Elektrodynamik – Übung 08

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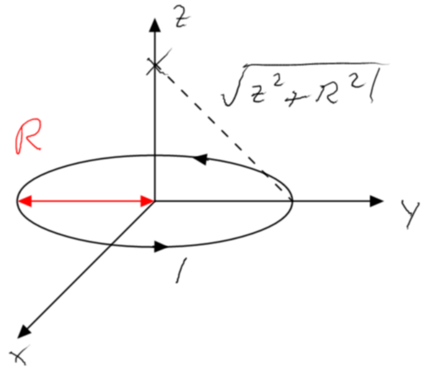
Aufgabe 21



es gilt in diesem Falle das Biot-Savart-Gesetz:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{s} \times (\vec{r} - \vec{s})}{|\vec{r}-\vec{s}|^3} = \vec{B}(\vec{r})$$

Kreisstrom $\Longrightarrow \vec{s}(t) = R(\cos t\, \vec{e_x} + \sin t\, \vec{e_y})$

$$\Longrightarrow \vec{s}\,'(t) = R(-\sin t\, \vec{e_x} + \cos t\, \vec{e_y})$$

weiterhin gilt: $|\vec{r}-\vec{s}|^3 \overset{(\vec{r}=z\vec{e_z})}{=} \sqrt{z^2+R^2}^{\,3}$

$$\Longrightarrow \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{\vec{s}\,' \times (z\vec{e_z} - s)}{\sqrt{z^2+R^2}^{\,3}}\, dt$$

$$\vec{s}\,' \times (\vec{r}-\vec{s}) = \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix} \times \begin{pmatrix} -R\cos t \\ -R\sin t \\ z \end{pmatrix} = \begin{pmatrix} zR\cos t \\ zR\sin t \\ R^2 \end{pmatrix}$$

$\Longrightarrow$ Integral über x- und y-Koordinate wird Null (aufgrund der Symmetrie auch verständlich)

$$\Longrightarrow \vec{B} = B(r)\,\vec{e_z} = \vec{e_z}\, \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R^2}{\sqrt{z^2+R^2}^{\,3}}\, dt$$

$$\Longrightarrow \boxed{\vec{B}(0,0,z) = \vec{e_z}\, \frac{\mu_0 I R^2}{2\sqrt{z^2+R^2}^{\,3}}}$$

Aufgabe 22

es herrschen stationäre Felder $\Longrightarrow \rho = \vec{\nabla}\cdot\vec{D} = \varepsilon\, \vec{\nabla}\cdot\vec{E}$ (*)

$\Longrightarrow \vec{j} = \vec{\nabla}\times\vec{H} = \mu^{-1}\, \vec{\nabla}\times\vec{B}$ (**)

$$\frac{\partial}{\partial x_j} T_{ij} = \frac{\partial}{\partial x_j}\left[\varepsilon E_i E_j + \frac{1}{\mu} B_i B_j - \frac{1}{2}\delta_{ij}\left(\varepsilon E^2 + \frac{1}{\mu} B^2\right)\right]$$

$$= \varepsilon E_j \frac{\partial E_i}{\partial x_j} + \varepsilon E_i \frac{\partial E_j}{\partial x_j} + \frac{1}{\mu} B_j \frac{\partial B_i}{\partial x_j} + \frac{1}{\mu} B_i \frac{\partial B_j}{\partial x_j}$$

$$- \delta_{ij}\, \varepsilon \sum_{k=1}^{3} E_k \frac{\partial E_k}{\partial x_j} - \delta_{ij} \frac{1}{\mu} \sum_{k=1}^{3} B_k \frac{\partial B_k}{\partial x_j}$$

$$\sum_{j=1}^{3} \varepsilon E_i \frac{\partial E_j}{\partial x_j} = E_i \cdot \varepsilon \cdot (\vec{\nabla} \cdot \vec{E}) \overset{(*)}{=} \rho E_i$$

$$\sum_{j=1}^{3} \frac{1}{\mu} B_i \frac{\partial B_j}{\partial x_j} = B_i \cdot \frac{1}{\mu} (\vec{\nabla} \cdot \vec{B}) \overset{(\vec{\nabla}\cdot\vec{B}=0)}{=} 0$$

$$\sum_{j=1}^{3} \varepsilon E_j \frac{\partial E_i}{\partial x_j} - \sum_{j=1}^{3} \delta_{ij} \varepsilon \sum_{k=1}^{3} E_k \frac{\partial E_k}{\partial x_j} = \sum_{j=1}^{3} \varepsilon E_j \frac{\partial E_i}{\partial x_j} - \varepsilon \sum_{k=1}^{3} E_k \frac{\partial E_k}{\partial x_i}$$

$$= \sum_{j=1}^{3} \left[\varepsilon \left(E_j \frac{\partial E_i}{\partial x_j} - E_j \frac{\partial E_j}{\partial x_i} \right) \right]$$

$$= \varepsilon \left[\sum_{j=1}^{3} \sum_{k,l=1}^{3} \delta_{jk} \delta_{il} E_j \frac{\partial E_l}{\partial x_k} - \sum_{j=1}^{3} \sum_{k,l=1}^{3} \delta_{jl} \delta_{ik} E_j \frac{\partial E_l}{\partial x_k} \right]$$

$$= \varepsilon \left[\sum_{j,k,l=1}^{3} (\delta_{jk} \delta_{il} - \delta_{jl} \delta_{ik}) E_j \frac{\partial E_l}{\partial x_k} \right]$$

$$= \varepsilon \left[\sum_{j,k,l,m=1}^{3} \varepsilon_{mji} \varepsilon_{mkl} E_j \frac{\partial E_l}{\partial x_k} \right] = \left[\sum_{j,k,l,m=1}^{3} \varepsilon_{ijm} \varepsilon_{mlk} E_j \frac{\partial E_l}{\partial x_k} \right] \cdot \varepsilon$$

$$= \varepsilon \left[\sum_{j,m=1}^{3} \varepsilon_{ijm} E_j \left(\sum_{k,l=1}^{3} \varepsilon_{mlk} \frac{\partial E_l}{\partial x_k} \right) \right] = \left[\sum_{j,m=1}^{3} \varepsilon_{ijm} E_j (-\vec{\nabla} \times \vec{E})_m \right] \cdot \varepsilon$$

$$= \varepsilon \left[\vec{E} \times (-\vec{\nabla} \times \vec{E}) \right]_i = \varepsilon \left[(\vec{\nabla} \times \vec{E}) \times \vec{E} \right]_i \overset{(\vec{\nabla}\times\vec{E}=0)}{=} 0$$

$$\Longrightarrow \sum_{j=1}^{3} \frac{1}{\mu} B_j \frac{\partial B_i}{\partial x_j} - \sum_{j=1}^{3} \delta_{ij} \frac{1}{\mu} \sum_{k=1}^{3} B_k \frac{\partial B_k}{\partial x_j} = \frac{1}{\mu} \left[(\vec{\nabla} \times \vec{B}) \times \vec{B} \right]_i$$

$$= \left[(\vec{\nabla} \times \vec{H}) \times \vec{B} \right]_i \overset{(**)}{=} \left[\vec{j} \times \vec{B} \right]_i$$

$$\Longrightarrow \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} = \rho E_i + [\vec{j} \times \vec{B}]_i = k_i \qquad \square$$