Masters Degree Study of Conservation Laws for Single Dimensional Dynamics of Inviscid Fluids

J. M. Lyons

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Declaration & Acknowledgements

This work is presented for the degree of Master of Science in the field of applied mathematics. The work described in this thesis is entirely my own, with exception of assistance recognised in the references, and has not been submitted, in whole or in part, in any previous application for a degree.

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Contents

Introduction

This work aims to establish and explore the equations of motion describing one-dimensional flow for an inviscid fluid; these are commonly known as the Euler equations, [1].

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{bmatrix}_x = 0 \tag{0.1}$$

The first of the Euler equations to be examined will be the equation for conservation of mass, also known as the continuity equation, this will be the subject of § 1. In § 1.2 it will be shown how the continuity equation evolves from the divergence theorem and how the equation works for a fluid moving with a constant velocity, a spatially dependent velocity and how they compare with the inviscid Burgers' equation.

In § 2 we will look at how momentum can be conserved for an ideal isentropic gas and how it reduces to a conservation equation for the velocity of the fluid. In this way the non-linear Euler equations can be transformed to a quasi linear hyperbolic system of conservation equations; solutions to this system of equations can be found if the eigensystem of the Jacobian matrix can be computed. We will also look at the integral curves which result from this conservation equation, later it will be shown that these curves are important for finding solutions to the Riemann problem. In § 2.7, the same analysis will be applied to a wave propagating in shallow water, there are significant differences for this situation and the conservation equations will not be in the same variables but the analysis is very similar. In § 2.8 the definition of a new variable will reduce the quasilinear hyperbolic system of conservation equations with a known eigensystem to a decoupled system of advection equations whose solutions are easy to calculate for given initial data.

In § 3 the solutions and properties for a special set of initial conditions will be investigated, a discontinuous jump in the value of one or more of the variable under analysis; this is known as the Riemann problem. In reality this does not happen but it is a good approximation for the very realistic and ubiquitous situations in which the quantity undergoes a very large, smooth change over a physically narrow region. This problem is not difficult for a linear system but in a non linear system discontinuities in the initial data can result in shock waves. In § 3.2 we will derive the conditions required for the propagation of these shock waves. § 3.3-§ 3.6 will discuss the conditions for a simple solution to the non-linear problem and the remainder of § 3 will be dedicated to finding these simple wave solutions to the system of equations introduced in the previous sections.

With solutions obtained for the Riemann problem for conservation of mass and momentum in an ideal gas the Euler equations are completed by deriving a conservation law for the energy of the fluid, this is done in § 4. In § 4.4 it will be shown that the conservation law for energy reduces to a conservation law for pressure. By combining the conservation laws for velocity and pressure with the continuity equation the Euler equations can be expressed as a quasilinear hyperbolic system of equations. Riemann problem solutions to this system will be discussed in the remainder of § 4. In particular we will look at the exact solution for two gases of different mass and pressure coming into contact and the process of cavitation and see how the exact solution compares with a linear solution. This text will end with a discussion of the effects of real gases on Riemann problem solutions.

1   The Continuity Equation

1.1   Introduction

Mass is the quantitative measure of a particle's inertia, calculated as the ratio of net external force to acceleration. For an homogeneous media one can think of mass as a measure of the amount of matter in a body. It is intuitive then that the mass of a fluid will not change unless there is matter being added to or taken from the fluid, i.e. mass is a conserved quantity. For an incompressible fluid flowing in a cylindrical tube the mass of fluid, dm_1, that passes through a segment with cross section area A_1 at velocity v_1 over the infinitesimal time dt is $dm_1 = \rho A_1 v_1 dt$ and at the mass of fluid passing through a different segment with cross section area A_2 at velocity v_2 is $dm_2 = \rho A_2 v_2 dt$. Because mass is a conserved quantity $dm_1 = dm_2$ so the continuity equation for the one dimensional flow of an incompressible fluid in a can be stated as

$$A_1 v_1 = A_2 v_2 \tag{1.1}$$

However the continuity equation is more generally stated in it's differential form which evolves naturally from the Divergence theorem. It will be shown how the continuity equation is derived from the divergence theorem and find analytical solutions to the continuity equation for a one dimensional fluid flowing with a constant velocity and spatially dependent velocity. Both will be examples of the linear continuity equation, to investigate the non linear form of the continuity equation will look into Burgers' equation and find analytical solutions to the non homogeneous form and numerical solutions to the homogeneous inviscid Burgers' equation.
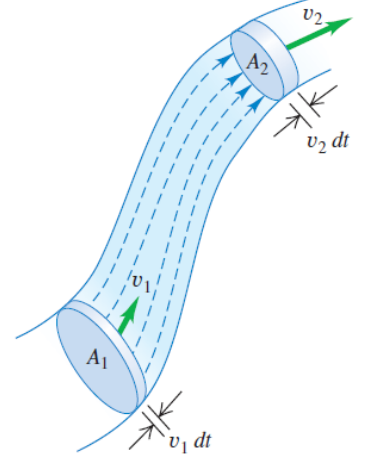

Figure 1.1: An incompressible fluid flowing through a tube with a variable cross section area, [3].

1.2   The Divergence Theorem

Let $\vec{F}$ be a vector field whose components have continuous derivatives and S be a piecewise smooth oriented closed surface. The Flux of $\vec{F}$ across S in the direction of the surface's outward unit normal field $\vec{n}$ equals the integral of $\nabla \cdot \vec{F}$ over the region D enclosed by the surface [2]:

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \iiint_D \nabla \cdot \vec{F} dV \tag{1.2}$$

Let F be the flux of fluid mass density in the region D. The flux of a physical parameter can be defined as the product of parameter's density in space and velocity. For the particular case of mass the flux will be defined as

$$\vec{F}(x, y, z, t) = \vec{u}(x, y, z, t)\rho(x, y, z, t) \tag{1.3}$$

The volume, ΔV, of fluid that crosses $x = a$ with cross section area, ΔA, in the time interval Δt is

$$\Delta V \approx \vec{u} \cdot \vec{n} \Delta A \Delta t$$

So the mass of fluid is

$$\Delta m \approx \rho \vec{u} \Delta A \Delta t$$

$$\frac{\sum \Delta m}{\Delta t} = \sum \rho \vec{u} \cdot \vec{n} \Delta A$$

In the limit as $\Delta t \rightarrow 0$:

$$\frac{dm}{dt} = \iint \rho \vec{u} \cdot \vec{n} dA = \iint \vec{F} \cdot \vec{n} dA$$

This is the LHS of (1.2) which is equal to the time rate of change on fluid mass. If V is the total volume of gas the average value of the divergence of $\vec{F}$ is

$$\mathbf{div}(\vec{F})_{av} = \frac{1}{V}\iiint \nabla \cdot \vec{F} dV = \frac{1}{V}\iint \vec{F}\cdot\vec{n}dA = \frac{1}{V}\frac{dm}{dt} \tag{1.4}$$

But $1/V\,dm = d\rho$. If the gas is centred at some point q then in the limit in which the volume of gas approaches zero the far LHS of (1.4) converges to $(\nabla\cdot\vec{F})_q$ and the RHS converges to $-\frac{\partial\rho}{\partial t}_q$. The equality of these two yields the continuity equation.

$$\nabla\cdot\vec{F} + \frac{\partial\rho}{\partial t} = 0 \tag{1.5}$$

Since we are only considering a one dimensional gas the gradient can be replaced with a partial derivative in x. For the conservation of mass equation the flux function is $F(\rho) = \rho u$

$$\rho_t + (\rho u)_x = 0 \tag{1.6}$$

(1.6) is the one-dimensional continuity equation and applies to any quantitative physical parameter, that is to say ρ can be replaced by any conserved quantity. The continuity equation is a particular form of the more general conservation law equation, this states that the temporal profile of the density function of some conserved physical quantity is determined by the spatial profile of the flux of that quantity.

1.3   Constant Velocity Fluid Flow

Consider the special case for fluid flow with $u(x,t) = c \in \mathbf{R}$; in this case the equality of mixed derivatives can be exploited. Using (1.6)

$$\rho_t = -c\rho_x \Rightarrow \quad \rho_{tx} = -c\rho_{xx}$$
$$\rho_x = -\frac{1}{c}\rho_t \Rightarrow \rho_{xt} = -\frac{1}{c}\rho_{tt}$$
$$\rho_{tt} - c^2\rho_{xx} = 0 \tag{1.7}$$

This is the wave equation and can be solved via the method of separation of variables. Assume

$$\rho(x,t) = X(x)T(t)$$

(1.7) is rewritten as

$$XT'' - c^2 X''T = 0$$
$$\frac{T''}{T} - c^2\frac{X''}{X} = 0$$
$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = \lambda^2$$
$$\frac{d^2T}{dt^2} = c^2\lambda^2 T, \quad \frac{d^2X}{dx^2} = \lambda^2 X$$

Ansantz: $X(x) = Ae^{\alpha x}$. Works for $\alpha = \pm\lambda$. So $X(x)$ must have the form

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

It is expected that $T(t)$ has the same form; therefor

$$\rho(x,t) = A\exp[\lambda(x+ct)] + B\exp[-\lambda(x+ct)] + C\exp[\lambda(x-ct)] + D\exp[-\lambda(x-ct)]; \quad A,B,C,D,c,\lambda \in \mathbf{R} \tag{1.8}$$

[note that A,B,C,D are arbitrary constants and not necessarily the same as the ones used above.] This is the form of a combination of left and right traveling plane waves moving at a constant velocity c, this is the expected solution for (1.7). Assuming that the fluid is flowing to the right the solution becomes

$$\rho(x,t) = C\exp[\lambda(x-ct)] + D\exp[-\lambda(x-ct)] = \rho(x\hat{-}ct) \tag{1.9}$$

The value of the constants may be determined from the initial and boundary conditions. It is clear that $\rho(x,t)$ is constant along any ray for which $x - ct$ is constant, the rays in space-time $\chi(t) = x_0 + ct$ are known as the characteristics of the equation because $\rho(\chi, t) = \rho(\hat{x_0})$. Suppose the mass has some initial distribution

$$\rho(x, 0) = \rho_0(x) \tag{1.10}$$

But the solution must be constant along each characteristic so we are led to conclude

$$\rho(x, t) = \rho_0(x - ct); \quad t > 0 \tag{1.11}$$

that is to say the initial mass distribution remains constant in shape and translates through space at a constant velocity c.

Supposing a fluid travelling in a one dimensional vessel to the left with a constant velocity of $c = 1$ has an initial distribution $\rho_0(x) = 1 - \cos x$, with (1.11)

$$\rho(x, t) = 1 - \cos(x + t) \tag{1.12}$$

A graphical representation of (1.12) can be seen in part a) of Figure 1.2.

1.4 Spatially Variant Velocity

Consider now a fluid flowing with a velocity which is dependent on the position in space which for a one dimensional flowing fluid would be $u(x, t) = u(x)$. Using this in (1.6)

$$\rho_x u + \rho u_x + \rho_t = 0$$

$$u\rho_x + \rho_t = -u_x\rho \tag{1.13}$$

The left hand side of (1.13) has the form of a first order hyperbolic differential equation. The right hand side is a source term.

Consider the same fluid as earlier with the initial distribution $\rho_0(x) = 1 - \cos x$. only now the velocity is no longer a constant $c = 1$ to the left but a spatially variable $u(x) = x$, meaning the velocity increases linearly with distance along the vessel. Using these conditions in (1.13)

$$\rho_t + x\rho_x = -\rho; \quad \rho(x, 0) = \rho_0(x) = 1 - \cos x$$

In this situation the method of characteristics is most appropriate.

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = x, \quad \frac{d\rho}{ds} = -\rho$$

$$t = s + c_1, \quad \ln x = s + c_2, \quad \rho = Ae^{-s}; \quad A, c_1, c_2 \in \mathbf{R}$$

From the initial condition, $\rho(x_0, 0) = 1 - \cos(x_0)$ it is known

$$t(0) = 0 \Rightarrow c_1 = 0, \quad \ln x_0 = c_2 \Rightarrow x_0 = e^{c_2} = \exp(\ln x - s) = xe^{-s}$$

$$\rho(x_0, 0) = A = 1 - \cos(x_0) = 1 - \cos\left(xe^{-s}\right)$$

Hence the solution is

$$\rho(x, t) = \left[1 - \cos\left(xe^{-t}\right)\right] e^{-t} \tag{1.14}$$

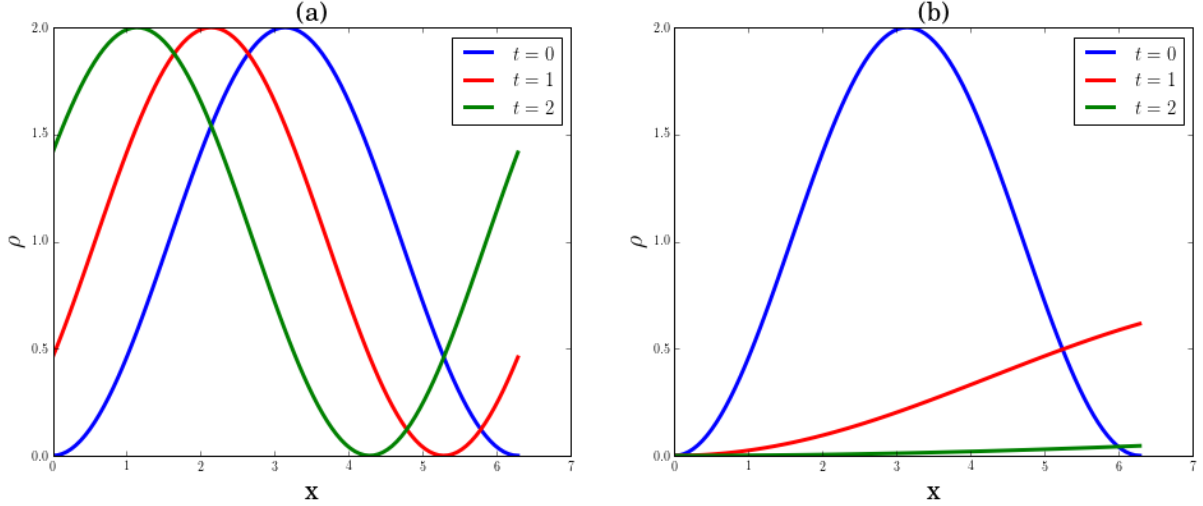See part b) in Figure 1.2 for a graphical representation of (1.14)

Figure 1.2: Solutions to the continuity equations of a one dimensional fluid for $0 \leq x \leq 2\pi$, $t = 0, 1, 2$; a) Fluid is moving with a constant velocity $u = -1$, b) Fluid velocity changes through space $u(x) = x$; [5].

1.5 Burgers' Equation

Thus far the only first order PDEs have been linear. The non linear form of the one dimensional continuity equation is

$$\rho_t + f(\rho)_x = 0 \tag{1.15}$$

For the Euler equations for Fluid dynamics the assumption of genuine nonlinearity holds, that is to say that the function $f(\rho)$ is smooth and $f''(\rho) > 0, \quad \forall u$, [4]. One of the more well known examples of the non linear conservation law is Burgers' equation, [4]

$$u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx} \tag{1.16}$$

Hoph and Cole showed (independently) that (1.16) can be reduced to the linear heat equation. The Hoph-Cole transformation requires rewriting Burgers' equation as

$$u_t = \left(\epsilon u_x - \frac{u^2}{2}\right)_x \tag{1.17}$$

and introduce a new variable

$$\phi(x,t) = \exp\left(-\frac{1}{2\epsilon}\int u dx\right) \tag{1.18}$$

so that $u(x,t)$ can be expressed as

$$u(x,t) = -2\epsilon\frac{\phi_x}{\phi}$$

$$\epsilon u_x = \epsilon\left[-2\epsilon\left(\frac{\phi\phi_{xx} - \phi_x^2}{\phi^2}\right)\right], \quad -\frac{1}{2}u^2 = -\frac{1}{2}\left(4\epsilon^2\frac{\phi_x^2}{\phi^2}\right) = -2\epsilon^2\frac{\phi_x^2}{\phi^2}$$

$$\left(\epsilon u_x - \frac{1}{2}u^2\right)_x = \left(-2\epsilon^2\frac{\phi_{xx}}{\phi}\right)_x = -2\epsilon^2\left(\frac{\phi\phi_{xxx} - \phi_{xx}\phi_x}{\phi^2}\right)$$

Using this in (1.17)

$$-2\epsilon\left(\frac{\phi\phi_{xt} - \phi_t\phi_x}{\phi^2}\right) = -2\epsilon^2\left(\frac{\phi\phi_{xxx} - \phi_{xx}\phi_x}{\phi^2}\right)$$

$$\phi^2\left(\frac{\phi_t}{\phi}\right)_x = \epsilon\phi^2\left(\frac{\phi_{xx}}{\phi}\right)_x$$

Cancel the ϕ^2s, integrate with respect to x and multiply by ϕ and one is left with the linear heat equation

$$\phi_t = \epsilon\phi_{xx}$$

This can be solved via separation of variables as shown earlier. However this solution was produced via a change of variable, (1.18) which requires $\epsilon \neq 0$. This is problematic if one wants the solution to the homogeneous form of (1.17), also known as the Inviscid Burgers' equation.

$$u_t + uu_x = 0 \tag{1.19}$$

With the method of characteristics assume $u = u(x(s), t(s))$ then

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u, \quad \frac{du}{ds} = 0$$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{u}{1} = u, \quad \frac{du}{dt} = \frac{du}{ds}\frac{ds}{dt} = 0$$

$$x = ut + C, \quad u = D; \quad C, D \in \mathbf{R}$$

$$D = f(C) = f(x - ut) = u(x, t)$$

One needs to know the initial conditions in order to know the exact form of f. Suppose we use the same initial distribution as in the cases of a constant velocity and a spatially dependent velocity, i.e. $\rho(x, 0) = 1 - \cos(x)$, then (going back to using $\rho(x, t)$ instead of u)

$$\rho(x, t) = 1 - \cos(x - \rho(x, t)t) \tag{1.20}$$

checking this with (1.19) shows (1.20) is a valid solution along the characteristic curves. The analytic solution to (1.19) may be more difficult to calculate explicitly, however it is not particularly complicated to find a numerical solution. Assuming $\rho(x, t) \approx w_{ij}$ the Inviscid Burgers' equation can be written as

$$\frac{w_{i,j+1} - w_{i,j}}{t_{j+1} - t_j} = -w_{i,j}\left[\frac{w_{i+1,j} - w_{i,j}}{x_{i+1} - x_i}\right]$$

assuming further that $x \approx x_i = ih$ and $t \approx t_j = kj$

$$\frac{w_{i,j+1} - w_{i,j}}{k} = -w_{i,j}\left[\frac{w_{i+1,j} - w_{i,j}}{h}\right]$$

Defining a new quantity $r = \frac{k}{h}$ and rearranging a few terms gives the finite difference equation for the Inviscid Burgers' equation

$$w_{i,j+1} = w_{i,j}[1 - rw_{i+1,j} + rw_{i,j}] \tag{1.21}$$

Using initial conditions $w_{i,0} = 1 - \cos x_i = 1 - \cos(ih)$ the numerical solution can be represented graphically for $0 \leq j \leq 4$, see Figure 1.3.
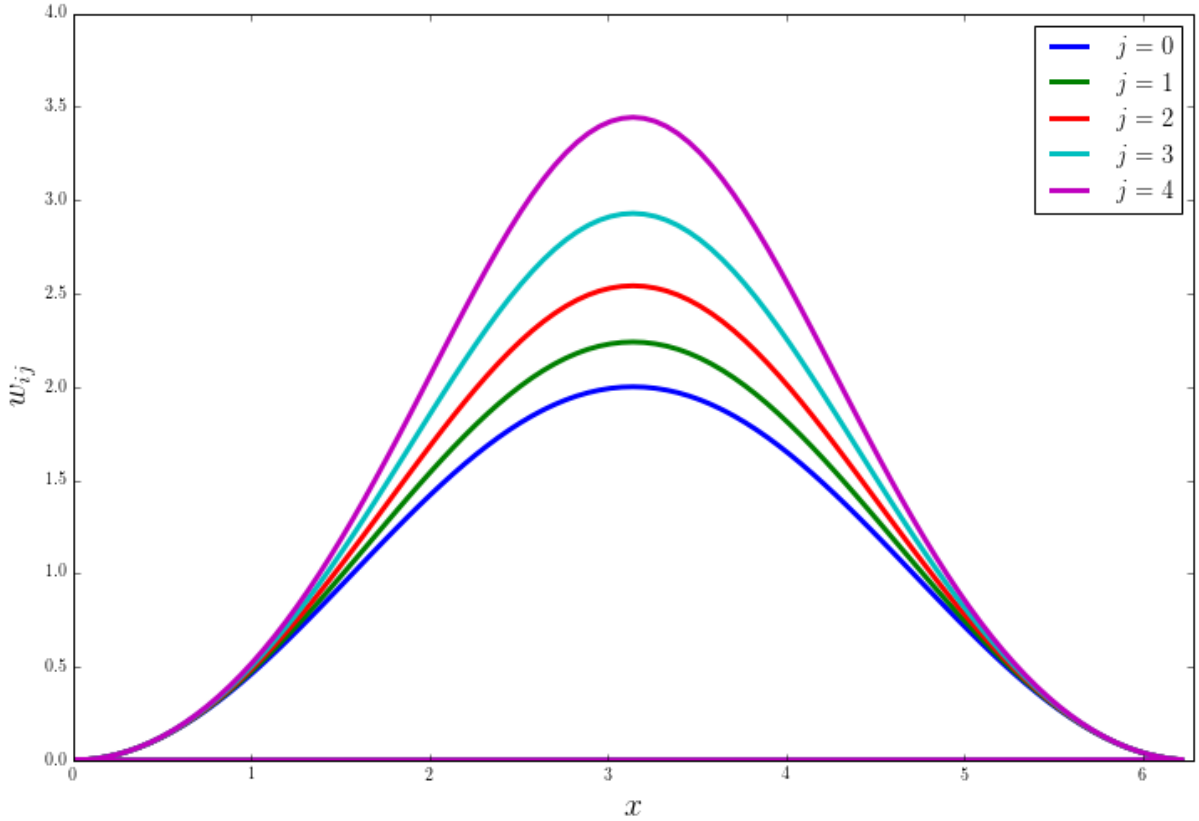
Figure 1.3: A numerical solution for the Inviscid Burgers' equation using (1.21), $0 \leq i \leq 2\pi$, $0 \leq j \leq 4$, $r = 0.06$; [5], see Appendix A.1.

1.6 Conclusions

We have seen that the divergence theorem gives rise to a general conservation law, (1.5), which we know as the continuity equation which governs the spatial and temporal dynamics of a physical quantity. When the dynamical fluid under investigation is restricted to one dimensional flow it is governed by the one dimensional continuity equation (1.6); for an incompressible fluid flowing at a constant velocity the equality of mixed partial derivatives reduces the continuity equation to the wave equations and thus has solutions in the form $g(x - ct)$. We also produced solutions for a particular case in which the fluid velocity is not constant in space but is proportional to the distance travelled relative to some reference position, $u(x) = x$, which had solutions given by (1.14). Visual representations of solutions to one dimensional continuity equation for a fluid moving at a) a constant velocity and b) a spatially dependent velocity were shown in Figure 1.2. In § 1.5 we looked at an example of a non linear example of the continuity equation, Burgers' equation. Using the Hoph-Cole transformation analytical solutions to the non homogeneous form of Burgers' equation can be produced by reducing it to the linear heat equation. These solutions could not work for the homogeneous form however using the method of characteristics an implicit solution was found for the given initial conditions. An explicit numerical solution to the Inviscid Burgers' equation was produced, (1.21), and graphed in Figure 1.3.

Mass is not the only conserved quantity, any physical parameter can be conserved under the right conditions. In fact for fluid in equilibrium the flux of mass, momentum, will be conserved as will be discussed next in § 2.

2   Conservation of Momentum

2.1   Introduction

From Newton's second law of motion it is known that the rate of change of momentum is equal to the net force on a body, thus for a fluid with no external unbalanced forces present momentum should be conserved. This next section will now apply some of the arguments used for finding the conservation equation for mass to momentum for a fluid flowing in one dimension. Just as the fluid's mass has a density function $\rho(x,t)$ which describes how the quantity is distributed in space and time so too will the fluid's momentum have a density function $q(x,t)$. Furthermore the momentum of the fluid will also have a flux function which describes the magnitude and direction of the flow of the quantity. Hence the one dimensional continuity equation (1.6) still holds with ρ replaced with q. In § 1 the flux function for a one dimensional flow was defined as the product of the density function and the velocity. Momentum is defined as the product of mass and velocity $q(x,t) = \rho(x,t)u(x,t)$. The momentum flux will be $(\rho u)u = \rho u^2$. However this flux only describes the bulk movement of the fluid, that is to say if every molecule was moving at the same speed. At the microscopic level molecules move in random directions over short distances before colliding with other molecules or container walls at large velocities. The random path each molecule traces drifts at the advective velocity u but the large velocities within each random walk are impossible to determine. For convenience we use a new quantity, pressure $p(x,t)$, to represent the microscopic contribution to momentum flux. Thus the flux of momentum must include a term for pressure $p(x,t)$. The conservation law for momentum will be

$$(\rho u)_t + (p + \rho u^2)_x = 0 \tag{2.1}$$

This equation can be coupled with (1.6) but if we are to find an explicit analytic solution it will be necessary to express p in terms of ρ; to this end we need a equation of state for the fluid.

