Chapter 4

Fourier Analysis

4.1 Existence of Test Functions

Definition 4.1.1. *Let X be a metric space and $f : X \to \mathbb{C}$. Then we define the* **support**

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}}$$

Remark 4.1.1. We note that the notion of support can more generally be defined for X a topological space.

Lemma 4.1.1. *There exists a non-negative function $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\phi(0) > 0$. For example the function*

$$\varphi : \mathbb{R}^d \to \mathbb{R}, \quad \varphi(x) := \begin{cases} \exp(-\frac{1}{1-|x|^2}) & , \quad |x| < 1 \\ 0 & , \quad |x| \geq 1, \end{cases}$$

is in $C_c^\infty(\mathbb{R}^n)$, with $\operatorname{supp}\varphi \subset B_1(0)$, $\varphi \geq 0$, $\varphi(0) = e^{-1} > 0$.

Proof. Let $f(t) = \exp(1-/t)$, for$t > 0$, and $f(t) = 0$, for $t \leq 0$. We know that $f \in C^\infty(\mathbb{R})$. Hence $\varphi(x) = f(1 - |x|^2)$ has the required properties. □

4.2 L^p spaces in $\mathbb{R}^d$

Definition 4.2.1. *Let f and g be measurable functions on $\mathbb{R}^d$. The* **convolution** *of f and g is the function $f * g$ defined by*

$$f * g(x) = \int f(x-y)g(y)dy, \tag{4.2.1}$$

for all x such that the integral exists.

There are very many different situations, in which the convolution exists. Next we consider two situations in which the convolution exists.

Definition 4.2.2.

$$L^p_{\text{loc}}(U) := \{f : U \to \mathbb{C} : \text{meas}, \ f|K \in L^p(K) \ \forall \text{ compact } K \subset U\}.$$

Lemma 4.2.1. *If $f \in C_c(\mathbb{R}^d)$ and $g \in L^1_{\text{loc}}$ or $g \in C(\mathbb{R}^d)$, then $f * g(x)$ is defined for all x and $f * g$ is continuous.*

Proof. The function $y \mapsto f(x-y)$ has compact support. Hence the integral in (4.2.1) exists. Continuity follows since f is uniformly continuous (by compactness) and hence $\sup_y |f(x-y) - f(x'-y)| \to 0$ as $x' \to x$. □

Theorem 4.2.2. *(Young Inequality) If $f \in L^1$ and $g \in L^p$ $(1 \leq p \leq \infty)$, then $f * g(x)$ exists for almost very x, $f * g \in L^p$, and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.*

Proof. The existence of the convolution follows from Hölder. Moreover, by Jensen' inequality with measure $\|f\|_1^{-1}|f(y)|dy$ we find

$$\begin{aligned} \|f * g\|_p^p &= \int \left| \int f(y)g(x-y)dy \right|^p dx \\ &\leq \int \|f\|_1^{p-1} \int |f(y)||g(x-y)|^p dy dx \leq \|f\|_1^p \|g\|_p^p. \end{aligned}$$

□

Proposition 4.2.3. *Assuming that all integrals in question exist, we have*

(a) $f * g = g * f$

(b) $(f * g) * h = f * (g * h)$.

(c) $\{x : f * g(x) \neq 0\} \subset \{x + y : f(x) \neq 0, g(y) \neq 0\}$.

$$f * g = g * f$$
$$(f * g) * h = f * (g * h)$$

Proof. (a) Substitution $z = x - y$ yields

$$f * g(x) = \int (x-y)g(y)dy = \int f(z)g(x-z)dz = g * f(x)$$

(b) Fubini's Theorem

$$\begin{aligned}\int (f * g) * h(x)dx &= \int\int f(y)g(x-z-y)h(z)dydz \\ &= \int f(y)g(x-y-z)h(z)dzdy = f * (g * h)(x).\end{aligned}$$

(c) Let A denote the set on the right hand side. Let $z \notin A$. Let y be arbitrary. If $g(y) = 0$, then trivially $f(z-y)g(y) = 0$. If $g(y) \neq 0$, then $f(z-y) = 0$, so $f(z-y)g(y) = 0$. So for any y we have $f(z-y)g(y) = 0$. Thus $f * g(z) = 0$. □

Convolution with a smooth function has a "smoothing" effect.

Proposition 4.2.4. *If* $f \in L^1$, $g \in C^k$, *and* $\partial^\alpha g$ *is bounded for* $|\alpha| \leq k$, *then* $f * g \in C^k$ *and* $\partial^\alpha(f * g) = f * \partial^\alpha g$ *for* $|\alpha| \leq k$.

