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Advanced Quantum Mechanics
Exercise Series 05

Problem 11

The coupled system can be described through $h_{j_1} \otimes h_{j_2}$. We define $\{ |m_1 m_2\rangle_S \mid m_1 m_2 \in \mathbb{Z}, |m_1| \leq j_1, |m_2| \leq j_2 \} =: S$ be the orthonormal basis of the Hilbert space with respect to the projections of the uncoupled angular momenta.

Analog we define $\{ |jm\rangle_T \mid j \in \mathbb{N}, |j_1 - j_2| \leq j \leq j_1 + j_2, m \in \mathbb{Z}, |m| \leq j \} =: T$

to be an orthonormal basis with respect to the total angular momentum and its projection.

In this case the Clebsch-Gordan-coefficients describe the basis transformation from S to T and the following recursion formulas hold

$$c^{\pm}_{jm} \langle m_1 m_2 |_S | j\, m \pm 1\rangle_T = c^{\mp}_{j_1 m_1} \langle m_1 \mp 1\, m_2 |_S | jm\rangle_T + c^{\mp}_{j_2 m_2} \langle m_1 m_2 \mp 1 |_S | jm\rangle_T$$

There is only one state in S which can contibute to the state in T with the maximum total angulor momentum. Here we use the normalization condition. $(j_1 = j_2 = 1)$

$$\Rightarrow \quad |22\rangle_T = |11\rangle_S \quad \Rightarrow \quad \langle 11|_S |22\rangle_T = 1$$

Now use second recursion:

$$c^{-}_{jm} \langle m_1 m_2 |_S | j\, m-1\rangle_T = c^{+}_{1 m_1} \langle m_1 + 1\, m_2 |_S | jm\rangle_T + c^{+}_{1 m_2} \langle m_1 m_2 + 1 |_S | jm\rangle_T$$

case $j = 2, m = 2$:

$$c^{-}_{22} \langle m_1 m_2 |_S | 21\rangle_T = c^{+}_{1 m_1} \langle m_1 + 1\, m_2 |_S | 22\rangle_T + c^{+}_{1 m_2} \langle m_1 m_2 + 1 |_S | 22\rangle_T$$

We want to compute coefficients which are not equal to zero:

$$\Rightarrow \quad m_1 + m_2 + 1 = 2 \quad \Rightarrow \quad m_2 = 1 - m_1$$

case $m_1 = 0,\ m_2 = 1$: $c_{22}^- \langle 01|_S |21\rangle_T = c_{10}^+ \langle 11|_S |22\rangle_T$

$$\Rightarrow \langle 01|_S |21\rangle_T = \frac{c_{10}^+}{c_{22}^-} \langle 11|_S |22\rangle_T = \frac{1}{\sqrt{2}}$$

case $m_1 = 1,\ m_2 = 0$: $c_{22}^- \langle 10|_S |21\rangle_T = c_{10}^+ \langle 11|_S |22\rangle_T$

$$\Rightarrow \langle 10|_S |21\rangle_T = \frac{c_{10}^+}{c_{22}^-} \langle 11|_S |22\rangle_T = \frac{1}{\sqrt{2}}$$

case $j = 2,\ m = 1$: (analog)

$m_1 + m_2 + 1 = 1 \Rightarrow m_1 + m_2 = 0 \Rightarrow m_2 = -m_1$

case $m_1 = 0 = m_2$:

$$c_{21}^- \langle 00|_S |20\rangle_T = c_{10}^+ \langle 10|_S |21\rangle_T + c_{10}^+ \langle 01|_S |21\rangle_T$$

(using previous equation) $\Rightarrow \langle 00|_S |20\rangle_T = \frac{c_{10}^+}{c_{21}^-} \left(\langle 10|_S |21\rangle_T + \langle 01|_S |21\rangle_T \right) = \frac{\sqrt{2}}{\sqrt{3}}$

case $m_1 = 1,\ m_2 = -1$: $c_{21}^- \langle 1\,-1|_S |20\rangle_T = c_{1-1}^+ \langle 10|_S |21\rangle_T$

$$\Rightarrow \langle 1\,-1|_S |20\rangle_T = \frac{c_{1-1}^+}{c_{21}^-} \langle 10|_S |21\rangle_T = \frac{1}{\sqrt{6}}$$

case $m_1 = -1,\ m_2 = 1$: $c_{21}^- \langle -11|_S |20\rangle_T = c_{1-1}^+ \langle 01|_S |21\rangle_T$

$$\Rightarrow \langle -11|_S |20\rangle_T = \frac{c_{1-1}^+}{c_{21}^-} \langle 01|_S |21\rangle_T = \frac{1}{\sqrt{6}}$$

case $j = 1,\ m = 1$: $\Rightarrow m_1 + m_2 = 1 \Rightarrow m_2 = 1 - m_1$

$\Rightarrow |11\rangle_T$ can only depend on $|10\rangle_S$ and $|01\rangle_S$

$$\Rightarrow |11\rangle_T = \langle 10|_S |11\rangle_T \,|10\rangle_S + \langle 01|_S |11\rangle_T \,|01\rangle_T$$

