

### Aufgabe 1      *Integrale mit Delta-Funktionen II*

Berechnen Sie die Integrale:

$$(i) \quad \int_{-\infty}^{\infty} f(x) \delta(-ax + b) \, dx \qquad (iv) \quad \int_{-\infty}^{\infty} \cos x \delta(x^2 - \pi^2) \, dx$$

$$(ii) \quad \int_0^{\infty} \ln x \delta(x^2 - 1) \, dx \qquad (v) \quad \int_{-\infty}^{\infty} f(x) \delta(e^x - 1) \, dx$$

$$(iii) \quad \int_{-\pi}^{\pi} (x + 1)^2 e^x \delta(\sin \pi x) \, dx \qquad (vi) \quad \int_{-\infty}^{\infty} f(x) \delta(x^2 + a^2) \, dx$$

LÖSUNG:

6 Punkte Lösung: Delta-Funktionen von komplizierteren Argumenten

$$\delta[g(x)] = \sum_{k=1}^n \frac{\delta(x-x_{0k})}{|g'(x_{0k})|} \quad \text{mit} \quad g(x_{0k})=0, g'(x_{0k}) \neq 0$$

i)  $\int_{-\infty}^{\infty} f(x) \delta(-ax+b) dx$

$$g(x) = -ax + b \quad g'(x) = -a \neq 0 \quad \forall x \quad [\text{obdA: } a \neq 0]$$

$$-ax_0 + b = 0 \quad \Rightarrow \quad x_0 = \frac{b}{a} \quad \& \quad g'(\frac{b}{a}) \neq 0$$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} f(x) \delta(-ax+b) dx &= \int_{-\infty}^{\infty} f(x) \cdot \frac{1}{|a|} \delta(x - \frac{b}{a}) dx \\ &= \frac{1}{|a|} f(\frac{b}{a}) // \end{aligned}$$

ii)  $\int_0^{\infty} \ln x \delta(x^2-1) dx$

$$g(x) = x^2 - 1 \quad g'(x) = 2x$$

$$x_0^2 - 1 = 0$$

$$\Rightarrow x_{01} = 1, x_{02} = -1 \quad \rightarrow \quad x_{02} \notin (0, \infty): \text{muss nicht ber\ddot{u}cksichtigt werden}$$

$$g'(x_{01}) = 2 \neq 0$$

$$\Rightarrow \int_0^{\infty} \ln x \cdot \frac{1}{2} \cdot \delta(x-1) dx = \frac{1}{2} \ln(1) = 0 //$$

iii)  $\int_{-\pi}^{\pi} (x+1)^2 \delta(\sin(\pi x)) dx$

$$g(x) = \sin(\pi x) \quad \Rightarrow \quad x_{0k} = k$$

$$x_{0k} \in (-\pi, \pi) \quad \text{f\ddot{u}r} \quad k = 0, \pm 1, \pm 2, \pm 3$$

$$g'(x) = \pi \cos(\pi x)$$

$$g'(0) = \pi, g'(\pm 1) = -\pi, g'(\pm 2) = \pi, g'(\pm 3) = -\pi$$

$$\begin{aligned} \int_{-\pi}^{\pi} (x+1)^2 \delta(\sin(\pi x)) dx &= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} (x+1)^2 \delta(x+3) dx \right. \\ &\quad + \int_{-\pi}^{\pi} (x+1)^2 \delta(x+2) dx + \int_{-\pi}^{\pi} (x+1)^2 \delta(x+1) dx + \int_{-\pi}^{\pi} (x+1)^2 \delta(x) dx \\ &\quad \left. + \int_{-\pi}^{\pi} (x+1)^2 \delta(x-1) dx + \int_{-\pi}^{\pi} (x+1)^2 \delta(x-2) dx + \int_{-\pi}^{\pi} (x+1)^2 \delta(x-3) dx \right] \\ &= \frac{1}{\pi} [(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2] = \frac{35}{\pi} // \end{aligned}$$



$$iv) \int_{-\infty}^{\infty} \cos x \delta(x^2 - \pi^2) dx$$

$$g(x) = x^2 - \pi^2$$

$$g'(x) = 2x$$

$$x_0^2 = \pi^2 \Rightarrow x_{01} = \pi, x_{02} = -\pi; \pi, -\pi \in (-\infty, \infty) \checkmark$$

$$g'(x_{01}) = 2\pi, g'(x_{02}) = -2\pi$$

$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} \cos x \delta(x^2 - \pi^2) dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos x [\delta(x - \pi) + \delta(x + \pi)] dx \\ &= \frac{1}{2\pi} (\cos \pi + \cos(-\pi)) = \frac{\cos \pi}{\pi} = -1/\pi // \end{aligned}$$

$$v) \int_{-\infty}^{\infty} f(x) \delta(e^x - 1) dx$$

$$g(x) = e^x - 1$$

$$g'(x) = e^x \neq 0 \quad \forall x$$

$$x_0 = \ln(1) = 0 \in (-\infty, \infty) \checkmark$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0) //$$

$$vi) \int_{-\infty}^{\infty} f(x) \delta(x^2 + a^2) dx$$

$$g(x) = x^2 + a^2 \Rightarrow \text{keine reelle Nullstelle}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \delta(x^2 + a^2) dx = 0 //$$