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Problems in Advanced Quantum Theory

Sheet 6

Problem 10: Interaction picture 1+1+4 points

Consider a forced harmonic oscillator with Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) + f(t) a + f(t)^* a^\dagger \,. \tag{1}$$

Assume $f(0) = 0$.

1. Write the ladder operators $a_I(t)$ and $a_I^\dagger(t)$ in the interaction picture in terms of t and of the corresponding Schrödinger-picture operators a and $a^\dagger$.
   Hint: you might recall some results from Problem 1
2. Let $|n, t\rangle_I$ be the interaction-picture state which at $t = 0$ coincides with the $|n\rangle$ state of the pure harmonic oscillator. Write the time-evolution differential equation fulfilled by this state.
3. Use the results to the previous two tasks to compute the time-dependent matrix element
$$_I\langle n', t | H_I(t) | n, t\rangle_I \tag{2}$$
   for infinitesimal time $t = \delta t$, at first order in δt.

Problem 11: Variational method 3+3+1(+3) points

A pointlike particle of mass m inside a one-dimensional double-well potential

$$V(x) = -\frac{m\omega^2}{2} x^2 + \frac{\lambda}{4} x^4 \,, \tag{3}$$

with $\lambda > 0$, has an energy spectrum which is bounded from below and discrete. We want to apply the variational method to estimate the ground state energy. For this purpose we choose the test function

$$\psi(x; k, v) = \sqrt{k}\ \varphi(k(x - v)) \,, \tag{4}$$

where

$$\varphi(y) = \frac{1}{\pi^{1/4}} e^{-\frac{y^2}{2}} \,, \tag{5}$$

while $k > 0$ and v are two real parameters. The variational principle can then be used to determine the values k_0 and v_0 which best approximate the exact ground-state wave function.

1. Compute the expectation value of the Hamiltonian in the state $\psi(x; k, v)$.
   Hint: you might need the following integrals:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}y \, y^2 e^{-(y-y_*)^2} = \frac{1}{2} + y_*^2 \,, \tag{6}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}y \, y^4 e^{-(y-y_*)^2} = \frac{3}{4} + 3y_*^2 + y_*^4 \,. \tag{7}$$

2. Determine k_0 and v_0 at leading order in λ, for λ small (i.e. λ^{-1} big). You will find two real values $v_0^{\pm}$ with opposite signs.

3. Compute the corresponding energy at leading order in λ, for λ small.

4. *Bonus question:* The presence of the two solutions $\psi_{\pm}(x) = \psi\big(x; k_0, v_0^{\pm}\big)$ can be interpreted by parity considerations. Since $V(x) = V(-x)$, parity is a good quantum number, therefore the even wave function

$$\psi_S(x) = \frac{1}{\sqrt{2}} \left(\psi_+(x) + \psi_-(x)\right) \tag{8}$$

   is an approximation to the ground state, while the odd wave function

$$\psi_A(x) = \frac{1}{\sqrt{2}} \left(\psi_+(x) - \psi_-(x)\right) \tag{9}$$

   is an approximation to the first excited state. Evaluate the splitting between these two energy levels at leading order in λ, for λ small.

Submission date: Thursday, 13 December 2018