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Advanced Quantum Theory
Exercise Sheet 10

Problem 16

(1) Within the first-order of the Born approximation, we know the scattering amplitude:

$$f(k,k') = -\frac{m}{2\pi\hbar^2}\int V(x)\, e^{i(k-k')x}\, dx^3 =: \tilde{f}(k-k')$$

$$\tilde{f}(k) = -\frac{m}{2\pi\hbar^2}\int e^{ikx}\, V(x)\, dx^3 \qquad \text{(Choose now spherical coordinates with } k = \|k\| e_z)$$

$$= -\frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{i\|k\| r\cos\vartheta}\, V_0 e^{-\alpha r^2}\, r^2 \sin\vartheta\, d\varphi\, d\vartheta\, dr$$

$$= -\frac{mV_0}{\hbar^2}\int_0^\infty r^2 e^{-\alpha r^2} \underbrace{\int_0^\pi e^{i\|k\| r\cos\vartheta}\sin\vartheta\, d\vartheta}\, dr$$

(Substitution with $z(\vartheta) := \cos\vartheta$, $z'(\vartheta) = -\sin\vartheta$)

$$-\int_{\cos(0)}^{\cos(\pi)} e^{i\|k\| r z}\, dz = \int_{-1}^{1} e^{i\|k\| r z}\, dz = \frac{1}{i\|k\| r}\left[e^{i\|k\| r} - e^{-i\|k\| r}\right] = \frac{2\sin(\|k\| r)}{\|k\| r}$$

$$= -\frac{2mV_0}{\hbar^2\|k\|}\int_0^\infty r e^{-\alpha r^2}\sin(\|k\| r)\, dr$$

Doing some extra work:

$$\int_0^\infty r e^{-\alpha r^2}\sin(\beta r)\, dr \overset{\text{(part. Int.)}}{=} \int \tilde{r} e^{-\alpha\tilde{r}^2} d\tilde{r}\Big|_r \sin(\beta r)\Big|_{r=0}^{\infty} - \int_0^\infty \int \tilde{r} e^{-\alpha\tilde{r}^2} d\tilde{r}\Big|_r \cdot \beta\cos(\beta r)\, dr$$

(here we use: $\int \tilde{r} e^{-\alpha\tilde{r}^2} d\tilde{r}\Big|_r = \frac{-e^{-\alpha r^2}}{2\alpha} + C$)

$$= \underbrace{\frac{-e^{-\alpha r^2}}{2\alpha}\sin(\beta r)\Big|_{r=0}^{\infty}}_{=0} + \frac{\beta}{2\alpha}\int_0^\infty e^{-\alpha r^2}\cos(\beta r)\, dr$$

$\left(\cos(\beta r) = \frac{1}{2}\left[e^{i\beta r} + e^{-i\beta r}\right]\right)$

$$= \frac{\beta}{2\alpha}\int_0^\infty e^{-\alpha r^2}\cos(\beta r)\, dr$$

$$= \frac{\beta}{4\alpha}\left[\int_0^\infty \exp(-\alpha r^2 + i\beta r)\, dr + \int_0^\infty \exp(-\alpha r^2 - i\beta r)\, dr\right]$$

more extra work: (analog for second term)

$$\int_0^\infty \exp(-\alpha r^2 + i\beta r)\, dr = \int_0^\infty \exp\left[-\left(\sqrt{\alpha} r - \frac{i\beta}{2\sqrt{\alpha}}\right)^2 - \frac{\beta^2}{4\alpha}\right] dr$$

$\left(\mu(r) := \sqrt{\alpha} r - \frac{i\beta}{2\sqrt{\alpha}} \Rightarrow \mu'(r) = \sqrt{\alpha}\right)$

$$= e^{-\frac{\beta^2}{4\alpha}}\int_0^\infty \exp\left[-\left(\sqrt{\alpha} r - \frac{i\beta}{2\sqrt{\alpha}}\right)^2\right] dr$$

$$= \frac{1}{\sqrt{\alpha}} e^{-\beta^2/4\alpha}\int_0^\infty e^{-\mu^2(r)}\mu'(r)\, dr$$

