Aufgabe 5 Dreireihige Determinante

$$\boxed{(\det A)\, \varepsilon_{lmn} = A_{li} A_{mj} A_{nk}\, \varepsilon_{ijk}}$$

setzen $l=1,\ m=2,\ n=3$, so daß $\varepsilon_{lmn} = 1$

damit

$$\det A = A_{1i} A_{2j} A_{3k}\, \varepsilon_{ijk} = A_{11}\, \varepsilon_{1jk} A_{2j} A_{3k} + A_{12}\, \varepsilon_{2jk} A_{2j} A_{3k} + A_{13}\, \varepsilon_{3jk} A_{2j} A_{3k}$$

$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) + A_{12}(-A_{21}A_{33} + A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

- zum Vergleich: entwickeln $\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$ nach erster Zeile $\longrightarrow$ Übereinstimmung

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Aufgabe 6 Vektoroperator-Identitäten

a) $\boxed{\mathrm{div}(\lambda \mathfrak{a})} = \dfrac{\partial}{\partial x_i}(\lambda a_i) = \dfrac{\partial \lambda}{\partial x_i} a_i + \lambda \dfrac{\partial a_i}{\partial x_i} = \boxed{\mathfrak{a}\, \mathrm{grad}\, \lambda + \lambda\, \mathrm{div}\, \mathfrak{a}}$ 0,5

„Produktregel"

b) $\boxed{\mathrm{rot}(\lambda \mathfrak{a})} \longrightarrow \varepsilon_{ijk} \dfrac{\partial}{\partial x_j}(\lambda a_k) = \varepsilon_{ijk} \dfrac{\partial \lambda}{\partial x_j} a_k + \lambda\, \varepsilon_{ijk} \dfrac{\partial a_k}{\partial x_j}$

$= \left[(\mathrm{grad}\, \lambda) \times \mathfrak{a}\right]_i + \lambda (\mathrm{rot}\, \mathfrak{a})_i \longrightarrow \boxed{(\mathrm{grad}\, \lambda) \times \mathfrak{a} + \lambda\, \mathrm{rot}\, \mathfrak{a}}$ 1,5

c) $\boxed{\mathrm{grad}(UV)} \longrightarrow \dfrac{\partial}{\partial x_i}(UV) = V \dfrac{\partial U}{\partial x_i} + U \dfrac{\partial V}{\partial x_i} \longrightarrow \boxed{V\, \mathrm{grad}\, U + U\, \mathrm{grad}\, V}$ 0,5

d) $\boxed{\mathrm{div}(\mathfrak{a} \times \mathfrak{b})} = \dfrac{\partial}{\partial x_i}(\mathfrak{a} \times \mathfrak{b})_i = \dfrac{\partial}{\partial x_i} \varepsilon_{ijk} a_j b_k = \varepsilon_{ijk} \dfrac{\partial a_j}{\partial x_i} b_k + \varepsilon_{ijk} a_j \dfrac{\partial b_k}{\partial x_i}$

$= b_k \varepsilon_{kij} \dfrac{\partial a_j}{\partial x_i} - a_j \varepsilon_{jik} \dfrac{\partial b_k}{\partial x_i}$

$= b_k (\mathrm{rot}\, \mathfrak{a})_k - a_j (\mathrm{rot}\, \mathfrak{b})_j = \boxed{\mathfrak{b}\, \mathrm{rot}\, \mathfrak{a} - \mathfrak{a}\, \mathrm{rot}\, \mathfrak{b}}$ 1,5

- speziell: $\left.\begin{array}{l} \mathfrak{a} = \mathrm{grad}\, U, \quad \mathrm{rot}\, \mathfrak{a} = \mathrm{rot}\, \mathrm{grad}\, U = 0 \\ \mathfrak{b} = \mathrm{grad}\, V, \quad \mathrm{rot}\, \mathfrak{b} = \mathrm{rot}\, \mathrm{grad}\, V = 0 \end{array}\right\}$ $\underline{\mathrm{div}(\mathrm{grad}\, U \times \mathrm{grad}\, V) = 0}$

