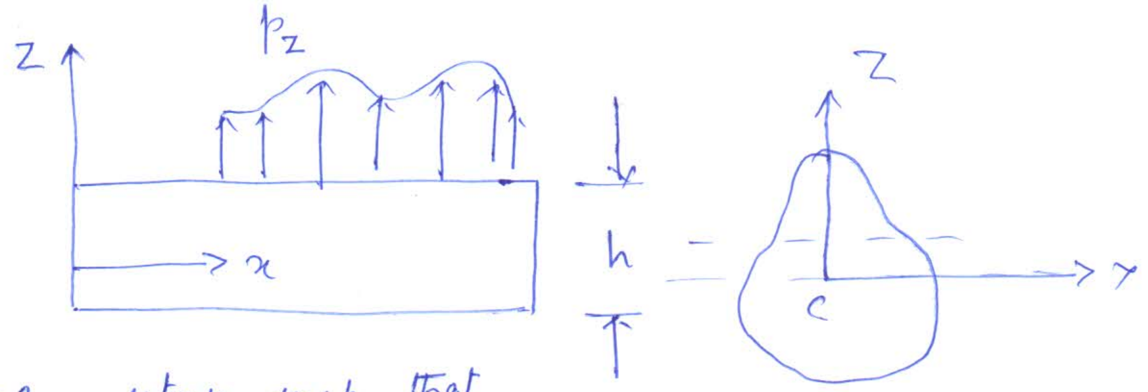
Bending and flexural shear



Coordinates are setup such that the x-axis coincides with the centroidal axis of the cross-sections along the beam and the z-axis coincides with the vertical line of symmetry. P_z is the transverse load (N/m) is applied in the x-z plane.

If width of the beam is small, then the state of stress due to transverse loading can be approximated by plane stress parallel to the x-z plane and u & w can be assumed to be functions of x & z only.

Expand u & w ~~upto linear term~~ as power series in z as

$$u(x,z) = u_0(x) + z\,u_1(x) + z^2 u_2(x) + \cdots$$

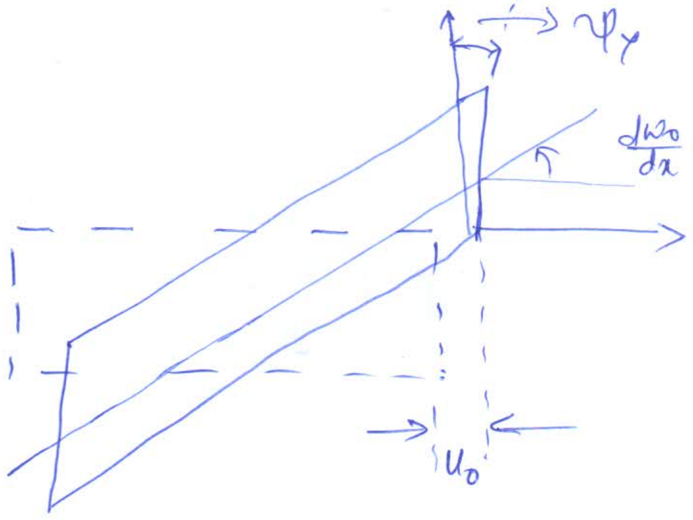
$$w(x,z) = w_0(x) + z\,w_1(x) + z^2 w_2(x) + \cdots$$

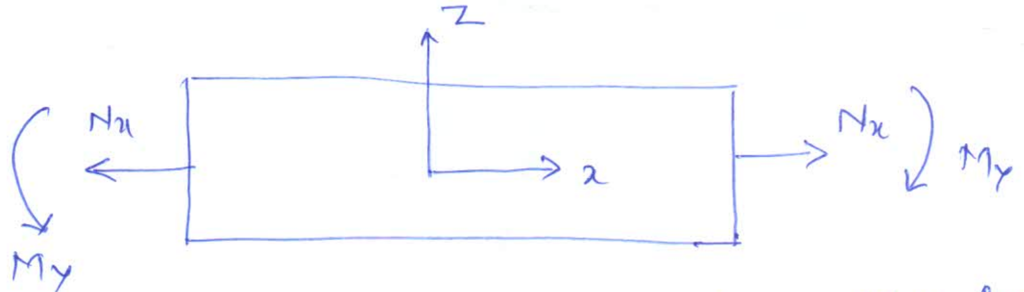
For slender beams, the depth is small compared to length. In other words, the range of z is small and the higher order terms in z make insignificant contributions. Hence as first order approximations we take:

$$u = u_0(x) + z\,\psi_y(x)$$

$$w = w_0(x)$$

$u_1(x) = \psi_y(x)$ $\longrightarrow$ rotation of c/s after deformation





Note: $u(x,z)$ is a linear function of z. This implies that plane cross-sections remain plane after deformation but may not be perpendicular to the centroidal axis.

