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Winter term 2018/19

Problems in Advanced Quantum Theory

Sheet 4

Problem 7: Addition of angular momenta: an operator's view 2+3+2(+3) points

Consider a set of $(2j_1+1)$ operators O_{m_1}, with $m_1 = -j_1, -j_1+1, \ldots, j_1-1, j_1$. Assume they fulfill the following commutation relations with the angular momentum operator $\boldsymbol{J}_1$:

$$[J_{1z}, O_{m_1}] = \hbar m_1 O_{m_1} \,, \tag{1}$$

$$[J_{1\pm}, O_{m_1}] = \hbar c^{\pm}_{j_1 m_1} O_{m_1 \pm 1} \,, \tag{2}$$

where $c^{\pm}_{jm} = \sqrt{j(j+1) - m(m \pm 1)}$. Let O_{m_2} be a similar set of $(2j_2+1)$ operators fulfilling the same commutation relations with $\boldsymbol{J}_2$. Assume that O_{m_1} commutes with $\boldsymbol{J}_2$ (and O_{m_2} with $\boldsymbol{J}_1$). Then consider a set of $(2j+1)$ operators O_m defined as

$$O_m = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle O_{m_1} O_{m_2} \,, \tag{3}$$

where the $\langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$ are Clebsch-Gordan coefficients.

1. Prove that the O_m fulfill a commutation relation analogous to Eq. (1) where J_{1z} and m_1 are replaced by $J_z = J_{1z} + J_{2z}$ and $m = m_1 + m_2$ respectively.

2. Prove that the O_m fulfill a commutation relation analogous to Eq. (2) where $J_{1\pm}$ and $j_1 m_1$ are replaced by $J_{\pm} = J_{1\pm} + J_{2\pm}$ and jm respectively.
   Hint: you need to use the recursion relations enjoyed by the Clebsch-Gordan coefficients.

3. Show that
$$\langle j'm' | O_{m_1} | jm \rangle = 0 \,, \qquad \text{if} \qquad m' \neq m + m_1 \,. \tag{4}$$

   Bonus question:

4. *By computing $\langle j'm' | \, [J_{\pm}, O_{m_1}] \, | jm \rangle$ show that the coefficients $\langle j'm' | O_{m_1} | jm \rangle$ fulfill the same kind of recursion relation as the one enjoyed by the Clebsch-Gordan coefficients $\langle j j_1 m m_1 | j j_1 j' m' \rangle$. This implies that these two very different coefficients must be proportional to each other, through a constant of proportionality which is independent of m_1, m and m'.*

Submission date: Thursday, 29 November 2018