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Winter term 2017/18

Problems in Advanced Quantum Mechanics

Problem Sheet 5

Problem 11: Clebsch-Gordan coefficients — 6 points

Consider a system composed of two subsystems with angular momenta $j_1 = 1$ and $j_2 = 1$. The z-components of these angular momenta are denoted by m_1 and m_2. The states of the total system are characterized by the total angular momentum j and corresponding magnetic quantum number m. Compute the Clebsch-Gordan coefficients for the quantum numbers given in the following tables (the cases $m = 0$ and $m = 1$) and in addition for $m = 2$.

$m = 0$	$j = 2$	$j = 1$	$j = 0$
$m_1 = 1, m_2 = -1$	$1/\sqrt{6}$	$1/\sqrt{2}$	$1/\sqrt{3}$
$m_1 = 0, m_2 = 0$	$\sqrt{2}/\sqrt{3}$	0	$-1/\sqrt{3}$
$m_1 = -1, m_2 = 1$	$1/\sqrt{6}$	$-1/\sqrt{2}$	$1/\sqrt{3}$

$m = 1$	$j = 2$	$j = 1$
$m_1 = 1, m_2 = 0$	$1/\sqrt{2}$	$1/\sqrt{2}$
$m_1 = 0, m_2 = 1$	$1/\sqrt{2}$	$-1/\sqrt{2}$

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Hint: Use both the normalization condition and the recursive formulas given in the lectures for the Clebsch-Gordan coefficients.

Problem 12: Coupling of three angular momenta — 2+2 = 4 points

Consider the eigenvectors of the total angular momentum

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3,$$

where the individual angular momenta are all 1. An eigenvalue of $\boldsymbol{J}^2$ is written as $\hbar^2 j(j+1)$.

1. What are the possible values of j? How many linearly independent eigenstates belong to each possible value of j?
   Hint: the tensor product and the direct sum are associative and the tensor product distributes over the direct sum, $\mathfrak{h}_1 \otimes (\mathfrak{h}_2 \oplus \mathfrak{h}_3) = (\mathfrak{h}_1 \otimes \mathfrak{h}_2) \oplus (\mathfrak{h}_1 \otimes \mathfrak{h}_3)$
2. Construct the state with $j = 0$ explicitly. If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are vectors in $\mathbb{R}^3$, then there is one multilinear scalar, namely $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. Find a connection between this fact and your result for the state with $j = 0$.
   Hint: you may use the relation between cartesian and spherical components of a vector given in the lecture.

Submission date: Thursday, 23. November 2017, before the lecture begins