A Unified Approach to Balanced ANOVA Models

We can develop a unified approach to obtaining orthogonal projection operators in arbitrary balanced k-way ANOVA models by exploiting the structure of the design matrix. The structure of the design matrix can be easily examined using Kronecker products. Therefore, before we proceed further, we need to establish some more properties of Kronecker products.

Recall the definition of a Kronecker product.

$$A \otimes B = (a_{ij}B).$$

Defn

Suppose A is an $n \times p$ matrix, written as $A = (a_1, \ldots, a_p)$, where a_i is the ith column of A. Then

$$\text{Vec}(A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix}.$$

Thus $\text{Vec}(A)$ is an $np \times 1$ vector. The Vec operator stacks the columns of A into one long vector.

Properties of Kronecker Products

1) $A \otimes 0 = 0 \otimes A = 0$.

2) $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$.

3) $B \otimes (A_1 + A_2) = B \otimes A_1 + B \otimes A_2$.

4) $(\alpha A) \otimes (\beta B) = (\alpha\beta)(A \otimes B)$, where α and β are scalars.

5) $(A_1 A_2) \otimes (B_1 B_2) = (A_1 \otimes B_1)(A_2 \otimes B_2)$.

6) $(A \otimes B)^- = A^- \otimes B^-$.

7) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

8) $(A \otimes B)' = A' \otimes B'$.

9) $\mid A \otimes B \mid = \mid A \mid^m \mid B \mid^n$, where $A_{n\times n}$ and $B_{m\times m}$.

10) $tr(A \otimes B) = tr(A)tr(B)$.

11) The eigenvalues of $A \otimes B$ are $\lambda_{ia}^m \lambda_{jb}^n$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where λ_{ia}, $i = 1, \ldots, n$ are the eigenvalues of A, and λ_{jb}, $j = 1, \ldots, m$ are the eigenvalues of B, and $A_{n\times n}$ and $B_{m\times m}$ are symmetric. The eigenvector of $\lambda_{ia}^m \lambda_{jb}^n$ is $v_{ia} \otimes v_{jb}$, where v_{ia} is the eigenvector of λ_{ia} and v_{jb} is the eigenvector of λ_{jb}.

12) $\text{Vec}(ABC) = (A \otimes C')\text{Vec}(B)$.

Consider the balanced two-way ANOVA model with interaction. This model is given by

$$Y_{ijk} = \mu + \alpha_i + \eta_j + \gamma_{ij} + \epsilon_{ijk} \ ,$$

where $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, N$, and $n = abN$.

We want to write

$$C(M) = C(M_\mu) + C(M_\alpha) + C(M_\eta) + C(M_\gamma)$$

and be able to compute the orthogonal projection operators in an easy and unified way.

We can represent each subspace making up $C(M)$ in terms of Kronecker products. Once we do this, we can easily obtain the orthogonal projection operator for that space.

Notation: Let s be an arbirary index. Define J_s as the $s \times 1$ vector of ones, $P_s = \frac{1}{s} J_s J_s'$ and $Q_s = I_s - P_s$, where I_s is the $s \times s$ identity matrix. Thus, P_s is the orthogonal projection operator onto $C(J_s)$ and Q_s is the orthogonal projection operator onto $C(J_s)^\perp$.

Facts

1) Recall that the orthogonal projector operator onto $C(A)$ is always given by $A(A'A)^- A'$.

2) If M is an orthogonal projection operator, then $M^- = M$.

Kronecker Product forms for the Orthogonal Projection Operators

1) Computing M_μ.

We can write $J_n = J_a \otimes J_b \otimes J_N$, so that M_μ is the orthogonal projection operator onto $C(J_a \otimes J_b \otimes J_N)$. Thus by Fact 1 above, we have

$$\begin{aligned}
&M_\mu \\
&= (J_a \otimes J_b \otimes J_N) \left((J_a' \otimes J_b' \otimes J_N')(J_a \otimes J_b \otimes J_N) \right)^- (J_a' \otimes J_b' \otimes J_N') \\
&= (J_a \otimes J_b \otimes J_N)(J_a' J_a \otimes J_b' J_b \otimes J_N' J_N)^- (J_a' \otimes J_b' \otimes J_N') \\
&= (J_a \otimes J_b \otimes J_N)(abN)^- (J_a' \otimes J_b' \otimes J_N') \\
&= \frac{1}{a} J_a J_a' \otimes \frac{1}{b} J_b J_b' \otimes \frac{1}{N} J_N J_N' \\
&= P_a \otimes P_b \otimes P_N .
\end{aligned}$$

Using the properties of Kronecker products, it can easily be shown that

$$M = I_a \otimes I_b \otimes P_N.$$

The error space is $C(I - M)$ and

$$\begin{aligned} I - M &= I_{abN} - M \\ &= (I_a \otimes I_b \otimes I_N) - (I_a \otimes I_b \otimes P_N) \\ &= (I_a \otimes I_b) \otimes (I_N - P_N) \\ &= I_a \otimes I_b \otimes Q_N. \end{aligned}$$

Observe that

$$\begin{aligned} M + I - M &= (I_a \otimes I_b \otimes P_N) + I_a \otimes I_b \otimes Q_N \\ &= (I_a \otimes I_b) \otimes (P_N + Q_N) \\ &= (I_a \otimes I_b) \otimes I_N \\ &= I_a \otimes I_b \otimes I_N \\ &= I_n. \end{aligned}$$

We can summarize the subspaces and the orthogonal projection operators for the two-way ANOVA model as follows.

Effect	Subspace	Orthogonal Projection Operator
μ	$C(J_a \otimes J_b \otimes J_N)$	$P_a \otimes P_b \otimes P_N$
α	$C(Q_a \otimes J_b \otimes J_N)$	$Q_a \otimes P_b \otimes P_N$
η	$C(J_a \otimes Q_b \otimes J_N)$	$P_a \otimes Q_b \otimes P_N$
γ	$C(Q_a \otimes Q_b \otimes J_N)$	$Q_a \otimes Q_b \otimes P_N$
Error	$C(I - M)$	$I_a \otimes I_b \otimes Q_N$
Total		$I_a \otimes I_b \otimes I_N$

Exercise

Consider the three-way ANOVA model

$$\begin{aligned} Y_{ijkl} &= \mu + \alpha_i + \eta_j + \gamma_k + (\alpha\eta)_{ij} + (\alpha\gamma)_{ik} + (\eta\gamma)_{jk} \\ &+ (\alpha\eta\gamma)_{ijk} + \epsilon_{ijkl} \end{aligned}$$

where $i = 1, \dots, a$, $j = 1, \dots, b$, $k = 1, \dots, c$, and $l = 1, \dots, N$.

a) Write out the subspaces and all orthogonal projection operators corresponding to each term in the ANOVA model completely in terms of Kronecker products.

b) Find the simplest expression for $M_\mu + M_\alpha + M_\eta$.