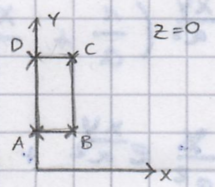
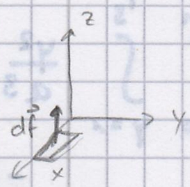
MMP2 2. Übungsserie : Der Satz von Stokes

Aufgabe 1: Verifikation — 9 Punkte

$$\vec{F} = F_0 \left[\left(\frac{y^3}{3a^3} + \frac{y}{a} e^{xy/a^2} + 1 \right) \vec{i} + \left(\frac{xy^2}{a^3} + \frac{x+y}{a} e^{xy/a^2} \right) \vec{j} + \frac{z}{a} e^{xy/a^2} \vec{k} \right]$$

(Skizze) 1 gesucht: $\oint_c \vec{F}\, d\vec{r}$





1. Weg — mit $d\vec{r} = \vec{v}\, dt$

2

$$\begin{cases} \vec{r}_1 = t\vec{i} + \vec{j} & 0 \le t \le 1 & \vec{v} = \vec{i} \\ \vec{r}_2 = \vec{i} + (1+t)\vec{j} & 0 \le t \le 2 & \vec{v} = \vec{j} \\ \vec{r}_3 = (1-t)\vec{i} + 3\vec{j} & 0 \le t \le 1 & \vec{v} = -\vec{i} \\ \vec{r}_4 = (3-t)\vec{j} & 0 \le t \le 2 & \vec{v} = -\vec{j} \end{cases}$$

$$\oint_c \vec{F}\, d\vec{r} = F_0 \Big(\int_0^1 \frac{1}{3a^3} + \frac{1}{a} e^{t/a^2} + 1 \, dt$$

$$+ \int_0^2 \frac{(1+t)^2}{a^3} + \frac{1+1+t}{a} e^{(1+t)/a^2} \, dt$$

$$- \int_0^1 \frac{9}{a^3} + \frac{3}{a} e^{3(1-t)/a^2} + 1 \, dt$$

1

$$- \int_0^2 \frac{(3-t)}{a} \, dt \Big)$$

$$= \frac{1}{3a^3} + \left[a e^{t/a^2} \right]_0^1 + \left[\frac{(1+t)^3}{3a^3} + a(2+t-a^2) e^{(1+t)/a^2} \right]_0^2$$

1

$$- \frac{9}{a^3} + \left[a e^{3(1-t)/a^2} \right]_0^1 + \left[\frac{(3-t)^2}{2a} \right]_0^2$$

$$= \frac{1}{a^3}\left(\cancel{\frac{1}{3}} - \cancel{9}\right) + a e^{1/a^2} - \cancel{a} + \cancel{\frac{9}{a^3}} + 4a e^{3/a^2} - \cancel{\frac{1}{3a^3}} - 2a e^{1/a^2}$$

$$+ \cancel{a} - a e^{3/a^2} + \frac{1}{2a} - \frac{9}{2a} - a^3 e^{3/a^2} + a^3 e^{1/a^2}$$

$$= -a e^{1/a^2} + \underbrace{4a e^{3/a^2}} \underbrace{- a e^{3/a^2}} - \frac{8}{2a} - a^3 e^{3/a^2} + a^3 e^{1/a^2}$$

1

$$= \underline{a\left[e^{3/a^2}(3 - a^2) + e^{1/a^2}(a^2 - 1) \right] - \frac{4}{a}}$$

$d\vec{f} = \vec{k}\, dx\, dy$ [2. Weg] 1

$$\iint \operatorname{rot} \vec{F}\, d\vec{f} = \iint \left[\operatorname{rot}\vec{F}\right]_3 dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=1}^{3} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy$$

$$= \int_{x=0}^{1} \int_{y=1}^{3} \frac{y^2}{a^3} + e^{\frac{xy}{a^2}} \left(\frac{1}{a} + \frac{x+y}{a} \cdot \frac{y}{a^2} \right) - \left(\frac{y}{a} \cdot \frac{x}{a^2}\, e^{\frac{xy}{a^2}} + \frac{y^2}{a^3} + \frac{1}{a}\, e^{xy/a^2} \right)$$

$$= \int_{x=0}^{1} \int_{y=1}^{3} \frac{y^2}{a^3} \cdot e^{xy/a^2}\, dx\, dy$$

$$= \int_{y=1}^{3} \left[e^{xy/a^2} \cdot \frac{y^2}{a^3} \cdot \frac{a^2}{y} \right]_{x=0}^{1} dy$$ 1

$$= \int_{y=1}^{3} \frac{y}{a} e^{y/a^2} - \frac{y}{a}\, dy$$

$$= \frac{1}{a} \left[\left(\frac{1}{a^2} \right)^{-2} e^{y/a^2} \left(\frac{y}{a^2} - 1 \right) \right]_{y=1}^{3} - \left[\frac{y^2}{2a} \right]_1^3$$

$$= a^3 \left(e^{3/a^2} \left(\frac{3}{a^2} - 1 \right) - e^{1/a^2} \left(\frac{1}{a^2} - 1 \right) \right) - \frac{9}{2a} - \frac{1}{2a}$$

$$\underline{= a\left[e^{3/a^2} (3 - a^2) + e^{1/a^2} (a^2 - 1) \right] - \frac{4}{a}}$$ 1