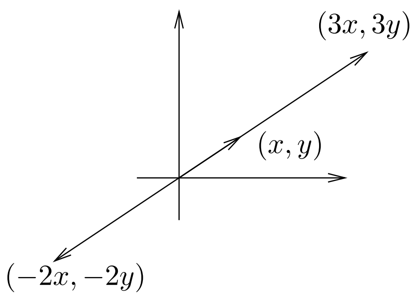

Figure 3
Scalar multiples of
a vector

Geometrically, two vectors v, w and their sum $v + w$ form two sides and a diagonal of a parallelogram; for this reason, equation (1.2.1) is called the *parallelogram law.* See Figure 2.

Given a vector $v = (x, y)$ and a real number k, the *scalar product* of k times v, denoted kv, is defined to be (kx, ky). The vector kv is $|k|$ times as long as v and points in the same direction as v if k is positive and in the opposite direction from v if k is negative. See Figure 3.

Vector addition and scalar multiplication obey the *distributive law.* For any two vectors v, w and any real number k, we have

$$k(v + w) = kv + kw. \tag{1.2.2}$$

Notice that vector operations say how to add two points in the plane, and how to multiply a point in the plane by a real number, but *not* how to multiply two vectors. We explain how to multiply complex numbers in §1.5 below.

1.3 Points, vectors and scalars as complex numbers

While points and vectors may seem to be different types of objects, the set which represents them both is the same, namely $\mathbf{R}^2$. We refer to an ordered pair (x, y) alternately as a point or as a vector, depending on convenience. This takes some getting used to, but it turns out to be useful to think of complex numbers both ways.

Similarly, a scalar $k \in \mathbf{R}$ and a vector $v \in \mathbf{R}^2 = \mathbf{C}$ seem to be different kinds of things. It is also natural, however, to think of the real number line as a *subset* of the plane. The x-axis, that is, the set of points of the form $\{(x, 0) \mid x \in \mathbf{R}\}$, is in one-to-one correspondence with the real numbers via $(x, 0) \leftrightarrow x$. In this way, we think of the real number x as the complex number $(x, 0)$. This is what we mean when we say that that the Euclidean plane *extends* the Euclidean line.

To summarize this subsection: Complex numbers may be thought of as either points or vectors in the plane; real numbers are also complex numbers since points on the real line (the x-axis) are also points in the plane.

1.4 Polar coordinates

The *norm* of a point $p = (x, y)$, denoted $|p|$, is defined to be

$$|p| = \sqrt{x^2 + y^2}.$$

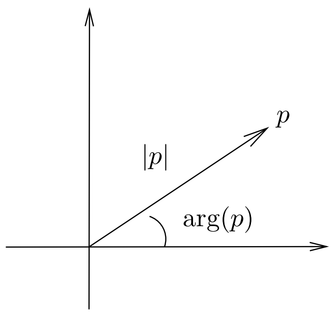

Figure 4
Norm and argument

Geometrically, the norm of a point is its length as a vector, which is the same as its distance from the origin $(0, 0)$. The *argument* of p, denoted $\arg(p)$, is the measure in radians of the directed angle with vertex at the origin, whose initial ray is the positive x-axis and whose terminal ray passes through p. See Figure 4. We agree that two arguments are the same if they differ by an integer multiple of 2π.

The norm and argument of a point are called the *polar coordinates* of that point. By contrast, the numbers x, y in the ordered pair $(x, y) \in \mathbf{R}^2$ are called the *rectangular* or *Cartesian* coordinates of the point. In these notes, we shall use the notation $P_{(r,\theta)}$ to denote the point whose polar coordinates are (r, θ).

1.5 Multiplication

The "obvious" way to attempt multiplication of vectors is to mimic vector addition by defining $(a, b) \cdot (c, d) = (ac, bd)$. However, this leads to an unacceptable conflict, as we shall describe.

Let k be a real number and let $v = (x, y)$ be a vector. As a point in the plane, the real number k is $(k, 0)$, so we would have $kv = (k, 0) \cdot (x, y) = (kx, 0)$. On the other hand, we have already defined (scalar multiplication) kv to be $k \cdot (x, y) = (kx, ky)$. But $(kx, 0)$ does not equal (kx, ky) unless one or both of k, y are zero. This example says that if we want scalar multiplication of vectors to be compatible with multiplying pairs of complex numbers, we must look for a different way to multiply points than component-wise.