2.2   Ideal Gas

An equation of state is a functional relationship relating the physical parameters of a body. The physical parameters most relevant and most understood are temperature, T, pressure, p, and volume, V. For a fluid an equation of state will have the general form

$$f(p, V, T) = 0$$
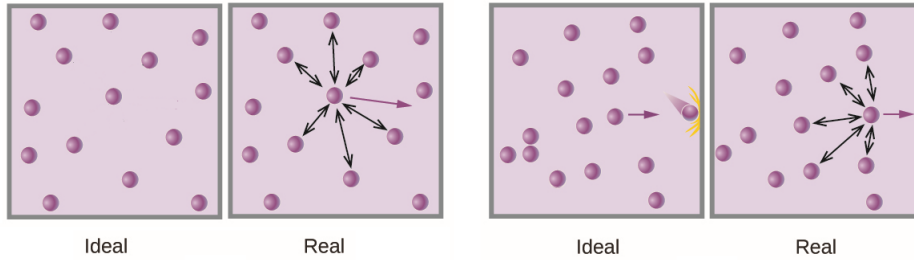
Figure 2.1: The ideal gas model makes some unrealistic assumptions about the behaviour of gases but is a good approximation at high temperatures. In the ideal gas collisions between molecules and walls and other molecules are completely elastic, Coulomb forces are zero and molecules are treated a point particles with zero volume.

The best known of all equations of state is the ideal gas equation. The ideal gas, as the name suggests, is a theoretical gas; one in which all particles in the gas have zero volume no intermolecular interactions, see Figure 2.1. The ideal gas is realistically impossible but for gases at very low pressures and high temperature it is a good approximation. The ideal gas equation of state states that the volume of a mole of gas will be directly proportional to temperature and inversely proportional to pressure [7], i.e. for n moles of ideal gas

$$pV = nRT \tag{2.2}$$

R is a constant of proportionality known as the gas constant it's value is $8.314 \text{ J} \cdot \text{mole}^{-1} \cdot \text{K}^{-1}$. If we assume that the gas is homogeneous with a molecular mass, M, then the mass of n moles of gas will be $m = nM$. The density function can be expressed as $\rho(x,t) = n(x,t)M$ and the ideal gas equation of state is rewritten as

$$p = \rho \hat{R} T \tag{2.3}$$

where $\hat{R} = \frac{R}{M}$, this can be used in (2.1)

$$(\rho u)_t + (\rho u^2 + \rho \hat{R} T)_x = 0 \tag{2.4}$$

$$\rho_t u + \rho u_t + \rho_x (u^2 + \hat{R}T) + \rho(2uu_x + \hat{R}T_x) = 0$$

$$(-\rho_x u - \rho u_x)u + \rho u_t + \rho_x(u^2 + \hat{R}T) + \rho(2uu_x + \hat{R}T_x) = 0$$

$$\rho u_t + \rho u u_x + \rho_x \hat{R}T + \rho \hat{R}T_x = 0$$

$$u_t + uu_x + \frac{\hat{R}T}{\rho}\rho_x + \hat{R}T_x = 0 \tag{2.5}$$

For the case of constant velocity flow momentum is just a scalar multiple of mass and so (2.4) reduces to the continuity equation, (1.6), and the solution os also a scalar multiple of $\rho(x,t)$. Note that (2.5) is a conservation equation for velocity in an ideal gas but T_x is not known; to complete the conservation equation we must impose some conditions on the ideal gas. The first condition we will look at is constant entropy.

2.3 Isentropic Ideal gas

Entropy is a quantitative measure of the disorder within a system or how much it is changing. If there are no shock waves, or more realistically the volume of the system is sufficiently large that any shock waves are negligible, the data for the system is smooth and the entropy is considered constant. For a gas in which the entropy is constant the pressure is related to mass by

$$p = \kappa \rho^\gamma \tag{2.6}$$

κ is a physical parameter related to entropy and is thus constant under isentropic conditions, γ is the ratio of heat capacities. The relationship can be used in (2.1)

$$(\rho u)_t + (\kappa \rho^\gamma + \rho u^2)_x = 0$$

$$\rho_t u + \rho u_t + \rho_x u^2 + 2uu_x \rho + \kappa \gamma \rho^{\gamma-1}\rho_x = 0$$

but $\rho_t u + \rho_x u^2 = u(-\rho u_x)$ and in an isentropic media the speed of sound is

$$c^2 = \frac{\partial p}{\partial \rho} = \kappa \gamma \rho^{\gamma-1} = \frac{\gamma p}{\rho} \tag{2.7}$$

$$u_t + uu_x + c^2\frac{\rho_x}{\rho} = 0 \tag{2.8}$$

When this equation is coupled with the continuity equation they may be expressed in matrix form

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} u & \rho \\ c^2/\rho & u \end{bmatrix}\begin{bmatrix} \rho \\ u \end{bmatrix}_x = 0 \tag{2.9}$$

This can be expressed as $q_t + f'(q)q_x = 0$. In many investigations of dynamical systems we are interested in small perturbations about some fixed state. In such cases it more practical to linearize the system about a fixed constant state q_0. By letting

$$q(x,t) = q_0 + \tilde{q}(x,t)$$

here $q_0 = (\rho_0, u_0)$ and $\tilde{q}$ is the perturbation about q_0. Hence the isentropic system can be expressed as $\tilde{q}_t + f'(q_0)\tilde{q}_x = 0$ and the isentropic system (2.9) becomes

$$\begin{bmatrix} \tilde{\rho} \\ \tilde{u} \end{bmatrix}_t + \begin{bmatrix} u_0 & \rho_0 \\ c_0^2/\rho_0 & u_0 \end{bmatrix} \begin{bmatrix} \tilde{\rho} \\ \tilde{u} \end{bmatrix}_x = 0 \tag{2.10}$$

The equation is said to be hyperbolic if the Jacobian matrix $A = f'(q_0)$ can diagonalized and has real eigenvalues.

$$det(A - \lambda I) = \begin{vmatrix} u_0 - \lambda & \rho_0 \\ c_0^2/\rho_0 & u_0 - \lambda \end{vmatrix} = (u_0 - \lambda)^2 - c_0^2 = 0$$

$$\lambda_1 = u_0 - c_0, \quad \lambda_2 = u_0 + c_0 \tag{2.11}$$

$$B_1 = A - \lambda_1 I = \begin{bmatrix} c_0 & \rho_0 \\ c_0^2/\rho_0 & c_0 \end{bmatrix}, \quad B_2 = A - \lambda_2 I = \begin{bmatrix} -c_0 & \rho_0 \\ c_0^2/\rho_0 & -c_0 \end{bmatrix}$$

If r_1 and r_2 are the eigenvectors then: $B_1 r_1 = 0 = B_2 r_2$.

$$\begin{bmatrix} c_0 & \rho_0 \\ c_0^2/\rho_0 & c_0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow c_0 x + \rho_0 y = 0 \Rightarrow x = -\frac{\rho_0}{c_0} y$$

repeating this method for r_2 will give us the eigenvectors

$$r_1 = \begin{bmatrix} -\rho_0/c_0 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} \rho_0/c_0 \\ 1 \end{bmatrix} \tag{2.12}$$

2.4 Integral Curves and Riemann Invariants

Consider the general quasilinear hyperbolic system with a Jacobian matrix $f'(q) : \mathbf{R}^m \to \mathbf{R}^{m \times m}$

$$q_t + f'(q)q_x = 0; \quad q \in \mathbf{R} \times (0, \infty) \tag{2.13}$$

it is possible to generate solutions, $v : \mathbf{R} \to \mathbf{r}^m$, in the form of travelling waves propagating a speed σ, i.e.

$$q(x,t) = v(x - \sigma t); \quad x \in \mathbf{R}, \quad t > 0 \tag{2.14}$$

By substituting the wave solution expression (2.14) into the PDE (2.13)

$$-\sigma v'(x - \sigma t) + f'(v(x - \sigma t))v'(x - \sigma t) = 0$$

$$f'(v')v' = \sigma v'$$

Keeping in mind that f' is a matrix then it will have eigenvalues σ and eigenvectors v'. Hence there exists wave solutions of the PDE found by integrating the right eigenvectors. Considering the isentropic system introduced in § 2.3 there will be set of parametric curves in ρ-u space over which pressure and velocity are constant. These curves are defined by the ordinary differential equation,

$$\frac{dv_i}{ds} = r^i \tag{2.15}$$

For the linearised system this is quite simple

$$dv^1 = \begin{bmatrix} d\rho \\ u \end{bmatrix} = \begin{bmatrix} -\rho_0/c_0 \\ 1 \end{bmatrix} ds$$

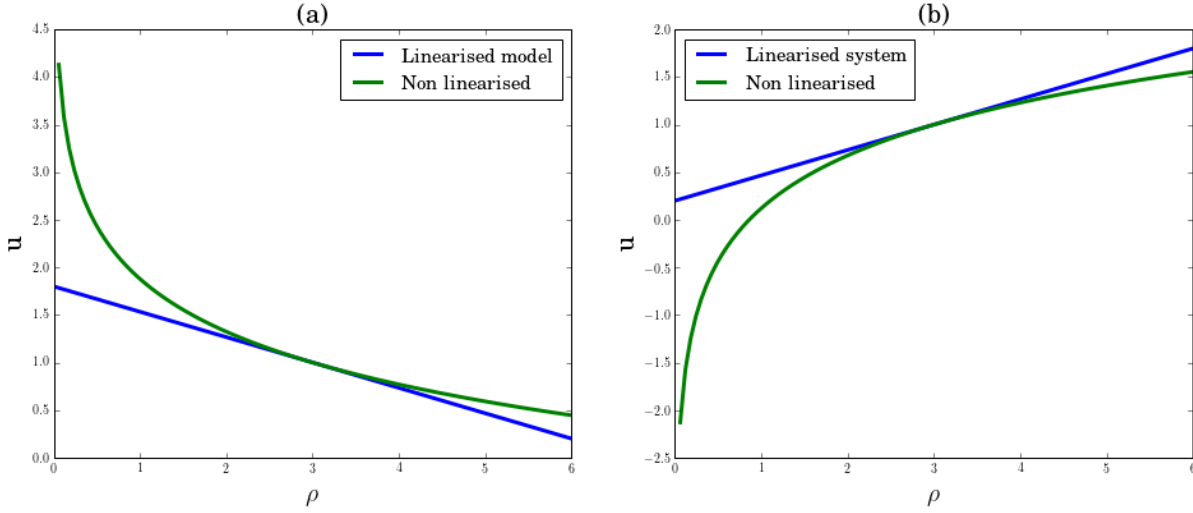$$\rho = \frac{c_0}{\rho_0} s + \text{constant}$$

Figure 2.2: A comparison of integral curves in $\rho - u$ space of the Euler equations under isentropic conditions when the state variables are linearised about some fixed state and not. In $a)$ we see the locus a states which can be connected to $(\rho_0, u_0) = (3, 1)$ by a simple 1-wave and in $b)$ a simple 2-wave.

$$u = s + \text{constant}$$

$$u = -\frac{c_0}{\rho_0}\rho + \text{constant}$$

but at the fixed state (ρ_0, u_0)

$$u_0 = -\frac{c_0}{\rho_0}\rho_0 + \text{constant} \Rightarrow \text{constant} = u_0 + c_0$$

$$u = -\frac{c_0}{\rho_0}\rho + u_0 + c_0$$

The integral curves for the linearised system are just straight lines with a slope $\pm c_0/\rho_0$ and intercept $u_0 \pm c_0$, i.e.

$$v^1 = \begin{bmatrix} \rho \\ -\frac{c_0}{\rho_0}\rho + u_0 + c_0 \end{bmatrix} \tag{2.16a}$$

$$v^2 = \begin{bmatrix} \rho \\ \frac{c_0}{\rho_0}\rho + u_0 - c_0 \end{bmatrix} \tag{2.16b}$$

v^i is in fact the locus of all states (ρ, u) which can be connected to a reference state (ρ_0, u_0) by simple i-wave. It should be kept in mind that it is assumed that c is constant, but it actually determined by the relation (2.7); if $c(\rho_0) = c_0$ then $c = c_0(\rho/\rho_0)^{\frac{\gamma-1}{2}}$. For the non linear isentropic system the solution to this ODE, (2.15) is a little more complex

$$dv^1 = \begin{bmatrix} d\rho \\ du \end{bmatrix} = \begin{bmatrix} -\rho/c \\ 1 \end{bmatrix} ds$$

Integrating both sides gives

$$\begin{bmatrix} \int \frac{d\rho}{\rho} \\ \int du \end{bmatrix} = \begin{bmatrix} -\frac{1}{c}\int ds \\ ds \end{bmatrix} \Rightarrow \begin{bmatrix} \ln \rho \\ u \end{bmatrix} = \begin{bmatrix} -\frac{1}{c}s + C_1 \\ s + C_2 \end{bmatrix}; \quad C_1, C_2 \in \mathbf{R}$$

$$\ln \rho = -\frac{1}{c}(u - C_2) + C_1$$

$$\ln \rho = D - \frac{u}{c}; \quad D \in \mathbf{R}$$

Taking some reference point (ρ^*, u^*)

$$\ln \rho^* = D - \frac{u^*}{c}$$

$$\ln \rho = \ln \rho^* + \frac{u^*}{c} - \frac{u}{c}$$

The integral curve v_1 and v_2 will be

$$v_1 = \begin{bmatrix} \rho \\ u^* - c \ln \left(\frac{\rho}{\rho^*} \right) \end{bmatrix} \tag{2.17a}$$

$$v_2 = \begin{bmatrix} \rho \\ u^* + c \ln \left(\frac{\rho}{\rho^*} \right) \end{bmatrix} \tag{2.17b}$$

Figure 2.2 shows a comparison of the integral curves for the linearised and non-linearised isentropic system taking $(\rho^*, u^*) = (\rho_0, u_0) = (3, 1)$; see that the linearised model is a good approximation near the reference state. In any hyperbolic PDE $q_t + f(q)_x = 0$ there are functions of q that are constant along any integral curve of each family, these functions are Riemann invariants. The integral curves are defined by the Riemann invariants, w_i, orthogonal to the right eigenvectors. The Riemann invariants are defined as such, [8]

$$\nabla_v w_i \cdot r_i = 0 \tag{2.18}$$

which can be rewritten as

$$\frac{\partial w_i}{\partial v_i} \cdot \frac{dv_i}{ds} = 0$$

$$\frac{dw_i}{ds} = 0$$

So these Riemann invariants are constant across the same parametrization of the integral curves. The Riemann invariants for the system (2.9) can be obtained by rearranging (2.17a) and (2.17b)

$$u + c \ln \rho = u^* + c \ln \rho^*$$
$$u - c \ln \rho = u^* - c \ln \rho^*$$

$$w_1 = u + c \ln \rho \tag{2.19a}$$

$$w_2 = u - c \ln \rho \tag{2.19b}$$

Keeping in mind that c is given by (2.7). It is easy to check that these choice of Riemann invariants satisfy (2.18)

$$\begin{bmatrix} w_\rho^1 & w_u^1 \end{bmatrix} \cdot r_1 = \begin{bmatrix} c/\rho & 1 \end{bmatrix} \begin{bmatrix} -\rho/c \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} w_\rho^2 & w_u^2 \end{bmatrix} \cdot r_2 = \begin{bmatrix} -c/\rho & 1 \end{bmatrix} \begin{bmatrix} \rho/c \\ 1 \end{bmatrix} = 0$$

If the columns of the matrix R are called the right eigenvectors, r^i, for the matrix $f'(q)$; it will also have left eigenvectors which are the rows of R^{-1}. In the case studied here

$$l_1 = \frac{1}{2} \begin{bmatrix} -c/\rho \\ 1 \end{bmatrix}, \quad l_2 = \frac{1}{2} \begin{bmatrix} c/\rho \\ 1 \end{bmatrix} \tag{2.20}$$

Notice that $l_i^T \cdot r_j = \delta_{ij}$. The eigenvalues are said for form a complete orthonormal set.

The characteristic curves of the isentropic system are defined by the ordinary differential equation

$$\frac{dx}{dt} = \lambda_i$$

So taking the dot product of the left eigenvectors with the ith characteristic curve

$$i = 1 : l_1 \cdot \frac{dq}{dt} = \frac{1}{2} \begin{bmatrix} -\frac{c}{\rho} & 1 \end{bmatrix} \begin{bmatrix} d\rho/dt \\ du/dt \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\frac{c}{\rho} & 1 \end{bmatrix} \begin{bmatrix} -\cancel{u\rho_x} - \rho u_x + \cancel{u\rho_x} - c\rho_x \\ -c^2/\rho\rho_t - \cancel{uu_x} + \cancel{uu_x} - cu_x \end{bmatrix}$$

$$= \frac{1}{2} \left[cu_x + c^2/\rho\rho_x - c^2/\rho\rho_x - uu_x + uu_x - cu_x \right] = 0$$

This can be shown to be true for $i = 2$, in general

$$l_i^T \cdot (q_t - \lambda_i q_x) = 0; \quad \forall i$$

The ith characteristic curve is orthogonal to the ith right eigenvector. But from (2.18) we know that r_i is orthogonal to the surface defined by the Riemann invariant, w_i. It is intuitive that the characteristic curve lies in that surface, i.e. along the ith characteristic curve

$$l_i \frac{dv_i}{dt} \propto \frac{dw_i}{dt} \tag{2.21}$$

We will prove this for the case of the isentropic system now

$$i = 1 : \quad \frac{d}{dt} v_1 = \begin{bmatrix} d\rho/dt \\ -c/\rho d\rho/dt \end{bmatrix} = \begin{bmatrix} \rho_t + \lambda_1 \rho_x \\ -c/\rho(\rho_t + \lambda_1 \rho_x) \end{bmatrix} = \begin{bmatrix} -\cancel{u\rho_x} - \rho u_x + \cancel{u\rho_x} - c\rho_x \\ -c/\rho(-\rho u_x - c\rho_x) \end{bmatrix}$$

$$l_1 \frac{dv_1}{dt} = \frac{1}{2} \begin{bmatrix} -c/\rho & 1 \end{bmatrix} \begin{bmatrix} -\rho u_x - c\rho_x \\ cu_x + c^2/\rho\rho_x \end{bmatrix} = \frac{1}{2} [cu_x + c^2/\rho\rho_x + cu_x + c^2/\rho\rho_x] = cu_x + \frac{c^2}{\rho}\rho_x$$

$$\frac{dw_1}{dt} = w_t^1 + \lambda^1 w_x^1 = \left[u_t + \frac{c}{\rho}\rho_t \right] + (u - c) \left[u_x + \frac{c}{\rho}\rho_x \right] =$$

$$-uu_x - \frac{c^2}{\rho}\rho_x + \frac{c}{\rho}(-u\rho_x - u_x\rho) + uu_x - cu_x + \frac{uc}{\rho}\rho_x - \frac{c^2}{\rho}\rho_x = -2cu_x - 2\frac{c^2}{\rho}\rho_x$$

$$\Rightarrow \frac{dw^1}{dt} = -2c \left(u_x + \frac{c}{\rho}\rho_x \right) = -2c \left(l^1 \cdot \frac{dv^1}{dt} \right)$$

and if we used $i = 2$ we would have found

$$\frac{dw^2}{dt} = -2c \left(l^2 \cdot \frac{dv^2}{dt} \right)$$

so (2.21) holds for the isentropic system.

2.5 Isothermal ideal gas

One of the important assumptions of the ideal gas is the absence of intermolecular interactions. If the electrostatic forces between gas molecules are neglected then the potential energy arising from these forces disappears and the internal energy is purely kinetic. According the Boltzmann the average kinetic energy of a gas with α degrees of freedom is

$$\frac{1}{2} m v_{\text{average}}^2 = \frac{\alpha}{2} N_A k_b T = \frac{\alpha}{2} RT \tag{2.22}$$

N_A is Avogadro's number and k_b is Boltzmann's constant. Later in § 4.2 we will encounter a new parameter, the specific heat capacity which is the energy per unit mass per unit volume required to raise the temperature by $1K$.

$$C_V \equiv \left(\frac{\partial e}{\partial T} \right)_V = \frac{de}{dT} \tag{2.23}$$

The first law of thermodynamic states that

$$de = dQ + dW \tag{2.24}$$

for an infinitesimal reversible process

$$dW = -PdV \quad \text{and} \quad dQ = TdS$$

so the first law of thermodynamics can be stated as

$$TdS = de + PdV$$

$$TdS = C_V dT + PdV$$

$$TdS = C_V dT + \frac{RT}{V} dV$$

$$dS = C_V \frac{dT}{T} + R \frac{dV}{V}$$

Integrating, we have an expression for the entropy per mole

$$S = C_V \ln T + R \ln V + S_0 = C_V \ln \left(\frac{p}{\rho^\gamma} \right) + S_0 \tag{2.25}$$

In we are considering a fixed volume then $S = S(T)$ so for an ideal gas the isentropic conditions require constant temperature for an ideal gas. For a fixed volume of ideal gas with α degrees of freedom (2.22) and (2.23) give us the specific heat capacity

$$c_v = \frac{\alpha}{2} R \tag{2.26}$$
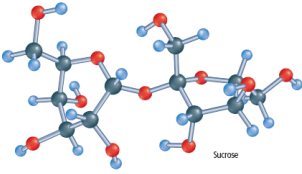
Figure 2.3: Sucrose molecule

All gas molecules have a minimum of three degrees of freedom, one for each axis in three dimensional space. Additionally simple molecules will also be able to rotate about an axis and vibrate along intermolecular bonds. For more complex molecules such as Sucrose, Caffeine and most organic molecules the number of available bonds and axes available for individual atoms to rotate and vibrate increase significantly. As molecules become increasingly complex $\alpha \to \infty$ and from (2.26) the energy required to increase temperature increases. For these complex molecules small changes in internal energy produce little change in temperature and the gas is considered isothermal.

But according to the equation of state, (2.3), for an ideal gas at constant temperature, pressure will be a scalar multiple of mass.

$$p = \rho RT \tag{2.27}$$

The speed of sound in an ideal gas at constant temperature, a, is

$$a^2 = \frac{\partial p}{\partial \rho} = RT \tag{2.28}$$

So for a constant temperature the speed of sound in the media will be constant; this is the real difference between isentropic and isothermal conditions. Using (2.27) in the equation for conservation of momentum the equations for isothermal flow are

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}_x = 0 \tag{2.29}$$

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or in the form of a quasilinear hyperbolic system

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} u & \rho \\ a^2/\rho & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = 0 \tag{2.30}$$

Note that the conservation equation for velocity is (2.5) with $T_x = 0$ as expected for isothermal conditions. (2.30) can be linearised as we did in § 2.3 to give a system of equations similar to (2.10) but with c replaced with a (the units of a will also be in m/s). Hence we can evaluate the eigenvalues, eigenvectors and integral curves for this system just as was done in § 2.3.

$$\lambda_1 = u - a, \quad \lambda_2 = u + a \tag{2.31a}$$

$$r_1 = \begin{bmatrix} -\rho/a \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} \rho/a \\ 1 \end{bmatrix} \tag{2.31b}$$

$$u = u_0 - a \ln\left(\frac{\rho}{\rho_0}\right) \tag{2.32a}$$

$$u = u_0 - a \ln\left(\frac{\rho}{\rho_0}\right) \tag{2.32b}$$

(2.32a) and (2.32b) are the set of states which can be connected to the reference state (ρ_0, u_0) via a 1-wave and 2-wave respectively. Because, mathematically, the difference between the isothermal and isentropic systems is a matter of using a^2 instead of c^2 Recall for an isentropic gas the speed of sound is

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{\partial}{\partial \rho}(\kappa \rho^\gamma) = \frac{\kappa \gamma \rho^\gamma}{\rho} = \gamma \frac{p}{\rho}$$

If the gas is ideal then

$$c^2 = \gamma \frac{\rho R T}{\rho} = \gamma \hat{R} T = \gamma a^2$$

So as $\gamma \to 1$ an isentropic system approaches an isothermal system (for an ideal gas that is). In fact

$$\lim_{\gamma \to 1} p_{\text{isentropic}} = \kappa \rho$$

$$\lim_{\gamma \to 1} \kappa = RT$$

2.6   Isobaric Ideal gas

Another possible condition to consider is that the ideal gas is at a constant pressure, $p = p_0$. Using this condition in (2.1)

$$(\rho u)_t + (p_0 + \rho u^2)_x = 0$$

$$\rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x = 0$$

$$\rho u_t + \rho u u_x = 0$$

$$u_t + u u_x = 0 \tag{2.33}$$

Combining (2.33) with (1.6)

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} u & \rho \\ 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \end{bmatrix}_x = 0$$

This system of conservation equations has an eigensystem

$$\lambda_1 = \lambda_2 = u \tag{2.34}$$

$$r^1 = r^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{2.35}$$

2.7 The Shallow Water Equations

Consider if we have a liquid fluid in vessel for which the width is equal to unity and assume that the direction of flow is restricted to one dimension. Because the fluid is incompressible, i.e. the mass density is constant, the total mass between two points x_1 and x_2 at time t will be

$$\int dm = \int_{x_1}^{x_2} \rho(x,t)h(x,t)dx$$

$h(x,t)$ is the height of the fluid in the vessel; this quantity is variable over space and time because although the direction of flow is only in one dimension mechanical waves can travel through the media in the vertical direction.

