Chapter 5

Unbounded Operators

We shall use the following theorem about

Theorem 5.0.1. *(Closed graph theorem) Let X and Y be Banach spaces and T a linear map of X into Y. Then T is bounded if and only if $\Gamma(T) = \{(x, Tx) : x \in X\}$, the graph of T, is a closed subset pf $X \times Y$.*

5.1 Definitions and Basic Properties

Let F stand for $\mathbb{C}$ or $\mathbb{R}$.

Definition 5.1.1. *Let X and Y be normed spaces. We say that T is a* **linear operator** **<u>in</u>** *X (with values in Y), if T is a linear mapping from a linear subspace $D(T) \subset X$, called the domain of T, to Y. We write*

$$T : D(T) \subset X \to Y.$$

We say that T is densely defined in X if $D(T)$ is dense in X. We say that T is bounded if the map $T : D(T) \to Y$ is bounded, otherwise T is called unbounded.

Definition 5.1.2. *Let X und Y be vectorspaces and $T : D(T) \subset X \to Y$ a linear operator. The the graph* **of** *T is the subset of $X \times Y$ which is defined by*

$$\Gamma(T) := \{(x, Tx) : x \in D(T)\} \tag{5.1.1}$$

$$:= \{(x, y) \in X \times Y : x \in D(T), y = Tx\}. \tag{5.1.2}$$

Lemma 5.1.1. *Let X and Y be vectorspaces and L a linear subspace of $X \times Y$. Then the followig statements are equivalent.*

(a) If $(0, y) \in L$, then $y = 0$.

(b) If $(x, y_1) \in L$ and $(x, y_2) \in L$, then $y_1 = y_2$.

(c) There exists a linear operator T from X to Y, such that $\Gamma(T) = L$.

Proof. (a)$\Rightarrow$ (b). If $(x, y_j) \in \Gamma(T)$ for $j = 1, 2$, then by linearity $(0, y_1 - y_2) \in L$. So by (a) $y_1 - y_2 = 0$.
(b) $\Rightarrow$ (c) . Define $D(T) = \{x \in X : \exists y \in Y, (x, y) \in L\}$. If $x \in D(T)$ then there exists by definition of $D(T)$ a $y \in Y$, such that $(x, y) \in L$. This y is unique by (b) and we deinfe $Tx = y$. It is straight forward to verify that $\Gamma(T) = L$.
(c) $\Rightarrow$ (a) Let $(0, y) \in \Gamma(T)$. Then $y = T0 = 0$. $\square$

Definition 5.1.3. *Let T be linear operator. A linear operator T' is called an extension of T, written $T \subset T'$, iff*

$$D(T) \subset D(T') \quad \text{and} \quad Tx = T'x \ , \ \forall x \in D(T) \ .$$

Theorem 5.1.2. *$T \subset T'$ iff $\Gamma(T) \subset \Gamma(T')$.*

Definition 5.1.4. *We define the* **addition** *and* **composition** *of unbounded operators as follows.*

- *Let $S : D(S) \subset X \to Y$ and $T : D(T) \subset X \to Y$. Define $D(S+T) := D(S) \cap D(T)$ and*

$$S + T : D(S+T) \to Y, \quad x \mapsto (Tx + Sx).$$

- *Let $S : D(S) \subset X \to Y$ and $T : D(T) \subset Y \to Z$. Define $D(ST) := \{x \in D(S) : Sx \in D(T)\}$ and*

$$TS : D(ST) \to Z, \quad x \mapsto T(Sx).$$

Remark 5.1.1. Note that addition of operators is associative and commutative. And composition of operators is associative. However, the distributive law does not necessarily need to be satisfied for the corresponding domains.

Closed Operators

Definition 5.1.5. *An operator $T : D(T) \subset X \to Y$ is called* **closed** *if $\Gamma(T)$ is a closed subset of $X \times Y$.*

Theorem 5.1.3. *Let a linear operator $T : D(T) \subset X \to Y$ be given. Then the following statements are equivalent.*

(a) T is closed.

(b) *If* $(x_n)_{n=0}^{\infty}$ *is a sequence in* $D(T)$, $x_n \to x$, *and* $Tx_n \to v$ *for some* $v \in Y$, *then* $x \in D(T)$ *and* $Tx = v$.

Proof. By properties of closed sets in metric spaces, see (**??**), we have

$$(x,v) \in \overline{\Gamma(T)} \quad \Leftrightarrow \quad \exists (x_n)_{n=0}^{\infty} \in D(T), s.t. (x_n, Tx_n) \to (x,v).$$

(a) $\Rightarrow$ (b). Thus we assume T is closed. Then

$$\begin{aligned} &(x_n)_{n=0}^{\infty} \text{ in } D(T), v \in Y, \text{ with } (x_n, Tx_n) \to (x,v) \\ &\Rightarrow (x,v) \in \overline{\Gamma(T)} = \Gamma(T) \\ &\Rightarrow x \in D(T), v = Tx\ . \end{aligned}$$

(b) $\Rightarrow$ (a). Thus assume the following statement is true.

$$(x_n)_{n=0}^{\infty} \text{ in } D(T), \text{ with } (x_n, Tx_n) \to (x,v) \Rightarrow x \in D(T), v = Tx\ .$$

Then

$$\begin{aligned} &(x,v) \in \overline{\Gamma(T)} \\ &\Rightarrow \exists (x_n)_{n=0}^{\infty} \text{ in } D(T), s.t. (x_n, Tx_n) \to (x,v) \\ &\Rightarrow x \in D(T), v = Tx \\ &\Rightarrow (x,v) \in \Gamma(T)\ . \end{aligned}$$

Thus we have shown $\overline{\Gamma(T)} \subset \Gamma(T)$. Since the opposite inclusion is trivial, we find $\overline{\Gamma(T)} = \Gamma(T)$, and hence T is closed. $\square$

Corollary 5.1.4. *If* $T : X \to Y$ *is a bounded operator, then it is closed.*

Definition 5.1.6. *A linear operator* T *is called* **closable**, *if it has a closed extension.*

Theorem 5.1.5. *If* T *is closable, then the following holds.*

(a) There exists a unique linear operator in X, which we denote by $\overline{T}$, such that

$$\Gamma(\overline{T}) = \bigcap_{T' :\ T' \text{ is a closed extension of } T} \Gamma(T'). \tag{5.1.3}$$

(b) The operator $\overline{T}$, defined in (a), is the smallest closed extension of T.