Proof. This follows from the theorem about interchanging limits (and hence derivatives) with integrals. □

Theorem 4.2.5. *Suppose* $\phi \in L^1(\mathbb{R}^n)$ *and* $\int \phi(x)dx = a$. *For* $t > 0$ *let*

$$\phi_t(x) = t^{-n}\phi(x/t).$$

Then the following holds.

(a) If $f \in L^p$ for $p \in [1,\infty)$, then $f * \phi_t \to af$ in L^p as $t \to 0$.

(b) If f is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$.

Proof.

(a) Setting $y = tz$, we have

$$\begin{aligned} f * \phi_t(x) - af(x) &= \int [f(x-y) - f(x)]\phi_t(y)dy \\ &= \int [f(x-tz) - f(x)]\phi(z)dz. \end{aligned} \tag{4.2.2}$$

Now applying Jensen's inequality, for we the measure $\|\phi\|_1^{-1}|\phi(z)|dz$

$$\begin{aligned} \|f * \phi_t - af\|_p^p &\leq \int \left(\int |f(x-tz) - f(x)||\phi(z)|dz \right)^p dx \\ &\leq \int \int |f(x-tz) - f(x)|^p \|\phi\|_1^{p-1} |\phi(z)|dzdx \\ &= \int \|f(\cdot - tz) - f(\cdot)\|_p^p \|\phi\|_1^{p-1} |\phi(z)|dz \\ &\to 0, \end{aligned}$$

as $t \downarrow 0$ by dominated convergence (where we used that $\|f(\cdot - tz) - f(\cdot)\|_p^p \to 0$ again by dominated convergence).

(b) Using (4.2.2), we find

$$\sup_x |f * \phi_t(x) - af(x)| \leq \int \sup_x |f(x-tz) - f(x)||\phi(z)|dz.$$

Now the claim follows from dominated convergence and $\sup_x |f(x-tz) - f(x)| \to 0$ as $t \downarrow 0$ by uniform convergence. □

Let $C_0(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{C} : f \text{ continuous }, \forall \epsilon > 0, \exists R > 0, \sup_{x:|x| \geq R} |f(x)| < \epsilon\}$.

Corollary 4.2.6. *C_c^∞ is dense in L^p for $p \in [1,\infty)$ and in C_0.*

Proof. Suppose first $f \in L^p$. Let $\epsilon > 0$. Then there exists an $R > 0$, such that for $g = f1_{B_R(0)}$ we have $\|g - f\|_p < \epsilon/2$. Let ϕ be a function in C_c^∞ such that $\int \phi = 1$. Then $g * \phi_t \in C_c^\infty$ and $\|g * \phi_t - g\|_p < \epsilon/2$ for sufficiently small t. This shows the claim in that case.

Now suppose $f \in C_0$. Then the argument is similar. Let $\epsilon > 0$. Then there exists an $R > 0$, such that for $g = f1_{B_R(0)}$ we have $\|g - f\|_\infty < \epsilon/2$. Let ϕ be a function in C_c^∞ such that $\int \phi = 1$. Then $g * \phi_t \in C_c^\infty$ and $\|g * \phi_t - g\|_\infty < \epsilon/2$ for sufficiently small t. This shows the claim in that case. □

Theorem 4.2.7. *Suppose $U \subset \mathbb{R}^d$ is open and $f \in L^1_{\text{loc}}(U)$. Then*

$$\int_U \varphi(x) f(x) dx = 0, \quad \forall \varphi \in C_c^\infty(U),$$

if and only if $f(x) = 0$ a.e..

Proof. The if part is trivial. To show the only if part, we proceed as follows. Define

$$\text{sign}(f) := \begin{cases} \overline{f}/|f| & , \quad f(x) \neq 0 \\ 1 & , \quad f(x) = 0 \end{cases}$$

Let K be a compact subset of U. Then $\psi := 1_K \text{sign}(f)$ is measurable and bounded by 1. For $\phi \in C_c^\infty$ with $\int \phi(x)dx = 1$ and $0 \leq \phi \leq 1$ we let $\phi_n = n^d \phi(nx)$. Then $\phi_n * \psi \in C_c^\infty$ $\|\phi_n * \psi\|_\infty \leq 1$ and for n sufficiently large $\text{supp}(\phi_n * \psi) \subset U$. Moreover,

$$\|\phi_n * \psi - \psi\|_1 \to 0, \quad (n \to \infty)$$

Thus there exists a subsequence $(\phi_{n_k} * \psi)_k$ which converges pointwise to ψ a.e. and has support conained in U. Hence by dominated covergence