This information can be used in (1.6):

$$(\rho h)_t + (\rho u h)_x = 0$$

$$\rho_t h + \rho h_t + \rho h u_x + \rho h_x u + \rho_x h u = 0 \tag{2.36}$$

Similarly conservation of momentum is now

$$(\rho h u)_t + (p + \rho h u^2)_x = 0$$

If the fluid is incompressible then $\rho(x,t) = \bar{\rho}$ and hence $\rho_t = \rho_x = 0$. By dividing all terms in (2.36) by ρ we have a new conservation equation

$$h_t + (hu)_x = 0 \tag{2.37}$$

To analyse the conservation of momentum equation further it will be necessary to get a relation for pressure, $p(x,t)$. Pressure is defined as the force per unit area; the force in this case is simply the weight of the overlying fluid.

$$p = \frac{W}{A} = \frac{mg}{A} = \frac{\bar{\rho}Ah(x,t)g}{A} = \bar{\rho}gh(x,t)$$

The pressure at a vertical column at position x and time t is calculated by integrating pressure over $h(x,t)$

$$p = \int_0^{h(x,t)} \bar{\rho}gh'dh' = \frac{1}{2}\bar{\rho}gh^2(x,t)$$

The conservation of momentum becomes

$$(\bar{\rho}hu)_t + \left(\frac{1}{2}\bar{\rho}gh^2 + \bar{\rho}hu^2\right)_x = 0$$

This can be reduced to a conservation equation for fluid velocity as such

$$\bar{\rho}hu_t + \bar{\rho}h_t u + g\bar{\rho}hh_x + \bar{\rho}h_x u^2 + 2\bar{\rho}huu_x = 0$$

Dividing all terms by $\bar{\rho}$ and using $h_t u + u^2 h_x = -huu_x$

$$hu_t + h_t u + ghh_x + huu_x = 0$$

By dividing all terms by h we have a new conservation equation for fluid velocity
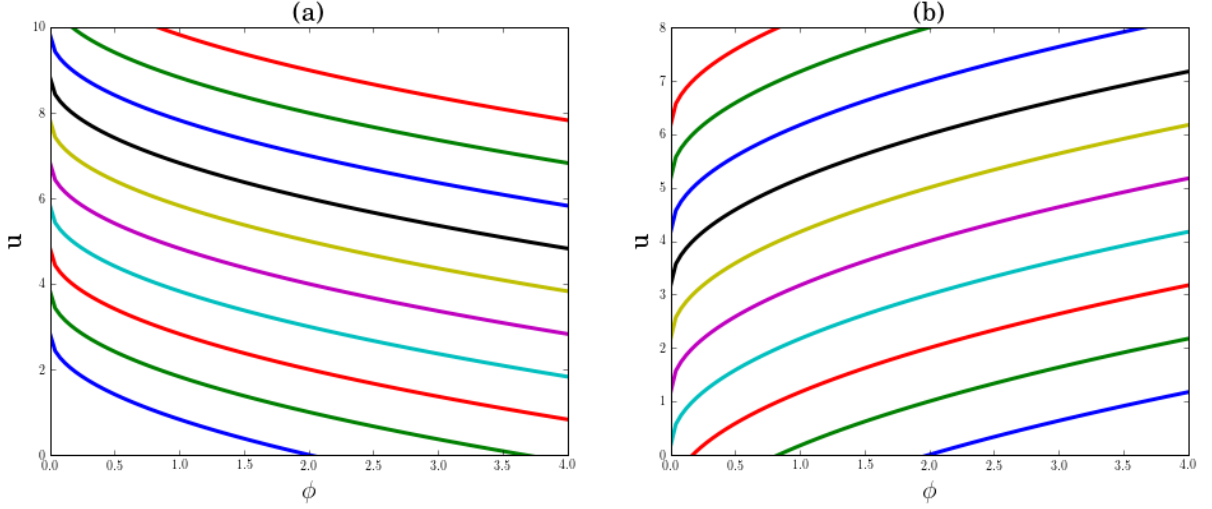
Figure 2.4: Integral curves of the shallow water equations in $\phi - u$ space. In a) we see the locus of all states which can be connected to some reference state (ϕ_0, u_0) by a simple 1-wave and in b) a simple 2-wave.

$$u_t + gh_x + uu_x = 0 \tag{2.38}$$

This equation can be combined with (2.36) to form a quasilinear hyperbolic equation

$$\begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} u & h \\ g & u \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix}_x = 0 \tag{2.39}$$

The eigenvalues and eigenvectors for this system are

$$\lambda_1 = u - gh, \quad \lambda_2 = u + gh \tag{2.40}$$

But if we try to get the eigenvectors of (2.39) we need to find an x and y such that

$$\begin{bmatrix} gh & h \\ g & gh \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

but there is no such combination of x and y. In order to find the correct eigenvalues and eigenvectors of the shallow water system it is necessary to make a transformation $\phi = gh$. By multiplying each term in (2.37) by g the system (2.39) can be rewritten as

$$\begin{bmatrix} \phi \\ u \end{bmatrix}_t + \begin{bmatrix} u & \phi \\ 1 & u \end{bmatrix} \begin{bmatrix} \phi \\ u \end{bmatrix}_x = 0 \tag{2.41}$$

For this system it can be shown that the eigenvalues and eigenvectors are

$$\lambda_1 = u - \sqrt{\phi}, \quad \lambda_2 = u + \sqrt{\phi} \tag{2.42}$$

$$r_1 = \begin{bmatrix} -\sqrt{\phi} \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} \sqrt{\phi} \\ 1 \end{bmatrix} \tag{2.43}$$

The integral curves for this system will be

$$v_1 = \begin{bmatrix} \phi \\ u^* + 2(\sqrt{\phi^*} - \sqrt{\phi}) \end{bmatrix}, \quad v_2 = \begin{bmatrix} \phi \\ u^* + 2(\sqrt{\phi} - \sqrt{\phi^*}) \end{bmatrix} \tag{2.44}$$

Figure 2.4 shows the integral curves for the shallow water equations. These curves will be defined by the Riemann variables

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$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u + 2\sqrt{\phi} \\ u - 2\sqrt{\phi} \end{bmatrix} \tag{2.45}$$

One of the assumptions upon which these equations were based was that the width of the container was fixed at unity. If the units of measurement of distance in the problem are defined to be equal to the width of the container then this assumption holds true. Thus the shallow water equation can describe the fluid dynamics of a liquid in a container with a fixed width.

2.8 Characteristic Variables

It is convenient to write the variables of the system as a product of the eigenvectors and some function, ω, which will be known as the characteristic variables of the system so that $q^i = w^i r^i$. The isentropic system (2.9) can be reduced to a system of advection equations by expressing it in terms of the characteristic variables $\omega = R^{-1}q$, here R is the adjoint matrix of the eigenvectors.

$$\omega_t + \Lambda \omega_x = 0 \tag{2.46}$$

Because Λ is the diagonal matrix with the diagonal entry of row i equal to λ_i, (2.46) represents a decoupled system of advection equations, i.e. the original system has been decoupled into simple waves, w^i, propagating at speeds given by the ith eigenvalue. Note that $\Lambda = R^{-1}AR \Rightarrow A = R\Lambda R^{-1}$, the matrix A is said to diagonalizable. Hence (2.9) represent a hyperbolic system.

$$\omega(x,0) = R^{-1}q(x,0)$$

The solution of the initial value problem in terms of the characteristic variables is quite simple, the *pth* equation of (2.46) is

$$\omega^p(x,t) = \omega(x - \lambda^p t, 0)$$

The solution can now be expressed in terms of the original conservative variables with the transformation $q = R\omega$:

$$q(x,t) = \sum_{p=1}^{m} \omega^p(x - \lambda^p t, 0) r_p \tag{2.47}$$

2.9 Conclusions

We began this section by assuming that the momentum a fluid is a conserved quantity is should thus be governed by the continuity equation. Although momentum is in fact the flux function for mass the conservation equation for momentum does not simply reduce to the equation for conservation of mass because there is an additional microscopic component to it's flux at the endpoints of the region of interest which is represented by the fluid pressure. This led us to the equation for conservation of momentum, (2.1). We applied this equation first to an ideal gas, an unrealistic but good approximation model for the behaviour of gases under conditions of low pressure characterised by the equation of state (2.3). In § 2.3 we showed how the equation for conservation of momentum can be reduced to conservation of velocity under conditions of constant entropy. Coupling this new conservation law for velocity with the continuity equation the system of equation can be represented in matrix form as a quasilinear hyperbolic system of equations (2.9) with eigenvalues, (2.11), and eigenvectors, (2.12). Once the eigensystem was obtained it was possible to generate the characteristic curves of the isentropic system, (2.17a), (2.17b); these curves were shown in Figure 2.2. It was also shown how the Riemann invariants are related to the integral curves. A special case of an isentropic ideal gas is an isothermal gas which will have it's own quasilinear hyperbolic system of equations identical to the isentropic case but with the speed of sound, c, replaced with $a = a(T)$. In § 2.7 we dropped the assumption that the fluid under inspection was compressible and applied our conservation laws to a simple case of an incompressible fluid: the shallow water equations. It was shown that the shallow water equations could be represented by the quasilinear hyperbolic system (2.41) with eigenvalues (2.42), eigenvectors (2.43) and Riemann invariants (2.45); the characteristics curves of the shallow water equations can be seen in Figure 2.4. Finally we saw how a system of hyperbolic equations could be decoupled into a system of advection equations, (2.46), by defining a new quantity called the characteristic variables

and shown how the solution to the original equations can be expressed in terms of these characteristic variables. It has been shown that the initial data is important in determining the time evolution of the solution. Hitherto only smooth distributions have been considered at $t = 0$. In reality most physical systems will not have perfectly continuous data throughout space for any instance in time but will have jumps and falls. In § 3 a simple case of solution to a system with a discontinuity in the initial data will be discussed.

3   The Riemann problem

3.1   Introduction

Riemann's problem concerns a system of conservation laws with piecewise-constant initial data

$$q_t + f(q)_x = 0 \tag{3.1a}$$

$$q(x,0) = \begin{cases} q_l & \text{if} \quad x < 0 \\ q_r & \text{if} \quad x > 0 \end{cases} \tag{3.1b}$$

$q_l \neq q_r \in \mathbf{R}$ are the left and right initial states. It is possible to find a unique state q^* such that

$$\min(q_l, q_r) < q^* < \max(q_l, q_r)$$

but first it is necessary to define a criteria called the Rankine-Hugoniot jump condition. Once this condition has been defined we will discuss the requirements for a particularly simple solution to the Riemann problem which we refer to as simple waves. These simple waves will come in two flavours: rarefaction and shock waves which can be illustrated as a "spreading out" and "squeezing together" respectively of characteristics. q^* will be connected to a state $q_1 > q^*$ by a rarefaction wave and to a state $q_0 < q^*$ by a shock wave. Thus the state q^* can be found from the intersection of the Hugoniot loci of the rarefaction ans shock waves. The early parts of this section will discuss the conditions which must be met for the Hugoniot locus of these waves for three different assumptions: isentropic, isothermal and shallow water. With these conditions defined we can start to look at finding exact solutions to the Riemann problem.

3.2   The Rankine-Hugoniot Jump Condition

Consider if we were to multiply the one dimensional continuity equation, (1.6), by some smooth test function v and integrate by parts to obtain the equality

$$\int_0^\infty \int_{-\infty}^\infty \rho \cdot v_t + (\rho u) \cdot v_x \, dx dt + \int_{-\infty}^\infty \rho(x,0) \cdot v dx|_{t=0} = 0 \tag{3.2}$$

Any ρ that satisfies (3.2) for all v is said to be an integral solution of the initial value problem. If ρ is smooth on either side of some curve C along which ρ has simple jump discontinuities and if C cuts some region V, over which ρ is smooth, into two regions V_l and V_r then the first term in (3.2) becomes, [10]

$$\iint_{V_l} \rho \cdot v_t + (\rho u) \cdot v_x \, dx dt + \iint_{V_r} \rho \cdot v_t + (\rho u) \cdot v_x \, dx dt = 0$$

Because v has compact support within V

$$\iint_{V_l} \rho \cdot v_t + (\rho u) \cdot v_x \, dx dt = -\iint_{V_l} [\rho_t + (\rho u)_x] \cdot v dx dt + \int_C (\rho_l \nu^2 + (\rho_l u_l) \nu^1) \cdot v dl = \int_C (\rho_l \nu^2 + (\rho_l u_l) \nu^1) \cdot v dl$$

Here $\nu = (\nu^1, \nu^2)$ is the unit outward normal to the curve C. Similarly for the region V_r

$$\iint_{V_l} \rho \cdot v_t + (\rho u) \cdot v_x \, dx dt = \int_C (\rho_l \nu^2 + (\rho_l u_l) \nu^1) \cdot v dl$$

So over the entire region of V

$$\int_C [(\rho_l u_l - \rho_r u_r) \nu^1 + (\rho_l - \rho_r) \nu^2] \cdot v dl = 0$$

This integral can equal zero if and only if the integrand is equal to zero

$$(\rho_l u_l - \rho_r u_r) \nu^1 + (\rho_l - \rho_r) \nu^2 = 0; \quad \text{along } C \tag{3.3}$$

Now suppose that the curve C is parametrized as $(s(t), t)$ then this means

$$\nu = (\nu^1, \nu^2) = \left(\frac{1}{\sqrt{1 + \dot{s}}}, \frac{-\dot{s}}{\sqrt{1 + \dot{s}}} \right)$$

Thus in the region V along the curve C (3.3) reads

$$\rho_l u_l - \rho_r u_r = \dot{s}(\rho_l - \rho_r) \tag{3.4}$$

Using a new notation: $[[q]]$ represents the jump in some quantity q across the curve C which is moving at a speed $\dot{s} = \sigma$

$$[[\rho u]] = \sigma [[\rho]] \tag{3.5}$$

If we are interested in some general conserved quantity q with a flux $F(q)$ then the Rankine-Hugoniot jump condition is

$$[[F(q)]] = \sigma [[q]] \tag{3.6}$$

The curve C is called a shock wave and the value of $u_{l,r}$, $F(u_{l,r})$ and the speed of it's propagation will vary along C but their balance described by (3.6) will remain.

3.3 Simple Waves

Consider the hyperbolic quasilinear form of the continuity equation (1.6), [11].

$$q_t + f'(q) q_x = 0 \tag{3.7}$$

there exist a real valued smooth eigenpair $(\lambda^i, r^i) = (\lambda^i(q), r^i(q))$ (λ and r are known as scalar and vector fields) such that

$$f'(q^i) r^i(q) = \lambda^i r^i(q); \quad \forall q$$

Keeping in mind that $r(q) \in \mathbf{R}^m$, $\lambda(q) \in \mathbf{R}$ and the Jacobian matrix $f'(q) \in \mathbf{R}^{m \times m}$. For every λ-characteristic family of integral curves it is possible to construct a simple waves solution for (3.7).

The simple wave principle states that: *a λ-simple wave solution for the quasilinear hyperbolic continuity equation is constructed from the integral curves of $r(q)$ if each state q propagates in the $x - t$ plane with velocity $dx/dt = \lambda(q)$.* Another way of stating this is to say that the conservation law will have solutions of the form

$$q(x, t) = \tilde{q}(s(x, t)) \tag{3.8}$$

As in § 2.4 it is assumed that the integral curves are parametrized by some quantity s; (2.15) can be expressed differently as

$$\frac{dq}{ds} = v^i(s) = \alpha(s) r^i(s) \tag{3.9}$$

If λ changes monotonically along the integral curve then we can take $s = \lambda$ and the integral curve can be parametrized by $v^i = v^i(\lambda^i)$, in such a case the eigenpair (λ, r) is said to be genuinely non-linear. This will require that the directional derivative of the eigenvalue in the direction of the eigenvector be non-zero everywhere. When the opposite is true, that is to say the directional derivative is equal to zero everywhere then λ is constant along the integral curve and the eigenpair is called linearly degenerate. If a system is linear degenerate then the parameter s must solve the linear transport equation

$$s_t + \lambda^i s_x = 0 \tag{3.10}$$

The variation of λ^i with the parameter s along the integral curve is

$$\frac{d\lambda^i}{ds} = \frac{\partial \lambda^i}{\partial q}\frac{\partial q}{\partial s} = \alpha(s)\nabla\lambda^i \cdot r^i(s)$$

If $\nabla\lambda^i \cdot r^i(s) = 0$ then $\lambda^i \in \mathbf{R}$, $\forall\, x, t$ and the field is said to be linearly degenerate, for now we will ignore this possibility and return to it in § 4.7. But if $\nabla\lambda^i \cdot r^i(s) \neq 0$ then the wave is genuinely non-linear. If initial conditions are such that

$$\frac{d}{ds}\lambda^i(s(x,0)) > 0; \qquad \forall x \tag{3.11}$$

then the characteristics will be spreading out, this is known as a rarefaction wave. But if

$$\frac{d}{ds}\lambda^i(s(x,0)) < 0; \qquad \forall x \tag{3.12}$$

then the characteristics will be converging, intersecting at a shock wave. Thus the solution to the Riemann problem will be, depending on the data, either a shock wave or a rarefaction wave. We will now look at the former case.

3.4   Shock Waves

The other form of simple waves solution to the Riemann problem are compression waves which are better known as shock waves. As with rarefaction waves, shock waves arise when $f(q)_x$ in

$$q_t + f(q)_x = 0$$

is non linear and consequently the eigenvalues will be functions of x and t, solutions cannot simple translate uniformly but instead deforms. By plotting characteristics of the solution for a range of values of x and t one can see shock formation where the characteristics collide, this is a defining feature of non-linear hyperbolic systems of equations. Of course physically realistic shock waves will not be sharp discontinuities but smooth transitions over narrow regions of space. Suppose we look at a shock wave over such a narrow region of space Δx and some time increment Δt. Now for a non linear flux function $f(q)$ the conservation law is
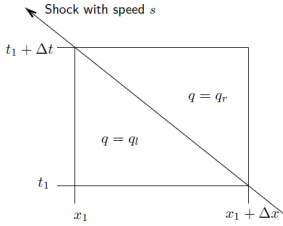

Figure 3.1: A shock wave over a narrow region of space, [1].

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = f(q(x_1,t) - f(q(x_2,t)) \tag{3.13}$$

Applying (3.13) to the situation seen in Figure 3.1.

$$\int_{x_1}^{x_1+\Delta x} q(x,t_1+\Delta t)dx - \int_{x_1}^{x_1+\Delta x} q(x,t_1)dx = \int_{t_1}^{t_1+\Delta t} f(q(x_1,t)dt - \int_{t_1}^{t_1+\Delta t} f(q(x_1+\Delta x,t)dt)$$

Because q is constant everywhere along the line representing the shock wave the integral equality can be written as

$$\Delta x q_r - \Delta x q_l = \Delta t f(q_l) - \Delta t f(q_r)$$

If the velocity of the shock wave is

$$v = -\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t}$$

(a $-$ is used in this definition because the shock in Figure 3.1 is propagating to the left) then the conservation law can be stated as

$$-v(q_r - q_l) = f(q_l) - f(q_r)$$

But this is actually the Rankine-Hugoniot jump condition (3.5) so the velocity of shock wave can be expressed as

$$v = \frac{f(q_l) - f(q_r)}{q_l - q_r} \tag{3.14}$$

If $q_{l,r} = q_{l,r}(x,t)$ the speed will change as time evolves, i.e. $v = v(x,t)$. But in the Riemann problem $q_{l,r}$ are constant so the shock wave will translate with constant speed. Note also that

$$\lim_{q_l \to q_r} v = f'(q_l)$$

This is the condition for weak shock waves. As an example let us apply (3.14) to the Burgers' equation for which $f(u) = 1/2u^2$

$$v = \frac{1/2u_l^2 - 1/2u_r^2}{u_l - u_r} = \frac{1}{2}(u_l + u_r)$$

and if $u_l \approx u_r$ then $v = u_l = f'(u_l)$.

If the problem involves a system of equations then $q_{l,r}$ will be a vector. For a linear system $f(q) = Aq$, in which case the speed of the shock wave will be an eigenvalue of the matrix A. As mentioned earlier the solution to the Riemann problem will either be a shock or rarefaction wave, the initial data will determine which. If $u_l > u_r$ the solution to (3.1) will be

$$q(x,t) = \begin{cases} q_l & x < vt \\ q_r & x > vt \end{cases}; \quad x \in \mathbf{R}, t > 0 \tag{3.15}$$

here v is given by (3.14). This solution is the unique entropy solution to the Riemann problem.

Shock waves are illustrated as a convergence of characteristics. Let us continue with the example of the Inviscid Burgers' equation with smooth initial data

$$u_t + uu_x = 0$$

the characteristic equations will be $X'(t) = u(x,t)$. If $x(t)$ is a solution to this ODE then $u(X(t),t)$ is the restriction of u to this curve.
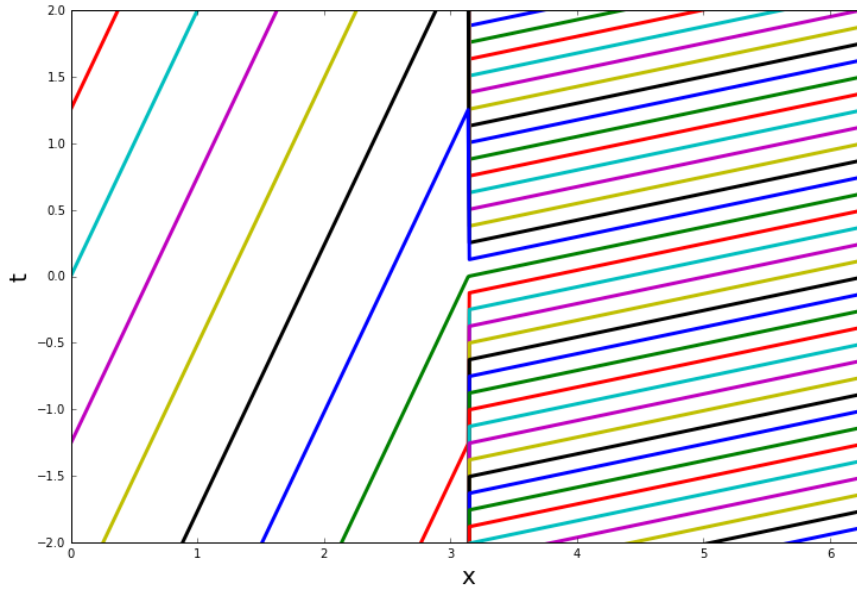

Figure 3.2: Characteristics in the $x - t$ plane for the Inviscid Burgers' equation. The characteristics are converging due to the discontinuity in the initial data creating shock waves.

$$\frac{d}{dt}u(X(t),t) = u_x\frac{dX}{dt} + u_t = uu_x + u_t = 0$$

Thus $u(X(t),t) = $ constant $\quad \forall t$. But this means that $u(X(t),t) = u(X(0),0)$ so if the initial data is known then the characteristic equation is $X'(t) = u(x_0,0)$. The solution to this ODE will be $X(t) = x_0 + u(x_0,0)t$. It is preferred to have the x on the horizontal so this will be rearranged as

$$t = \frac{1}{u(x_0,0)}(x - x_0) \tag{3.16}$$

Supposing we use Riemann problem initial data

$$u(x,0) = \begin{cases} 0.5 & \text{if} \quad x < \pi \\ 5 & \text{if} \quad x > \pi \end{cases} \tag{3.17}$$

Figure 3.2 shows the characteristics of the Inviscid Burgers' equation using (3.17).

