2020321170 송종호 Assignment 3

1.

If U is unitary, then $UU^\dagger = I$. Thus,

$$U|v\rangle = \lambda|v\rangle \Rightarrow \langle v|U^\dagger = \langle v|U^*$$

and vice versa.

Combining both leads to

$$\langle v|v\rangle = \langle v|U^\dagger U|v\rangle = \langle v|\lambda^*\lambda|v\rangle = |\lambda|^2\langle v|v\rangle$$

Assuming $\lambda ! = 0$, we thus have $|\lambda|^2 = 1$.

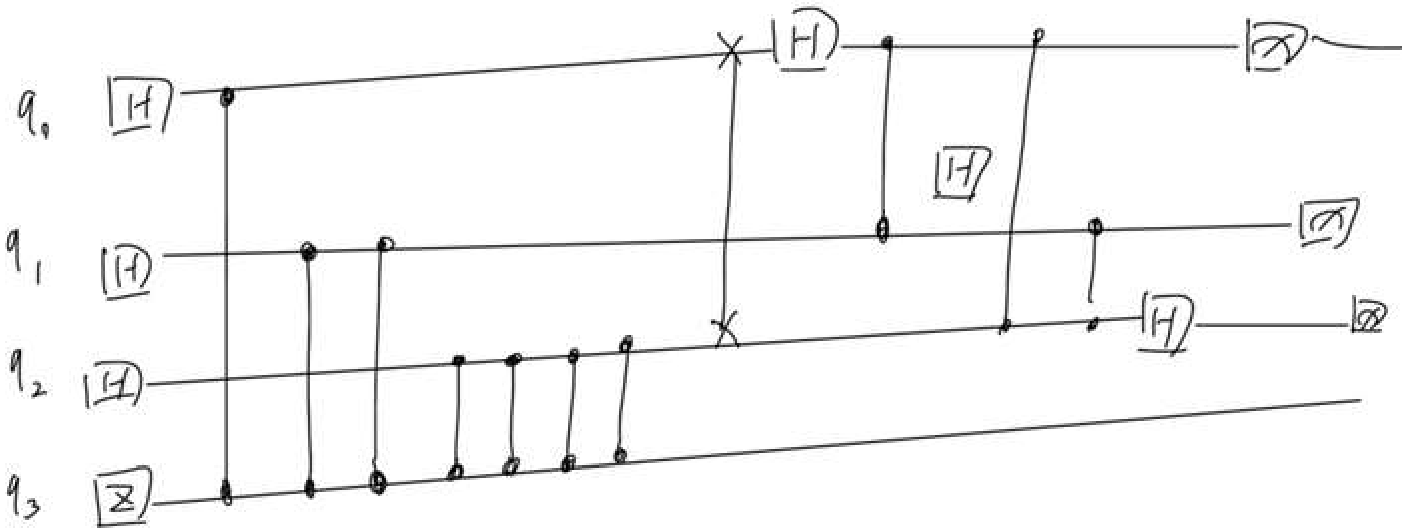
2.

a.

$$\sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

at here, eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, and eigenvalue of each vectors are $1, -1, -1, 1$.

b.



c.

Starting from $Ax = \lambda x$, we have

$$ABx = BAx = B\lambda x = \lambda Bx$$

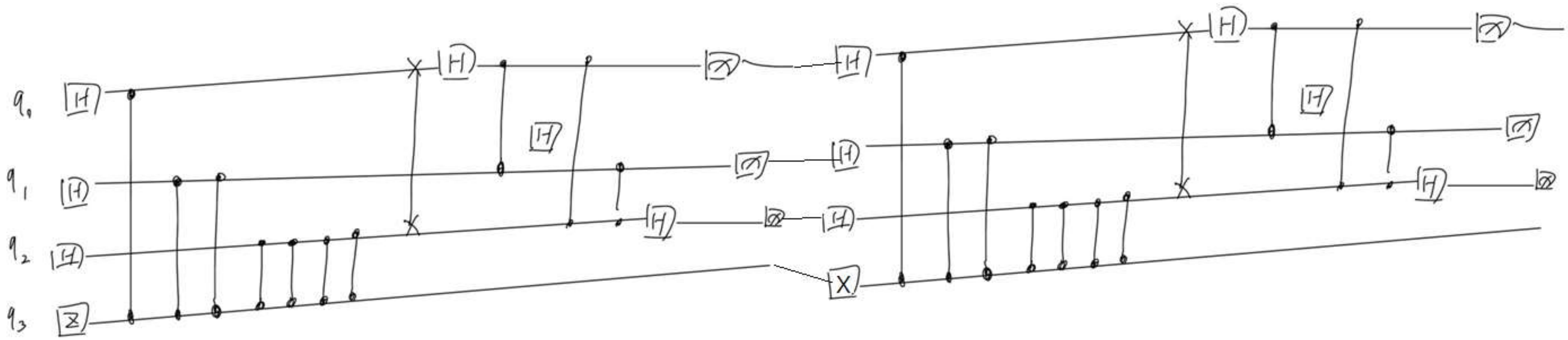
Thus x and Bx are both eigenvectors of A, sharing the same λ (or else $Bx = 0$). If we assume for convenience that the eigenvalues of A are distinct – the eigenspaces are one dimensional – then Bx must be a multiple of x. In other words, x is an eigenvector of B as well as A.

d.

$$A = \sigma_x \otimes \sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ and } BA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

e.

