1 Preliminaries

1.1 Metric Spaces

- Definition: Metric and Metric Space
- Definition: Induced Metric
- Examples: d_1, d_∞, d_2 for $\mathbb{R}^n$ and $C(I)$
- Definition: Convergent Sequences and Limits
- Lemma: Limit is Unique
- Definition: Cauchy Sequence
- Proposition: Convergent Sequence is Cauchy Sequence
- Remark: d_1 restricted to $\mathbb{Q}$ is not dense.
- Definition: Complete Metric Spaces
- Definition: Dense Subsets
- Definition: Continuity of Functions
- Proposition: Characterization of Continuity
- Definition: Isometry and Isometric Metric Spaces
- Theorem: Unique Completion of Isometric Dense Metric Subspaces
- Definition: Open and Closed Ball, Open Sets, Neighborhood and Interior Points
- Definition: Set of Limit Points, Closed Sets and Closures
- Lemma: Characterization of Limit Points and Closed Sets
- Lemma: Closed Subspaces of Complete Spaces are Complete
- Remark: Characterization of Density of Subspace
- Theorem: Rules for Open and Closed Sets
- Theorem: Characterization of Continuous Functions
- Theorem: Product Metric

1.2 Normed Vector Spaces

- Definition: Linear Operator
- Definition: Norm, Seminorm and Normed Spaces
- Remark: Normed Spaces Induce Metric Spaces
- Example: Minkowski-Norm for $\mathbb{R}^n$ and $C(I)$
- Lemma: Young's Inequality
- Theorem: Hölder's Inequalities
- Theorem: Minkowski's Inequalities
- Theorem: Second Version of Triangle Inequality
- Definition: Equivalent Norms
- Remark: Finite Dimensional Norms are Equivalent, Indistinguishability
- Remark: Product Metric
- Proposition: $+$ and $\cdot$ of Normed Spaces are Continuous
- Definition: Bound linear Transformations
- Lemma: Normed Space of Bounded Linear Operators and Norm Characterizations
- Theorem: Characterization of Bounded Linear Operators
- Definition: Banach Space
- Theorem: Completion of Normed Vector Spaces
- Remark: $L^p(I)$ Spaces
- Theorem: BLT-Theorem
- Remark: Lebesgue-Integral as Extension of $C_c(\mathbb{R}^n)$
- Theorem: Characterization of Completeness

2 Measure Theory

2.1 Measure Theory

- Measurable Spaces and Measure Spaces
  - Definition: Algebra, σ-Algebra, Measurable Space, Measurable Sets
  - Example: Trivial σ-Algebras
  - Lemma: Properties of Algebras and σ-Algebras
  - Definition: Generator of σ-Algebra
  - Remark: Existence and Uniqueness of Smallest σ-Algebra
  - Definition: Borel-σ-Algebra
  - Lemma: Monotonicity of Generator
  - Example: Generators of Borel-σ-Algebra
  - Example, Proposition, Lemma, Corollary: Generator of Product Borel-σ-Algebra
  - Definition: Product σ-Algebra
  - Proposition, Example: Elementary Family Induces Algebra
  - Definition: Measure, Measure Spaces, finite and σ-finite
  - Example: Dirac Point Measure
  - Theorem: Monotonicity, Subadditivity, Continuity from Below and Above
  - Definition: Null Set, Almost Everywhere Holding Statements
  - Definition: Complete Measures
  - Theorem, Definition: Completion of Measure Space
- Lebesgue-Measure
  - Definition: Lebesgue Outer Measure
  - Remark: Lebesgue Outer Measure is well defined
  - Theorem: Restriction of Lebesgue Outer Measure to Borel-σ-Algebra is a Measure
  - Definition: Lebesgue Measure and Lebesgue-measurable Sets
  - Theorem: Measurable Sets
  - Theorem: Characterization Lebesgue-Measurability
- Construction of Measures from Outer Measures
  - Definition: Outer Measure
  - Lemma: Construction of Outer Measure
  - Example: Lebesgue Outer Measure is Outer Measure
  - Definition, Remark: Measurable Set with Respect to Outer Measure
  - Theorem: Caratheodory, Existence of Measures
  - Lemma: Open Intervals are outer-measurable
  - Theorem: Characterization of Lebesgue-Measurability
- Uniqueness of Measures and Monotone Classes
  - Definition: Monotone Class and Generator
  - Theorem: Monotone Class Theorem
  - Theorem: Uniqueness of Measures