To see another way, we examine some multiplications of real numbers using polar coordinates.

$$\begin{array}{rclclcl} P_{(2,0)} \cdot P_{(3,0)} & = & 2 \cdot 3 & = & 6 & = & P_{(6,0)} \\ P_{(2,0)} \cdot P_{(3,\pi)} & = & 2 \cdot -3 & = & -6 & = & P_{(6,\pi)} \\ P_{(2,\pi)} \cdot P_{(3,\pi)} & = & -2 \cdot -3 & = & 6 & = & P_{(6,2\pi) = P(6,0)} \end{array}$$

These examples suggest that in polar coordinates, norms multiply and arguments add. That is the inspiration for the following definition of multiplication.

$$P_{(r,\theta)} \cdot P_{(s,\varphi)} = P_{(rs,\theta+\phi)} \tag{1.5.3}$$

This multiplication does indeed extend the ordinary multiplication of real numbers, and resolves the conflict from the previous paragraph. It is not immediately obvious, but it turns out that other desirable properties of the algebra of the real numbers also hold in the plane. In particular, extended addition and multiplication satisfy the distributive law

$$z(u+v) = zu + zv \tag{1.5.4}$$

for all z, u, v in the plane. Note that we normally omit the dot between two points and write zw to denote the product of two points z and w, just as we do for real numbers.

1.6 Real and imaginary parts, rectangular form

The complex number $(0,1) = P_{(1,\pi/2)}$ is denoted by the symbol i. It has the curious property that its square is negative one. Note that the number $z = (x,y)$ is equal to $x(1,0) + y(0,1) = x + yi$. The real number x is called the *real part* and the real number y is called the *imaginary part* of $z = x + yi$, and we write $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. A number of the form $(0,y) = yi$ is called *pure imaginary*. The coordinates (x,y) of z are called the *rectangular* or *Cartesian* coordinates, and the expression $x + yi$ is called the *rectangular form* of z.

Next we demonstrate how to multiply two complex numbers in rectangular form without converting to polar coordinates. Let $(x,y) = x+yi$ and let $(u,v) = u+vi$ be two complex numbers. Using the distributive law, we have

$$\begin{array}{rcll} (x,y) \cdot (u,v) & = & (x+yi)(u+vi) & \\ & = & xu + yui + xvi + yvi^2 & \text{(distributing)} \\ & = & xu + yui + xvi - yv & (i^2 = -1) \\ & = & (xu - yv) + (yu + xv)i & \text{(collecting real and imaginary parts).} \end{array}$$

1.7 Exponential notation and polar form

The elementary functions of a single real variable, including polynomials, rational functions, sine, cosine, and the natural exponential function can be extended to functions of a complex variable. This is done in the subject of *complex analysis*. It turns out that the natural exponential function (that is, the function that sends the complex number z to the complex number e^z) has the following formula for pure imaginary exponents.

(1.7.5) $$e^{it} = \cos t + i \sin t \qquad \text{for } t \in \mathbf{R}$$

Equation (1.7.5) is not supposed to be obvious; it takes a bit of work to even define what e^z means for a complex number z. However, it is not necessary to know the theory of the complex exponential function to use (1.7.5). For now, you may think of (1.7.5) as a definition of the symbols e^{it}.

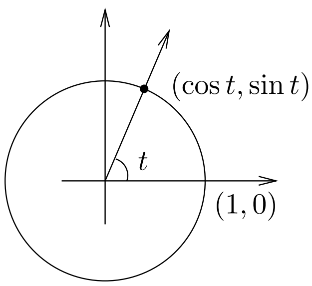

Figure 5
Point on
the unit circle

Recall that $p = (\cos t, \sin t)$ is the point on the unit circle intersected by the terminal ray of a directed angle of t radians in standard position. See Figure 5. It follows that $e^{it} = P_{(1,t)}$ has norm 1 and argument t. If z is an arbitrary complex number with norm r and argument θ, then we have

(1.7.6) $$z = P_{(r,\theta)} = rP_{(1,\theta)} = re^{i\theta}.$$

The expression on the right-hand side of the above equation is called the *polar form* of the complex number z.

