zu Aufgabe 4 — Vektoroperator-Identitäten (I)

(vii)

• $$\left(a_i \frac{\partial}{\partial x_i}\right) a_k = a_i \frac{\partial a_k}{\partial x_i} \qquad (*)$$

• $$-\varepsilon_{ijk}\, a_j\, (\text{rot}\, a)_k = -\varepsilon_{ijk}\, a_j\, \varepsilon_{klm} \frac{\partial}{\partial x_l} a_m = -\varepsilon_{ijk}\,\varepsilon_{klm}\, a_j \frac{\partial a_m}{\partial x_l}$$

$$= -\varepsilon_{kij}\,\varepsilon_{klm}\, a_j \frac{\partial a_m}{\partial x_l} = -(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j \frac{\partial a_m}{\partial x_l}$$

$$= -a_j \frac{\partial a_j}{\partial x_i} + a_j \frac{\partial a_i}{\partial x_j}$$

für $a^2 = a_j a_j = \text{const}$ ist $\frac{\partial}{\partial x_k}(a_j a_j) = 2 a_j \frac{\partial a_j}{\partial x_k} = 0 \longrightarrow a_j \frac{\partial a_j}{\partial x_i} = 0$

damit $\boxed{-\varepsilon_{ijk}\, a_j\, (\text{rot}\, a)_k = a_j \frac{\partial a_i}{\partial x_j}}$, Übereinstimmung mit (*) 1

(viii)

$$\frac{\partial^2}{\partial x_i \partial x_i}(UV) = \frac{\partial}{\partial x_i}\left(U \frac{\partial V}{\partial x_i} + V \frac{\partial U}{\partial x_i}\right) = 2 \frac{\partial U}{\partial x_i}\frac{\partial V}{\partial x_i} + U \frac{\partial^2 V}{\partial x_i \partial x_i} + V \frac{\partial^2 U}{\partial x_i \partial x_i}$$

$$\boxed{\Delta(UV) = 2\, \text{grad}\, U \cdot \text{grad}\, V + U \Delta V + V \Delta U}$$ 1

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Aufgabe 5 — Vektoroperator-Identitäten (II)

(i)

$$\left.\begin{aligned} &\bullet\; c_i \frac{\partial}{\partial x_i}(a_k b_k) = c_i \frac{\partial a_k}{\partial x_i} b_k + c_i a_k \frac{\partial b_k}{\partial x_i} \\ &\bullet\; a_i \left(c_j \frac{\partial}{\partial x_j}\right) b_i + b_i \left(c_j \frac{\partial}{\partial x_j}\right) a_i = a_i c_j \frac{\partial b_i}{\partial x_j} + b_i c_j \frac{\partial a_i}{\partial x_j} \end{aligned}\right\} \text{Übereinstimmung } (i \to j,\; k \to i)$$

(ii)

• $$\left(c_i \frac{\partial}{\partial x_i}\right) \varepsilon_{klm}\, a_l b_m = \varepsilon_{klm}\, c_i \frac{\partial}{\partial x_i}(a_l b_m) = \varepsilon_{klm}\, c_i \frac{\partial a_l}{\partial x_i} b_m + \varepsilon_{klm}\, c_i a_l \frac{\partial b_m}{\partial x_i}$$

• $$\varepsilon_{klm}\, a_l \left(c_i \frac{\partial}{\partial x_i}\right) b_m - \varepsilon_{klm}\, b_l \left(c_i \frac{\partial}{\partial x_i}\right) a_m$$

$$= \varepsilon_{klm}\, a_l\, c_i \frac{\partial b_m}{\partial x_i} - \varepsilon_{klm}\, b_l\, c_i \frac{\partial a_m}{\partial x_i}$$

$$= \varepsilon_{klm}\, a_l\, c_i \frac{\partial b_m}{\partial x_i} - \underbrace{\varepsilon_{kml}}_{-\varepsilon_{klm}}\, b_m\, c_i \frac{\partial a_l}{\partial x_i}$$, Übereinstimmung

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$\boxed{(iii)}$ • $\left(\frac{\partial}{\partial x_j} a_i\right) b_j = \frac{\partial}{\partial x_i}(a_i b_j)$, wenn $\vec{\nabla}$ auf $\mathfrak{a}$ und $\mathfrak{b}$ wirken soll (!)

$$= \frac{\partial a_i}{\partial x_i} b_j + a_i \frac{\partial b_j}{\partial x_i}$$

• $\left(a_i \frac{\partial}{\partial x_i}\right) b_j + b_j \frac{\partial a_i}{\partial x_i} = a_i \frac{\partial b_j}{\partial x_i} + b_j \frac{\partial a_i}{\partial x_i}$, Übereinstimmung

$\boxed{(iv)}$ • $\varepsilon_{ijk} a_j b_k \varepsilon_{imn} \frac{\partial}{\partial x_m} c_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_j b_k \frac{\partial c_n}{\partial x_m}$

$$= a_m b_n \frac{\partial c_n}{\partial x_m} - a_n b_m \frac{\partial c_n}{\partial x_m}$$

