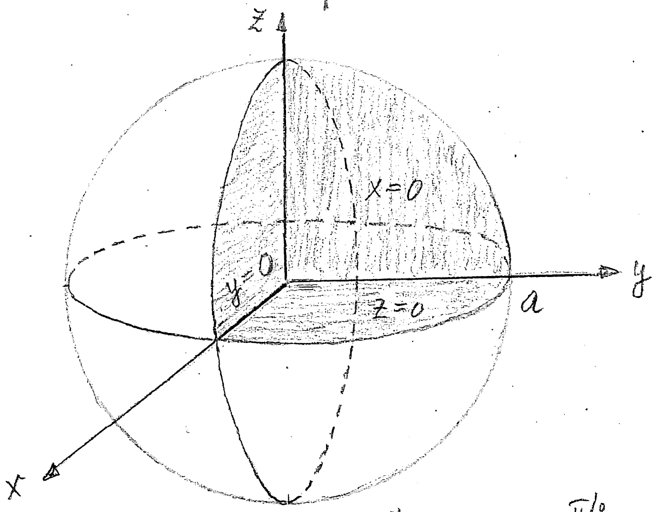
Aufgabe 3 Ein Dreifachintegral (II)

$$I = \iiint_V x z^2 \exp \frac{x^2+y^2+z^2}{a^2} \, dx\,dy\,dz$$

• Kugelkoordinaten:

$$x = r \cos\varphi \sin\vartheta$$
$$y = r \sin\varphi \sin\vartheta$$
$$z = r \cos\vartheta$$

$$dx\,dy\,dz = r^2 \sin\vartheta \, dr\,d\vartheta\,d\varphi$$



Integrationsgrenzen:

$$\left.\boxed{\begin{array}{l} 0 \le r \le a \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \vartheta \le \frac{\pi}{2} \end{array}}\right\} \text{Oktant}$$

$$I = \int_0^a r^2 dr \int_0^{\pi/2} \sin\vartheta \, d\vartheta \int_0^{\pi/2} d\varphi \; r\cos\varphi \sin\vartheta \cdot r^2 \cos^2\vartheta \, e^{r^2/a^2}$$

$$= \int_0^a r^5 e^{r^2/a^2} dr \underbrace{\int_0^{\pi/2} \sin^2\vartheta \cos^2\vartheta \, d\vartheta}_{\left(\frac{\vartheta}{8} - \frac{\sin 4\vartheta}{32}\right)\Big|_0^{\pi/2}} \underbrace{\int_0^{\pi/2} \cos\varphi \, d\varphi}_{\sin\varphi\big|_0^{\pi/2} = \sin\frac{\pi}{2} = \underline{\underline{1}}}$$

$$= \underline{\underline{\frac{\pi}{16}}}$$

Bronstein/S., S. 320, № 355

Substitution: $t = \frac{r^2}{a^2}$, $r^2 = a^2 t$; $r=0:\ t=0$, $r=a:\ t=1$

$$dt = \frac{2}{a^2} r\,dr$$

$$\int_0^1 \frac{a^2}{2} dt \cdot a^4 t^2 e^t = \frac{a^6}{2} \int_0^1 t^2 e^t dt = \frac{a^6}{2} (t^2 - 2t + 2) e^t \Big|_0^1$$

$$= \underline{\underline{\frac{a^6}{2}(e-2)}}$$

(1/6) Resultat: $$\underline{\underline{I}} = \frac{a^6}{2}(e-2)\cdot\frac{\pi}{16}\cdot 1 = \underline{\underline{\frac{\pi a^6}{32}(e-2)}}$$