1.8 Conjugation

Let $z = (x, y) = P_{(r,\theta)} = x + iy = re^{i\theta}$ be a complex number. The *conjugate* of z, denoted $\overline{z}$ is defined to be

(1.8.7) $$\overline{z} = (x, -y) = P_{(r,-\theta)} = x - iy = re^{-i\theta}.$$

Geometrically, the conjugate of a point is the reflection of that point across the x-axis. Here is a useful relation that involves the conjugate.

(1.8.8) $$z\overline{z} = |z|^2$$

Exercises

1. What is the difference between polar coordinates and polar form? What is the difference between rectangular coordinates and rectangular form? Write formulas for converting from polar to rectangular coordinates and vice-versa.

2. Express each of the following in rectangular and polar form.
   (a) $3(2 - i) + 6(1 + i)$
   (b) $\left(2e^{i\pi/6}\right)\left(3e^{-i\pi/3}\right)$
   (c) $(2 + 3i)(4 - i)$
   (d) $(1 + i)^3$

3. Prove the following property of norm, for all complex numbers z, w.
$$|zw| = |z||w|$$
   Do the proof using rectangular and polar forms. Which is easier?

4. Prove the following property of norm, called the *triangle inequality*. For any two complex numbers z, w, we have
$$|z + w| \leq |z| + |w|.$$

5. Prove (1.8.8).

6. Let p and q be complex numbers. Prove that the distance (ordinary distance between points in the plane) between p and q is $|p-q|$. Hint: Use rectangular form.

7. Verify the distributive law (1.5.4). Suggestion: First prove case (i) where z is a real number. Next prove case (ii) where z is has norm 1 (use the fact that the diagonal of a rotated parallelogram is the rotation of the diagonal of the original parallelogram. Finally prove the general case where $z = re^{i\theta}$.

8. This exercise outlines the definitions and properties of complex division.

   (a) Notice that the real number 1, considered as the complex number (1,0), has the property that $1z = z$ for any complex number z. For this reason, 1 is called a *multiplicative identity* for **C**. Are there any other multiplicative identities? That is, does there exist any other complex number u with the property that $uz = z$ for every complex number z? If so, find one. If not, explain why none exists.

   (b) Given two complex numbers u and v with $v \neq 0$, the quotient of u divided by v, denoted u/v, is defined to be the complex number z with the property that $u = vz$. Write expressions for $z = u/v$ and $w = 1/v$ in polar form if $u = re^{i\theta}$ and $v = se^{i\varphi}$.

   Note a new definition: we call $1/v$ the *reciprocal* or *multiplicative inverse* of v, and also write it as v^{-1}. Notice that multiplicative identity, division and multiplicative inverse are defined the same as for real numbers.

   (c) Suppose that z, w are nonzero complex numbers. Prove that

   $$(1/z)(1/w) = 1/(zw).$$

   (d) Find the multiplicative inverse of $z = x + iy$ in rectangular form (assume $z \neq 0$). Hint: Multiply $1/z$ by $\overline{z}/\overline{z}$ and use the previous exercise. This is called *rationalizing the denominator*.

9. Express each of the following in rectangular and polar form.

   (a) $\dfrac{2+i}{3-i}$

   (b) $\dfrac{1+2i}{1-2i}$

   (c) $\dfrac{2e^{i\pi/4}}{3e^{-i\pi/2}}$

10. Verify the following formulas. For any complex number z, we have

    (a) $\mathrm{Re}(z) = \dfrac{z+\overline{z}}{2}$, and

    (b) $\mathrm{Im}(z) = \dfrac{z-\overline{z}}{2i}$.

11. Given a nonzero complex number z, explain why z has exactly two square roots, and explain how to find them.

12. Find all complex solutions of the following equations.
    (a) $z^2 + 3z + 5 = 0$
    (b) $(z - i)(z + i) = 1$
    (c) $\dfrac{2z + i}{-z + 3i} = z$

13. Derive the double angle identities for $\cos 2\theta$ and $\sin 2\theta$ by computing $(e^{i\theta})^2$ two ways: in polar form and in rectangular form. Then compare real and imaginary parts.

