Lyapunov optimization 简介

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Outline

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适用任务

现实中，我们做决策时常常受到随机因素的影响：random events, time variation, and uncertainty。解决这种问题会受益于某些应用：

- wireless mesh networks with opportunistic scheduling
- cognitive radio networks
- ad-hoc mobile networks
- internets with peer-to-peer communication
- sensor networks with joint compression and transmission
- ...

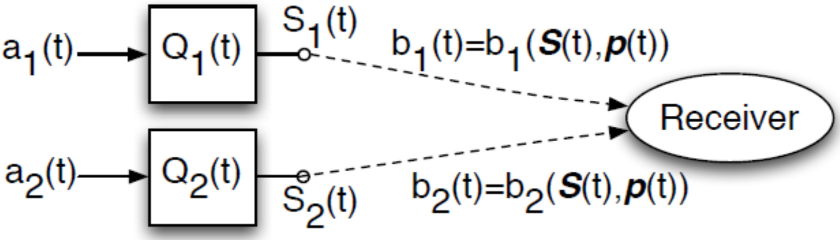
在这些应用中，要解决的问题通常可以形式化为：optimize the time averages of certain quantities subject to time average constraints on other quantities.

多时槽决策系统

考虑离散的时槽（time-slotted）系统，总共有 T 个时槽，每个时槽通常会持续一段时间，在每个时槽前会观察到一些参数的值，然后需要做一个决策，从而得到一个目标函数值。

例子: Opportunistic scheduling

Consider a 2-user wireless uplink that operates in $t \in \{0, 1, 2, ...\}$,



- State (Parameters): the channel conditions $\boldsymbol{S}(t) = (S_1(t), S_2(t))$, such as fading coefficients and/or noise ratios
- Decision: power allocation vector $\boldsymbol{p}(t) = (p_1(t), p_2(t)) \in \mathcal{P}$.
- Constraints: Queues are stable.

问题形式化

- Arrival of queues: $a_1(t), a_2(t)$
- Departure of queues: $b_1(t) = \hat{b}_1(\boldsymbol{p}(t), \boldsymbol{S}(t))$; $b_2(t) = \hat{b}_2(\boldsymbol{p}(t), \boldsymbol{S}(t))$
- Queues dynamics:

$$Q_k(t+1) = \max[Q_k(t) - \hat{b}_k(\boldsymbol{p}(t), \boldsymbol{S}(t)), 0] + a_k(t), \forall k \in \{1,2\}, \forall t \in \{0,1,2,...\}$$

Let $\bar{p}_k$ be the time average power expenditure of user k under a scheduling algorithm. The scheduling problem can be formulated as,

$$\begin{aligned} \min_{\boldsymbol{p}(1),\boldsymbol{p}(2),\ldots,\boldsymbol{p}(T)} \quad & \bar{p}_1 + \bar{p}_2 \\ \text{s.t.} \quad & \text{Queues } Q_k(t) \text{ are stable}, \forall k \in \{1,2\} && \text{(1a)} \\ & \boldsymbol{p}(t) \in \mathcal{P}, \forall t \in \{0,1,2,...\}. && \text{(1b)} \end{aligned}$$

Outline

随机优化问题: optimizing a time average (or a function of time averages) subject to time average constraints.

- operates in discrete time, $t \in \{0, 1, 2, ...\}$
- There are K queues, $\boldsymbol{Q}(t) = (Q_1(t), ..., Q_K(t))$.
- In round t, observing random event $\omega(t)$ and taking action $\alpha(t) \in \mathcal{A}_{\omega(t)}$, which affects attribute vectors $\boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{e}(t)$:

$$\begin{aligned}
\boldsymbol{x}(t) &= (x_1(t), ..., x_M(t)), & x_m(t) &= \hat{x}_m(\omega(t), \alpha(t)), \forall m \in \{1, ..., M\}, \\
\boldsymbol{y}(t) &= (y_0(t), y_1(t), ..., y_L(t)), & y_l(t) &= \hat{y}_l(\omega(t), \alpha(t)), \forall l \in \{0, 1, ..., L\}, \\
\boldsymbol{e}(t) &= (e_1(t), ..., e_J(t)). & e_j(t) &= \hat{e}_j(\omega(t), \alpha(t)), \forall j \in \{1, ..., J\}.
\end{aligned}$$

- Let $\bar{x}_m, \bar{y}_l, \bar{e}_j$ represent the time average of $x_m(t), y_l(t), e_j(t)$ under a control algorithm.

