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Winter term 2017/18

Problems in Advanced Quantum Mechanics

Problem Sheet 6

Problem 13: H-atom between the plates of a capacitor 4 points

A hydrogen atom in its ground state is placed between two parallel plates of a capacitor. A impulse voltage produces a spatially homogeneous electric pulse

$$\boldsymbol{E}(t) = -E_0\,\theta(t)\,\mathrm{e}^{-t/\tau}\,\boldsymbol{e}_z, \quad \tau > 0,$$

between the plates, and orthogonal to them (parallel to the z-axis with unit vector $\boldsymbol{e}_z$). Calculate in first order perturbation theory the transition probability, that the atom at $t > 0$ is

1. in the $2s$ state
2. in one of the $2p$ states.

Hint: You might need the explicit form of some of the following wave functions of the Hydrogen atom $\langle \boldsymbol{r}|n\ell m\rangle = R_{n\ell}(r)Y_{\ell m}(\theta,\varphi)$:

$$R_{10}(r) = \frac{2}{a^{3/2}}\mathrm{e}^{-r/a}, \quad R_{20}(r) = \frac{2}{(2a)^{3/2}}(1 - r/2a)\mathrm{e}^{-r/2a}, \quad R_{21}(r) = \frac{1}{\sqrt{3}(2a)^{3/2}}\frac{r}{a}\mathrm{e}^{-r/2a},$$

$$Y_{00}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}, \qquad Y_{10}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta, \qquad Y_{1\pm1}(\theta,\varphi) = \mp\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{\pm i\varphi}.$$

Problem 14: Two-states system with a periodic perturbation 2+3+1 = 6 points

A stationary quantum system with discrete energies, $H_0|n\rangle = E_n|n\rangle$, is subject for $t > 0$ to a time-dependent perturbation $V(t)$.

1. Derive the system of differential equations for the expansion coefficients in

$\partial_t|\psi(t)\rangle = \sum_n c_n'(t) \frac{-iE_n}{\hbar} e^{-i\frac{E_n t}{\hbar}}|n\rangle$

$[H_0 + V(t)]|\psi(t)\rangle = -i\hbar\,\partial_t|\psi(t)\rangle$

$$|\psi(t)\rangle = \sum_n c_n(t)\,\mathrm{e}^{-\mathrm{i}E_n t/\hbar}|n\rangle\,.$$

$H_0|\psi(t)\rangle = \sum_n E_n c_n(t) e^{-\frac{iE_n t}{\hbar}}|n\rangle$

from the Schrödinger equation with Hamiltonian $H(t) = H_0 + V(t)$. Show that in first order perturbation theory the solution is

$\Rightarrow$ comparison of coefficients

$-E_n c_n'(t) e^{-\frac{iE_n t}{\hbar}}$

$= E_n c_n(t) e^{-\frac{iE_n t}{\hbar}}$

$+ V(t) c_n(t) e^{-\frac{iE_n t}{\hbar}}$

$$c_n(t) = c_n(0) - \frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t' \sum_m \langle n|V_W(t')|m\rangle c_m(0)\,.$$

2. Now assume that the unperturbed system has only two possible energy eigenstates $|1\rangle$ and $|2\rangle$, with energy difference $\hbar\omega_{21}$. In this basis the matrix $V_{nm} = \langle n|V(t)|m\rangle$ reads

$$\big(V_{nm}(t)\big) = \begin{pmatrix} 0 & \hbar\omega_0\mathrm{e}^{\mathrm{i}\omega t} \\ \hbar\omega_0\mathrm{e}^{-\mathrm{i}\omega t} & 0 \end{pmatrix}$$

$\partial_t|\psi(t)\rangle = \sum_{n\in 1}\left(c_n'(t)\, e^{-\frac{iE_n t}{\hbar}}|n\rangle + c_n(t)\cdot\frac{-iE_n}{\hbar}\, e^{-\frac{iE_n t}{\hbar}}|n\rangle\right)$

Initially at $t = 0$ the system is in the state $|1\rangle$. The time evolution can be determined exactly and analytically as follows: From the coupled differential equations for the $c_n(t)$ obtained in point (1) of this exercise, you should derive the equation

$$\ddot{c}_2(t) - 2i\Omega_1\dot{c}_2(t) + \omega_0^2 c_2(t) = 0$$

with $\Omega_1 = (\omega_{21} - \omega)/2$. Then solve it with the appropriate initial conditions. Discuss the occupation probability of state $|2\rangle$ as a function of t and ω.

3. Calculate the occupation probability in first order perturbation theory and discuss the domain of validity of this approximation via a comparison with the exact result obtained in point (2).

Submission date: Thursday, 30. November 2017, before the lecture begins.