3.5 Entropy Conditions

There are some stability conditions which shock waves must meet. A propagating discontinuity with characteristics coming out of it would be unstable, the shock needs to have characteristics coming into it. For a convex scalar conservation law (3.1a), such as Burgers' equation, a propagating discontinuity with velocity given by the Rankine-Hugoniot jump condition (3.14) must satisfy the Lax entropy condition

$$f'(q_l) < v < f'(q_r) \tag{3.18}$$

Recall that $f'(q)$ is the characteristic speed λ^i. But if v satisfies the Rankine-Hugoniot jump condition then the Lax entropy condition reduces to $f'(q_l) < f'(q_r)$. In the particular case of Burgers' equation

$$uu_x = (\frac{1}{2}u^2)_x = f(u)_x \Rightarrow f''(u) = 1 > 0; \qquad \forall u$$

so Lax requires $u_l > u_r$, such as the case in (3.17). We can now consider the problem of determining all the states q that can be connected to a fixed state q_0 by a shock wave. Considering the system of equations for the flow of an ideal gas in one dimension

$$v(\rho - \rho_0) = \rho u - \rho_0 u_0$$
$$v(\rho u - \rho_0 u_0) = \rho u^2 + p - \rho_0 u_0^2 - p_0$$

By rearranging the jump condition for ρ

$$v = \frac{\rho u - \rho_0 u_0}{\rho - \rho_0}$$

and substituting into the condition for momentum yields, see Appendix B

$$u_1 = u_0 - \sqrt{\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(p - p_0)} \tag{3.19a}$$

$$u_2 = u_0 + \sqrt{\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(p - p_0)} \tag{3.19b}$$

An isentropic ideal gas, by definition, has no shock waves although in the Riemann problem the initial data is constant on either side of the discontinuity so the gas may have constant entropy across the jump but a different constant entropy. For this reason we will use the isentropic relation $p = \rho c^2/\gamma = \kappa \rho^\gamma$. Gases composed of different molecules will have different values of γ. The parameter, γ, known as the adiabatic exponent or heat capacity ration is related to the degrees of freedom. Without getting into the specifics

$$\gamma \approx \begin{cases} 1.67 & \text{for monatomic gases} \\ 1.4 & \text{for diatomic gases} \\ 1.33 & \text{for polytomic gases} \end{cases}$$

Recall also from § 2.5 that an isentropic gas approaches isothermal conditions as $\gamma \to 1$.
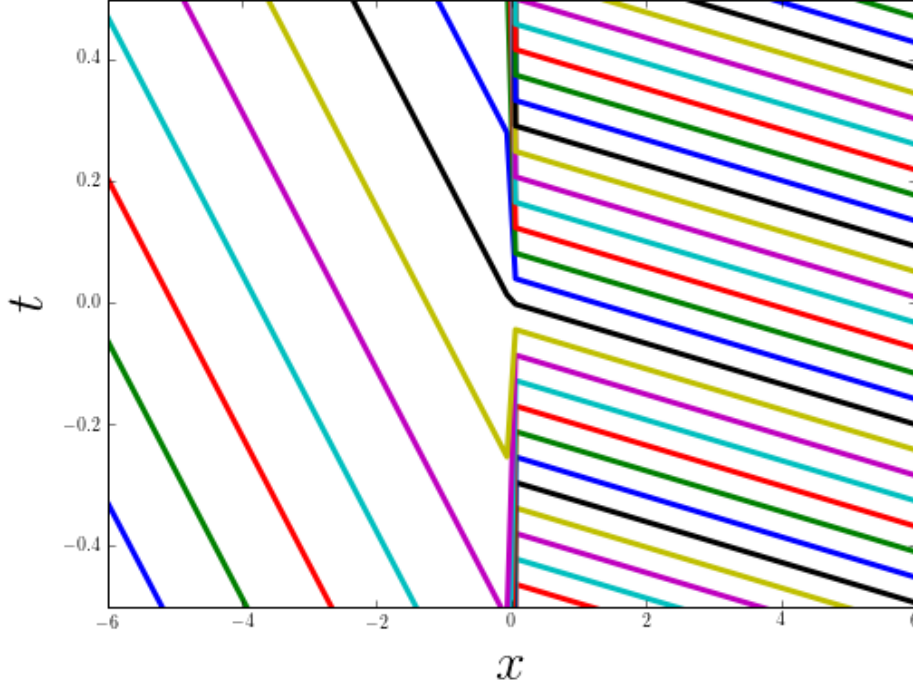
Figure 3.3: Characteristics of the Euler equations in the $x - t$ plane for an ideal gas under isothermal conditions. The converging of the lines is due to the shock wave generated by initial data.

The Lax entropy conditions for an isothermal system will also be of the form (3.19a)-(3.19b) with the difference that $p = \rho R T$. If there is a known relation between p and ρ and the initial conditions for mass, $\rho(x, 0)$, are given then the entropy conditions, (3.19), can be used to determine the slope of the characteristics in (3.16). Figure 3.3 shows these characteristics in the $t - x$ plane for the initial data

$$\rho(x, 0) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$$

Shock waves can also appear in shallow water systems. The analysis used in the isothermal system an be applied to the shallow water equations, i.e.

$$h_t + (hu)_x = 0$$
$$(hu)_t + (hu^2 + 1/2gh^2)_x = 0$$

Using the same process as in § 3.5 then Lax entropy condition for the shallow water equations will be

$$u = u^* \pm \sqrt{\frac{g}{2}\left(\frac{h}{h^*} - \frac{h^*}{h}\right)(h - h^*)} \tag{3.20}$$

See Appendix B for the derivation.

3.6   Rarefaction waves

Rarefaction is the opposite of compression this occurs when the distance between molecules increases. Fluid flow can become rarefied when the characteristic speed, λ^i, is decreasing with increasing x. Because the characteristic speed is an eigenvalue of the non-linear Jacobian, $f'(q)$, it should be expected that the characteristics are not straight lines. When we look at families of λ-characteristics there will be a visible line where the curves will change, this is the rarefaction wave. For rarefaction waves the solution is constant along any ray x/t, hence the solution can be expressed as

$$q(x,t) = \tilde{q}(x/t)$$

Using this in (3.7)

$$-\frac{x\tilde{q}'}{t^2} + \frac{f'(\tilde{q})\tilde{q}'}{t} = 0; \quad t \neq 0$$

Unless $\tilde{q}$ is a constant solution

$$f'(\tilde{q}(x/t)) = \lambda^i(\tilde{q}(x/t)) = x/t$$

But rarefaction waves require a genuinely non-linear system which as was mentioned in § 3.3 requires $\lambda = s$

$$s = \lambda^i(\tilde{q}(s))$$

differentiating with respect to s

$$1 = \nabla\lambda^i(\tilde{q}(s)) \cdot \frac{d\tilde{q}}{ds}$$

But from (3.9)

$$1 = \nabla\lambda^i(\tilde{q}(s)) \cdot \alpha(s)r^i(s)$$

$$\alpha(s) = \frac{1}{\nabla\lambda^i(\tilde{q}(s)) \cdot r^i(s)}$$

Using this expression for α in (3.9)

$$\frac{dq}{ds} = \frac{1}{\nabla\lambda^i(\tilde{q}(s)) \cdot r^i(s)}r^i(s) \tag{3.21}$$

3.6.1   Rarefaction waves for the Isentropic System

We can apply the analysis from § 3.6 to the non-linearized isentropic eigenpair. Taking the gradient of the eigenvalues (2.11)

$$\nabla\lambda^1 = \begin{bmatrix} \lambda^1_\rho & \lambda^1_u \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad \nabla\lambda^2 = \begin{bmatrix} \lambda^2_\rho & \lambda^2_u \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

With (2.12)

$$\nabla\lambda^1 \cdot r^1 = \nabla\lambda^2 \cdot r^2 = 1$$

Thus according to (3.21) the rarefaction curves are

$$\frac{dq^1}{ds} = r^1; \quad \frac{dq^2}{ds} = r^2$$

$$\begin{bmatrix} d\rho \\ du \end{bmatrix} = \begin{bmatrix} -\rho/cds \\ ds \end{bmatrix}; \quad \begin{bmatrix} d\rho \\ du \end{bmatrix} = \begin{bmatrix} \rho/cds \\ ds \end{bmatrix}$$

$$du = ds = \frac{c}{\rho}d\rho = \sqrt{\kappa\gamma}\rho^{\frac{\gamma-3}{2}}d\rho$$

$$u = u^* + \sqrt{\kappa\gamma}\frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}} = u^* + \frac{2\sqrt{\kappa\gamma}}{\gamma-1}\left[\rho^{\frac{\gamma-1}{2}}\right]^\rho_{\rho^*}$$

Therefore the rarefaction conditions for an isentropic system

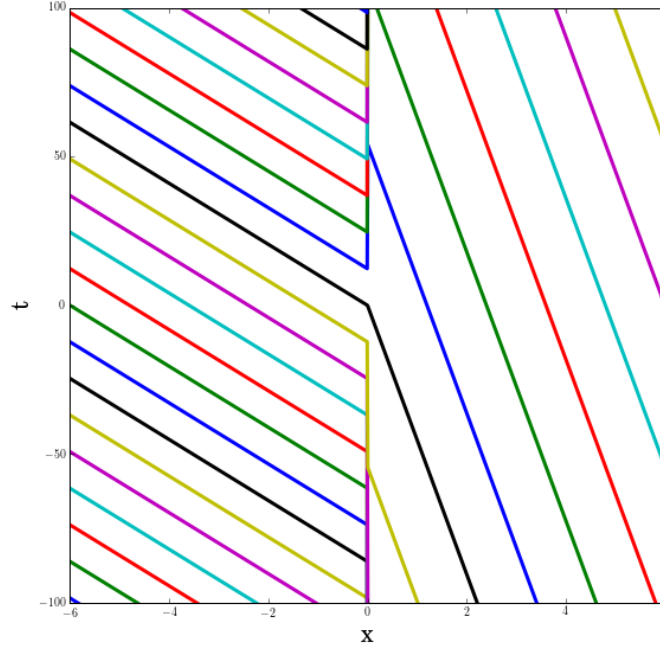$$u_1 = u^* - \frac{2}{\gamma-1}[c(\rho) - c(\rho^*)] \tag{3.22a}$$

Figure 3.4: Characteristics of the Euler equations in the $x - t$ plane under isentropic conditions. The spreading out of the lines is due to the rarefaction wave generated by the initial data; see Appendix A.2.

$$u_2 = u^* + \frac{2}{\gamma - 1}[c(\rho) - c(\rho^*)] \tag{3.22b}$$

For given initial data

$$\rho(x, 0) = \begin{cases} 2 & x < 0 \\ 1 & x > 0 \end{cases}$$

the isentropic conditions (3.22a), (3.22b) can be used in (3.16) to see the characteristics of the system shown in Figure 3.4 which shows the characteristics are spreading out downstream of the rarefaction, as opposed to in Figure 3.3 in which the characteristics are compressed downstream of the shock wave.

3.6.2   Rarefaction Waves for the Shallow Water Equations

Our analysis of simple rarefaction waves can be extended to the shallow water equations:

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + (hu^2 + 1/2gh^2)_x &= 0 \end{aligned} \tag{3.23}$$

It was shown in § 2.7 that this system of equations could be expressed as a quasilinear hyperbolic system

$$\begin{bmatrix} \phi \\ u \end{bmatrix}_t + \begin{bmatrix} u & \phi \\ 1 & u \end{bmatrix} \begin{bmatrix} \phi \\ u \end{bmatrix}_x = 0$$

with eigenvalues $\lambda_{1,2} = u \mp \sqrt{\phi}$ and eigenvectors

$$r_1 = \begin{bmatrix} -\sqrt{\phi} \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} \sqrt{\phi} \\ 1 \end{bmatrix}$$

We will now perform the same analysis on this system as with the isentropic system in § 3.6.1.

$$\nabla \lambda^1 = \begin{bmatrix} \lambda^1_\phi & \lambda^1_u \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\sqrt{\phi}} & 1 \end{bmatrix}$$

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$$\nabla\lambda^2 = \begin{bmatrix} \lambda_\phi^2 & \lambda_u^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{\phi}} & 1 \end{bmatrix}$$

$$\nabla\lambda^1 \cdot r^1 = \nabla\lambda^2 \cdot r^2 = \frac{3}{2} \tag{3.24}$$

Using (3.24) and (2.43) in (3.21) the 1-rarefaction waves are

$$\frac{d\phi}{ds} = -\frac{2}{3}\sqrt{\phi}$$

$$\frac{du}{ds} = \frac{2}{3}$$

$$d\phi = -\frac{2}{3}\sqrt{\phi}ds = -\sqrt{\phi}du$$

$$-\phi^{-1/2}d\phi = du$$

$$u = u^* - 2\left[\phi^{1/2}\right]_{\phi^*}^{\phi}$$

Thus the conditions for a rarefaction wave in the shalllow water equations are

$$u_1 = u^* - 2\sqrt{g}(\sqrt{h} - \sqrt{h^*}) \tag{3.25a}$$

$$u_1 = u^* + 2\sqrt{g}(\sqrt{h} - \sqrt{h^*}) \tag{3.25b}$$

3.6.3   Rarefaction Waves for an Isothermal System

We have seen in § 2.5 that the conservation laws for an ideal gas at constant temperature can be represented by a quasilinear hyperbolic system of equations (2.30) with an eigensystem given by (2.31a), (2.31b). Since, as discussed earlier, the isothermal system of equations is very similar to the isentropic system we know that the Riemann invariants are

$$w_1 = u + a\ln\rho \tag{3.26a}$$

$$w_2 = u - a\ln\rho \tag{3.26b}$$

$$\nabla\lambda^1 = \begin{bmatrix} \lambda_\rho^1 & \lambda_u^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} = \nabla\lambda^2$$

$$\nabla\lambda^1 \cdot r^1 = \nabla\lambda^2 \cdot r^2 = 1$$

The family-1 and family-2 rarefaction curves for the isothermal system of equations are

$$\text{Family-1:}\quad \frac{d}{ds}\begin{bmatrix} \rho \\ u \end{bmatrix} = \begin{bmatrix} -\rho/a \\ 1 \end{bmatrix}, \qquad \text{Family-2:}\quad \frac{d}{ds}\begin{bmatrix} \rho \\ u \end{bmatrix} = \begin{bmatrix} \rho/a \\ 1 \end{bmatrix}$$

Recall that $a^2 = RT$ is constant for constant temperature. The conditions for a rarefaction wave in an isothermal ideal gas are

$$u_1 = u^* - a\ln\left(\frac{\rho}{\rho^*}\right) \tag{3.27a}$$

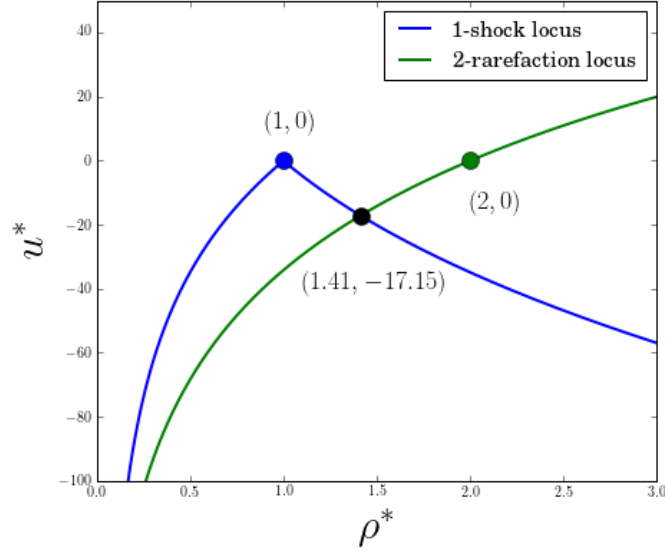$$u_2 = u^* + a\ln\left(\frac{\rho}{\rho^*}\right) \tag{3.27b}$$

Figure 3.6: The Hugoniot loci of the Euler equations applied to an ideal gas under isothermal conditions for the 1-rarefaction and 2-shock waves. The intersection of the curves gives us the intermediary state.
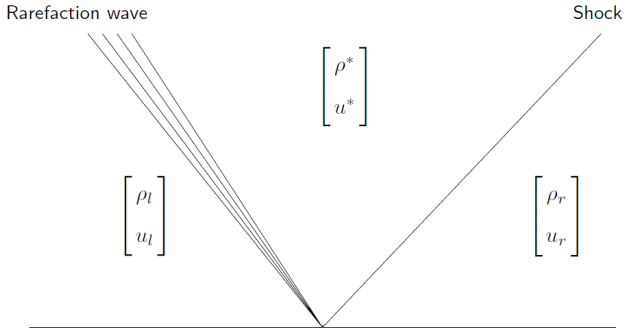
3.7 Exact Solutions to the Riemann Problem

Figure 3.5: The general form of a solution for of the Riemann problem defined by the Euler equations. Although the graphic shows a rarefaction-shock solution it will be the same for the other three types of solution.

As mentioned earlier the solution to the Riemann problem will be a combination of a shock wave and rarefaction wave. There will be four possible solutions: rarefaction-rarefaction, rarefaction-shock, shock-rarefaction and shock-shock, Figure 3.5 shows the general form of the rarefaction-shock solution. Which of the four depends on the initial data, if the initial data for mass is given then the solution will be

$$\begin{cases} \text{rarefaction-shock} & \rho_l < \rho_r \\ \text{shock-rarefaction} & \rho_l > \rho_r \end{cases}$$

and the state (ρ^*, u^*) will be the intersection of the curves derived in the earlier parts of this section. In Figure 3.5 the state (ρ^*, u^*) is the state which is connected to (ρ_l, u_l) by a 1-rarefaction wave and to the state (ρ_r, u_r) by a 2-shock wave. Consider the case of an ideal gas at a constant temperature of 293 K and the initial distribution of mass and velocity is

$$\rho(x,0) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}; \qquad u(x,0) = 0 \tag{3.28}$$

As the solution will be shock-rarefaction the state (ρ^*, u^*) will solve the equations

$$u^* = 0 + \sqrt{(8.314)(293)} \ln\left(\frac{\rho^*}{2}\right)$$

$$u^* = 0 - \sqrt{(8.314)(293)} \sqrt{\left(\frac{1}{1} - \frac{1}{\rho^*}\right)(\rho^* - 1)}$$

Solutions are plotted in Figure 3.6 and show that for the initial data given the state $(\rho^*, u^*) \approx (1.41, -17.15)$. Once this state has been found we can start to think about how the data changes with time. For this it is necessary to know the speed of the wave fronts; along a shock wave the speed is given by the Rankine-Hugoniot jump condition (3.5), the speed of the rarefaction wave front is the speed of the fluid relative to the speed of the media, i.e. the speed of some wave front v^i connecting (ρ_0, u_0) to (ρ^*, u^*) will be

$$v^i = \begin{cases} \frac{\rho_0 u_0 - \rho^* u^*}{\rho_0 - \rho^*} & \rho_0 < \rho^* \\ u_0 - c & \rho_0 > \rho^* \end{cases} \tag{3.30}$$

This is easy to compute in the isothermal case because the speed of sound will be constant but this will not hold in isentropic conditions or the shallow water equations. To the left of the 1-wave front the data will take the value q_l, along the 1-wave the data will be q_1 which can be deduced from the Riemann invariants of the shock or rarefaction wave. Between the 1- and 2-wave the data will be q^*, along the 2-wave the data will be q_2 and to the right of the 2-wave the data will be q_r. If there exists v_l, v_r such that

$$v_l = \frac{\rho_l u_l - \rho^* u^*}{\rho_l - \rho^*}; \qquad v_r = \frac{\rho_r u_r - \rho^* u^*}{\rho_r - \rho^*}$$

Then a rarefaction-shock solution will be

$$q(x, t) = \begin{cases} q_l & \frac{x}{t} \leq u_l - c_l \\ q_1 & u_l - c_l < \frac{x}{t} < u^* - c^* \\ q^* & u^* - c^* \leq \frac{x}{t} < v_r \\ q_2 & \frac{x}{t} = v_r \\ q_r & \frac{x}{t} > v_r \end{cases} \tag{3.31}$$

and the shock-rarefaction solution will be

$$q(x, t) = \begin{cases} q_l & \frac{x}{t} < v_l \\ q_1 & \frac{x}{t} = v_l \\ q^* & v_l < \frac{x}{t} \leq u^* + c^* \\ q_2 & u^* + c^* < \frac{x}{t} < u_r + c_r \\ q_r & \frac{x}{t} \geq u_r + c_r \end{cases} \tag{3.32}$$

Returning to the case of the isothermal equations, $q_i = (\rho_i, u_i)$. Values for ρ^* and u^* are already found. The speed of sound in the ideal gas is $c_l = c_r = c^* = \sqrt{RT} = 49.36 = a$ all that remains is to find expressions for the data along the rarefaction and shock waves; let the eigenvalue equal x/t:

$$\lambda_1 = u - a$$

$$u_1 = a + \frac{x}{t} \tag{3.33a}$$

Using this with the rarefaction condition (3.27a)

$$u_1 = u_l - a \ln\left(\frac{\rho_1}{\rho_l}\right)$$

$$\rho_1 = \rho_l \exp\left(\frac{u_l - u_1}{a}\right) \tag{3.33b}$$

Because a shock wave propagating to the right is equivalent to a rarefaction wave propagating to the left the expressions for u_2 and ρ_2 can be found the same way

$$u_2 = \frac{x}{t} - a \tag{3.34a}$$

$$\rho_2 = \rho_r \exp\left(\frac{u - 2 - u_r}{a}\right) \tag{3.34b}$$

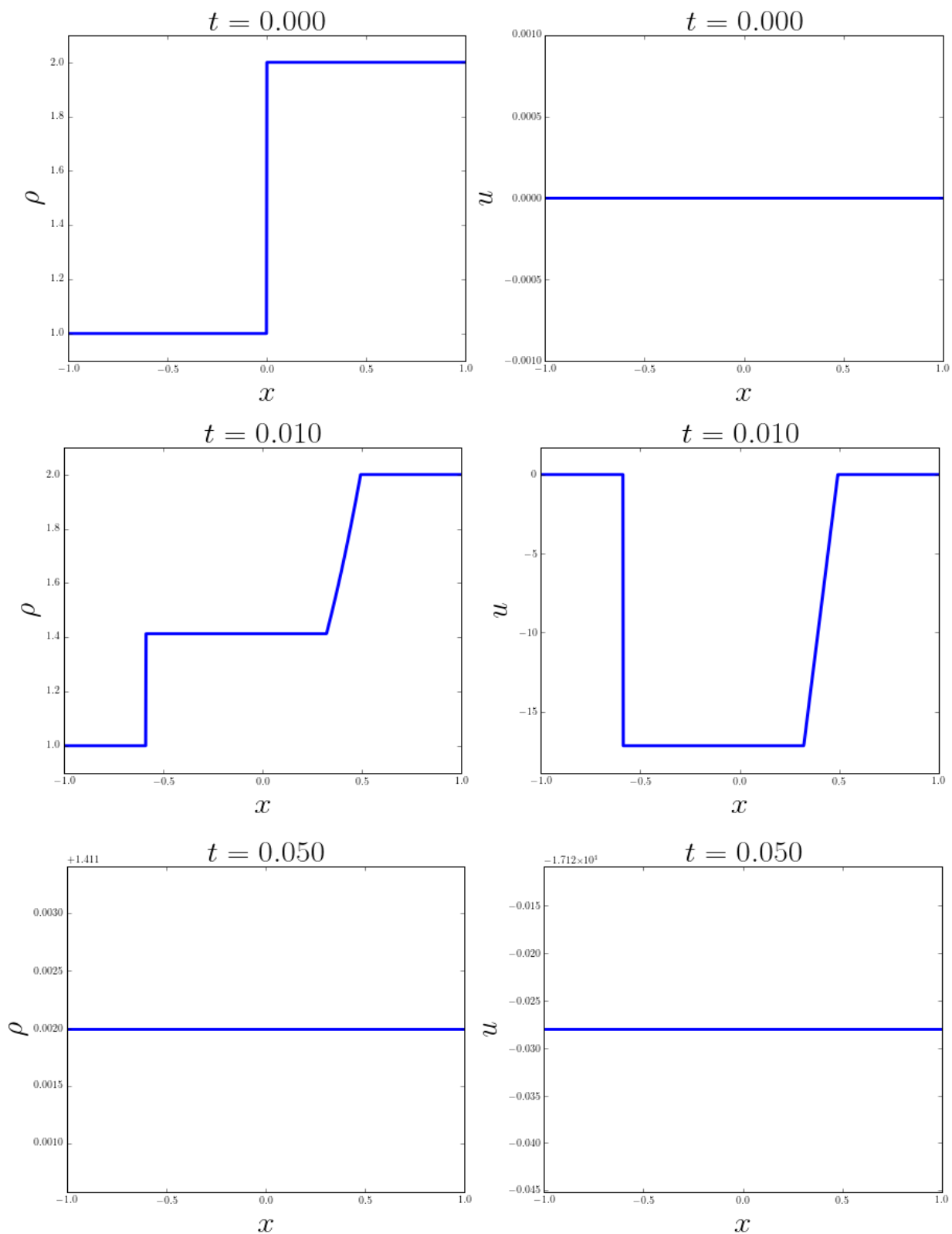Using these value in (3.31) gives the solution shown is Figure 3.7.

