Markus Pawellek
144645

Advanced Quantum Theory

Exercise Sheet 9

Problem 14

(1) Lippmann - Schwinger equation

$$|\psi_k^{(+)}\rangle = |k\rangle + \hat{G}_+ U |\psi_k^{(+)}\rangle$$

$$\Longrightarrow \langle x|\psi_k^{(+)}\rangle = \psi_k^{(+)}(x) = \langle x|k\rangle + \langle x|\hat{G}_+ U|\psi_k^{(+)}\rangle$$

We know: $\langle x|k\rangle = \int e^{ikx'} \langle x|x'\rangle \, dx' = \int e^{ikx'} \delta(x-x') \, dx' = e^{ikx}$

Additionally: $$\langle x|\hat{G}_+ U|\psi_k^{(+)}\rangle = \iint \underbrace{\langle x|\hat{G}_+|x'\rangle}_{G_+(x,x')} \underbrace{\langle x'|U|x''\rangle}_{U(x',x'')} \underbrace{\langle x''|\psi_k^{(+)}\rangle}_{\psi_k^{(+)}(x'')} \, dx'' \, dx'$$

$$= \int G_+(x,x') \int U(x',x'') \, \psi_k^{(+)}(x'') \, dx'' \, dx'$$

$$\Rightarrow \underline{\underline{\psi_k^{(+)}(x) = e^{ikx} + \int G_+(x,x') \int U(x',x'') \, \psi_k^{(+)}(x'') \, dx'' \, dx'}}$$

case of a local potential: $U(x,x') = U(x)\,\delta(x-x')$

$$\Rightarrow \int U(x',x'') \, \psi_k^{(+)}(x'') \, dx'' = U(x') \int \delta(x'-x'') \, \psi_k^{(+)}(x'') \, dx''$$

$$= U(x') \, \psi_k^{(+)}(x')$$

$$\Rightarrow \psi_k^{(+)}(x) = e^{ikx} + \int G_+(x,x') \, U(x') \, \psi_k^{(+)}(x') \, dx'$$

$$= e^{ikx} - \frac{2m}{4\pi\hbar^2} \int \frac{e^{ik\|x-x'\|}}{\|x-x'\|} V(x') \, \psi_k^{(+)}(x') \, dx'$$

(2) case of nonlocal potential: $U(x,x') = \lambda g(x) g^*(x')$

$$\Rightarrow \int U(x',x'') \, \psi_k^{(+)}(x'') \, dx'' = \lambda g(x') \int g^*(x'') \, \psi_k^{(+)}(x'') \, dx''$$

We know: $|g\rangle = \int g(x) \, |x\rangle \, dx \quad \Rightarrow \quad \langle g| = \int g^*(x) \, \langle x| \, dx$

$$\Rightarrow \int g^*(x'') \, \psi_k^{(+)}(x'') \, dx'' = \int g^*(x'') \, \langle x''|\psi_k^{(+)}\rangle \, dx''$$

$$= \int \int g^*(x''') \underbrace{\delta(x''-x''')}_{\langle x'''|x''\rangle} \, dx''' \, \langle x''|\psi_k^{(+)}\rangle \, dx'' \qquad (*)$$

$$= \int \underbrace{\int g^*(x''') \, \langle x'''| \, dx'''}_{\langle g|} \, |x''\rangle\langle x''|\psi_k^{(+)}\rangle \, dx''$$

$$= \int \langle g|x''\rangle\langle x''|\psi_k^{(+)}\rangle \, dx'' = \langle g|\psi_k^{(+)}\rangle$$

$$\Rightarrow \int U(x',x'')\,\psi_k^{(+)}(x'')\,dx'' = \lambda g(x')\,\langle g|\psi_k^{(+)}\rangle$$

Solve Lippmann-Schwinger equation for $\langle g|\psi_k^{(+)}\rangle$:

$$\psi_k^{(+)}(x) = e^{ikx} + \lambda \langle g|\psi_k^{(+)}\rangle \int G_+(x,x')\,g(x')\,dx'$$

$$\Rightarrow \underbrace{\int g^*(x)\,\psi_k^{(+)}(x)\,dx}_{\langle g|\psi_k^{(+)}\rangle} = \underbrace{\int g^*(x)\,e^{ikx}\,dx}_{\int g^*(x)\langle x|k\rangle dx \overset{(*)}{=} \int \langle g|x\rangle\langle x|k\rangle dx = \langle g|k\rangle} + \lambda \langle g|\psi_k^{(+)}\rangle \underbrace{\iint g^*(x)\,G_+(x,x')\,g(x')\,dx'dx}_{\overset{(*)}{=} \iint \langle g|x\rangle\langle x|\hat{G}_+|x'\rangle\langle x'|g\rangle dx'dx = \langle g|\hat{G}_+|g\rangle}$$

$$\Rightarrow \langle g|\psi_k^{(+)}\rangle = \langle g|k\rangle + \lambda \langle g|\hat{G}_+|g\rangle\langle g|\psi_k^{+}\rangle$$

$$\Rightarrow \boxed{\langle g|\psi_k^{(+)}\rangle = \frac{\langle g|k\rangle}{1-\lambda\langle g|\hat{G}_+|g\rangle}} \qquad (**)$$