We call $\overline{T}$ the **closure** *of T.*

Proof. (a): Since T is closable, it has at least one closed extension. It follows that the r.h.s. of (5.1.3) contains $\Gamma(T)$. Since intersections of closed sets are closed, it follows that the r.h.s. of (5.1.3) is closed. Since each $\Gamma(T')$ in the r.h.s. of (5.1.3) is a linear graph, and intersections of linear graphs are linear graphs, it follows that the r.h.s. of (5.1.3) is a linear graph as well. Thus (a) now follows from Lemma 5.1.1.
(b) Follows as a direct consequence of (a). □

Theorem 5.1.6. *If T is closable, then $\overline{\Gamma(T)} = \Gamma(\overline{T})$.*

Proof. Clearly $\Gamma(T) \subset \Gamma(\overline{T})$, thus taking the closure and using that $\overline{T}$ has a closed graph we find

$$\overline{\Gamma(T)} \subset \overline{\Gamma(\overline{T})} = \Gamma(\overline{T}) . \tag{5.1.4}$$

It remains to show the opposite inclusion. By (5.1.4), we see that $\overline{\Gamma(T)}$ is a linear graph. Thus there exists a linear operator R such that $\Gamma(R) = \overline{\Gamma(T)}$. Now R is closed since its graph is closed and moreover it is an extension of T. Since $\overline{T}$ is the smallest closed extension of T, it follows that $\Gamma(\overline{T}) \subset \Gamma(R) = \overline{\Gamma(T)}$. □

Theorem 5.1.7. *(Sequence criterion) Let a linear operator $T : D(T) \subset X \to Y$ be given. The following statements are equivalent.*

(i) T is closable.

(ii) If $(x_n)_{n=1}^{\infty}$ is a sequence in $D(T)$, $x_n \to 0$, and $Tx_n \to v$ for some $v \in Y$, then $v = 0$.

Proof. (i) $\Rightarrow$ (ii): Let $\overline{T}$ be the closure of T. Let $(x_n)_{n=0}^{\infty}$ be sequence a in $D(T)$ which converges to zero such that Tx_n converges to some vector v. Then of course $x_n \in D(\overline{T})$, and $\overline{T}x_n = Tx_n \to v$. Since $\overline{T}$ is closed $v = T0 = 0$.

(ii) $\Rightarrow$ (i): By the definition of the closure of a set

$$\begin{aligned} \overline{\Gamma(T)} &= \{(x,v) | \exists ((x_n, v_n))_{n=0}^{\infty} \in \Gamma(T), s.t. (x_n, v_n) \to (x,v)\} \\ &= \{(x,v) | \exists (x_n)_{n=0}^{\infty} \in D(T), s.t. (x_n, Tx_n) \to (x,v)\} \,. \end{aligned}$$

By assumption this is a linear graph. Thus there is an R such that $\Gamma(R) = \overline{\Gamma(T)}$. This R is a closed extension of T. □

Remark 5.1.2. If $T : D(T) \subset X \to Y$ is a bounded operator, then it is closable.

Theorem 5.1.8. *(B.L.T.-theorem) Let $T : D(T) \subset X \to Y$ be a bounded linear operator. Then T is closable, its closure $\overline{T}$ is bounded with bound $\|\overline{T}\| = \|T\|$ and furthermore, $D(\overline{T}) = \overline{D(T)}$. In particular, if $D(T)$ is dense in X, then $D(\overline{T}) = X$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ in $D(T)$ and $x_n \to 0$ and $Tx_n \to y$. Then by continuity $Tx_n \to 0 = y$. Thus T is closable. Moreover,

$$D(\overline{T}) = \{\ x \mid \exists (x_n)_{n=1}^{\infty} \text{ in } D(T),\ \text{s.t. } x_n \to x,\ Tx_n \text{ converges}\} = \overline{D(T)}$$

To show the norm bound note that for any sequence $(x_n)_{n\in\mathbb{N}}$ in $D(T)$ which converges to $x \in X$ one has $\|\overline{T}x\| = \lim_{n\to\infty} \|Tx_n\| \leq \|T\| \lim_{n\to\infty} \|x_n\| \leq \|T\|\|x\|$. Thus $\|\overline{T}\| \leq \|T\|$. Since the operator norm can only increase for an extension, the opposite inequality also holds. □

Remark 5.1.3. The B.L.T.-theorem allows to extend a densely defined bounded operator to the whole space. In particular if an operator is defined on a space D which is not complete one can embed D in its completion.

Lemma 5.1.9. *Let $T : D(T) \subset X \to Y$ be an injective linear operator and let T^{-1} : $\mathrm{Ran}(T) \subset Y \to X$ be its inverse. Then, T is closed if and only if T^{-1} is closed.*

Proof. Note that $\Gamma(T^{-1}) = \{(y, x) \in Y \times X : (x, y) \in \Gamma(T)\}$. Thus $\Gamma(T^{-1})$ is closed if and only if $\Gamma(T)$ is closed. $\square$

Lemma 5.1.10. *Let $T : D(T) \subset X \to Y$ be a closed linear operator. Then*

(a) If $A \in \mathcal{L}(X, Y)$, then the map

$$T + A : D(T) \to Y \quad , \quad x \mapsto Tx + Ax$$

is closed.

(b) If $B \in \mathcal{L}(W, X)$, then $D(TB) = \{x \in W | Bx \in D(T)\}$ and the map

$$TB : D(TB) \to Y \quad , \quad x \mapsto T(Bx)$$

is closed.

Proof. (a) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $D(T)$ with $x_n \to x$ and $(T + A)x_n \to v$. Then $Tx_n \to v - Ax$. Since T is closed, $x \in D(T)$ and $Tx = v - Ax$. Thus $(T + A)x = v$. Therefore, $T + A$ is closed.
(b) Let $(x_n)_{n=1}^{\infty}$ be sequence in $D(TB)$, with $x_n \to x$ and $TBx_n \to v$. Since B is bounded, $Bx_n \to Bx$, $(Bx_n)_{n=1}^{\infty}$ is a sequence in $D(T)$ and $T(Bx_n) \to v$. Since T is closed, $Bx \in D(T)$ and $TBx = v$. It follows that TB is closed. $\square$

Definition 5.1.7. *Let $T : D(T) \subset X \to Y$ be a linear operator. The* **graph norm** *of T is the norm on $D(T)$, defined by*

$$\|x\|_T = \|x\| + \|Tx\| \quad , \quad \forall x \in D(T) \; .$$

Theorem 5.1.11. *Let X and Y be Banach spaces. An operator $T : D(T) \subset X \to Y$ is closed if and only if $(D(T), \|\cdot\|_T)$ is complete.*

Proof. "only if" Assume T is closed. Assume $(x_n)_{n=1}^{\infty}$ in $D(T)$ is a Cauchy sequence w.r.t. the graph norm:

$$\|x_n - x_m\|_T \to 0 \quad (n, m \to \infty) \; .$$

This implies for $n, m \to \infty$,

$$\|x_n - x_m\| \to 0 \quad , \quad \|Tx_n - Tx_m\| \to 0 \; .$$

Since X and Y are Banach spaces there exist $x \in X$ and $y \in Y$ such that $x_n \to x$ (w.r.t. the norm in X) and $Tx_n \to y$ (w.r.t. the norm in Y). Since T is closed $x \in D(T)$ and $y = Tx$. Therefore,

$$\|x_n - x\|_T = \|x_n - x\| + \|Tx_n - Tx\| \to 0 \; .$$

and $(D(T), \|\cdot\|_T)$ is complete.

