February 7, 2022

Problem 1): Let $f : [0,1] \to \mathbf{R}$ be continously differentiable and define

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

prove that $\lim_{n\to\infty} B_n(x) = f(x)$ for each $x \in [0,1]$.

solution):

We will prove this results using four intermediary results. Let define

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

RESULT 1: $\sum_{k=0}^{n} b_{n,k} = 1 \qquad \forall x \in [0,1]$

By binomial theorem,

$$((1-x)+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \qquad \forall x \in [0,1]$$

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$$((1-x)+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \qquad \forall x \in [0,1]$$

$$\implies \qquad \sum_{k=0}^{n} b_{n,k} = 1^n = 1$$

end of the proof of result 1.

RESULT 2: $\sum_{k=0}^{n} \left(\frac{k}{n}\right) b_{n,k}(x) = x \qquad \forall\ x \in [0,1]$

By binomial theorem,

$$(p+q)^n = \sum^{n} \binom{n}{\ } p^k q^{n-k}$$

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By binomial theorem,

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

Differentiating both sides with respect to p:

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = n(p+q)^{n-1}$$

$\implies$

$$\sum_{k=0}^{n}\binom{n}{k}kp^k q^{n-k} = n(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n}\left(\frac{k}{n}\right)b_{n,k}(x) = x \qquad \forall\, x \in [0,1]$$

end of the proof of result 2.

RESULT 3: $\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)b_{n,k}(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x \qquad \forall\, x \in [0,1]$

end of the proof of result 2.

RESULT 3: $\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)b_{n,k}(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x \qquad \forall\, x \in [0,1]$

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$\Longrightarrow$

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}\right)p^k q^{n-k} = (p+q)^{n-1}p$$

Again differentiating both sides with respect to p, and multiplied by p we have:

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

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$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k^2}{n^2}\right) p^k q^{n-k} = \frac{n-1}{n}(p+q)^{n-2}p^2 + \frac{1}{n}(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k^2}{n^2}\right) p^k q^{n-k} = \frac{n-1}{n}x^2 + \frac{1}{n}x$$

$\Longrightarrow$

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k}{n}\Big)p^k q^{n-k} = (p+q)^{n-1}p$$

Again differentiating both sides with respect to p, and multiplied by p we have:

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k^2}{n^2}\Big)p^k q^{n-k} = \frac{n-1}{n}(p+q)^{n-2}p^2 + \frac{1}{n}(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k^2}{n^2}\Big)p^k q^{n-k} = \frac{n-1}{n}x^2 + \frac{1}{n}x$$

end of the proof of result 3.

RESULT 4: $\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_{n,k}(x)=\frac{x(1-x)}{n} \qquad \forall\, x\in[0,1]$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_{n,k}(x)=\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)b_{n,k}(x)-2x\sum_{k=0}^{n}\left(\frac{k}{n}\right)b_{n,k}(x)+x^2\sum_{k=0}^{n}b_{n,k}$$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_{n,k}(x)=\frac{n-1}{n}x^2+\frac{1}{n}x-2x^2+x^2=\frac{x(1-x)}{n} \qquad \forall\, x\in[0,1]$$

end of the proof of result 4

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Finally we use the fact that f is continuous on $[0,1]$. i.e,

$$\exists\, M>0 \text{ such that } |f(x)|\le M \qquad \forall x\in[0,1]$$

For a given point x and $\epsilon>0$,

$$\exists\, \delta>0 \text{ such that } |f(x)-f(y)|\le\epsilon \qquad |x-y|\le\delta$$

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For a given point x and $\epsilon > 0$,

$$\exists\, \delta > 0 \text{ such that } |f(x) - f(y)| \leq \epsilon \qquad |x - y| \leq \delta$$

$$|B_n(x) - f(x)| =$$

$$\leq \left| \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} - f(x) \right|$$

$$\leq \left| \sum_{k=0}^{n} \binom{n}{k} \left(f\left(\frac{k}{n}\right) - f(x) \right) x^k (1-x)^{n-k} \right|$$

