Bits of Complex Analysis for Modern Geometry
revised: 2 November 2018

1 Complex exponential function

A sequence of complex numbers

$$a_1, a_2, a_3, \ldots$$

is said to *converge* to the number L if, for every real $\epsilon > 0$, there is a natural number N, so that

$$n \geq N \Longrightarrow |a_n - L| < \epsilon.$$

A sequence that does not converge to any complex number is said to *diverge*.

A series

$$a_1 + a_2 + a_3 + \cdots$$

with complex summands a_i is said to *converge* to the complex number S if the complex sequence $s_1, s_2, s_3, \ldots$ converges to S, where $s_n = a_1 + a_2 + \cdots + a_n$ is the nth partial sum, for $n \geq 1$. A series that does not converge to any complex number is said to *diverge*.

A power series

$$a_0 + a_1 z + a_2 z^2 + \cdots,$$

with complex coefficients a_i, is said to *converge at $z = z_0$* if the series

$$a_0 + a_1 z_0 + a_2 z_0^2 + \cdots$$

converges.

Facts/Definitions: The power series

$$1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

converges at all complex numbers. The function defined by this series is called the *complex exponential* function, whose value at z is denoted e^z or $\exp(z)$. It can be shown (using Taylor series) that

$$e^{it} = \cos t + i \sin t$$

for all real t. This is *Euler's formula*.

2 Extending complex functions to $\mathbf{C}^+$

Complex analysis gives a rigorous foundation for important things like limits for functions of complex variables. Here's a sample of some limit definitions. Let f be a function of a complex variable and let a, L be complex numbers.

We write

$$\lim_{z \to a} f(z) = L$$

to mean that, for any (real) $\epsilon > 0$, there is a (real) $\delta > 0$, so that

$$0 < |z - a| < \delta \Rightarrow |f(z) - L| < \epsilon.$$

We write

$$\lim_{z \to \infty} f(z) = L$$

to mean that, for any (real) $\epsilon > 0$, there is a (real) $M > 0$, so that

$$M < |z| \Rightarrow |f(z) - L| < \epsilon.$$

If the above holds, we write "$f(\infty) = L$".

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If the above holds, we write "$f(a) = \infty$".

Suppose that the domain of f is the entire complex plane minus only a finite number of points, and that $\lim_{z \to a} f(z) = \infty$ for each point a outside of the domain of f. Suppose further that $\lim_{z \to \infty} f(z) = u$ for some $u \in \mathbf{C}$ or $u = \infty$. Then it is natural to extend f to a function $f \colon \mathbf{C}^+ \to \mathbf{C}^+$ by defining $f(a) = \infty$ for each point a not in the original domain of f, and set $f(\infty) = u$.

Fact: For any complex numbers a, b, c, d, not all of which are zero, the function $f(z) = \dfrac{az + b}{cz + d}$ can be defined on the entire extended complex plane $\mathbf{C}^+$. For the case when a, c are both nonzero, one can show that the appropriate limits exist, so that we have $f(-d/c) = \infty$ and $f(\infty) = a/c$.

3 Differentiability and Conformality

Derivatives of complex functions are defined exactly as for real functions. The derivative $f'(z_0)$ of a function $f \colon \mathbf{C} \to \mathbf{C}$ is defined to be

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{1}$$

if the limit exists. If the limit exists, we say f is *differentiable at* z_0, and if $f'(z)$ exists for all z in the domain of f, we say f is *differentiable.*

A key piece of complex geometry is the analysis of how complex functions transform curves in the plane. An important detail is how lines tangent to curves are transformed. Here is how it works: A curve in the plane parameterized by a function $z = \gamma(t)$ has a tangent line at $z_0 = \gamma(t_0)$ determined by the tangent vector

$$\gamma'(t_0) = \lim_{h \to 0} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h} \tag{2}$$

if that limit exists (if the limit does not exist, then there is no well-defined tangent line). In what follows, we assume that γ is differentiable at t_0 and that f is differentiable at $z_0 = f(t_0)$. The transformed curve $f \circ \gamma$ has a tangent vector $(f \circ \gamma)'(t_0)$ at the point $w_0 = f(z_0)$ given by the following chain rule formula.

$$(f \circ \gamma)'(t_0) = f'(z_0)\, \gamma'(t_0). \tag{3}$$

Now we write $f'(z_0)$ in polar form $f'(z_0) = re^{i\theta}$. The formula (3) now reads

$$(f \circ \gamma)'(t_0) = re^{i\theta}\gamma'(t_0).$$

From this we see the effect of transforming the curve γ by f. The tangent vector $(f \circ \gamma)'(t_0)$ is a rescaling and rotation of the tangent vector $\gamma'(t_0)$.

Now consider *two* curves γ_1, γ_2 that intersect $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$. The transformation f takes these curves to $f \circ \gamma_1, f \circ \gamma_2$ that intersect at $w_0 = f(z_0)$. The tangent lines to γ_1, γ_2 at z_0 are both rotated by the same angle $\theta = \arg(f'(z_0))$, so the angle made by the tangent vectors $\gamma_1'(t_0), \gamma_2'(t_0)$ at z_0 is *equal to* the angle made by the tangent vectors $(f \circ \gamma_1)'(t_0), (f \circ \gamma_2)'(t_0)$ at w_0. This angle-preserving property is called *conformality.*

4 Derivatives for $\hat{f} \colon \mathbf{R}^2 \to \mathbf{R}^2$

Let $\phi \colon \mathbf{R}^2 \to \mathbf{C}$ denote the canonical identification $(x, y) \to x + iy$. Given $f \colon \mathbf{C} \to \mathbf{C}$, let $\hat{f} \colon \mathbf{R}^2 \to \mathbf{R}^2$ denote the lift $\hat{f} = \phi^{-1} \circ f \circ \phi$ of f, and let

$$(u, v) = \hat{f}(x, y).$$

Assume f is differentiable at $z_0 = x_0 + iy_0$, and write $f'(z_0) = re^{i\theta}$. It can be shown that $\hat{f}$ is differentiable at (x_0, y_0), and that the following relationships between the derivatives $D\hat{f}(x_0, y_0)$ and $f'(z_0)$ hold.

Let $D\hat{f}(x_0, y_0)$ be given by the matrix

$$D\hat{f}(x_0, y_0) = \left[\begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right]_{(x_0, y_0)}$$

and let $J\hat{f}$ denote the Jacobian

$$J\hat{f} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Then we have the following.

$$D\hat{f}(x_0, y_0) = r \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right] \tag{4}$$

$$J\hat{f}(x_0, y_0) = r^2 = |f'(z_0)|^2. \tag{5}$$

Notes: Equation (4) can be used directly to show conformality of f. Equation (5) can be used for a shortcut in Chapter 10 Exercise 2.