Quantum Field Theory
Exercise Sheet 4

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Problem 9: Lorentz transformations

If not defined otherwise we will use the Latin alphabet $\{a, b, c, \dots\}$ as a set of dummy indices in the set $I_0 \coloneqq \{0, 1, 2, 3\}$ and the Greek alphabet $\{\alpha, \beta, \gamma, \dots\}$ as a set of dummy indices in the set $I \coloneqq \{1, 2, 3\}$ for the sum convention of Einstein. We denote the Minkowski metric as follows.

$$\eta \coloneqq (\eta_{mn}) \coloneqq (\eta^{mn}) \coloneqq \operatorname{diag}(-1, 1, 1, 1)$$

We define the Lorentz group $\mathbb{L}$ as a subset of the 4-dimensional invertible matrices.

$$\mathbb{L} \coloneqq \left\{ L \in \mathrm{GL}_4(\mathbb{R}) \;\middle|\; L^{\mathrm{T}} \eta L = \eta \right\}$$

(1): Choose $L \in \mathbb{L}$ with the following definition for an arbitrary $\omega \in \mathbb{R}^{4\times 4}$. We define $\mathrm{d}L$ as the infinitesimal change of L.

$$L^m{}_n = \delta^m{}_n\,, \qquad \mathrm{d}L^m{}_n = \omega^m{}_n$$

From the definition of the Lorentz group we do know that the following condition must hold.

$$\eta_{mn} = \eta_{pq} L^p{}_m L^q{}_n$$

By computing the infinitesimal alteration of this equation, we can directly proof the lemma.

$$\begin{aligned} 0 &= \eta_{pq} \left(L^q{}_n \mathrm{d}L^p{}_m + L^p{}_m \mathrm{d}L^q{}_n \right) \\ &= \eta_{pq} \delta^q{}_n \omega^p{}_m + \eta_{pq} \delta^p{}_m \omega^q{}_n \\ &= \eta_{pn} \omega^p{}_m + \eta_{mq} \omega^q{}_n \\ &= \omega_{nm} + \omega_{mn} \end{aligned}$$

(2): Let $L \in \mathbb{L}$ be an arbitrary Lorentz transformation. Then from the definition of $\mathbb{L}$ we do know the following.

$$\eta = L^{\mathrm{T}} \eta L$$

Now we use some rules for the computation of determinants. The determinant of a matrix product is the same as the product of the determinants of these matrices. Furthermore the determinant does not change if we transpose the given matrix.

$$\begin{aligned} \det \eta &= \det \left(L^{\mathrm{T}} \eta L \right) \\ &= \det L^{\mathrm{T}} \cdot \det \eta \cdot \det L \\ &= \det \eta \cdot (\det L)^2 \end{aligned}$$

We divide this equation by $\det \eta$ and get the first proposition which should be shown.

$$(\det L)^2 = 1 \qquad \Longrightarrow \qquad \det L = \pm 1$$

For the second proposition we again use the definition of the Lorentz group $\mathbb{L}$. This time the equation will be formulated in index notation.

$$\eta_{mn} = \eta_{pq} L^p{}_m L^q{}_n$$

In particular we can state the following because η is a diagonal matrix.

$$\begin{aligned} -1 = \eta_{00} &= \eta_{pq} L^p{}_0 L^q{}_0 \\ &= \eta_{00} \left(L^0{}_0 \right)^2 + \eta_{\alpha\beta} L^\alpha{}_0 L^\beta{}_0 \\ &= -\left(L^0{}_0 \right)^2 + \left(L^1{}_0 \right)^2 + \left(L^2{}_0 \right)^2 + \left(L^3{}_0 \right)^2 \end{aligned}$$

By shifting $L^0{}_0$ to the left of this equation and applying the inequality $x^2 \geq 0$ for any real number $x \in \mathbb{R}$ the equation becomes

$$\left(L^0{}_0 \right)^2 = 1 + \left(L^1{}_0 \right)^2 + \left(L^2{}_0 \right)^2 + \left(L^3{}_0 \right)^2 \geq 1$$

This equation is equivalent to the second proposition which should be shown.

$$L^0{}_0 \geq 1 \quad \vee \quad L^0{}_0 \leq -1$$

(3): We consider the Lorentz transformation L with the following definition. Here we define $\eta \in \mathbb{R}$ to be a real scalar value.