(Using Result 1)

$$\leq \sum_{k=0}^{n} \binom{n}{k} \left| \left(f\left(\frac{k}{n}\right) - f(x) \right) \right| x^k (1-x)^{n-k}$$

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$$\leq \sum_{\substack{k=0 \\ |\frac{k}{n}-x| \leq \delta}}^{n} \binom{n}{k} \left| \left(f\left(\frac{k}{n}\right) - f(x) \right) \right| x^k (1-x)^{n-k} + \sum_{\substack{k=0 \\ |\frac{k}{n}-x| > \delta}}^{n} \binom{n}{k} \left| \left(f\left(\frac{k}{n}\right) - f(x) \right) \right| x^k (1-x)^{n-k}$$

$$\leq \epsilon + \frac{2M}{\delta^2} \sum_{\substack{k=0 \\ |\frac{k}{n}-x|>\delta}}^{n} \binom{n}{k} \left(\left(\frac{k}{n}\right) - x\right)^2 x^k (1-x)^{n-k}$$

$$\leq \epsilon + \frac{2M}{\delta^2} \frac{x(1-x)}{n} \leq \epsilon + \frac{2M}{n\delta^2}$$

Taking $n \geq \left[\frac{2M}{\epsilon\delta^2}\right] + 1$, we have:

$$|B_n(x) - f(x)| \leq 2\epsilon$$

$\Longrightarrow$

$$\lim_{n\to\infty} B_n(x) = f(x) \text{ for each } x \in [0,1]$$

Problem 2): If $f(x) = x^2$ then show that $B_n(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x.$

solution):

$$\lim_{n\to\infty} B_n(x) = f(x) \text{ for each } x \in [0,1]$$

Problem 2): If $f(x) = x^2$ then show that $B_n(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x$.

solution):

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k^2}{n^2}\right) x^k (1-x)^{n-k}$$

Again we divide this into three intermediary results. Let define

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

RESULT 1: $\sum_{k=0}^{n} b_{n,k} = 1 \qquad \forall x \in [0,1]$

By binomial theorem,

$$((1-x)+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \qquad \forall x \in [0,1]$$

$$\implies \qquad \sum_{k=0}^{n} b_{n,k} = 1^n = 1$$

end of the proof of result 1

end of the proof of result 3.

As, $B_n(x) = \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) b_{n,k}(x)$; we have: $B_n(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x$.

something's missing

idk how, i must've spaced out (mess coffee quality has declined)

Problem 3): Use the above $B_n(x)$ to determine n such that $|B_n(x) - x^2| < 10^{-2}$ for all $x \in [0,1]$.

solution):

$$|B_n(x) - x^2| \leq \left|\frac{n-1}{n}x^2 + \frac{1}{n}x - x^2\right| \leq \frac{1}{n}\left|x - x^2\right|$$

Now $x - x^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$, is maximize at $x = 2$, and the maximum value is $\frac{1}{4}$. Using this,

$$|B_n(x) - x^2| \leq \frac{1}{4n}$$

So, it is enough to solve for the n, for which, $\frac{1}{4n} \leq 10^{-2}$, i.e $n > 25 \implies n = 26$.

Problem 4): Use Neville's method to approximate $\sqrt{3}$ with $f(x) = 3^x$ and $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1$ and $x_4 = 2$. Find the absolute and relative errors.

Problem 4): Use Neville's method to approximate $\sqrt{3}$ with $f(x) = 3^x$ and $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1$ and $x_4 = 2$. Find the absolute and relative errors.

solution): The nth order polynomial in Neville's method:

$$P_{0,1,2,\cdots,n}(x) = \frac{(x-x_0)P_{1,2,\cdots,n} - (x-x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

