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Artistic painting: A fractional calculus perspective

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ABSTRACT

This paper studies artistic paintings by means of information theory and fractional calculus. A set of artworks created by 100 artists in a time scale covering the last seven centuries of the human kind, is characterized through different entropy indices. First, the paintings are converted into digital format and discretized. Afterwards, the data for color and grayscale images are processed by means of the Shannon and the fractional entropies, both for 1- and 2-dim histograms. The results are correlated with artistic movements that took place throughout the history of painting. In a second phase, the similarities between artists are analyzed for various periods of time. A sample of 10 artworks per artist is considered and their 2-dim fractional entropy is calculated for time scales of 700 and 130 years. The data are processed by a hierarchical clustering algorithm and the results interpreted under the light of the emerging clusters. Finally, the works of a single artist are compared during a short time scale of 47 years.

1. Introduction

Throughout time musicians, painters, sculptors and architects attempted to build harmonies of sounds, colors and forms in their works [1]. Artworks transmit the artist's sensitivity and creativity and represent the dimension of man's spirit and culture [2]. In some cases, it is also perceptible the existence of a mathematical exercise when producing the objects. The *Moorish* tessellations, mandalas, rose windows of the stained glass in the *Gothic* cathedrals, and many other master pieces reveal the use of ancient geometric knowledge [3]. Evidences are also visible during the *Renaissance*, when several artists used techniques based on the principle of linear perspective and projective geometry to accurately portray 3-dim scenes on a flat surface [4]. This period is a striking example of the reciprocal link between mathematics and art [5], being patent their contributions to the progress of human kind [6–14]. The fascination of many artists for mathematics, such as Escher, Thomas, Klee, Kandinsky and Le Corbusier motivated them to explore new optical possibilities and geometries [8,15–17]. Nowadays, artists and scientists can use computers to produce and study artworks using advanced techniques and algorithms to reinforce the synergies between art and science.

During the evolution of art, several artistic movements and cycles took place. The quantitative analysis of an artwork may help the expert to identify a copy of an original work, distinguish the type of brushstroke, or understand the trends and historical traditions, social movements, cultural values, ways of interpreting the world and the human being through time. Present day computational analysis of painting, such as X-ray fluorescence spectroscopy, reveal important information. These techniques are able to uncover layers of paint covered by an artist [18,19]. Dodgson [20] adopted the entropy

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and correlation to characterize Riley's stripe painting. In [3], the author proposes a quantitative metric of the aesthetic quality of an artwork. Taylor et al. [21] showed Pollock's dripped patterns to be fractals. They used a mesh to analyze Pollock's patterns and verified that the artist adopted nature's pattern-generating processes on his canvas. Wallraven et al. [22] addressed the issue of whether there are mathematical features that allow us to characterize artistic movements.

The concept of entropy first appeared in the early 1860s in physics as the characterization of the second law of thermodynamics by Clausius. Later, in 1877, Boltzmann, developed a statistical mechanical evaluation of entropy. In 1948 and 1957, the term was used in information theory by Shannon and Jaynes, respectively [23–25]. Since then, several generalizations of entropy have been applied to the study of complex systems [26–34].

Fractional Calculus (FC) generalizes the classical differential operations to non-integer orders [35–39]. The area of FC dates back to year 1695, in the follow-up of several letters between de l'Hôpital and Leibniz about the meaning and apparent paradox of the nth-order derivative $\frac{d^n f(t)}{dt^n}$, for $n = \frac{1}{2}$. However, only during the last decades FC was recognized to play an important role in the modeling and control of many physical phenomena. FC emerged as a key tool in the area of dynamical systems. Nowadays the FC concepts are applied in different scientific fields, namely mathematics, physics, biology, finance and geophysics [40–45]. Indeed, fractional derivatives capture memory effects and hereditary properties providing a more insightful description of the phenomena [46–49].

In this paper, we adopt the information and FC theories for studying the evolution of painting since the 13th century and to unveil possible similarities between painters. Therefore, in a first phase, a total of 1296 artworks created in the last seven centuries by 100 artists are characterized through entropy indices. The paintings, in number between 10 and 38 per artist, are converted into the digital format and discretized. The data for grayscale and color images are processed by computing the Shannon and the fractional entropies for 1- and 2-dim statistical distributions. The results are correlated with artistic movements that took place throughout the history of painting. In a second phase, similarities between artists are analyses in two distinct time scales, namely 700 and 130 years. In this phase, a set of 10 artworks per artist is considered and their 2-dim fractional entropy is calculated. The data is processed by a hierarchical clustering (HC) algorithm and the results are interpreted in the perspective of the generated clusters. Finally, the same procedure is adopted for comparing and visualizing artworks produced by an individual artist in a small time scale of 47 years.

In this line of ideas, the paper is organized as follows. Section 2 presents the fundamental mathematical concepts. Section 3 studies the evolution of the artist painting in the perspective of the Shannon and fractional entropies, for 1- and 2-dim probability distributions, and for grayscale and color images, while encompassing a time scale of 7 centuries. Section 4 explores distinct numbers of painters and three time scales. The section analyses similarities between 100 and 26 artists for large and medium time scales, namely 700 and 130 years, respectively. After, the artworks by a single painter are compared for a short period of 47 years. The 2-dim fractional entropy, several distance measures, and HC algorithms are adopted for unveiling patterns. Finally, Section 5 draws the main conclusions.

2. Mathematical background

2.1. Entropy

An event with probability of occurrence $P(x)$ has information content, I, given by:

$$I[P(x)] = -\log P(x). \tag{1}$$

The expected value of I is the Shannon entropy:

$$S(X) = E[-\log P(x)] = \sum_{x\in X}[-\log P(x)]P(x), \tag{2}$$

where X is a random variable and $E(\cdot)$ denotes the expected value operator.

Expression (2) satisfies the four Khinchin axioms [24,25]. Generalizations of the Shannon entropy [31,33] have been proposed during the last decades, being less strict and obeying only to some of those axioms.

The entropy, $S(X)$, measures the uncertainty in a distribution of a 1-dim random variable X. The joint entropy quantifies the shape of the distribution associated with a set of random variables [50]. The joint Shannon entropy of 2-dim discrete random variables (X, Y) is defined as

$$S(X,Y) = \sum_{x\in X}\sum_{y\in Y} -P(x,y)\log P(x,y) = E_{X,Y}[I(X,Y)], \tag{3}$$

where $P(x, y)$ represents a joint probability distribution.

If X and Y are independent, then their joint entropy is the sum of the individual entropies, meaning that $S(X,Y) = S(X) + S(Y)$.

2.2. A fractional calculus approach to information measures

In the scope of the Shannon approach, we note that the information, $I[P(x)] = -\log P(x)$, is a function between the cases $D^{-1}I[P(x)] = P(x)[1 - \log P(x)]$ and $D^1 I[P(x)] = -\frac{1}{P(x)}$. The interpretation of these expressions in the perspective of FC led to the formulation of information and entropy of order $\alpha \in \mathbb{R}$ [51,52]:

$$I_\alpha[P(x)] = D^\alpha I[P(x)] = -\frac{P(x)^{-\alpha}}{\Gamma(\alpha+1)}[\log P(x) + \psi(1) - \psi(1-\alpha)], \tag{4}$$

$$S_\alpha(X) = \sum_{x\in X}\left\{-\frac{P(x)^{-\alpha}}{\Gamma(\alpha+1)}[\log P(x) + \psi(1) - \psi(1-\alpha)]\right\}P(x), \tag{5}$$

where $\Gamma(\cdot)$ and $\psi(\cdot)$ represent the gamma and digamma functions, respectively.