$$0 = \int (\phi_{n_k} * \psi) f \to \int \psi f = \int 1_K |f|.$$

Since K is arbitrary this implies that $|f| = 0$ a.e.. □

4.3 The Fourier Transform

Definition 4.3.1. *For $f \in L^1(\mathbb{R}^d)$, define the Fouriertransform*

$$\hat{f}(k) = (\mathcal{F}f)(k) = \frac{1}{(2\pi)^{d/2}} \int f(x) e^{-ikx} dx$$

and the Inverse Fourier transform by

$$\check{f}(k)(\mathcal{F}^{-1}f)(k) = \frac{1}{(2\pi)^{d/2}} \int f(x) e^{ikx} dx$$

Lemma 4.3.1. *Let $f \in L^1$. Then the following holds.*

(a) $\check{f}(k) = \hat{f}(-k)$, $\overline{\hat{f}(k)} = \hat{\overline{f}}(-k)$, $\overline{\check{f}(k)} = \check{\overline{f}}(-k)$.

(b) $\|\hat{f}\|_\infty \leq \frac{1}{(2\pi)^{d/2}} \|f\|_1$ *and* $\|\check{f}\|_\infty \leq \frac{1}{(2\pi)^{d/2}} \|f\|_1$

(c) $\hat{f}, \check{f} \in C_\infty(\mathbb{R}^d)$ *(Riemann Lebesgue), where*

$$C_\infty(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} : f \text{ continuous, } \lim_{R\to\infty} \sup_{|x|>R} |f(x)| = 0\}$$

Proof. (a) and (b) follow directly from the definition.
(c). To show that the (Inverse) Fourier transform is continuous, we use the Theorem about parameter dependent integrals. Next we show that the Fourier transforms vanishes at infinity. For this we use a density argument. We can either use (V1) that characterstic functions of bounded intervals dense in L^1 or (V2) that smooth functions with compact support are dense in L^1.
Let $f \in L^1$. Let $\epsilon > 0$. Then there exists a smooth function with compact support h such that for any $\epsilon > 0$ we have $\|f - h\|_1 < \epsilon$. Hence by (b) $\|\hat{f} - \hat{h}\|_\infty \leq \frac{1}{(2\pi)^{d/2}}\epsilon$. Now

$$\begin{aligned}
||k|^2 \hat{h}(k)| &= \frac{1}{(2\pi)^{d/2}} |\int (-\Delta_x e^{-ikx}) h(x) dx| \\
&= \frac{1}{(2\pi)^{d/2}} |\int (e^{-ikx}(-\Delta_x h)(x) dx| \\
&\leq \frac{1}{(2\pi)^{d/2}} \| - \Delta_x h\|_1.
\end{aligned}$$

Thus

$$|\hat{h}(k)| \leq (1+|k|)^{-1}\frac{1}{(2\pi)^{d/2}}(\|-\Delta_x h\|_1 + \|h\|_1).$$

Hence there exists an $R > 0$ such that $|\hat{h}(k)| \leq \epsilon$ for $|k| \geq R$. Thus for $|k| \geq R$ we have

$$|\hat{f}(k)| \leq |\hat{h}(k)| + \|\hat{h} - \hat{f}\|_\infty \leq \epsilon + \frac{\epsilon}{(2\pi)^{d/2}}.$$

□

Lemma 4.3.2. *(Parseval) For $f, g \in L^1(\mathbb{R}^d)$ we have*

(a) $\int \hat{f}(x)g(x)dx = \int f(x)\hat{g}(x)dx$.

Proof. (a) With Fubini we find

$$\int \left(\int e^{-ikx} f(x)dk\right) g(x)dx = \int \left(\int e^{-ikx} g(x)dx\right) f(k)dk.$$

□

Theorem 4.3.3. *(Convolution and Fourier Transform) Let $f, g \in L^1$. Then*

$$\widehat{f * g} = (2\pi)^{n/2}\hat{f}\hat{g}.$$

Proof. Note that by Young's inquality Theorem 4.2.2 we have $f * g \in L^1$.

$$\begin{aligned}
\widehat{f * g} &= \frac{1}{(2\pi)^{n/2}} \int e^{-ikx} \int f(y)g(x-y)dydx \\
&= \int e^{-iky} \frac{1}{(2\pi)^{n/2}} \int f(y)e^{-ik(x-y)} g(x-y)dxdy \\
&= (2\pi)^{n/2}\hat{f}(k)\hat{g}(k)
\end{aligned}$$