We have the following table for $x = \frac{1}{2}$:

i	x_i	$P_i(x)$	$P_{i-1,i}(x)$	$P_{i-2,i-1,i}(x)$	$P_{i-3,i-2,i-1,i}(x)$	$P_{i-4,i-3,i-2,i-1,i}(x)$
0	-2	$\frac{1}{9}$				
1	-1	$\frac{1}{3}$	$\frac{2}{3}$			
2	0	1	$\frac{4}{3}$	$\frac{3}{2}$		
3	1	3	2	$\frac{11}{6}$	$\frac{16}{9}$	
4	2	9	0	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{41}{24}$

$$\text{Absolute error} = \left|\sqrt{3} - \frac{41}{24}\right| = 0.023717$$

$$\text{Relative error} = \frac{\left|\sqrt{3} - \frac{41}{24}\right|}{} = 0.013693$$

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1	-1	$\frac{1}{3}$	$\frac{2}{3}$			
2	0	1	$\frac{4}{3}$	$\frac{3}{2}$		
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$$\text{Relative error} = \frac{\left|\sqrt{3} - \frac{41}{24}\right|}{\sqrt{3}} = 0.013693$$

Problem 5): Use Neville's method to approximate $\sqrt{3}$ with $f(x) = \sqrt{3}$ and $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4$ and $x_4 = 5$. Find the absolute and relative errors.

solution): The nth order polynomial in Neville's method:

$$P_{0,1,2,\cdots,n}(x) = \frac{(x - x_0)P_{1,2,\cdots,n} - (x - x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

We have the following table for $x = 3$:

i	x_i	$P_i(x)$	$P_{i-1,i}(x)$	$P_{i-2,i-1,i}(x)$	$P_{i-3,i-2,i-1,i}(x)$	$P_{i-4,i-3,i-2,i-1,i}(x)$
0	0	0				
1	1	1	3			
2	2	1.4142	1.4244	1.2426		
3	4	2	1.7071	1.7475	1.6213	
4	5	2.2341	1.7439	1.7260	1.7368	1.6906

$$\text{Absolute error} = \left|\sqrt{3} - 1.6906\right| = |1.73205 - 1.6906 = 0.04145$$

$$\text{Relative error} = \frac{\left|\sqrt{3} - \frac{41}{24}\right|}{[illegible]} = \frac{|1.73205 - 1.6906|}{[illegible]} = 0.02393$$

$$P_{0,1,2,\cdots,n}(x) = \frac{(x-x_0)P_{1,2,\cdots,n} - (x-x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

We have the following table for $x = 3$:

i	x_i	$P_i(x)$	$P_{i-1,i}(x)$	$P_{i-2,i-1,i}(x)$	$P_{i-3,i-2,i-1,i}(x)$	$P_{i-4,i-3,i-2,i-1,i}(x)$
0	0	0				
1	1	1	3			
2	2	1.4142	1.4244	1.2426		
3	4	2	1.7071	1.7475	1.6213	
4	5	2.2341	1.7439	1.7260	1.7368	1.6906

$$\text{Absolute error} = \left|\sqrt{3} - 1.6906\right| = |1.73205 - 1.6906 = 0.04145$$

$$\text{Relative error} = \frac{\left|\sqrt{3} - \frac{41}{24}\right|}{\sqrt{3}} = \frac{|1.73205 - 1.6906|}{1.73205} = 0.02393$$

One thing to note: the approximation is better if we choose $f(x) = 3^x$ instead of $f(x) = \sqrt{3}$, as in problem 4; as both the absolute and relative error is smaller in comparison to problem 5.

Problem 6): If $P_3(x)$ is the interpolating polynomial for the following data then we use Neville's method to find y if $P_3(1.5) = 0$.

x	0	.5	1	2
$f(x)$	0	y	3	2

solution): Here we define $P_{0,1,2,3}$ as the Neville's P_3 polynomial. So we have: $P_{0,1,2,3}(1.5) = 0$.

$$P_0 = f(0) = 0; P_1 = f(.5) = y; P_2 = f(1) = 3; P_3 = f(2) = .2$$

The nth order polynomial in Neville's method:

$$P_{0,1,2,\cdots,n}(x) = \frac{(x - x_0)P_{1,2,\cdots,n} - (x - x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

