6 Punkte **Lösung:** Delta-Funktionen von komplizierteren Argumenten

$$\delta[g(x)] = \sum_{k=1}^{n} \frac{\delta(x-x_{0k})}{|g'(x_{0k})|} \quad \text{mit} \quad g(x_{0k}) = 0,\ g'(x_{0k}) \neq 0$$

i) $\int_{-\infty}^{\infty} f(x)\,\delta(-ax+b)\,dx$

$g(x) = -ax+b \qquad g'(x) = -a \neq 0 \quad \forall x \quad [\text{OBdA}: a \neq 0]$

$-ax_0 + b = 0 \quad \Rightarrow x_0 = \frac{b}{a} \quad \& \ g'(\frac{b}{a}) \neq 0$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(-ax+b)\,dx = \int_{-\infty}^{\infty} f(x) \cdot \frac{1}{|a|}\,\delta(x-\frac{b}{a})\,dx$$

$$= \frac{1}{|a|} f(\frac{b}{a}) \ //$$

ii) $\int_{0}^{\infty} \ln x \, \delta(x^2-1)\,dx$

$g(x) = x^2 - 1 \qquad g'(x) = 2x$

$x_0^2 - 1 = 0$

$\Rightarrow x_{01} = 1, \ x_{02} = -1 \quad \rightarrow \quad x_{02} \notin (0, \infty)$: muss nicht berücksichtigt werden

$g'(x_{01}) = 2 \neq 0$

$$\Rightarrow \int_{0}^{\infty} \ln x \cdot \frac{1}{2} \cdot \delta(x-1)\,dx = \frac{1}{2}\ln(1) = 0 \ //$$

iii) $\int_{-\pi}^{\pi} (x+1)^2\,\delta(\sin(\pi x))\,dx$

$g(x) = \sin(\pi x) \quad \Rightarrow x_{0k} = k$

$x_{0k} \in (-\pi, \pi)$ für $k = 0, \pm 1, \pm 2, \pm 3$

$g'(x) = \pi \cos(\pi x)$

$g'(0) = \pi, \ g'(\pm 1) = -\pi, \ g'(\pm 2) = \pi, \ g'(\pm 3) = -\pi$

$$\int_{-\pi}^{\pi} (x+1)^2 \delta(\sin(\pi x))\,dx = \frac{1}{\pi}\Big[\int_{-\pi}^{\pi} (x+1)^2\,\delta(x+3)\,dx$$

$$+ \int_{-\pi}^{\pi} (x+1)^2\,\delta(x+2)\,dx + \int_{-\pi}^{\pi} (x+1)^2\,\delta(x+1)\,dx + \int_{-\pi}^{\pi} (x+1)^2\,\delta(x)\,dx$$

$$+ \int_{-\pi}^{\pi} (x+1)^2\,\delta(x-1)\,dx + \int_{-\pi}^{\pi} (x+1)^2\,\delta(x-2)\,dx + \int_{-\pi}^{\pi} (x+1)^2\,\delta(x-3)\,dx\Big]$$

$$= \frac{1}{\pi}\left[(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2\right] = \frac{35}{\pi} \ //$$

iv) $\int_{-\infty}^{\infty} \cos x \; \delta(x^2-\pi^2)\, dx$

$g(x) = x^2-\pi^2 \qquad g'(x) = 2x$

$x_0^2 = \pi^2 \Rightarrow x_{01} = \pi \;,\; x_{02} = -\pi \quad ; \quad \pi, -\pi \in (-\infty, \infty) \checkmark$

$g'(x_{01}) = 2\pi \;,\; g'(x_{02}) = -2\pi$

$$\Rightarrow \int_{-\infty}^{\infty} \cos x \, \delta(x^2-\pi^2)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos x \left[\delta(x-\pi) + \delta(x+\pi)\right] dx$$

$$= \frac{1}{2\pi}\left(\cos\pi + \cos(-\pi)\right) = \frac{\cos\pi}{\pi} = -1/\pi \;//$$

v) $\int_{-\infty}^{\infty} f(x)\, \delta(e^x - 1)\, dx$

$g(x) = e^x - 1 \qquad g'(x) = e^x \neq 0 \quad \forall x$

$x_0 = \ln(1) = 0 \in (-\infty, \infty) \checkmark$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0) \;//$$

vi) $\int_{-\infty}^{\infty} f(x)\, \delta(x^2+a^2)\, dx$

$g(x) = x^2 + a^2 \quad \Rightarrow$ keine reelle Nullstelle

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, \delta(x^2+a^2)\, dx = 0 \;//$$