2.2 Integration Theory

- Integration
  - Measurable Functions
  - Lemma: Characterization, Composition, Continuity
  - Proposition: Measurable Functions on Product Spaces
  - Corollary: Coordinates, Sum and Product are Measurable
  - Theorem: sup, inf, limsup, liminf, lim are Measurable
  - Definition: Simple Functions
  - Lemma: Simple Function Formulation with Linear Combination, Characterization Measurability
  - Theorem: Monotone Convergence of Simple Functions to Measurable Functions
  - Definition: Integral of Simple Functions
  - Proposition: Measure, Linearity and Monotonicity of Positive Simple Function Integrals
  - Definition: Lebesgue Integral for Non-negative Functions
  - Remark: Properties of Lebesgue Integral
  - Theorem: Monotone Convergence Theorem by Lebesgue
  - Corollary: Linearity, Interchange of Sum and Integral
  - Theorem: Lemma by Fatou
  - Definition: Lebesgue Integral
  - Proposition, Definition: L^1 Space
  - Theorem: Dominated Convergence
  - Theorem: Parameter dependent Integrals
- Product Measures
  - Proposition: Section Property
  - Theorem: Product Measure
  - Theorem: Fubini-Tonelli
- L^p Spaces
  - Definition: L^p Spaces
  - Lemma: L^p is a Vector Space
  - Theorem: Hölder's Inequality
  - Theorem: Minkowski's Inequality
  - Theorem: L^p-Norm Implies almost everywhere Pointwise Convergence of Subsequence
  - Corollary: Cauchy Sequence
  - Theorem: Set of Simple Functions is Dense in L^p
  - Corollary: L^p has countable dense subset, $C_c(\mathbb{R}^n)$ is Dense in $L^p(\mathbb{R}^n)$
  - Definition: Essential Supremum and L^∞
  - Theorem: Properties of L^∞

3 Hilbert Spaces

3.1 Sesquilinearforms

- Definition: Sesquilinearform, Symmetry, Non-negativity, Positive Definite, Bilinear Form
- Lemma: $s(0, x) = s(x, 0) = 0$
- Proposition: Parallelogram Identity
- Proposition: Polarization
- Corollary: Positive Sesquilinearforms are symmetric

3.2 Vector Spaces with Semi-Inner Products

- Definition: Semi-Inner Product and Inner Product
- Lemma: Antilinearity in First Argument
- Remark: Characterization Semi-Inner Product and Inner Product
- Examples
- Proposition: Cauchy-Schwarz-Bunyakowski
- Definition: Semi-Norm
- Lemma: Semi-Norm Properties
- Proposition: Induced Semi-Norm and Norm
- Definition: Orthogonal and Orthogonal Complement
- Definition: Linear Span
- Lemma: Algebraic Properties of Orthogonal Complement
- Definition: Orthonormal System
- Proposition: Pythagoras
- Definition: General Sum
- Lemma: Sum over uncountable elements
- Theorem: Bessel
- Proposition: Gram-Schmidt Algorithm