Strain components:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} + z\frac{\partial \psi_y}{\partial x}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{dw_0}{dx} + \psi_y$$

Resultant force & moments:

$$N_x = \iint_A \sigma_{xx}\, dA \qquad M_y = \iint_A z\,\sigma_{xx}\, dA$$

For slender beams transverse shear strain is small,

$$\Rightarrow \gamma_{xz} \approx 0 \quad \Rightarrow \psi_y = -\frac{dw_0}{dx}$$

The above relation implies that the plane cross-section remains perpendicular to the centroidal axis after deformation, and the amount of rotation of the c/s is equal to the slope of deflection

NOW,

$$\sigma_{xx} = E\,\varepsilon_{xx} = E\left[\frac{du_0}{dx} - z\frac{d^2 w_0}{dx^2}\right]$$

$\Rightarrow$

$$N_x = EA\frac{du_0}{dx} - E\frac{d^2 w_0}{dx^2}\iint_A z\,dA$$

$$M_y = E\frac{du_0}{dx}\iint_A z\,dA - E\frac{d^2 w_0}{dx^2}\iint z^2\,dA$$

But origin of coordinates coincide with centeroidal axis $\Rightarrow \iint_A z\,dA = 0$.

$$\therefore\ N_x = EA\frac{du_0}{dx} \qquad M_y = -EI_y\frac{d^2 w_0}{dx^2}$$

$$\boxed{I_y = \iint_A z^2\,dA}$$

moment of inertia about y-axis

No axial force $\Rightarrow N_x = 0$

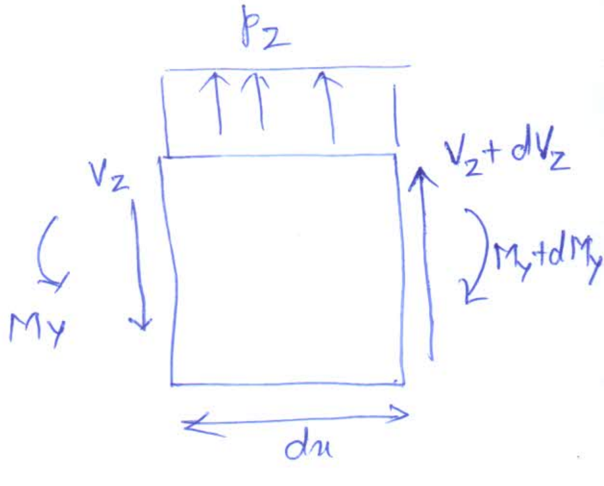
$\frac{du_0}{dx} = 0 = \varepsilon_{xx}$ along x-axis

$\therefore$ x-axis is the neutral axis

& x-y plane is the neutral plane.

Resultant transverse shear stress

$$V_z = \iint_A \tau_{xz}\,dA$$



Force equillibrium along z-axis

$$dV_z + p_z\,dx = 0$$

$$\Rightarrow \frac{dV_z}{dx} = -p_z$$

Moment equillibrium yields

$$-dM_y + p_z\,dx\,\frac{dx}{2} + (V_z + dV_z)\,dx = 0 \qquad \text{as } dx \to 0$$

$$\Rightarrow \frac{dM_y}{dx} = V_z$$

Note: If beam is subjected to a pure constant moment then

$$V_z = \frac{dM_y}{dx} = 0 \implies \underline{\underline{\tau_{xz} = 0}}.$$

$\therefore$ assumption $\gamma_{xz} = 0$ is exact only when moment is constant.

Now substituting $M_y = -EI_y \frac{d^2 w_0}{dx^2}$ into above eqns we get,

$$EI_y \frac{d^4 w_0}{dx^4} = p_z \rightarrow \text{Euler - Bernoulli beam equation}$$

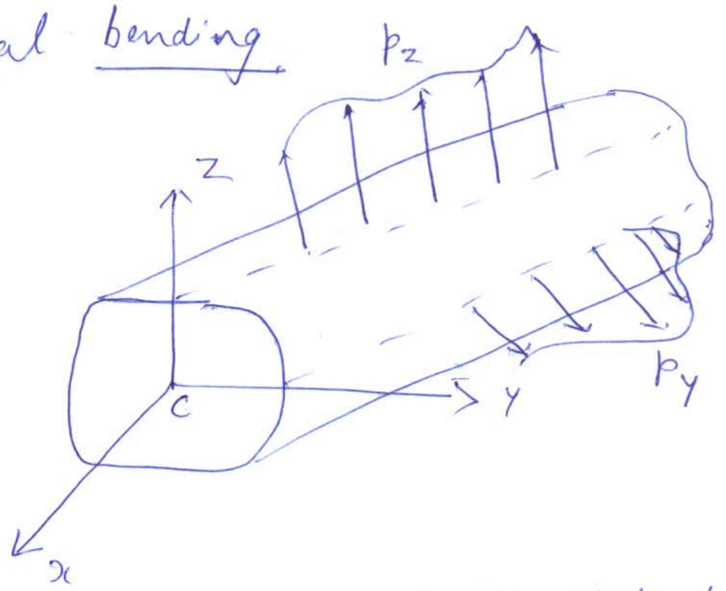
also when $N_x = 0$ & $\frac{du_0}{dx} = 0$,