$|11\rangle_T$ has to be orthogonal to other states. If we want to gain some information we have to use $|21\rangle_T$.

$$\Rightarrow 0 = \langle 11|_T |21\rangle_T = \langle 10|_s|21\rangle_T \langle 10|_s|11\rangle_T + \langle 01|_s|21\rangle_T \langle 01|_s|11\rangle_T$$

Of course $|11\rangle_T$ should be normalized: $\langle 11|_T|11\rangle_T = 1$

$$\Rightarrow 1 = |\langle 10|_s|11\rangle_T|^2 + |\langle 01|_s|11\rangle_T|^2$$

$$\Rightarrow \langle 01|_s|11\rangle_T = -\frac{\langle 10|_s|21\rangle_T}{\langle 01|_s|21\rangle_T}\langle 10|_s|11\rangle_T = -\langle 10|_s|11\rangle_T$$

$$\Rightarrow |\langle 10|_s|11\rangle_T|^2 = \frac{1}{2} \overset{\text{(choose)}}{\Rightarrow} \langle 10|_s|11\rangle_T = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \langle 01|_s|11\rangle_T = -\frac{1}{\sqrt{2}}$$

After this construction we are able to use recursion again.

case $m_1 = 0 = m_2$:

$$c_{11}^- \langle 00|_s|10\rangle_T = c_{10}^+ \langle 10|_s|11\rangle_T + c_{10}^+ \langle 01|_s|11\rangle_T$$

$$\Rightarrow \langle 00|_s|10\rangle_T = 0$$

case $m_1 = 1, m_2 = -1$: $c_{11}^- \langle 1\,-1|_s|10\rangle_T = c_{1-1}^+ \langle 10|_s|11\rangle_T$

$$\Rightarrow \langle 1\,-1|_s|10\rangle_T = \frac{c_{1-1}^+}{c_{11}^-}\langle 10|_s|11\rangle_T = \frac{1}{\sqrt{2}}$$

case $m_1 = -1, m_2 = 1$: $c_{11}^- \langle -11|_s|10\rangle_T = c_{1-1}^+ \langle 01|_s|11\rangle_T$

$$\Rightarrow \langle -11|_s|10\rangle_T = \frac{c_{1-1}^+}{c_{11}^-}\langle 01|_s|11\rangle_T = -\frac{1}{\sqrt{2}}$$

case $j = 0, m = 0$:

Again we have to construct an orthogonal normalized state $|00\rangle_T$.
$|00\rangle_T$ can only depend on $|1\,-1\rangle_s$, $|-11\rangle_s$, $|00\rangle_s$.

$$\Rightarrow (1) \quad 1 = \langle 00|_T|00\rangle_T = |\langle 1\,-1|_s|00\rangle_T|^2 + |\langle -11|_s|00\rangle_T|^2 + |\langle 00|_s|00\rangle_T|^2$$

$$\begin{aligned}(2) \quad 0 = \langle 00|_T|10\rangle_T = &\langle 1\,-1|_s|00\rangle_T \langle 1\,-1|_s|10\rangle_T \\ &+ \langle -11|_s|00\rangle_T \langle -11|_s|10\rangle_T \\ &+ \langle 00|_s|00\rangle_T \langle 00|_s|10\rangle_T\end{aligned}$$

(3) $0 = \langle 00|_T |20\rangle_T = \langle 1\,{-1}|_S |00\rangle_T \; \langle 1\,{-1}|_S |20\rangle_T$
$+ \langle -1\,1|_S |00\rangle_T \; \langle -1\,1|_S |20\rangle_T$
$+ \langle 00|_S |00\rangle_T \; \langle 00|_S |20\rangle_T$

$\overset{(2)}{\Longrightarrow} \quad \langle 1\,{-1}|_S |00\rangle_T = \langle -1\,1|_S |00\rangle_T$

$\overset{(3)}{\Longrightarrow} \quad \langle 00|_S |00\rangle_T = - \langle 1\,{-1}|_S |00\rangle_T$

$\underset{\text{(choose)}}{\overset{(1)}{\Longrightarrow}} \quad \langle 1\,{-1}|_S |00\rangle_T = \frac{1}{\sqrt{3}} = \langle -1\,1|_S |00\rangle_T$