3.3 Hilbert Spaces

- Lemma: Induced Metric
- Proposition: Norm and Inner Product are Continuous
- Definition: Hilbert Space
- Lemma: Continuity Properties
- Lemma: Characterization of Norm and Operator Norm through Inner Product
- Proposition: $l^2(\mathbb{N})$
- Theorem: Approximation Theorem
- Theorem: Projection Theorem
- Remark: Direct Sum
- Lemma: Completeness Properties
- Definition: Projection
- Lemma: Projection Properties
- Definition: Orthogonal Projection
- Lemma: Orthogonal Projections are Bounded
- Proposition: Orthogonal Projection for Closed Subspace
- Definition: Orthonormal Basis
- Proposition: Orthonormal Basis in $l^2(\mathbb{N})$
- Definition: General Sum Convergence in Normed Space
- Theorem: Coefficient Representation
- Lemma: Characterization Orthonormal Basis
- Corollary: Physics Identity
- Theorem: Every Hilbert Space has an Orthonormal Basis
- Proposition: Orthogonal Projection Coefficients for Closed Subspace

3.4 Separable Hilber Spaces

- Definition: Dense, Separable
- Definition: Unitary Mapping, Unitarily Equivalence
- Lemma: Characterization Unitarity
- Proposition: Characterization Separability
- Definition: Completion
- Theorem: Existence and Uniqueness of Completion

3.5 Riesz's Theorem

- Lemma: Continuous Linear Mappings l_v
- Theorem: Riesz's Theorem
- Corollary: Unique Bounded Linear Transformation for Sesquilinearforms

3.6 Hilbert Adjoint for Bounded Operators

- Lemma: Inner Product Identity
- Lemma: Existence of unique Hilbert Adjoint
- Lemma: Properties of Adjoint
- Definition: Selfadjointness and Normality
- Lemma: Selfadjoint Operators are normal
- Lemma: Characterization of Normal Operators
- Lemma: Characterization of Selfadjointness
- Corollary: Zero Identity for Selfadjoint Operators
- Definition: Unitary Operator
- Lemma: Characterization Unitary Operator
- Lemma: Characterization Orthogonal Projection

3.7 Construction of Hilbert Spaces

- Proposition: Direct Sums of Hilbert Spaces
- Proposition: Hilbert Space of Vector Valued Functions
- Definition, Lemma: Tensor Product of Hilbert Spaces
- Proposition: Orthonormal Basis of Tensor Product
- Example: General Fock Space, Symmetric Fock Space, Antisymmetric Fock Space
- Theorem: Tensor Product of L^2 Spaces
- Examples

4 Fourier Analysis

4.1 Existence of Test Functions

- Definition: Support
- Lemma: Existence of $C_C^\infty(\mathbb{R}^n)$ Function

4.2 L^p Spaces in $\mathbb{R}^n$

- Definition: Convolution
- Definition: $L^p_{\text{loc}}(U)$
- Lemma: Convolution is continuous for C_c and L_{loc}
- Theorem: Young Inequality
- Proposition: Convolution Properties
- Proposition: Convolution Preserves Differentiability
- Theorem: ϕ_t Test Functions
- Corollary: C_c^∞ is dense in L^p and C_0
- Theorem: Zero Identity for C_c^∞ Test Functions

4.3 The Fourier Transform

- Definition: Fourier Transformation and Inverse
- Lemma: Fourier Transform Properties
- Lemma: Parseval (Interchange of Convolution)
- Theorem: Convolution and Fourier Transform
- Lemma: Fourier Transform of Gaussians

4.4 Schwartz Space

- Definition: Schwartz Space $\mathcal{S}$
- Lemma: C_c^∞ is Dense in $\mathcal{S}$
- Lemma: Fourier Transform and Inverse are Linear and Interchange Multiplication with Differentiation
- Theorem: Fourier Transform is Bijection on $\mathcal{S}$
- Definition: Fourier Transform on L^2
- Theorem: Definitions of Fourier Transform agree on L^1 and L^2
- Theorem: Fourier Transform is Bijection on L^2

4.5 Dynamics of Free Schrödinger Equation

- Theorem: Solutions ψ_t of Schrödinger Equation
- Lemma: Reformulation of Solution
- Corollary: Decay of Wave Packet
- Corollary: Solution can be Splitted into Sum
- Theorem: Norm Equality