Similarly, for the pair of random variables (X, Y) the joint fractional entropy of order $\alpha \in \mathbb{R}$ can be written as

$$S_\alpha(X,Y) = \sum_{x\in X}\sum_{y\in Y}\left\{-\frac{P(x,y)^{-\alpha}}{\Gamma(\alpha+1)}[\log P(x,y) + \psi(1) - \psi(1-\alpha)]\right\}P(x,y). \tag{6}$$

Expressions (4)–(6) lead to the Shannon information and entropies when $\alpha = 0$.

2.3. Distance functions

The mathematical notion of metric, or distance function, was formulated one century ago by Fréchet and Hausdorff [53]. Let us consider a set $\mathcal{C}$. A function $d : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a metric of $\mathcal{C}$ if, for all elements $P, Q, M \in \mathcal{C}$ the following conditions hold:

1. $d(P, Q) \geq 0$ (non-negativity);
2. $d(P, Q) = 0$ if and only if $P = Q$ (identity of indiscernibles);
3. $d(P, Q) = d(Q, P)$ (symmetry);
4. $d(P, Q) \leq d(P, M) + d(M, Q)$ (triangle inequality).

Conditions 1 and 2 together establish $d(P, Q)$ as a positive-definite function, while the first condition is implied by the others.

The metric $d(P, Q)$ corresponds to the distance between each pair of elements (P, Q) of $\mathcal{C}$. The set $\mathcal{C}$ with the metric $d(P, Q)$ is called a metric space.

In the follow-up the four distances {Canberra, Cosine, Jaccard, Jeffreys } = $\{Ca, Co, Ja, Je\}$ are considered. The Ca distance was proposed, and latter modified, by Lance and Williams [54,55]. Given 2 points, $P = (p_1, \ldots, p_K)$ and $Q = (q_1, \ldots, q_K)$, in a K-dim space, the Ca distance between P and Q is given by:

$$Ca(P,Q) = \sum_{k=1}^{K}\frac{|p_k - q_k|}{|p_k| + |q_k|}. \tag{7}$$

Eq. (7) is a metric often used for quantifying data scattered around an origin. The Ca has several interesting properties, namely it is unitary when the arguments are of opposite sign, biased for measures around the origin, and highly sensitive for values close to zero.

The Co distance is calculated by means of the expression [56]:

$$Co(P,Q) = \frac{1}{\pi}\cdot\arccos\left(\frac{\sum_{k=1}^{K} p_k q_k}{\sqrt{\sum_{k=1}^{K} p_k^2}\sqrt{\sum_{k=1}^{K} q_k^2}}\right). \tag{8}$$

The Co has been widely used, for example in text document clustering [57], and is very useful when comparing items characterized by vectors with different magnitudes.

The Ja distance was introduced by Jaccard [58] and can be calculated by means of the expression [56]:

$$Ja(P,Q) = 1 - \frac{\sum_{k=1}^{K}(p_k - q_k)^2}{\sum_{k=1}^{K} p_k^2 + \sum_{k=1}^{K} q_k^2 - \sum_{k=1}^{K} p_k q_k}. \tag{9}$$

The Ja has several practical applications, namely in information retrieval, data mining and machine learning [59].

The Je distance is given by [56]:

$$Je(P,Q) = \sum_{k=1}^{K}(p_k - q_k)^2 \log\frac{p_k}{q_k}. \tag{10}$$

The Je was derived empirically and it is numerically stable and robust with respect to noise [60].

2.4. Hierarchical clustering

Clustering is a data analysis technique [61] that groups similar items. In the scope of HC, two possible iterative strategies generate a hierarchy of clusters, and consist of (i) agglomerative and the (ii) divisive clustering strategies. With strategy (i) each item starts in its own cluster and the algorithm merges the two most similar clusters until there is one single cluster. With strategy (ii) all items start in a single cluster and the algorithm removes the “outsiders” from the least cohesive cluster, until each item is in its own cluster. In both strategies, it is required a linkage criterion, that is a function of the distances between pairs of items, for quantifying the dissimilarity between clusters.

For 2 clusters, R and S, the distance $d(x_R, x_S)$ between items $x_R \in R$ and $x_S \in S$ is based on metrics such as the maximum, minimum and average linkages given by [62]:

$$d_{max}(R,S) = \max_{x_R \in R, x_S \in S} d(x_R, x_S), \tag{11}$$

$$d_{min}(R,S) = \min_{x_R \in R, x_S \in S} d(x_R, x_S), \tag{12}$$

$$d_{ave}(R,S) = \frac{1}{\parallel R \parallel \parallel S \parallel} \sum_{x_R \in R, x_S \in S} d(x_R, x_S). \tag{13}$$

After opting for one of the algorithms, the results of HC are presented in a graphical locus such as a dendrogram or a hierarchical tree.

To assess the “quality” of the clustering, the cophenetic correlation (CC) coefficient is often used [63]. The CC gives a measure of how well the generated graphical object preserves the original pairwise distances. If the clustering is successful, the links between items in the graphical object have a strong correlation with those in the original data set. The closer the CC value to 1, the better the clustering result. The quality assessment is plotted in a Shepard diagram that compares the original and the cophenetic distances. A good clustering leads to a layout of points close to the 45 degree line.

3. Evolution of artistic painting

This section studies the evolution of the artistic painting in the perspective of entropy. Section 3.1 starts with the data characterization and the translation of the raw information into digital format. Section 3.2 assesses the use of the Shannon and fractional entropies, both for the 1- and 2-dim cases, and when adopting grayscale and color images.

3.1. Data characterization

The characterization of the artistic painting evolution is supported by the works of 100 painters created from year 1297–1987, as listed in Table A.1, in the appendix. For each painter we study between $s = 10$ and $s = 38$ artworks, making a total of 1296 paintings during the seven centuries under consideration.

Each painting is edited so that cracks, borders, or other artifacts are eliminated, and the image is converted into a square matrix, A, 500×500 dimensional. The matrix elements represent the red, green and blue (RGB) intensities, when considering color images, or values, Gr, between black and white, obtained by the ITU-R BT.601 (formerly CCIR 601) RGB to gray conversion formula, $Gr = 0.2989R + 0.5870G + 0.1140B$, when considering grayscale images. In both cases, the values are integers in the range between 0 and 255. Fig. 1 depicts an example of one original painting, its grayscale image and their corresponding 500×500 discretized versions [64]. This pre-processing results in a loss of information present in the originals, namely in terms of fine detail, color and texture, since it reduces the resolution, compresses and clamps the color values (losing the brightest whites and the darkest blacks), and neglects any information about the 3-dim textural surface of the painting. However, it is necessary for reducing the global volume of information and to put all works in an uniform format.