bending strain

$$\varepsilon_{xx} = -z \frac{d^2 w_0}{dx^2} = \frac{M_y z}{EI_y}$$

$$\therefore \sigma_{xx} = \frac{M_y z}{I_y}$$

Bidirectional bending



For beams with arbitrarily shaped cross-sections, we set up the coordinates as shown above. x-axis coincides with the centroidal axis.

Typical displacement field for this problem is

$$u = u_0(x) + z\,\psi_y(x) + y\,\psi_z(x)$$
$$v = v_0(x)$$
$$w = w_0(x)$$

Corresponding strains are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{du_0}{dx} + z\frac{d\psi_y}{dx} + y\frac{d\psi_z}{dx}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{dv_0}{dx} + \psi_z$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{dw_0}{dx} + \psi_y$$

Assume no shear strain $\Rightarrow$

$$\gamma_{xy} = \gamma_{xz} = 0$$

$$\Rightarrow \quad \psi_z = -\frac{dv_0}{dx}$$

$$\psi_y = -\frac{dw_0}{dx}$$

$$\therefore \varepsilon_{xx} = \frac{du_0}{dx} - y\frac{d^2v_0}{dx^2} - z\frac{d^2w_0}{dx^2}$$

suppose $N_x = 0 \implies \frac{du_0}{dx} = 0$ then

$$M_y = \iint_A z\,\sigma_{xx}\,dA = -E\iint_A \left[yz\frac{d^2v_0}{dx^2} + z^2\frac{d^2w_0}{dx^2}\right]dA.$$

$$= -EI_{yz}\frac{d^2v_0}{dx^2} - EI_y\frac{d^2w_0}{dx^2} \longrightarrow (1)$$

$$M_z = \iint_A y\,\sigma_{xx}\,dA = -EI_z\frac{d^2v_0}{dx^2} - EI_{yz}\frac{d^2w_0}{dx^2} \longrightarrow (2)$$

where,

$$I_y = \iint_A z^2\,dA \qquad I_z = \iint_A y^2\,dA$$

$$I_{yz} = \iint_A yz\,dA$$

Solving (1) & (2) we have,

$$-E\frac{d^2v_0}{dx^2} = \frac{1}{I_yI_z - I_{yz}^2}\left(-I_y M_y + I_y M_z\right)$$

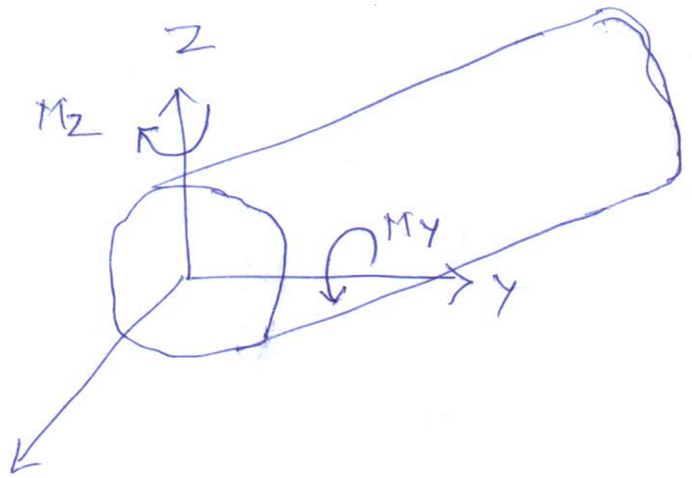
$$-E\frac{d^2w_0}{dx^2} = \frac{1}{I_yI_z - I_{yz}^2}\left(I_z M_y - I_{yz} M_z\right)$$

$$\therefore \sigma_{xx} = E\varepsilon_{xx} = -yE\frac{d^2v_0}{dx^2} - zE\frac{d^2w_0}{dx^2}$$

$$\sigma_{xx} = \frac{I_y M_z - I_{yz} M_y}{I_yI_z - I_{yz}^2}\,y + \frac{I_z M_y - I_{yz} M_z}{I_yI_z - I_{yz}^2}\,z$$

