Probabilistic Circuits: Marginal Maximum a Posteriori Queries

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Outline

Introduction

Background

Marginal Determinism

Tractable Computation

Sufficient Conditions

Conclusions

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- Marginal maximum a posteriori (MMAP) queries combine marginal (MAR) and maximum a posteriori (MAP) inference

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- Here, notations varies a little bit: no boldface letters

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$$\operatorname*{arg\,max}_{q \in \mathrm{val}(Q)} \; p\,(Q = q \mid E = e, Z \in \mathfrak{I}) \;=\; \operatorname*{arg\,max}_{q \in \mathrm{val}(Q)} \int_{\mathfrak{I}} p(q, e, z)\, \mathrm{d}Z$$

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$$\operatorname*{arg\,max}_{q \in \mathrm{val}(Q)} \int_{\mathcal{J}} p(q, e, z) \, \mathrm{d}Z$$

- $\textbf{case } Q = \emptyset :$ MAR Query

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- **case** $Q = \emptyset$: MAR Query

$$\int_{\mathcal{J}} p(e, z)\, \mathrm{d}Z$$

- **case** $Z = \emptyset$: MAP Query

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$\mathcal{C}$ is tractable for both MAR and MAP queries $\iff$ $\mathcal{G}$ is decomposable, smooth, and deterministic

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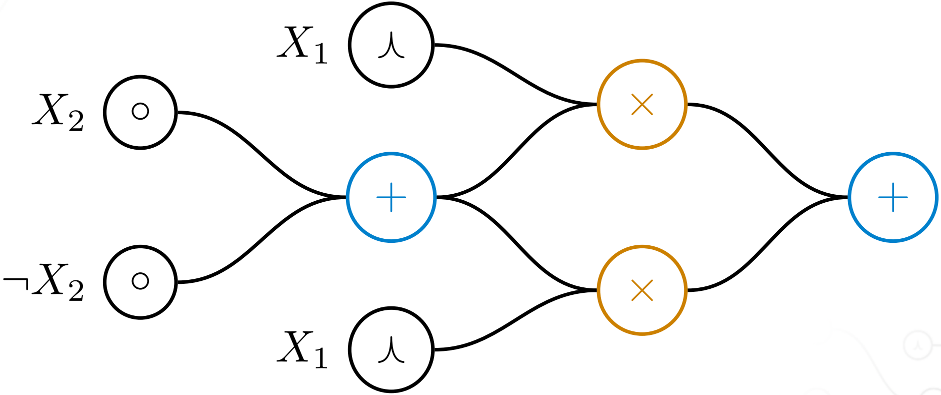
$\mathcal{G}$ is Q-marginal deterministic

$:\iff$ for all sum units $n \in \mathcal{G}$ with $\varphi(n) \cap Q \neq \emptyset$,

n is marginal deterministic with respect to Q

Marginal Determinism: Example



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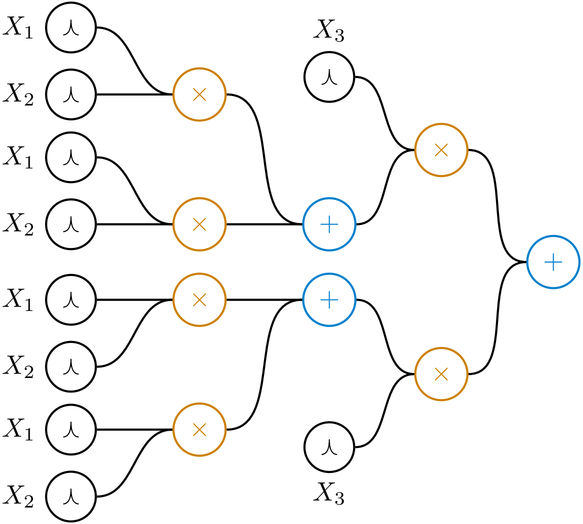

Tractable Computation

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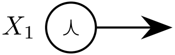

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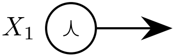
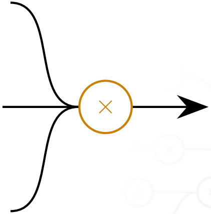
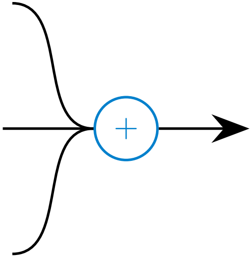