$$L := \exp\left(\mathrm{i}\eta^j \left({K_j}^\alpha{}_\beta\right)_{\alpha,\beta}\right)$$

$$\eta^j := \eta\delta_3^j$$

$${K_j}^\alpha{}_\beta := -\mathrm{i}\left(\delta_0^\alpha\delta_{j\beta} + \delta_{0\beta}\delta_j^\alpha\right)$$

First, we compute the argument of the exponent of the supposed Lorentz boost L.

$$\begin{aligned} S_\beta^\alpha &:= \mathrm{i}\eta^j {K_j}^\alpha{}_\beta \\ &= \eta\delta_3^j\left(\delta_0^\alpha\delta_{j\beta} + \delta_{0\beta}\delta_j^\alpha\right) \\ &= \eta\left(\delta_0^\alpha\delta_{3\beta} + \delta_{0\beta}\delta_3^\alpha\right) \end{aligned}$$

Based on this expression we see that components with the indices 1 and 2 can be omitted. Because of the exponential they appear as an identity in the actual result.

$$S := \left(S_\beta^\alpha\right)_{\alpha,\beta\in\{0,3\}} = \eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To express the exponent we need to compute S^n for every $n \in \mathbb{N}_0$. Let $k \in \mathbb{N}_0$ be arbitrary. Then the following formulas can be easily shown by mathematical induction.

$$S^{2k} = \eta^{2k}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S^{2k+1} = \eta^{2k+1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we use these relations and the definition of the exponential to compute L.

$$\begin{aligned} \left(L_\beta^\alpha\right)_{\alpha,\beta\in\{0,3\}} &= \sum_{n=0}^\infty \frac{S^n}{n!} \\ &= \sum_{k=0}^\infty \frac{S^{2k}}{(2k)!} + \sum_{k=0}^\infty \frac{S^{2k+1}}{(2k+1)!} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\sum_{k=0}^\infty \frac{\eta^{2k}}{(2k)!} \\ &+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sum_{k=0}^\infty \frac{\eta^{2k+1}}{(2k+1)!} \\ &= \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \end{aligned}$$

Now let $v := \tanh\eta$ be the velocity related to the rapidity η. Then the relations for the relative velocity β and the Lorentz factor γ can be written in the following form by using some rules for computing hyperbolic functions.

$$\begin{aligned} \beta &= v = \tanh\eta \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\eta}} = \cosh\eta \\ \beta\gamma &= \tanh\eta\cosh\eta = \sinh\eta \end{aligned}$$

Let $(t', x', y', z')^\mathrm{T} := L(t, x, y, z)^\mathrm{T}$. In this case we get the following.

$$\begin{aligned} t' &= \gamma\left(t + \beta z\right) \\ x' &= x \\ y' &= y \\ z' &= \gamma\left(z + \beta t\right) \end{aligned}$$

By definition this is an inverse Lorentz boost with velocity $\vec{v} := (0, 0, v)$. Hence, L is a Lorentz boost with velocity $-\vec{v}$.

(4): The Noether currents $\mathcal{J}^{ijk}$ were given by the symmetric energy-momentum tensor $\mathcal{T}^{pq}$.

$$\mathcal{J}^{ijk} = \frac{1}{2}\left(\mathcal{T}^{ij}x^k - \mathcal{T}^{ik}x^j\right)$$

From the interpretation of the energy-momentum tensor we do know that $\vec{p} := \left(\mathcal{T}^{0\alpha}\right)_\alpha$ is the density of the linear momentum. Furthermore we define the following.

$$l^\alpha := -\varepsilon^{\alpha\beta\gamma}\mathcal{J}^{0\beta\gamma}$$

To make things a little bit easier, we recompute this expression by interchanging β and γ.

$$\varepsilon^{\alpha\beta\gamma}\mathcal{T}^{0\beta}x^\gamma = \varepsilon^{\alpha\gamma\beta}\mathcal{T}^{0\gamma}x^\beta = -\varepsilon^{\alpha\beta\gamma}\mathcal{T}^{0\gamma}x^\beta$$

Therefore we can write l^α in the following form.

$$l^\alpha = -\varepsilon^{\alpha\beta\gamma}\mathcal{T}^{0\beta}x^\gamma$$

Using the definition of the Levi-Civita symbol and the cross product we obtain

$$\vec{l} = -\vec{p} \times \vec{x} = \vec{x} \times \vec{p}$$

Therefore we can interpret $\vec{l}$ as density of the angular momentum. Hence, the following integral L^α has to be the total angular momentum.

$$L^\alpha = \int_{\mathbb{R}^4} l^\alpha(x)\,\mathrm{d}\lambda(x)$$