(2/4) oder direkt: $\operatorname{div}(\operatorname{grad} U \times \operatorname{grad} V) = \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial U}{\partial x_j} \frac{\partial V}{\partial x_k}$

$$= \underbrace{\varepsilon_{ijk}}_{\substack{\text{antisymmetr.}\\ \text{in } i,j}} \underbrace{\frac{\partial^2 U}{\partial x_i \partial x_j}}_{\text{symmetrisch}} \frac{\partial V}{\partial x_k} + \underbrace{\varepsilon_{ijk}}_{\substack{\text{antisymmetr.}\\ \text{in } i,k}} \frac{\partial U}{\partial x_j} \underbrace{\frac{\partial^2 V}{\partial x_i \partial x_k}}_{\text{symmetrisch}} = 0$$

e) $\boxed{\operatorname{grad}(\mathfrak{a}\mathfrak{b})} \longrightarrow \frac{\partial}{\partial x_i}(a_j b_j) = \underline{\frac{\partial a_j}{\partial x_i} b_j} + \underline{a_j \frac{\partial b_j}{\partial x_i}}$

andererseits:

$\boxed{(\mathfrak{b}\operatorname{grad})\mathfrak{a} + (\mathfrak{a}\operatorname{grad})\mathfrak{b} + \mathfrak{a} \times \operatorname{rot}\mathfrak{b} + \mathfrak{b} \times \operatorname{rot}\mathfrak{a}} \longrightarrow$

$$b_j \frac{\partial}{\partial x_j} a_i + a_j \frac{\partial}{\partial x_j} b_i + \varepsilon_{ijk} a_j \varepsilon_{klm} \frac{\partial b_m}{\partial x_l} + \varepsilon_{ijk} b_j \varepsilon_{klm} \frac{\partial a_m}{\partial x_l}$$

$$= b_j \frac{\partial a_i}{\partial x_j} + a_j \frac{\partial b_i}{\partial x_j} + a_j(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial b_m}{\partial x_l} + b_j(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial a_m}{\partial x_l}$$

$$= \cancel{b_j \frac{\partial a_i}{\partial x_j}} + \cancel{a_j \frac{\partial b_i}{\partial x_j}} + \underline{\frac{\partial b_j}{\partial x_i} a_j} - \cancel{\frac{\partial b_i}{\partial x_j} a_j} + \underline{\frac{\partial a_j}{\partial x_i} b_j} - \cancel{\frac{\partial a_i}{\partial x_j} b_j}$$, Übereinstimmung

1,5 • speziell $\mathfrak{b} = \mathfrak{a}$: $\operatorname{grad}(\mathfrak{a}\mathfrak{a}) = 2(\mathfrak{a}\operatorname{grad})\mathfrak{a} + 2\mathfrak{a} \times \operatorname{rot}\mathfrak{a}$

f) $\boxed{\operatorname{rot}(\mathfrak{a} \times \mathfrak{b})} \longrightarrow \varepsilon_{ijk} \frac{\partial}{\partial x_j}(\mathfrak{a} \times \mathfrak{b})_k = \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_j}(a_l b_m)$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left(\frac{\partial a_l}{\partial x_j} b_m + a_l \frac{\partial b_m}{\partial x_j} \right)$$

$$= \frac{\partial a_i}{\partial x_j} b_j + a_i \frac{\partial b_j}{\partial x_j} - \frac{\partial a_j}{\partial x_j} b_i - a_j \frac{\partial b_i}{\partial x_j}$$

$$= b_j \frac{\partial a_i}{\partial x_j} - a_j \frac{\partial b_i}{\partial x_j} + a_i \operatorname{div}\mathfrak{b} - b_i \operatorname{div}\mathfrak{a}$$

1,5 $\longrightarrow \boxed{(\mathfrak{b}\operatorname{grad})\mathfrak{a} - (\mathfrak{a}\operatorname{grad})\mathfrak{b} + \mathfrak{a}\operatorname{div}\mathfrak{b} - \mathfrak{b}\operatorname{div}\mathfrak{a}}$