

Numerical Relativity: Simulating the Evolution of Binary Black Holes

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Introduction

Introduction

Example:



▶ SXS: Gravitational Lensing of GW150914

▶ Web Version

What is Numerical Relativity?

- ▶ science about numerical solutions to the Einstein field equations

Introduction

Example:

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What is Numerical Relativity?

- ▶ science about numerical solutions to the Einstein field equations

Why?

- ▶ prediction and explanation of cosmic phenomena and relativistic instabilities

Background

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Vacuum ($T_{\mu\nu} = 0$):

$$R_{\mu\nu} = 0$$

Analytical Solutions

Schwarzschild Metric:

$$(g_{\mu\nu})(t, r, \vartheta, \varphi) = \text{diag} \left(- \left(1 - \frac{2M}{r} \right), \left(1 - \frac{2M}{r} \right)^{-1}, r^2, r^2 \sin^2 \vartheta \right)$$

Other Solutions:

- ▶ Kerr metric
- ▶ Kerr-Newman metric

$$\ddot{x}^i + \Gamma_{mn}^i \dot{x}^m \dot{x}^n = 0$$

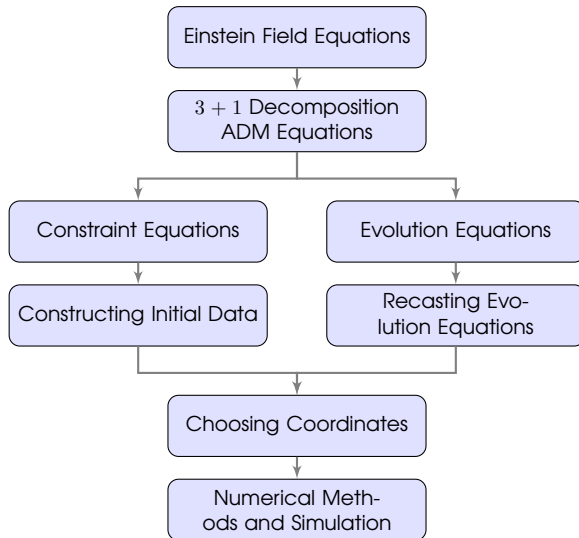
- ▶ freely-falling test particles and light move along geodesic curves

Examples:

- ▶ Newton versus Schwarzschild trajectories
- ▶ Kerr-Newman-Orbit

The Basic Idea

The Basic Idea



The Details

3 + 1 Decomposition

3 + 1 Decomposition: The ADM Equations

Constraint Equations:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$

$$D_j \left(K^{ij} - \gamma^{ij} K \right) = 8\pi S^i$$

$$\rho := n_a n_b T^{ab} \ , \quad S^i := -\gamma^{ij} n^a T_{aj}$$

3 + 1 Decomposition: The ADM Equations

Evolution Equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\begin{aligned} \partial_t K_{ij} = & \alpha \left(R_{ij} - 2K_{ik} K^k_j + K K_{ij} \right) - D_i D_j \alpha - 8\pi \alpha \left[S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho) \right] \\ & + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k \end{aligned}$$

$$S_{ij} := \gamma_{ia} \gamma_{jb} T^{ab} , \quad S := \gamma^{ij} S_{ij}$$

Constructing Initial Data

Constructing Initial Data: CTT Decomposition

Solve Constraint Equations for W^i and ψ :

$$\begin{aligned}(\bar{\Delta}_L W)^i - \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K &= 8\pi\psi^{10} S^i \\ 8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} &= -16\pi\psi^5 \rho\end{aligned}$$

$$\bar{A}^{ij} = \bar{A}_{\text{TT}}^{ij} + (\bar{L}W)^{ij}$$

Constructing Initial Data: CTT Decomposition

Construct Physical Solution:

$$\begin{aligned}\gamma_{ij} &= \psi^4 \bar{\gamma}_{ij} \\ K_{ij} &= A_{ij} + \frac{1}{3} \gamma_{ij} K \\ &= \psi^{-2} \bar{A}_{ij} + \frac{1}{3} \gamma_{ij} K\end{aligned}$$

Recasting The Evolution Equations

Recasting the Evolution Equations: The BSSN Formalism

Idea:

$$\gamma_{ij} = e^{4\varphi} \bar{\gamma}_{ij}$$

$$K_{ij} = e^{4\varphi} \tilde{A}_{ij} + \frac{1}{3} \gamma_{ij} K$$

$$\bar{\Gamma}^i := \bar{\gamma}^{jk} \bar{\Gamma}_{jk}^i = -\partial_j \bar{\gamma}^{ij}$$

Choosing Coordinates

Choosing Coordinates: The Lapse and Shift

Geodesic Slicing:

$$\alpha = 1, \quad \beta^i = 0$$

Choosing Coordinates: The Lapse and Shift

Geodesic Slicing:

$$\alpha = 1, \quad \beta^i = 0$$

Maximal Slicing:

$$K = 0 = \partial_t K \quad \implies \quad D^2 \alpha = \alpha \left[K_{ij} K^{ij} + 4\pi (\rho + S) \right]$$

Numerical Methods

Numerical Methods: Classification of PDE

$$A\partial_1^2\varphi + 2B\partial_1\partial_2\varphi + C\partial_2^2\varphi = \rho$$

Elliptic:

$$AC - B^2 > 0$$

Parabolic:

$$AC - B^2 = 0$$

Hyperbolic:

$$AC - B^2 < 0$$

Numerical Methods: Finite Difference Method

Examples of Stencils:

$$[\partial_1 \varphi]_{i,j} = \frac{[\varphi]_{i+1,j} - [\varphi]_{i-1,j}}{2h} + \mathcal{O}(h^2)$$

$$[\partial_1^2 \varphi]_{i,j} = \frac{-[\varphi]_{i-2,j} + 16[\varphi]_{i-1,j} - 30[\varphi]_{i,j} + 16[\varphi]_{i+1,j} - [\varphi]_{i+2,j}}{12h^2} + \mathcal{O}(h^4)$$

Results

Visualizing the results:

- ▶ [SXS: BBH head-on collision](#) ▶ [Web Version](#)
- ▶ [SXS: BBH orbit and collision](#) ▶ [Web Version](#)
- ▶ [SXS: Highly precessing BBH](#) ▶ [Web Version](#)

Going Further:

- ▶ [NASA: Colliding Neutron Stars](#) ▶ [Web Version](#)

References

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