WeiYa's Work Yard

A dog, who fell into the ocean of statistics, tries to write down his ideas and notes to save himself.

Bernstein Bounds

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Tags: Bernstein

I noticed that the papers of matrix/tensor completion always talk about the Bernstein inequality, then I picked the Bernstein Bounds discussed in Wainwright (2019).

A random variable X with mean $\mu=\mathbb{E}[X]$ is sub-exponential if there are non-negative parameter (ν,α) such that

$$\mathbb{E}[e^{\lambda(X-\mu)}]\le e^{\frac{\nu^2\lambda^2}{2}}\quad\text{for all }|\lambda|<\frac{1}{\alpha}\,.$$

Given a random variable X with mean $\mu=\mathbb{E}[X]$ and variance $\sigma^2=\mathrm{E}[X^2]-\mu^2$, the Bernstein's condition with parameter b holds if

$$|\mathbb{E}[(X-\mu)^k]|\le\frac{1}{2}k!\sigma^2b^{k-2}\quad\text{for }k=3,4,\ldots$$

One sufficient condition for Bernstein's condition to hold is that X be bounded.

For any random variable satisfying the Bernstein condition, we have

$$\mathbb{E}[e^{\lambda(X-\mu)}]\le e^{\frac{\lambda^2\sigma^2}{1-b|\lambda|}}\ \text{for all }|\lambda|<\frac{1}{b}\,,$$

and moreover, the concentration inequality

$$P[|X-\mu|\ge t]\le 2e^{-\frac{t^2}{2(\sigma^2+bt)}}\ \text{for all}t\ge 0\,.$$

A zero-mean symmetric random matrix Q satisfies a Bernstein condition with parameter $b>0$ if

$$\mathbb{E}[Q^j]\preceq\frac{1}{2}j!b^{j-2}\mathrm{var}(Q)\quad\text{for }j=3,4,\ldots$$

Let $Q_1,\ldots,Q_n$ be a sequence of independent, zero-mean, symmetric random matrices that satisfy the Bernstein condition with parameter $b>0$. Then for all $\delta>0$, the operator norm satisfies the tail bound

$$P\left[\|\frac{1}{n}\sum_{i=1}^nQ_i\|_2\ge\delta\right]\le 2\mathrm{rank}(\sum_{i=1}^n\mathrm{var}(Q_i))\exp\left\{-\frac{n\delta^2}{2(\sigma^2+b\delta)}\right\}.$$

Reference

Wainwright MJ. High-dimensional statistics: A non-asymptotic viewpoint. Cambridge University Press; 2019 Feb 21.