• $b_i \left(a_j \frac{\partial}{\partial x_j}\right) c_i - a_i \left(b_j \frac{\partial}{\partial x_j}\right) c_i = a_m b_n \frac{\partial c_n}{\partial x_m} - a_n b_m \frac{\partial c_n}{\partial x_m}$, Übereinstimmung

$\boxed{(v)}$ • $\varepsilon_{ijk} (\mathfrak{a} \times \mathrm{grad})_j b_k = \varepsilon_{ijk} \varepsilon_{jlm} a_l \frac{\partial}{\partial x_m} b_k = -\varepsilon_{jik} \varepsilon_{jlm} a_l \frac{\partial b_k}{\partial x_m}$

$$= -(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) a_l \frac{\partial b_k}{\partial x_m} = -a_i \frac{\partial b_k}{\partial x_k} + a_k \frac{\partial b_k}{\partial x_i}$$

• $\left(a_k \frac{\partial}{\partial x_k}\right) b_i + \varepsilon_{ijk} a_j (\mathrm{rot}\, b)_k - a_i \frac{\partial b_k}{\partial x_k}$

$$= a_k \frac{\partial b_i}{\partial x_k} + \varepsilon_{ijk} a_j \varepsilon_{klm} \frac{\partial}{\partial x_l} b_m - a_i \frac{\partial b_k}{\partial x_k}$$

$$= a_k \frac{\partial b_i}{\partial x_k} + \varepsilon_{kij} \varepsilon_{klm} a_j \frac{\partial b_m}{\partial x_l} - a_i \frac{\partial b_k}{\partial x_k}$$

$$= a_k \frac{\partial b_i}{\partial x_k} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j \frac{\partial b_m}{\partial x_l} - a_i \frac{\partial b_k}{\partial x_k}$$

$= \cancel{a_k \frac{\partial b_i}{\partial x_k}} + a_m \frac{\partial b_m}{\partial x_i} - \cancel{a_j \frac{\partial b_i}{\partial x_j}} - a_i \frac{\partial b_k}{\partial x_k}$, Übereinstimmung

$\boxed{(vi)}$ • $\varepsilon_{ijk} (\vec{\nabla} \times \mathfrak{a})_j b_k = \varepsilon_{ijk} \varepsilon_{jlm} \frac{\partial}{\partial x_l}(a_m b_k)$, wenn $\vec{\nabla}$ auf $\mathfrak{a}$ und $\mathfrak{b}$ wirken soll (!)

$$= -\varepsilon_{jik} \varepsilon_{jlm} \left(\frac{\partial a_m}{\partial x_l} b_k + a_m \frac{\partial b_k}{\partial x_l}\right)$$

$$= -(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) \left(\frac{\partial a_m}{\partial x_l} b_k + a_m \frac{\partial b_k}{\partial x_l}\right)$$

$$= -\frac{\partial a_k}{\partial x_i} b_k - a_k \frac{\partial b_k}{\partial x_i} + \frac{\partial a_i}{\partial x_k} b_k + a_i \frac{\partial b_k}{\partial x_k}$$

• $$a_i \frac{\partial b_k}{\partial x_k} - \left(a_k \frac{\partial}{\partial x_k}\right) b_i - \varepsilon_{ijk}\, a_j (\mathrm{rot}\, \mathfrak{b})_k - \varepsilon_{ijk}\, b_j (\mathrm{rot}\, \mathfrak{a})_k$$

$$= a_i \frac{\partial b_k}{\partial x_k} - a_k \frac{\partial b_i}{\partial x_k} - \varepsilon_{ijk}\, a_j\, \varepsilon_{klm} \frac{\partial}{\partial x_l} b_m - \varepsilon_{ijk}\, b_j\, \varepsilon_{klm} \frac{\partial}{\partial x_l} a_m$$

$$= a_i \frac{\partial b_k}{\partial x_k} - a_k \frac{\partial b_i}{\partial x_k} - \varepsilon_{kij}\, \varepsilon_{klm} \left(a_j \frac{\partial b_m}{\partial x_l} + b_j \frac{\partial a_m}{\partial x_l} \right)$$

$$= a_i \frac{\partial b_k}{\partial x_k} - a_k \frac{\partial b_i}{\partial x_k} - \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \left(a_j \frac{\partial b_m}{\partial x_l} + b_j \frac{\partial a_m}{\partial x_l} \right)$$

$$= a_i \frac{\partial b_k}{\partial x_k} - \cancel{a_k \frac{\partial b_i}{\partial x_k}} - a_m \frac{\partial b_m}{\partial x_i} - b_m \frac{\partial a_m}{\partial x_i} + \cancel{a_j \frac{\partial b_i}{\partial x_j}} + b_j \frac{\partial a_i}{\partial x_j}\,,$$ Übereinstimmung