- Product units are handled identically in MAR and MAP

$$r_n \longleftarrow \prod_{c \in \mathrm{in}(n)} r_c$$



Tractable Computation: Sum Units



Tractable Computation: Sum Units

```
if $\varphi(n) \cap Q \neq \emptyset$ then
    $r_n \longleftarrow \max_{c \in \mathrm{in}(n)} \vartheta_{nc} r_c$
else
    $r_n \longleftarrow \sum_{c \in \mathrm{in}(n)} \vartheta_{nc} r_c$
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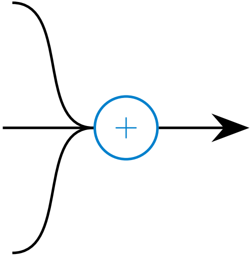

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Theorem:

$\mathcal{G}$ be smooth, decomposable, and Q-marginal deterministic

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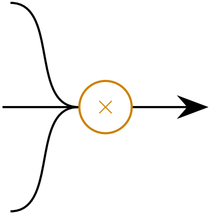
$$\mathcal{Q}(e, \mathcal{J}) = \max_{q \in \mathrm{val}(Q)} \int_{\mathcal{J}} \mathcal{C}(Z, q, e) \, \mathrm{d}Z$$

Sufficient Conditions: Product Units

- Apply decomposability and partition Z, Q, and E



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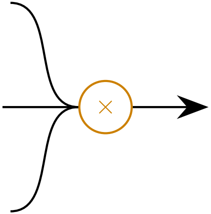

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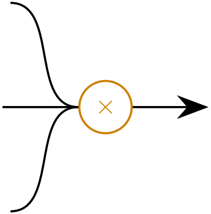

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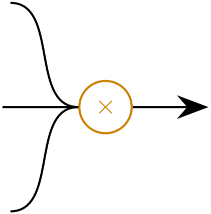

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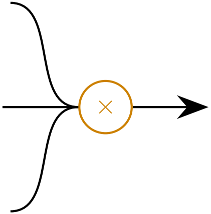

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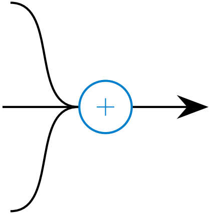
- Sum units with $\varphi(n) \cap Q = \emptyset$ reduce to MAR queries



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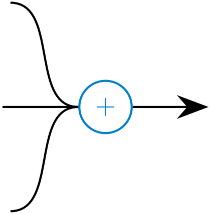

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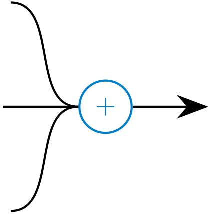

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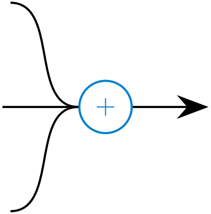

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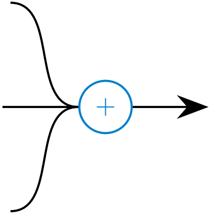

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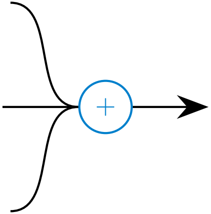

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- Sum units compute maxima if support contains query variables

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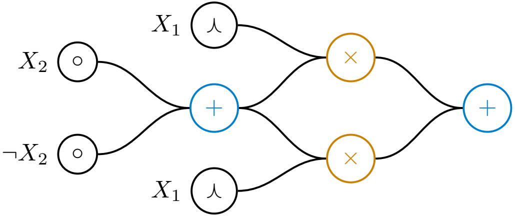

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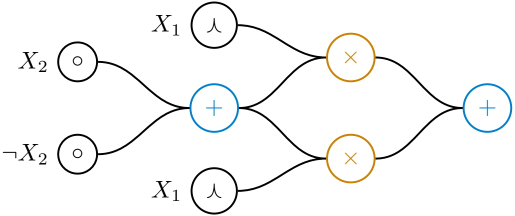

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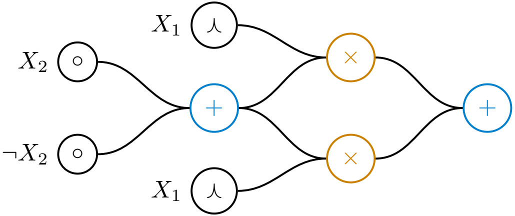

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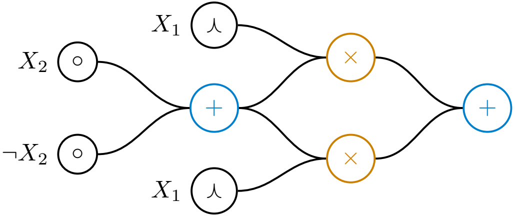

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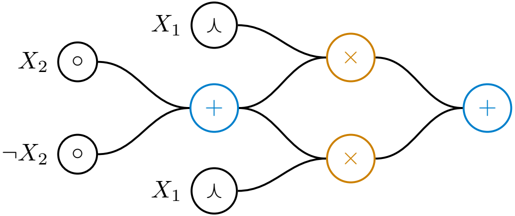
Thank you for Your Attention!



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References

[1] YooJung Choi, Antonio Vergari, and Guy Van den Broeck. "Probabilistic Circuits: A Unifying Framework for Tractable Probabilistic Models". In: (October 2020). URL: http://starai.cs.ucla.edu/papers/ProbCirc20.pdf (visited on 01/19/2022).