14. Graph the solutions to the following complex equations.
    (a) $|z - 2| = 3$
    (b) $|4z - 2i| = 3$
    (c) $\mathrm{Im}(z) = 3$
    (d) $\mathrm{Im}(2e^{i\pi/4}z - 2 + 3i) = 0$

1.9 Solutions

1. Let z be a complex number, let $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $r = |z|$ and $\theta = \arg(z)$. The pair (r, θ) is called the polar coordinates for z, while the expression $re^{i\theta}$ is called the polar form for z. The pair (x, y) is called the rectangular coordinates for z, while the expression $x + iy$ is called the rectangular form for z.

   To convert from polar to rectangular, use the equations $x = r\cos\theta, y = r\sin\theta$ (show sketches to explain these formulas). To convert from rectangular to polar, use $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. For the last equation, you must use judgment when $x = 0$ to decide whether θ should be $\pi/2$ or $-\pi/2$. You must also use judgment to interpret any machine computation of the arctangent function to decide which quadrant θ should be in.

2. (a) $12 + 3i = \sqrt{153}\, e^{i\arctan(1/4)}$
   (b) $6e^{-i\pi/6} = 3\sqrt{3} - 3i$
   (c) $11 + 10i = \sqrt{221}\, e^{i\arctan(10/11)}$
   (d) $-2 + 2i = 2\sqrt{2}\, e^{i3\pi/4}$

3. Let $z = x + iy = re^{i\theta}$ and $w = u + iv = se^{i\varphi}$. In rectangular form, we have

$$
\begin{aligned}
|zw| &= |(x+iy)(u+iv)| \\
&= |(xu - yv) + i(xv + yu)| \quad \text{(multiply)} \\
&= \sqrt{(xu-yv)^2 + (xv+yu)^2} \quad \text{(definition of norm)} \\
&= \sqrt{x^2u^2 - 2xyuv + y^2v^2 + x^2v^2 + 2xyuv + y^2u^2} \quad \text{(multiply)} \\
&= \sqrt{(x^2+y^2)(u^2+v^2)} \quad \text{(simplify and factor)} \\
&= |z||w|.
\end{aligned}
$$

   Alternatively, and more simply, in polar coordinates we have

$$
\begin{aligned}
|zw| &= |re^{i\theta}se^{i\varphi}| \\
&= |rse^{i(\theta+\varphi)}| \quad \text{(definition(!) of multiplication)} \\
&= rs \quad \text{(definition of norm)} \\
&= |z||w|.
\end{aligned}
$$

   Q.E.D.

4. (This is a challenge problem. The simplest approach is geometric: sketch the parallelogram for vector subtraction and use the fact from elementary geometry that the total length for any two sides of a triangle is more than the length of the remaining side. It is harder, but a fun exercise, to do an algebraic proof.)

5. Let $z = re^{i\theta}$. Then $\overline{z} = re^{-i\theta}$, and we have

$$
\begin{aligned}
z\overline{z} &= re^{i\theta}re^{-i\theta} \\
&= r^2e^0 \\
&= r^2 = |z|^2.
\end{aligned}
$$

6. Let $p = a + ib$ and $q = c + id$. We have

$$\begin{aligned} |p - q| &= |(a + ib) - (c + id)| \\ &= |(a - c) + i(b - d)| \\ &= \sqrt{(a - c)^2 + (b - d)^2}. \end{aligned}$$

   The latter expression is the distance from p to q, so we are done.

7. We prove (4.5.2) in two stages. First we prove the special case where z is a real number. Second we prove the case where $z = e^{i\theta}$ is a complex number with norm 1. Then we put the two results together to get the distributive law.

   First, if $z = k$ is a real number, then (4.5.2) becomes (4.2.2). Here is the proof. Let $u = (a, b)$ and $v = (c, d)$ be complex numbers. We have

$$\begin{aligned} z(u + v) &= k((a, b) + (c, d)) \\ &= k((a + c, b + d)) \quad \text{(definition of addition)} \\ &= (k(a + c), k(b + d)) \quad \text{(definition of scalar multiplication)} \\ &= (ka + kc, kb + kd) \quad \text{(distributive law for real numbers)} \\ &= (ka, kb) + (kc, kd) \quad \text{(definition of addition)} \\ &= k(a, b) + k(c, d) \quad \text{(definition of scalar multiplication)} \\ &= ku + kv. \end{aligned}$$

   This proves the first case.