$$\longrightarrow (3)$$

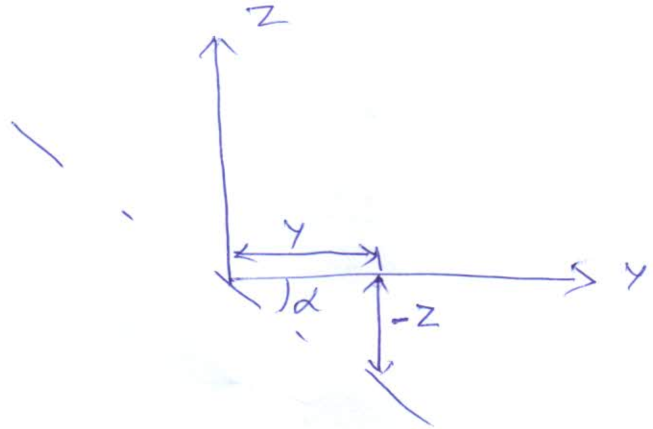
Sign convention based on derivation.



The location of neutral axis along which $\sigma_{xx} = 0$ can be found from (3) as

$$\sigma_{xx} = 0 = \frac{I_y M_z - I_{yz} M_y}{I_y I_z - I_{yz}^2}\, y + \frac{I_z M_y - I_{yz} M_z}{I_y I_z - I_{yz}^2}\, z$$

$$\Rightarrow \tan\alpha = \frac{-z}{y} = \frac{I_y M_z - I_{yz} M_y}{I_z M_y - I_{yz} M_z}$$



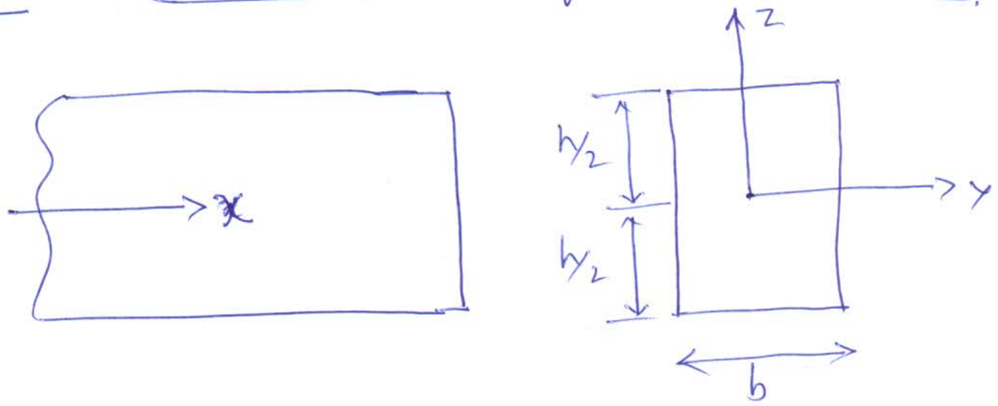
If y-axis or z-axis is an axis of symmetry then $I_{yz} = 0$ and (3) reduces to

$$\sigma_{xx} = \frac{M_z}{I_z}\, y + \frac{M_y}{I_y}\, z$$

If $I_{yz} \neq 0$ & $M_z = 0$, we have,

$$\sigma_{xx} = \frac{-I_{yz} M_y}{I_y I_z - I_{yz}^2}\, y + \frac{I_z M_y}{I_y I_z - I_{yz}^2}\, z$$

Transverse shear stress in symmetric sections



Consider a narrow rectangular section as shown above.
The resultant transverse shear force is

$$V_z = b\int_{-h/2}^{h/2} \tau_{xz}\, dz$$

Note: τ_{xz} assumed to be uniform across the width of the section. Only true if $h >> b$

Now,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{M_y z}{I_y}\right) + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\Rightarrow \frac{z}{I_y}\frac{\partial}{\partial x} M_y + \frac{\partial \tau_{xz}}{\partial z} = 0$$

But $\frac{\partial M_y}{\partial x} = V_z$

$$\Rightarrow -\frac{z V_z}{I_y} = \frac{\partial \tau_{xz}}{\partial z}$$

$$\int_{-h/2}^{z} \frac{\partial \tau_{xz}}{\partial z} = \int_{-h/2}^{z} -\frac{z V_z}{I_y}$$

to see the profile of τ_{xz} across thickness we integrate from $-h/2$ to any z'

$$\Rightarrow \tau_{xz}(z) - \tau_{xz}(-h/2) = \frac{-V_z}{2 I_y}\left(z^2 - \frac{h^2}{4}\right)$$

From boundary condition we have

$$\tau_{xz}(\pm h/2) = 0$$

$$\therefore \quad \tau_{xz}(z) = \frac{V_z c^2}{2 I_y}\left(1 - \frac{z^2}{c^2}\right) \quad \text{where } c = h/2$$

$$(\tau_{xz})_{max} = \frac{V_z c^2}{2 I_y} \quad \because (\tau_{xz})_{max} \text{ @ } z = 0.$$