RCS 103

Introduction

Lecture 1

1. Numbers
2. Numbers and real line
3. Counting, Natural, Integers, Real numbers,
4. Rational Numbers
5. Intervals
6. Inequalities and Equations
7. Absolute Value

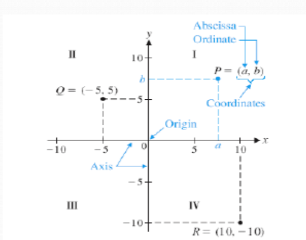
Lecture 2

1. Cartesian Coordinates
2. Increment and Distance
3. Functions and their graphs
4. Straight Lines
5. Functions

Cartesian Coordinates and axis

Recall that to form a **Cartesian or rectangular coordinate system, we select two real** number lines one horizontal and one vertical and let them cross through their origins as indicated in following Figure.

Cartesian Coordinates and axis



The Cartesian (rectangular) coordinate system

Cartesian Coordinates and axis

Now we want to assign *coordinates to each point in the plane. Given an arbitrary point P in the plane, pass horizontal and vertical lines through the point*

The vertical line will intersect the horizontal axis at a point with coordinate *a, and the horizontal line will intersect the vertical axis at a point with coordinate b.*

Cartesian Coordinates and axis

These two numbers, written as the **ordered pair** (a, b)* ***form the coordinates of the*** point P. *The first coordinate, a, is called the* ***abscissa of P; the second coordinate, b,*** is called the **ordinate of P.**
The abscissa of Q in Figure 1 is -5 and the ordinate of Q is 5.The coordinates of a point can also be referenced in terms of the axis labels.

The ***x coordinate of R in Figure 1 is 10, and the y coordinate of R is The point with*** coordinates (0, 0) is called the **origin.**

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Increment and distance

When a particle moves from one point to another, the net change in its coordinate are called increment.
They are calculated by subtracting the coordinates of the starting point from those of the end point
An increment in a variable is the net change in the value of the variable. If x changes from x1 to x2 the increment in x is

$$\Delta x = x_2 - x_1.$$

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Increment and distance

Example 2

Find the increments in the coordinates of a particle that moves from $A(3, -3)$ to $B(-1, 2)$.

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Increment and distance

Solution 2
The increment are

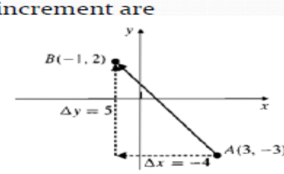

Figure P.11 Increments in x and y

$$\Delta x = -1 - 3 = -4 \quad \text{and} \quad \Delta y = 2 - (-3) = 5.$$

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Increment and distance

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, the straight line segment PQ is the hypotenuse of a right triangle PCQ, as shown in Figure P.12. The sides PC and CQ of the triangle have lengths

$$|\Delta x| = |x_2 - x_1| \quad \text{and} \quad |\Delta y| = |y_2 - y_1|.$$

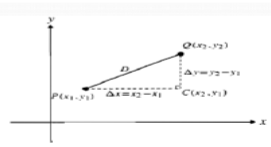

Figure P.12 The distance from P to Q is

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Increment and distance

These are the *horizontal distance* and *vertical distance* between P and Q. By the Pythagorean Theorem, the length of PQ is the square root of the sum of the squares of these lengths.

Distance formula for points in the plane

The distance D between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

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Increment and distance

Example 3
Distance between

$A(3, -3)$ and $B(-1, 2)$

in P.11, is

$$\sqrt{(-1-3)^2 + (2-(-3))^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41} \text{ units.}$$

Increment and distance

Example 4

The distance from the origin $O(0, 0)$ to a point $P(x, y)$ is

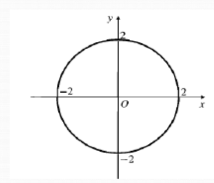
Functions and their Graphs

The **graph** of an equation (or inequality) involving the variables x and y is the set of all points $P(x, y)$ whose coordinates satisfy the equation (or inequality).