Figure 3.7: An exact solution to the Riemann problem defined by the Euler equations applied to an ideal gas under isothermal conditions and initial data given by (3.28) for $0 \leq t \leq 0.05$.

3.7.1 Exact Solutions for an Isentropic Ideal Gas

Now let us consider the isentropic ideal gas with $\gamma = 1.4$, $c_l = 1$

$$\rho(x,0) = \begin{cases} 1 & x < 0 \\ 0.1 & x > 0 \end{cases}; \qquad u(x,0) = 0 \tag{3.35}$$

$\rho_l > \rho_r$ so this solution will be a rarefaction-shock of the form (3.31). Unlike the isothermal gas $c_l \neq c_r \neq c^*$ but this is not a difficult issue, we know $c^2 = \kappa\gamma\rho^{\gamma-1}$ so if the speed of sound at some mass ρ_0 is known to be $c(\rho_0) = c_0$ then

$$c(\rho) = c_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \tag{3.36}$$

The second wave front will be a rarefaction wave so by $c = x/t - u_2$ in (3.22b)

$$u_2 = u_r + \frac{2}{\gamma - 1}\left(\frac{x}{t} - u_2 - c_r\right)$$

$$u_2\left(1 + \frac{2}{\gamma - 1}\right) = u_r + \frac{2}{\gamma - 1}\left(\frac{x}{t} - c_r\right)$$

$$u_2\left(\frac{\gamma + 1}{\gamma - 1}\right) = u_r + \frac{2}{\gamma - 1}\left(\frac{x}{t} - c_r\right)$$

$$u_2 = u_r\left(\frac{\gamma - 1}{\gamma + 1}\right) + \frac{2}{\gamma + 1}\left(\frac{x}{t} - c_r\right) \tag{3.37a}$$

with some rearranging one can obtain an equation for c_2

$$c_2 = c_r + \frac{\gamma - 1}{2}(u_2 - u_r) \tag{3.37b}$$

using this with (3.36) gives an equation for ρ_2, if $c_l = c(\rho_l)$ is known then

$$\rho_2 = \rho_l \left(\frac{c_2}{c_l}\right)^{\frac{2}{\gamma-1}} \tag{3.37c}$$

Repeating this process on the fist wave gives

$$\rho_1 = \rho_l \left(\frac{c_1}{c_l}\right)^{\frac{2}{\gamma-1}} \tag{3.38a}$$

$$c_1 = c_l - \frac{\gamma - 1}{2}(u_1 - u_l) \tag{3.38b}$$

$$u_1 = \left(\frac{\gamma - 1}{\gamma + 1}\right)u_l + \frac{2}{\gamma + 1}\left(\frac{x}{t} + c_l\right) \tag{3.38c}$$

If we have piecewise smooth explicit functions for ρ and c then it is possible to also have a solution for pressure using $p = \rho c^2/\gamma$. Figure 3.8 shows the solutions to the problem defined by the data (3.35) at time $t = 0.33$.
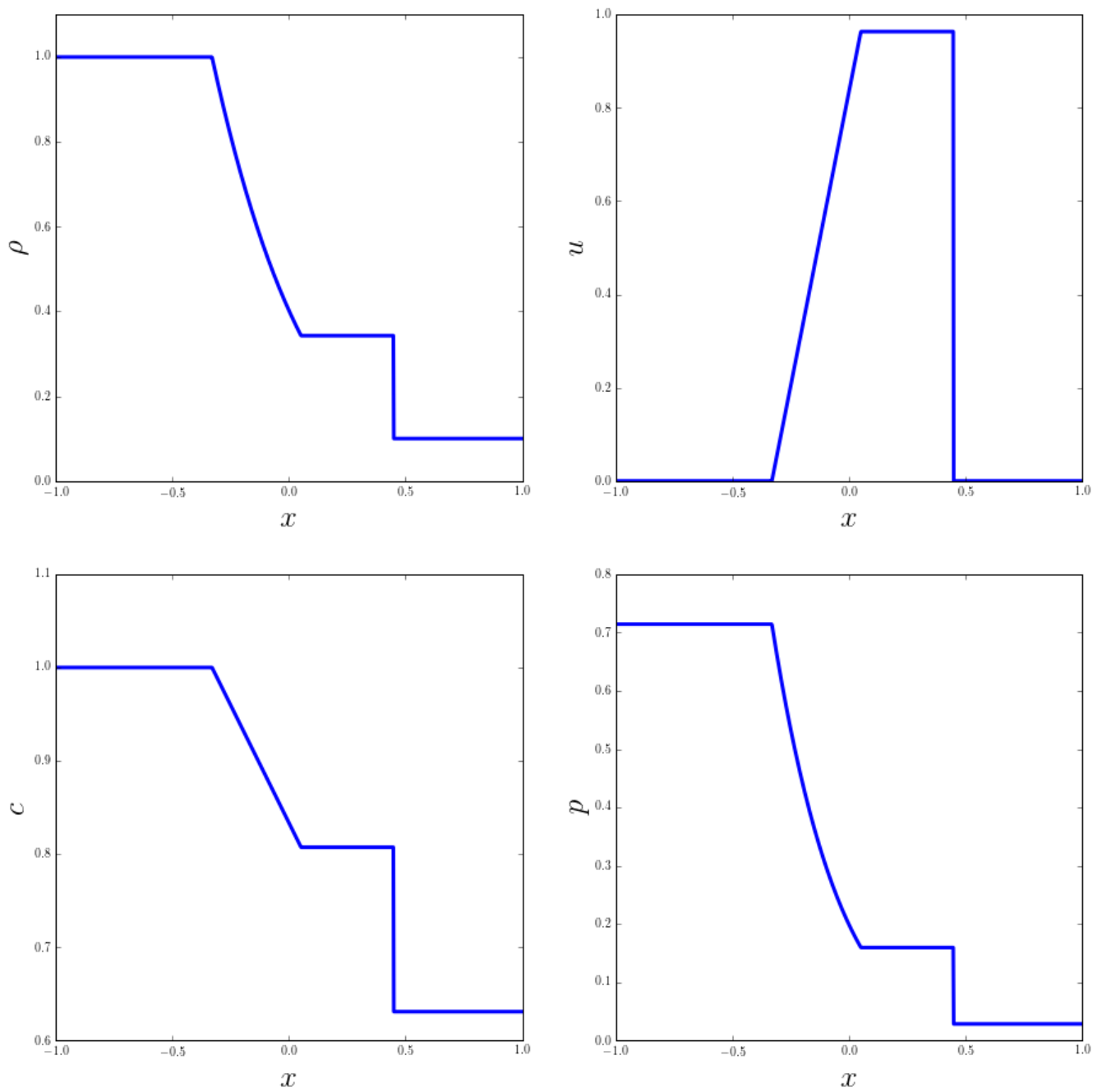
Figure 3.8: An exact solution to the Riemann problem defined by the Euler equations applied to an ideal gas under isentropic conditions and initial data given by (3.35) at $t = 0.33$.

3.7.2   Exact Solutions for Shallow Wave equations

Supposing there is a shallow layer of water at rest in some container and the fluid is disturbed, this may described with the Riemann problem

$$
\begin{bmatrix} gh \\ u \end{bmatrix}_t + \begin{bmatrix} u & gh \\ 1 & u \end{bmatrix} \begin{bmatrix} gh \\ u \end{bmatrix}_x = 0 \tag{3.39a}
$$

$$
h(x,0) = 0.5; \qquad u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \tag{3.39b}
$$

The solution to this problem will be rarefaction-rarefaction. It will be necessary to find the state (h^*, u^*) which can satisfy the conditions

$$
u^* = u_l - 2\sqrt{g}\left[\sqrt{h^*} - \sqrt{h_l}\right]
$$

$$
u^* = u_r + 2\sqrt{g}\left[\sqrt{h^*} - \sqrt{h_r}\right]
$$

With some simple manipulation

$$
h^* = \left[\frac{\sqrt{h_r} + \sqrt{h_l}}{2} + \frac{u_l - u_r}{4\sqrt{g}}\right]^2 \tag{3.40}
$$

$$
u^* = \frac{u_l + u_r}{2} + \frac{\sqrt{h_l} - \sqrt{h_r}}{\sqrt{g}} \tag{3.41}
$$

$$
q(x,t) = \begin{cases} q_l & \frac{x}{t} \leq u_l - c_l \\ q_1 & u_l - c_l < \frac{x}{t} < u^* - c^* \\ q^* & u^* - c^* \leq \frac{x}{t} \leq u^* + c^* \\ q_2 & u^* + c^* < \frac{x}{t} < u_r + c_r \\ q_r & \frac{x}{t} \geq u_r + c_r \end{cases} \tag{3.42}
$$

As in § 3.7 and § 3.7.1 the data along the waves is determined by letting the eigenvalue equal the x/t. For a system of shallow waves in one dimension the eigenvalues are

$$
\lambda_{1,2} = u \mp \sqrt{gh}
$$

with some manipulation of the Riemann invariants

$$
u_1 = \frac{u_l}{3} + \frac{2}{3}\sqrt{gh_l} + \frac{2}{3}\frac{x}{t} \tag{3.43a}
$$

$$
h_1 = \frac{1}{g}\left[u_1 - \frac{x}{t}\right]^2 \tag{3.43b}
$$

$$
u_2 = \frac{u_r}{3} - \frac{2}{3}\sqrt{gh_r} + \frac{2}{3}\frac{x}{t} \tag{3.43c}
$$

$$
h_2 = \frac{1}{g}\left[u_2 - \frac{x}{t}\right]^2 \tag{3.43d}
$$

The solution to the problem defined by (3.39) is show in Figure 3.9; note that in a liquid pressure is $p = \rho gh$ so taking $g = 9.8$ and $\rho = 1$ the solution for pressure is simply a scalar multiple of $h(x,t)$.
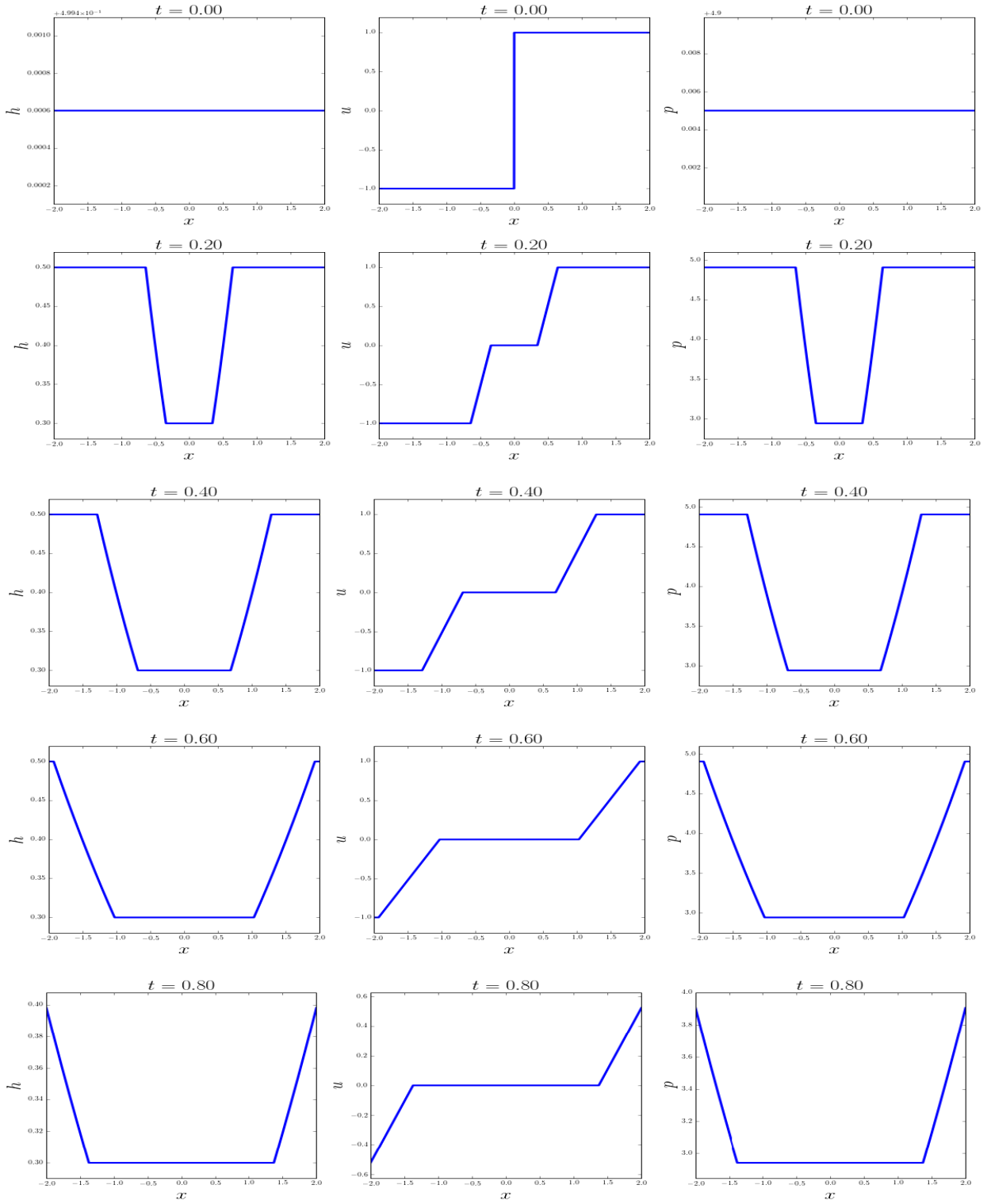
Figure 3.9: An exact solution to the Riemann problem defined by the shallow water equations with the initial data given by (3.39).

3.7.3 Linear Riemann Solutions

As mentioned in § 2.3 and § 2.4 a quasi linear hyperbolic system can be linearised about some fixed state.

$$q(x,t) = q_0 + \tilde{q}; \qquad \tilde{q}_t + f'(q_0)\tilde{q}_x = 0; \qquad \tilde{q}(x,0) = \begin{cases} q_l & x < 0 \\ q_r & x > 0 \end{cases}$$

The Riemann problem for such a system is best solved by decomposing $q_{l,r}$ into decoupled advection equations, as mentioned in § 2.8,with the relation

$$w^p(x,t) = l^p q(x,t)$$

the pth advection equation will have Riemann data

$$w^p(x,0) = \hat{w}^p(x) = \begin{cases} w_l^p & x < 0 \\ w_r^p & x > 0 \end{cases}$$

But discontinuities in linear systems simply propagate at speed λ^p, the solution can be expressed as

$$q(x,t) = \sum_{p=1}^{m} \hat{w}^p(x - \lambda^p t) r^p$$

$$q(x,t) = \sum_{p:\lambda^p < x/t} w_r^p r^p + \sum_{p:\lambda^p > x/t} w_l^p r^p \tag{3.44}$$

The solution will be the same at any point between characteristic p and characteristic $p + 1$, as it crosses a characteristic the characteristic variable jumps from w_l^p to w_r^p. This jump must meet the Rankine-Hugoniot condition, i.e.

$$(w_r^p - w_l^p) r^p = \alpha^p r^p$$

It is unusual for given Riemann data satisfy this condition so it is necessary to break up the discontinuity into a series of jumps which can satisfy Rankine-Hugoniot

$$q_r - q_l = \sum_{p=1}^{m} \alpha^p r^p$$

The solution can be written as either

$$q(x,t) = q_l + \sum_{p:\lambda^p < x/t} (w_r^p - w_l^p) r^p$$

$$q(x,t) = q_r - \sum_{p:\lambda^p \geq x/t} (w_r^p - w_l^p) r^p$$

If we define two step functions

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}; \qquad G(x) = \begin{cases} -1 & x \leq 0 \\ 0 & x > 0 \end{cases} \tag{3.45}$$

then the solution becomes

$$q(x,t) = q_l + \sum_{p=1}^{m} H(x - \lambda^p t)(w_r^p - w_l^p) r^p \tag{3.46a}$$

$$q(x,t) = q_r + \sum_{p=1}^{m} G(x - \lambda^p t)(w_r^p - w_l^p) r^p \tag{3.46b}$$

If we are considering the Riemann problem for an ideal isothermal gas linearised about the state
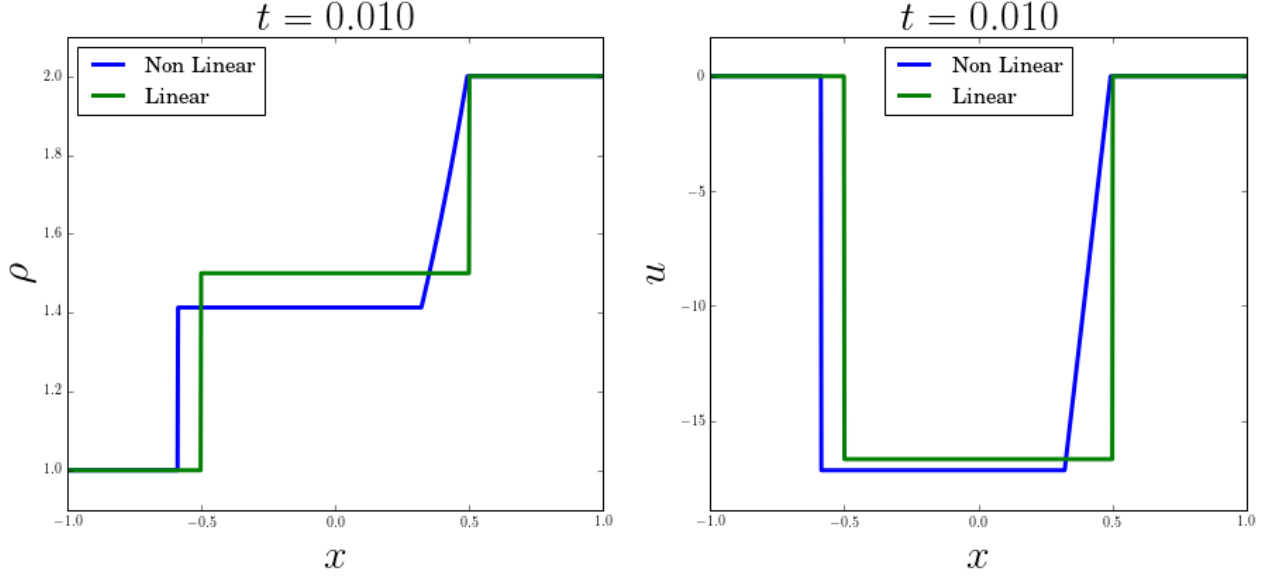
Figure 3.10: A comparison if the the linear and exact solution to the Riemann problem defined by the Euler equations applied to an ideal gas under isothermal conditions and initial data given by (3.28) at $t = 0.01$.

$$(\rho_0, u_0) = \left(\frac{\rho_l + \rho_r}{2}, \frac{u_l + u_r}{2} \right)$$

at temperature T then given that $\lambda^{1,2} = u_0 \mp a$ the solutions can be evaluated by $q = Rw$. If we consider the example of an ideal gas at a constant temperature of 293 K ($a \approx 50$) with the initial data given by (3.28)

$$u_0 = 0, \qquad \rho_0 = 3/2$$

$$\lambda^{1,2} = \mp 50$$

$$r^1 = \begin{bmatrix} -3/100 \\ 1 \end{bmatrix}, \qquad r^2 = \begin{bmatrix} 3/100 \\ 1 \end{bmatrix}$$

$$w_l^1 = \frac{u_l}{2} - \frac{a}{2}\frac{\rho_l}{\rho_0} = -\frac{50}{3}$$

$$w_l^2 = \frac{u_l}{2} + \frac{a}{2}\frac{\rho_;}{\rho_0} = \frac{50}{3}$$

$$w_r^1 = \frac{u_r}{2} - \frac{a}{2}\frac{\rho_r}{\rho_0} = -\frac{100}{3}$$

$$w_r^2 = \frac{u_r}{2} + \frac{a}{2}\frac{\rho_r}{\rho_0} = \frac{100}{3}$$

A snapshot of the linear solution to the isothermal ideal gas at $t = 0.01$ is shown in Figure 3.10, this can be compared with the non linear solution at the same time in Figure 3.7. Both linear and non linear solutions are to the same scale and the shape is generally similar. The linear model could be judged a good approximation for this initial data. Larger discontinuities in the initial data the linear model will not work.

3.8   Conclusions

Once the Rankine-Hugoniot jump condition was defined we could begin looking for solutions to the Riemann problem. The first form of solutions that we saw were the simple waves which can be constructed from the eigenvectors of some hyperbolic system of equations. These simple wave solutions are a combination of shock waves and rarefaction waves. In § 3.4-§ 3.5 we discussed how shock waves arise from the Riemann problem, how the waves can be illustrated in the characteristics of both isothermal and shallow water systems and the entropy conditions which must be satisfied for the the propagation of shock waves to be stable.

In § 3.6 a condition, (3.21), was introduced to define the propagation of rarefaction waves. This condition was applied to the system of equations for an ideal isentropic, isothermal gases and shallow water conditions. Once the conditions for the stable propagation of shock waves and rarefaction waves were defined we could begin looking for exact solutions for the Riemann problem in § 3.7. We began by looking for exact solutions for the isothermal ideal gas, this is the most simple condition because the speed of sound in the media remains constant. Riemann problems solutions for an isothermal gas were shown in Figure 3.7 in the domain $\{(x,t) : x \in [-1,1], t \in [0,0.02]\}$. In § 3.7.1 the same analysis was applied to an isentropic gas with solutions for the given problem displayed in Figure 3.8 at $t = 0.33$. Exact solutions for the shallow water equations were also evaluated in § 3.7.2. This section finished with a analysis and comparison of the linear Riemann solver model with the full non linear model.

Before moving on it should be noted that shock waves in an isentropic gas require an input of energy to remove the entropy created in this process. Hence isentropic shocks require the assumption that conservation of energy does not hold. This raises the question of how a system of equations for one dimensional dynamics of fluids is governed when conservation of energy does hold; this will discussed in the following section.

4   The Euler Equations and Solutions

4.1   Introduction

In this section we will look at the last conserved physical quantity for a fluid: energy. A discussion on the forms of energy relevant to fluid dynamics, namely kinetic and thermal or internal energy, for an ideal gas will make up the early part of this section. By introducing new variables enthalpy and adiabatic exponent the gamma law gas equation of state is derived, this equation will describe the total energy of an ideal gas with a given mass, velocity and pressure. Once this equation of state is defined, the conservation law for energy can be stated, it will be similar to the conservation law for momentum and it will be shown that conservation of energy reduces to a conservation law for pressure. The conservation law for pressure can be combined with the coupled systems whose properties were the subject of investigation in § 2 and § 3 to create a new quasilinear hyperbolic system of equations commonly known as the Euler equations. The properties and Riemann problem solutions of this new system of equations will be the subject of the remainder of the section. We will end this section with a discussion of how the solutions are affected when the assumptions associated with ideal gases are dropped from our model.

4.2   Energy in a Fluid

Energy is commonly defined as the ability to perform work. This is really a definition for mechanical energy of a system as the energy of a body is not necessarily available to perform work. The definition of energy really depends on it's form. There are several forms in which energy can observed: kinetic, thermal, potential (of which there are several forms again: gravitational, electrical, elastic, ...), chemical, nuclear, etc. In the present discussion of one dimensional fluid dynamics the only forms of energy which are important are kinetic and thermal; we will assume that the total energy of the fluid will be composed of kinetic and thermal energy. According to the work-energy theorem the difference in kinetic energy is equal to the work performed by the body; hence kinetic energy is expressed mathematically as

$$K = \frac{1}{2}\rho u^2 = \frac{(\rho u)^2}{2\rho} \tag{4.1}$$

this equation should imply that if both mass and momentum are conserved then kinetic energy is also conserved.

