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Advanced Quantum Mechanics exercise series 2

Problem 3

Let $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ be the underlying hilbert space. Then we define the Hamiltonian $H : \mathcal{H} \to \mathcal{H}$ through

$$H := \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1, x_2)$$

with $V(x_1, x_2)$ as given potential operator and the property

$$V(x_1, x_2) = V(x_2, x_1)$$

Now assume $|\varphi\rangle \in \mathcal{H}$ is an eigenfunction of H with non-degenerated ✓ Energy $E \in \mathbb{R}$. In other words:

$$H|\varphi\rangle = E|\varphi\rangle$$

and for every other $|\psi\rangle \in \mathcal{H}$ with

$$H|\psi\rangle = E|\psi\rangle$$

it holds that $|\psi\rangle = \alpha|\varphi\rangle$ for an $\alpha \in \mathbb{C}$ ✓

Let us further define π_{12} as the transposition which interchanges 1 and 2 and $P_{12} := P(\pi_{12})$.

From definition we know the momentum operator $\frac{p_1^2}{2m} + \frac{p_2^2}{2m}$ and the potential operator $V(x_1, x_2)$ are symmetric. Therefore the Hamiltonian H has to be symmetric and

$$P_{12} H = H P_{12} \quad ✓$$

$\Longrightarrow$ $P_{12}(E|\varphi\rangle) = E P_{12}|\varphi\rangle = P_{12} H|\varphi\rangle = H P_{12}|\varphi\rangle$

$\Longrightarrow$ $P_{12}|\varphi\rangle$ is an eigenstate of H with energy E

$\Longrightarrow$ $P_{12}|\varphi\rangle = \alpha|\varphi\rangle$ for a constant $\alpha \in \mathbb{C}$

$\Longrightarrow$ $|\varphi\rangle$ is an eigenstate of P_{12} with eigenvalue $\alpha \in \mathbb{C}$ ✓

We know $\sigma(P_{12}) = \{-1, 1\}$. $\Longrightarrow$ $P_{12}|\varphi\rangle = \pm|\varphi\rangle$ $\square$

Auf 3: $\frac{2.0}{2.0}$

PERFECT!

P_{12} is a unitary operator. Therefore it preserves the inner product $\langle \cdot | \cdot \rangle$ of the hilbert space.

$$\langle x | P_{12}^{+} P_{12} | y \rangle = \langle x | y \rangle \quad \text{for all } x, y \in \mathcal{H}$$

$$\Rightarrow \quad \| P_{12} | \varphi \rangle \|^2 = \langle \varphi | P_{12}^{+} P_{12} | \varphi \rangle = \langle \varphi | \varphi \rangle = \| | \varphi \rangle \|^2$$

On the other hand: $\| P_{12} | \varphi \rangle \|^2 = \| \alpha | \varphi \rangle \|^2 = | \alpha |^2 \| | \varphi \rangle \|^2$
for $\alpha \in \mathbb{C}$

$$\Rightarrow \quad \| | \varphi \rangle \|^2 = | \alpha |^2 \| | \varphi \rangle \|^2 \quad \text{for an } \alpha \in \mathbb{C}$$

Now we choose $\alpha \in \mathbb{C}$ with $P_{12} | \varphi \rangle = \alpha | \varphi \rangle$ and $\| | \varphi \rangle \|^2 = | \alpha |^2 \| | \varphi \rangle \|^2$.

$$\Rightarrow \quad | \alpha |^2 = 1$$

<u>Problem 4</u>

Let $| \varphi(0) \rangle$ be the wave function at time 0 with

$$P(\pi) | \varphi(0) \rangle = (\operatorname{sgn} \pi)^{\alpha} | \varphi(0) \rangle \quad \text{for } \alpha \in \{0, 1\}, \pi \in S_n, n \in \mathbb{N}$$

We assume the time evolution is given by the operators $U(t)$ for all times $t \in (0, \infty)$, so that

$$| \varphi(t) \rangle := U(t) | \varphi(0) \rangle \quad \text{for all } t \in (0, \infty)$$

is the wave function of the system at time t.

Then we know that $U(t)$ is symmetric for all $t \in (0, \infty)$, hence $P(\pi) U(t) = U(t) P(\pi)$ for all $t \in (0, \infty)$, $\pi \in S_n$.

$$\Rightarrow \quad P(\pi) | \varphi(t) \rangle = P(\pi) U(t) | \varphi(0) \rangle = U(t) P(\pi) | \varphi(0) \rangle$$
$$= U(t) \left[(\operatorname{sgn} \pi)^{\alpha} | \varphi(0) \rangle \right] = (\operatorname{sgn} \pi)^{\alpha} U(t) | \varphi(0) \rangle$$
$$= (\operatorname{sgn} \pi)^{\alpha} | \varphi(t) \rangle$$

$\Rightarrow$ $| \varphi(t) \rangle$ is symmetric or antisymmetric for all $t \in (0, \infty)$

PERFECT

Auf 4: $1.0 / 1.0$

<u>Problem 5</u>

(1) Let $L^2(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid \int |f|^2 d\lambda < \infty \}$ be the space of square-integrable functions with domain $\mathbb{R}$. For three particles with spin equal to 0 we need

$$\mathcal{H} := L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \cong L^2(\mathbb{R}^3)$$

as a superset of our Hilbert space to denote the position of every particle. Because the spin is 0 the three identical particles are Bosons. Therefore their wave functions have to be symmetric.