3.2. Quantitative analyses of the grayscale and color images

The Shannon and fractional entropies of the artworks are explored, both for the 1- and 2-dim formulations, and the grayscale and color images.

The order, α, of the fractional entropies (4) and (6) has to be tuned carefully in the perspective of yielding a good sensitivity to the data of the indices $S_\alpha(X)$ and $S_\alpha(X, Y)$. Several numerical experiments led to $\alpha = 0.75$, a value that is in accordance with the results published in the literature [65].

For the analysis using 1-dim entropies the Eqs. (2) and (4) are adopted, where $x = \{0, 1, \ldots, 255\}$ corresponds to gray intensities. The probability distribution $P(x)$ is estimated from the histogram of relative frequencies of the image grayscale levels.

When using the 2-dim entropies (4) and (6), $P(x, y)$ is obtained from the matrix $A = [a_{ij}]$, $i, j = 1, 2, \ldots, 500$, that is interpreted as a 2-dim discrete probability distribution. In this perspective, x and y stand for row i and column j, respectively, while the probability is approximated with the histogram by means of the proportion $\frac{a_{ij}}{\sum_{i=1}^{500} \sum_{j=1}^{500} a_{ij}}$.

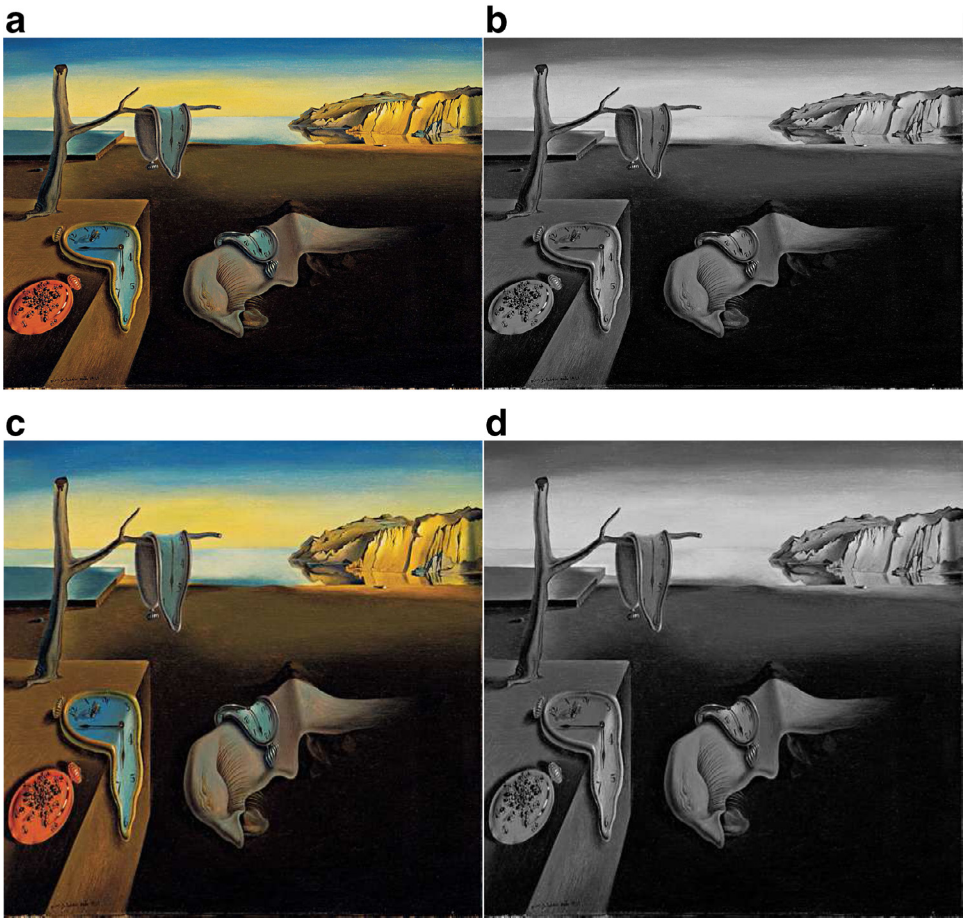

Fig. 1. The painting 'The Persistence of Memory' (c. 1931) by Salvador Dalí: (a) color image, (b) grayscale image, (c) color matrix A with 500×500 resolution and (d) grayscale matrix A with 500×500 resolution.

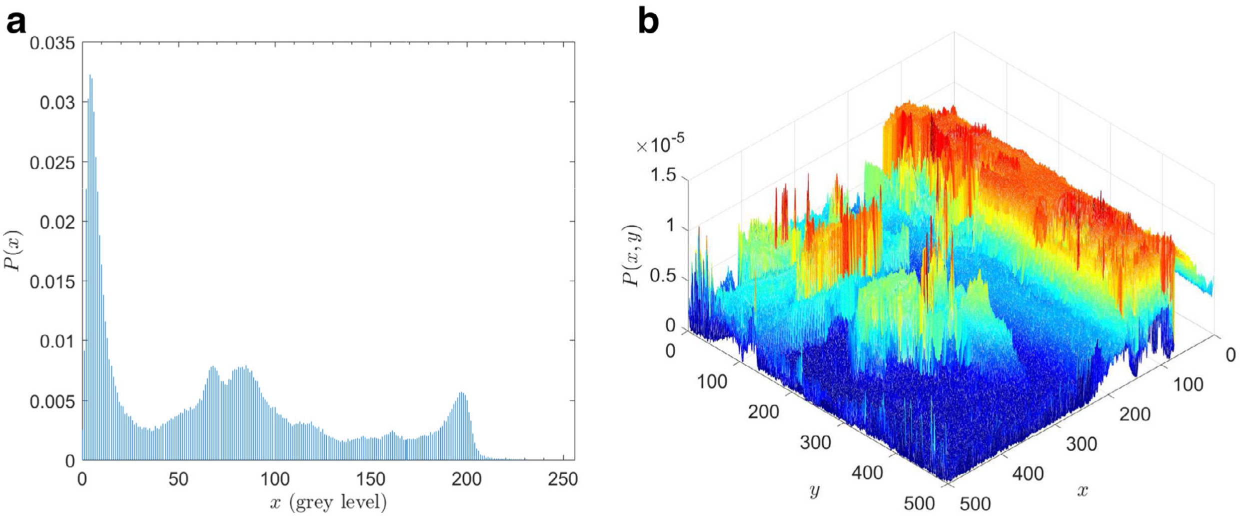

Fig. 2. The histograms of relative frequencies of the grayscale version of the painting 'The Persistence of Memory' by Salvador Dalí: (a) 1-dim; (b) 2-dim.

Fig. 2 presents as an illustrative example the 1- and 2-dim histograms of relative frequencies for the grayscale version of the painting 'The Persistence of Memory' by Salvador Dalí. These histograms serve for approximating $P(x)$ and $P(x, y)$, respectively.

The numerical experiments with different combinations of entropy indices, space dimensions, and color/grayscale painting versions demonstrate that:

- The entropies for the grayscale and *RGB* images are strongly correlated, meaning that the information about color is not the most relevant issue in the perspective of the adopted indices;
- The 2-dim entropy is more sensitive than the 1-dim entropy;
- The fractional entropy is more assertive for representing artworks than the Shannon entropy.