Thermal energy is not exactly the same thing as heat energy; heat is not strictly speaking a form of energy but rather the transfer of energy and is governed by the heat equation. Thermal energy is the internal energy of the fluid which results from the translational, rotational and vibrational motions at the microscopic level; for more complex molecules there are additional degrees of freedom. The temperature of a fluid is a quantitative measure of the average kinetic energy of the molecules. The principle of equipartition states that the average kinetic energy of each molecule per degree of freedom is directly proportional to the temperature. The constant of proportionality is known as the Boltzmann constant, $k_B = 1.38 \times 10^{-23} \mathrm{J \cdot K^{-1}}$. Thus the internal energy for n moles of ideal gas with α degrees of freedom at temperature T is

$$e = \frac{\alpha}{2}nRT$$

The internal energy has a density function given by

$$e(x,t;T,\alpha) = \frac{\alpha}{2}\rho(x,t)RT \tag{4.2}$$

Hence the total energy of an isothermal ideal gas is

$$E = \frac{1}{2}[\alpha p + \rho u^2] \tag{4.3}$$

This will also work for isobaric conditions, if the pressure is at a constant $p = p_0$

$$E = \frac{1}{2}[\alpha p_0 + \rho u^2] \tag{4.4}$$

4.3   Gamma law gas

Define two new parameters: the specific heat capacities at constant volume and constant pressure, [7]

$$c_v = \left(\frac{\partial e}{\partial T}\right)_v \tag{4.5a}$$

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{4.5b}$$

h is a quantity known as enthalpy defined as

$$h = e + pv \tag{4.6}$$

For an infinitesimal process

$$dh = de + pdv + vdp$$

but according to the first law of thermodynamics $de = dq - pdv$

$$dh = dq + vdp$$

$$\Rightarrow \left(\frac{\partial h}{\partial T}\right)_p = \frac{dq}{dT} = c_p$$

Enthalpy is the heat added to a system under constant pressure conditions. For an ideal gas the internal energy is a function of only the temperature, consequently the constant volume heat capacity is

$$c_v = \frac{de}{dT} \Rightarrow de = c_v dT$$

Rearranging the infinitesimal form of the first law

$$dq = c_v dT + pdv$$

If the process is performed at constant pressure then dividing this equation by dT

$$\frac{dq_p}{dT} = c_v + p\left(\frac{\partial v}{\partial T}\right)_p = c_p$$

But for an ideal gas:

$$p\left(\frac{\partial v}{\partial T}\right)_p = \hat{R}$$

$$c_p = c_v + \hat{R} \tag{4.7}$$

$$e = c_v T = c_v\left(\frac{p}{\hat{R}\rho}\right) = \frac{pc_v}{(c_p - c_v)\rho} = \left(\frac{1}{\frac{c_p}{c_v} - 1}\right)\frac{p}{\rho} \tag{4.8}$$

We now define a new parameter: the adiabatic exponent, also known as the heat capacity ratio

$$\gamma = \frac{c_p}{c_v} \tag{4.9}$$

Combining (4.9) with (4.8), the total internal energy is ρe. Including the kinetic energy, the total energy of the system is

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \tag{4.10}$$

(4.10) is known as the γ-law equation of state.

4.4   Conservation law for Energy

Energy is a quantitative physical parameter and is subject to the continuity equation, (1.6), just like mass and momentum. Energy has a density function which for most ideal gases will be given by (4.10). The flux for energy is similar in form to that of momentum. There is a macroscopic component equal to the product of the density function and the velocity; and a microscopic component equal to the product of the pressure and the velocity, i.e.

$$E_t + [(E + p)u]_x = 0 \tag{4.11}$$

Using (4.10) in (4.11) and multiplying each term by $\gamma - 1$ gives

$$p_t + \frac{\gamma}{2}\rho_t u^2 - \frac{1}{2}\rho_t u^2 + \gamma \rho u u_t - \rho u u_t + \cancel{up_x} + \frac{\gamma}{2}\rho_x u^3$$
$$- \frac{1}{2}\rho_x u^3 + \gamma \rho u^2 u_x - \rho u^2 u_x + \gamma p_x u - \cancel{up_x} + \gamma p u_x - \cancel{pu_x} + \cancel{pu_x} + \frac{\gamma}{2}\rho u^2 u_x - \frac{1}{2}\rho u^2 u_x = 0 \tag{4.12}$$

With some manipulation (4.12) reduces to a conservation equation for pressure.

$$p_t + \gamma p u_x + u p_x = 0 \tag{4.13}$$

if instead the isothermal EOS was used in (4.11) we get

$$p_t + \left(1 + \frac{2}{\alpha}\right)p u_x + u p_x = 0 \tag{4.14}$$

The derivation for this is in Appendix B. Under isobaric conditions the pressure conservation equation would obviously be $p_t = 0$.

4.5   Eigensystem of an Isothermal Gas

By coupling the conservation equation for pressure in an isothermal gas, (4.14), to the conservation equations for mass and momentum, (2.30) gives us a hyperbolic quasilinear system of equations

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & pb^2 & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0 \tag{4.15}$$

Here b is a constant defined as $b^2 = 1 + 2\alpha^{-1}$. It is not difficult to find the eigensystem of this system of equations. By recalling that $a^2 = p/\rho$

$$\lambda_1 = u, \quad \lambda_2 = u - ab, \quad \lambda_3 = u + ab \tag{4.16}$$

$$r^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix} -\rho/ab \\ 1 \\ \rho ab \end{bmatrix}, \quad r^3 = \begin{bmatrix} \rho/ab \\ 1 \\ \rho ab \end{bmatrix} \tag{4.17}$$

Another system of Euler equations one could consider are the isobaric conditions

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0 \tag{4.18}$$

However this system of PDEs does not yield any non zero eigenvalues or eigenvectors leading to the conclusion that there are no isobaric solutions to the Euler equations. The remainder of this section will focus on generating solutions for the Euler equations under conditions of constant entropy.

4.6   The Eigensystem of an Isentropic Gas

Combining the conservation equations for mass, (1.6), momentum, (2.1) and energy, (4.11) we get the Euler equations mentioned earlier, (0.1). As seen in § 2 and § 4.2 the equations for conservation of momentum and energy can be reduced to equations for conservation of velocity and pressure respectively. Hence (0.1) can be expressed as a quasilinear hyperbolic system of equations in the form $q_t + Aq_x = 0$

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0 \tag{4.19}$$

The eigenvalues and eigenvectors of the matrix can be evaluated just like in § 2.

$$|A - \lambda I| = (u - \lambda) \begin{vmatrix} u - \lambda & 1/\rho \\ \gamma p & u - \lambda \end{vmatrix} = 0$$

$$(u - \lambda)[(u - \lambda)^2 - \gamma p/\rho] = 0$$

But for an isentropic fluid $c^2 = \gamma p/\rho$

$$(u - \lambda)[(u - \lambda)^2 - c^2] = 0$$

$$(u - \lambda)(u - \lambda - c)(u - \lambda + c) = 0$$

The eigenvalues of the system are:

$$\lambda_1 = u, \quad \lambda_2 = u - c, \quad \lambda_3 = u + c \tag{4.20}$$

Using these eigenvalues in $A - \lambda_i I$

$$(A - \lambda_1 I)r_1 = \begin{bmatrix} 0 & \rho & 0 \\ 0 & 0 & 1/\rho \\ 0 & \gamma p & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Repeating this procedure with $\lambda_{2,3}$ gives the other two right eigenvectors

$$r_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} -\rho/c \\ 1 \\ -\rho c \end{bmatrix}, \quad r_3 = \begin{bmatrix} \rho/c \\ 1 \\ \rho c \end{bmatrix} \tag{4.21}$$

By computing the inverse of the ajoint matrix of the right eigenvectors the left eigenvectors are

$$l_1 = \frac{1}{c^2} \begin{bmatrix} c^2 \\ 0 \\ -1 \end{bmatrix}, \quad l_2 = \frac{1}{2\rho c} \begin{bmatrix} 0 \\ \rho c \\ -1 \end{bmatrix}, \quad l_3 = \frac{1}{2\rho c} \begin{bmatrix} 0 \\ \rho c \\ 1 \end{bmatrix} \tag{4.22}$$

4.7   Contact Discontinuities

From simple inspection it is clear that $\nabla \lambda_1 \cdot r_1 = 0$. This means that along any integral curve of r_1 the eigenvalue λ_1 is constant. Simple waves in which the variation in $[\rho \quad u \quad p]^T$ is only in this field will behave like solutions to linear hyperbolic equations; for this reason the 1st field is called linear degenerate. Because λ_1 is constant throughout wave-1 it will not distort as it propagates but rather translate with constant speed; if there is a discontinuity in the initial data, such as in (3.1b), and both values either side of the jump are on an integral curve of field-1 then the solution will be a discontinuity propagating at a constant speed λ_1. If the characteristic speed is the same for all of the initial data then characteristics will be parallel to the wave in the $x - t$ plane rather than impinging it so this is not in fact a shock wave nor a rarefaction wave. We refer to waves of this form as contact discontinuities. If the coefficients of diffusion are not sufficiently small so that diffusive effects cannot be ignored then discontinuities in the initial data will be smeared out, the analysis in this work will assume that these coefficients are sufficiently small.
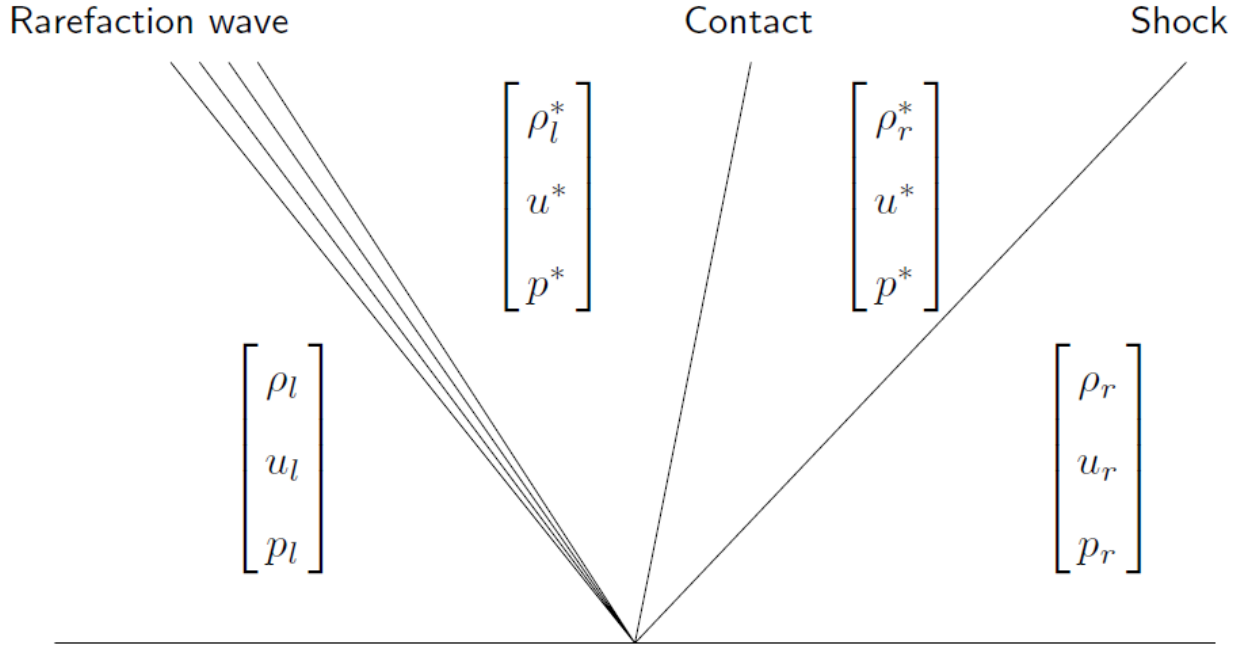
Figure 4.1: The general form of a solution to a Riemann problem defined by the Euler equations and a discontinuity in the initial data, [1].

Consider the integral curves of r^1

$$\frac{d}{ds}\begin{bmatrix} \rho \\ u \\ p \end{bmatrix} = r^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Clearly u and p are constant along integral curves of r^1 and only ρ varies. As mentioned in § 2.4 variables over which the integral curves are constant are known as Riemann invariants, so u and p will be the 1-Riemann invariants. Because u and p are constant across the contact discontinuity it is more convenient to perform our analysis in the variables (ρ, u, p)

4.8   Riemann Problem Solutions for Euler Equations

Before we look at solutions to the Euler equations let the Riemann problem for (4.19) be defined

$$q(x,0) = \begin{bmatrix} \rho & u & p \end{bmatrix}^T = \begin{cases} \begin{bmatrix} \rho_l & u_l & p_l \end{bmatrix}^T & \text{if} \quad x < 0 \\ \begin{bmatrix} \rho_r & u_r & p_r \end{bmatrix}^T & \text{if} \quad x > 0 \end{cases} \tag{4.23}$$

Figure 4.1 shows a typical Riemann problem solution. there will be a contact discontinuity and two non linear waves. The two non linear waves may be two shocks or two rarefactions or a shock and a rarefaction, the initial data will determine which one. The characteristic fields of all the gas systems considered in § 3 were genuinely non linear so the behaviour of the second and third fields should be similar to these systems. For the initial data (4.23) the solution requires two new constant states $q_l^* = (\rho_l^*, u^*, p^*)$ which can be connected to q_l via a 2-wave and via a 1-wave to $q_r^* = (\rho_r^*, u^*, p^*)$ which can be connected via a 3-wave to q_r. The constant states u^* and p^* can be obtained by taking the intersection of the integral curves of families 2 and 3 in $p - u$ space. At this point it necessary to consider the Riemann invariants of the family-2 and family-3 waves.

In § 2.3 a relationship between pressure and mass was stated that at constant entropy

$$p = \kappa \rho^\gamma$$

This relation derives from an explicit function for entropy, (2.25)

$$S = C_v \ln\left(\frac{p}{\rho^\gamma}\right) + S_0; \qquad S_0 \in \mathbf{R} \tag{4.24}$$

By rearranging (4.24) to express p in terms of ρ

$$p = \exp\left(\frac{-S_0}{C_v}\right) e^{S/C_v} \rho^\gamma = \hat{\kappa} e^{S/C_v} \rho^\gamma \tag{4.25}$$

By applying (4.13) with (4.25) reduces to an advection equation for entropy, see Appendix B

$$S_t + uS_x = 0 \tag{4.26}$$

(4.26) is satisfied everywhere along the particle paths except at the contact discontinuity. Entropy will in fact be constant through any simple wave in the 2 or 3 families and thus is a Riemann invariant for these families. There is another Riemann invariant which can be found by calculating expressions for the integral curves of family 2.

$$\begin{bmatrix} d\rho \\ du \\ dp \end{bmatrix} = \begin{bmatrix} -\rho/cds \\ ds \\ -\rho cds \end{bmatrix} \Rightarrow du = -\frac{c}{\rho} d\rho \tag{4.27}$$

But for a polytropic gas $c\rho^{\frac{\gamma-1}{2}} = \text{constant}$. Hence is there is some reference state ρ_0 then

$$c = c_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \tag{4.28}$$

Using this expression for c in (4.27) gives a new Riemann invariant

$$u + \frac{2c}{\gamma - 1} \tag{4.29}$$

and repeating this with the family-3 curves would give

$$u - \frac{2c}{\gamma - 1}$$

To summarise

$$\begin{array}{lll} \text{1-Riemann invariants} & u, & p \\ \text{2-Riemann invariant} & S, & u + \frac{2c}{\gamma-1} \\ \text{3-Riemann invariants} & S, & u - \frac{2c}{\gamma-1} \end{array} \tag{4.30}$$

For some reference state: (ρ_0, u_0, p_0) and $c_0 = c(\rho_0)$ the locus of states in $p - u$ space that can be connected to the reference state by a 2-rarefaction wave, that is to say the integral curve of the family-2 waves is

$$u = u_0 + \frac{2c_0}{\gamma - 1}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{4.31a}$$

and for the family-3 waves

$$u = u_0 - \frac{2c_0}{\gamma - 1}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right] \tag{4.31b}$$

The derivation of equations (4.29)-(4.31b) is in Appendix B. If however we need the locus of states which can be connected to the point (ρ_0, u_0, p_0) by a 2-shock and 3-shock wave the condition will be (3.19a), (3.19b) respectively

$$u = u_0 - \sqrt{\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(p - p_0)}$$

$$u = u_0 + \sqrt{\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right)(p - p_0)}$$

For some polytropic gas with the initial data:

$$\rho(x,0) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases}; \qquad u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}; \qquad p(x,0) = \begin{cases} p_l & x < 0 \\ p_r & x > 0 \end{cases}$$

The pressure and velocity, (p^*, u^*), across the discontinuity will be the intersection of these curves. The values of ρ_l^* and ρ_r^* can be evaluated by

$$\rho_l^* = \rho_l \left(\frac{p^*}{p_l}\right)^{\frac{1}{\gamma}}; \qquad \rho_r^* = \rho_r \left(\frac{p^*}{p_r}\right)^{\frac{1}{\gamma}} \tag{4.32}$$

4.9   A Linear Solution

Let us now consider a specific example of a Riemann problem for the Euler equations of an ideal polytropic gas; $\gamma = 1.4$ and initial data:

$$\rho(x,0) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases}; \qquad u(x,0) = \begin{cases} 0 & x < 0 \\ 0 & x > 0 \end{cases}; \qquad p(x,0) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases} \tag{4.33}$$

The linear solution will be given by (3.46a) or (3.46b), there are some important differences from § 3.7.3. The system will be be linearised about the state (ρ_0, u_0, p_0). In § 3.7.3 this reference state was taken to be the mean of the upstream states q_l and q_r, we will continue using this choice of reference state

$$(\rho_0, u_0, p_0) = \left(\frac{\rho_l + \rho_r}{2}, \frac{u_l + u_r}{2}, \frac{p_l + p_r}{2}\right)$$

the eigensystem will be a linearisation of (4.20) and (4.21) about the point (ρ_0, u_0, p_0) and the speed of sound in the media will be given by (3.36), i.e.

$$c_0(\rho_0) = c_l \left(\frac{\rho_0}{\rho_l}\right)^{\frac{\gamma-1}{2}}$$

$$\lambda^1 = u_0; \quad \lambda^2 = u_0 - c_0; \quad \lambda^3 = u_0 + c_0$$

$$r^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix} -\rho_0/c_0 \\ 1 \\ -\rho_0 c_0 \end{bmatrix}, \quad r^3 = \begin{bmatrix} \rho_0/c_0 \\ 1 \\ \rho_0 c_0 \end{bmatrix}$$

The linear solution to the Euler equations, shown in Figure 4.2 at $t = 0.4$, will be given by

$$\rho(x,t) = \rho_l + H(x - \lambda^1 t)(w_r^1 - w_l^1) - H(x - \lambda^2 t)(w_r^2 - w_l^2)\frac{\rho_0}{c_0} + H(x - \lambda^3 t)(w_r^3 - w_l^3)\frac{\rho_0}{c_0} \tag{4.34a}$$

$$u(x,t) = u_l + H(x - \lambda^2 t)(w_r^2 - w_l^2) + H(x - \lambda^3 t)(w_r^3 - w_l^3) \tag{4.34b}$$

$$p(x,t) = p_l - H(x - \lambda^2 t)(w_r^2 - w_l^2)\rho_0 c_0 + H(x - \lambda^2 t)(w_r^3 - w_l^3)\rho_0 c_0 \tag{4.34c}$$

Here H is the Heaviside step function and the energy is given by the γ-law equation of state (4.10). The values for w_l^i and w_r^i are evaluated by

$$w_{l,r} = \begin{bmatrix} w_{l,r}^1 \\ w_{l,r}^2 \\ w_{l,r}^3 \end{bmatrix} = \begin{bmatrix} 1 & -\rho_0/c_0 & \rho_0/c_0 \\ 0 & 1 & 1 \\ 0 & -\rho_0 c_0 & \rho_0 c_0 \end{bmatrix}^{-1} \begin{bmatrix} \rho_{l,r} \\ u_{l,r} \\ p_{l,r} \end{bmatrix}$$
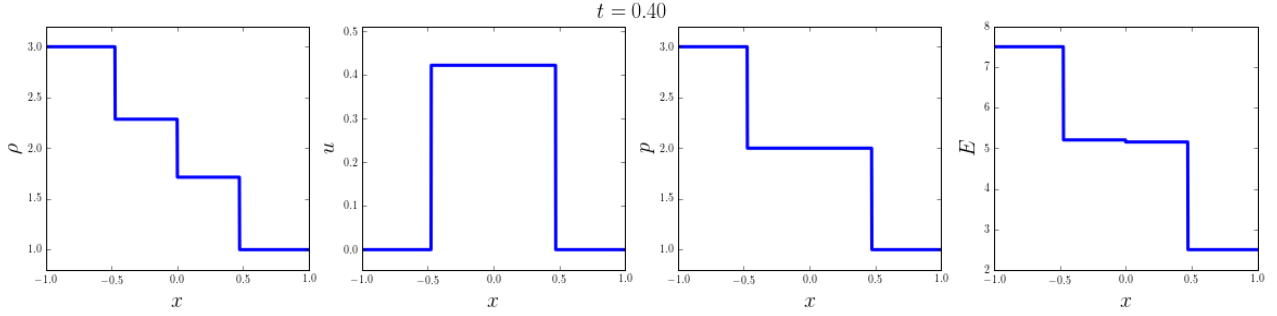
Figure 4.2: A linear solution to the Riemann problem defined by the Euler equations applied to an ideal gas under isentropic conditions and initial data given by (4.33).

4.10   Exact Solutions

We now have all the theory and information required to evaluate a full non linear solution to the Riemann problem in the Euler equations. By amending the analysis in § 3.7.1 with the conditions discussed in § 4.8. Consider the case of two polytropic ideal gases with different chemical compositions and hence different mass and pressure, coming into contact; the situation can be represented as a Riemann problem

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0 \tag{4.35a}$$

$$\rho(x,0) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases}; \qquad u(x,0) = 0 \quad \forall x; \qquad p(x,0) = \begin{cases} 3 & x < 0 \\ 1 & x > 0 \end{cases} \tag{4.35b}$$

Figure 4.3 shows the Hugoniot loci of states (p, u) which can be connected to the states $(3, 0)$ and $(1, 0)$ by a 1-rarefaction and a 3-shock wave respectively. The intersection will be the state (p^*, u^*).

$$\rho_l^* = 3 \left(\frac{1.69}{3} \right)^{\frac{1}{1.4}} \approx 1.99; \qquad \rho_r^* = 1 \left(\frac{1.69}{1} \right)^{\frac{1}{1.4}} \approx 1.45$$

The exact solution to the Riemann problem at $t = 0.4$ is shown in Figure 4.4; this describes how the two gases expand into one another on their own energy, that is to say in the absence of diffusion. E is given by the γ-law equation of state (4.10); alongside is the linear solution from Figure 4.2. It may be argued that the linear solution is a reasonably good model for this problem.