问题形式化

设计一个控制算法对应要解决的问题如下:

$$\min_{\alpha(1),\alpha(2),\ldots,\alpha(T)} \quad \bar{y}_0$$

$$\text{s.t.} \quad \bar{y}_l \leq 0, \forall l \in \{1, \ldots, L\}, \tag{2a}$$

$$\bar{e}_j = 0, \forall j \in \{1, \ldots, J\}, \tag{2b}$$

$$\alpha(t) \in \mathcal{A}_{\omega(t)}, \forall t \in [T], \tag{2c}$$

$$\text{Queues } Q_k(t) \text{ are stable}, \forall k \in [K]. \tag{2d}$$

- A solution is an algorithm for choosing control actions over time in reaction to the existing network state, such that all of the constraints are satisfied and the quantity to be minimized is as small as possible
- Even if there are no underlying queues in the original problem, we can introduce virtual queues as a strong method for ensuring that the required time average constraints are satisfied.
- Inefficient control actions incur larger backlog in certain queues.
- These backlogs act as "sufficient statistics" on which to base the next control decision. This enables algorithms that do not require knowledge of the probabilities associated with the random network events $\omega(t)$

Solving Problem (2) can be done with a simple and elegant theory of Lyapunov drift and Lyapunov optimization.

- Transformation: transform the time-average constraints in Eq. (2a)–(2b) into virtual queues
- Define a Lyapunov function $L(t)$: the sum of squares of backlog in all virtual and actual queues on slot t
  - it is a scalar measure of network (queue) congestion.
  - If $L(t)$ is small, then all queues are small
  - Otherwise, at least one queue is large.
- Define Lyapunov drift: $\Delta(t) = L(t+1) - L(t)$.
- 思路 1 (minimizing the Lyapunov drift): If control decisions are made every slot t to greedily minimize $\Delta(t)$, then backlogs are consistently pushed towards a lower congestion state, which intuitively maintains network stability.
- 思路 2 (drift-plus-penalty): jointly stabilizing queues and minimizing objective. The objective function is mapped to an appropriate function penalty(t). Greedily minimize the following drift-plus-penalty expression,

$$\Delta(t) + V\text{penalty}(t),$$

  where V is a non-negative control parameter.

算法特点与结果

特点：

- Online decision
- Without knowledge of the probabilities associated with the random network

理论结果：

- a smooth tradeoff between backlog reduction and penalty minimization
- $[\mathcal{O}(1/V), \mathcal{O}(V)]$ performance-delay tradeoff: the time average objective function deviates by at most $\mathcal{O}(1/V)$ from optimality, with a time average queue backlog bound of $\mathcal{O}(V)$.

Outline

两队列无线下行链路传输调度：建模

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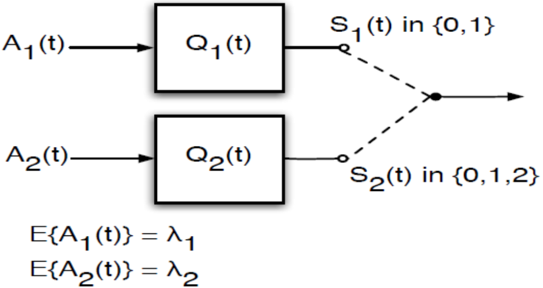
The 2-queue wireless downlink example with time-varying channels



- State (Parameters): the channel conditions $\boldsymbol{S}(t) = (S_1(t), S_2(t))$. $S_i(t)$ is a non-negative integer that represents the number of packets that can be transmitted over channel i on slot t (for $i \in \{1, 2\}$).
- Decision: transmission decision $\alpha(t) \in \{$"Transmit over channel 1", "Transmit over channel 2", "Idle"$\}$.
- Constraints: Queues are stable.

建模：续

- Arrival of queues: $A_1(t), A_2(t)$ i.i.d. $\lambda_1 \triangleq \mathbb{E}\{A_1(t)\}$ and $\lambda_2 \triangleq \mathbb{E}\{A_2(t)\}$.
- Departure of queues: $b_1(t) = \hat{b}_1(\alpha(t), \boldsymbol{S}(t))$; $b_2(t) = \hat{b}_2(\alpha(t), \boldsymbol{S}(t))$

$$b_i(t) = \hat{b}_i(\alpha(t), \boldsymbol{S}(t)) \triangleq \begin{cases} S_i(t), & \text{if } \alpha(t) = \text{Transmit over channel } i, \\ 0, & \text{otherwise.} \end{cases}$$

- Queues dynamics:

$$Q_i(t+1) = \max[Q_i(t) - \hat{b}_i(\alpha(t), \boldsymbol{S}(t)), 0] + A_i(t), \forall i \in \{1, 2\}, \forall t \in \{0, 1, 2, ...\}$$