$\langle 00|_S |00\rangle_T = -\frac{1}{\sqrt{3}}$

Conclusion:

$m=2$	$j=2$
$m_1 = m_2 = 1$	1

$m=1$	$j=2$	$j=1$
$m_1 = 1,\ m_2 = 0$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$m_1 = 0,\ m_2 = 1$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

$m=0$	$j=2$	$j=1$	$j=0$
$m_1 = 1,\ m_2 = -1$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$
$m_1 = 0,\ m_2 = 0$	$\frac{\sqrt{2}}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$
$m_1 = -1,\ m_2 = 1$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$

Problem 12

(1) The hilbert space is given by $h_{j_1} \otimes h_{j_2} \otimes h_{j_3}$ with $j_1 = j_2 = j_3 = 1$.

We define the orthonormal basis

$$S := \{ |m_1 m_2 m_3\rangle_S \mid m_1, m_2, m_3 \in \mathbb{Z},\ |m_1| \leq j_1,\ |m_2| \leq j_2,\ |m_3| \leq j_3 \}$$

with respect to the projections of the angular momenta of the uncoupled system.

We already know $h_{j_1} \otimes h_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} h_j$

$$\Rightarrow h_{j_1} \otimes h_{j_2} \otimes h_{j_3} = h_{j_1} \otimes \left[\bigoplus_{k=|j_2-j_3|}^{j_2+j_3} h_k \right]$$

$$= \bigoplus_{k=|j_2-j_3|}^{j_2+j_3} h_{j_1} \otimes h_k = \bigoplus_{k=|j_2-j_3|}^{j_2+j_3} \bigoplus_{j=|j_1-k|}^{j_1+k} h_j$$

$j_1 = j_2 = j_3 = 1$:

$$\Rightarrow h_{j_1} \otimes h_{j_2} \otimes h_{j_3} = \bigoplus_{k=0}^{2} \bigoplus_{j=|1-k|}^{1+k} h_j$$

Think of k as the total angular momentum number of particles 2 and 3!

$$= \bigoplus_{j=1}^{1} h_j \oplus \bigoplus_{j=0}^{2} h_j \oplus \bigoplus_{j=1}^{3} h_j$$

$$= h_0 \oplus [h_1 \oplus h_1 \oplus h_1] \oplus [h_2 \oplus h_2] \oplus h_3$$

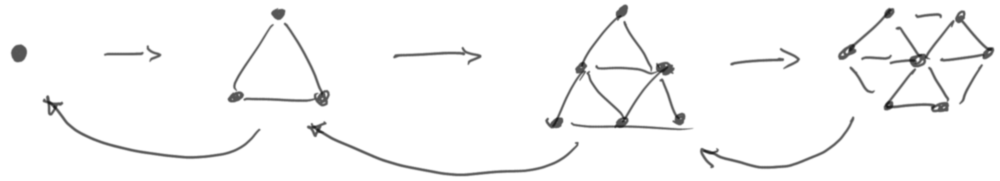
$\Rightarrow j \in \mathbb{N}_0,\ j \leq 3$ possible values of j

$j = 0$: only one h_0 $\Rightarrow$ $2j+1$ linear independent eigenstates $\Rightarrow$ 1 linear independent eigenstates

$j = 1$: 3 times h_1 $\Rightarrow$ $3(2j+1) = 9$ linear indep. eigenstates

$j = 2$: 2 times h_2 $\Rightarrow$ $2(2j+1) = 10$ linear indep. eigenstates

$j = 3$: only one h_3 $\Rightarrow$ $2j+1 = 7$ linear indep. eigenstates



1 3 6 7

cutting planes for 3 dimensional cube

(2) Now define orthonormal basis with respect to total angular momentum and total angular momentum of particles 2 and 3 and projection of total angular momentum:

$$T := \{ |jkm\rangle_T \mid j,k \in \mathbb{N}_0,\ m \in \mathbb{Z},\ |j_2-j_3| \le k \le j_2+j_3,\ |j_1-k| \le j \le j_1+k,\ |m| \le j \}$$

$j=0 \implies m=0,\ k=1$

$$\implies \text{state} \quad |010\rangle_T = \sum_{\substack{m_1,m_2,m_3 \in \{-1,0,1\} \\ m_1+m_2+m_3=0}} \langle m_1 m_2 m_3|_S |010\rangle_T \, |m_1 m_2 m_3\rangle_S$$

I am sorry. I do not understand what is meant by explicit construction. Therefore it is not possible for me to find a connection between $\mathbb{R}^3$ and $h_{j_1} \otimes h_{j_2} \otimes h_{j_3}$.