(contour integration: $\int_{\mu(\mathbb{R}^+)} e^{-z^2} d\mu(z) = \int_0^\infty e^{-t^2} dt$)

$$= \frac{1}{\sqrt{\alpha}} e^{-\beta^2/4\alpha}\int_{\mu(\mathbb{R}^+)} e^{-z^2}\, d\mu(z)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{\alpha}} e^{-\frac{\beta^2}{4\alpha}}$$

inserting 'more work' into 'extra work':

$$\Rightarrow \int_0^\infty r e^{-\alpha r^2} \sin(\beta r)\, dr = \frac{\sqrt{\pi}\,\beta}{4\alpha^{3/2}} e^{-\frac{\beta^2}{4\alpha}}$$

inserting 'extra work' into computation:

$$\Rightarrow \tilde{f}(k) = -\frac{\sqrt{\pi}\, m V_0}{2\hbar^2 \alpha^{3/2}} e^{-\|k\|^2/4\alpha} \quad \text{(here we used: } \beta = \|k\|\text{)}$$

$$\Rightarrow \tilde{f}(k) = \tilde{\tilde{f}}(\|k\|^2)$$

Additionally:
$$\|k-k'\|^2 = \|k\|^2 - 2k\cdot k' + \|k'\|^2 = \|k\|^2 + \|k'\|^2 - 2\|k\|\,\|k'\|\cos\vartheta$$
$$(\|k\| = \|k'\|) \quad = 2\|k\|^2(1-\cos\vartheta) = 4\|k\|^2 \sin^2\frac{\vartheta}{2}$$

$$\Rightarrow f(\vartheta) = \tilde{\tilde{f}}(\|k-k'\|^2) = \tilde{\tilde{f}}\left(2\|k\|^2(1-\cos\vartheta)\right)$$

$$\Rightarrow \underline{\underline{f(\vartheta) = -\frac{\sqrt{\pi}\, m V_0}{2\hbar^2\alpha^{3/2}} \exp\left[-\frac{\|k\|^2}{2\alpha}(1-\cos\vartheta)\right] = -\frac{\sqrt{\pi}\, m V_0}{2\hbar^2 \alpha^{3/2}} \exp\left(-\frac{\|k\|^2}{\alpha}\sin^2\frac{\vartheta}{2}\right)}}$$

(2)
$$\underline{\underline{\frac{d\sigma}{d\Omega}(\vartheta) \overset{(def)}{=} |f(\vartheta)|^2 = \frac{\pi m^2 V_0^2}{4\hbar^4 \alpha^3} \exp\left[-\frac{\|k\|^2}{\alpha}(1-\cos\vartheta)\right]}}$$

(3)
$$\sigma \overset{(def)}{=} \int \frac{d\sigma}{d\Omega}(\vartheta)\, d\Omega(\vartheta) = 2\pi \int_0^\pi \frac{d\sigma}{d\Omega}(\vartheta) \sin\vartheta\, d\vartheta$$

$$= \frac{\pi^2 m^2 V_0^2}{2\hbar^4 \alpha^3} \int_0^\pi \exp\left[-\frac{\|k\|^2}{\alpha}(1-\cos\vartheta)\right] \sin\vartheta\, d\vartheta$$

$$\begin{pmatrix} z(\vartheta) := -\cos\vartheta \\ z'(\vartheta) = \sin\vartheta \end{pmatrix} \quad = \frac{\pi^2 m^2 V_0^2}{2\hbar^4\alpha^3} \underbrace{\int_{z(0)}^{z(\pi)} \exp\left[-\frac{\|k\|^2}{\alpha}(1+s)\right] ds}$$

$$e^{-\|k\|^2/\alpha} \int_{-1}^{1} \exp\left(-\frac{\|k\|^2}{\alpha} s\right) ds = \frac{\alpha}{\|k\|^2} e^{-\frac{\|k\|^2}{\alpha}} \left(e^{\frac{\|k\|^2}{\alpha}} - e^{-\frac{\|k\|^2}{\alpha}}\right)$$
$$= \frac{\alpha}{\|k\|^2}\left(1 - e^{-2\|k\|^2/\alpha}\right)$$

$$\Rightarrow \underline{\underline{\sigma = \frac{\pi^2 m^2 V_0^2}{2\hbar^4 \alpha^2 \|k\|^2}\left[1 - \exp\left(-\frac{2\|k\|^2}{\alpha}\right)\right]}}$$