Fig. 3 depicts using box plots the 2-dim Shannon, $S(X, Y)$, and fractional, $S_\alpha(X, Y)$, $\alpha = 0.75$, entropies versus time. The statistics are calculated for paintings within 30-year windows centered at the time stamp. We verify that $S_\alpha(x, y)$ is more sensitive than $S(x, y)$, leading to a better discrimination between periods. Moreover, we can relate the time evolution of the entropy by associating the periods of increasing and decreasing trends exhibited by the entropy with the different historical artistic movements, even knowing that the classifications and dates are neither strictly defined, nor consensual within the artistic arena [66]. Just regarding the contemporary art, there have been several movements post-1900. For example, Pollack, Rothko and Riley created artworks that are vastly different from an information theoretical point of view. Therefore, we highlight 10 periods:

- until 1470 – corresponds to the *Gothic*, or *Medieval*, movement. The entropy $S_\alpha(X, Y)$ exhibits slight oscillations, but unveils a clear decreasing trend, reaching a local minimum;
- 1470–1580 – includes the *Renaissance* and the *Mannerism*. In this period $S_\alpha(X, Y)$ increase considerably;
- 1580–1640 – coincides with the *Baroque* and $S_\alpha(X, Y)$ decreases;
- 1640–1720 – is related to the *Late-Baroque*, where $S_\alpha(X, Y)$ increases and reaches a new local maximum;
- 1720–1765 – is the *Rococo*, characterized by a slight decrease of $S_\alpha(X, Y)$;
- 1765–1815 – coincides with the *Neoclassicism* and $S_\alpha(X, Y)$ remains almost constant;
- 1815–1870 – includes the *Romanticism* and the *Realism*. In this period $S_\alpha(X, Y)$ exhibits a positive grow step and then stays almost constant;
- 1870–1890 – agrees with the *Impressionism*. The value of $S_\alpha(X, Y)$ increases quickly until a local maximum;
- 1890–1945 – includes the *Post-Impressionism* and the *Modernism*, being characterized by a decreasing $S_\alpha(X, Y)$;
- 1945 onward – matches to the *Contemporary* art, where $S_\alpha(X, Y)$ reveals a very slight increasing trend.

In synthesis, the 2-dim entropy, $S_\alpha(X, Y)$, produces a slightly more assertive identification of different artistic movements than the Shannon entropy. However, the wide dispersion and the presence of outliers may rise questions, such as:

- What is it about the outliers that makes them to exhibit a low, or high, value of $S_\alpha(X, Y)$?
- Are the outliers typical exemplars of the period, or are they anomalies?
- What about the quality of the original images, since they were took from publicly available digital databases?
- In what extent they may bias the results obtained?

4. Hierarchical clustering of painters

We explore similarities between artists and between artworks, for distinct periods of time and for different cardinalities of the number of painters and paintings. In Sections 4.1 and 4.2 100 and 26 artists are compared considering time scales of 700 and 130 years, respectively. In Section 4.3 the same methodology is adopted for analyzing artworks produced by a single artist during 47 years.

4.1. Analysis in a long time scale

We start by considering 100 painters in a period of 7 centuries. Firstly, we consider a sample of 10 masterpieces from the portfolio of the kth artist and we calculate the 1×10 vector $P_k = [S_\alpha^1, S_\alpha^2, \ldots, S_\alpha^{10}]$, where S_α^m, represents the 2-dim fractional entropy of the mth masterpiece extracted from its 500×500 matrix A. The vectors P_k include values for artworks distributed along the work life span of each artist. Secondly, we compute the distance matrices $D = [d(P_k, P_l)]$, $k, l \in \{1, 2, \ldots, 100\}$, where $d(P_k, P_l)$ represents one distance of the set $\{Ca, Co, Ja, Je\}$ between the vectors P_k and P_l. Thirdly, we process the matrices D by means of HC for generating loci of objects representing painters.

We adopt $\alpha = 0.75$ for the order of the fractional entropy, and the successive (agglomerative) clustering with the average-linkage method for constructing the HC [67].

Fig. 4a depicts the dendrogram generated by the HC based on the *Ca* distance, where the numbers at the leaves denote the kth painter (see Table A.1, in the appendix). Fig. 4b represents the corresponding Shepard plot, that assesses the reliability of the clustering by comparing the original and the cophenetic distances [63]. From the chart we verify that the dendrogram represents well the original data in matrix D, since the points are close to the 45 degrees line. For the other distances in the set we obtain similar results.

