Aufgabe 1 *Integrale mit Delta-Distributionen II*

Berechnen Sie die Integrale:

(i) $$\int_{-\infty}^{\infty} f(x)\delta(-ax+b)\,\mathrm{d}x$$

(ii) $$\int_{0}^{\infty} \ln x\,\delta(x^2-1)\,\mathrm{d}x$$

(iii) $$\int_{-\pi}^{\pi} (x+1)^2\delta(\sin \pi x)\,\mathrm{d}x$$

(iv) $$\int_{-\infty}^{\infty} \cos x\,\delta(x^2-\pi^2)\,\mathrm{d}x$$

(v) $$\int_{-\infty}^{\infty} f(x)\delta(e^x-1)\,\mathrm{d}x$$

(vi) $$\int_{-\infty}^{\infty} f(x)\delta(x^2+a^2)\,\mathrm{d}x \quad a \in \mathbb{R}\,,$$

LÖSUNG:

bitte wenden

6 Punkte **Lösung:** Delta-Funktionen von komplizierteren Argumenten

$$\delta[g(x)] = \sum_{k=1}^{n} \frac{\delta(x-x_{0k})}{|g'(x_{0k})|} \quad \text{mit} \quad g(x_{0k})=0,\ g'(x_{0k})\neq 0$$

i) $\int_{-\infty}^{\infty} f(x)\, \delta(-ax+b)\, dx$

$g(x) = -ax+b \qquad g'(x) = -a \neq 0 \quad \forall x \quad [\text{OBdA}: a\neq 0]$

$-ax_0+b=0 \quad \Rightarrow x_0 = \frac{b}{a} \quad \& \ g'(\frac{b}{a}) \neq 0$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(-ax+b)\,dx = \int_{-\infty}^{\infty} f(x)\cdot \frac{1}{|a|}\,\delta(x-\frac{b}{a})\,dx$$

$$= \frac{1}{|a|} f(\frac{b}{a}) \,//$$

ii) $\int_{0}^{\infty} \ln x\, \delta(x^2-1)\, dx$

$g(x) = x^2-1 \qquad g'(x) = 2x$

$x_0^2 - 1 = 0$

$\Rightarrow x_{01} = 1,\ x_{02} = -1 \quad \rightarrow \quad x_{02} \notin (0,\infty)$: muss nicht berücksichtigt werden

$g'(x_{01}) = 2 \neq 0$

$$\Rightarrow \int_{\cancel{-\infty}\,0}^{\infty} \ln x \cdot \frac{1}{2}\cdot \delta(x-1)\,dx = \frac{1}{2}\ln(1) = 0 \,//$$

iii) $\int_{-\pi}^{\pi} (x+1)^2\, \delta(\sin(\pi x))\, dx$

$g(x) = \sin(\pi x) \quad \Rightarrow x_{0k} = k$

$x_{0k} \in (-\pi,\pi)$ für $k = 0, \pm 1, \pm 2, \pm 3$

$g'(x) = \pi\cos(\pi x)$

$g'(0) = \pi,\ g'(\pm 1) = -\pi,\ g'(\pm 2) = \pi,\ g'(\pm 3) = -\pi$

$$\int_{-\pi}^{\pi} (x+1)^2\delta(\sin(\pi x))\,dx = \frac{1}{\pi}\Big[\int_{-\pi}^{\pi}(x+1)^2\,\delta(x+3)\,dx$$

$$+ \int_{-\pi}^{\pi}(x+1)^2\,\delta(x+2)\,dx + \int_{-\pi}^{\pi}(x+1)^2\,\delta(x+1)\,dx + \int_{-\pi}^{\pi}(x+1)^2\,\delta(x)\,dx$$

$$+ \int_{-\pi}^{\pi}(x+1)^2\,\delta(x-1)\,dx + \int_{-\pi}^{\pi}(x+1)^2\,\delta(x-2)\,dx + \int_{-\pi}^{\pi}(x+1)^2\,\delta(x-3)\,dx\Big]$$

$$= \frac{1}{\pi}\Big[(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2\Big] = \frac{35}{\pi} \,//$$

iv) $\int_{-\infty}^{\infty} \cos x \; \delta(x^2 - \pi^2)\, dx$

$g(x) = x^2 - \pi^2 \qquad g'(x) = 2x$

$x_0^2 = \pi^2 \Rightarrow x_{01} = \pi \;,\; x_{02} = -\pi \quad ; \quad \pi, -\pi \in (-\infty, \infty) \checkmark$

$g'(x_{01}) = 2\pi \;,\; g'(x_{02}) = -2\pi$

$$\Rightarrow \int_{-\infty}^{\infty} \cos x \; \delta(x^2 - \pi^2)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos x \left[\delta(x-\pi) + \delta(x+\pi) \right] dx$$

$$= \frac{1}{2\pi} \left(\cos \pi + \cos(-\pi) \right) = \frac{\cos \pi}{\pi} = -1/\pi \;//$$

v) $\int_{-\infty}^{\infty} f(x)\, \delta(e^x - 1)\, dx$

$g(x) = e^x - 1 \qquad g'(x) = e^x \neq 0 \quad \forall x$

$x_0 = \ln(1) = 0 \in (-\infty, \infty) \checkmark$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0) \;//$$

vi) $\int_{-\infty}^{\infty} f(x)\, \delta(x^2 + a^2)\, dx$

$g(x) = x^2 + a^2 \quad \Rightarrow$ keine reelle Nullstelle

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, \delta(x^2 + a^2)\, dx = 0 \;//$$