□

Lemma 4.3.4. *(Fourier Transfrom of Gaussians) For $a \in \mathbb{C}$ define $f_a(x) = \exp(-ax^2)$. If $\mathrm{Re} a > 0$, then*

$$\hat{f}_a(k) = (2a)^{-d/2}\exp(-\frac{k^2}{4a}).$$

Proof. First consider the case $a > 0$. Suppose fist $d = 1$. Then

$$\widehat{f_a}'(k) = \frac{1}{i}\widehat{xf_a}(k) = \frac{-1}{2ia}\widehat{\frac{d}{dx}f_a}(k) = -\frac{1}{2a}k\widehat{f_a}(k)$$

Hence

$$\frac{d}{dk}\left[e^{\frac{k^2}{4a}}\hat{f}_a(k)\right] = 0.$$

We find

$$\widehat{f_a}(k) = \widehat{f_a}(0)e^{-\frac{k^2}{4a}}.$$

A well known calculation shows $\widehat{f_a}(0) = (2a)^{-1/2}$. The case $d > 1$ follows by taking the d-fold product of the one dimensional case. Since we have shown that both sides of the final equation hold for $a > 0$ and both sides are analytic in a on $i\mathbb{R} + [0, \infty)$, they must agree for $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. $\square$

4.4 Schwartz Space

We we will use the following notation. For $\alpha \in \mathbb{N}_0^d$ define

$$
\begin{aligned}
|\alpha| &:= \sum_{i=1}^{d} \alpha_i \\
\partial^\alpha &:= \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}} \\
c^\alpha &:= {c_1}^{\alpha_1} \cdots {c_d}^{\alpha_d}, \quad c = (c_1, \dots, c_d) \in \mathbb{C}^d \\
m_\alpha &: \mathbb{R}^d \to \mathbb{R}, \quad x \mapsto m_\alpha(x) = x^\alpha.
\end{aligned}
$$

Definition 4.4.1. *The functions of rapid decrease $\mathcal{S}(\mathbb{R}^d)$ (a.k.a. Schwartz spaces) is the set of infinitely differentiable complex valued functions φ on $\mathbb{R}^n$ for which*

$$
\|\varphi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta \varphi(x) \right| < \infty
$$

for all $\varphi, \beta \in \mathbb{N}_0^d$. We say that a sequence $(f_n)_{n\in\mathbb{N}}$ converges in $\mathcal{S}$ to $f \in \mathcal{S}$, if

$$
\|f - f_n\|_{\alpha,\beta} \to 0, \quad (n \to \infty)
$$

for all $\alpha, \beta \in \mathbb{N}_0^d$.

Lemma 4.4.1. *$C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.*

Proof. Choose a smooth cutoff function χ with compact support and $\chi(0) = 1$, for example

$$
\chi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}+1}, & \text{if } |x| < 1 \\ 0, & \text{otherwise.} \end{cases}
$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then Then $f_n(x) := f(x)\chi(x/n)$ is a sequence in $C_0^\infty(\mathbb{R}^d)$, which converges in all seminorms $\|\cdot\|_{\alpha,\beta}$ to f. □

Lemma 4.4.2. *The maps $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear. They map $\mathcal{S}$ into $\mathcal{S}$. For $\alpha, \beta \in \mathbb{N}_0^d$ we have*

$$
\begin{aligned}
m_\alpha \partial^\beta (\mathcal{F} f) &= (-i)^{|\alpha|+|\beta|} \mathcal{F}(\partial^\alpha m_\beta f) \\
m_\alpha \partial^\beta (\mathcal{F}^{-1} f) &= i^{|\alpha|+|\beta|} \mathcal{F}^{-1}(\partial^\alpha m_\beta f)
\end{aligned}
$$

Proof. The given equations follow from a straight forward calculation. The statement that $\mathcal{F}$ and $\mathcal{F}^{-1}$ map Schwarz functions into Schwarz functions follows from the above two identities. □

Theorem 4.4.3. *$\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is bijective with inverse $\mathcal{F}^{-1}$. Moreover, for all $f, g \in \mathcal{S}$ we have*

$$\begin{aligned} &\|\mathcal{F}f\|_2 = \|f\|_2 \qquad (4.4.1)\\ &\langle \mathcal{F}f, \mathcal{F}g\rangle = \langle f, g\rangle = \langle \mathcal{F}^{-1}f, \mathcal{F}^{-1}g\rangle \\ &\int \hat{f}(x)g(x)dx = \int f(x)\hat{g}(x)dx \\ &\int \check{f}(x)g(x)dx = \int f(x)\check{g}(x)dx \end{aligned}$$