"if". Assume $(D(T), \|\cdot\|_T)$ is complete. Let $(x_n)_{n=0}^{\infty} \subset D(T)$ be a sequence such that for some $x \in X$ and $y \in Y$,

$$x_n \to x \quad , \quad Tx_n \to y.$$

Then for $n, m \to \infty$,

$$\|x_n - x_m\|_T = \|x_n - x_m\| + \|Tx_n - Tx_m\| \to 0 \; .$$

Hence $(x_n)_{n=1}^{\infty}$ in $D(T)$ is a Cauchy sequence w.r.t. to the graph norm. Since $(D(T), \|\cdot\|_T)$ is complete, there exists an $x \in D(T)$ such that

$$\|x_n - x\| + \|Tx_n - Tx\| = \|x_n - x\|_T \to 0 \; .$$

It follows that $y = Tx$ for $x \in D(T)$. $\square$

5.2 Spectral Theory

In this sections $\mathbb{F} = \mathbb{C}$ and X will denote a Banach space over the complex numbers. We set $\mathcal{L}(X) = \mathcal{L}(X, X)$. For any number $\lambda \in \mathbb{C}$, we shall denote the operator $\lambda \mathrm{id}_X : X \to X, x \mapsto \lambda x$ also by the symbol λ.

Definition 5.2.1. *Let $T : D(T) \subset X \to X$ be a linear operator. The* **resolvent set** *of T is a subset of $\mathbb{C}$ denoted by $\rho(T)$ and defined by*

$$\rho(T) := \{z \in \mathbb{C} : (T - z) : D(T) \to X \text{ is bij. and } (T - z)^{-1} \text{ is bounded}\} .$$

If $z \in \rho(T)$, then

$$R := R_z := R_z(T) := (T - z)^{-1}$$

is called the **resolvent** *of T. The* **spectrum** *is defined by*

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

Definition 5.2.2. *A number $\lambda \in \mathbb{C}$ is called an* **eigenvalue** *of $T : D(T) \subset X \to X$, if there exists a nonzero vector $x \in D(T)$ such that*

$$Tx = \lambda x. \tag{5.2.1}$$

The nonzero vector $x \in D(T)$ satisfying (5.2.1) *is called an* **eigenvector** *of T with eigenvalue λ. The set $\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}$ is called the* **point spectrum** *of T.*

Lemma 5.2.1. *Let T be a linear operator. Then $\sigma_p(T) \subset \sigma(T)$.*

Lemma 5.2.2. *If $z \in \rho(T)$, then*

(i) $(T - z)R_z = 1$ *on X,*

(ii) $R_z(T - z) = 1$ *on $D(T)$.*

(iii) $R_zT = TR_z$ *on $D(T)$;*

Proof. (i) and (ii) follow directly from the definition. (iii) follows from (i) and (ii). $\square$

Remark 5.2.1. If T is closed and $T - z : D(T) \to X$ is bijective, then $(T - z)^{-1}$ is bounded. Proof: Since T is closed the inverse $(T - z)^{-1} : X \to X$ is closed. Since the domain of the inverse is a Banach space, it follows by the closed graph theorem that $(T - z)^{-1} : X \to X$ is bounded.

Remark 5.2.2. If $z \in \rho(T)$, then $(T - z)^{-1} : X \to X$ is bounded and hence closed. Thus also $T - z : D(T) \to X$ is closed. We conclude that an operator with nonempty spectrum must be closed.

Lemma 5.2.3. *Let $T : D(T) \subset X \to X$. Then for $z, w \in \rho(T)$,*

(i) $R_w - R_z = (w - z)R_z R_w$ *(first resolvent identity)*

(ii) $R_z R_w = R_w R_z$.

Proof. (i) Let $x \in \mathcal{H}$. We want to show that (i) holds on x. There exists a unique $y \in D(T)$ such that $x = (T - z)y$. Then

$$(R_w - R_z)x = R_w(T - w)y + R_w(w - z)y - y = (w - z)R_z R_w x.$$

(ii) is a direct consequence of (i). $\square$

Theorem 5.2.4. *Neumann Series. Let $A \in \mathcal{L}(X)$ and $\|A\| < 1$. Then $(1 - A) : X \to X$ is bijective and its inverse is given by the convergent series*

$$(1 - A)^{-1} = \sum_{n=0}^{\infty} A^n \, .$$

Moreover,

$$\|(1 - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

Proof. An algebraic calculation shows that

$$(1 - A)\sum_{n=0}^{N} A^n = 1 - A^{N+1} = \sum_{n=0}^{N} A^n(1 - A).$$

Taking the limit $N \to \infty$ and using $\|A^N\| \leq \|A\|^N \to 0$, we obtain the result. The bound on the resolvent follows from its power series expansion. $\square$

Lemma 5.2.5. *Let X be a Banach space and $T : D(T) \subset X \to X$ a linear operator. If $w \in \rho(T)$, then the following disk is contained in the resolvent*

$$D = \{z \in \mathbb{C} : |z - w| < \|(T - w)^{-1}\|^{-1}\} \subset \rho(T)$$

and for all $z \in D$

$$(T - z)^{-1} = \sum_{n=0}^{\infty} (z - w)^n \left[(T - w)^{-1}\right]^{n+1} .$$

Proof.

$$T - z = (T - w) - (z - w) = \underbrace{\left(1 - (z - w)(T - w)^{-1}\right)}_{\substack{\text{is bijective if} \\ |z-w|\|(T-w)^{-1}\|<1}} \underbrace{(T - w)}_{\text{bijective}}$$

$\square$

Theorem 5.2.6. *Let $T : D(T) \subset X \to X$ be a linear operator.*

(a) Then $\rho(T)$ is open and $\sigma(T)$ closed. On the resolvent set $\rho(T)$ the function

$$z \mapsto (z - T)^{-1} = R_z(T)$$

is an analytic $\mathcal{L}(X)$ valued function.

(b) For $z \in \rho(T)$,

$$\|(z - T)^{-1}\|^{-1} \leq \operatorname{dist}(z, \sigma(T)) \quad \text{and} \quad \frac{1}{\operatorname{dist}(z, \sigma(T))} \leq \|(z - T)^{-1}\| .$$

Proof. This follows directly from the above lemma. $\square$

Theorem 5.2.7. *Let X be a Banach space and $T : D(T) \subset X \to X$ a closed linear operator. Let $\lambda \in \mathbb{C}$. If there exists a sequence $(x_n)_{n=1}^{\infty}$ in $D(T)$ with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(T - \lambda)x_n\| = 0$, then $\lambda \in \sigma(T)$.*

Proof. If $T-\lambda$ is not bijective, then clearly $\lambda \in \sigma(T)$ and we are done. Suppose now that $T-\lambda$ is bijective. Then $y_n := (T-\lambda)x_n \neq 0$, and

$$\left\| (T-\lambda)^{-1} \frac{y_n}{\|y_n\|} \right\| = \frac{1}{\|y_n\|} \to \infty.$$

Hence $(T-\lambda)^{-1}$ is unbounded, and therefore $\lambda \in \sigma(T)$. $\square$

Remark 5.2.3. A sequence as in the above theorem is called Weyl sequence. To prove only if one needs some additional property such as selfadjointness.