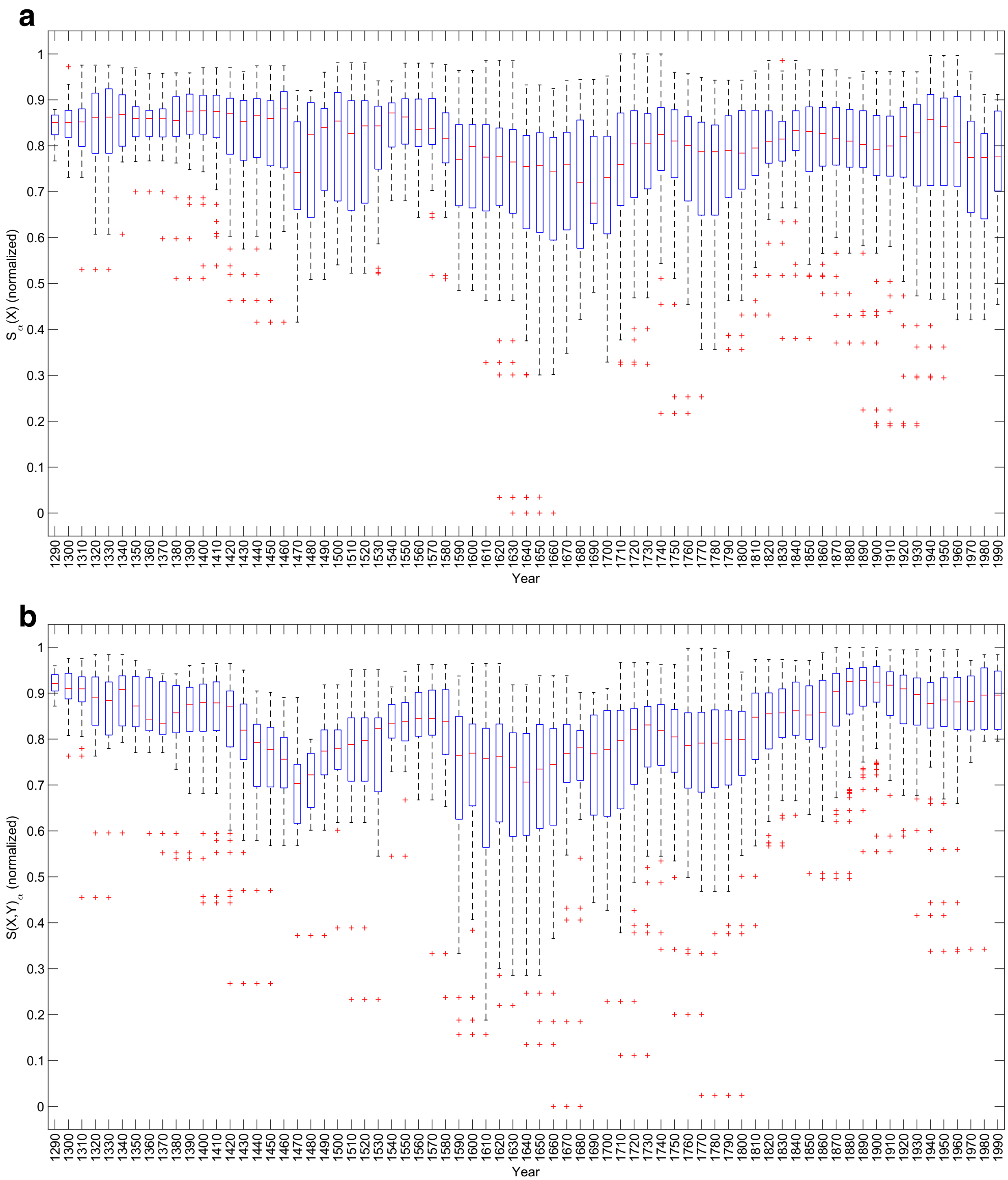

Fig. 3. The 2-dim entropies versus time during the period 1297–1987: (a) Shannon; (b) fractional S_α, $\alpha = 0.75$.

The dendrograms reveal the relationships embedded in the data. However, some details are better perceived when using other representations. Fig. 5 depicts the four loci, each generated by the HC and one distance in the set {*Ca*, *Co*, *Ja*, *Je*} using visualization trees. To enrich the visualization we superimpose a contour map, where the z-coordinate is the arithmetic mean of the birth and death years of the kth painter (e.g., time information representing his work period), at the (x, y) coordinates of the tree leaves, and the global coordinates are derived by interpolation. We note the emergence of the same type of patterns for the 4 distances, which reflect the relative positioning of the artists in terms of the artworks considered.

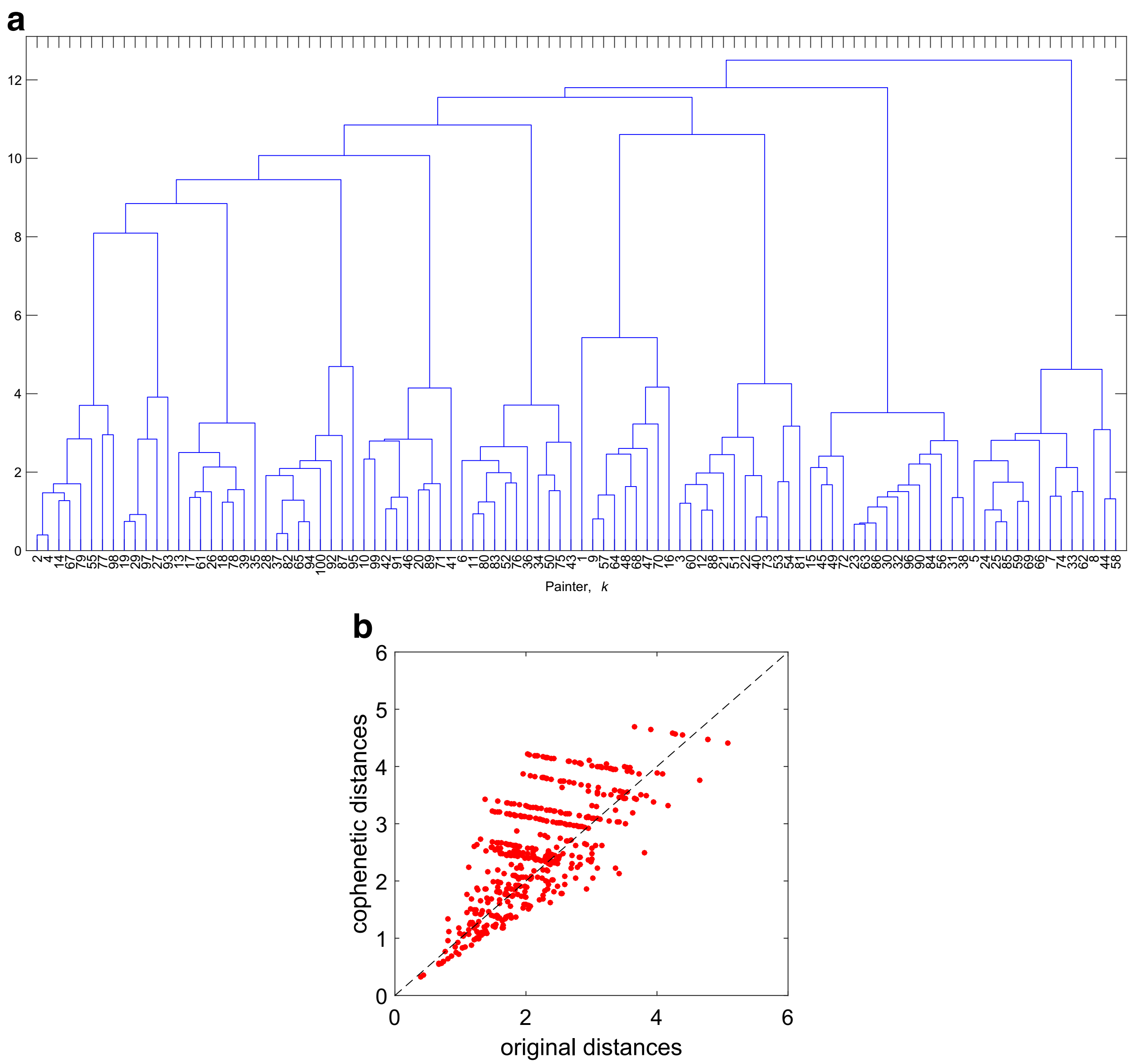

Fig. 4. Clustering generated by the HC based on the 2-dim fractional entropy, $\alpha = 0.75$, during the time scale of 7 centuries, for the *Ca* distance between the two 1×10 vectors P_k, $k = 1, \ldots, 100$: (a) dendrogram; (b) Shepard plot. The numbers in the dendrogram leaves denote the painter index, k, at Table A.1.

Moreover, we verify that the emerging clusters are loosely correlated with time. This means that, in the perspective of the entropy measure, art has a quasi-timeless nature and reflects mainly deep-seated constructions of the human mind. Nevertheless, we can also discuss if the adopted abstract mathematical indices can capture adequately artistic concepts.

4.2. Analysis in a medium time scale

We consider a time scale of approximately 130 years, that corresponds roughly to the *Impressionism, Post-Impressionism, Modernism* and *Contemporary* artistic movements. A total of 26 artists, $k = 75, \ldots, 100$, (see Table A.1, in the appendix) are compared by means of the procedure described in Section 4.1.

Fig. 6 depicts the loci generated by the HC based on the 2-dim fractional entropy, $\alpha = 0.75$, for the {*Ca, Ja*} distances between the two 1×10 vectors P_k, $k = 75, \ldots, 100$.

We verify the emergence of identical clusters independently of the adopted distance. Again the time information is not of main relevance, since we find several cases of non-consecutive indices k in the same cluster. Probably similar ideas or concepts can be found underlying distinct epochs and different painters. If such characteristics of the human creativity are captured in fact by the proposed technique is still an aspect to be further validated.