Proof.

$$\begin{aligned} [\mathcal{F}^{-1}(\mathcal{F}f)](x) &= \frac{1}{(2\pi)^{d/2}} \int e^{ikx}(\mathcal{F}f)(k)dk \\ &= \lim_{a\downarrow 0} \frac{1}{(2\pi)^{d/2}} \int e^{ikx} f_a(k)(\mathcal{F}f)(k)dk \\ &= \lim_{a\downarrow 0} \frac{1}{(2\pi)^{d/2}} \int e^{ikx} f_a(k) \int e^{-iky} f(y)dydk \\ &= \lim_{a\downarrow 0} \frac{1}{(2\pi)^{d/2}} \int\int e^{-ik(y-x)} f_a(k)f(y)dydk \\ &= \lim_{a\downarrow 0} \frac{1}{(2\pi)^{d/2}} \int a^{-d/2} e^{-\frac{(y-x)^2}{2a}} f(y)dy \\ &= \lim_{a\downarrow 0} \frac{1}{(2\pi)^{d/2}} \int e^{-\frac{y^2}{2}} f(\sqrt{a}y + x)dy \\ &= f(x), \end{aligned}$$

where we used dominated convergence in the last line (uniform convergence would also work). Similarly, one shows

$$\mathcal{F}\mathcal{F}^{-1}f = f$$

This implies the bijectivity. The isometric properties follow from Parseval's identity and Lemma 4.3.1 (a). □

Definition 4.4.2. Fourier transform on L^2. *Let* $f \in L^2$.

$$\mathcal{F}f := \lim_{n\to\infty} \mathcal{F}f_n \tag{4.4.2}$$

for any sequence $(f_n)_{n\in\mathbb{N}}$ *in* $\mathcal{S}$ *with* $f_n \to f$. *(Such a limiting sequence* (f_n) *always exists since* $\mathcal{S}$ *is dense in* L^2*, moreover the limit* (4.4.2) *exists and does not depend on the sequence because of* (4.4.1).*) Analogously one defines the inverse Fourier transform* $\mathcal{F}^{-1}$ *on* L^2.

Theorem 4.4.4. *If* $f \in L^1 \cap L^2$. *Then the definition of the Fourier transform (and its inverse) on* L^1 *and* L^2 *agree.*

Proof. Let $\hat{f}^{L^1}$ and $\hat{f}^{L^2}$ denote the Fourier transform of f as an L^1 function and an L^2 function, respectively. Let $\varphi \in C_c^\infty$. Then for any sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ which converges f we have

$$\int \varphi(x)\hat{f}^{L^1}(x)dx = \int \hat{\varphi}(x)f(x) = \lim_n \int \hat{\varphi}(x)f_n(x) = \lim_n \int \varphi(x)\hat{f}_n(x) = \int \varphi(x)\hat{f}^{L^2}(x).$$

Hence $\hat{f}^{L^1} = \hat{f}^{L^2}$. □

Henceforth, we shall not distinguish between the two different defintions for functions in $L^1 \cap L^2$.

Remark 4.4.1. For any $f \in L^2$ we have

$$\hat{f} = \text{l.i.m.}_{R\to\infty} \int_{|x|<R} e^{-ikx} f(x)dx,$$

where l.i.m. stands for the limit in L^2. This follows from Plancherel and the fact that $\mathcal{F}$ is continuous (+ dominated convergence).

Theorem 4.4.5. *Let* $f \in L^2$. *Then we have* $\mathcal{F}f, \mathcal{F}^{-1}f \in L^2$,

$$\mathcal{F}^{-1}\mathcal{F}f = f = \mathcal{F}\mathcal{F}^{-1}f.$$

For $f, g \in L^2$ we have

$$\begin{aligned}
&\|\mathcal{F} f\|_2 = \|f\|_2 \text{ (Plancherel)} && (4.4.3)\\
&\langle \mathcal{F} f, \mathcal{F} g\rangle = \langle f, g\rangle = \langle \mathcal{F}^{-1} f, \mathcal{F}^{-1} g\rangle\\
&\int \hat{f}(x) g(x) dx = \int f(x) \hat{g}(x) dx\\
&\int \check{f}(x) g(x) dx = \int f(x) \check{g}(x) dx
\end{aligned}$$

Proof. This follows directly from Theorem 4.4.3, since identities carry over to limits. □