Using this:

$$
\begin{aligned}
P_{0,1}(1.5) &= \frac{(1.5-0)y-(1.5-.5)\times 0}{.5} = 3y \\
P_{1,3}(1.5) &= \frac{(1.5-0.5)\times 3-(1.5-1)\times y}{.5} = 6-y \\
P_{2,3}(1.5) &= \frac{(1.5-1)\times 2-(1.5-2)\times 3}{1} = 2.5 \\
P_{0,1,2}(1.5) &= \frac{(1.5-0)\times(6-y)-(1.5-1)\times 3y}{1} = 9-3y \\
P_{1,2,3}(1.5) &= \frac{(1.5-.5)\times 2.5-(1.5-2)\times(6-y)}{1.5} = \frac{11-y}{3} \\
P_{0,1,2,3}(1.5) &= \frac{(1.5-0)\times\frac{(11-y)}{3}-(1.5-2)\times(9-3y)}{2} = 5-y = 0 \\
\implies &\qquad y = 5
\end{aligned}
$$

Problem 7): Use the forward difference formula to construct interpolating

Problem 7): Use the forward difference formula to construct interpolating polynomials of degree one,two, and three for the following data and approximate $f(-\frac{1}{3})$.

x	-0.75	-0.5	-0.25	0
$f(x)$	-0.07181250	-0.02475000	-0.33493750	1.10100000

solution): Note that $$-.75 < -.5 < -\tfrac{1}{3} < -.25 < 0.$$
From the given data data we have the given divided difference table, where divided difference formula is give by ,

$$f[x_i] = f(x_i);\ \text{and}\ f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}\ :$$

x	$f(x)$			
$x_0 = -.75$	$-.0718125$			
		$.18825$		
$x_1 = -.5$	$-.02475$		2.501	
		1.43875		1
$x_2 = -.25$	$.3349375$		3.251	

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$f(x)$	-0.07181250	-0.02475000	-0.33493750	1.10100000

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$$f[x_i] = f(x_i); \text{ and } f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} :$$

x	$f(x)$			
$x_0 = -.75$	$-.0718125$			
		$.18825$		
$x_1 = -.5$	$-.02475$		2.501	
		1.43875		1
$x_2 = -.25$	$.3349375$		3.251	
		3.06425		
$x_3 = 0$	1.101			

For polynomial of degree 1, we use the subtable corresponding to $[x_1, x_2]$.

$$P_1(x) = -.02475 + 1.43875(x + .5)$$

$$P_1(-\frac{1}{3}) = -.02475 + 1.43875(-\frac{1}{3} + .5) = .215041$$

For polynomial of degree 2 we use subtable corresponding to $[x_0, x_2]$ and denoted it by $P_2^1(x)$, and the subtable corresponding to $[x_1, x_2]$, denoting it by $P_2^2(x)$.

$$P_2^1(x) = -.0718125 + .18825(x + .75) + 2.501(x + .75)(x + .5)$$

$$P_2^1(-\frac{1}{3}) = -.0718125 + .18825(-\frac{1}{3} + .75) + 2.501(-\frac{1}{3} + .75)(-\frac{1}{3} + .5) = .180306$$

Similarly:

$$P_2^2(x) = -.02475 + 1.43875(x + .5) + 3.251(x + .5)(x + .25)$$

$$P_2^2(-\frac{1}{3}) = -.02475 + 1.43875(-\frac{1}{3} + .5) + 3.251(-\frac{1}{3} + .5)(-\frac{1}{3} + .25) = .169895$$

For polynomial of degree 3, we will use the entire table corresponding to the interval $[x_0, x_3]$.

$$P_3(x) = P_2^1(x) + (x + .75)(x + .5)(x + .25)$$

$$P_3(-\frac{1}{3}) = P_2^1(-\frac{1}{3}) + (-\frac{1}{3} + .75)(-\frac{1}{3} + .5)(-\frac{1}{3} + .25) = .174519$$

For polynomial of degree 1, we use the subtable corresponding to $[x_1, x_2]$.

