Numerical Relativity: Simulating the Evolution of Binary Black Holes

Markus Pawellek

3. Juli 2018

Outline

Introduction

Introduction

Example:

- ▸ SXS: Gravitational Lensing of GW150914 ▸ Web Version

What is Numerical Relativity?

- science about numerical solutions to the Einstein field equations

Introduction

Example:

- ▸ SXS: Gravitational Lensing of GW150914 ▸ Web Version

What is Numerical Relativity?

- science about numerical solutions to the Einstein field equations

Why?

- prediction and explanation of cosmic phenomena and relativistic instabilities

Background

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Einstein Field Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Vacuum ($T_{\mu\nu} = 0$):

$$R_{\mu\nu} = 0$$

Analytical Solutions

Schwarzschild Metric:

$$(g_{\mu\nu})(t, r, \vartheta, \varphi) = \operatorname{diag}\left(-\left(1 - \frac{2M}{r}\right),\ \left(1 - \frac{2M}{r}\right)^{-1},\ r^2,\ r^2 \sin^2 \vartheta \right)$$

Other Solutions:

- Kerr metric
- Kerr-Newman metric

Geodesics

$$\ddot{x}^i + \Gamma^i_{mn} \dot{x}^m \dot{x}^n = 0$$

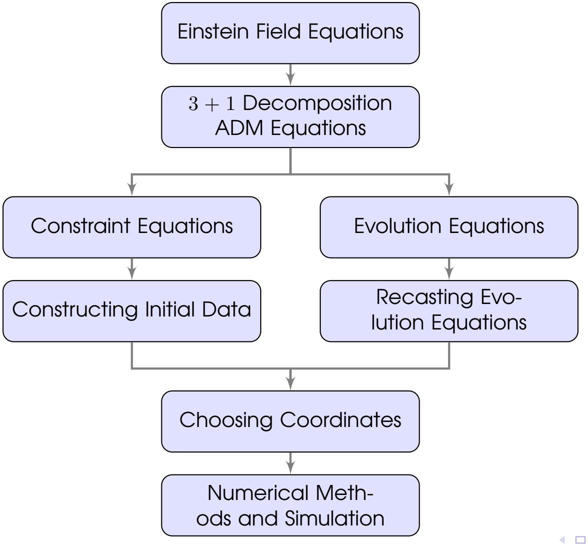
- freely-falling test particles and light move along geodesic curves

Examples:

- Newton versus Schwarzschild trajectories
- Kerr-Newman-Orbit

The Basic Idea

The Basic Idea



The Details

$3+1$ Decomposition

$3+1$ Decomposition: The ADM Equations

Constraint Equations:

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho$$
$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi S^i$$

$$\rho := n_a n_b T^{ab}\ , \qquad S^i := -\gamma^{ij} n^a T_{aj}$$

$3+1$ Decomposition: The ADM Equations

Evolution Equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i$$

$$\partial_t K_{ij} = \alpha \left(R_{ij} - 2K_{ik}K^k{}_j + K K_{ij} \right) - D_i D_j \alpha - 8\pi\alpha \left[S_{ij} - \frac{1}{2}\gamma_{ij} \left(S - \rho \right) \right]$$
$$+ \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k$$

$$S_{ij} := \gamma_{ia}\gamma_{jb} T^{ab} \ , \qquad S := \gamma^{ij} S_{ij}$$

Constructing Initial Data

Constructing Initial Data: CTT Decomposition

Solve Constraint Equations for W^i and ψ:

$$\left(\bar{\Delta}_L W\right)^i - \frac{2}{3}\psi^6 \bar{\gamma}^{ij} \bar{D}_j K = 8\pi\psi^{10} S^i$$

$$8\bar{D}^2\psi - \psi\bar{R} - \frac{2}{3}\psi^5 K^2 + \psi^{-7}\bar{A}_{ij}\bar{A}^{ij} = -16\pi\psi^5\rho$$

$$\bar{A}^{ij} = \bar{A}^{ij}_{\mathrm{TT}} + \left(\bar{L}W\right)^{ij}$$

Constructing Initial Data: CTT Decomposition

Construct Physical Solution:

$$
\begin{aligned}
\gamma_{ij} &= \psi^4 \bar{\gamma}_{ij} \\
K_{ij} &= A_{ij} + \frac{1}{3}\gamma_{ij} K \\
&= \psi^{-2} \bar{A}_{ij} + \frac{1}{3}\gamma_{ij} K
\end{aligned}
$$