离线静态随机调度策略

- Channel domain $\mathcal{S} = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}$
- S-only scheduling algorithms: characterized by two probabilities $q_1(S_1,S_2)$ and $q_2(S_1,S_2)$, where $q_i(S_1,S_2)$ is the probability of transmitting over channel i if $\boldsymbol{S}(t)=(S_1,S_2)$.
- Let $\alpha^*(t)$ represent the transmission decisions under a particular S-only policy and define $b_1^*(t)=\hat{b}_1(\alpha^*(t),\boldsymbol{S}(t))$; $b_2^*(t)=\hat{b}_2(\alpha^*(t),\boldsymbol{S}(t))$. By definition, we have,

$$\mathbb{E}\{b_1^*(t)\} = \sum_{(S_1,S_2)\in\mathcal{S}} \text{Pr}[S_1,S_2]S_1q_1(S_1,S_2)$$

$$\mathbb{E}\{b_2^*(t)\} = \sum_{(S_1,S_2)\in\mathcal{S}} \text{Pr}[S_1,S_2]S_2q_2(S_1,S_2)$$

- S-only policy is characterized by $q_1(S_1, S_2)$ and $q_2(S_1, S_2)$ that are solved from the following maximization,

$$
\begin{aligned}
\max_{q_1(S_1,S_2),q_2(S_1,S_2)} \quad & \epsilon \\
\text{s.t.} \quad & \lambda_1 + \epsilon \leq \sum_{(S_1,S_2)\in\mathcal{S}} \mathrm{Pr}[S_1, S_2] S_1 q_1(S_1, S_2), && \text{(3a)} \\
& \lambda_2 + \epsilon \leq \sum_{(S_1,S_2)\in\mathcal{S}} \mathrm{Pr}[S_1, S_2] S_2 q_2(S_1, S_2), && \text{(3b)} \\
& q_1(S_1, S_2) + q_2(S_1, S_2) \leq 1, \forall (S_1, S_2) \in \mathcal{S}, && \text{(3c)} \\
& q_1(S_1, S_2) \geq 0, q_2(S_1, S_2) \geq 0, \forall (S_1, S_2) \in \mathcal{S}. && \text{(3d)}
\end{aligned}
$$

- There are 8 known parameters: λ_1, λ_2, $\mathrm{Pr}[S_1, S_2], \forall (S_1, S_2) \in \mathcal{S}$
- There are 13 unknowns: ϵ, $q_1(S_1, S_2)$ and $q_2(S_1, S_2), \forall (S_1, S_2) \in \mathcal{S}$
- S-only policy: require a-priori knowledge of the arrival rates and channel probabilities
- we pursue queue stability via an algorithm that makes decisions based on both the current channel states and the current queue backlogs

- Let $\boldsymbol{Q}(t) = (Q_1(t), Q_2(t))$ be the vector of current queue backlogs, and define a Lyapunov function $L(\boldsymbol{Q}(t))$ as follows:

$$L(\boldsymbol{Q}(t)) = \frac{1}{2}[Q_1(t)^2 + Q_2(t)^2].$$

- Now define $\Delta(\boldsymbol{Q}(t))$ as the conditional Lyapunov drift for slot t:

$$\Delta(Q(t)) \triangleq \mathbb{E}\{L(\boldsymbol{Q}(t+1)) - L(\boldsymbol{Q}(t))|\boldsymbol{Q}(t)\}.$$

- Recall queue dynamics: $Q_i(t+1) = \max[Q_i(t) - \hat{b}_i(\alpha(t), \boldsymbol{S}(t)), 0] + A_i(t)$
- Using $(\max[Q-b, 0] + A)^2 \leq Q^2 + A^2 + b^2 + 2Q(A-b)$ for all $Q \geq 0, b \geq 0, A \geq 0$, we have,

$$\begin{aligned} L(\boldsymbol{Q}(t+1)) - L(\boldsymbol{Q}(t)) &= \frac{1}{2}\sum_{i=1}^{2}\left[Q_i(t+1)^2 - Q_i(t)^2\right] \\ &= \frac{1}{2}\sum_{i=1}^{2}\left[(\max[Q_i(t) - \hat{b}_i(\alpha(t), \boldsymbol{S}(t)), 0] + A_i(t))^2 - Q_i(t)^2\right] \qquad (4) \\ &\leq \sum_{i=1}^{2}\frac{[A_i(t)^2 + \hat{b}_i(\alpha(t), \boldsymbol{S}(t))^2]}{2} + \sum_{i=1}^{2} Q_i(t)[A_i(t) - \hat{b}_i(\alpha(t), \boldsymbol{S}(t))] \end{aligned}$$