$$P_1(x) = -.02475 + 1.43875(x + .5)$$

$$P_1(-\frac{1}{3}) = -.02475 + 1.43875(-\frac{1}{3} + .5) = .215041$$

For polynomial of degree 2 we use subtable corresponding to $[x_0, x_2]$ and denoted it by $P_2^1(x)$, and the subtable corresponding to $[x_1, x_2]$, denoting it by $P_2^2(x)$.

$$P_2^1(x) = -.0718125 + .18825(x + .75) + 2.501(x + .75)(x + .5)$$

$$P_2^1(-\frac{1}{3}) = -.0718125 + .18825(-\frac{1}{3} + .75) + 2.501(-\frac{1}{3} + .75)(-\frac{1}{3} + .5) = .180306$$

Similarly:

$$P_2^2(x) = -.02475 + 1.43875(x + .5) + 3.251(x + .5)(x + .25)$$

$$P_2^2(-\frac{1}{3}) = -.02475 + 1.43875(-\frac{1}{3} + .5) + 3.251(-\frac{1}{3} + .5)(-\frac{1}{3} + .25) = .169895$$

For polynomial of degree 3, we will use the entire table corresponding to the interval $[x_0, x_3]$.

$$P_3(x) = P_2^1(x) + (x + .75)(x + .5)(x + .25)$$

$$P_3(-\frac{1}{3}) = P_2^1(-\frac{1}{3}) + (-\frac{1}{3} + .75)(-\frac{1}{3} + .5)(-\frac{1}{3} + .25) = .174519$$

Problem 8): Use the backward difference formula to construct interpolating polynomials of degree one,two, and three for the following data and

Problem 8): Use the backward difference formula to construct interpolating polynomials of degree one,two, and three for the following data and approximate $f(0.25)$.

x	0.1	0.2	0.3	0.4
$f(x)$	-0.62049958	-0.28398668	0.00660095	0.24842440

solution): Note that $$.1 < .2 < .25 < .3 < .4.$$
From the given data data we have the given divided difference table, where divided difference formula is give by ,

$$f[x_i] = f(x_i); \text{ and } f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0};$$

x	$f(x)$			
$x_0 = .1$	$-.62049958$			
		3.651286		
$x_1 = .2$	$-.28398668$		-2.2962615	
		2.9058763		$-.4731583$
$x_2 = .3$	.00660095		-2.438209	
		2.4182345		
$x_3 = .4$	.24842440			

lating polynomials of degree one,two, and three for the following data and approximate $f(0.25)$.

x	0.1	0.2	0.3	0.4
$f(x)$	-0.62049958	-0.28398668	0.00660095	0.24842440

solution): Note that $$.1 < .2 < .25 < .3 < .4.$$
From the given data data we have the given divided difference table, where divided difference formula is give by ,

$$f[x_i] = f(x_i); \text{ and } f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}:$$

x	$f(x)$			
$x_0 = .1$	$-.62049958$			
		3.651286		
$x_1 = .2$	$-.28398668$		-2.2962615	
		2.9058763		$-.4731583$
$x_2 = .3$	.00660095		-2.438209	
		2.4182345		
$x_3 = .4$	.24842440			

For polynomial of degree 1, we use the subtable corresponding to $[x_1, x_2]$.

$$P_1(x) = .006095 + 2.9058763(x - .3)$$

$$P_1(.25) = .006095 + 2.9058763(.25 - .3) = -.138692365$$

For polynomial of degree 2 we use subtable corresponding to $[x_1, x_3]$ and denoted it by $P_2^1(x)$, and the subtable corresponding to $[x_0, x_2]$, denoting it by $P_2^2(x)$.

$$P_2^1(x) = .24842440 + 2.4182345(x - .4) - 2.438209(x - .4)(x - .3)$$

$$P_2^1(.25) = .24842440 + 2.4182345(.25 - .4) - 2.438209(.25 - .4)(.25 - .3) = -.13279734$$

$$P_2^2(x) = .00660095 + 2.9058763(x - .3) - 2.2962615(x - .3)(x - .2)$$

$$P_2^2(.25) = .00660095 + 2.9058763(.25 - .3) - 2.2962615(.25 - .3)(.25 - .2) = -.132952$$

For polynomial of degree 3, we will use the entire table corresponding to the interval $[x_0, x_3]$.