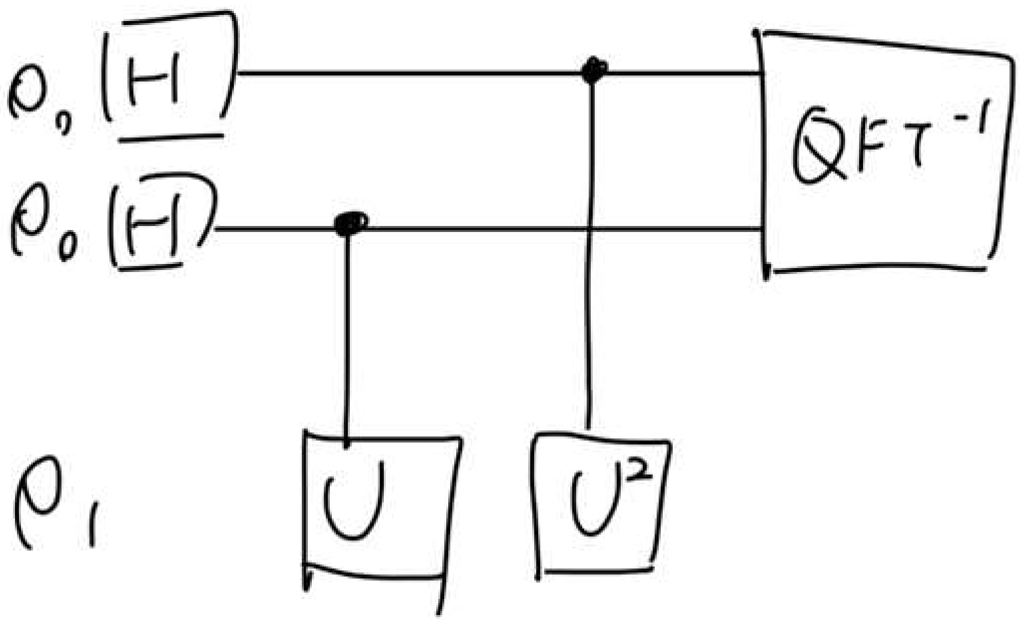


3.

a.

$$\langle \sigma_x \rangle = \frac{1}{2}(\langle 0| + e^{-2\pi}\langle 1|)\sigma_x(|0\rangle + e^{-2\pi}|1\rangle) = 2e^{-2\pi}$$

b.



c.

$$
\begin{aligned}
\frac{1}{2}\langle\sigma_x\rangle + \frac{1}{2}\langle\sigma_z\rangle &= \frac{1}{2}\left\{\frac{1}{2}(\langle 0| + e^{-4\pi}\langle 1|)\sigma_x(|0\rangle + e^{-4\pi}|1\rangle)\frac{1}{2}(\langle 0| + e^{-2\pi}\langle 1|)\sigma_z(|0\rangle + e^{-2\pi}|1\rangle)\right\} && (1)\\
&\quad + \frac{1}{2}\left\{\frac{1}{2}(\langle 0| + e^{-4\pi}\langle 1|)\sigma_z(|0\rangle + e^{-4\pi}|1\rangle)\frac{1}{2}(\langle 0| + e^{-2\pi}\langle 1|)\sigma_x(|0\rangle + e^{-2\pi}|1\rangle)\right\} && (2)\\
&= \frac{1}{2}\{\frac{1}{2}(2e^{-4\pi}) + \frac{1}{2}(2e^{-2\pi})\} + \frac{1}{2}\{\frac{1}{2}(1 - e^{-8\pi}) + \frac{1}{2}(1 - e^{-4\pi})\} && (3)\\
&= \frac{1}{4}\left(2 - e^{-8\pi} + e^{-4\pi} + 2e^{-2\pi}\right) && (4)
\end{aligned}
$$

4.

