Beweis Leibniz'sche Produktregel

$$\frac{d^n}{dx^n}(u\cdot v) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\,\frac{d^k u}{dx^k}\cdot\frac{d^{n-k} v}{dx^{n-k}} \qquad (*)$$

Induktionsanfang: $n=0$

$$\frac{d^{(0)}}{dx^{(0)}}(u\cdot v) = u\cdot v = \binom{0}{0}\frac{d^{(0)}u}{dx^{(0)}}\cdot\frac{d^{(0)}v}{dx^{(0)}} \qquad \checkmark$$

Induktionsvoraussetzung: $(*)$ gelte für festes n

Induktionsschritt: $n \to n+1$

$$\frac{d^{(n+1)}}{dx^{(n+1)}}(u\cdot v) = \frac{d}{dx}\left(\frac{d^{(n)}}{dx^{(n)}}(u\cdot v)\right)$$

$$\overset{IV}{=} \frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{d^k u}{dx^k}\cdot\frac{d^{(n-k)}v}{dx^{(n-k)}}\right)$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left[\frac{d^{(k+1)}u}{dx^{(k+1)}}\cdot\frac{d^{(n-k)}v}{dx^{(n-k)}} + \frac{d^{(k)}u}{dx^{(k)}}\cdot\frac{d^{(n+1-k)}v}{dx^{(n+1-k)}}\right]$$

neuer Index:
$k' = k+1$
$(k = k'-1)$

$$= \sum_{k'=1}^{n+1} \frac{d^{k'}u}{dx^{k'}}\cdot\frac{d^{(n+1-k')}v}{dx^{(n+1-k')}}\cdot\frac{n!}{(k'-1)!(n+1-k')!}$$

$$+ \sum_{k=0}^{n}\binom{n}{k}\frac{d^{(k)}u}{dx^{(k)}}\cdot\frac{d^{(n+1-k)}v}{dx^{(n+1-k)}}$$

Anpassung der Grenzen

$$= \sum_{k'=0}^{n+1}\binom{n}{k'-1}\frac{d^{k'}u}{dx^{k'}}\cdot\frac{d^{(n+1-k')}v}{dx^{(n+1-k')}} + \sum_{k=0}^{n+1}\binom{n}{k}\frac{d^{(k)}u}{dx^{(k)}}\cdot\frac{d^{(n+1-k)}v}{dx^{(n+1-k)}}$$

da $\binom{n}{n+1} = 0 = \binom{n}{-1}$

Zusammenziehen der Summen

$$= \sum_{k=0}^{n+1}\frac{d^k u}{dx^k}\cdot\frac{d^{(n+1-k)}v}{dx^{(n+1-k)}}\left[\frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(n-k)!\,k!}\right]$$

$$= \sum_{k=0}^{n+1}\frac{d^k u}{dx^k}\cdot\frac{d^{(n+1-k)}v}{dx^{(n+1-k)}}\,\frac{n!}{k!}\left[\frac{k}{(n-k+1)!} + \frac{n-k+1}{(n-k+1)!}\right]$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k u}{dx^k}\cdot\frac{d^{(n+1-k)}v}{dx^{(n+1-k)}} \qquad \square$$