4.5 Dynamics of the free Schrödinger Equation

Theorem 4.5.1. *Let $\varphi \in \mathcal{S}$ or $\varphi \in C_c^\infty$. Define*

$$\psi_t = \mathcal{F}^{-1} e^{-i\frac{1}{2}|\cdot|^2 t} \mathcal{F}\varphi \tag{4.5.1}$$

Then $\|\psi_t\| = \|\varphi\|$ and ψ_t solves the free Schrödinger equation in the following sense. The function $\psi_\cdot : \mathbb{R} \to L^2(\mathbb{R}^d)$, $t \mapsto \psi_t$ is differentiable. Let $\frac{d}{dt}\psi_t$ denote its derivative. Then

$$i\frac{d}{dt}\psi_t = -\frac{1}{2}\Delta\psi_t, \quad \psi_0 = \varphi.$$

The theorem follows for example directly from the spectral theorem. We give a more direct proof however, using only properties of the Fourier transform.

Proof. Define

$$\dot{\psi}_t := \mathcal{F}^{-1}(-i\frac{1}{2}|\cdot|^2) e^{-i\frac{1}{2}|\cdot|^2 t} \mathcal{F}\psi_\circ.$$

Using that the Fourier transform preserves the L^2-norm, we find

$$\left\| h^{-1}\left(\psi_{t+h} - \psi_t\right) - \dot{\psi}_t \right\| = \left\| \left(h^{-1}(e^{-i\frac{1}{2}|\cdot|^2 h} - 1) - (-i\frac{1}{2}|\cdot|^2) \right) e^{-i\frac{1}{2}|\cdot|^2 t}\mathcal{F}\varphi \right\| \to 0, \quad (h \to \infty),$$

where in the limit we used dominated (or uniform convergence would also work). □

We note that (4.5.1) in fact makes sense for a larger class of functions.

Lemma 4.5.2. *Let $\varphi \in \mathcal{S}$. Then for $t \neq 0$*

$$[\mathcal{F}^{-1} e^{-i\frac{1}{2}|\cdot|^2 t} \mathcal{F}\varphi](x) = \frac{1}{(2\pi i t)^{d/2}} \int e^{i\frac{|x-y|^2}{2t}} \varphi(y) dy$$

Proof.

$$\begin{aligned}
[\mathcal{F}^{-1} e^{-i\frac{1}{2}|\cdot|^2 t} \mathcal{F}\varphi](x) &= \lim_{\epsilon \downarrow 0} [\mathcal{F}^{-1} e^{-i\frac{1}{2}|\cdot|^2 (t - i\epsilon)} \mathcal{F}\varphi](x) \\
&= \lim_{\epsilon \downarrow 0} \frac{1}{(2\pi i (t - i\epsilon))^{d/2}} \int e^{i\frac{|x-y|^2}{2(t-i\epsilon)}} \varphi(y) dy \\
&= \frac{1}{(2\pi i t)^{d/2}} \int e^{i\frac{|x-y|^2}{2t}} \varphi(y) dy,
\end{aligned}$$

where in the second equation we used Theorem 4.3.3 and the expression for the Fourier transform of a Gaussian. □

Corollary 4.5.3. *We have for $\psi_\circ \in \mathcal{S}$ that for all $x \in \mathbb{R}^d$ and $t \neq 0$*

$$|\psi(x,t)| \leq \frac{C}{|t|^{d/2}} \|\psi_\circ\|_1 \to 0, \quad (t \to \infty).$$

(decay of wave packet.)

A refined estimate yields.

Corollary 4.5.4. *Let $\varphi \in \mathcal{S}$. Then*

$$\psi_t(x) = \frac{1}{(it)^{d/2}} e^{i\frac{1}{2}\left(\frac{x}{t}\right)^2 t} \hat{\varphi}(x/t) + r(t,x)$$

for some function $r(t,x)$ satisfying $\lim_{t\to\infty} \|r(t,\cdot)\|_2 = 0$.

Proof.