$$P_3(x) = P_2^1(x) - .4731583(x - .4)(x - .3)(x - .2)$$

$$P_3(.25) = P_2^1(.25) - .4731583(.25 - .4)(.25 - .3)(.25 - .2) = -.13297478$$

Problem 9): A fourth degree polynomial $P(x)$ satisfies $\Delta^4 P(0) = 24, \Delta^3 P(0) = 6$, and $\Delta^2 P(0) = 0$, where $\Delta P(x) = P(x+1) - P(x)$.

Problem 9): A fourth degree polynomial $P(x)$ satisfies $\Delta^4 P(0) = 24, \Delta^3 P(0) = 6,$ and $\Delta^2 P(0) = 0,$ where $\Delta P(x) = P(x+1) - P(x).$ Compute $\Delta^2 P(10)$.

solution): We have the following divided difference table:

x	$f(x)$				B
$x_0 = 0$	$P(0)$				
		$\Delta P(0)$			
$x_1 = 1$	$P(1)$		$\Delta^2 \frac{P(0)}{2} = 0$		
		$\Delta P(1)$		$\Delta^3 \frac{P(0)}{6} = 1$	
$x_2 = 2$	$P(2)$		$\Delta^2 \frac{P(1)}{2}$		$\Delta^4 \frac{P(0)}{24} = 1$
		$\Delta P(2)$		$\Delta^3 \frac{P(1)}{6}$	
$x_3 = 3$	$P(3)$		$\Delta^2 \frac{P(2)}{2}$		$\Delta^4 \frac{P(1)}{24}$
		$\Delta P(3)$		$\Delta^2 \frac{P(2)}{2}$	
$x_4 = 4$	$P(4)$				

So our required polynomial will be:

$$P_5(x) = P(0) + \Delta P(0)(x-0) + \Delta^2 \frac{P(0)}{}(x-0)(x-1) + \Delta^3 \frac{P(0)}{}(x-0)(x-1)(x-2)$$

Compute $\Delta^2 P(10)$.

solution): We have the following divided difference table:

x	$f(x)$				B
$x_0 = 0$	$P(0)$				
		$\Delta P(0)$			
$x_1 = 1$	$P(1)$		$\Delta^2 \frac{P(0)}{2} = 0$		
		$\Delta P(1)$		$\Delta^3 \frac{P(0)}{6} = 1$	
$x_2 = 2$	$P(2)$		$\Delta^2 \frac{P(1)}{2}$		$\Delta^4 \frac{P(0)}{24} = 1$
		$\Delta P(2)$		$\Delta^3 \frac{P(1)}{6}$	
$x_3 = 3$	$P(3)$		$\Delta^2 \frac{P(2)}{2}$		$\Delta^4 \frac{P(1)}{24}$
		$\Delta P(3)$		$\Delta^2 \frac{P(2)}{2}$	
$x_4 = 4$	$P(4)$				

So our required polynomial will be:

$$P_5(x) = P(0) + \Delta P(0)(x-0) + \Delta^2 \frac{P(0)}{2}(x-0)(x-1) + \Delta^3 \frac{P(0)}{6}(x-0)(x-1)(x-2)$$
$$+ \Delta^4 \frac{P(1)}{24}(x-0)(x-1)(x-2)(x-3)$$

$\Longrightarrow$

$$P_5(x) = P(0) + \Delta P(0)x + x(x-1)(x-2) + x(x-1)(x-2)(x-3)$$

$$\Delta^2 P(10) = \Delta P(11) - \Delta P(10) = P(12) - 2P(11) - P(10)$$

$$= 12\times 11\times 10 + 12\times 11\times 10\times 9 + 10\times 9\times 8 + 10\times 9\times 8\times 7 - 2\times 11\times 10\times 9 - 2\times 11\times 10\times 9\times 8$$

$$= 1140$$