5.3 Operators in Hilbert Spaces

Multiplication Operators

Definition 5.3.1. *(Multiplication operator) Let (Ω, μ) be a measure space. Let $F : \Omega \to \mathbb{C}$ be a measurable function. We define a the multiplication operator M_F by*

$$D(M_F) := \{\varphi \in L^2(\Omega, d\mu) : F\varphi \in L^2(\Omega, d\mu)\}.$$

and

$$M_F\varphi := F\varphi$$

for all $\varphi \in D(M_F)$.

Lemma 5.3.1. *Let (Ω, ν) be a measure space. Let $F, G : \Omega \to \mathbb{C}$ be measurable. Then*

$$M_F M_G \subset M_{FG}.$$

Proof. We set $L^2 := L^2(\Omega, \nu)$. Let $f \in D(M_F M_G)$. Then $f \in L^2$, $Gf \in L^2$, and $GFf \in L^2$. Thus $f \in D(M_{GF})$ and $M_F M_G f = FGf = M_{FG}f$. □

Definition 5.3.2. *Let μ be Borel measure on Ω with measure μ. Then*

$$\operatorname{supp}(\mu) = \{\omega \in \Omega : \forall N, x \in N \text{ and } N \text{ open} \Rightarrow \mu(N) > 0\}.$$

is called the support of a measure. Let $F : \Omega \to \mathbb{C}$ be a measurable function. Define

$$\|F\|_\infty := \operatorname{ess\,sup}_{x\in\Omega}|f(x)| := \inf\{c \in \mathbb{R} : \mu(\{x \in \Omega : |F(x)| > c\}) = 0\}.$$

Proposition 5.3.2. *Let (Ω, μ) be a measure space. Let $F : \Omega \to \mathbb{C}$ be a measurable function.*

(a) If $\|F\|_\infty < \infty$, then M_F is bounded and

$$\|M_F\| = \|F\|_\infty.$$

(b) If $|F| < \infty$ a.e., then $D(M_F)$ is dense.

(c) M_F is closed.

(d) Define the measure $\mu_F := \mu \circ F^{-1}$. Then

$$\sigma(M_F) = \mathrm{supp}\mu_F.$$

(e) If $z \in \rho(M_F)$, then

$$(M_F - z)^{-1} = M_{(F-z)^{-1}}.$$

Proof. (a): For $r \geq 0$, define $E_r := \{x \in \Omega : |F(x)| > r\}$. Suppose for some $r \geq 0$ we have $\mu(E_r) = 0$. Then

$$\|M_F\varphi\|^2 = \int |F\varphi|^2 d\mu = \int_{E_r^c} |F\varphi|^2 d\mu \leq |r|^2 \int |\varphi|^2 d\mu = |r|^2 \|\varphi\|^2,$$

and hence $\|M_F\| \leq |r|$. Thus $\|M_F\| \leq \|F\|_\infty$. To show the opposite inequality, let $\|F\|_\infty > r'$. Then $E_{r'}$ has positive measure, i.e., $\mu(E_{r'}) > 0$ and $\|M_F 1_{E_{r'}}\| \geq r' \|1_{E_{r'}}\|$. Hence $\|M_F\| \geq r'$. This implies $\|M_F\| \geq \|F\|_\infty$.
(b): Let $\varphi \in L^2(\Omega, d\mu)$ and $A_n := \{x \in \Omega : |F(x)| \leq n\}$. Since $F < \infty$ a.e. we have by dominated convergence

$$\lim_{n\to\infty} \|\varphi - 1_{A_n}\varphi\| = 0.$$

Since $1_{A_n}\varphi \in D(M_F)$ the claim follows.
(c): $L^2(\Omega, (1 + |F|^2)d\mu)$ is complete. Thus $D(M_F)$ equipped with the graph norm (i.e., $\|f\|^2_{M_F} = \int |f|^2(1 + |F|^2)d\mu$) is complete. Thus M_F is closed.
(d): Let $\lambda \in \mathrm{supp}\mu_F$. For $n \in \mathbb{N}$ define $D_{1/n}(\lambda) := \{z \in \mathbb{C} : |z - \lambda| < 1/n\}$ and $A_n := F^{-1}(D_{1/n}(\lambda))$. Then $\mu(A_n) > 0$ and $x_n := \mu(A_n)^{-1/2} 1_{A_n}$ is normalized a sequence with

$$\|(M_F - \lambda)x_n\| \leq 1/n.$$

Hence $\lambda \in \sigma(M_F)$. Let $\lambda \notin \mathrm{supp}\mu_F$. Then for some $\epsilon > 0$ we have $\mu(F^{-1}(D_\epsilon(\lambda)) = 0$. Thus $\|(F - \lambda)^{-1}\|_\infty \leq \frac{1}{\epsilon}$ and hence by (a)

$$\|M_{(F-\lambda)^{-1}}\| = \|(F - \lambda)^{-1}\|_\infty < \infty.$$

Since by a straight forward calculation, $M_{(F-\lambda)^{-1}}(M_F-\lambda) = 1_{D(M_F)}$ and $(M_F-\lambda)M_{(F-\lambda)^{-1}} = 1$, it follows that $\lambda \in \rho(M_F)$.
(e): The claim follows from the proof of (d).

$\square$

Example 20. We define the **Laplacian** by $-\Delta = \mathcal{F} M_{|\cdot|^2} \mathcal{F}^{-1}$. By definition of the composition of operators the domain of $-\Delta$ given by

$$D(-\Delta) = \{\psi \in L^2(\mathbb{R}^d) : M_{|\cdot|^2} \hat{\psi} \in L^2(\mathbb{R}^d)\} = H^2(\mathbb{R}^d),$$

where the last equality is shown in the appendix. Since $\mathcal{F}$ is unitary, we see that $-\Delta$ is closed.

Example 21. We have $\overline{-\Delta|_{\mathcal{S}(\mathbb{R}^d)}}$. Proof: $M_{|\cdot|^2}$ is a closed operator. Moreover $\overline{M_{|\cdot|}|_{\mathcal{S}}} = M_{|\cdot|}$. (Which follows, since for $f \in D(M_{|\cdot|^2})$ we have $e^{-\epsilon|\cdot|^2} f \to f$ and $|\cdot|^2 e^{-\epsilon|\cdot|^2} f \to |\cdot|^2 f$ in L^2.)

5.4 Selfadjoint Operators

Below we shall generalize the definition of the Hilbert adjoint to unbounded operators.

