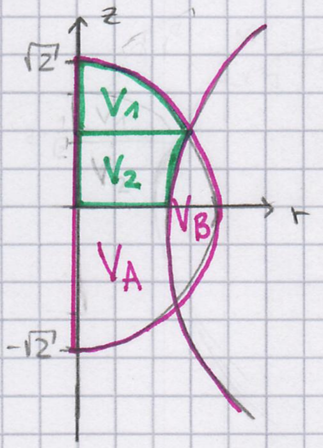
Aufgabe: Volumenberechnung   6 Punkte

$x^2+y^2 = 1+z^2$ Hyperboloid

$x^2+y^2 = 2-z^2 \quad \Rightarrow \quad x^2+y^2+z^2=(\sqrt{2})^2$ Kugel mit Radius $\sqrt{2}$

$\Rightarrow$ Axialsymmetrisches Problem $\rightsquigarrow$ verwende Zylinderkoordinaten



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Variante 1: (grüne Zerlegung)

$$V = \iiint r\,dr\,d\varphi\,dz = 2\cdot V_1 + 2 V_2$$

1 $$= 2\cdot \int_{\varphi=0}^{2\pi}\int_{r=0}^{\sqrt{3}/\sqrt{2}}\int_{z=1/\sqrt{2}}^{\sqrt{2-r^2}} r\,dr\,d\varphi\,dz$$

1 $$+2\cdot\int_{\varphi=0}^{2\pi}\int_{z=0}^{+\sqrt{1/2}}\int_{r=0}^{\sqrt{1+z^2}} r\,dr\,d\varphi\,dz$$

$$= 4\pi\int_{r=0}^{\sqrt{3}/\sqrt{2}} r\left[z\right]_{1/\sqrt{2}}^{\sqrt{2-r^2}} dr \; + 4\pi\int_{z=0}^{1/\sqrt{2}}\left[\frac{1}{2}r^2\right]_{r=0}^{\sqrt{1+z^2}} dz$$

1 $$= 4\pi\int_0^{\sqrt{3}/\sqrt{2}} r\sqrt{2-r^2} - \frac{1}{\sqrt{2}}\, r\,dr \; + 2\pi\int_{z=0}^{1/\sqrt{2}} 1+z^2\,dz$$

1 $$= 4\pi\left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{1}{2\sqrt{2}}\,r^2\right]_0^{\sqrt{3}/\sqrt{2}} + 2\pi\left[z+\frac{1}{3}z^3\right]_0^{1/\sqrt{2}}$$

$$= 4\pi\left[\left(-\frac{1}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{3}{\sqrt{2}\cdot 4}\right) - \left(-\frac{1}{3}\,2\sqrt{2} - \frac{1}{2\sqrt{2}}\cdot 0\right)\right]$$

$$+ 2\pi\left[\frac{1}{\sqrt{2}} + \frac{1}{3}\cdot\frac{1}{\sqrt{2}\,2}\right]$$

1 $$= 4\pi\left[-\frac{1}{12\sqrt{2}} - \frac{1}{4\sqrt{2}} + \frac{2}{3}\sqrt{2}\right] = 4\pi\,\frac{\sqrt{2}}{3}\cdot\frac{3}{2} = \underline{2\sqrt{2}\,\pi}$$

Variante 2 : (lila Zerlegung)

$$V = V_A - V_B$$

$$V_A = \int_{r=0}^{\sqrt{2}} \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi$$ 1

$$= 4\pi \cdot \frac{1}{3} \left[r^3 \right]_0^{\sqrt{2}}$$

$$= \frac{4}{3} \pi \, 2\sqrt{2}$$

$$= \frac{8\pi}{3} \cdot \sqrt{2}$$ 1

$$V_B = \int_{\varphi=0}^{2\pi} \int_{z=-1/\sqrt{2}}^{1/\sqrt{2}} \int_{r=\sqrt{1+z^2}}^{\sqrt{2-z^2}} r \, dr \, d\varphi \, dz$$ 1

$$= 2\pi \int_{z=-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1}{2} \left[r^2 \right]_{\sqrt{1+z^2}}^{\sqrt{2-z^2}} dz$$

$$= \pi \int_{z=-1/\sqrt{2}}^{1/\sqrt{2}} (2 - z^2) - (1 + z^2) \, dz$$

$$= \pi \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 1 - 2z^2 \, dz$$

$$= \pi \left[z - \frac{2}{3} z^3 \right]_{-1/\sqrt{2}}^{1/\sqrt{2}} = \pi \left[\left(\frac{1}{\sqrt{2}} - \frac{2}{3} \frac{1}{2\sqrt{2}} \right) - \left(-\frac{1}{\sqrt{2}} + \frac{2}{3} \frac{1}{2\sqrt{2}} \right) \right]$$ 1

$$= \pi \left[\frac{2}{\sqrt{2}} - \frac{2}{3} \frac{1}{\sqrt{2}} \right] = \frac{2\pi}{\sqrt{2}} \left[1 - \frac{1}{3} \right] = \frac{4\pi}{3\sqrt{2}} = \frac{2\sqrt{2}\pi}{3}$$

$$\Rightarrow V = \frac{8\pi}{3} \sqrt{2} - \frac{2\pi}{3} \sqrt{2} = \frac{6\pi}{3} \sqrt{2} = \underline{2\sqrt{2}\,\pi}$$ 1