Probabilistic Circuits: Marginal Maximum a Posteriori Queries

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January 21, 2022

Outline

Introduction

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- Marginal maximum a posteriori (MMAP) queries combine marginal (MAR) and maximum a posteriori (MAP) inference

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- Here, notations varies a little bit: no boldface letters

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- Joint distribution p over X

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- Partial state $e \in E$
- Generalized interval $\mathcal{I} \subset \mathrm{val}(Z)$

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$$\underset{q \in \mathrm{val}(Q)}{\arg\max}\; p\left(Q = q \mid E = e, Z \in \mathcal{I}\right) = \underset{q \in \mathrm{val}(Q)}{\arg\max} \int_{\mathcal{I}} p(q, e, z)\, \mathrm{d}Z$$

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$$\arg\max_{q \in \mathrm{val}(Q)} \int_{\mathcal{J}} p(q, e, z)\, \mathrm{d}Z$$

- **case** $Q = \emptyset$: MAR Query

$$\int_{\mathcal{J}} p(e, z)\, \mathrm{d}Z$$

- **case** $Z = \emptyset$: MAP Query

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$\mathcal{C}$ is tractable for both MAR and MAP queries
$\iff$ $\mathcal{G}$ is decomposable, smooth, and deterministic

Marginal Determinism

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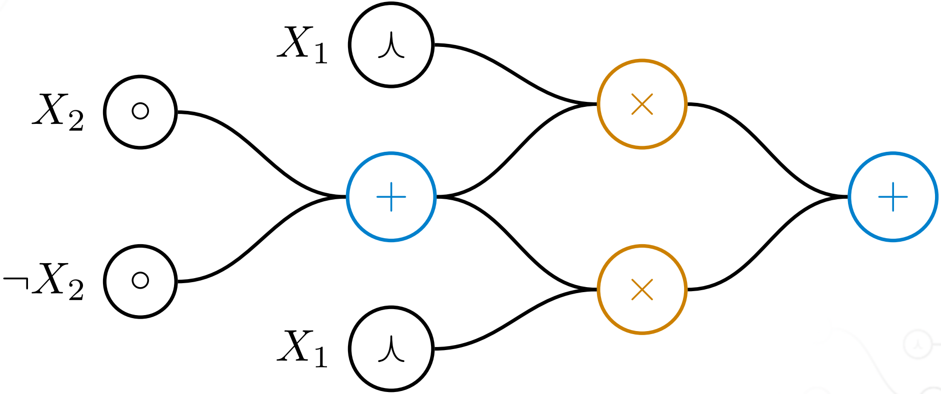
$\mathcal{G}$ is Q-marginal deterministic

$:\iff$ for all sum units $n \in \mathcal{G}$ with $\varphi(n) \cap Q \neq \emptyset$,

n is Q-marginal deterministic

Marginal Determinism: Example



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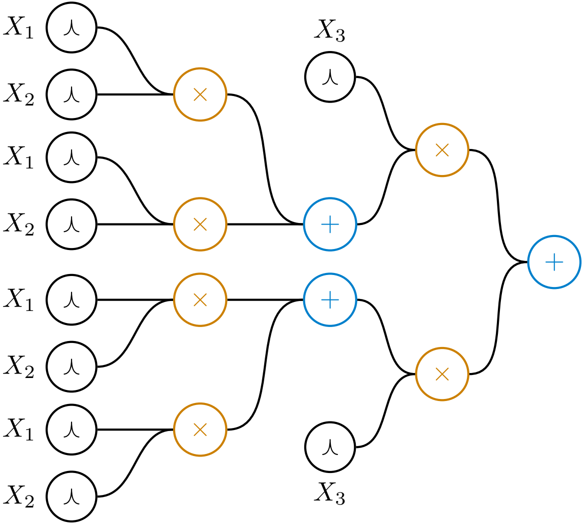

Tractable Computation

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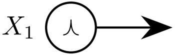

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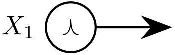
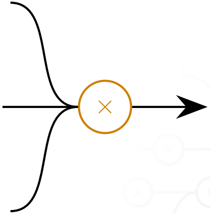
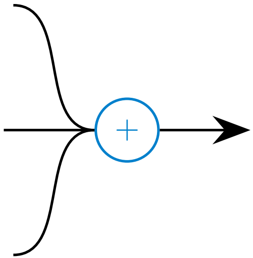