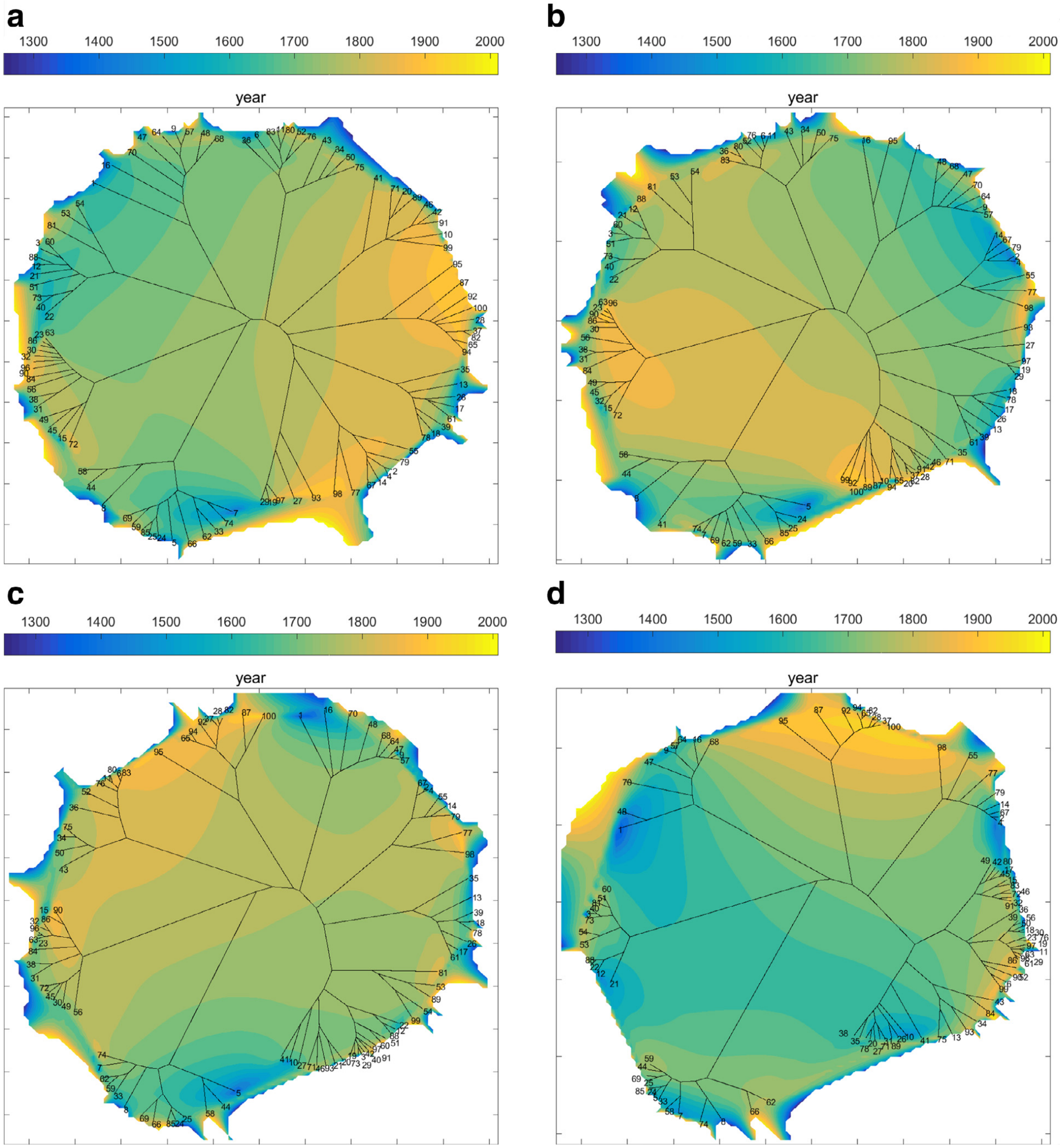

Fig. 5. Loci generated by the HC based on the 2-dim fractional entropy, $\alpha = 0.75$, during the time scale of 7 centuries, for the distance between the two 1×10 vectors P_k, $k = 1, \ldots, 100$: (a) Canberra, *Ca*; (b) Cosine, *Co*; (c) Jaccard, *Ja*; (d) Jeffreys, *Je*. The numbers in the trees leaves denote the painter index, k, at Table A.1, and the color bars represent time obtained by interpolation between information at the leaves.

4.3. Analysis in a small time scale

We analyze paintings by Cézanne during a period of 47 years. Cézanne was an important French painter of the transition between *Impressionism* and *Cubism* of the late 19th and early 20th centuries, respectively, and one of the main precursors of *Modern Art*. In his artistic career Cézanne produced about 900 oil paintings and 400 watercolors, besides many incomplete works.

We consider the sample of 50 paintings during the period 1860–1906 listed in Table A.2, in the appendix. First, we calculate the 2-dim fractional entropy, S_α^m, of the mth artwork from the corresponding 500×500 matrix A. Second, we

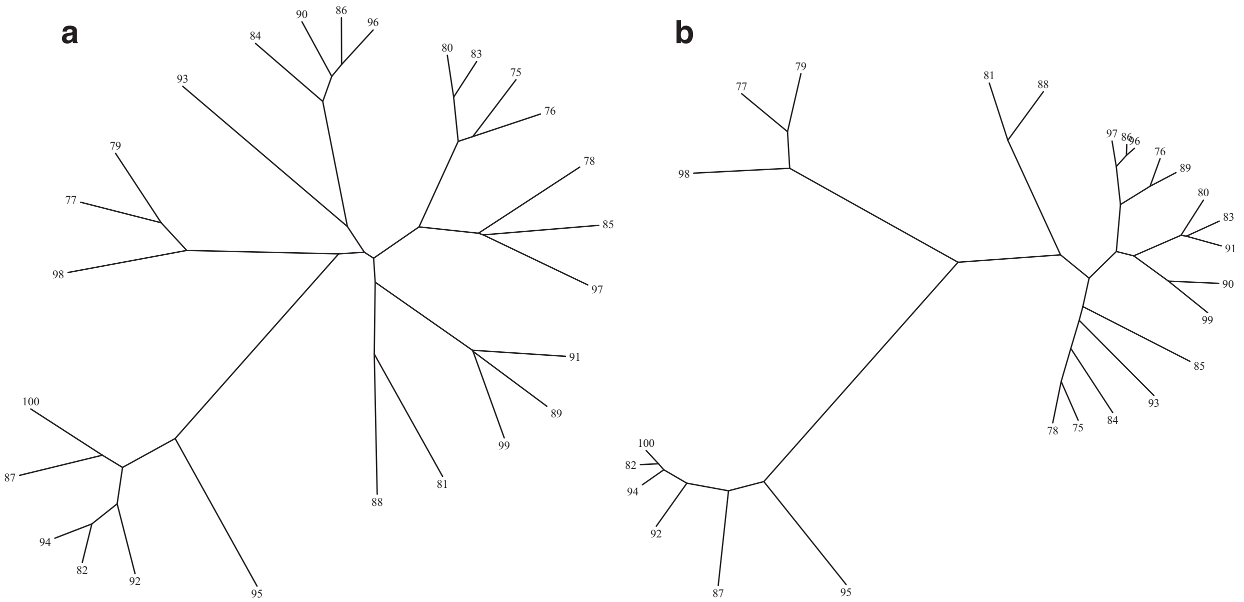

Fig. 6. Loci generated by the HC based on the 2-dim fractional entropy, $\alpha = 0.75$, during the time scale of 130 years, for the distance between the two 1×10 vectors P_k, $k = 75, \ldots, 100$: (a) Canberra, *Ca*; (b) Jaccard, *Ja*. The numbers in the trees leaves denote the painter index, *k*, at Table A.1.