$\Rightarrow$ $\mathcal{H}_s := \{ |\psi\rangle \in \mathcal{H} \mid P(\pi)|\psi\rangle = |\psi\rangle$ for all $\pi \in S_3 \} \subset \mathcal{H}$ ✓

with S_3 as the three dimensional permutation group and $P(\pi)$ as the related operator for $\pi \in S_3$

is the hilbert space of the system. ✓

(2) The particles do not interact.

$\Rightarrow$ $H = \sum_{k=1}^{3} H^{(k)}$ is the Hamiltonian of the system

with $H : \mathcal{H}_s \to \mathcal{H}_s$.

Now we assume $\{ |n\rangle \in L^2(\mathbb{R}) \mid n \in \mathbb{N} \}$ is an orthonormal eigenbasis of $H^{(1)}$. Therefore it is an eigenbasis of $H^{(2)}, H^{(3)}$ because they are identical.

$\Rightarrow$ $\{ |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \mid n_1, n_2, n_3 \in \mathbb{N} \}$ forms orthonormal eigenbasis of H with domain $\mathcal{H}$. But these states are not symmetric.

$\rightarrow$ $|n_1 n_2 n_3\rangle_S := \sum_{\pi \in S_3} P(\pi) |n_1 n_2 n_3\rangle = \sum_{\pi \in S_3} |n_{\pi(1)} n_{\pi(2)} n_{\pi(3)}\rangle$ ✓

is symmetric and an eigenvector of H with domain $\mathcal{H}_s$.

$\rightarrow$ $\{ |n_1 n_2 n_3\rangle_S \mid n_1, n_2, n_3 \in \mathbb{N} \}$ is an eigenbasis which is not normalized ✓

case $n_1 = n_2 = n_3$: $\quad |n_1 n_2 n_3\rangle_{\tilde{S}} := \frac{1}{3!} |n_1 n_2 n_3\rangle_S = |n_1 n_2 n_3\rangle$

$$\Rightarrow \quad \| |n_1 n_2 n_3\rangle_{\tilde{S}} \| = 1 \qquad ✓$$

case $n_1 = n_2 \neq n_3$: $\quad |n_1 n_2 n_3\rangle_{\tilde{S}} := \frac{1}{\sqrt{3! \cdot \cancel{2!}}} |n_1 n_2 n_3\rangle_S \qquad (-0{,}25)$

$$\Rightarrow \quad \| |n_1 n_2 n_3\rangle_{\tilde{S}} \| = 1$$

case $n_1 \neq n_2 \neq n_3 \neq n_1$: $\quad |n_1 n_2 n_3\rangle_{\tilde{S}} := \frac{1}{\sqrt{3!}} |n_1 n_2 n_3\rangle_S$

$$\Rightarrow \quad \| |n_1 n_2 n_3\rangle_{\tilde{S}} \| = 1$$

$\Rightarrow \quad \{ |n_1 n_2 n_3\rangle_{\tilde{S}} \mid n_1, n_2, n_3 \in \mathbb{N} \}$ is an orthonormal eigenbasis of H with domain $\mathcal{H}_S$

(3) We know there are constants $E_0, \varphi_0, \omega \in \mathbb{R}$ such that

$$H^{(1)} |n\rangle = E_n |n\rangle \quad \text{with} \quad E_n = E_0 n^2$$

$$\varphi_n(x) = \varphi_0 \sin(\omega n x) \mathbb{1}_{[-a,a]}(x), \quad \varphi_n = |n\rangle$$

for all $n \in \mathbb{N}$

$$\Rightarrow \quad H |n_1 n_2 n_3\rangle_{\tilde{S}} = E_{n_1 n_2 n_3} |n_1 n_2 n_3\rangle_{\tilde{S}}$$

with $\quad E_{n_1 n_2 n_3} = E_{n_1} + E_{n_2} + E_{n_3} = E_0 (n_1^2 + n_2^2 + n_3^2)$

for all $n_1, n_2, n_3 \in \mathbb{N}$

five eigenstates with lowest energies:

