Friedrich-Schiller-Universität Jena Winter term 2018/19
Prof. Dr. Silvana Botti
Dr. Luca Zambelli
MSc. Alessandro Ugolotti

Problems in Advanced Quantum Theory

Sheet 1

Problem 1: Harmonic Oscillator in the Heisenberg Picture 2+2+2 (+2+2) points

Consider a one-dimensional harmonic oscillator with mass m and frequency ω.

1. Write the time-evolution equations for the operators $\hat{x}(t)$ and $\hat{p}(t)$ in the Heisenberg picture.
2. Solve the equations previously obtained, by writing $\hat{x}(t)$ and $\hat{p}(t)$ as functions of $\hat{x}(0)$, $\hat{p}(0)$ and t.
3. Write the time-evolution equation for the annihilation operator $\hat{a}(t)$ in the Heisenberg picture, and solve it.

Bonus question: Assume the oscillator has two point-like charges at its endpoints: $-e < 0$ at $x = 0$ and $+e > 0$ at x. Then, it has an electric dipole moment ex. In presence of a uniform but non-stationary electric field $E(t)$ in the x direction, its Hamiltonian contains also an interaction term of the form

$$H_{\text{int}} = -eE(t)x \,.$$

Compute again the answer to the previous tasks 1. and 2., for the case $E(t) = E_0 \cos \omega_0 t$.

Problem 2: Path Integral Propagator by Fourier Series 2+3+3 points

The path-integral representation of the propagator for the one-dimensional harmonic oscillator reads

$$K(x'', T; x', 0) = \int_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(t)\, e^{\mathrm{i}S[x(t)]/\hbar} \,,$$

where the action

$$S[x(t)] = \frac{m}{2} \int_0^T \mathrm{d}t\, x(t) \hat{A} x(t)$$

can be written in terms of the so-called fluctuation operator

$$\hat{A} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega^2 \,.$$

Call $x_{\text{cl}}(t)$ the classical path, fulfilling the classical equations of motion, with endpoints $x_{\text{cl}}(0) = x'$ and $x_{\text{cl}}(T) = x''$. Change variable of integration in the path integral from $x(t)$ to $q(t)$ by means of $x(t) = x_{\text{cl}}(t) + q(t)$.

1. Show that

$$K(x'', T; x', 0) = F(T) e^{\mathrm{i}S[x_{\mathrm{cl}}(t)]/\hbar},$$

where

$$F(T) = \int_{q(0)=0}^{q(T)=0} \mathcal{D}q(t)\, e^{\mathrm{i}S[q(t)]/\hbar}.$$

2. Find the eigenvalues λ_n and the orthonormal eigenfunctions $f_n(t)$ of the operator $\hat{A}$, fulfilling the Dirichlet boundary conditions $f_n(0) = f_n(T) = 0$.

3. Define $F_N(T)$ as the $F(T)$ given above, provided the integration is computed over the subset of functions of the form

$$q^{(N)}(t) = \sum_{n=1}^{N} a_n f_n(t), \quad a_n \in \mathbb{R}.$$

More precisely, we give the operational definition

$$F_N(T) = C_N \prod_{n=1}^{N} \int_{\mathbb{R}} \mathrm{d}a_n\, e^{\mathrm{i}S[q^{(N)}(t)]},$$

with C_N taken to reproduce the well-known result for the free particle (i.e. for $\omega = 0$)

$$C_N = \left[F_N^{-1}(T)\right]_{\omega=0} \sqrt{\frac{m}{2\pi \mathrm{i}\hbar T}}.$$

Then, compute $F(T) = \lim_{N\to\infty} F_N(T)$.

Hint: Each integral over a_n becomes Gaussian. To evaluate the limit, use the formula

$$\prod_{n=1}^{\infty} \left(1 - \left(\frac{x}{\pi n}\right)^2\right) = \frac{\sin x}{x}.$$

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