Example 4

The equation $x^2 + y^2 = 4$ represents all points $P(x, y)$ whose distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{4} = 2$. These points lie on the **circle** of radius 2 centred at the origin. This circle is the graph of the equation $x^2 + y^2 = 4$. (See Figure P.13(a).)

Functions and their Graphs



Functions and their Graphs

Example 5

Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$.

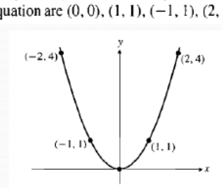


Figure P.14 The parabola $y = x^2$

Straight lines

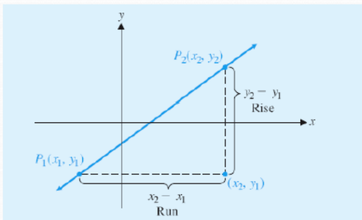
Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$, respectively, the **run** and the **rise** between P_1 and P_2. Two such points always determine a unique **straight line** (usually called simply a **line**) passing through them both. We call the line P_1P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The constant $m = \Delta y / \Delta x$ is called the **slope** of the nonvertical line.

Straight lines



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Straight lines

Example 6

The slope of the line joining $A\ (3, -3)$ and $B\ (-1, 2)$ is

?

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Straight lines

Exercise 2

Find the slope of the line through each pair of points.

(A) $(-2, 4), (3, 4)$ (B) $(-2, 4), (0, -4)$

(C) $(-1, 5), (-1, -2)$ (D) $(-1, -2), (2, 1)$

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Straight lines

The equation

$$y = mx + b \quad m = \text{slope}, b = y\text{intercept}$$

is called the **slope-intercept form** of an equation of a line.

An equation of a line with slope m that passes through (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

which is called the **point-slope form** of an equation of a line.

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Straight lines

Exercise 3

1. Find an equation for the line that has slope $\frac{1}{2}$ and passes through $(-4, 3)$. Write the final answer in the form $Ax + By = C$.
2. Find an equation for the line that passes through the points $(-3, 2)$ and $(-4, 5)$. Write the resulting equation in the form $y = mx + b$.

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Straight lines

Applications

Cost Equation The management of a company that manufactures skateboards has fixed costs (costs at 0 output) of $300 per day and total costs of $4,300 per day at an output of 100 skateboards per day. Assume that cost *C is linearly related to* output *x*.

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Straight lines

1. Find the slope of the line joining the points associated with outputs of 0 and 100; that is, the line passing through (0, 300) and (100, 4,300).
2. Find an equation of the line relating output to cost. Write the final answer in the form $C = mx + b$.
3. Graph the cost equation from part (B) for $0 \le x \le 200$.

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Straight lines

Solution:

(1)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4{,}300 - 300}{100 - 0} = \frac{4{,}000}{100} = 40$$

(2) We must find an equation of the line that passes through (0, 300) with slope 40. We use the slope-intercept form:

$$C = mx + b$$
$$C = 40x + 300$$

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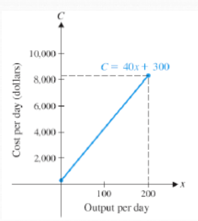
Straight lines

Solution:

(3)



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Straight lines

In this Example, the *fixed cost of \$300 per day covers plant cost, insurance, and* so on. This cost is incurred whether or not there is any production. The *variable cost is 40x, which depends on the day s output. Since increasing production from x* to will increase the cost by \$40 (from), the slope 40 can be interpreted as the **rate of change of the cost function with respect to** production *x*.

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Straight lines

Exercise 4

Supply and Demand At a price of \$9.00 per box of oranges, the supply is 320,000 boxes and the demand is 200,000 boxes. At a price of \$8.50 per box, the supply is 270,000 boxes and the demand is 300,000 boxes.

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Straight lines

1. Find a price–supply equation of the form $p = mx + b$, where p is the price in dollars and x is the corresponding supply in thousands of boxes.
2. Find a price–demand equation of the form $p = mx + b$, where p is the price in dollars and x is the corresponding demand in thousands of boxes.
3. Graph the price–supply and price–demand equations in the same coordinate system and find their point of intersection.

