# Probabilistic Circuits: Marginal Maximum a Posteriori Queries

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# Outline

# Introduction

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- Marginal maximum a posteriori (MMAP) combine marginal (MAR) and maximum a posteriori (MAP) inference
- We need sufficient conditions for tractability
- We use MAR and MAP algorithm
- Notations varies a little bit: no boldface letters

# Background

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$X \ldots$ finite set of random variables

$p \ldots$ joint probability distribution over $X$

$\mathcal{C} \ldots$ PC over $X$ with $\mathcal{C} = (\mathcal{G}, \vartheta)$

$Q \ldots$ set of query variables with $Q \subset X$

# Background: MMAP Queries

- query variables $Q$, evidence variables $E$, marginal variables $Z$ form partition of $X$
- $e \in E$, intevals $\mathcal{I} \subset \mathrm{val}(Z)$

$$\operatorname*{arg\,max}_{q \in \mathrm{val}(Q)} \; p\left(Q = q \mid E = e, Z \in \mathcal{I}\right)$$

$$\operatorname*{arg\,max}_{q \in \mathrm{val}(Q)} \int_{\mathcal{I}} p(q, e, z) \, \mathrm{d}Z$$

# Background: MMAP Connection

$$\arg\max_{q \in \mathrm{val}(Q)} \int_{\mathcal{J}} p(q, e, z) \, \mathrm{d}Z$$

- **case** $Q = \emptyset$ : MAR Query

$$\int_{\mathcal{J}} p(e, z) \, \mathrm{d}Z$$

- **case** $Z = \emptyset$ : MAP Query

$$\arg\max_{q \in \mathrm{val}(Q)} p(q, e)$$

# Background: Review

$\mathcal{C}$ is tractable for MAR queries

$\iff$ $\mathcal{G}$ is decomposable and smooth

$\mathcal{C}$ is tractable for MAP queries

$\iff$ $\mathcal{G}$ is consistent and deterministic

$\mathcal{G}$ is decomposable $\implies$ $\mathcal{G}$ is consistent

$\mathcal{C}$ is tractable for both MAR and MAP queries

$\iff$ $\mathcal{G}$ is decomposable, smooth, and deterministic

# Marginal Determinism

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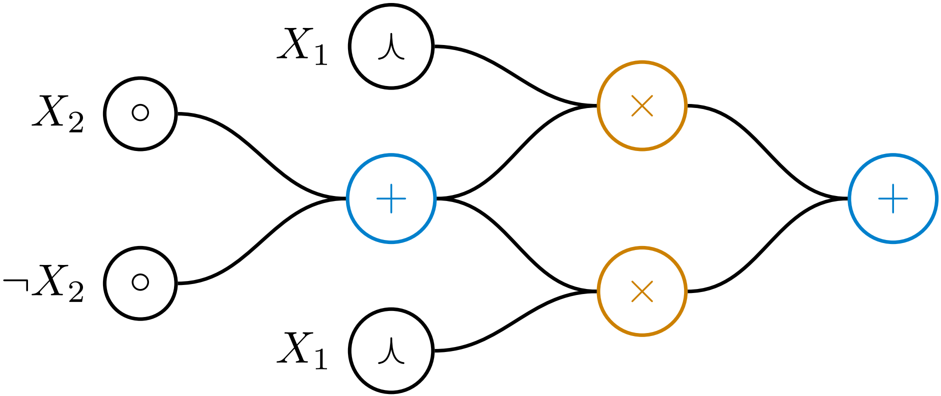
A sum node is marginal deterministic with respect to $Q$ if for all partial states $q \in Q$ at most one of its inputs is non-zero.

A circuit structure $\mathcal{G}$ is marginal deterministic with respect to $Q$ if for all sum units $n \in \mathcal{G}$ with $\varphi(n) \cap Q \neq \emptyset$, $n$ is marginal deterministic with respect to $Q$.

- Simple generalization of determinism
- Structural property about the support of input units
- Defined with respect to $Q$

# Marginal Determinism: Example



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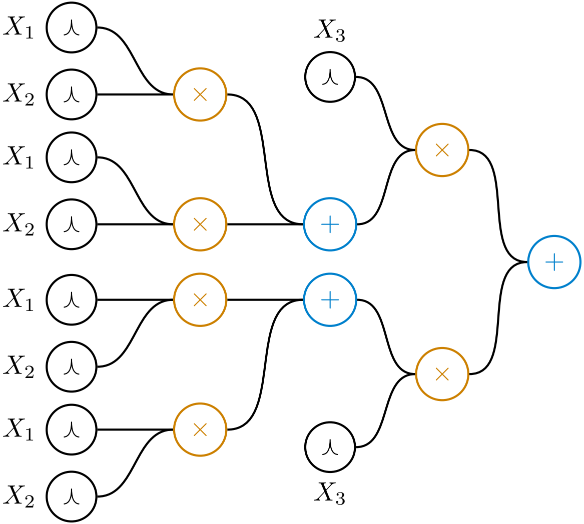

