Theoretische Mechanik – Übung 10

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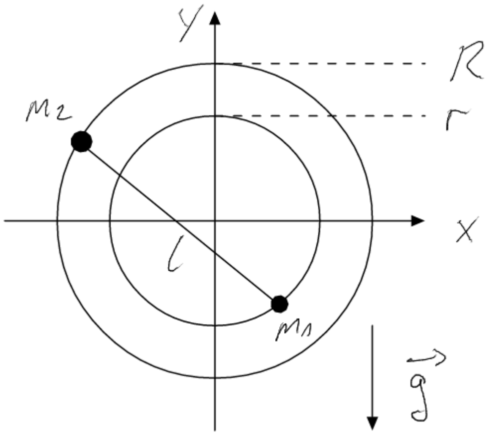
Aufgabe 1



$m_1 = m_2 = m$

$R - r < l < R + r$

$\vec{g} = -g\,\vec{e_y}$

a) Zwangsbedingungen:

$$g_1(\vec{r_1}) = x_1^2 + y_1^2 - r^2 = 0$$
$$g_2(\vec{r_2}) = x_2^2 + y_2^2 - R^2 = 0$$

konstante Kreisbahn

$$g_3(\vec{r_1}, \vec{r_2}) = (x_1 - x_2)^2 + (y_1 - y_2)^2 - l = 0$$

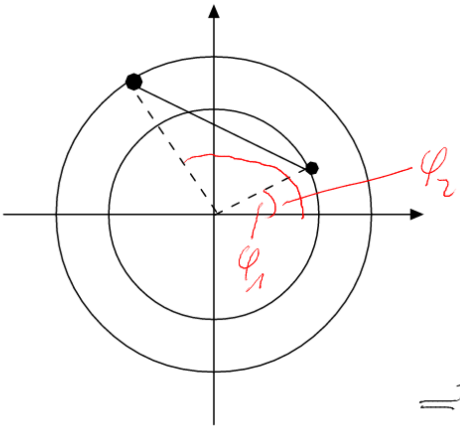
$\longrightarrow$ konstante Länge der Stange

$$\Longrightarrow \quad \mathrm{grad}_1\, g_1 = 2x_1 \vec{e_x} + 2y_1 \vec{e_y}$$
$$\mathrm{grad}_2\, g_2 = 2x_2 \vec{e_x} + 2y_2 \vec{e_y}$$
$$\mathrm{grad}_1\, g_3 = 2(x_1 - x_2)\vec{e_x} + 2(y_1 - y_2)\vec{e_y}$$
$$\mathrm{grad}_2\, g_3 = -2(x_1 - x_2)\vec{e_x} - 2(y_1 - y_2)\vec{e_y}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} m\ddot{x}_1 &= 2\lambda_1 x_1 + 2\lambda_3(x_1 - x_2) \\ m\ddot{y}_1 &= 2\lambda_1 y_1 + 2\lambda_3(y_1 - y_2) - mg \\ m\ddot{x}_2 &= 2\lambda_2 x_2 - 2\lambda_3(x_1 - x_2) \\ m\ddot{y}_2 &= 2\lambda_2 y_2 - 2\lambda_3(y_1 - y_2) - mg \end{aligned}}$$

Lagrange-Gleichungen I. Art

Polarkoordinaten:



$$g_1(\vec{r_1}) = \rho_1 - r = 0$$

$$g_2(\vec{r_2}) = \rho_2 - R = 0$$

Kosinussatz: $l^2 = r^2 + R^2 - 2rR\cos(\varphi_1 - \varphi_2)$

$$\Rightarrow \cos \varphi_1 - \varphi_2 = \frac{r^2+R^2-l^2}{2rR} = \text{konst}$$

$$\Rightarrow |\varphi_1 - \varphi_2| =: \Delta\varphi = \arccos \frac{r^2+R^2-l^2}{2rR}$$

o.E.: $\varphi_1 < \varphi_2 \Longrightarrow \varphi_2 - \varphi_1 = \Delta\varphi$

$$\Rightarrow g_3(\vec{r_1}, \vec{r_2}) = \varphi_2 - \varphi_1 - \Delta\varphi = 0$$

$$\Rightarrow \text{grad}_1\, g_1 = \vec{e_\rho} \qquad \text{grad}_2\, g_2 = \vec{e_\rho}$$

$$\text{grad}_1\, g_3 = -\vec{e_\varphi} \qquad \text{grad}_2\, g_3 = \vec{e_\varphi}$$

$$\Rightarrow \vec{g} = -g\,\vec{e_y} = -g\left(\sin\varphi\,\vec{e_\rho} + \cos\varphi\,\vec{e_\varphi}\right)$$

$$\Rightarrow \boxed{\begin{aligned} m(\ddot{\rho}_1 - \rho_1\dot{\varphi}_1^2) &= -mg\sin\varphi_1 + \lambda_1 \\ m(\rho_1\ddot{\varphi}_1 + 2\dot{\rho}_1\dot{\varphi}_1) &= -mg\cos\varphi_1 - \lambda_3 \\ m(\ddot{\rho}_2 - \rho_2\dot{\varphi}_2^2) &= -mg\sin\varphi_2 + \lambda_2 \\ m(\rho_2\ddot{\varphi}_2 + 2\dot{\rho}_2\dot{\varphi}_2) &= -mg\cos\varphi_2 + \lambda_3 \end{aligned}}$$

b) – im Gleichgewicht herrschen keine Beschleunigung und Geschwindigkeiten:

$$\Rightarrow 0 = -mg\cos\varphi_1 - \lambda_3 = -mg\cos\varphi_2 + \lambda_3$$

$$\Rightarrow \cos\varphi_1 + \cos\varphi_2 = 0$$

$$\varphi_2 = \varphi_1 + \Delta\varphi$$

$$\Longrightarrow \quad \cos\varphi_1 + \cos\Delta\varphi\cos\varphi_1 - \sin\Delta\varphi\sin\varphi_1 = 0$$

