Exam 1(100 points)

1. (10 points) Let's consider two matrices $A(n \times p)$ and $B(p \times n)$. Prove or disprove that the non-zero eigenvalues of AB are the same as those of BA.

2. (20 points) Let A be an $n \times n$ square matrix and a $n \times 1$ random vector $y \sim N_n(0, \Sigma)$ with a known Σ. Let $D = \{y | y \in \mathbf{R}^n, ||y|| = 1\}$.

(a) (15 points) Show that

$$y^T A y \quad \sim \quad \sum_{j=1}^{r} \lambda_j \chi_j^2(1)$$

where $\chi_1^2(1), \chi_2^2(1) \ldots, \chi_r^2(1)$ are independent Chi-squared random variables with degrees of freedom, df =1, $\chi^2(1)$. Specify r and λ_j exactly.

(b) (5 points) Find $\max_D(y^T A y)$ and $\min_D(y^T A y)$.

3. (30 points) Let's consider the following model

$$\begin{aligned} y_1 &= \alpha_1 - \alpha_2 + \epsilon_1 \\ y_2 &= -\alpha_2 - \alpha_3 + \epsilon_2 \\ y_3 &= \alpha_2 + \alpha_3 + \epsilon_3 \end{aligned}$$

where

$$\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^T \sim N(0, \sigma^2 \mathrm{I}_3)$$

Find the BLUE of $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ based on the model subject to the constraint $\alpha_2 = -\alpha_1$.

4. (40 points) Consider the one-way model

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \ i = 1, 2, 3, 4, \quad j = 1, 2 \quad \epsilon_{ij} \sim N(0, \sigma^2)$$

where all the ϵ_{ij} are independent. Suppose that we want to test the hypothesis H_0 that simultaneously $\alpha_1 - 2\alpha_2 + \alpha_3 = 0$ and $\alpha_1 + \alpha_2 + \alpha_3 - 3\alpha_4 = 0$.

(a) (10 points) Write the hypothesis as a general parametric hypothesis, in other words, if $\psi = A\beta$, find A. Is ψ estimable?

(b) (15 points) Let $\mathcal{C}$ denote the estimation space and $\mathcal{C}_0$ the estimation space for H_0 with $\mathcal{C}_0 \subset \mathcal{C}$. Express $\mathcal{C}$, $\mathcal{C}_0$ and $\mathcal{C} - \mathcal{C}_0$.

(c) (15 points) Develop the hypothesis-testing and suggest the test statistic.