Recasting The Evolution Equations

Recasting the Evolution Equations: The BSSN Formalism

Idea:

$$\gamma_{ij} = e^{4\varphi}\bar{\gamma}_{ij}$$

$$K_{ij} = e^{4\varphi}\tilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

$$\bar{\Gamma}^i := \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk} = -\partial_j\bar{\gamma}^{ij}$$

Choosing Coordinates

Choosing Coordinates: The Lapse and Shift

Geodesic Slicing:

$$\alpha = 1\,, \qquad \beta^i = 0$$

Choosing Coordinates: The Lapse and Shift

Geodesic Slicing:

$$\alpha = 1\,, \qquad \beta^i = 0$$

Maximal Slicing:

$$K = 0 = \partial_t K \qquad \Longrightarrow \qquad D^2\alpha = \alpha\left[K_{ij}K^{ij} + 4\pi\left(\rho + S\right)\right]$$

Numerical Methods

Numerical Methods: Classification of PDE

$$A\partial_1^2\varphi + 2B\partial_1\partial_2\varphi + C\partial_2^2\varphi = \rho$$

Elliptic:

$$AC - B^2 > 0$$

Parabolic:

$$AC - B^2 = 0$$

Hyperbolic:

$$AC - B^2 < 0$$

Numerical Methods: Finite Difference Method

Examples of Stencils:

$$[\partial_1 \varphi]_{i,j} = \frac{[\varphi]_{i+1,j} - [\varphi]_{i-1,j}}{2h} + \mathcal{O}\left(h^2\right)$$

$$\left[\partial_1^2 \varphi\right]_{i,j} = \frac{-[\varphi]_{i-2,j} + 16\,[\varphi]_{i-1,j} - 30\,[\varphi]_{i,j} + 16\,[\varphi]_{i+1,j} - [\varphi]_{i+2,j}}{12h^2} + \mathcal{O}\left(h^4\right)$$

Results

Visualizing the results:

- SXS: BBH head-on collision | Web Version
- SXS: BBH orbit and collision | Web Version
- SXS: Highly precessing BBH | Web Version

Going Further:

- NASA: Colliding Neutron Stars | Web Version

References

(1) *Colliding Neutron Stars Create Black Hole and Gamma-ray Burst*, April 2011. https://www.nasa.gov/topics/universe/features/gamma-ray-engines.html#.WA4phz_VTZE.

(2) *SXS Project: Simulating Extreme Spacetimes*, Juli 2018. https://www.black-holes.org/explore/movies.

(3) Baumgarte, Thomas W. and Stuart L. Shapiro: *On the numerical integration of einstein's field equations*. Physical Review D, 59(2):024007, 1998.

(4) Baumgarte, Thomas W. and Stuart L. Shapiro: *Numerical Relativity: Solving Einstein's Equations on the Computer*. Cambridge University Press, 2010.

(5) Hehl, F. W., R. A. Puntigam, and H. Ruder: *Relativity and Scientific Computing: Computer Algebra, Numerics, Visualization*. Springer, 1996.

(6) Meinel, Reinhard: *Spezielle und allgemeine Relativitätstheorie für Bachelorstudenten*. Springer Spektrum, 2016.

(7) Oloff, Rainer: *Geometrie der Raumzeit: Eine mathematische Einführung in die Relativitätstheorie*. Vieweg-Teubner, 5. Auflage, 2010.

(8) Pretorius, Frans: *Evolution of binary black-hole spacetimes*. Physical review letters, 95(12):121101, 2005.