Exercise 1 (100 points): November 10, 2020

1. (50 points) Let's consider a liner model

$$Y = X\beta + \epsilon = X_0\beta_0 + X_1\beta_1 + \epsilon, \qquad \epsilon \sim N_n(0, \sigma^2 I_n) \tag{1}$$

where the $n \times p (p < n)$ matrix X is full rank. Let's decompose X into $X = (X_0, X_1)$ where $X_0 : n \times q$ and $X_1 : n \times (p-q)$. Let M and M_0 denote the orthogonal projection operator onto $\mathcal{C}(X)$ and $\mathcal{C}(X_0)$ respectively.

(a) (20 points) $(X^TX)^{-1}$ is given by

$$(X^TX)^{-1} =$$
$$= \begin{bmatrix} (X_0^TX_0)^{-1} + (X_0^TX_0)^{-1}X_0^TX_1BX_1^TX_0(X_0^TX_0)^{-1} & -(X_0^TX_0)^{-1}X_0^TX_1B \\ -BX_1^TX_0(X_0^TX_0)^{-1} & B \end{bmatrix}$$

where

$$B = \left[X_1^TX_1 - X_1^TX_0(X_0^TX_0)^{-1}X_0^TX_1\right]^{-1}$$

Prove or disprove algebraically that

$$M = M_0 + M_{1*}$$

where M_{1*} is the orthogonal projection operator onto $\mathcal{C}(X_{1*})$ with $X_{1*} = (I - M_0)X_1$.

(b) (15 points) Consider U and P for $\Lambda^T\beta$ in textbook, prove or disprove that

$$\mathcal{C}(MP) = \mathcal{C}(X_{1*})$$

(c) (15 points) Let $p = 2q$. Find condition(s) so that $\beta_0 + 2\beta_1$ is estimable. If $\beta_0 + 2\beta_1$ and $\text{Cov}(\epsilon) = 5I_n$, develop the test statistic for $H_0 : \beta_0 + 2\beta_1 = b$ vs $H_1 : \beta_0 + 2\beta_1 \neq b$ where b is known.

2. (10 points) **Exercise 3.7**
Show that $\rho^TMY = \rho^T[M\rho(\rho^TM\rho)^-\rho^TM]Y$ so that to estimate $\rho^TX\beta$, one only needs the perpendicular projection of Y onto $\mathcal{C}(M\rho)$.

3. (20 points) **Exercise 3.9.1**

4. (20 points) **Exercise 3.9.3**
Consider a set of seemingly unrelated regression equations

$$Y_i = X_i\beta_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma^2 I), \quad i = 1, \ldots, r$$

where X_i is an $n_i \times p$ matrix and the ϵ_is are independent.
Find the test for $H_0 : \beta_1 = \ldots = \beta_r$.