在线调度策略：续

- Based on the result of Eq. (4) and the definition of drift $\Delta(\boldsymbol{Q}(t))$, we have,

$$\Delta(\boldsymbol{Q}(t)) \leq \mathbb{E}\left\{\sum_{i=1}^{2} \frac{[A_i(t)^2 + \hat{b}_i(\alpha(t), \boldsymbol{S}(t))^2]}{2} |\boldsymbol{Q}(t)\right\} + \sum_{i=1}^{2} Q_i(t)\lambda_i$$
$$- \mathbb{E}\left\{\sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t))|\boldsymbol{Q}(t)\right\}$$

- Let B be a finite constant satisfying $\mathbb{E}\left\{\sum_{i=1}^{2} \frac{[A_i(t)^2+\hat{b}_i(\alpha(t),\boldsymbol{S}(t))^2]}{2} |\boldsymbol{Q}(t)\right\} \leq B$, we further have,

$$\Delta(\boldsymbol{Q}(t)) \leq B + \sum_{i=1}^{2} Q_i(t)\lambda_i - \mathbb{E}\left\{\sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t))|\boldsymbol{Q}(t)\right\}$$

- **得到了调度策略**: maximizes the drift expression, $\mathbb{E}\left\{\sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t))|\boldsymbol{Q}(t)\right\}$.
- opportunistically maximizing an expectation: The above expression is maximized by the algorithm that observes the current queues $(Q_1(t), Q_2(t))$ and channel states $(S_1(t), S_2(t))$, and chooses $\alpha(t)$ to maximize $\sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t))$.

在线调度策略：理论保证

- Recall that $\Delta(\boldsymbol{Q}(t)) \le B + \sum_{i=1}^{2} Q_i(t)\lambda_i - \mathbb{E}\left\{\sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t)) | \boldsymbol{Q}(t)\right\}$
- Using $\lambda_i + \epsilon_{\text{max}} \le \hat{b}_i(\alpha(t), \boldsymbol{S}(t))$, we further have $\Delta(\boldsymbol{Q}(t)) \le B - \epsilon_{\text{max}} \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\}$
- By definition of $\Delta(\boldsymbol{Q}(t))$, we have $\mathbb{E}\{L(\boldsymbol{Q}(t+1)) - L(\boldsymbol{Q}(t))\} \le B - \epsilon_{\text{max}} \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\}$, holds for all $t \in \{0, 1, 2, ...\}$
- Summing over $t \in \{0, 1, 2, ..., T-1\}$, yields, $\mathbb{E}\{L(\boldsymbol{Q}(T)) - L(\boldsymbol{Q}(0))\} \le BT - \epsilon_{\text{max}} \sum_{t=0}^{T-1} \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\}$
- Rearranging terms, dividing by $\epsilon_{\text{max}} T$, and using the fact that $L(\boldsymbol{Q}(T)) \ge 0$ yields:

$$\frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\} \le \frac{B}{\epsilon_{\text{max}}} + \frac{\mathbb{E}\{L(\boldsymbol{Q}(0))\}}{\epsilon_{\text{max}} T}$$

- Assuming that $\mathbb{E}\{L(\boldsymbol{Q}(0))\} < \infty$ and taking a $\limsup$ yields

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\} \le \frac{B}{\epsilon_{\text{max}}}$$

变体问题: 稳定性和平均功率最小化

- consider the same system, but define $p(t)$ as the power expenditure incurred by the transmission decision $\alpha(t)$ on slot t
- power is a function of $\alpha(t)$, i.e., $p(t) = \hat{p}(\alpha(t))$, which can be expressed as,

$$\hat{p}(\alpha(t)) = \begin{cases} 1, & \text{if } \alpha(t) \in \{\text{“Transmit over channel 1”, “Transmit over channel 2”}\} \\ 0, & \text{if } \alpha(t) = \text{“Idle”}. \end{cases}$$

- 调度思路: Instead of taking a control action to minimize a bound on $\Delta(\boldsymbol{Q}(t))$, we minimize a bound on the following drift-plus-penalty expression:

$$\Delta(\boldsymbol{Q}(t)) + V\mathbb{E}\{p(t)|\boldsymbol{Q}(t)\}$$

- Expanding the above expression, we have,

$$\begin{aligned} \Delta(\boldsymbol{Q}(t)) + V\mathbb{E}\{p(t)|\boldsymbol{Q}(t)\} \leq B + V\mathbb{E}\{\hat{p}(\alpha(t))|\boldsymbol{Q}(t)\} + \sum_{i=1}^{2} Q_i(t)\lambda_i \\ - \mathbb{E}\left\{\sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t))|\boldsymbol{Q}(t)\right\} \end{aligned} \tag{5}$$

