Aufgabe 3: Lösung

~~7~~ 7 Punkte

$$u = \frac{x}{x^2+y^2}, \quad v = \frac{-y}{x^2+y^2}, \quad z = z$$

Zusatz

a) $\underline{u = \text{const.}} \equiv C \, (\neq 0)$

$$C = \frac{x}{x^2+y^2} \Rightarrow C(x^2+y^2) - x = 0 \Rightarrow x^2 - \frac{1}{C}x + y^2 = 0$$

quadratische Ergänzung: $x^2 - \frac{1}{C}x + \frac{1}{4C^2} + y^2 = \frac{1}{4C^2}$

$$\left(x - \frac{1}{2C}\right)^2 + y^2 = \left(\frac{1}{2C}\right)^2 \qquad 1$$

→ Schar von Kreisen, Mittelpunkt $M = \left(\frac{1}{2C}, 0\right)$,
Radius $R = \frac{1}{2C}$

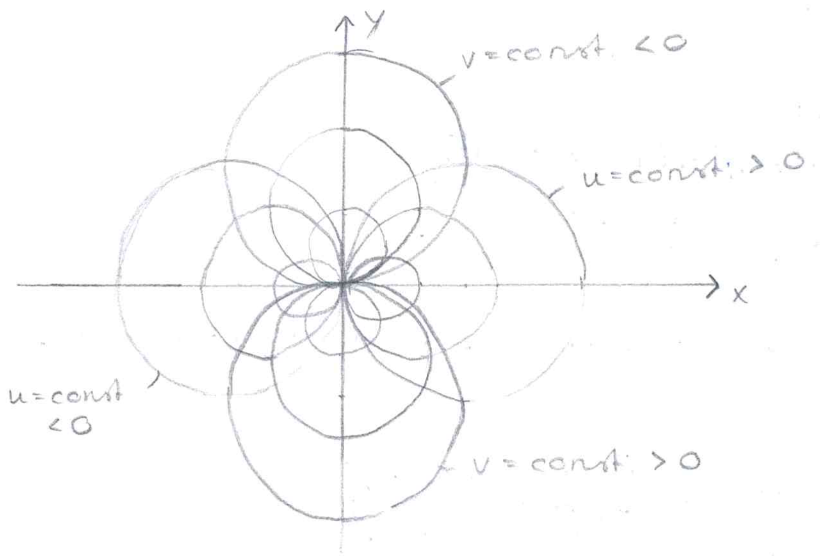
$\underline{v = \text{const.}} = D \, (\neq 0)$

$$D = \frac{-y}{x^2+y^2} \Rightarrow x^2 + y^2 + \frac{1}{D}y = 0$$

$$x^2 + y^2 + \frac{1}{D}y + \frac{1}{4D^2} = \frac{1}{4D^2}$$

$$\left(y + \frac{1}{D}\right)^2 + x^2 = \left(\frac{1}{2D}\right)^2 \qquad 1$$

→ Schar von Kreisen, Mittelpunkt $M\left(0, -\frac{1}{2D}\right)$, Radius $R = \frac{1}{2D}$



1 (vollständige Skizze, gute Beschriftung)

b) $u^2 + v^2 = \frac{1}{x^2+y^2}$

und: $\frac{u}{x} = -\frac{v}{y} \Rightarrow x = -\frac{u}{v}y$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} u^2+v^2 = \frac{1}{\frac{u^2}{v^2}y^2 + y^2} = \frac{v^2}{(u^2+v^2)y^2}$$

$$\Rightarrow y^2 = \frac{v^2}{(u^2+v^2)^2} \quad \Rightarrow x^2 = \frac{u^2}{(u^2+v^2)^2}$$

Wurzelziehen ergibt zwei Lösungen, um das richtige Vorzeichen zu bestimmen werden x^2 & y^2 oben eingesetzt.