4.11   Physical Applications of the Riemann Problem

One practical application of the Euler equations is in the physical description of the process of cavitation. [13] Cavitation is defined as the process of formation of the vapor phase of a liquid when it is subjected to reduced pressures at constant ambient temperatures. The drop in pressure can occur because the fluid is accelerated faster than it can react resulting in shock and rarefaction waves, typically leaving behind a gas bubble. In manufacturing, mechanical components involved with fluids need to be designed to minimise the damage caused by cavitation. A fluid must be subjected to some abrupt change in velocity for cavitation to occur, this can be expressed as a Riemann problem

$$\rho(x,0) = 1, \quad p(x,0) = 1, \quad u(x,0) = \begin{cases} -3 & x < 0 \\ 3 & x > 0 \end{cases} \tag{4.36}$$

The exact solution to this Riemann problem, using the methods derived in § 3 and § 4 is shown in Figure 4.5. It can be seen in the solution that as time progresses the mass, pressure and energy will reach a minimum below which they cannot go. This is because physically these parameters cannot be negative but they do not reach zero as there is still some physical matter remaining. For larger discontinuities in velocity the cavitation process will create a
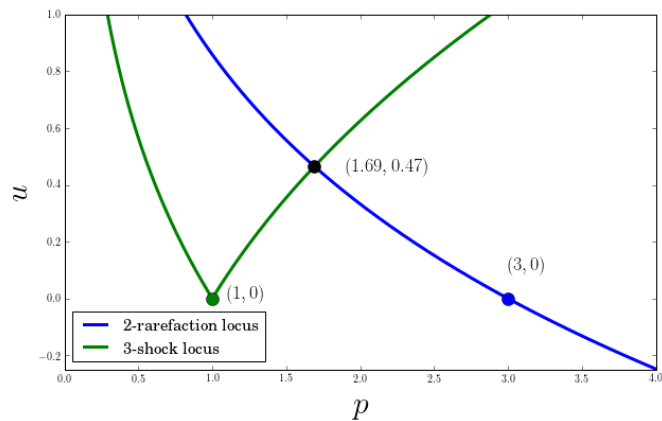
Figure 4.3: The Hugoniot loci of the Euler equations applied to an ideal gas under isentropic conditions. The blue curve is the locus of states in $p - u$ space which can be connected to $(3, 0)$ by a 2-rarefaction wave, the green curve is the locus of states which can be connected to $(1, 0)$ by a 3-shock wave. The intersection of the curves gives us the intermediary state.; see Appendix A.3.

closer approximation to a vacuum. It should still be kept in mind that these solutions neglect the effects of molecular diffusion.

$$t = 0.40$$

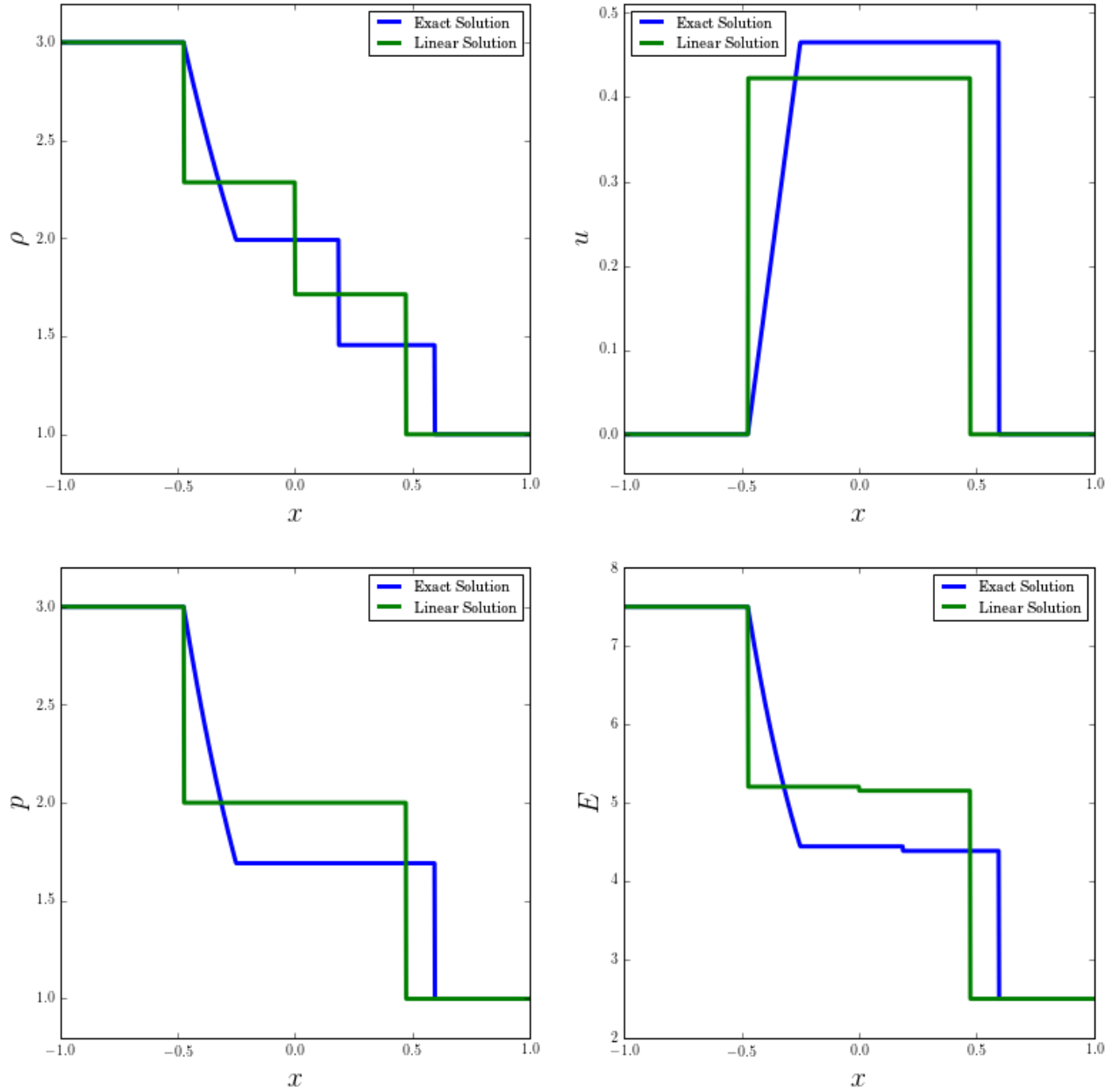Figure 4.4: A comparison of the linear and exact solution to the Riemann problem defined by the Euler equations applied to an ideal gas under isentropic conditions with initial data given by (4.35b) at $t = 0.4$; see Appendix A.4.
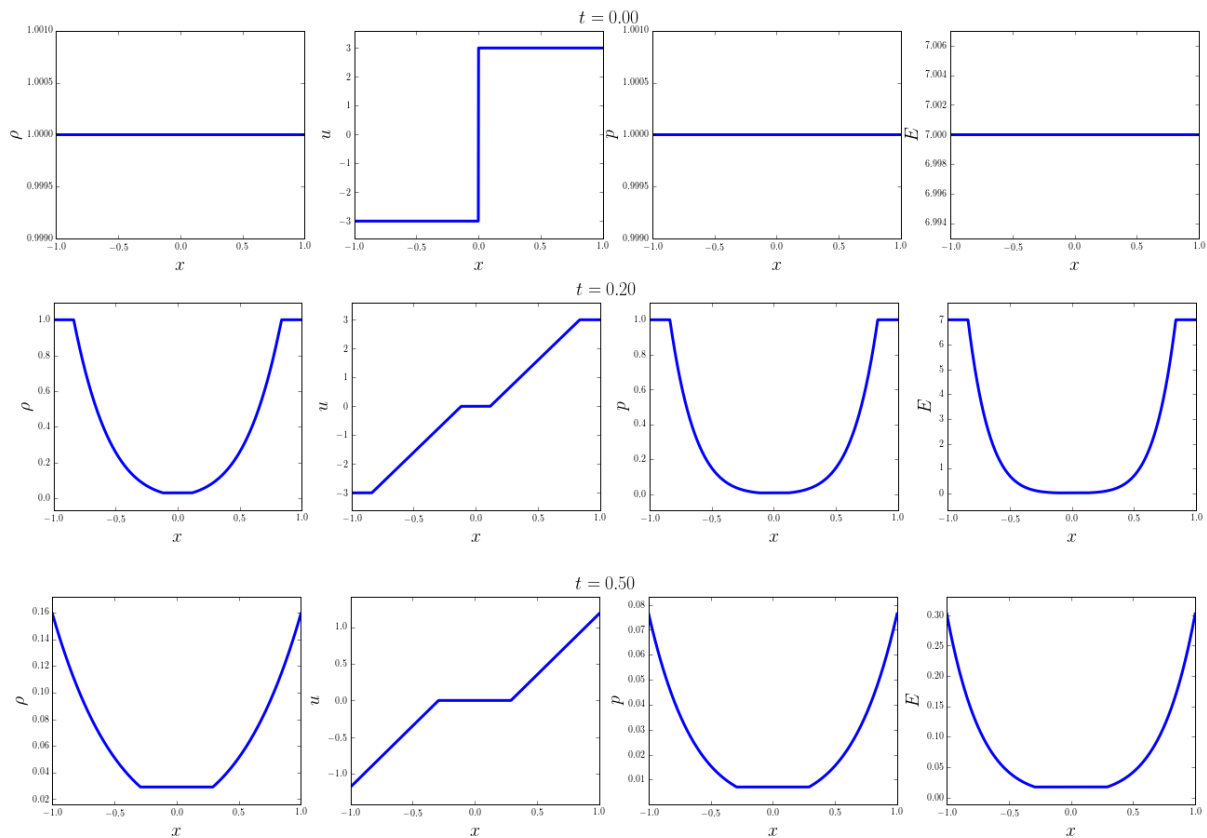
Figure 4.5: An exact solution to the Riemann problem defined by the Euler equations applied to an ideal gas under isothermal conditions and initial data given by (4.36) for $0 \leq t \leq 0.5$.
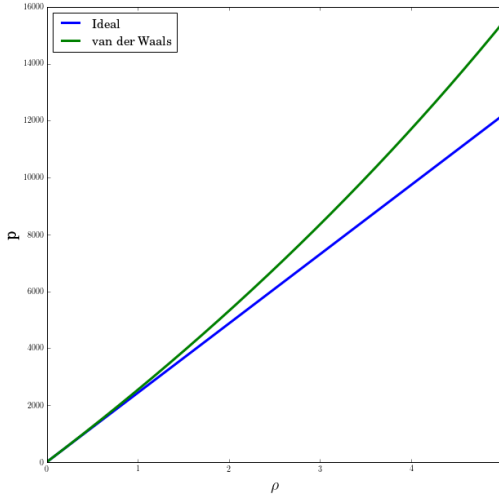
4.12   Real Gases

Figure 4.6: A comparison of the relation of pressure to mass for an ideal gas model with the van der Waals correction.

To finish this section we will look at how the analysis followed thus far is applied when more realistic equations of state are assumed and compare it to the ideal gas model. The most obvious candidate model is the van der Waals equation of state as this is more an improvement or alteration of the ideal gas model than a completely different model such as the Virial equation of state. Of course the conservation law is independent of the equation of state so the non linear form of the Euler equations (0.1) is unchanged however the derivation of the γ-law EOS worked on the assumption of an ideal gas. Consequently the conservation law for energy cannot be reduced to the conservation equation for pressure and furthermore the Euler equations cannot be expressed in the quasilinear hyperbolic form (4.19). It will be necessary to derive a new relation between energy, mass, pressure and velocity before any comparison with the ideal gas model can be started.

As mentioned in § 2.2 the ideal gas model works on the assumption that gas molecules do not occupy any actual volume and that there are no intermolecular interactions present. This is of course not realistic but sufficiently high temperatures and/or low pressures the volumes of the constituent molecules are negligible relative to the volume of the gas and the electromagnetic potential is not strong enough to significantly influence the molecular kinetic energy. The van der Waals equation of state makes corrections for these assumptions. Realistically the intermolecular electromagnetic forces will reduce the pressure, pulling them together as they push against the walls of the container, let us call this reduction α. The gas molecules will of course occupy some physical volume, however infinitesimal, which we will call β. Hence the ideal gas model becomes

$$(p + \alpha)(v - \beta) = nRT$$

Let us assume that the volume occupied by a mole of gas molecules is b so that $\beta = nb$. We can think of b as the volume of 1 mole of gas molecules. The decrease in pressure is proportional to the number of molecules per unit volume in the layer near the wall and also to the number per unit volume in the next layer beyond the wall hence $\alpha \propto n^2/v^2$; let us choose the constant of proportionality as a. The van der Waals equation of state is

$$\left(p + \frac{an^2}{v^2} \right)(v - nb) = nRT \tag{4.37}$$

the constants a and b experimentally derived and characteristic of the gas. We need $p = p(\rho)$ to follow the analysis from the earlier sections so let $v = \rho^{-1}$. Through some simple manipulation (see Appendix B)

$$p = \frac{\rho RT}{1 - \rho b} - a\rho^2 \tag{4.38}$$

As mentioned a and b represent the strength of intermolecular interactions and the volume of a mole of the constituent molecules respectively. Table 1 shows some values of a and b for different gases measured in $\text{Pa} \cdot (\text{m}^3)^2/\text{mol}^2$ and m^3/mol respectively, [12]

In the ideal gas model a, $b \to 0$ and (4.38) is reduced to the ideal gas equation of state. Figure 4.6 shows a comparison of the ideal CO_2 gas model with the van der Waals correction. One conclusion that is clear from the graphic is that the ideal gas model is a good approximation at low pressures but at higher pressure fails increasingly, this is true at all temperatures.

Now we want the total energy, E, of a real gas in terms of p and ρ we will continue with the assumption that the total energy is a combination of kinetic and internal energy. The kinetic energy will be the same as the ideal case but the assumption that internal energy is solely a function of T must be dropped. For a real gas $e = e(T, v)$

Gas	Symbol	a	b
Hydrogen	H_2	0.02453	0.02651
Helium	He	0.00346	0.0238
Water (steam)	H_2O	0.5537	0.03049
Oxygen	O_2	0.1382	0.03186
Carbon Dioxide	CO_2	0.3658	0.0486
Carbon Monoxide	CO	0.147	0.039
Nitrogen	N_2	0.137	0.039
Methane	CH_4	0.230	0.043

Table 1: Van der Waals constants.

$$de = \left(\frac{\partial e}{\partial T}\right)_v dT + \left(\frac{\partial e}{\partial v}\right)_T dv$$

But the Helmholtz equation states that

$$\left(\frac{\partial e}{\partial v}\right)_T = T^2 \left(\frac{\partial}{\partial T}\frac{p}{T}\right)_v \tag{4.39}$$

$$e = e_{\text{ideal}} + \int_{v_0}^{v} T^2 \left(\frac{\partial}{\partial T}\frac{p}{T}\right)_v dv'$$

$$\frac{\partial}{\partial T}\frac{p}{T} = \frac{\partial}{\partial T}\left(\frac{\rho R}{1-\rho b} - \frac{a}{T}\rho^2\right)_v = \frac{a}{T^2}\rho^2$$

but because $\lim_{v\to\infty} e = e_{\text{ideal}}$ so let $v_0 = \infty$ and as $dv = \rho^{-2}d\rho$

$$\int_{\infty}^{v} T^2 \left(\frac{\partial}{\partial T}\frac{p}{T}\right)_v dv' = \int_{0}^{\rho} a d\rho' = a\rho$$

$$e = \frac{p}{\rho(\gamma - 1)} + a\rho$$

The total energy for the real gas is

$$E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} + a\rho^2 \tag{4.40}$$

Note that as $a \to 0$ (4.40) approaches the γ-law gas equation of state, (4.10). The Euler equations can be amended for the van der Waals equation of state by using this new relationship between energy, pressure, mass and velocity, with some manipulation (see Appendix B) the new conservation equation for pressure is

$$p_t - c^2(\gamma - 1)u\rho_x + (\gamma p - a(\gamma - 1)\rho^2)u_x + \gamma u p_x = 0 \tag{4.41}$$

The Euler equations of the real gas are

$$\begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ c^2 u(1-\gamma) & \gamma p + a(1-\gamma)\rho^2 & \gamma u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}_x = 0 \tag{4.42}$$

The eigenvalues of this new system of partial differential equations can be evaluated if one can solve the characteristic equation of the Jacobian

$$\lambda^3 + [(2+\gamma)u - a\rho]\lambda^2 + [(a\rho + c^2)(1+u) - (1+2\gamma)u^2]\lambda - [2uc^2 + a\rho u] = 0$$

this cubic equation can be solved, if we write it as $\lambda^3 + d\lambda^2 + e\lambda + f = 0$, by using the transformation

$$\lambda = y - \frac{d}{3}$$

$$A = e - \frac{1}{3}d^2$$

$$B = \frac{1}{3}ed - f - \frac{2}{27}d^3$$

The characteristic equation becomes

$$y^3 + Ay = B$$

his cubic has a solution $y = s - t$ if we define s and t such that

$$3st = A \tag{4.43a}$$

$$s^2 - t^3 = B \tag{4.43b}$$

These simultaneous equations have solutions

$$s_1 = \left[\frac{1}{2}\sqrt{B^2 + \frac{4A^3}{27}}\right]^{\frac{1}{3}}, \quad s_2 = \left[-\frac{1}{2}\sqrt{B^2 + \frac{4A^3}{27}}\right]^{\frac{1}{3}} \tag{4.44a}$$

$$t_1 = \left[-\frac{B}{2} - \frac{1}{2}\sqrt{B^2 + \frac{4A^3}{27}}\right]^{\frac{1}{3}}, \quad t_2 = \left[-\frac{B}{2} + \frac{1}{2}\sqrt{B^2 + \frac{4A^3}{27}}\right]^{\frac{1}{3}} \tag{4.44b}$$

An alternative approach to finding the eigenvalues and eigenvectors of this system of partial differential equations is to take the solution for the real gas as the solution to an ideal gas at a higher pressure; that is to say if, at time $t = 0$, the left state is (ρ_l, u_l, p_l) and the right upstream state is (ρ_r, u_r, p_r) use the ideal solution with the upstream states (ρ_l, u_l, p_{il}) and (ρ_r, u_r, p_{ir}) such that

$$p_{il} = (p_l + a\rho_l^2)(1 - \rho_l b) \tag{4.45a}$$

$$p_{ir} = (p_r + a\rho_r^2)(1 - \rho_r b) \tag{4.45b}$$

Such a solution is shown in Figure 4.7.

4.13   Conclusions

Our discussion of the Euler equations began with the assumption that for the present analysis only kinetic and internal energy will be significant. This assumption is quite reasonable for an ideal gas in which intermolecular interactions are considered negligible. By introducing new physical parameters, namely the specific heat capacities at constant pressure and volume and the adiabatic exponent it was shown the total energy will determined by (4.10), known as the γ-law equation of state. Using this equation of state with the conservation law we deduced the conservation equation for energy which through some manipulation can be reduced to a conservation equation for pressure, (4.13). This last relation can be coupled with the conservation equations for mass and velocity to form a quasilinear hyperbolic system of equations. With the eigensystem for the Euler equations stated it was discussed how a linear degenerate field is interpreted in the solution as a contact discontinuity and how in particular for the Euler equations shows that velocity and pressure will constant across the discontinuity. To analyse the Riemann problem for the Euler equations it was necessary to define new intermediary states on either side of the discontinuity, $q_{l,r}^*$, and derive an advection equation for entropy, (4.26). This advection equation is satisfied everywhere except at the jump in the initial data, hence we could state all the Riemann invariants of the system and derive the equations of the integral curves of the non linear simple waves. By taking the intersection of these the curves and with some manipulation expressions were derived for the intermediary states. With all the theory necessary to solve the Riemann problem
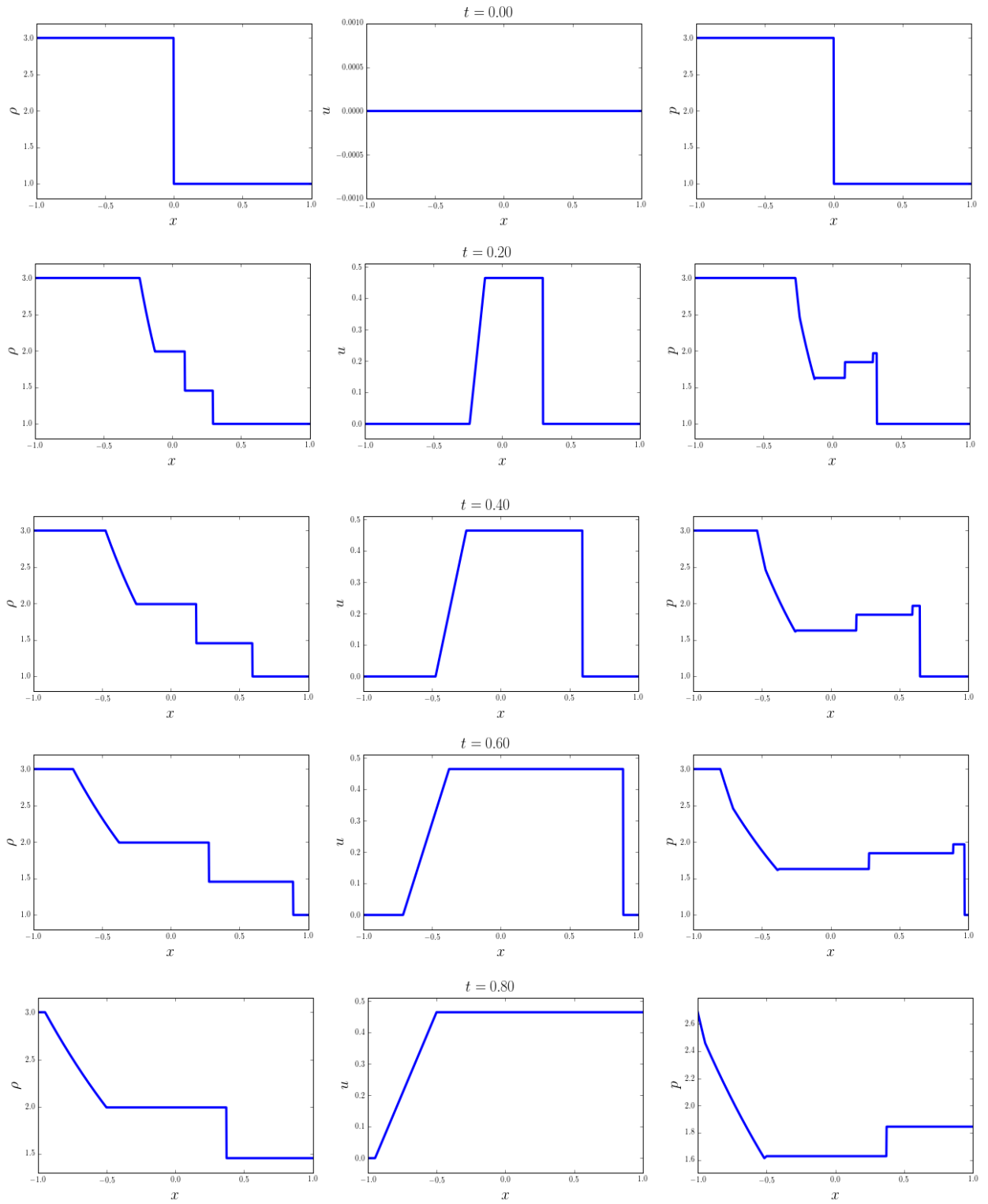
Figure 4.7: The exact solution to the Riemann problem defined by the Euler equations applied to a van der Waals oxygen gas under isentropic conditions and initial conditions as shown at $t = 0$

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for the Euler equations derived a linear and non linear solution to a problem involving two gases coming into contact with one another and describing the process of cavitation was shown in § 4.9, § 4.10 and § 4.11 respectively. We ended this section by discussing how the solutions will be affected is the assumptions associated with the ideal gas are dropped; the effects of non negligible intermolecular forces and finite molecular size increase the total energy of the gas and reduce the pressure.

Closing Remarks

As stated the purpose of this work was to derive a system of conservation equations to describe the evolution of the physical parameters of a fluid in time and space and find solutions to satisfy these equations. In § 1, § 2 and § 4 we derived the conservation equations in mass, momentum and energy respectively which collectively are known as the Euler equations. It as also shown that if this non-linear system of equations could be expressed as a quasi-linear hyperbolic equation whose variable parameters are vectors in 3-space then there exists simple wave solutions which as shown can be generated from the eigensystem of the Jacobian. It was also considered, in § 3, how these simple wave solutions are affected by discontinuities in the initial data; a situation known as the Riemann problem. The simple wave solution to the Riemann problem was shown to be a combination of rarefaction waves and shock waves; the combination determined by the initial data. It was also shown that there exists a intermediary state between these waves, this intermediary state was shown to be the solution to the equations describing the locus of states which can be connected to the upstream states by rarefaction and shock waves; these equations were derived via the Riemann invariants and the Rankine-Hugoniot condition respectively. Solutions in one dimensional space and time were found for ideal gases under isothermal conditions, Figure 3.7, isentropic conditions, Figure 4.4, a disturbed shallow liquid, Figure 3.9, and gases described by the van der Waals equation os state, Figure 4.7.

Further work on this topic may include solutions for the Navier-Stokes equations, these are essentially the inhomogeneous form of the Euler equations and account for the affects of viscosity in the fluid. Other possibilities would be to generalise the Euler equations for three dimensions.

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \nabla \cdot \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix} = 0$$

Finding exact solutions to these equations may not possible but a numerical solution could certainly be realised with approximations $q(x, y, z, t) \approx q_{ijkl}$; alternatively it may be more convenient to express the physical parameters in spherical or cylindrical coordinates as opposed to Cartesian.

