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Problems in Advanced Quantum Theory

Sheet 2

Problem 3: Polarization states 1+2+2 (+2+2) points

We model the phase of a photon as a two-level system (q-bit) whose Hilbert space is spanned by a orthonormal basis of two states: $|1\rangle$ and $|2\rangle$. They correspond, for instance, to two orthogonal linear polarizations, with observable polarization operators:

$$P_1 = |1\rangle\langle 1| \,, \qquad\qquad P_2 = |2\rangle\langle 2| \,. \tag{1}$$

1. Provide an example of a normalized pure state for which the expectation value of both P_1 and P_2 equals $1/2$, and a two-by-two matrix representation of the corresponding density-matrix operator in the given basis.

2. Find the density matrix for the most general normalized pure state with the properties specified in the previous task.

3. The state with partial polarization has the general representation

$$\rho = \frac{1}{2}\left(\mathbb{1} + \boldsymbol{\xi}\cdot\boldsymbol{\sigma}\right), \tag{2}$$

where $\mathbb{1}$ is the two-by-two identity matrix, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices, and $\boldsymbol{\xi} \in \mathbb{R}^3$. For which $\boldsymbol{\xi}$ this state is a pure state?

Bonus questions.

4. *For which $\boldsymbol{\xi}$ is $\langle P_1\rangle = 1$? For which $\boldsymbol{\xi}$ is $\langle P_2\rangle = 1$?*

5. *Define the states with circular polarizations:*

$$|\pm\rangle = \frac{1}{2}\left(\,|1\rangle \pm \mathrm{i}|2\rangle\right) . \tag{3}$$

For which $\boldsymbol{\xi}$ is $\langle P_+\rangle = 1$? For which $\boldsymbol{\xi}$ is $\langle P_-\rangle = 1$?

Problem 4: Canonical probability density: the harmonic oscillator 2+2+2+2 points

Consider a one dimensional quantum harmonic oscillator with mass m and frequency ω in a canonical ensemble such that the density matrix at some temperature T is given by

$$\hat{\rho}(T) = \sum_{n=0}^{\infty} e^{-\frac{E_n}{k_B T}} |n\rangle\langle n| \tag{4}$$

where $|n\rangle$ is the eigenstate of the Hamiltonian with energy $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$. The partition function is given by the trace of the density-matrix operator

$$Z(T) = \mathrm{Tr}[\hat{\rho}(T)]. \tag{5}$$

1. Knowing that the energy-eigenstate wave functions $\psi_n(x)$ are real, find the canonical probability density in configuration space $P(x)$.

2. Show that, up to some normalization pre-factor C, the derivative of $P(x)$ is

$$\frac{dP(x)}{dx} = C\sqrt{\frac{2m\omega}{\hbar}}\left(e^{-\frac{\hbar\omega}{k_B T}} - 1\right)\sum_{n=0}^{\infty} e^{-\frac{E_n}{k_B T}}\sqrt{n+1}\,\psi_n(x)\psi_{n+1}(x)\,. \tag{6}$$

   Hint: express the momentum operator $\hat{p}$ in terms of the lowering $\hat{a}$ and raising $\hat{a}^\dagger$ operators.

3. Similarly show that, for the same constant as in the previous task,

$$xP(x) = C\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-\frac{\hbar\omega}{k_B T}} + 1\right)\sum_{n=0}^{\infty} e^{-\frac{E_n}{k_B T}}\sqrt{n+1}\,\psi_n(x)\psi_{n+1}(x)\,. \tag{7}$$

   Hint: express the position operator $\hat{x}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$.

4. Show that

$$\frac{1}{xP(x)}\frac{dP(x)}{dx} = -\frac{2m\omega}{\hbar}\tanh\left(\frac{\hbar\omega}{2k_B T}\right), \tag{8}$$

   and integrate the previous differential equation in the variable x, i.e. solve for $P(x)$.

Submission date: Thursday, 15 November 2018