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Advanced Quantum Theory
Exercise Sheet 7

Problem 12

(1)

$t<0: \quad |n(t)\rangle = |n^{(0)}\rangle$

$$\Rightarrow D(t) = \langle n^{(0)}|\vec{d}_I(t)|n^{(0)}\rangle = \langle n^{(0)}| e^{+\frac{i}{\hbar}H_0 t}\vec{d}\, e^{-\frac{i}{\hbar}H_0 t}|n^{(0)}\rangle$$
$$= e^{i\omega_n t} e^{-i\omega_n t}\langle n^{(0)}|\vec{d}|n^{(0)}\rangle = \vec{d}_{nn} = 0$$

$t\geq 0: \quad |n(t)\rangle = |n^{(0)}\rangle + \underbrace{|n^{(1)}(t)\rangle}_{\text{first-order correction}}, \quad |n^{(1)}(t)\rangle = \sum_{m\in I} c_m^{(1)}(t)|m^{(0)}\rangle$

$$\Rightarrow D(t) = \underbrace{\langle n^{(0)}|\vec{d}_I(t)|n^{(0)}\rangle}_{=0} + 2\langle n^{(0)}|\vec{d}_I(t)|n^{(1)}(t)\rangle + \underbrace{\langle n^{(1)}(t)|\vec{d}_I(t)|n^{(1)}(t)\rangle}$$

(will be neglected, because it belongs to second-order approximation)

$$\approx 2\langle n^{(0)}|\vec{d}_I(t)|n^{(1)}(t)\rangle$$
$$= 2\sum_{m\in I} c_m^{(1)}(t)\langle n^{(0)}|\vec{d}_I(t)|m^{(0)}\rangle$$
$$= 2\sum_{m\in I} c_m^{(1)}(t)\langle n^{(0)}| e^{i/\hbar H_0 t}\vec{d}\, e^{-i/\hbar H_0 t}|m^{(0)}\rangle$$
$$= 2\sum_{m\in I} c_m^{(1)}(t)\, e^{i(\omega_n-\omega_m)t}\langle n^{(0)}|\vec{d}|m^{(0)}\rangle$$
$$= 2\sum_{m\in I} c_m^{(1)}(t)\, e^{i\omega_{nm}t}\vec{d}_{nm}$$

(2) Based on time-dependent perturbation theory: $c_m^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{mn}\tilde{t}}\langle m^{(0)}|V(\tilde{t})|n^{(0)}\rangle\, d\tilde{t}$

$V(t) = \vartheta(t)\left[F^{+}e^{i\omega t} + Fe^{-i\omega t}\right], \quad F := -\frac{1}{2}\vec{d}\cdot\vec{E}_0 \Rightarrow F^{+} = F$

$$\Rightarrow V(t) = \vartheta(t)\, F\left[e^{i\omega t} + e^{-i\omega t}\right]$$
$$\Rightarrow \langle m^{(0)}|V(t)|n^{(0)}\rangle = \vartheta(t)\left[e^{i\omega t}+e^{-i\omega t}\right]\langle m^{(0)}|F|n^{(0)}\rangle$$
$$= -\frac{1}{2}\vartheta(t)\left[e^{i\omega t}+e^{-i\omega t}\right](\vec{d}_{mn}\cdot\vec{E}_0)$$

$(t_0<0)$

$$\Rightarrow c_m^{(1)}(t) = \frac{i}{2\hbar}\int_0^t e^{i\omega_{mn}\tilde{t}}\left(e^{i\omega\tilde{t}}+e^{-i\omega\tilde{t}}\right)d\tilde{t}\,\left[\vec{d}_{mn}\cdot\vec{E}_0\right]$$
$$= \frac{i}{2\hbar}\left[\vec{d}_{mn}\cdot\vec{E}_0\right]\cdot\frac{1}{i}\left[\frac{e^{i(\omega_{mn}+\omega)t}-1}{\omega_{mn}+\omega} + \frac{e^{i(\omega_{mn}-\omega)t}-1}{\omega_{mn}-\omega}\right]$$

$$\Rightarrow D(t) = \frac{1}{\hbar}\sum_{m\in I}\vec{d}_{nm}\left[\vec{d}_{mn}\cdot\vec{E}_0\right]\left[\frac{e^{i\omega t}-e^{-i\omega_{mn}t}}{\omega_{mn}+\omega} + \frac{e^{-i\omega t}-e^{-i\omega_{mn}t}}{\omega_{mn}-\omega}\right]$$

(3) $\frac{\omega_{mn}}{\omega} \ll 1 \quad \Longrightarrow \quad D(t) \approx \frac{1}{\hbar} \sum_{m \in I} \vec{d}_{nm} \left[\vec{d}_{mn} \cdot \vec{E}_0 \right] \left[\frac{e^{i\omega t} - e^{-i\omega_{mn} t}}{\omega} + \frac{e^{-i\omega t} - e^{-i\omega_{mn} t}}{-\omega} \right]$

$$\Longrightarrow \quad D(t) \approx \frac{1}{\hbar\omega} \sum_{m \in I} \vec{d}_{nm} \underbrace{\left[\vec{d}_{mn} \cdot \vec{E}_0 \right]}_{= \sum_{j=1}^{3} d_{mn_j} \cdot E_{0_j}} \underbrace{\left(e^{i\omega t} - e^{-i\omega t} \right)}_{= 2i \sin(\omega t)}$$

$$\Longrightarrow \quad D_i(t) \approx \frac{1}{\hbar\omega} \sum_{m \in I} d_{nm_i} \, 2i \sin(\omega t) \sum_{j=1}^{3} d_{mn_j} \cdot E_{0_j}$$

$$= \sum_{j=1}^{3} \underbrace{E_{0_j} \cdot \sin(\omega t)}_{E_j(t)} \cdot \underbrace{\frac{2i}{\hbar\omega} \sum_{m \in I} d_{nm_i} \cdot d_{mn_j}}_{\alpha_{ij}} = \sum_{j=1}^{3} \alpha_{ij} \cdot E_j(t)$$