变体问题：理论保证

- 调度策略: greedily minimizing $V\hat{p}(\alpha(t)) - \sum_{i=1}^{2} Q_i(t)\hat{b}_i(\alpha(t), \boldsymbol{S}(t))$
- By definition of $\Delta(\boldsymbol{Q}(t))$ and the result of Eq. (5), we have,

$$\mathbb{E}\{L(\boldsymbol{Q}(t+1)) - L(\boldsymbol{Q}(t))\} + V\mathbb{E}\{p(t)\} \leq B + Vp^* - \epsilon \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\}.$$

- Summing over $t \in \{0, 1, 2, ..., T-1\}$, yields,
  $\mathbb{E}\{L(\boldsymbol{Q}(T)) - L(\boldsymbol{Q}(0))\} + V\sum_{t=0}^{T-1} \mathbb{E}\{p(t)\} \leq BT + VTp^* - \epsilon \sum_{t=0}^{T-1} \sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\}$
- Rearranging terms in the above and neglecting non-negative quantities where appropriate yields the following two inequalities:

$$\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}\{p(t)\} \leq p^* + \frac{B}{V} + \frac{\mathbb{E}\{L(\boldsymbol{Q}(0))\}}{VT}$$

$$\frac{1}{T}\sum_{t=0}^{T-1}\sum_{i=1}^{2} \mathbb{E}\{Q_i(t)\} \leq \frac{B + V(p^* - \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}\{p(t)\})}{\epsilon} + \frac{\mathbb{E}\{L(\boldsymbol{Q}(0))\}}{\epsilon T}$$

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- N built dams, denoted by $\mathcal{D} = \{1, 2, ..., N\}$.
- part $X_{1,i}(t)$ is used to meet the water demand of the surrounding area
- part $X_{2,i}(t)$ is used for hydropower generation

$$\max \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N} U(X_{1,i}(t); X_{2,i}(t)) \tag{6}$$

$$\text{s.t.} \quad X_{1,i}(t) \geq C(t), \tag{7}$$

$$X_{2,i}(t) \geq 0, \tag{8}$$

$$X_{2,i-1}(t) \leq d, \tag{9}$$

$$\frac{1}{T}\sum_{t=1}^{T} k_i \cdot X_{2,i}(t) \geq B_i, \tag{10}$$

$$X_{1,i}(t) + X_{2,i}(t) \leq X_{2,i-1}(t) + C_i \tag{11}$$

问题特征

- 时间平均目标函数: $\sum_{t=1}^{T}\sum_{i=1}^{N} U(X_{1,i}(t); X_{2,i}(t))$
- 时间平均约束: $\frac{1}{T}\sum_{t=1}^{T} k_i \cdot X_{2,i}(t) \geq B_i$
- For any dam i, introduce a virtual queue $Q_i(t)$ with update equation

$$Q_i(t+1) = \max[Q_i(t) - k_i X_{2,i}(t), 0] + B_i.$$

水资源调度策略

- 根据 drift-plus-penalty, 我们的调度策略是： $\Delta(\boldsymbol{Q}(t)) - V\mathbb{E}\{\sum_{i=1}^{N} U(X_{1,i}(t); X_{2,i}(t))|\boldsymbol{Q}(t)\}$
- 由于 $U(\cdot;\cdot)$ 是线性的，调度策略变为以下线性规划问题:

$$\min \Delta(\boldsymbol{Q}(t)) - V\mathbb{E}\{\sum_{i=1}^{N} U(X_{1,i}(t); X_{2,i}(t))|\boldsymbol{Q}(t)\} \tag{12}$$

$$\text{s.t.} \quad X_{1,i}(t) \geq C(t), \tag{13}$$

$$X_{2,i}(t) \geq 0, \tag{14}$$

$$X_{2,i-1}(t) \leq d, \tag{15}$$

$$X_{1,i}(t) + X_{2,i}(t) \leq X_{2,i-1}(t) + C_i \tag{16}$$

性能保证

Let U^* be the optimal value and $\delta = U^* - \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N} U(X_{1,i}(t); X_{2,i}(t))$. For any control parameter $V > 0$ and a finite constant B, we have,

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N} U(X_{1,i}(t); X_{2,i}(t)) \leq U^* + \frac{B}{V},$$

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N} \mathbb{E}\{Q_i(t)\} \leq \frac{B + V\delta}{\epsilon}.$$