This circuit estimates the phase of a unitary operator U. It estimates θ in $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$, where $|\psi\rangle$ is an eigenvector and $e^{2\pi i\theta}$ is the corresponding eigenvalue. The circuit operates in the following steps:

1. Setup: $|\psi\rangle$ is in one set of qubit registers. An additional set of n qubits form the counting register on which we will store the value $2^n\theta$:

$$|\psi_0\rangle = |0\rangle^{\otimes n}|\psi\rangle$$

2. Superposition: Apply a n-bit Hadamard gate operation $H^{\otimes n}$ on the counting register:

$$|\psi_1\rangle = \frac{1}{2^{\frac{n}{2}}}(|0\rangle + |1\rangle)^{\otimes n}|\psi\rangle$$

3. Controlled Unitary Operations: We need to introduce the controlled unitary CU that applies the unitary operator U on the target register only if its corresponding control bit is $|1\rangle$. Since U is a unitary operator with eigenvector $|\psi\rangle$ such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$, this means:

$$U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{2\pi i\theta}|\psi\rangle = \cdots = e^{2\pi i 2^j\theta}|\psi\rangle$$

Applying all the n controlled operations CU^{2^j} with $0 \le j \le n-1$, and using the relation

$$|0\rangle \otimes |\psi\rangle + |1\rangle \otimes e^{2\pi i\theta}|\psi\rangle = \Big(|0\rangle + e^{2\pi i\theta}|1\rangle\Big) \otimes |\psi\rangle\$$$

we could get below.

$$\begin{aligned}|\psi_2\rangle &= \frac{1}{2^{\frac{n}{2}}}\Big(|0\rangle + e^{2\pi i\theta 2^{n-1}}|1\rangle\Big) \otimes \cdots \otimes \Big(|0\rangle + e^{2\pi i\theta 2^1}|1\rangle\Big) \otimes \Big(|0\rangle + e^{2\pi i\theta 2^0}|1\rangle\Big) \otimes |\psi\rangle \\ &= \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle \otimes |\psi\rangle\end{aligned}$$

where k denotes the integer representation of n-bit binary numbers.

1. Inverse Fourier Transform: Notice that the above expression is exactly the result of applying a quantum Fourier transform as we derived in the notebook on Quantum Fourier Transform and its Qiskit Implementation. Recall that QFT maps an n-qubit input state $|x\rangle$ into an output as

$$QFT|x\rangle = \frac{1}{2^{\frac{n}{2}}}\Big(|0\rangle + e^{\frac{2\pi i}{2}x}|1\rangle\Big) \otimes \Big(|0\rangle + e^{\frac{2\pi i}{2^2}x}|1\rangle\Big) \otimes \ldots \otimes \Big(|0\rangle + e^{\frac{2\pi i}{2^{n-1}}x}|1\rangle\Big) \otimes \Big(|0\rangle + e^{\frac{2\pi i}{2^n}x}|1\rangle\Big)$$

Replacing x by $2^n\theta$ in the above expression gives exactly the expression derived in step 2 above. Therefore, to recover the state $|2^n\theta\rangle$, apply an inverse Fourier transform on the auxiliary register. Doing so, we find

$$|\psi_3\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{2\pi i\theta k}|k\rangle \otimes |\psi\rangle \xrightarrow{\mathcal{QF}_n^{-1}} \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{2\pi i k}{2^n}(x-2^n\theta)}|x\rangle \otimes |\psi\rangle$$

Let's assume that there is a problem that time complexity is $O(f(n))$. In traditional computers, if there are n resources, the processing capacity is n, so time complexity is linear. In quantum computers, if n resources exist, the processing capacity is 2^n, so the time complexity is nonlinear and will be proportional to $1/2^n$. However, quantum computers do not solve problems that existing computers cannot solve well because they are just increase of processing power, not change of processing power. This means that only P-problems can be solved, and quantum computers only solve problems that can be solved before, but not new problems at all.

5.

a.

original state can be rewritten as $|\Psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle$,where $|\alpha\rangle \equiv \frac{1}{\sqrt{N-M}}\sum_{x=f^{-1}(0)}|x\rangle, |\beta\rangle \equiv \frac{1}{\sqrt{M}}\sum_{x=f^{-1}(1)}|x\rangle$.

let $G = V * U_f$. then

1. the oracle U_f, performs a reflection about the vector $|\alpha\rangle$ which is orthogonal to $|\beta\rangle$:

$$U(\cos\theta|\alpha\rangle + \sin\theta|\beta\rangle) = \cos\theta|\alpha\rangle - \sin\theta|\beta\rangle$$

2. the vector $\cos\theta|\alpha\rangle - \sin\theta|\beta\rangle$ is by $V = 2|\Psi\rangle\langle\Psi| - I$ flipped about $|\Psi\rangle$

3. after k iteration the state, the product of two reflection is a rotation:

$$G^k|\Psi\rangle = \cos(2k+1)\theta|\alpha\rangle + \sin(2k+1)\theta|\beta\rangle$$

b.

모르겠습니다....

6.