- Product units are handled identically in MAR and MAP

$$r_n \longleftarrow \prod_{c \in \mathrm{in}(n)} r_c$$



Tractable Computation: Sum Units



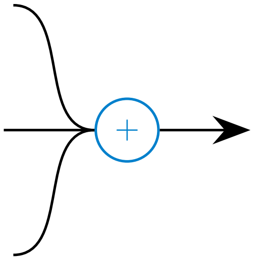
Tractable Computation: Sum Units

if $\varphi(n) \cap Q \neq \emptyset$ **then**

$$r_n \longleftarrow \max_{c \in \mathrm{in}(n)} \vartheta_{nc} r_c$$

else

$$r_n \longleftarrow \sum_{c \in \mathrm{in}(n)} \vartheta_{nc} r_c$$



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$\mathcal{G}$ be smooth, decomposable, and Q-marginal deterministic
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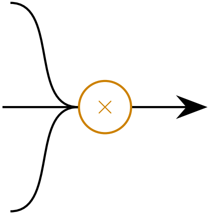
$$\mathcal{Q}(e, \mathcal{I}) = \max_{q \in \mathrm{val}(Q)} \int_{\mathcal{I}} \mathcal{C}(Z, q, e) \, \mathrm{d}Z$$

Sufficient Conditions: Product Units

- Apply decomposability and partition Z, Q, and E



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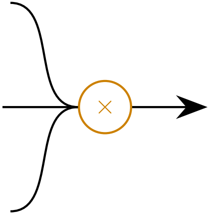

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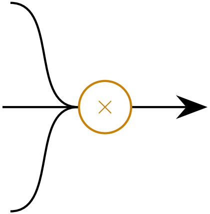

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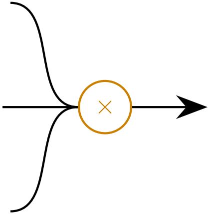

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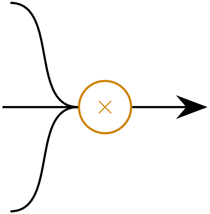

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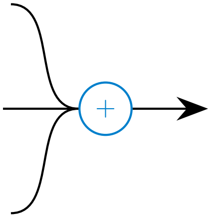
- Sum units with $\varphi(n) \cap Q = \emptyset$ reduce to MAR queries



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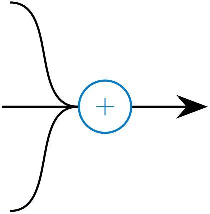

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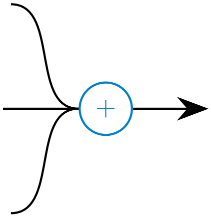

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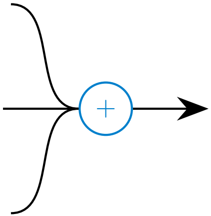

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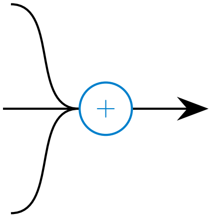

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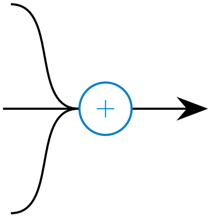

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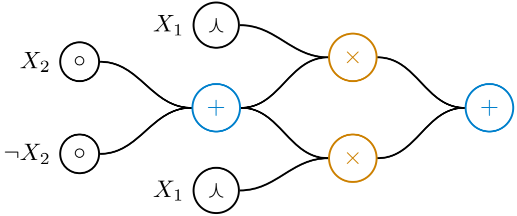
- Sum units compute maxima if support contains query variables

Conclusions: Outlook



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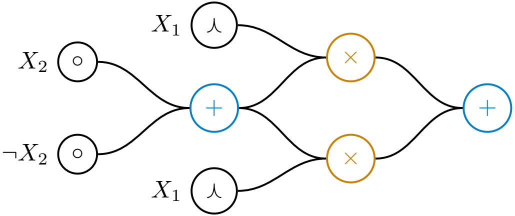

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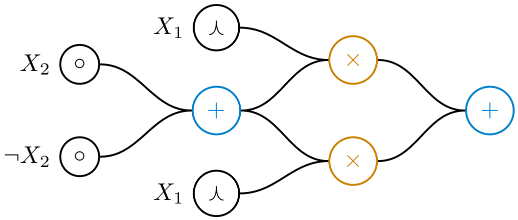

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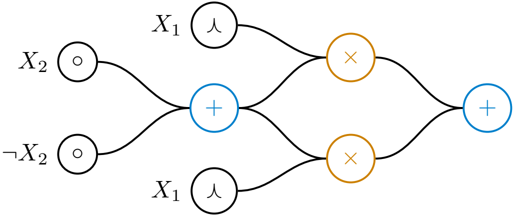

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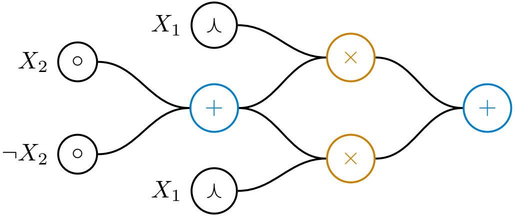
Thank you for Your Attention!



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References

(1) YooJung Choi, Antonio Vergari, and Guy Van den Broeck. "Probabilistic Circuits: A Unifying Framework for Tractable Probabilistic Models". In: (October 2020). URL: http://starai.cs.ucla.edu/papers/ProbCirc20.pdf (visited on 01/19/2022).