Exam 1(100 points)

1. (15 points) State and prove **Gauss-Markov Theorem**.

2. (15 points) Let A be symmetric and a $p \times 1$ random vector $Y \sim N_p(0, \Sigma)$. Show that

$$Y^T AY \quad \sim \quad \sum_{j=1}^{r} \lambda_j \chi_j^2(1)$$

where $\chi_1^2(1), \chi_2^2(1) \ldots, \chi_r^2(1) r$ are independent Chi-squared random variable with degrees of freedom, df =1, $\chi^2(1)$. Specify r and λ_j exactly.

3. (20 points) Consider $Y = X\beta + \epsilon$ where X is of full rank and $\mathbf{E}(\epsilon) = 0$.
Let $\hat{\beta} =$ the ordinary least squares estimate of β=$(X^T X)^{-1} X^T Y$ with $\text{Cov}(\epsilon) = \sigma^2 I$ and $\tilde{\beta} =$ the generalized least squares estimate of β=$\tilde{\beta} = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1} y$ with $\text{Cov}(\epsilon) = \sigma^2 \Sigma$.

Prove that

$$\hat{\beta} = \tilde{\beta} \quad \text{if and only if} \quad \mathcal{C}(\Sigma^{-1} X) = \mathcal{C}(X)$$

4. (50 points) Let's consider a liner model

$$Y = X\beta + \epsilon, \qquad \epsilon \sim N_n(0, \sigma^2 I_n) \qquad (1)$$

where the $n \times p (p < n)$ matrix X is full rank. Let's decompose X into $X = (X_0, X_1)$ where $X_0 : n \times q$ and $X_1 : n \times (p - q)$. Let M and M_0 denote the orthogonal projection operator onto $\mathcal{C}(X)$ and $\mathcal{C}(X_0)$ respectively.

(a) (5 points) The model (1) can be rewritten as

$$Y = W\alpha + \epsilon, \qquad \epsilon \sim N_n(0, \sigma^2 I_n)$$

where $\mathcal{C}(W) = \mathcal{C}(X)$. Let M_W denote the orthogonal projection operator onto $\mathcal{C}(W)$. If any, find (a) condition(s) so that $M_W = M$.

(b) (5 points) Show that

$$\mathcal{C}(M - M_0) = \mathcal{N}(X_0^T) \qquad \text{with respect to } \mathcal{C}(X).$$

(c) (10 points) Explain in detail that in general,

$$M \neq M_0 + M_1 \qquad \text{where } M_1 \text{ is the orthogonal projection operator onto } \mathcal{C}(X_1).$$

What is(are) condition(s) so that $M_0 + M_1$ is orthogonal projection?
Suggest X_{1*} so that $\mathcal{C}(X) = \mathcal{C}(X_0) \oplus \mathcal{C}(X_{1*})$ and

$$M = M_0 + M_{1*} \qquad \text{where } M_{1*} \text{ is the orthogonal projection operator onto } \mathcal{C}(X_{1*}).$$

(d) (15 points) Find condition(s) so that β_0 is estimable. If β_0 is estimable, derive the test statistic for $H_0 : \beta_0 = 0$ vs $H_1 : \beta_0 \neq 0$.

(e) (15 points) Let $p = 2q$. Find condition(s) so that $\beta_0 + 2\beta_1$ is estimable. If $\beta_0 + 2\beta_1$ and $\text{Cov}(\epsilon) = 5$, develop the test statistic for $H_0 : \beta_0 + 2\beta_1 = b$ vs $H_1 : \beta_0 + 2\beta_1 \neq b$ where b is known.