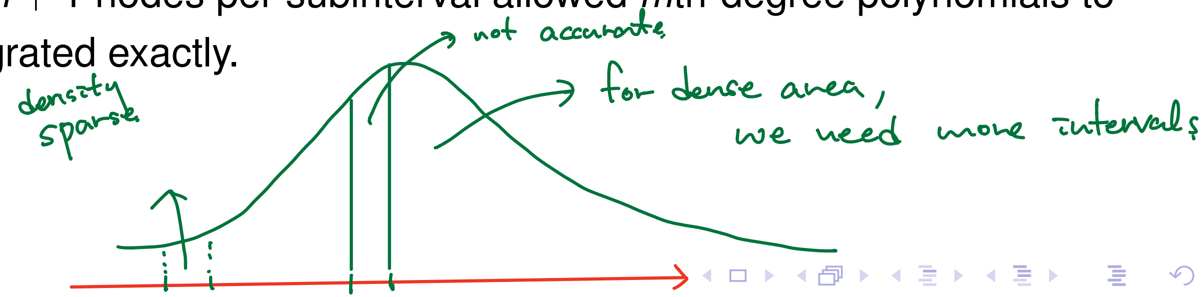
Gaussian Quadrature

Newton - Cote's rule ⇒ Assume all subintervals have equal length.

- All the Newton-Cotes rules are based on subintervals of equal length. The estimated integral is a sum of weighted evaluations of the integrand on a regular grid of points. Can we choose the grid point flexibly?
- For a fixed number of subintervals and nodes, only the weights may be flexibly chosen; we have limited attention to choices of weights that yield exact integrand of polynomials.
- Using $m+1$ nodes per subinterval allowed mth-degree polynomials to be integrated exactly.



Gaussian Quadrature

↳ Adaptively choose the grid ~~trade-off~~ between efforts parameter specification and accuracy

- An important question is the amount of improvement that can be achieved if the constraint of evenly spaced nodes and subintervals is removed.

interval. f(x_i). → equal length. ↳ remove ⇒ specify weights.

- By allowing both the weights and the nodes to be freely chosen, we have twice as many parameters to use in the approximation of f.

→ two types of parameter.

- If we consider that the value of an integral is predominantly determined by regions where the magnitude of the integrand is large, then it makes sense to put more nodes in such regions.
- With a suitably flexible choice of $m+1$ nodes, $x_0, \cdots, x_m$, and corresponding weights, $A_0, \cdots, A_m$, exact integration of $2(m+1)$th-degree polynomials can be obtained using

$\int_a^b f(x)\,dx = \sum_{i=0}^{m} A_i f(x_i)$

adaptively choose.

$$\int_a^b f(x)dx = \sum_{i=0}^{m} A_i f(x_i).$$

We can use Gaussian quadrature

Gaussian Quadrature

$\int_a^b f(x)\underbrace{w(x)}_{\text{non-negative function (assume density)}}dx = E_w[f(x)]\Big|_a^b$ $\quad \int_a^b x^k w(x)dx = E[x^k]\Big|_a^b$

- This approach, called Gaussian quadrature, can be extremely effective for integrals like $\int_a^b f(x)w(x)dx$ where w is a nonnegative function and $\int_a^b x^k w(x)dx < \infty$ for all $k \geq 0$.
- These requirements are reminiscent of density function with finite moments.
- It is often useful to think of w as a density, in which case integrals like expected values and Bayesian posterior normalizing constants are natural candidates for Gaussian quadrature.
- This method is more generally applicable, however, by defining $f^*(x) = f(x)/w(x)$ and applying the method to $\int_a^b f^*(x)w(x)dx$.

$$\int_a^b f^*(x)w(x)dx = \int_a^b \frac{f(x)}{w(x)} w(x)dx.$$

5.1, 5.3, 5.4

Excluding from HW. (Do not solve 5.6)

Gaussian Quadrature

(Orthogonal Polynomial)

$P_k(x)$: A generic polynomial of degree K.

- Let $p_k(x)$ denote a generic polynomial of degree k. For convenience in what follows, assume that the leading coefficient of $p_k(x)$ is positive.

$\int_a^b f(x)^2 w(x) < \infty$ f: square-integrable w.r.t w on [a.b]

- If $\int_a^b f(x)^2 w(x) dx < \infty$, then the function f is said to be square-integrable with respect to w on $[a, b]$. For any f and g in square-integrable w.r.t w on $[a, b]$, there inner product w.r.t. w on $[a, b]$ is defined to be

$< f, g >_{w,[a.b]} = \int_a^b f(x) g(x) w(x) dx. = 0$

function $f(x) \perp g(x)$ w.r.t w on [a.b]

$$< f, g >_{w,[a,b]} = \int_a^b f(x)g(x)w(x)dx.$$

- If $< f, g >_{w,[a,b]} = 0$, then f and g are said to be orthogonal w.r.t. w on $[a, b]$.

$< f. f >_{w,[a.b]} = \int_a^b (f(x))^2 w(x) dx = 1$

f. g orthonormal

- If also f and g are scaled so that $< f, f >_{w,[a,b]} = < g, g >_{w,[a,b]} = 1$, then f and g are orthonormal w.r.t. w on $[a, b]$.

$< g. g >_{w.[a.b]} = \int_a^b (g(x))^2 w(x) dx = 1$

Gaussian Quadrature

W : nonnegative [a.b]
$\Rightarrow$ there exist a sequence of polynomial $P_k(x)$ that are orthogonal w.r.t w on [a.b]

- Given any w that is nonnegative on $[a, b]$, there exists a sequence of polynomials $\{p_k(x)\}_{k=0}^{\infty}$ that are orthogonal w.r.t w on $[a, b]$.