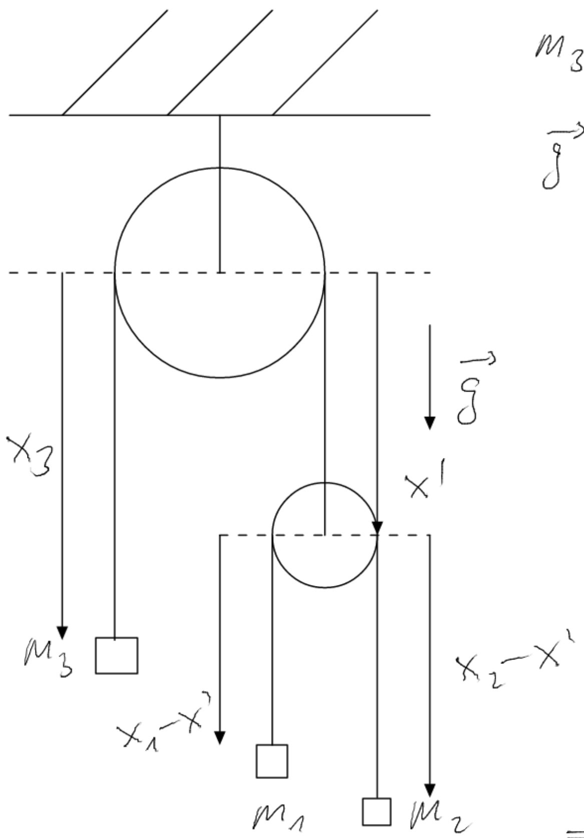
$$\Longrightarrow \quad \tan\varphi_1 = \frac{1+\cos\Delta\varphi}{\sin\Delta\varphi}$$

$$\Longrightarrow \quad \cos\Delta\varphi\cos\varphi_2 + \sin\Delta\varphi\sin\varphi_2 + \cos\varphi_2 = 0$$

$$\Longrightarrow \quad \tan\varphi_2 = -\frac{1+\cos\Delta\varphi}{\sin\Delta\varphi}$$

$$\Longrightarrow \quad \boxed{\begin{array}{ll} \rho_1 = r & \varphi_1 = \arctan\dfrac{1+\cos\Delta\varphi}{\sin\Delta\varphi} \\ \rho_2 = R & \varphi_2 = -\arctan\dfrac{1+\cos\Delta\varphi}{\sin\Delta\varphi} \end{array}}$$

Aufgabe 2



$$m_3 = m_1 + m_2$$

$$\vec{g} = -g\vec{e_x}$$

Zwangsbedingung:

$x_3 + x' = l_1$ (konstante Seillänge der ersten Rolle)

$x_1 + x_2 - 2x' = l_2$ (konst. Seillänge der zweiten Rolle)

$$\Longrightarrow \quad x_1 + x_2 + 2x_3 = l = \text{konst}$$

$$\Longrightarrow \quad g(x_1, x_2, x_3) = x_1 + x_2 + 2x_3 - l = 0$$

$$\Longrightarrow \quad \text{grad}_1\, g = \vec{e_x} = \text{grad}_2\, g \qquad \text{grad}_3\, g = 2\vec{e_x}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} m_1 \ddot{x}_1 &= -m_1 g + \lambda \\ m_2 \ddot{x}_2 &= -m_2 g + \lambda \\ m_3 \ddot{x}_3 &= -m_3 g + 2\lambda \end{aligned}}$$

Lagrange – Gleichungen 1. Art

b) Ableitung Zwangsbedingung: $\ddot{x}_1 + \ddot{x}_2 + 2\ddot{x}_3 = 0$

$$\Longrightarrow \quad \ddot{x}_2 = -\ddot{x}_1 - 2\ddot{x}_3$$

$$\Longrightarrow \quad m_2 \ddot{x}_2 = -m_2 \ddot{x}_1 - 2 m_2 \ddot{x}_3 = -m_2 g + \lambda$$

$$\Longrightarrow \quad -\ddot{x}_1 - 2\ddot{x}_3 = -g + \frac{\lambda}{m_2}$$

$$= g - \frac{\lambda}{m_1} + 2g - \frac{4\lambda}{m_3} = -g + \frac{\lambda}{m_2}$$

$$\Longrightarrow \quad 4g = \lambda \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{4}{m_3} \right) \Longrightarrow \lambda = 4\mu g$$

mit $\mu := \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{4}{m_3} \right)^{-1} = \left(\frac{m_2 m_3 + m_1 m_3 + 4 m_1 m_2}{m_1 m_2 m_3} \right)^{-1}$

$$= \frac{m_1 m_2 (m_1 + m_2)}{m_1^2 + 6 m_1 m_2 + m_2^2}$$

$$\Longrightarrow \quad \boxed{\ddot{x}_3 = -g + 8 \frac{\mu}{m_3} g = g \left(8 \frac{\mu}{m_1 + m_2} - 1 \right)}$$