$$\begin{aligned}
\psi(t,x) &= \frac{1}{(2\pi i t)^{d/2}} e^{i\frac{x^2}{2t}} \int e^{i\frac{y^2}{2t}} e^{-i\frac{xy}{t}} \varphi(y) dy \\
&= \frac{1}{(2\pi i t)^{d/2}} e^{i\frac{1}{2}\left(\frac{x}{t}\right)^2 t} \int e^{-i\frac{x}{t}y} e^{i\frac{y^2}{2t}} \varphi(y) dy \\
&= \frac{1}{(2\pi i t)^{d/2}} e^{i\frac{1}{2}\left(\frac{x}{t}\right)^2 t} \int e^{-i\frac{x}{t}y} \left(\varphi(y) - \varphi(y) + e^{i\frac{y^2}{2t}} \varphi(y) \right) dy \\
&= \frac{1}{(it)^{d/2}} e^{i\frac{1}{2}\left(\frac{x}{t}\right)^2 t} \hat{\varphi}(x/t) + r(t,x),
\end{aligned}$$

where

$$r(t,x) := \frac{1}{(it)^{d/2}} e^{i\frac{1}{2}\left(\frac{x}{t}\right)^2 t} \hat{h}_t(x/t), \quad h_t(y) = e^{i\frac{y^2}{2t}} \varphi(y) - \varphi(y).$$

Since

$$\|r(t,\cdot)\|_2^2 = \|\hat{h}_t\|_2^2 = \|h_t\|_2^2 \to 0, \quad (t \to \infty),$$

by dominated covergence. □

Theorem 4.5.5. *Let the situation be as above. Let $\Omega \subset \mathbb{R}^d$ be measurable. Then*

$$\lim_{t\to\infty} \int_{t\Lambda} |\psi_t(x)|^2 dx = \int_\Lambda |\hat{\varphi}(p)|^2 dp$$

Proof. Let $m_t := \psi_t - r_t$.

$$\begin{aligned}\int_{t\Lambda} |\psi_t(x)|^2 dx &= \|1_{t\Lambda}\psi_t\|_2^2 \\ &= \|1_{t\Lambda} m_t\|_2^2 + R_t.\end{aligned}$$

where $R_t = 2\mathrm{Re}\langle 1_{t\Lambda} m_t, 1_{t\Lambda} r_2\rangle + \|1_{t\Lambda} r_t\|_2^2$

$$|R_t| \leq \|1_{t\Lambda} m_t\|_2 \|r_t\|_2 + \|r_t\|_2^2$$

This shows the claim. $\square$

4.6 Weak Derivatives

In the following U shall always denote an open subset of $\mathbb{R}^d$. We start by substantially weakening the notion of partial derivatives.

Remark 4.6.1. (Notation.) Let $C_c^\infty(U)$ denote the space of infinitely differentiable-functions $\phi : U \to \mathbb{R}$, with compact support in U. We will sometimes calla function ϕ belonging to $C_c^\infty(U)$ a test function.

Remark 4.6.2. (Motivation for the definition of weak derivative.) Assume we are given a function $u \in C^1(U)$. Then if $\phi \in C_c^\infty(U)$, we see from integration by parts that

$$\int_U u\phi_{x_i} dx = -\int_U u_{x_i}\phi dx \quad (i = 1, ..., n).$$

There are no boundary terms, since ϕ has compact support in U and thus vanishes near ∂U. More generaly now, if k is a positive integer, $u \in C^k(U)$, and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$, then

$$\int_U u\partial^\alpha \phi dx = (-1)^{|\alpha|} \int_U (\partial^\alpha u)\phi dx.$$

This equality holds since we can apply the previous one $|\alpha|$ times.

Definition 4.6.1. *Let $f, g \in L^1_{\text{loc}}(U)$ and $\alpha \in \mathbb{N}_0^d$. Then g is called the α^{th}-weak partial derivative of f, written*

$$\partial^\alpha f = g$$

provided

$$\int f(x)(\partial^\alpha \varphi)(x)dx = (-1)^{|\alpha|} \int g(x)\varphi(x)dx$$

for all functions $\varphi \in C_c^\infty(U)$.

Remark 4.6.3. We note that $L^p_{\text{loc}}(U) \subset L^1_{\text{loc}}(U)$. (Hölder: Let $f \in L^p_{\text{loc}}(U)$. Then $f \in L^p(K)$ for some compact subset $K \subset U$. So $\|1_K f\|_1 \leq \|1_K f\|_p \|1_K\|_q$ for $p^{-1} + q^{-1} = 1$.) So we can apply the above definition of course also to funtions in $L^p_{\text{loc}}(U)$.

Lemma 4.6.1. *(Uniqueness of weak derivatives). A weak α^{th}-partial derivativeof u if it exists is uniquely definedup to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{\text{loc}}(U)$ satisfy

$$\int_U u\partial^\alpha \phi dx = (-1)^{|\alpha|} \int_U v\phi dx = (-1)^{|\alpha|} \int \tilde{v}\phi dx$$

for all $\phi \in C_c^\infty(U)$; whence $v - \tilde{v} = 0$ a.e. by Theorem 4.2.7. $\square$

Example 19. Let $d = 1$, $U = (0, 2)$ and $u(x) = x$ if $0 < x \leq 1$ and $u(x)1$ if $1 < x < 2$, and $v(x) = 1$ if $0 < x \leq 1$ and $v(x) = 0$ if $1 < x < 2$. Then $u' = v$ in the weak sense.