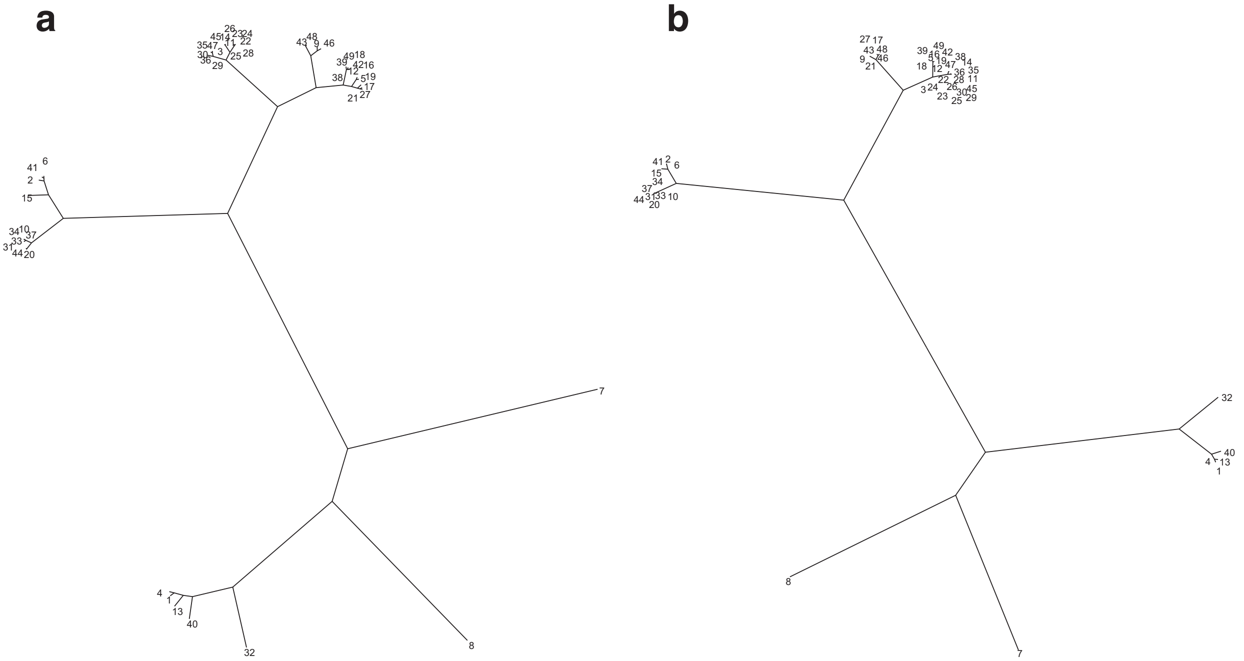

Fig. 7. Loci generated by the HC based on the 2-dim fractional entropy, $\alpha = 0.75$, of the sample of 50 artworks by Cézanne during a time scale of 47 years, between two scalar values: (a) Canberra, *Ca*; (b) Jaccard, *Ja*. The numbers in the trees leaves denote the painting index, *m*, at Table A.2.

compute the matrices $D = [d(S_\alpha^m, S_\alpha^n)]$, $m, n \in \{1, 2, \ldots, 50\}$, between the scalars S_α^m and S_α^n, where $d(S_\alpha^m, S_\alpha^n)$ represents one distance in the set $\{Ca, Ja\}$. Since we are comparing individual paintings, the distance *D* is evaluated between two scalars values, instead of adopting vectors distributed in time, as considered in Sections 4.1 and 4.2. Third, we process the matrices *D* by means of HC for generating and visualizing the loci of paintings.

Fig. 7 depicts the loci generated by the HC. We have identical clusters for the $\{Ca, Ja\}$ distances, while the emerging patterns are loosely correlated with time. We observe two main clusters $\mathcal{A}$ in the top left and $\mathcal{B}$ in the bottom right. In both cases there is a sub-division into in 3 smaller groups. Clusters in $\mathcal{A}$ include most of the works, while clusters in $\mathcal{B}$ correspond to a restricted number of paintings. It is not clear if the clusters map into some particular known period of the painter, if they reveal some hidden information not acknowledge by art experts, or if they are simply numerical results without a direct correspondence with artistic human concepts. Nevertheless, it is clear that art produced in different locations, periods and artists share some common patterns. Therefore, we believe that further study is required to overcome the gap between mathematics based on an abstract logic, and the human intellect, supported by biological objects such as the human brain, eyes and hands. Analyzing art requires constructing a bridge between two worlds that have evolved separately, but that have man as the common architect.

In summary, the entropy-based analysis just scratched the surface of a complex problem, since the concept of art resides not only in work itself, but also in the human being. Therefore, the information flow starting with a source represented by the artist (with his own vision and cultural bias), having for the transmission media the artwork itself (with the limitations associated with the adopted technique) and for reception the scholar, or the art enthusiast (with his personal perspective), requires a set of mathematical and computational tools encompassing the 3 parts. The entropy is a limited tool to handle such formidable challenge of mathematical modeling. Nonetheless, we can say that this work may be, perhaps, a beginning exercise in the topic.

5. Conclusions

This paper analyzed artistic paintings by means of information theory and FC. A set of artworks by 100 artists created in the last 7 centuries was processed using several techniques, namely adopting grayscale and color, 1- and 2-dim histograms, and using measures based on the integer and fractional entropies. The results revealed some correlation with the historical artistic movements. The similarities between painters were explored by analyzing a set of masterpieces for each artist. Each painting was characterized in terms of 2-dim fractional entropy, and the distance measures between painters tackled by HC. The visualization by means of trees with time embedded information revealed clusters of artists with some correlation in time. The time scale of analysis was reduced, first for a period of 130 years involving 26 painters and, finally, for a single painter capturing a time scale of 47 years. In the three time scales, time and the related artistic period, seem not the main issue underlying the embedded information. Indeed, the adopted numerical and computational treatment is insufficient to describe clearly all aesthetic information. Further processing based on complementary techniques traversing the labyrinths of the human mind may unravel other properties of artistic works.

Appendix A. Tables of painters and paintings

Table A1
List of the 100 painters in the time scale of 7 centuries.