Definition 5.4.1. *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined operator. Then one defines the adjoint as the operator T^* with domain*

$$D(T^*) = \{x \in \mathcal{H} : \exists \eta \in \mathcal{H}, \forall y \in D(T),\ \langle x, Ty\rangle = \langle \eta, y\rangle\}$$

and for $x \in D(T^*)$ its value is defined by

$$T^* x = \eta,$$

where $\eta \in \mathcal{H}$ is the unique element (by density of $D(T)$) such that $\langle x, Ty\rangle = \langle \eta, y\rangle$ for all $y \in D(T)$.

Remark 5.4.1. Note that by Riez lemma we have the following equivalences. Let T be a densely defined operator in a Hilbert space. Let $x \in \mathcal{H}$. Then the following statements are equivalent:

(a) $x \in D(T^*)$

(b) $\exists \eta \in \mathcal{H}$ such that $\langle x, Ty\rangle = \langle \eta, y\rangle, \quad \forall y \in D(T)$.

(c) $D(T) \to \mathbb{C} : y \mapsto \langle x, Ty\rangle$ is bounded.

(d) $\sup_{\|y\| \leq 1, y \in D(T)} |\langle x, Ty\rangle| < \infty$.

Remark 5.4.2. If $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is densly defined. Then for all $x \in D(T^*)$ and $y \in D(T)$ we have

$$\langle x, Ty\rangle = \langle T^* x, y\rangle.$$

(This follows directly from the definition).

For a Hilbert space $\mathcal{H}$ we define the following operator

$$V : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}, \quad (x, y) \mapsto (-y, x).$$

It follows from the definition that V is linear, isometric, $V^2 = -1$, and $V^* = -V$.

Lemma 5.4.1. *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined operator. Then*

$$\Gamma(T^*) = [V\Gamma(T)]^\perp = V[\Gamma(T)]^\perp.$$

Proof. First observe that the second inequality follows since V is isometric. To show the first equality, observe that

$$\begin{aligned}(x, y) \in \Gamma(T^*) \Leftrightarrow &\quad y = T^*x,\ x \in D(T^*)\\ \Leftrightarrow &\quad \langle y, z\rangle = \langle T^*x, z\rangle,\ \forall z \in D(T),\ x \in D(T^*)\\ \Leftrightarrow &\quad \langle y, z\rangle = \langle x, Tz\rangle,\ \forall z \in D(T)\\ \Leftrightarrow &\quad (-y, x) \in \Gamma(T)^\perp.\end{aligned}$$

□

Lemma 5.4.2. *If T is a densely defined operator in $\mathcal{H}$. Then the following holds:*

(a) T^* is closed.

(b) T is closable if and only if $D(T^*)$ is dense, in which case $\overline{T} = T^{**}$.

(c) If T is closable, then $(\overline{T})^* = T^*$.

Proof. (a) Follows directly from the previous lemma.
(b) Since $\Gamma(T)$ is a linear subset of $\mathcal{H} \times \mathcal{H}$ we have

$$\begin{aligned}\overline{\Gamma(T)} &= (\Gamma(T)^\perp)^\perp = (V^2\Gamma(T)^\perp)^\perp\\ &= (V(V\Gamma(T))^\perp)^\perp\\ &= (V(\Gamma(T^*))^\perp. \qquad (5.4.1)\end{aligned}$$

If $D(T^*)$ is dense, then

$$(V(\Gamma(T^*))^\perp = \Gamma(T^{**}),$$

and hence T is closable and $\overline{T} = T^{**}$. On the other hand if T is closable,

$$\Gamma(\overline{T}) = \overline{\Gamma(T)}. \qquad (5.4.2)$$

Let $x \in \mathcal{H}$, and suppose $\langle x, y\rangle = 0$ for all $y \in D(T^*)$. Since

$$\langle x, y\rangle = \langle (x,0),(y,T^*y)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle (0,x), V(y,T^*y)\rangle_{\mathcal{H}\times\mathcal{H}},$$

this implies that $(0,x) \in (V(\Gamma(T^*))^\perp$. By (5.4.1) and (5.4.2) it follows that $x = 0$, and hence $D(T^*)$ is dense.

(c) If T is closable we have by (a) and then applying (b) twice, that $T^* = \overline{T^*} = T^{***} = \overline{T}^*$. □

Proposition 5.4.3. *Let T be a densely defined operator. Then*

$$\mathrm{Ker}T^* = (\mathrm{Im}T)^\perp$$

Proof. $x \in \mathrm{Ker}T^*$ iff $x \in D(T^*)$, $T^*x = 0$ iff $\langle x, Ty\rangle = 0, \forall y \in D(T)$. □

Lemma 5.4.4. *Let T be a densely defined operator with $z \in \rho(T)$. Then $\overline{z} \in \rho(T^*)$ and*

$$[(T-z)^{-1}]^* = (T^* - \overline{z})^{-1}.$$

Proof. Note that a linear operator $S : D(S) \to \mathcal{H}$ is bijective iff there exists a linear operator S' such that $\Gamma(S') = V\Gamma(-S)$, in which case $S' = S^{-1}$. Using this, we find

$$\begin{aligned}\Gamma([(T-z)^{-1}]^*) &= [V\Gamma((T-z)^{-1})]^\perp = [VV\Gamma(z-T)]^\perp \\ &= V[V\Gamma(z-T)]^\perp = V\Gamma(\overline{z}-T^*) = \Gamma([(T^*-\overline{z})^{-1}]).\end{aligned}$$

Hence the claim follows. □

Lemma 5.4.5. *Let T and S be denslely defined operators on a Hilbert space. Then*

(a) $T^* + S^* \subset (T+S)^*$. Equality holds if $T \in \mathcal{L}(\mathcal{H})$.

(b) $T^*S^* \subset (ST)^*$. Equality holds if $S \in \mathcal{L}(\mathcal{H})$.

(c) If $S \subset T$, then $T^* \subset S^*$.