4.7 Sobolev Spaces

Let U be an open subset of $\mathbb{R}^d$. Fix $1 \leq p \leq \infty$ and let k be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

Definition 4.7.1. *The* **Sobolev space**

$$W^{k,p}(U)$$

consists of all locally integrable functions $f : U \to \mathbb{C}$ such that for all multi-indices α with $|\alpha| \leq k$ we have $D^\alpha u \in L^p(U)$. We write

$$H^k(U) = W^{k,2}(U)$$

We define the **Sobolev norm** *for $u \in W^{k,p}(U)$ by*

$$\|u\|_{W^{k,p}} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p dx\right)^{1/p} & , \quad 1 \leq p < \infty \\ \sum_{|\alpha|\leq k} \text{ess sup}_{x\in U} |D^\alpha u(x)| & , \quad p = \infty \end{cases}$$

We define

$$W_0^{k,p}(U)$$

as the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. We set

$$H_0^k(U) = W_0^{k,2}(U).$$

Theorem 4.7.1. *The space $W^{k,p}$ with the norm $\|\cdot\|_{W^{k,p}}$ is a Banach space. $W_0^{k,p}$ is a a Banach space.*

Proof. Suppose that (f_n) is a Cauchy sequence in $W^{k,p}(U)$. Then $\partial^\alpha f_n$ is a Caucy sequence in $L^p(U)$ for all $|\alpha| \leq k$. By the completness of $L^p(U)$, there exists $g_\alpha \in L^p(U)$ such that $g_\alpha = L^p - \lim_{n\to\infty} \partial^\alpha f_n$ for all $|\alpha| \leq k$. Therefore, for all $\phi \in C_c^\infty(U)$ we have

$$\langle f, \partial^\alpha \phi\rangle = \lim_{n\to\infty} \langle f_n, \partial^\alpha \phi\rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle \partial^\alpha f_n, \phi\rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle g_\alpha, \phi\rangle.$$

This shows $\partial^\alpha f$ exists weakly and $g_\alpha = \partial^\alpha f$ a.e. $\square$

Lemma 4.7.2. *We have for* $n \in \mathbb{N}$

$$H^n(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) : k \mapsto (1+|k|)^n \hat{f}(k) \text{ is in } L^2(\mathbb{R}^d)\}. \tag{4.7.1}$$

Moreover, for $f \in H^n$ *and* $|\alpha| \le n$ *we have*

$$\widehat{\partial^\alpha f}(k) = (ik)^\alpha \hat{f}(k) \qquad a.e.k.$$

Proof. Let $f \in L^2(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then by Theorem 4.4.5 and Lemma 4.4.2

$$\int (\partial^\alpha \varphi)(x) f(x) dx = \int (-ik)^\alpha \check{\varphi}(k) \hat{f}(k) dk = \int \check{\varphi}(k)(-ik)^\alpha \hat{f}(k) dk. \tag{4.7.2}$$

$\supset$: If $f \in$ r.h.s. of (4.7.1), then we obtain from (4.7.2)

$$\int (\partial^\alpha \varphi)(x) f(x) dx = \int \varphi(x)(-i)^{|\alpha|} [M_{\mathrm{id}_{\mathbb{R}}^\alpha} \hat{f}]\check{}(x) dx,$$

and hence $f \in H^n(\mathbb{R}^d)$.
$\subset$: Let $f \in H^n(\mathbb{R}^d)$, $|\alpha| \le n$, and $g = \partial^\alpha f$. then for all $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\int (\partial^\alpha \varphi)(x) f(x) dx = (-1)^{|\alpha|} \int \varphi(x) g(x) dx.$$

Using a limiting argument (as in the proof of Lemma 4.4.1), one can show that the above equation holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Hence by (4.7.2) and again Theorem 4.4.5 we find

$$\int \check{\varphi}(k)(-1)^{|\alpha|}(-ik)^\alpha \hat{f}(k) dk = \int \varphi(x) g(x) dx = \int \check{\varphi}(k) \hat{g}(k) dk$$

This implies $(-1)^{|\alpha|}(-ik)^\alpha \hat{f}(k) = \hat{g}(k)$ a.e. by Theorem 4.2.7. Thus we conclude that $f \in$ r.h.s. of (4.7.1). $\square$