4.6 Weak Derivatives

- Remark: Motivation
- Definition: Weak Partial Derivative
- Remark: $L^p_{\text{loc}}(U) \subset L^1_{\text{loc}}(U)$
- Lemma: Uniqueness of Weak Derivatives

4.7 Sobolev Spaces

- Definition: Sobolev Space and Sobolev Norm
- Theorem: Sobolev Spaces are Banach Spaces
- Lemma: Characterization H^n and Application of Fourier Transform

5 Unbounded Operators

- Theorem: Closed Graph Theorem

5.1 Definitions and Basic Properties

- Definition: General Linear Operator
- Definition: Graph of Linear Operator
- Lemma: Equivalences for Graph
- Definition: Extension of a Linear Operator
- Theorem: Characterization of Extension
- Definition: Addition and Composition

5.2 Closed Operators

- Definition: Closed Linear Operator
- Theorem: Characterization of Closed Operator
- Corollary: Bounded Operators are Closed Operators
- Definition: Closable Operators
- Theorem: Existence of Unique Smallest Closed Extension
- Theorem: $\overline{\Gamma(T)} = \Gamma(\overline{T})$
- Theorem: Sequence Criterion
- Theorem: BLT Theorem
- Lemma: Inverse is closed iff Operator is closed
- Lemma: Addition and Right Composition Preserve Closedness
- Definition: Graph Norm
- Theorem: Characterization Closedness by Completeness

5.3 Spectral Theory

- Definition: Resolvent Set, Resolvent and Spectrum
- Definition: Eigenvalue, Eigenvector and Point Spectrum
- Lemma: Point Spectrum is Subset of Spectrum
- Lemma: Properties of Resolvent
- Remark: Closedness implies Boundness of Inverse
- Remark: Operator with Nonempty Spectrum must be Closed
- Lemma: Resolvent Identities
- Theorem: Neumann Series
- Lemma: Disk Contained in Resolvent
- Theorem: Resolvent Set is Open and Spectrum is Closed, Resolvent-valued function is Analytic
- Theorem: Spectrum for Closed Linear Operators

5.4 Operators in Hilbert Spaces

- Definition: Multiplication Operator
- Lemma: $M_f M_g \subset M_{fg}$
- Definition: Measure Support and Essential Supremum
- Proposition: Properties of Multiplication Operator
- Example: Laplacian

5.5 Adjoint Operators

- Definition: Adjoint Operator
- Remark: Characterizations for Domain of Definition
- Remark: Inner Product Identity and V function
- Lemma: Graph of Dense Operator
- Lemma: Closability Properties of Dense Operator
- Proposition: Kernel of Adjoint is equal to Orthogonal Complement of Image of Dense Operator
- Lemma: Adjoint of Inverse of Resolvent
- Lemma: Addition and Composition Properties
- Lemma: Adjoint of Multiplication Operator

5.6 Selfadjoint and Symmetric Operators

- Definition: Selfadjoint Operator
- Remark: Selfadjointness implies Closedness
- Example: Laplacian on H^2 is selfadjoint
- Theorem: Properties of Selfadjoint Operators
- Definition: Symmetric Operator
- Remark: Selfadjoint Operators are Symmetric
- Lemma: Properties of Symmetric Operators
- Theorem: Basic Criterion for Selfadjointness
- Definition: Essentially Selfadjoint Operators
- Example: Laplacian restricted to Schwartz Space is essentially selfadjoint

5.7 Kato Rellich Theorem

- Definition: T-Boundness and Relative Bound
- Theorem: Kato-Rellich
- Lemma: Boundness by Laplacian
- Corollary: Laplacian summed with sum over Functions is selfadjoint
- Example

6 Spectral Theorem

- Theorem: Spectral Theorem - Multiplication Operator Form
- Theorem: Spectral Theorem - Multiplication Operator Form - Separable Case
- Remark: Physics Notation