Proof. (a). Let $x \in D(T^* + S^*)$. Then by definition $x \in D(T^*) \cap D(S^*)$ and

$$\langle x, Ty\rangle = \langle T^*x, y\rangle, \forall y \in D(T), \quad \langle x, Sy\rangle = \langle S^*x, y\rangle, \forall y \in D(S)$$

Hence $\langle x, (S+T)y\rangle = \langle T^*x + S^*x, y\rangle)$ for all $y \in D(T) \cap D(S)$. Therefore, $x \in D((S+T)^*)$ and

$$(S+T)^*x = S^*x + T^*y.$$

Now suppose $T \in \mathcal{L}(\mathcal{H})$. Let $x \in D((T+S)^*)$. Then there exists a constant C such that for all $y \in D(S+T) = D(S)$ we have

$$|\langle x, Sy\rangle| = |\langle x, (S+T)y\rangle - \langle x, Ty\rangle| \leq C\|y\|.$$

Hence $x \in D(S^*) = D(S^* + T^*)$.
(b) Let $x \in D(T^*S^*)$. Then by definition $x \in D(S^*)$ and $S^*x \in D(T^*)$, and

$$\langle x, STy\rangle = \langle S^*x, Ty\rangle = \langle T^*S^*x, y\rangle, \quad \forall y \in D(ST).$$

Therefore $x \in D((ST)^*)$ and $(ST)^*x = T^*S^*x$.
Now suppose that $S \in \mathcal{L}(\mathcal{H})$. If $x \in D((ST)^*)$, then

$$\langle x, STy\rangle = \langle (ST)^*x, y\rangle, \quad \forall y \in D(ST) = D(T).$$

Furthermore since $\langle x, STy\rangle = \langle S^*x, Ty\rangle$, we have $S^*x \in D(T^*)$, and hence $x \in D(T^*S^*)$.
(c) Let $x \in D(T^*)$. Then for all $y \in D(T)$ we have

$$\langle x, Ty\rangle = \langle T^*x, y\rangle$$

Hence restricting y to $D(S)$ we obtain $x \in D(S^*)$ and $S^*x = T^*x$.

$\square$

Lemma 5.4.6. *Let (Ω, μ) be a measure space. Let $F : \Omega \to \mathbb{C}$ be a measurable function for which $|F| < \infty$ a.e. Then $M_F^* = M_{\overline{F}}$. In particular, $M_F^* M_F = M_F M_F^*$.*

Proof. Let $\varphi \in D(M_F^*)$. Then $|(\varphi, M_F\psi)| \leq C\|\psi\|$ for all $\psi \in D(M_F)$. Thus inserting $\psi = 1_{|F\varphi|\leq n} M_F\varphi$ we find

$$\int |F|^2|\varphi|^2 1_{|F\varphi|\leq n} d\mu \leq C \int (|F|^2|\varphi|^2 1_{|F\varphi|\leq n} d\mu)^{1/2}$$

Hence $\int (|F|^2|\varphi|^2 1_{|F\varphi|\leq n} d\mu)^{1/2} \leq C$. By monotone convergence we find $\int |F|^2|\varphi|^2 d\mu \leq C^2$. Thus $\varphi \in M_{\overline{F}}$. Suppose now $\varphi \in M_{\overline{F}}$. Then for all $\psi \in D(M_F)$ we have $(\varphi, M_F\psi) = (M_{\overline{F}}\varphi, \psi)$. Thus $\varphi \in D(M_F^*)$ and $M_F^*\varphi = M_{\overline{F}}\varphi$. This yields the first claim. The inparticular statement follows from the first claim. $\square$

5.5 Selfadjoint and Symmetric Operators

Definition 5.5.1. *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined operator. Then T is called* **selfadjoint** *if $T^* = T$.*

Remark 5.5.1. Selfadjoint operators are always closed since adjoints are closed. For a seldadjoint operator T we always have

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in D(T).$$

Example 22. $-\Delta : H^2 \subset L^2 \to L^2$ is selfadjoint. Proof: Follows from $-\Delta = \mathcal{F}^{-1} M_{|\cdot|^2} \mathcal{F}$ and Lemma 5.4.6.

Theorem 5.5.1. *Let T be self-adjoint. Then the following holds.*

(a) $\|(T - z)x\| = \|(T - \overline{z})x\|$ *for all $x \in D(T)$.*

(b) $\ker(T - z) \perp \ker(T - w)$ *if $w \neq z$.*

(c) If for some $\delta > 0$ we have

$$\|(T - \lambda)x\| \geq \delta\|x\| \tag{5.5.1}$$

for all $x \in D(T)$, then $\lambda \in \rho(T)$.

(d) $\lambda \in \sigma(T)$ *iff there exists a normalized sequence if $(x_n)_{n\in\mathbb{N}} \in D(T)$ with*

$$\|(T - \lambda)x_n\| \to 0.$$

Proof. (a) For $x \in D(T)$ we have $\|(T-\overline{z})x\|^2 = \langle (T - \overline{z})x, (T - \overline{z})x\rangle = \langle (T - z)x, T - z)x\rangle = \|(T - z)x\|^2$
(b) Suppose for $\lambda \in \{z, w\}$ we are given a vector $x_\lambda \in \ker(T - \lambda)$. Then using (a) we find

$$(z - w)\langle x_z, x_w\rangle = \langle \overline{z}x_z, x_w\rangle - \langle x_z, wx_w\rangle = \langle Tx_z, x_w\rangle - \langle x_z, Tw_w\rangle = 0.$$

(c) By assumption $(T - \lambda)$ is injective as well as $T - \overline{\lambda}$ by (a). Since $\operatorname{ran}(T - \lambda)^\perp = \ker(T - \lambda)^* = \ker(T - \overline{\lambda}) = \{0\}$ (as a straightforward calculation shows), $\operatorname{ran}(T - \lambda)$ is

dense in $\mathcal{H}$. On the other hand it must be closed. To see this, let (y_n) be a sequence in $\operatorname{ran}(T-\lambda)$ which converges to say, y. Then for some x_n we have $y_n = (T-\lambda)x_n$. By (5.5.1) (x_n) must converge as well, to say x. Since $T-\lambda$ is closed $y = (T-\lambda)x \in \operatorname{ran}(T-\lambda)$, and hence we have shown that $\operatorname{ran}(T-\lambda)$ is closed. The boundedness of the resolvent follows from (5.5.1)
(d) We have already shown the proof in one direction. It remains to show the proof in the opposite direction. But this is merely the contraposition of (c). □

Definition 5.5.2. *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be an operator. Then T is called* **symmetric** *if it is densely defined (this is not always required) and*

$$\langle x, Ty\rangle = \langle Tx, y\rangle, \quad \forall x, y \in D(T).$$

Proposition 5.5.2. *T is symmetric if and only if $T \subset T^*$.*

Proof. This follows directly from the defining relations. □

Remark 5.5.2. A selfadjoint operator is always symmetric.

Lemma 5.5.3. *Let T be a symmetric operator in $\mathcal{H}$ and let $z \in \mathbb{C}$. Then the following statements hold.*

(a) $\|(T-z)x\|^2 = \|(T-\mathrm{Re}z)x\|^2 + (\mathrm{Im}z)^2\|x\|^2$ *for all $x \in D(T)$.*

(b) $\ker(T-z) = \{0\}$, *if* $\mathrm{Im}z \neq 0$.

(c) $\mathrm{Ran}(T-z)$ *is closed, if T is closed and* $\mathrm{Im}z \neq 0$.