Index, k	Paintings, s	Name	Period	Index, k	Paintings, s	Name	Period
1	27	Giotto di Bondone	1267–1337	51	11	Corrado Gianquinto	1703–1766
2	17	Pietro Lorenzetti	1280–1348	52	10	Pompeo Batoni	1708–1787
3	11	Simone Martini	1284–1344	53	11	Joshua Reynolds	1723–1792
4	10	Giovanni da Milano	1324–1360	54	10	Thomas Gainsborough	1727–1788
5	10	Bartolo di Fredi	1330–1410	55	10	Anton Raphael Mengs	1728–1779
6	10	Andrea da Firenze	1346–1379	56	19	Jean-Honoré Fragonard	1732–1806
7	10	Broederlam	1350–1409	57	11	Joseph Wright of Derby	1734–1797
8	10	Jacquemart de Hesdin	1355–1414	58	12	George Romney	1734–1802
9	10	Andrei Rublev	1360–1430	59	10	Francisco Goya	1746–1828
10	20	Lorenzo Monaco	1370–1425	60	10	Luis Paret y Alcázar	1746–1799
11	11	Gentile da Fabriano	1370–1427	61	16	Jacques-Louis David	1748–1825
12	11	Limburg brothers Herman, Paul, and Johan	1385–1416	62	10	Henry Raeburn	1756–1823
13	22	Jan van Eyck	1390–1441	63	10	Thomas Lawrence	1769–1830
14	27	Rogier van der Weyden	1399–1464	64	10	Caspar David Friedrich	1774–1840
15	14	Leonardo da Vinci	1452–1519	65	12	Joseph Turner	1775–1851
16	10	Quentin Matsys	1466–1530	66	10	John Constable	1776–1837
17	11	Albrecht Durer	1471–1528	67	13	Jean-Auguste Dominique Ingres	1780–1867
18	11	Ticiano Vecelli	1477–1576	68	11	Jean-Baptiste Camille Corot	1796–1875
19	15	Raphael Sanzio	1483–1520	69	10	Eugène Delacroix	1798–1863
20	10	Juan Correa de Vivar	1510–1566	70	10	Constant Troyon	1810–1865
21	15	Pieter Bruegel	1525–1569	71	10	Gustave Courbet	1819–1877
22	10	Giuseppe Arcimboldo	1527–1593	72	10	William Powell Frith	1819–1909
23	10	El Greco	1541–1614	73	10	Dante Gabriel Rossetti	1828–1882
24	10	Annibale Carracci	1560–1609	74	10	John Everett Millais	1829–1896
25	11	Michelangelo Merisi (Caravaggio)	1571–1610	75	16	Camille Pissarro	1830–1903
26	17	Guido Reni	1575–1642	76	11	Édouard Manet	1832–1883
27	19	Peter Paul Rubens	1577–1640	77	14	Cézanne	1839–1906

(continued on next page)

Table A1 (*continued*)

Index, k	Paintings, s	Name	Period	Index, k	Paintings, s	Name	Period
28	10	José de Ribera	1591–1652	78	15	Claude Monet	1840–1926
29	12	Georges de La Tour	1593–1652	79	19	Gauguin	1848–1903
30	10	Artemisia Gentileschi	1593–1653	80	38	Vincent van Gogh	1853–1890
31	10	Jacob Jordeans	1593–1678	81	14	Ferdinand Hodler	1853–1918
32	12	Francisco de Zurbarán	1598–1664	82	15	Georges Seurat	1859–1891
33	11	Antoon van Dyck	1599–1641	83	14	Gustav Klimt	1862–1918
34	33	Diego Velázquez	1599–1660	84	12	Edvard Munch	1863–1944
35	12	Rembrandt van Rijn	1606–1669	85	20	Toulouse-Lautrec	1864–1901
36	13	Bartolomé Esteban Murillo	1617–1682	86	11	Wassily Kandinsky	1866–1944
37	10	Charles Le Brun	1619–1690	87	14	Henri Matisse	1869–1954
38	10	Juan de Valdés Leal	1622–1690	88	12	Pieter Mondrian	1872–1944
39	12	Josefa de Óbidos	1630–1684	89	12	Kasimir Malevich	1878–1935
40	13	Johannes Vermeer	1632–1675	90	10	Pablo Picasso	1881–1973
41	15	Francesco Solimena	1657–1747	91	10	Amadeu de Souza-Cardoso	1887–1918
42	12	Rachel Ruysch	1664–1750	92	11	Juan Gris	1887–1927
43	11	Giuseppe Maria Crespi	1665–1747	93	12	Giorgio de Chirico	1888–1978
44	13	Alessandro Magnasco	1667–1749	94	12	René Magritte	1898–1967
45	10	Giovanni Battista Piazzetta	1682–1754	95	14	Tamara de Lempicka	1898–1980
46	22	Jean-Antoine Watteau	1684–1721	96	10	Salvador Dalí	1904–1989
47	10	Nicolas Lancret	1690–1743	97	10	Victor Vasarely	1906–1997
48	14	Giovanni Battista Tiepolo	1696–1770	98	14	Maria Helena Vieira da Silva	1908–1992
49	16	Giovanni Canal (Canaletto)	1697–1768	99	11	Jackson Pollock	1912–1956
50	10	Francois Boucher	1703–1770	100	10	Jean-Michel Basquiat	1960–1988

Table A2
Sample list of Cézanne artworks in a time scale of 47 years.

Index, m	Date	Name	Index, m	Date	Name
1	1860	The Kiss of the Muse	26	1887	Jas de Bouffan
2	1864	Judgment of Paris	27	1887	Self-Portrait with a Palette
3	1865	Landscape in the Ile-de-France	28	1888	Banks of the Marne
4	1867	The Kidnapping	29	1889	Montagne Sainte-Victoire
5	1869	Nature Morte à La Bouilloire Still Life with Kettle, 1869, Detail	30	1889	Still Life with Milk Jug and Fruit
6	1869	The Picnic	31	1890	Bathsheba
7	1870	A Modern Olympia (Pasha)	32	1890	Mont Sainte-Victoire
8	1871	Portrait of Antony Valabrègue	33	1890	Still Life With Flowers And Fruit
9	1874	A Modern Olympia (Une Moderne Olympia)	34	1890	Houses in Provence, near Gardanne
10	1875	Afternoon In Naples With A Black Servant	35	1892	Path from Mas Jolie to Château noir
11	1875	Self-Portrait with Pink Background	36	1892	The Cardplayer
12	1875	Un Peintre au Travail	37	1893	Straw Vase, Sugar Bowl, and Apples
13	1877	Madame Cézanne in a Red Armchair	38	1893	The Card Players
14	1878	Five Bathers	39	1894	Fruit and Jug on a Table
15	1879	Village Behind Trees	40	1895	Self-Portrait
16	1879	The Pond	41	1895	Self-Portrait in a Felt Hat
17	1880	Ile de France Landscape	42	1895	Woman with a Coffeepot
18	1880	Road Leading to the Lake	43	1896	Gustave Geffroy
19	1881	Turn in the Road	44	1896	Young Italian Girl Resting On Her Elbow
20	1882	Self-Portrait	45	1897	Peasant in a Blue Smock
21	1882	The Road Bridge at L'Estaque	46	1898	Still Life with Apples
22	1883	Houses in Provence	47	1904	Seated Woman in Blue
23	1884	Bottom of the Ravine	48	1905	The Bathers
24	1885	The Bay of Marseilles	49	1906	Góra Sainte-Victoire
25	1886	The Bare Trees at Jas de Bouffan	50	1906	The Garden at Les Lauves

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