$|111\rangle_{\tilde{S}}$: $\quad E_{1,1,1} = 3E_0$

$|112\rangle_{\tilde{S}}$: $\quad E_{1,1,2} = 6E_0$

$|122\rangle_{\tilde{S}}$: $\quad E_{1,2,2} = 9E_0$ ✓

$|113\rangle_{\tilde{S}}$: $\quad E_{1,1,3} = 11E_0$

$|222\rangle_{\tilde{S}}$: $\quad E_{2,2,2} = 12E_0$

Auf 5: $\frac{5{,}75}{6{,}00}$

Problem 6

Let $S_R^2 := \{ x \in \mathbb{R}^3 \mid \|x\| = R \}$ be the surface of the ball with Radius $R \in \mathbb{R}^+$ in $\mathbb{R}^3$. In that case the Hilbert space for one particle is given by

$$\mathcal{H} := L^2(S_R^2) \otimes \mathbb{C}^2$$

with $L^2(S_R^2) := \{ f : S_R^2 \to \mathbb{C} \mid \int |f|^2 d\sigma < \infty \}$

as the space of square-integrable functions on the sphere

The Hamiltonian of this particle can be defined by

$$H := \frac{p^2}{2m} = \frac{\hbar^2}{2m} \Delta$$

where $m \in \mathbb{R}^+$ is the mass of the particle. Now let $Y_{lp} : S_R^2 \to \mathbb{C}$ be the spherical harmonics for $l \in \mathbb{N}_0$ and $p \in \mathbb{Z}$, $|p| \le l$. Then we know

$$\Delta Y_{lp} = \frac{l(l+1)}{R^2} Y_{lp} \qquad \text{for all } l \in \mathbb{N}_0, p \in \mathbb{Z}, |p| \le l$$

$\Rightarrow$ Y_{lp} are eigenstates of H

$$\frac{\hbar^2}{2m} \Delta Y_{lp} = \frac{\hbar^2}{2mR^2} l(l+1) Y_{lp} =: E_{lp} Y_{lp} \checkmark$$

with $E_{lp} := \frac{\hbar^2}{2mR^2} l(l+1)$ for all $l \in \mathbb{N}, p \in \mathbb{Z}, |p| \le l$

note: for $l = 0$ we do not have a valid eigenstate because of $E_{00} = 0$ Warum? (−0,5)

Now consider the Hilbert space for $n \in \mathbb{N}$ identical particles. (fermions)

$$\mathcal{H}_a^n := \{ |\varphi\rangle \in \mathcal{H}^n \mid P(\pi)|\varphi\rangle = \mathrm{sgn}(\pi)|\varphi\rangle \text{ for all } \pi \in S_n \}$$

$\Rightarrow$ every fermion has to be in another state (Pauli Principle)

We want to find the smallest degree $l \in \mathbb{N}$ for the energy $E := 42 \frac{\hbar^2}{2mR^2}$:

$$E_f(n) := \sum_{l=1}^{n} \sum_{p=-l}^{l} 2 E_{lp} = \frac{\hbar^2}{2mR^2} \sum_{l=\cancel{1}0}^{n} \sum_{p=-l}^{l} 2l(l+1) \checkmark$$

$$= \frac{\hbar^2}{2mR^2} \sum_{l=\cancel{1}0}^{n} 2l(l+1)(2l+1) \checkmark = \frac{\hbar^2}{2mR^2} 2 \sum_{l=1}^{n} 2l^3 + 3l^2 + l$$

$$\Rightarrow E_f(n) = \frac{\hbar^2}{2mR^2} \, 2 \sum_{l=1}^{n} 2l^3 + 3l^2 + l$$

$$= \frac{\hbar^2}{2mR^2} \cdot 2 \cdot \left[\frac{1}{2} n^2(n+1)^2 + \frac{1}{2} n(n+1)(2n+1) + \frac{1}{2} n(n+1) \right]$$

$$= \frac{\hbar^2}{2mR^2} \cdot n(n+1) \left[n(n+1) + 2n+1 + 1 \right]$$

$$= \frac{\hbar^2}{2mR^2} \cdot n(n+1)^2(n+2)$$ ✓

$E_f(n)$ describes the energy of the system with the identical particles occupying the first n degrees.

$$\Rightarrow E_f(1) = \frac{12\hbar^2}{2mR^2}, \quad E_f(2) = \frac{72\hbar^2}{2mR^2}$$

$\Rightarrow E_f(1) < E < E_f(2) \Rightarrow$ first degree of system is occupied

$\Rightarrow$ second degree is partially occupied

In the second degree every Fermion of the system increments the energy by $E_{2m} = \frac{6\hbar^2}{2mR^2}$, $m \in \{-2,-1,0,1,2\}$

$\Rightarrow$ number of Fermions in second degree: $n_2 = \frac{E - E_f(1)}{E_{2m}}$

$$n_2 = \frac{30\hbar^2}{2mR^2} \Big/ \frac{6\hbar^2}{2mR^2} = 5$$

number of fermions in (full) first degree: $n_1 = 6$

$$\left(n_1 = \frac{E_f(1)}{E_{1m}}, \quad m \in \{-1,0,1\} \right)$$

$\Rightarrow$ number of fermions on sphere: $n = n_1 + n_2 = \underline{\underline{11}}$ ✓

+ 2 PARTICLES IN THE $l=0$ STATE

$$H = \frac{\vec{L}^2}{2mR}$$

Auf 6: $2{,}5/3{,}0$