(d) If $\mathrm{Im}z \neq 0$, *then $T-z : D(T) \to \mathrm{Ran}(T-z)$ is bijective and for all $y \in \mathrm{Ran}(T-z)$*

$$\|(T-z)^{-1}y\| \le \frac{1}{|\mathrm{Im}z|}\|y\|, \quad \|(T-\mathrm{Re}z)(T-z)^{-1}y\| \le \|y\|.$$

Proof. We set $z = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$.
(a). For $x \in D(T)$,

$$\begin{aligned} \|(T-z)x\|^2 &= \|(T-\alpha)x + i\beta x\|^2 \\ &= \|(T-\alpha)x\|^2 + \langle (T-\alpha)x, i\beta x\rangle + \langle (T-\alpha)x, i\beta x\rangle + |\beta|^2\|x\|^2 \\ &= \|(T-\alpha)x\|^2 + |\beta|^2\|x\|^2. \end{aligned}$$

(b). From (a) we find for all $x \in D(T)$,

$$\|(T-z)x\|^2 \geq |\beta|^2\|x\|^2. \tag{5.5.2}$$

This implies (b).
(c). Let (y_n) be a sequence in $\mathrm{Ran}(T-z)$ with $y_n \to y$. Then there exists a sequence (x_n) in $D(T)$ with $y_n = (T-z)x_n$. Because of (5.5.2) it follows that (x_n) is a Cauchy sequence. Hence it converges to some x. Since T is closed, it follows that $x \in D(T)$ and $y = (T-z)x \in \mathrm{Ran}(T-z)$. We conclude that $\mathrm{Ran}(T-z)$ is closed.
(d). Let $y \in \mathrm{Ran}(T-z)$. Then there exists a unique $x \in D(T)$ with $y = (T-z)x$. Inserting this into (5.5.2), we obtain

$$\|(T-z)^{-1}y\| = \|x\| \leq \frac{1}{|\beta|}\|(T-z)x\| = \frac{1}{|\beta|}\|y\|.$$

This implies the first estimate. The second follows likewise using (a)

$$\|(T-\mathrm{Re}z)(T-z)^{-1}y\| \leq \|(T-z)(T-z)^{-1}y\| = \|y\|.$$

□

Theorem 5.5.4. *(Basic Criterion for Selfadjointness) Let T be a symmetric operator in a Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

(a) T is self-adjoint.

(b) T is closed, and there exists a $z \in \mathbb{C}$ with $\mathrm{Im}z \neq 0$, such that $\ker(T^* - z) = \{0\}$ and $\ker(T^* - \overline{z}) = \{0\}$.

(c) There exists a $z \in \mathbb{C}$*, such that* $\mathrm{Ran}(T - z) = \mathcal{H}$ *and* $\mathrm{Ran}(T - \overline{z}) = \mathcal{H}$*.*

(d) $\sigma(T) \subset \mathbb{R}$*.*

Proof. (a) $\Rightarrow$ (b). Let $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$. Then using $T^* = T$, we find

$$\mathrm{Ker}(T^* - z) = \mathrm{Ker}(T - z) = \{0\},$$

where in the last equality we used Part (a) of the previous Lemma. Similarly one shows $\mathrm{Ker}(T^* - \overline{z}) = \{0\}$.
(b) $\Rightarrow$ (c). Let z be an Element of $\mathbb{C}$ with $\mathrm{Im} z \neq 0$, $\ker(T^* - z) = \{0\}$ and $\ker(T^* - \overline{z}) = \{0\}$. Then

$$\{0\} = \ker(T^* - z) = [\mathrm{Ran}(T - \overline{z})]^\perp, \quad \{0\} = \ker(T^* - \overline{z}) = [\mathrm{Ran}(T - z)]^\perp$$

We conclude that the ranges of $T - \overline{z}$ and $T - z$ are dense. Since $\mathrm{Im} z \neq 0$ and T is closed these ranges are in fact closed (by Part (c) of the previous Lemma). Hence $\mathrm{Ran}(T - \overline{z}) = \mathrm{Ran}(T - z) = \mathcal{H}$.
(c) $\Rightarrow$ (a). Let $x \in D(T^*)$. Then there exists by surjectivity a $y \in D(T)$ such that

$$(T^* - z)x = (T - z)y.$$

Since $T \subset T^*$, this implies

$$(T^* - z)x = (T^* - z)y.$$

Hence $(T^* - z)(x - y) = 0$. It follows that $(x - y) \in \ker(T^* - z) = [\mathrm{Ran}(T - \overline{z})]^\perp = \{0\}$. Hence $x = y \in D(T)$ and therefore $T^* = T$.
(d) $\Rightarrow$ (c). Since $\sigma(T) \subset \mathbb{R}$, we have $\pm i \in \rho(T)$. This implies (c).
(a) $\Rightarrow$ (d). From the proofs (a) $\Rightarrow$ (b) $\Rightarrow$ (c), we see that for all $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$, we have

$$\ker(T - z) = \{0\}, \qquad \mathrm{Ran}(T - z) = \mathcal{H}.$$

Thus the mapping $T - z : D(T) \to \mathcal{H}$ is bijective. (Boundedness of the resolvent follows from $\mathrm{Im} z \neq 0$ and the previous Lemma part (d), alternatively one could invoke the closed graph theorem.) We conlcude that $\sigma(T) \subset \mathbb{R}$. $\square$

Definition 5.5.3. *Let T be a symmetric operator in $\mathcal{H}$. T is called essentially selfadjoint if $\overline{T}$ is self adjoint.*

Example 23. $-\Delta|_{\mathcal{S}}$ *is essentially selfadjoint.*

5.6 Kato Rellich Theorem

Definition 5.6.1. *Let T and A be densely defined linear operators in $\mathcal{H}$. Suppose*

(i) $D(A) \supset D(T)$.

(ii) For some $a, b \in \mathbb{R}$ we have

$$\|Ax\| \leq a\|Tx\| + b\|x\|, \quad \forall x \in D(T). \tag{5.6.1}$$

Then A is said to be T-bounded. The infimum over all numbers a for which there is a $b \in \mathbb{R}$ such that (5.6.1) *holds, i.e.,*

$$\inf\{a \in \mathbb{R} : \exists b \in \mathbb{R}, (5.6.1)\}$$

is called the relative bound of A w.r.t. T.

In view of the theorem below the above definition is meaningful.

Theorem 5.6.1. *(Kato-Rellich) Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a selfadjoint operator and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ a symmetric operator such that*

(i) $D(A) \supset D(T)$.

(ii) There exists an $a \in [0, 1)$ and a $b \geq 0$ such that for all $x \in D(T)$ we have

$$\|Ax\| \leq a\|Tx\| + b\|x\|. \tag{5.6.2}$$

Then $T + A : D(T) \subset \mathcal{H} \to \mathcal{H}$ is self adjoint.

