Exam 1(100 points)

1. (15 points) Prove the following :
Let $\mathcal{M}$ be a vector space, and let $\mathcal{N}$ be a subspace of $\mathcal{M}$. The orthogonal complement of $\mathcal{N}$ with respect to $\mathcal{M}$ is a subspace of $\mathcal{M}$; and if $x \in \mathcal{M}$, x can be written uniquely as $x = x_0 + x_1$ with $x_0 \in \mathcal{N}$ and $x_1 \in \mathcal{N}_{\mathcal{M}}^{\perp}$. The ranks of these spaces satisfy the relation $\text{rank}(\mathcal{M}) = \text{rank}(\mathcal{N}) + \text{rank}(\mathcal{N}_{\mathcal{M}}^{\perp})$.

2. (15 points) Let X be an $n \times p$ matrix. Prove or disprove that every vector in $\mathcal{R}^n$ is in either $\mathcal{C}(X)$ or $\mathcal{C}(X)^{\perp}$ or both.

3. (15 points) Let M_1 and M_2 be perpendicular projection matrices on R^n. $(M_1 + M_2)$ is the perpendicular projection matrix onto $\mathcal{C}(M_1, M_2)$ if and only if $\mathcal{C}(M_1) \perp \mathcal{C}(M_2)$.

4. (15 points) Let M_1 and M_2 be perpendicular projection matrices, and let M_0 be a perpendicular projection operator onto $\mathcal{C}(M_1) \cap \mathcal{C}(M_2)$. Show that the following are equivalent:
(a) $M_1 M_2 = M_2 M_1$
(b) $M_1 M_2 = M_0$
(c) $\mathcal{C}(M_1 - M_0) \perp \mathcal{C}(M_2 - M_0)$

5. (40 points) Consider $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$.

(a) (10 points) Perform the Singular Value Decomposition (SVD) of X.

(b) (10 points) Find the eigenvalues of $X^T X$, $M = X(X^T X)^- X^T$ and $I - M$ respectively.

(c) (10 points) Find $\mathcal{C}(M)$, $\mathcal{C}\left((X^T X)^{-1}\right)$, $\mathcal{N}(XX^T)$ and $\mathcal{N}(X^T X)$.

(d) (10 points) Prove or disprove that $\mathcal{C}(I - M) = \mathcal{C}(X^T)^{\perp}$.