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Advanced Quantum Theory
Exercise Sheet 8

Problem 13

(1) The Hamiltonian of unperturbed Hydrogen atom is given by

$$H_0 := -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{4\pi\varepsilon_0 r} = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}$$

The Hamiltonian of the perturbed System is then given by (up to first order)

$$H = H_0 + V = H_0 - \frac{e}{m_e c} A\cdot p$$

with $$A := A_0 \varepsilon \left[\underbrace{e^{i\omega t - i\frac{\omega}{c} n\cdot x}}_{\text{emission part}} + \underbrace{e^{-i\omega t + \frac{i\omega}{c} n\cdot x}}_{\text{absorption part}} \right]$$

In our case V is a time-dependent harmonic perturbation of H_0. Therefore we can write the transition rate $\omega_{i\to f}$ for the photon emission in the following way (based on the lecture)

$$\omega_{i\to f} = \frac{2\pi}{\hbar} |\langle f|V_E|i\rangle|^2 \delta(E_f - E_i + \hbar\omega)$$
$$= \frac{2\pi}{\hbar^2} |\langle f|V_E|i\rangle|^2 \delta(\omega_{fi} + \omega)$$

with $V_E := -\frac{e}{m_e c} A_0 e^{i\omega t - i\frac{\omega}{c}n\cdot x} \varepsilon\cdot p \quad , \quad \omega_{fi} := \omega_f - \omega_i$

$$\Rightarrow \omega_{i\to f} = \frac{2\pi}{\hbar^2}\frac{e^2}{m_e^2 c^2} A_0^2 \underbrace{|\langle f| e^{i\omega t} e^{-i\frac{\omega}{c}n\cdot x} \varepsilon\cdot p|i\rangle|^2}_{} \delta(\omega_{fi}+\omega)$$
$$= |\langle f| e^{-i\frac{\omega}{c}n\cdot x} \varepsilon\cdot p|i\rangle|^2$$
$$\approx |\langle f|\varepsilon\cdot p|i\rangle|^2 = |\langle f|p|i\rangle\cdot\varepsilon|^2$$

$$\Rightarrow \omega_{i\to f} = \frac{2\pi}{\hbar^2}\frac{e^2}{m_e^2 c^2} A_0^2 \delta(\omega_{fi}+\omega) |\langle f|p|i\rangle\cdot\varepsilon|^2$$

(2) We compute: $[x_i, \partial_j^2] = \partial_j[x_i,\partial_j] + [x_i,\partial_j]\partial_j = -2\delta_{ij}\partial_j$

$$[x_i, \tfrac{1}{r}] = 0$$

$$\Rightarrow [x, H_0] = -\frac{\hbar^2}{2m}[x,\Delta] = -\frac{\hbar^2}{2m}\sum_{j=1}^{3}[x,\partial_j^2] = \frac{\hbar^2}{m}\nabla$$
$$= \frac{i\hbar}{m} p \quad \Rightarrow \quad p = \frac{m}{i\hbar}[x, H_0]$$

$$\Rightarrow \langle f|p|i\rangle = \frac{m}{i\hbar}\langle f|[x,H_0]|i\rangle = \frac{m}{i\hbar}\langle f|(xH_0 - H_0x)|i\rangle$$

$$= \frac{m}{i\hbar}\Big[\underbrace{\langle f|xH_0|i\rangle}_{E_i\langle f|x|i\rangle} - \underbrace{\langle f|H_0x|i\rangle}_{E_f\langle f|x|i\rangle}\Big] \qquad (|f\rangle, |i\rangle \text{ are eigenstates of } H_0)$$

$$= \frac{m}{i\hbar}(E_i - E_f)\langle f|x|i\rangle$$

$$= \frac{m}{i}(\omega_i - \omega_f)\langle f|x|i\rangle = im\,\omega_{fi}\langle f|x|i\rangle$$

$$\Rightarrow \omega_{i\to f} = \frac{2\pi}{\hbar^2}\frac{e^2}{m_e^2c^2}A_0^2\,\delta(\omega_{fi}+\omega)\,|\varepsilon\cdot im\,\omega_{fi}\langle f|x|i\rangle|^2$$

$$= \frac{2\pi}{\hbar^2}\frac{e^2m^2}{m_e^2c^2}A_0^2\,\omega_{fi}^2\,\delta(\omega_{fi}+\omega)\,|\varepsilon\cdot\langle f|x|i\rangle|^2$$

(3)

$$x = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

We already know:

$$Y_1^{-1}(\vartheta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\vartheta\, e^{-i\varphi}$$

$$Y_1^{0}(\vartheta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\vartheta$$

$$Y_1^{1}(\vartheta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\vartheta\, e^{i\varphi}$$

Based on

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

we use

$$\cos\varphi = \frac{1}{2}(e^{i\varphi}+e^{-i\varphi}), \quad \sin\varphi = \frac{1}{2i}(e^{i\varphi}-e^{-i\varphi})$$

$$\Rightarrow \cos\vartheta = 2\sqrt{\frac{\pi}{3}}\,Y_1^0(\vartheta,\varphi)$$

$$\sin\vartheta\cos\varphi = \frac{1}{2}\sin\vartheta\,e^{i\varphi} + \frac{1}{2}\sin\vartheta\,e^{-i\varphi}$$

$$= \sqrt{\frac{2\pi}{3}}\left[Y_1^{-1}(\vartheta,\varphi) - Y_1^{1}(\vartheta,\varphi)\right]$$

$$\sin\vartheta\sin\varphi = \frac{1}{2i}\sin\vartheta\,e^{i\varphi} - \frac{1}{2i}\sin\vartheta\,e^{-i\varphi}$$

$$= i\sqrt{\frac{2\pi}{3}}\left[Y_1^{-1}(\vartheta,\varphi) + Y_1^{1}(\vartheta,\varphi)\right]$$

$$\Rightarrow x = \sqrt{\frac{2\pi}{3}}\,r\begin{pmatrix} Y_1^{-1} - Y_1^{1} \\ iY_1^{-1} + iY_1^{1} \\ \sqrt{2}\,Y_1^0 \end{pmatrix}(\vartheta,\varphi)$$

(4) $[z,L_z] = 0 \Rightarrow 0 = \langle l'm'|[z,L_z]|lm\rangle = \hbar(m-m')\langle l'm'|z|lm\rangle$

$\Rightarrow (\langle l'm'|z|lm\rangle \neq 0 \Rightarrow m = m')$

$$\left.\begin{aligned}[x,L_z] &= -i\hbar y\\ [y,L_z] &= i\hbar x\end{aligned}\right\} \Rightarrow \begin{aligned}\langle l'm'|[x,L_z]|lm\rangle &= \hbar(m-m')\langle l'm'|x|lm\rangle\\ &= -i\hbar\langle l'm'|y|lm\rangle\\ \langle l'm'|[y,L_z]|lm\rangle &= \hbar(m-m')\langle l'm'|y|lm\rangle\\ &= i\hbar\langle l'm'|x|lm\rangle\end{aligned}$$

$$\Rightarrow i\underbrace{\langle l'm'|x|lm\rangle}_{\neq 0} = i(m-m')^2\langle l'm'|x|lm\rangle \Rightarrow (m-m')^2 = 1 \Rightarrow m' = m\pm 1$$

$\Rightarrow$ selection rules for m': $m' \in \{m-1, m, m+1\}$