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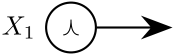

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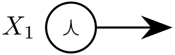
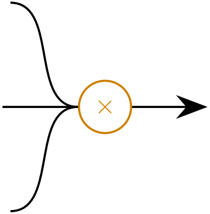



- Product units are handled identically in MAR and MAP

$$r_n \longleftarrow \prod_{c \in \mathrm{in}(n)} r_c$$



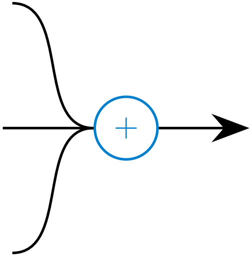
# Tractable Computation: Sum Units

**if** $\varphi(n) \cap Q \neq \emptyset$ **then**

$$r_n \longleftarrow \max_{c \in \mathrm{in}(n)} \vartheta_{nc} r_c$$

**else**

$$r_n \longleftarrow \sum_{c \in \mathrm{in}(n)} \vartheta_{nc} r_c$$



# Sufficient Conditions

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**Theorem:** Let $\mathcal{G}$ be smooth, decomposable, and $Q$-marginal deterministic. Then for any parameterization $\vartheta$ the above algorithm tractably computes MMAP queries of $\mathcal{C}$ over $Q$.

- Proof by induction
- Input units are correct by assumption

$$\mathcal{Q}(e, \mathcal{I}) = \max_{q \in \mathrm{val}(Q)} \int_{\mathcal{I}} \mathcal{C}(Z, q, e) \, \mathrm{d}Z$$

# Sufficient Conditions: Product Units

- Apply decomposability and partition $Z$, $Q$, and $E$

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&= \max_{i \in \mathrm{in}(n)} \vartheta_i \mathcal{Q}_i(e, \mathcal{I})
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- Complexity depends on set of query variables
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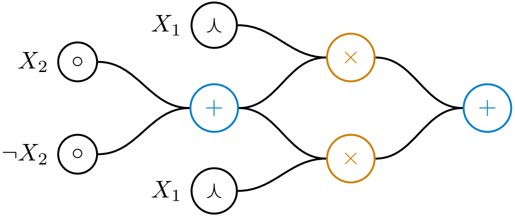
- sum units have to compute maxima when support contains query variables

# Conclusions: Outlook



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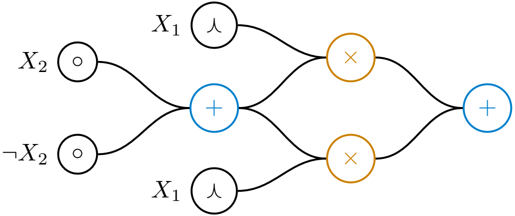


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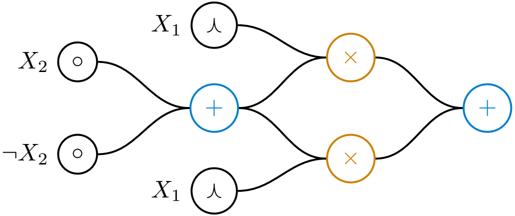


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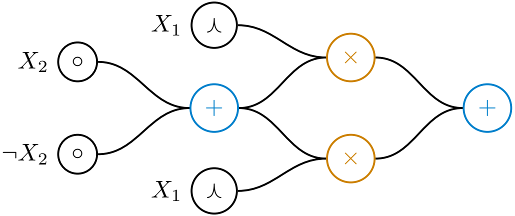


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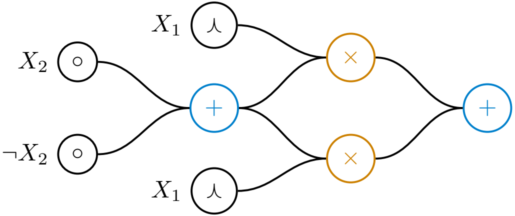
**Thank you for Your Attention!**

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# References

(1) YooJung Choi, Antonio Vergari, and Guy Van den Broeck. "Probabilistic Circuits: A Unifying Framework for Tractable Probabilistic Models". In: (October 2020). URL: http://starai.cs.ucla.edu/papers/ProbCirc20.pdf (visited on 01/19/2022).