  $<f.g>_{w,[a.b]} = 0 \Rightarrow <c.f, g>_{w,[a.b]} = 0$
- This sequence is not unique without some form of standardization because $< f, g >_{w,[a,b]} = 0$ implies $< cf, g >_{w,[a,b]} = 0$ for any constant c.
- A common choice is to set the leading coefficient of $p_k(x)$ equal to 1.
- For use in Gaussian quadrature, the randge of integration is also customarily transformed from $[a, b]$ to a range $[a^*, b^*]$ whose choice depends on w.

Gaussian Quadrature

- A set of standardized, orthogonal polynomials can be summarized by a recurrence relation

$$p_k(x) = (\alpha_k + x\beta_k)\, p_{k-1}(x) - \gamma_k p_{k-2}(x)$$

  for appropriate choices of α_k, β_k, and γ_k that vary with k and w.
- The roots of any polynomial in such a standardized set are all in (a^*, b^*). These roots will serve as nodes for Gaussian quadrature.

Gaussian Quadrature

Denote the roots of $p_{m+1}(x)$ by $a < x_0 < \cdots < x_m < b$. Then there exist weights $A_0, \cdots, A_m$ such that:

1. $A_i > 0$ for $i = 0, \cdots, m$.
2. $A_i = -c_{m+2} / \left[c_{m+1} p_{m+2}(x_i) p'_{m+1}(x_i)\right]$, where c_k is the leading coefficient of $p_k(x)$.
3. $\int_a^b f(x)w(x)dx = \sum_{i=0}^m A_i f(x_i)$ whenever f is a polynomial of degree not exceeding $2m+1$. In other words, the method is exact for the expectation of any such polynomial with respect to w.
4. If f is $2(m+1)$ times continuously differentiable, then

$$\int_a^b f(x)w(x)dx - \sum_{i=0}^m A_i f(x_i) = \frac{f^{(2m+2)}(\xi)}{(2m+2)!c_{m+1}^2}$$

for some $\xi \in (a, b)$.

Gaussian Quadrature

The way of choosing $\alpha_k, \beta_k, \gamma_k$ and $w(x)$ to use the recursive formula.

TABLE 5.6 Orthogonal polynomials, their standardizations, their correspondence to common density functions, and the terms used for their recursive generation. The leading coefficient of a polynomial is denoted c_k. In some cases, variants of standard definitions are chosen for best correspondence with familiar densities.

Name (Density)	$w(x)$	Standardization (a^*, b^*)	α_k β_k γ_k
Jacobi[a] (Beta)	$(1-x)^{p-q}x^{q-1}$	$c_k = 1$ $(0, 1)$	See [2, 516]
Legendre[a] (Uniform)	1	$p_k(1) = 1$ $(0, 1)$	$(1-2k)/k$ $(4k-2)/k$ $(k-1)/k$
Laguerre (Exponential)	$\exp\{-x\}$	$c_k = (-1)^k/k!$ $(0, \infty)$	$(2k-1)/k$ $-1/k$ $(k-1)/k$
Laguerre[b] (Gamma)	$x^r \exp\{-x\}$	$c_k = (-1)^k/k!$ $(0, \infty)$	$(2k-1+r)/k$ $-1/k$ $(k-1+r)/k$
Hermite[c] (Normal)	$\exp\{-x^2/2\}$	$c_k = 1$ $(-\infty, \infty)$	0 1 $k-1$

[a] Shifted.

[b] Generalized.

[c] Alternative form.

Frequently Encountered Problems

$\int_a^b f(x)\,dx \rightarrow \int_{-\infty}^{\infty} f(x)\,dx$ or $\int_0^{\infty} f(x)\,dx$

Easiest way to avoid : transformation

Useful transformation : $1/x$, $\frac{\exp(x)}{1+\exp(x)}$, $\exp(-x)$, $\frac{x}{1+x}$

$\Rightarrow$ May face sigularity issues.

- Range of Integration
- Integrands with Singularities or Other Extreme Behavior
- Multiple Integrals
- Adaptive Quadrature

$\hookrightarrow$ (Ex)

$$\int_0^1 \frac{\exp(x)}{\sqrt{x}}\,dx = 2\int_0^1 \exp(u^2)\,du$$

sigularity denominator = 0.

Find a proper transformation

$u = \sqrt{x}$.

Another way of handling singularity

: Subtract out the singularity

$$\int_{-\pi/2}^{\pi/2} \log(\sin^2 x)\, dx$$ singularity at $x = 0$

We cannot take log 0

$$= \underbrace{\int_{-\pi/2}^{\pi/2} \log(\sin^2 x / x^2)\, dx}_{\hookrightarrow \text{ Numerical Integration}} + \underbrace{\int_{-\pi/2}^{\pi/2} \log x^2\, dx}_{\hookrightarrow \text{ Analytically calculate}}$$

Multiple integral

$$\int_{a_1}^{b_1} \cdots \int_{a_p}^{b_p} f(x_1, \cdots, x_p)\, dx_p \cdots dx_1$$

include p times integration

let. each integration requires n intervals

$\Rightarrow$ We need n^p intervals

total

The number of interval increases exponentially

$\Rightarrow$ Monte Carlo Method

$\Rightarrow$ Adaptive Quadrature.