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Straight lines

Exercise 5

Cost analysis. A plant can manufacture 50 tennis rackets per day for a total daily cost of \$3,855 and 60 tennis rackets per day for a total daily cost of \$4,245.

(A) Assuming that daily cost and production are linearly related, find the total daily cost of producing x tennis rackets.

(B) Graph the total daily cost for $0 \le x \le 100$.

(C) Interpret the slope and y intercept of this cost equation.

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Straight lines

Exercise 6

Boiling point. The temperature at which water starts to boil is called its **boiling point** and is linearly related to the altitude. Water boils at 212°F at sea level and at 193.6°F at an altitude of 10,000 feet. (*Source*: biggreenegg.com)

(A) Find a relationship of the form $T = mx + b$ where T is degrees Fahrenheit and x is altitude in thousands of feet.

(B) Find the boiling point at an altitude of 3,500 feet.

(C) Find the altitude if the boiling point is 200°F.

(D) Graph T and illustrate the answers to (B) and (C) on the graph.

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Functions

In many practical situations, the value of one quantity may depend on the value of a second.

For example;

1. The consumer demand for beef may depend on the current market price;
2. The amount of air pollution in a metropolitan area may depend on the number of cars on the road;

Such relationships can often be represented mathematically as **functions.**

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Functions

A function consists of two sets and a rule that associates elements in one set with elements in the other.

For instance, suppose you want to determine the effect of price on the number of units of a particular commodity that will be sold at that price.

To study this relationship, you need to know the set of admissible prices, the set of possible sales levels, and a rule for associating each price with a particular sales level.

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Functions

Definition:

A **function is a rule that assigns to each object in a set** ***A exactly*** one object in a set *B. The set A is called the* ***domain of the function, and the set*** of assigned objects in *B is called the* ***range.***

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Functions

The domain and range will be collections of real numbers and the function itself will be denoted by a letter such as *f.*

The value that the function *f assigns to the number x in the domain is then denoted by f (x) (read as "f of x"), which is often given by a formula, such as*

$$f(x) = x^2 + 4.$$

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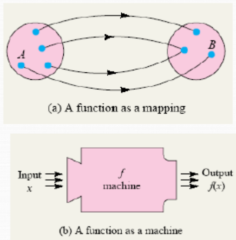
Functions

Interpretations of the function $f(x)$



(a) A function as a mapping

(b) A function as a machine

Functions

For instance, the function $f(x) = x^2 + 4$ *can be thought of as an "f machine" that accepts* an input x, *then squares it and adds* 4 to *produce an output* $y = x^2 + 4$.

Note: A *function assigns one and only one number in the range (output) to each number in the domain (input).*

Functions

Example 1

1 Find $f(3)$ if $f(x) = x^2 + 4$.

2. If $g(t) = (t - 2)^{1/2}$, find (if possible) $g(27)$, $g(5)$, $g(2)$, and $g(1)$.

3. Find $f\left(-\frac{1}{2}\right)$, $f(1)$, and $f(2)$ if

$$f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 1 \\ 3x^2 + 1 & \text{if } x \geq 1 \end{cases}$$

Functions

Solution 1

$f(3) = 3^2 + 4 = 13$

Functions

Solution 2

Rewrite the function as $g(t) = \sqrt{t-2}$.

Then

$$g(27) = \sqrt{27-2} = \sqrt{25} = 5$$
$$g(5) = \sqrt{5-2} = \sqrt{3} \approx 1.7321$$
$$g(2) = \sqrt{2-2} = \sqrt{0} = 0$$

Functions

Solution 3

Since $x = -\frac{1}{2}$ satisfies $x < 1$, use the top part of the formula to find

$$f\left(-\frac{1}{2}\right) = \frac{1}{-1/2 - 1} = \frac{1}{-3/2} = -\frac{2}{3}$$