   Second, let $z = e^{i\theta}$. Geometrically, u, v and $u + v$ are three corners of a parallelogram whose fourth corner is the origin. If we rotate this parallelogram θ radians counterclockwise, we have a new parallelogram with corners zu, zv, $z(u + v)$ and the origin. Since the diagonal of this new parallelogram is the sum of the two edge vectors, we have $z(u + v) = zu + zv$.

   Finally, let $z = re^{i\theta}, u, v$ be any complex numbers. Then

$$\begin{aligned} z(u + v) &= re^{i\theta}(u + v) \\ &= r(e^{i\theta}u + e^{i\theta}v) \quad \text{(by the second case above)} \\ &= re^{i\theta}u + re^{i\theta}v \quad \text{(by the first case above)} \\ &= zu + zv. \end{aligned}$$

   Q.E.D.

8. (a) Suppose that the complex number u has the property that $uz = z$ for all complex numbers z. Then we must have $|u||z| = |uz| = |z|$. Dividing by $|z|$ yields $|u| = 1$. We also have $\arg(u) + \arg(z) = \arg(uz) = \arg(z)$ (where it is understood that two arguments are considered equal if they differ by an integer multiple of 2π), so subtracting $\arg(z)$ yields $\arg(u) = 0$. The only complex number with norm 1 and argument 0 is the real number 1, so the multiplicative identity is indeed unique.

(b) Let $z = te^{i\psi}$. Since $re^{i\theta} = u = vz = ste^{i(\varphi+\psi)}$, we must have $r = st$ and $\theta = \varphi + \psi$ (plus possibly some multiple of 2π). Clearly then $z = (r/s)e^{i(\theta-\varphi)}$ satisfies the equation $u = vz$. It is also clear that this z is unique. Setting $u = 1$ gives $w = 1/v = (1/s)e^{-i\varphi}$.

(c) Let $z = re^{i\theta}, w = se^{i\varphi}$. Then $zw = rse^{i(\theta+\varphi)}$, $1/z = (1/r)e^{-i\theta}$, $1/w = (1/s)e^{-i\varphi}$, and $1/(zw) = (1/(rs))e^{-i(\theta+\varphi)}$. The identity $(1/z)(1/w) = 1/(zw)$ clearly holds.

(d) Let $z = x + iy$. Then

$$1/z = \frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

9. (a) $\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}e^{i\pi/4}$

(b) $-\frac{3}{5} + \frac{4}{5}i = e^{i(\arctan(-4/3)+\pi)}$

(c) $\frac{2}{3}e^{i3\pi/4} = -\frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3}i$

10. Let $z = x + iy$. Then we have

(a) $\frac{z + \overline{z}}{2} = \frac{2x}{2} = x = \mathrm{Re}(z)$, and

(b) $\frac{z - \overline{z}}{2i} = \frac{2iy}{2i} = y = \mathrm{Im}(z)$.

11. Since squaring a number squares the norm and doubles the argument, a square root can be found by taking the square root of the norm and dividing the argument by two. That is, for $z = re^{i\theta}$, a square root of z is $\sqrt{r}e^{i\theta/2}$. Another square root of z is the negative of that expression. Any other square root of z would have to have norm $\sqrt{|z|}$ and argument $\theta/2$ plus or minus an integer multiple of π, so these must be all the square roots of z.

12. (a) $-\frac{3}{2} \pm i\frac{\sqrt{11}}{2}$

(b) 0

(c) $(1/2)[(-2 \pm 281^{1/4}\cos\varphi) + i(3 \pm 281^{1/4}\sin\varphi)]$, where $\varphi = (\arctan(16/5) + \pi)/2$

13. On the one hand, we have $(e^{i\theta})^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$. On the other hand, we have $(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$. Comparing real and imaginary parts yields the double angle identities.

14. (a) (circle of radius 3 centered at 2)

(b) (circle of radius 3/4 centered at $(1/2)i$)

(c) (line $y = 3$)

(d) (line $y = -x - \frac{3\sqrt{2}}{2}$)