Friedrich-Schiller-Universität Jena Winter term 2018/19
Prof. Dr. Silvana Botti
Dr. Luca Zambelli
MSc. Alessandro Ugolotti

Problems in Advanced Quantum Theory

Sheet 9

Problem 14: Scattering Amplitude for a Nonlocal Potential 2+3+3+3(+5) points

The wavefunction of a particle which is scattered by the potential

$$U = \frac{2mV}{\hbar^2}, \tag{1}$$

fulfills the Lippmann-Schwinger equation

$$|\psi_{\boldsymbol{k}}^{(+)}\rangle = |\boldsymbol{k}\rangle + \hat{G}_+ U |\psi_{\boldsymbol{k}}^{(+)}\rangle, \tag{2}$$

where $|\boldsymbol{k}\rangle$ is an eigenstate of momentum, and G_+ is the inverse Helmholtz operator

$$\langle \boldsymbol{x}|\hat{G}_+|\boldsymbol{x}'\rangle = G_+(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{4\pi}\frac{e^{\mathrm{i}k|\boldsymbol{x}-\boldsymbol{x}'|}}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{3}$$

such that $G_+(\boldsymbol{x}, \boldsymbol{x}')$ is the corresponding Green's function. We correspondingly denote

$$\langle \boldsymbol{x}|U|\boldsymbol{x}'\rangle = U(\boldsymbol{x}, \boldsymbol{x}'). \tag{4}$$

In the lecture only local potentials were discussed, for which $U(\boldsymbol{x}, \boldsymbol{x}') = U(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}')$. We now generalize this discussion to nonlocal potentials, for which $U(\boldsymbol{x}, \boldsymbol{x}')$ is nonvanishing also when $\boldsymbol{x} \neq \boldsymbol{x}'$.

1. Write the Lippmann-Schwinger equation in position space, for a nonlocal potential.
2. In the special case of a separable nonlocal potential

$$U(\boldsymbol{x}, \boldsymbol{x}') = \lambda\, g(\boldsymbol{x}) g^*(\boldsymbol{x}') \tag{5}$$

   solve the Lippmann-Schwinger equation for $\langle g|\psi_{\boldsymbol{k}}^{(+)}\rangle$ as a function of $\langle g|\boldsymbol{k}\rangle$ and $\langle g|G_+|g\rangle$. Here the state $|g\rangle$ is defined as

$$|g\rangle = \int \mathrm{d}^3\boldsymbol{x}\, g(\boldsymbol{x})|\boldsymbol{x}\rangle. \tag{6}$$

3. The exact result of the previous task, can be cast in terms of the scattering amplitude

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{1}{4\pi}\langle \boldsymbol{k}'|U|\psi_{\boldsymbol{k}}^{(+)}\rangle, \tag{7}$$

where by energy conservation $k' = k$. Define the first-order Born-approximated scattering amplitude as

$$f_B(\boldsymbol{k}, \boldsymbol{k}') = -\frac{1}{4\pi} \langle \boldsymbol{k}' | U | \boldsymbol{k} \rangle, \tag{8}$$

and show that

$$f = \frac{f_B}{1 - \lambda \langle g | G_+ | g \rangle}. \tag{9}$$

4. Consider now the example of the Yamaguchi potential

$$g(\boldsymbol{x}) = \sqrt{\frac{m}{\pi \hbar^2}} \frac{1}{|\boldsymbol{x}|} e^{-\beta |\boldsymbol{x}|}. \tag{10}$$

   Compute f_B explicitly and show that this scattering amplitude is isotropic.

   Bonus question:

5. *Compute the exact scattering amplitude f for the case of the Yamaguchi potential.*

   Hint: you might perform the computation in momentum space. For this you need to recall from the lecture the momentum-space representation of G_+, i.e. $\langle \boldsymbol{k} | \hat{G}_+ | \boldsymbol{k}' \rangle$, and then use the result:

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \mathrm{d}k' \, \frac{k'^2}{(k^2 - k'^2 + \mathrm{i}\epsilon)(k'^2 + \beta^2)^2} = -\frac{\pi}{2\beta(\beta - \mathrm{i}k)^2}. \tag{11}$$

Submission date: Thursday, 17 January 2019