However, $x = 1$ and $x = 2$ satisfy $x \geq 1$, so $f(1)$ and $f(2)$ are both found by using the bottom part of the formula:

$$f(1) = 3(1)^2 + 1 = 4 \quad \text{and} \quad f(2) = 3(2)^2 + 1 = 13$$

Functions

It is often convenient to represent a functional relationship by an equation $y = f(x)$, *and in this context, x and y are called* ***variables.***

In particular, since the numerical value of y *is determined by that of x, we refer to y as the* ***dependent variable*** and to x *as the* ***independent variable.***

Functions

There are several ways to represent a function:

(a) by a formula such as $y = x^2$,

(b) by a formula such as $f(x) = x^2$

(c) by a mapping rule such as $x \longrightarrow x^2$.

(Read this as "x goes to x^2.")

Functions

Unless otherwise specified, if a formula (or several formulas, as in Example 1-3) is used to define a function f, *then we assume the domain of f to be the set of all numbers for which f(x) is defined (as a real number).*
We refer to this as the **natural domain of *f*.**

Functions

Determining the natural domain of a function often amounts to excluding all numbers *x that result in dividing by zero or in taking the square root of a negative number.*
This procedure is illustrated in the following Example.

Functions

<u>Example 2</u>
Find the domain and range of each of these functions.

a. $f(x) = \dfrac{1}{x-3}$

b. $g(t) = \sqrt{t-2}$

Functions

Solution 2 (a)
Since division by any number other than 0 is possible, the domain of *f* is *the set* of all numbers *x such that* $x - 3 \neq 0$; *that is,* $x \neq 3$. *The range of f is the set of* all numbers *y except 0, since for any* $y \neq 0$, *there is an x such that*

$y = \dfrac{1}{x-3}$; in particular, $x = 3 + \dfrac{1}{y}$.

Functions

Examples:
There are several functions associated with the marketing of a particular commodity:

1. The **demand function** ***D(x) for the commodity is the price p =D(x) that must be*** charged for each unit of the commodity if *x units are to be sold (demanded).*
2. The **supply function** ***S(x) for the commodity is the unit price p = S(x) at which producers***are willing to supply *x units to the market.*

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Functions

3. The **revenue** ***R(x) obtained from selling x units of the commodity is given by the*** product
   R(x) (number of items sold)(price per item)
   =x .p(x)

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Functions

Example 3
Market research indicates that consumers will buy *x thousand units of a particular* kind of coffee maker when the unit price is $p(x) = -0.27x + 51$

dollars. The cost of producing the *x thousand units is*

thousand dollars. $C(x) = 2.23x^2 + 3.5x + 85$

1. What are the revenue and profit functions, R(x) and P(x), for this production process?
2. For what values of x is production of the coffee makers profitable?

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Functions

Solution (a)

The revenue is

$$R(x) = xp(x) = -0.27x^2 + 51x$$

thousand dollars, and the profit is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.27x^2 + 51x - (2.23x^2 + 3.5x + 85) \\ &= -2.5x^2 + 47.5x - 85 \end{aligned}$$

thousand dollars.

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Functions

Solution (b)

Production is profitable when $P(x) > 0$. We find that

$$\begin{aligned} P(x) &= -2.5x^2 + 47.5x - 85 \\ &= -2.5(x^2 - 19x + 34) \\ &= -2.5(x - 2)(x - 17) \end{aligned}$$

Since the coefficient -2.5 is negative, it follows that $P(x) > 0$ only if the terms $(x - 2)$ and $(x - 17)$ have different signs; that is, when $x - 2 > 0$ and $x - 17 < 0$. Thus, production is profitable for $2 < x < 17$.

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Functions

Exercise 1
Suppose the total cost in dollars of manufacturing *q units of a certain commodity is* given by the function

$$C(q) = q^3 - 30q^2 + 500q + 200.$$

1. Compute the cost of manufacturing 10 units of the commodity.
2. Compute the cost of manufacturing the 10th unit of the commodity.

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Next Lecture

1. Composite functions
2. Limits

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END OF LECTURE 2

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