*computing difference of coefficients of in- and out-going wave

Problem 15

(1) A_0 should be regular $\Rightarrow$ $u_0(0) \overset{!}{=} 0$

$$u_0(r) = \alpha\left[\lambda\tanh(\lambda r) - ik\right]e^{ikr} + \beta\left[\lambda\tanh(\lambda r) + ik\right]e^{-ikr}$$

$$\Rightarrow 0 = u_0(0) = -ik\alpha + ik\beta \Rightarrow \alpha = \beta$$

$$\Rightarrow u_0(r) = 2\alpha\left[\lambda\tanh(\lambda r)\cos(kr) + k\sin(kr)\right]$$

$$\Rightarrow A_0(r) = \frac{2\alpha}{r}\left[\lambda\tanh(\lambda r)\cos(kr) + k\sin(kr)\right]$$

doing some extra work for scattering amplitude:* $x, y, a, b \in \mathbb{R}$ | computation of scattering coefficient

$$(x+iy)(a+ib) = -x+iy$$

$$\Rightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \Rightarrow a = \frac{y^2-x^2}{y^2+x^2},\ b = \frac{2xy}{x^2+y^2}$$

applying on $u_0(r)$: $\lambda\tanh(\lambda r) \xrightarrow{r\to\infty} \lambda =: x$, $k =: y$

$$\Rightarrow a+ib = \frac{k^2-\lambda^2}{\lambda^2+k^2} + i\,\frac{2k\lambda}{\lambda^2+k^2} = \frac{k^2-\lambda^2+2ik\lambda}{\lambda^2+k^2}$$

for scattering amplitude: $1 + 2ikf_0(k) = a+ib$

$$\Rightarrow f_0(k) = \frac{a+ib-1}{2ik} = \frac{k^2-\lambda^2+2ik\lambda-k^2-\lambda^2}{2ik(\lambda^2+k^2)}$$

$$= \frac{\lambda}{\lambda^2+k^2} - \frac{\lambda^2}{ik(\lambda^2+k^2)} = \frac{\lambda}{\lambda^2+k^2}\left(1 + i\frac{\lambda}{k}\right)$$

we know: $f_0(k) = \dfrac{1}{k(\cot\delta_0 - i)} \Rightarrow \cot\delta_0 = i + \dfrac{1}{kf_0(k)}$

$$\Rightarrow \cot\delta_0 = i + \frac{\lambda^2+k^2}{\lambda k}\left(1+i\frac{\lambda}{k}\right)^{-1} = i + \frac{\lambda^2+k^2}{\lambda}(k+i\lambda)^{-1}$$

$$= i + \frac{\lambda^2+k^2}{\lambda}\,\frac{k-i\lambda}{k^2+\lambda^2} = i + \frac{k}{\lambda} - i = \frac{k}{\lambda}$$

$$\Rightarrow \underline{\underline{\delta_0 = \arctan\left(\frac{\lambda}{k}\right)}}$$

(2) σ is dominated by s-wave $\Rightarrow$ $\sigma = \dfrac{4\pi}{k^2}\displaystyle\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \approx \dfrac{4\pi}{k^2}\sin^2\delta_0$

Additionally we know: $f_0(k) = \dfrac{1}{k}e^{i\delta_0}\sin\delta_0 \Rightarrow |f_0(k)|^2 = \dfrac{1}{k^2}\sin^2\delta_0$

$$\Rightarrow \underline{\underline{\sigma \approx 4\pi|f_0(k)|^2 = \frac{4\pi\lambda^2}{(\lambda^2+k^2)^2}\left(1+\frac{\lambda^2}{k^2}\right) = \frac{4\pi\lambda^2}{k^2(\lambda^2+k^2)}}}$$

(3) We know : $E \propto k^2 \Rightarrow (\lim E \to 0 \Leftrightarrow \lim k^2 \to 0)$

$$\Rightarrow \quad \sigma = \frac{4\pi\lambda^2}{k^2(\lambda^2+k^2)} \longrightarrow \infty, \; k^2 \to 0$$

This makes sense :

The given potential is a long - range potential because it will never be zero. For small energies one has long wavelengths which are then able to interact with the potential everywhere. Therefore the total cross section should be infinite.