Proof. The goal is to show that for some $\beta > 0$ we have $\mathrm{Ran}(T + A \pm i\beta) = \mathcal{H}$. We have

$$T + A \pm i\beta = \underbrace{(1 + A(T \pm i\beta)^{-1})}_{\substack{\text{is surjective if} \\ \|A(T \pm i\beta)^{-1}\| < 1}} \underbrace{(T \pm i\beta)}_{\text{surjective}} \tag{5.6.3}$$

It remains to show that $\|A(T \pm i\beta)^{-1}\| < 1$. Since T is symmetric we have for all $x \in D(T)$ that

$$\|(T \pm i\beta)x\|^2 = \|Tx\|^2 + \beta^2\|x\|^2.$$

This implies

$$\|T(T \pm i\beta)^{-1}\| \le 1, \quad \|(T \pm i\beta)^{-1}\| \le \frac{1}{|\beta|}.$$

Inserting this into (5.6.2) we find

$$\|A(T \pm i\beta)^{-1}\| \le a \underbrace{\|T((T \pm i\beta)^{-1}\|}_{\le 1} + b \underbrace{\|(T \pm i\beta)^{-1}\|}_{\le \frac{1}{|\beta|}} < 1,$$

where that last inequality holds for $|\beta|$ sufficiently large. Thus $T + A$ is selfadjoint.

□

Lemma 5.6.2. *Let $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in L^\infty(\mathbb{R}^3)$. Then for any $a > 0$ there exists a $b \in \mathbb{R}$ such that*

$$\|V\psi\| \le a\| - \Delta\psi\| + b\|\psi\|, \quad \forall \psi \in D(-\Delta).$$

Proof. The lemma will follows by inserting Step 2 into Step 1 and using that $\mathcal{S}$ is a core for $-\Delta$.
Step 1: $\|V\psi\|_2 \le \|V_1\|_2\|\psi\|_\infty + \|V_2\|_\infty\|\psi\|_2$.

Step 2: For all $a > 0$ there is a $b \in \mathbb{R}$ such that for all $\psi \in \mathcal{S}(\mathbb{R}^3)$ we have

$$\|\psi\|_\infty \le a\| - \Delta\psi\|_2 + b\|\psi\|_2.$$

For any $\epsilon > 0$ we have

$$\begin{aligned}
\|\psi\|_\infty &= \sup_x |\psi(x)| = \sup_x \left|(2\pi)^{-3/2} \int e^{-ipx}\hat{\psi}(p)d^3p\right| \\
&\leq (2\pi)^{-3/2} \int |\hat{\psi}(p)|d^3p \\
&= (2\pi)^{-3/2} \int \{1 + (\epsilon|p|)^2\}^{-1}\{1 + (\epsilon|p|)^2\}|\hat{\psi}(p)|d^3p \\
&\leq (2\pi)^{-3/2} \left(\int \{1 + (\epsilon|p|)^2\}^{-2}d^3p\right)^{1/2} \left[\epsilon^2\|p^2\hat{\psi}\| + \|\hat{\psi}\|\right] \\
&= (2\pi)^{-3/2} \left(\int \{1 + |p|^2\}^{-2}d^3p\right)^{1/2} \epsilon^{-3/2} \left[\epsilon^2\|p^2\hat{\psi}\| + \|\hat{\psi}\|\right].
\end{aligned}$$

This yields Step 2. □

Corollary 5.6.3. *Let $\{V_k\}_{k=1}^m$ be a collection of real valued measurable functions each of which is in $L^2(\mathbb{R}^3)+L^\infty(\mathbb{R}^3)$. Let $V_k(y_k)$ be the multiplication operator on $L^2(\mathbb{R}^3)$ obtained by choosing y_k to be three coordinates of $\mathbb{R}^{3n}$. Then*

$$-\Delta + \sum_{k=1}^m V(y_k)$$

is self-adjoint on $D(-\Delta)$, where Δ denotes the Laplacian on $\mathbb{R}^{3n}$ with domain $H^2(\mathbb{R}^{3n})$.

Proof. First consider one of the functions V_k separately. By rotation of variables we may assume the variables $y_k = (x_1, x_2, x_3)$. Let Δ_1 denote the Laplacian with respect to y_k. Then by the above theorem there exists for any $a > 0$ a b such that for all $\varphi \in H^2(\mathbb{R}^{3n})$

$$\|V_k\varphi\|^2 \leq a\| - \Delta_1\varphi\|^2 + b\|\varphi\|^2 \leq a\| - \Delta\varphi\|^2 + b\|\varphi\|^2,$$

where the last inequality follows for example by representing the Laplacian in terms of the Fourier transform. □

Example 24. Let $x_1, ..., x_n$ in $\mathbb{R}^3$ be orthogonal coordinates for $\mathbb{R}^3$. Then from the above corollary we conclude that

$$-\sum_{i=1}^n \Delta_i - \sum_{i=1}^n \frac{c_i}{|x_i|} + \sum_{i<j} \frac{c_{ij}}{|x_i - x_j|},$$

is self-adjoint on $H^2(\mathbb{R}^{3n})$ for any real numbers c_i, c_{ij}.

Chapter 6

Spectral Theorem

6.1 Spectral Theorem

Theorem 6.1.1. *(Spectral Theorem–Multiplication Operator Form) Every selfadjoint operator is unitarily equivalent to a Multiplication operator. More precisely: Let A be a self adjoint operator on a Hilbert space $\mathcal{H}$. Then there exists a measure space (Ω, μ) (with μ finite if $\mathcal{H}$ is separable), a measurable function $F : \Omega \to \overline{\mathbb{R}}$ which is finite a.e., and a unitary operator*

$$U : \mathcal{H} \to L^2(\Omega, \nu)$$

such that

$$UAU^{-1} = M_F.$$

Proof. Trivial. □

Theorem 6.1.2. *(Spectral Theorem–Multiplication Operator Form: separable case) Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Then there exists an $N \in \mathbb{N} \cup \{\infty\}$, a finite measure μ on $\sigma(A) \times \{1, ..., N\}$, and a unitary map*

$$U : \mathcal{H} \to L^2(\mathbb{R} \times \{1, ..., N\}, d\mu)$$

such that

$$(UAU^{-1}f)(n; \lambda) = \lambda f(\lambda, n), \quad n = 1, ..., N, \text{a.e. } \lambda \in \mathbb{R}$$

for all $f \in L^2(\mathbb{R} \times \{1, ..., N\}, d\mu)$.

Remark 6.1.1. We note that the above theorem is essentially a rigorous form of the physicists Dirac notation. If we write $\hat{\psi} = U\psi$

$$\langle \psi, \phi \rangle = \int \overline{\hat{\psi}(\lambda, n)} \hat{\phi}(\lambda, n) d\mu(\lambda, n) =: \sum\!\!\!\!\!\!\!\int_{\lambda,n} \langle \psi | \lambda, n \rangle \langle \lambda, n | \phi \rangle$$

$$\langle \psi, A\phi \rangle = \int \overline{\hat{\psi}(\lambda, n)} \lambda \hat{\phi}(\lambda, n) d\mu(\lambda, n) =: \sum\!\!\!\!\!\!\!\int_{\lambda,n} \langle \psi | \lambda, n \rangle \lambda \langle \lambda, n | \phi \rangle$$