(3) $f(k,k') = -\frac{1}{4\pi}\langle k'|U|\psi_k^{(+)}\rangle$, $f_B(k,k') = -\frac{1}{4\pi}\langle k'|U|k\rangle$

$$\Rightarrow f_B(k,k') = -\frac{1}{4\pi}\iint \langle k'|x\rangle \underbrace{\langle x|U|x'\rangle}_{U(x,x') = \lambda g(x) g^*(x')} \langle x'|k\rangle\,dx'dx$$

$$= -\frac{\lambda}{4\pi}\underbrace{\left[\int \langle k'|x\rangle g(x)\,dx\right]}_{\langle k'|g\rangle}\cdot\underbrace{\left[\int g^*(x')\langle x'|k\rangle\,dx'\right]}_{\langle g|k\rangle}$$

$$= -\frac{\lambda}{4\pi}\langle k'|g\rangle\langle g|k\rangle$$

$$\Rightarrow f(k,k') = -\frac{1}{4\pi}\iint \langle k'|x\rangle\langle x|U|x'\rangle \underbrace{\langle x'|\psi_k^{(+)}\rangle}_{\psi_k^{(+)}(x')}\,dx'dx$$

$$= -\frac{\lambda}{4\pi}\underbrace{\left[\int \langle k'|x\rangle g(x)\,dx\right]}_{\langle k'|g\rangle}\cdot\underbrace{\left[\int g^*(x')\,\psi_k^{(+)}(x')\,dx'\right]}_{\langle g|\psi_k^{(+)}\rangle \overset{(**)}{=} \frac{\langle g|k\rangle}{1-\lambda\langle g|\hat{G}_+|g\rangle}}$$

$$= -\frac{\lambda}{4\pi}\langle k'|g\rangle \frac{\langle g|k\rangle}{1-\lambda\langle g|\hat{G}_+|g\rangle} = \frac{f_B(k,k')}{1-\lambda\langle g|\hat{G}_+|g\rangle} \qquad \square$$

(4) $g(x) := \sqrt{\frac{m}{\pi\hbar^2}} \; \frac{1}{\|x\|} \; e^{-\beta\|x\|}$

$$\Rightarrow \quad \langle k|g\rangle = \int \langle k|x\rangle\langle x|g\rangle \, dx = \int g(x)\, e^{-ikx} \, dx$$

$$\stackrel{(def)}{=} \sqrt{\frac{m}{\pi\hbar^2}} \int \frac{1}{\|x\|} e^{-\beta\|x\|} e^{-ikx} \, dx$$

Now we use spherical coordinates and transform the integral. ($kx = \|k\| \, r \cos\vartheta$)

$$= \sqrt{\frac{m}{\pi\hbar^2}} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{r} e^{-\beta r} e^{-i\|k\| r \cos\vartheta} r^2 \sin\vartheta \, d\varphi \, d\vartheta \, dr$$

$$= \sqrt{\frac{m}{\pi\hbar^2}} \; 2\pi \int_0^\infty r e^{-\beta r} \underbrace{\int_0^\pi e^{-i\|k\| r \cos\vartheta} \sin\vartheta \, d\vartheta}_{} \, dr$$

$$\int_{-\cos(0)}^{-\cos(\pi)} e^{-i\|k\| r z} \, dz$$

(Substitution $z(\vartheta) := -\cos\vartheta$ $\Rightarrow z'(\vartheta) = \sin\vartheta$)

$$= \frac{-1}{i\|k\|r} e^{-i\|k\| r z} \Big|_{z=-\cos(0)}^{-\cos(\pi)} = \frac{1}{i\|k\|r}\left[e^{i\|k\|r} - e^{-i\|k\|r}\right]$$

$$= \frac{2\sin(\|k\| r)}{\|k\| r}$$

$$= \sqrt{\frac{m}{\pi\hbar^2}} \; \frac{4\pi}{\|k\|} \int_0^\infty \sin(\|k\| r) \, e^{-\beta r} \, dr$$

doing some extra work:

$$\int_0^\infty \sin(ax)\, e^{-bx} \, dx \stackrel{(part. Int.)}{=} -\frac{1}{b} e^{-bx} \sin(ax) \Big|_0^\infty + \frac{a}{b} \int_0^\infty \cos(ax)\, e^{-bx} \, dx$$

$$\int_0^\infty \cos(ax)\, e^{-bx} \, dx \stackrel{(part. Int)}{=} -\frac{1}{b} e^{-bx} \cos(ax) \Big|_0^\infty - \frac{a}{b} \int_0^\infty \sin(ax)\, e^{-bx} \, dx$$

$$\Rightarrow \quad \int_0^\infty \sin(ax)\, e^{-bx} \, dx = \frac{-e^{-bx}\left[a\cos(ax) + b\sin(ax)\right]}{a^2+b^2} \Bigg|_0^\infty$$

$$= \frac{a}{a^2+b^2}$$

$$\Rightarrow \quad \langle k|g\rangle = \sqrt{\frac{m}{\pi\hbar^2}} \; \frac{4\pi}{k^2+\beta^2}$$

$$\Rightarrow \quad f_B(k,k') = -\frac{\lambda}{4\pi} \langle k'|g\rangle\langle g|k\rangle = \frac{-4m\lambda}{\hbar^2} \; \frac{1}{(k^2+\beta^2)(k'^2+\beta^2)}$$

$$\Rightarrow \quad f_B(k,k') = \tilde{f}_B(\|k\|, \|k'\|) = -\frac{4m\lambda}{\hbar^2} \; \frac{1}{(\|k\|^2+\beta^2)(\|k'\|^2+\beta^2)}$$

$$= \tilde{f}_B(\|k'\|, \|k\|)$$

$\Rightarrow$ f_B is isotropic $\square$