$$u = \frac{x}{\frac{u^2+v^2}{(u^2+v^2)^2}} = x(u^2+v^2) \Rightarrow x = \frac{u}{u^2+v^2} \quad //$$ 1

$$v = \frac{-y}{\frac{u^2+v^2}{(u^2+v^2)^2}} = -y(u^2+v^2) \Rightarrow y = \frac{-v}{u^2+v^2} \quad //$$ 1

$$z = z \quad //$$

c) $$d_u\vec{r} = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}\right)du = \left(\frac{(v^2-u^2)\vec{i} + 2uv\vec{j}}{(u^2+v^2)^2}\right)du$$

$$\frac{\partial x}{\partial u} = \frac{1}{u^2+v^2} - \frac{u\cdot 2u}{(u^2+v^2)^2} = \frac{v^2-u^2}{(u^2+v^2)^2}$$

$$\frac{\partial y}{\partial u} = \frac{+2uv}{(u^2+v^2)^2}$$

analog:

$$\frac{\partial x}{\partial v} = -\frac{2uv}{(u^2+v^2)^2} \qquad \frac{\partial y}{\partial v} = \frac{v^2-u^2}{(u^2+v^2)^2}$$

$$\left|(v^2-u^2)\vec{i} + 2uv\vec{j}\right| = \sqrt{(v^2-u^2)^2 + 4u^2v^2} = \sqrt{(u^2+v^2)^2} = u^2+v^2$$

$$d_u\vec{r} = \vec{e_u}\cdot\frac{du}{u^2+v^2}$$

$$\Rightarrow \vec{e_u} = \frac{(v^2-u^2)\vec{i} + 2uv\vec{j}}{u^2+v^2}$$ 1

analog: $$\vec{e_v} = \frac{-2uv\vec{i} + (v^2-u^2)\vec{j}}{u^2+v^2} \qquad \left(d_v\vec{r} = \vec{e_v}\frac{dv}{u^2+v^2}\right)$$ 1

$$\vec{e_z} = \vec{k}$$

d) Orthogonalität:

$$\vec{e_u}\cdot\vec{e_v} = \frac{-2uv\cancel{(v^2-u^2)} + 2uv\cancel{(v^2-u^2)}}{(u^2+v^2)^2} = 0 \quad ✓$$

$$\vec{e_u}\cdot\vec{e_z} = \vec{e_v}\cdot\vec{e_z} = 0 \quad ✓$$

(u,v,z) ist ein orthogonales Koordinatensystem. 1/2

Händigkeit:

$$\vec{e_u}\times\vec{e_v} = \frac{\left[(v^2-u^2)^2 + 4u^2v^2\right]}{(u^2+v^2)^2}\vec{k} = \frac{(u^2+v^2)^2}{(u^2+v^2)^2}\vec{k} = \vec{e_z}$$

(u,v,z) ist ein rechtshändiges Koordinatensystem. 1/2

e) Linienelement: $ds^2 = d\vec{r}\cdot d\vec{r}$

$$d\vec{r} = d_u\vec{r} + d_v\vec{r} + d_z\vec{r} = \frac{\vec{e_u}du + \vec{e_v}dv}{u^2+v^2} + \vec{e_z}dz$$

$$ds^2 = \frac{(\vec{e_u}\cdot\vec{e_u})du^2 + (\vec{e_v}\cdot\vec{e_v})dv^2}{(u^2+v^2)^2} + (\vec{e_z}\cdot\vec{e_z})dz^2 = \frac{du^2+dv^2}{(u^2+v^2)^2} + dz^2 \quad //$$ 1

Volumenelement dV

$$\frac{\partial(x,y,z)}{\partial(u,v,z)} = \begin{vmatrix} \frac{v^2-u^2}{(u^2+v^2)^2} & \frac{-2uv}{(u^2+v^2)^2} & 0 \\ 2uv/(u^2+v^2)^2 & \frac{(v^2-u^2)}{(u^2+v^2)^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \frac{1}{(u^2+v^2)^4}\left((v^2-u^2)^2 + 4u^2v^2\right)$$

$$= \frac{(v^2+u^2)^2}{(u^2+v^2)^4} = \frac{1}{(u^2+v^2)^2}$$

$$\Rightarrow dV = \frac{1}{(u^2+v^2)^2}\, du\, dv\, dz \; //$$

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