Friedrich-Schiller-Universität Jena Winter term 2018/19
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Problems in Advanced Quantum Theory

Sheet 8

Problem 13: Photon emission by an atom 4+2+2+2(+5) points

To describe the spontaneous emission of a photon by an excited Hydrogen atom we write the Hamiltonian of the electron coupled to the electromagnetic potential $\boldsymbol{A}$ to linear order in the latter. Furthermore, we assume the photon corresponds to a monocromatic plane wave. Then for emission it suffices to take

$$\boldsymbol{A}(\boldsymbol{x}, t) = \hat{\boldsymbol{\epsilon}}\, A_0 e^{\mathrm{i}\omega t - \mathrm{i}\frac{\omega}{c}\hat{\boldsymbol{n}}\cdot\boldsymbol{x}}\,, \tag{1}$$

while the corresponding complex conjugate would contribute to the absorption of a photon. Here $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{\epsilon}}$ are unit vectors specifying the direction of propagation and the polarization of the photon respectively. In Coulomb gauge they fulfill

$$\hat{\boldsymbol{\epsilon}}\cdot\hat{\boldsymbol{n}} = 0\,. \tag{2}$$

At the leading order of the long-wavelength approximation one neglects

$$\frac{\omega}{c}\hat{\boldsymbol{n}}\cdot\boldsymbol{x} \approx 0\,, \tag{3}$$

in Eq. (1).

1. Express the transition rate $w_{i\to f}$ from the initial to the final state (two stationary states of the unperturbed Hydrogen atom with energies E_i and E_f) in terms of the matrix element
$$\langle f|\boldsymbol{p}|i\rangle\,. \tag{4}$$

2. Further manipulate the previous result and express the transition rate $w_{i\to f}$ in terms of the matrix element
$$\langle f|\boldsymbol{x}|i\rangle\,. \tag{5}$$
Hint: you might recall the form of the Hamiltonian H_0 for the unperturbed Hydrogen atom, and its commutators with $\boldsymbol{x}$ and $\boldsymbol{p}$.

3. By using spherical coordinates (r, θ, ϕ) express $\boldsymbol{x}$ in terms of the unit vectors $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$ and of the spherical harmonics $Y_1^m(\theta, \phi)$.

4. By specifying the angular momentum quantum numbers for the states
$$|i\rangle \propto |lm\rangle\,, \qquad |f\rangle \propto |l'm'\rangle\,, \tag{6}$$
and using the result of the previous task, deduce a selection rule for the transition rate. *Hint: you might recall the result of problem 7.3.*

Bonus question:

5. *Choose the emission plane to be the $\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}$ plane and thus parametrize*

$$\hat{\boldsymbol{n}} = \sin\theta\hat{\boldsymbol{x}} + \cos\theta\hat{\boldsymbol{z}}\,. \tag{7}$$

Then $\hat{\boldsymbol{\epsilon}}$ can be parametrized by θ and by one additional angle α. What is the angular distribution of the emitted photon, averaged over the polarization angle α, in the case $m' = m - 1$?

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