Exam 2(100 points)

1. (20 points) Let A be symmetric and a $p \times 1$ random vector $Y \sim N_p(0, \Sigma)$. Show that

$$Y^T A Y \quad \sim \quad \sum_{j=1}^{r} \lambda_j \chi_j^2(1)$$

where $\chi_1^2(1), \chi_2^2(1) \ldots, \chi_r^2(1)$ are the independent Chi-squared random variable with degrees of freedom, df $=1$, $\chi^2(1)$. Specify r and λ_j exactly.

2. (30 points) Let $Y = X\beta + \epsilon$ and $\epsilon \sim N_n(0, \sigma^2\Sigma)$ where X is an $n \times p$ full-rank matrix $(p < n)$ and Σ is a known positive definite matrix.

(a) (5 points) Find the best linear unbiased estimator(BLUE) $\hat{\beta}$ of β.

(b) (5 points) Find $\text{Cov}(\hat{\beta})$ from **(a)**.

(c) (20 points) Let's partition X into $X = (X_1, X_2, X_3)$ where $X_1 : \; n \times r$, $X_2 : \; n \times q$ and $X_3 : \; n \times (p - r - q)$. Let's partition β into $\beta^T = (\beta_1^T, \beta_2^T, \beta_3^T)$ where $\beta_1 : \; r \times 1$, $\beta_2 : \; q \times 1$ and $\beta_3 : \; (p - r - q) \times 1$. Then the model can be rewritten as

$$Y = X\beta + \epsilon = X_1\beta_1 + X_2\beta_2 + X_3\beta_3 + \epsilon$$

Suppose that $\mathcal{C}(X_2) \perp \mathcal{C}(X_1, X_3)$ so that

$$\mathcal{C}(X) = \mathcal{C}(X_2) \oplus \mathcal{C}(X_1, X_3)$$

Develop a test for $H_0 : \beta_1 = 0$ vs $H_a : \beta_1 \neq 0$.

3. (10 points) Let $y_i = \beta x_i + \epsilon_i, i = 1, 2$, where $\epsilon_1 \sim N(0, \sigma^2)$ and $\epsilon_2 \sim N(0, 2\sigma^2)$, and ϵ_1 and ϵ_2 are statistically independent. If $x_1 = 1$ and $x_2 = -1$, then obtain the BLUE of β and find the variance of your estimator.

4. (40 points) Suppose that $y \in \mathbf{R}^2$, $\text{Var}(y) = \sigma^2 I$ and $E(y_i) = \beta_1 - \beta_2, i = 1, 2$

(a) (10 points) Which of the following are estimable: $\beta_1, \beta_2, \beta_1 - \beta_2, \beta_1 + \beta_2$?

(b) (15 points) For the estimable functions, find the best linear unbiased estimators and their variances.

(c) (15 points) Now consider the following mean structure:

$$\begin{aligned} E(y_1) &= (1 - c)\beta_1 - (1 + c)\beta_2 \\ E(y_2) &= (1 + c)\beta_1 - (1 - c)\beta_2 \end{aligned}$$

where $|c|$ is a small number. Repeat **(a)** and **(b)**. What happens to the variances as $c \to 0$?