A Python Script

A.1 Figure 1.3

```
from matplotlib.pyplot import *
from numpy import linspace, sqrt,cos,sin,log
import warnings
warnings.filterwarnings("ignore")

rc('text', usetex=True)
rc('font', family='serif')

dx=0.02*pi; r=0.06
x_end=2*pi

N=int(x_end/dx); Nt=int(t_end/dt)
x=zeros(N+2); w0=zeros(N+2); w1=zeros(N+2); w2=zeros(N+2);
w3=zeros(N+2)
w4=zeros(N+2)
#Initial conditions
x[0]=0; w0[0]=0; w1[0]=0; w2[0]=0


for i in range(1,N+1):
x[i]=x[i-1]+dx

w0[i]=1-cos(x[i])
w1[i]=w0[i]*(1-r*w0[i+1]+r*w0[i])
w2[i]=w1[i]*(1-r*w1[i+1]+r*w1[i])
w3[i]=w2[i]*(1-r*w2[i+1]+r*w2[i])
w4[i]=w3[i]*(1-r*w3[i+1]+r*w3[i])

figure(figsize=(12,8))
plot(x,w0,x,w1,x,w2,x,w3,x,w4,linewidth=3)
legend((r'$j=0$','$j=1$','$j=2$','$j=3$','$j=4$'),loc='best',fontsize=15)
xlabel(r'$x$',fontsize=20); ylabel(r'$w_{ij}$',fontsize=20)
axis([0,2*pi,0,4])

show()
```

A.2 Figure 3.4

```
get_ipython().magic('matplotlib inline')
from numpy import linspace,zeros,sqrt,arange
from warnings import filterwarnings
from matplotlib.pyplot import *

filterwarnings("ignore")

rc('text', usetex=True)
rc('font', family='serif')

def rho(x):
value=zeros(npt)
for i in range(npt):
```

```
if x[i]<0:
value[i]=2
if x[i] >= 0:
value[i]=1
return value

def u_1(rho_star,u_star,kappa,gamma,x):
return u_star - (2/gamma-1)*sqrt(kappa*gamma)*(rho(x)**((gamma-1)/2)-rho_star**((gamma-1)/2))

def u_2(rho_star,u_star,kappa,gamma,x):
return u_star + (2/gamma-1)*sqrt(kappa*gamma)*(rho(x)**((gamma-1)/2)-rho_star**((gamma-1)/2))

def t_1(rho_star,u_star,kappa,gamma,x_star,x):
return (1/u_1(rho_star,u_star,kappa,gamma,x))*(x-x_star)

def t_2(rho_star,u_star,kappa,gamma,x_star,x):
return (1/u_2(rho_star,u_star,kappa,gamma,x))*(x-x_star)

xlim=6
npt=1000
dx=xlim/npt

x=linspace(-xlim,xlim,npt)

fig1=figure(figsize=(9,9))

#subplot(121)
for x_star in arange(-4*xlim,4*xlim,dx*200):
plot(x,t_1(0.8,0,1,1.4,x_star,x),linewidth=3)
axis([-6,6,-100,100])
xlabel('x',fontsize=20)
ylabel('t',fontsize=20)
```

A.3   Figure 4.3

```
get_ipython().magic('matplotlib inline')
from numpy import linspace,zeros,exp,arange,sqrt, cos
from warnings import filterwarnings
from math import pi
from matplotlib.pyplot import *

filterwarnings("ignore")


def v_2(rho_0,u_0,gamma,c_0,rho):
return u_0 + (2*c_0/(gamma-1))*(1-(rho/rho_0)**((gamma-1)/(2*gamma)))

def v_3(rho_0,u_0,gamma,c_0,rho):
return u_0 - (2*c_0/(gamma-1))*(1-(rho/rho_0)**((gamma-1)/(2*gamma)))

rho=linspace(0,4,401)

figure(figsize=(10,10))
```

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```
rc('text',usetex=True)
rc('font', family='serif')

plot(rho,v_2(1,0,1.4,0.7,rho),rho,v_3(0.5,0,1.4,0.7,rho),linewidth=3)
plot(0.7011,0.1731,marker='o',ms=12,color='black')
plot(1,0,'bo',ms=12)
plot(0.5,0,'go',ms=12)
axis([0,1.1,-2,2])
xlabel(r'p', fontsize=20)
ylabel(r'u', fontsize=20)
legend((r'v^2','v^3'),fontsize=20,loc='best')
annotate(r'$(0.701,0.173)$', xy=(0.65,0.4),fontsize=20)
annotate(r'$(1,0)$', xy=(1,-0.3),fontsize=20)
annotate(r'$\left( \frac{1}{2} , 0\right)$', xy=(0.5,-0.3),fontsize=20)
```

A.4 Figure 4.4

```
# -*- coding: utf-8 -*-
"""
Created on Fri Nov 18 08:58:46 2016

@author: lyons
"""

get_ipython().magic('matplotlib inline')
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve
import math as m
import warnings
warnings.filterwarnings("ignore")

plt.rc('text', usetex=True)
plt.rc('font', family='serif')

def rie_1d_isen_exact(rhol,ul,pl,rhor,ur,pr,gamma,t,x):
"""Exact solution to 1d Isentropic equations
"""


#obtain isentropic constant for internal use
kapl=pl/(rhol**gamma)
kapr=pr/(rhor**gamma)

def speed_l(p):
return np.sqrt(gamma*kapl**(1/gamma))*p**((gamma-1)/(2*gamma))
def press_l(rho):
return kapl*rho**gamma
def rho_l(p):
return (p/kapl)**(1/gamma)

def speed_r(p):
return np.sqrt(gamma*kapr**(1/gamma))*p**((gamma-1)/(2*gamma))
def press_r(rho):
```

```
return kapr*rho**gamma
def rho_r(p):
return (p/kapr)**(1/gamma)

#pl=press_l(rhol)
cl=speed_l(pl)
#pr=press_r(rhor)
cr=speed_r(pr)


#expressions for downstream velocity ustar as function of downstream density rhostar
shk1 = lambda pstar : ul - np.sqrt((((1/rho_l(pstar))-(1/rhol))*(pl-pstar)))
rar1 = lambda pstar : ul + (2*cl/(gamma-1))*(1-(pstar/pl)**((gamma-1)/(2*gamma)))

shk3 = lambda pstar : ur + np.sqrt((((1/rho_r(pstar))-(1/rhor))*(pr-pstar)))
rar3 = lambda pstar : ur - (2*cr/(gamma-1))*(1-(pstar/pr)**((gamma-1)/(2*gamma)))


#decide on whether wave is shock or rarefaction for given value of rhostar
def phil(pstar):
if pstar<pl: return rar1(pstar)
else: return shk1(pstar)

def phir(pstar):
if pstar<pr: return rar3(pstar)
else: return shk3(pstar)

#difference in values of downstream velocities is measure of error in rhostar
phi = lambda pstar : phil(pstar)-phir(pstar)

pstar,info, ier, msg = fsolve(phi, (pl+pr)/2.,full_output=True,factor=0.1,xtol=1.e-10)

#common upstream velocity
ustar = phil(pstar)


#def f(rholstar):
#   return press_l(rholstar)-pstar
rholstar=rho_l(pstar)


#g = lambda rhorstar : press_r(rhorstar)-pstar
rhorstar = rho_r(pstar)

#common upstream soundspeed
clstar=speed_l(pstar)
crstar=speed_r(pstar)

ws = np.zeros(5)

if pstar<pl:
ws[0] = ul - cl
ws[1] = ustar - clstar
else:
```

```python
ws[0] = (rhol*ul - rholstar*ustar)/(rhol - rholstar)
ws[1] = ws[0]

ws[2]=ustar

if pstar<pr:
ws[3] = ustar + crstar
ws[4] = ur + cr
else:
ws[3] = (rhor*ur - rhorstar*ustar)/(rhor - rhorstar)
ws[4] = ws[3]


xs = ws*t # Wave front positions

xi = x/t  # coordinate for rarefactions

def E(rho,u,p,gamma):
return  p/(gamma-1.) + 0.5*rho*u**2

El=E(rhol,ul,pl,gamma)
Elstar=E(rholstar,ustar,pstar,gamma)
Erstar=E(rhorstar,ustar,pstar,gamma)
Er=E(rhor,ur,pr,gamma)

u1=((gamma-1)/(gamma+1))*ul+(2/(gamma+1))*(xi+cl) #left moving rarefaction
c1=cl+0.5*(gamma-1)*(ul-u1)
rho1=rhol*(c1/cl)**(2/(gamma-1))
p1=(1/gamma)*rho1*c1**2
E1=E(rho1,u1,p1,gamma)

u2=((gamma-1)/(gamma+1))*ur+(2/(gamma+1))*(xi-cr) #right moving rarefaction
c2=cr-0.5*(gamma-1)*(ur-u2)
rho2=rhor*(c2/cr)**(2/(gamma-1))
p2=(1/gamma)*rho2*c2**2
E2=E(rho2,u2,p2,gamma)

cnd=[x<=xs[0], (x>xs[0]) & (x<xs[1]), (x>=xs[1]) & (x<=xs[2]), (x>xs[2]) & (x<xs[3]), (x>xs[



uout=np.select(cnd,[ul,u1,ustar,ustar,u2,ur])
cout=np.select(cnd,[cl,c1,clstar,crstar,c2,cr])
rhoout=np.select(cnd,[rhol,rho1,rholstar,rhorstar,rho2,rhor])
pout=np.select(cnd,[pl,p1,pstar,pstar,p2,pr])
#Eout=np.select(cnd,[El,E1,Elstar,Erstar,E2,Er])
Eout=E(rhoout,uout,pout,gamma)



return rhoout, uout, pout, Eout
```

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```python
def rie_1d_isen_lin(rhol,ul,pl,rhor,ur,pr,gamma,t,x):
"""Exact solution to 1d Isentropic equations
"""

npt=1000
#obtain isentropic constant for internal use
kapl=pl/(rhol**gamma)
kapr=pr/(rhor**gamma)

def speed_l(p):
return np.sqrt(gamma*kapl**(1/gamma))*p**((gamma-1)/(2*gamma))
def press_l(rho):
return kapl*rho**gamma
def rho_l(p):
return (p/kapl)**(1/gamma)

def speed_r(p):
return np.sqrt(gamma*kapr**(1/gamma))*p**((gamma-1)/(2*gamma))
def press_r(rho):
return kapr*rho**gamma
def rho_r(p):
return (p/kapr)**(1/gamma)

#pl=press_l(rhol)
cl=speed_l(pl)
#pr=press_r(rhor)
cr=speed_r(pr)


#expressions for downstream velocity ustar as function of downstream density rhostar
shk1 = lambda pstar : ul - np.sqrt((((1/rho_l(pstar))-(1/rhol))*(pl-pstar)))
rar1 = lambda pstar : ul + cl - (cl/rhol)*rho_l(pstar)

shk3 = lambda pstar : ur + np.sqrt((((1/rho_r(pstar))-(1/rhor))*(pr-pstar)))
rar3 = lambda pstar : ur - cr + (cr/rhor)*rho_r(pstar)


#decide on whether wave is shock or rarefaction for given value of rhostar
def phil(pstar):
if pstar<pl: return rar1(pstar)
else: return shk1(pstar)

def phir(pstar):
if pstar<pr: return rar3(pstar)
else: return shk3(pstar)

#difference in values of downstream velocities is measure of error in rhostar
phi = lambda pstar : phil(pstar)-phir(pstar)

pstar,info, ier, msg = fsolve(phi, (pl+pr)/2.,full_output=True,factor=0.1,xtol=1.e-10)

#common upstream velocity
ustar = phil(pstar)
```

```
#def f(rholstar):
#    return press_l(rholstar)-pstar
rholstar=rho_l(pstar)


#g = lambda rhorstar : press_r(rhorstar)-pstar
rhorstar = rho_r(pstar)

#common upstream soundspeed
clstar=speed_l(pstar)
crstar=speed_r(pstar)

def H(x):
v=np.zeros(npt)
for i in range(npt):
if x[i]<0:
v[i]=0
if x[i]>0:
v[i]=1
return v

def w_l(rhol,ul,pl,cl,rhor,ur,pr):
u0=ustar
rho0=(rhorstar+rholstar)/2
c0=(crstar+clstar)/2

return rhol-(pl/(c0**2)), 0.5*ul - (pl/(2*rho0*c0)), 0.5*ul + (pl/(2*rho0*c0))




def w_r(rhol,ul,pl,cl,rhor,ur,pr):
u0=ustar
rho0=(rhorstar+rholstar)/2
c0=(crstar+clstar)/2

return rhor-(pr/(c0**2)), 0.5*ur - (pr/(2*rho0*c0)), 0.5*ur + (pr/(2*rho0*c0))

def u_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t):

u0=ustar
rho0=(rhorstar+rholstar)/2
c0=(crstar+clstar)/2

lambda1=u0
lambda2=u0-c0
lambda3=u0+c0

w1r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[0]
w2r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[1]
w3r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[2]

w1l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[0]
```

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```python
w2l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[1]
w3l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[2]


ta = t*np.ones(npt)

return ul +H(x-lambda2*ta)*(w2r-w2l) + H(x-lambda3*ta)*(w3r-w3l)



def rho_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t):

u0=ustar
rho0=(rhorstar+rholstar)/2
c0=(crstar+clstar)/2

lambda1=u0
lambda2=u0-c0
lambda3=u0+c0

w1r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[0]
w2r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[1]
w3r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[2]

w1l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[0]
w2l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[1]
w3l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[2]

ta = t*np.ones(npt)

return rhol +H(x-lambda1*ta)*(w1r-w1l) - H(x-lambda2*ta)*(w2r-w2l)*(rho0/c0) + H(x-lambda3*t



def p_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t):

u0=ustar
rho0=(rhorstar+rholstar)/2
c0=(crstar+clstar)/2

lambda1=u0
lambda2=u0-c0
lambda3=u0+c0

w1r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[0]
w2r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[1]
w3r=w_r(rhol,ul,pl,cl,rhor,ur,pr)[2]

w1l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[0]
w2l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[1]
w3l=w_l(rhol,ul,pl,cl,rhor,ur,pr)[2]

ta = t*np.ones(npt)

return pl - H(x-lambda2*ta)*(w2r-w2l)*rho0*c0 + H(x-lambda3*ta)*(w3r-w3l)*rho0*c0
```

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```
def E_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t):

p=p_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)
gamma=1.4
rho=rho_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)
u=u_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)

return (p/(gamma-1)) + 0.5*rho*u**2

rhoout=rho_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)
uout=u_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)
pout=p_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)
Eout=E_lin_isen(rhol,ul,pl,cl,rhor,ur,pr,t)

return rhoout,uout,pout,Eout



def plot_rie_1d_euler(rhol,ul,pl,rhor,ur,pr,gamma,t):

x = np.linspace(-1.,1.,1000)

rholin, ulin, plin, Elin = rie_1d_isen_lin(rhol,ul,pl,rhor,ur,pr,gamma,t,x)
rho, u, p, E = rie_1d_isen_exact(rhol,ul,pl,rhor,ur,pr,gamma,t,x)

fig = plt.figure(figsize=(12,12))
sol = [rho,u,p,E]
linsol=[rholin,ulin,plin,Elin]
#names = ['Density','Velocity','Pressure','Energy']
ylabels = [r'$\rho$','$u$','$p$','$E$']
axes = [0]*4
fts=20
for i in range(4):
axes[i] = fig.add_subplot(2,2,i+1)
q = sol[i]
ql = linsol[i]
plt.plot(x,q,x,ql,linewidth=3)
qmax = max(q)
qmin = min(q)
qdiff = qmax - qmin
axes[i].set_ylim((qmin-0.1*qdiff,qmax+0.1*qdiff))
plt.xlabel(r'$x$',fontsize=fts)
plt.ylabel(ylabels[i],fontsize=fts)
plt.legend((r'Exact Solution','Linear Solution'),loc='best',fontsize=0.4*fts)

plt.suptitle(r'$t=%.2f$' % t, fontsize=fts)

plot_rie_1d_euler(3,0,3,1,0,1,1.4,0.4)
```

B   Derivations

B.1   Equation (3.19a)

$$\frac{\rho u - \rho^* u^*}{\rho - \rho^*}(\rho u - \rho^* u^*) = \rho u^2 + p - \rho u^{*2} - p^*$$

$$\frac{(\rho u - \rho^* u^*)^2}{\rho - \rho^*} = p - p^* + \rho u^2 - \rho^* u^{*2}$$

$$(\rho u - \rho^* u^*)^2 = (p - p^*)(\rho - \rho^*) + \rho^2 u^2 - \rho\rho^* u^{*2} - \rho\rho^* u^{*2} + \rho^{*2} u^{*2}$$

$$-2\rho\rho^* u u^* = (p - p^*)(\rho - \rho^*) - \rho\rho^*(u^2 - u^{*2})$$

$$2uu^* = u^2 + u^{*2} - \left(\frac{\rho - \rho^*}{\rho\rho^*}\right)(p - p^*)$$

$$u^2 - 2uu^* + u^{*2} - \left(\frac{1}{\rho^*} - \frac{1}{\rho}\right)(p - p^*) = 0$$

$$(u - u^*)^2 = \left(\frac{1}{\rho^*} - \frac{1}{\rho}\right)(p - p^*)$$

$$u = u^* \pm \sqrt{\left(\frac{1}{\rho^*} - \frac{1}{\rho}\right)(p - p^*)}$$

B.2   Equation (3.20)

$$v(h - h^*) = hu - h^* u^* \Rightarrow v = \frac{hu - h^* u^*}{h - h^*}$$

$$\frac{hu - h^* u^*}{h - h^*}(hu - h^* u^*) = hu^2 + \frac{1}{2}gh^2 - h^* u^{*2} - \frac{1}{2}gh^{*2}$$

$$(hu - h^* u^*)^2 = (hu^2 - h^* u^{*2})(h - h^*) + \frac{1}{2}g(h^2 - h^{*2})(h - h^*)$$

$$-2hh^* uu^* = -hh^* u^2 - hh^* u^{*2} + \frac{1}{2}g\left(\frac{h^2 - h^{*2}}{hh^*}\right)(h - h^*)$$

$$-2uu^* = -u^2 - u^{*2} + \frac{1}{2}g\left(\frac{h}{h^*} - \frac{h^*}{h}\right)(h - h^*)$$

$$u^2 - 2uu^* + u^{*2} - \frac{1}{2}g\left(\frac{h}{h^*} - \frac{h^*}{h}\right)(h - h^*) = 0$$

$$u^2 - 2uu^* + u^{*2} = \frac{1}{2}g\left(\frac{h}{h^*} - \frac{h^*}{h}\right)(h - h^*)$$

$$(u - u^*)^2 = \frac{1}{2}g\left(\frac{h}{h^*} - \frac{h^*}{h}\right)(h - h^*)$$

$$u = u^* \pm \sqrt{\frac{1}{2}g\left(\frac{h}{h^*} - \frac{h^*}{h}\right)(h - h^*)}$$

B.3   Equation (4.14)

$$E = \frac{1}{2}[\alpha p + \rho u^2]$$

$$E_t = \frac{1}{2}[\alpha p_t + \rho_t u^2 + 2\rho u u_t]$$

$$u_t = -\frac{a^2}{\rho}\rho_x - u u_x, \quad \rho_t = -\rho_x u - \rho u_x$$

$$E_t = \frac{1}{2}[\alpha p_t - \rho_x u^3 - \rho u^2 u_x - 2a^2 u \rho_x - 2\rho u^2 u_x]$$

$$E_t = \frac{1}{2}[\alpha p_t - \rho_x u^3 - 3\rho u^2 u_x - 2a^2 u \rho_x]$$

$$(E + p)u = \frac{1}{2}[(\alpha + 2)pu + \rho u^3]$$

$$[(E + p)u]_x = \frac{1}{2}[(\alpha + 2)p_x u + (\alpha + 2)pu_x + \rho_x u^3 + 3\rho u^2 u_x]$$

$$\alpha p_t - 2a^2 u \rho_x + (\alpha + 2)pu_x + (\alpha + 2)p_x u$$

$$p_t - 2\frac{a^2}{\alpha}u\rho_x + \left(1 + \frac{2}{\alpha}\right)pu_x + \left(1 + \frac{2}{\alpha}\right)up_x = 0$$

But $p = a^2\rho \Rightarrow p_x = a^2\rho_x$

$$p_t + \left(1 + \frac{2}{\alpha}\right)pu_x + up_x = 0$$

B.4   Equation (4.26)

$$p = \hat{\kappa}e^{S/C_v}\rho^\gamma$$

$$p_t = \frac{\hat{\kappa}}{C_v}e^{S/C_v}s_t\rho^\gamma + \hat{\kappa}e^{\Sigma/C_v}\rho_t$$

$$p_t = p\left(\frac{s_t}{C_v} - \frac{\gamma}{\rho}(\rho u)_x\right)$$

$$p_t = p\left(\frac{S_t}{C_v} - \frac{\gamma\rho_x u}{\rho} - \gamma u_x\right)$$

$$p_x = \frac{\hat{\kappa}}{C_v}e^{S/C_v}s_x\rho^\gamma + \hat{\kappa}e^{\Sigma/C_v}\rho_x$$

$$p_x = p\left(\frac{S_x}{C_v} + \frac{\gamma}{\rho}\rho_x\right)$$

$$up_x = p\left(\frac{uS_x}{C_v} + \frac{\gamma u \rho_x}{\rho}\right)$$

$$p_t + \gamma pu_x + up_x = 0$$

$$p\left[\frac{S_t}{C_v} - \frac{\gamma\rho_x u}{\rho} - \gamma u_x + \gamma u_x \frac{uS_x}{C_v}\frac{\gamma u\rho_x}{\rho}\right] = 0$$

$$\frac{p}{C_v}(S_t + uS_x) = 0$$

But $p, C_v \neq= 0$ therefore

$$S_t + uS_x = 0$$

B.5   Equation (4.29)

$$dq = r^2 ds$$

$$\begin{bmatrix} d\rho \\ du \\ dp \end{bmatrix} = \begin{bmatrix} -\rho/c \\ 1 \\ -\rho c \end{bmatrix} ds$$

$$\int -c\frac{d\rho}{\rho} = -\int c_0 \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} \cdot \rho^{-1} d\rho$$

$$= -c_0 \rho_0^{\frac{1-\gamma}{2}} \int \rho^{\frac{\gamma-3}{2}} d\rho$$

$$= -c_0 \rho_0^{\frac{1-\gamma}{2}} \left[\frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}}\right]$$

$$= -\frac{c_0}{\rho_0^{\frac{\gamma-1}{2}}} \left(\frac{2\rho^{\frac{\gamma-1}{2}}}{\gamma-1}\right)$$

$$= -\frac{2c}{\gamma-1}$$

$$du = -\frac{c}{\rho} d\rho$$

$$u = \text{constant} - \int \frac{c}{\rho} d\rho$$

$$u = \text{constant} - \frac{2c}{\gamma-1}$$

$$\Rightarrow \text{constant} = u + \frac{2c}{\gamma-1}$$

Thus $u + \frac{2c}{\gamma-1}$ is a Riemann invariant for the family-2 curves. Furthermore if there is some reference state (ρ_0, u_0) then the integral curve will be

$$u = u_0 + \frac{2c_0}{\gamma-1} - \frac{2c}{\gamma-1}$$

$$u = u_0 + \frac{2c_0}{\gamma-1}\left[1 - \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}}\right]$$

But across the 2 and 3-waves entropy is constant so the isentropic condition (2.6) and be used with (4.28) to express the integral curves in $p - u$ space.

$$\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \frac{c^2}{c_0^2} = \frac{\gamma}{c_0^2}\frac{p}{\rho}$$

$$p = \frac{c_0^2}{\gamma}\frac{\rho^\gamma}{\rho_0^{\gamma-1}}$$

$$p(\rho) = \frac{c_0^2\rho_0}{\gamma}\left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$p_0 = p(\rho_0) = \frac{c_0^2\rho_0}{\gamma}$$

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$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \Rightarrow \frac{\rho}{\rho_0} = \left(\frac{p}{p_0}\right)^{1/\gamma}$$

$$\left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}$$

$$u = u_0 + \frac{2c_0}{\gamma - 1}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}\right]$$

B.6   Equation (4.38)

$$(p + an^2\rho^2)(1/\rho - nb) = RT$$

$$\frac{p}{\rho} - bnp + an^2\rho - abn^3\rho^2 = RT$$

$$p - \rho bnp + an^2\rho^2 - abn^3\rho^3 = \rho RT$$

$$p = \frac{\rho RT + abn^3\rho^3 - an^2\rho^2}{1 - nb\rho}$$

$$p = \rho RT - an^2\rho^2(1 - nb\rho)$$

$$p = \frac{\rho RT}{1 - nb\rho} - an^2\rho^2$$

Because the present analysis is not interested in the extensive properties of the gas we will take $n = 1$.

B.7   Equation (4.41)

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 + a\rho^2$$

$$E_t = \frac{p_t}{\gamma - 1} + \rho u u_t + \frac{1}{2}\rho_t u^2 + 2a\rho\rho_t$$

$$E_t = \frac{p_t}{\gamma - 1} + \rho u(-uu_x - c^2/\rho\rho_x) + \frac{1}{2}(-\rho_x u - \rho u_x)u^2 + 2a\rho(-\rho_x u - \rho u_x)$$

$$E_t = \frac{p_t}{\gamma - 1} - \rho u^2 u_x - c^2 u\rho_x - \frac{1}{2}u^2 u_x\rho - \frac{1}{2}u^3\rho_x - 2a\rho^2 u_x - 2au\rho\rho_x$$

$$E + p = \frac{\gamma p}{\gamma - 1} + \frac{1}{2}\rho u^2 + a\rho^2$$

$$(E + p)u = \frac{\gamma pu}{\gamma - 1} + \frac{1}{2}\rho u^3 + a\rho^2 u$$

$$[(E + p)u]_x = \frac{\gamma p_x u}{\gamma - 1} + \frac{\gamma pu_x}{\gamma - 1} + \frac{1}{2}\rho_x u^3 + \frac{3}{2}\rho u^2 u_x + 2a\rho\rho_x u + a\rho^2 u_x$$

$$E_t + [(E + p)u]_x = 0$$

$$\frac{p_t}{\gamma - 1} - c^2 u\rho_x - a\rho^2 u_x + \frac{\gamma p_x u}{\gamma - 1} + \frac{\gamma pu_x}{\gamma - 1} = 0$$

$$p_t - c^2(\gamma - 1)u\rho_x - a(\gamma - 1)\rho^2 u_x + \gamma p_x u + \gamma p u_x = 0$$

$$p_t - c^2(\gamma - 1)u\rho_x + [\gamma p - a(\gamma - 1)\rho^2]u_x + \gamma u p_x = 0$$

References

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