Kepler's Equation Validity

Orbital Limits of Kepler's Equation

Conceptually, Kepler's equation is **valid** for all values of 'e' including **1.0.**

However, we note from **E - θ** relation that as value of '**e**' increases, value of '**E**' decreases, for same '**θ**'.

Orbital Limits of Kepler's Equation

We also **see** that for **e = 1**, Kepler's equation **reduces** to the following **form.**

$$\boxed{E - \sin E = 0 \quad (a = \infty; \quad n = 0; \quad M = n \cdot \Delta t = 0)}$$

Thus, the only **possible** solution is **E = 0**, for all 'Δt'.

In view of the **above**, we need an **alternate** approach.

TOF for Parabolic Trajectories

Parabolic Trajectory Time Solution

We know that **'Δt'** for any **conic** section can be **obtained** from the following **relation**.

$$TOF = t_B - t_A = \frac{h^3}{\mu^2} \int_{\theta_A}^{\theta_B} \frac{1}{(1+e\cos\theta)^2} d\theta$$

Parabolic Trajectory Time Solution

Thus, for **e = 1,** it can be **re-written** as,

$$t_B - t_A = \frac{h^3}{4\mu^2} \int_{\theta_A}^{\theta_B} \frac{d\theta}{\cos^4\left(\theta/2\right)}$$

'**Δt**' can be **obtained** by carrying out the **integration.**

Time Solution

Integration can be **performed** by using the trigonometric **identity** for $\mathbf{sec^2(\theta/2)}$, as shown below.

$$\sec^4\left(\theta/2\right) = \sec^2\left(\theta/2\right)\left[1+\tan^2\left(\theta/2\right)\right]$$

$$t_B - t_A = \Delta t = \frac{h^3}{2\mu^2}\left(\tan\frac{\theta}{2} + \frac{1}{3}\tan^3\frac{\theta}{2}\right)_{\theta_A}^{\theta_B}$$

‘**t’** is ∞ at apogee (θ = **180°**), where ‘**v**’ = ‘0’ and ‘**r**’ = ∞.

Nature of Parabolic Δt Solution

An important feature of **Δt solution** for a parabolic path is that **unlike** ellipse, angle θ is always less than **±180°**, due to apogee being **undefined** (or infinite).

Thus, **practical** solutions for '**Δt**' (or '$\Delta\theta$') on a **parabolic** path are for a **finite** distance, 'r'.

Summary

As was **noted** earlier, TOF **solution** for parabolic trajectories can be **obtained** by direct integration of the **time** equation.

We also **note** that Kepler's equation, as **formulated** for an elliptic orbit, **breaks** down for $e = 1$.

TOF for Hyperbolic Trajectories

Hyperbolic Trajectories

If **spacecraft** have **ε > 0,** these move on a **hyperbolic** path**.**

Similar to **parabolic** case, in these **cases** also, we **need** to know how **long** it would take to **reach** a specific point and thus, **need** a process to arrive at **hyperbolic Δt.**

Hyperbolic Trajectory Relation

Hyperbola is **characterized** by $\varepsilon > 0$**,** resulting in **'a' < 0** while, **'p',** which is a physical **distance,** is positive.

Thus, we need to **rewrite** the basic conic section **relations**, in the context of **hyperbola**.

Hyperbolic Trajectory Relation

The **applicable** conic relations are as **follows.**

$$p = (-a)(e^2 - 1), \quad r_p = (-a)(e - 1); \quad r_a = (a)(e + 1)$$

We **see** that while, both **'p' and 'r_p'** are positive, 'r_a' is **negative**, because 'r' **crosses** '∞' at cosθ **= (-1/e)**.

Hyperbolic Δt Solution Strategy

One way to **arrive** at the **solution** for hyperbola is to perform **explicit** integration of **time** equation for **e > 1**.

In this regard, we can **examine** the elliptic orbit **integral** for its **possible** re-interpretation, when **e > 1.**

Hyperbolic Δt Solution Strategy

Another option is to **examine** the possibility of using **Kepler's** equation in the context of a **hyperbolic** path.

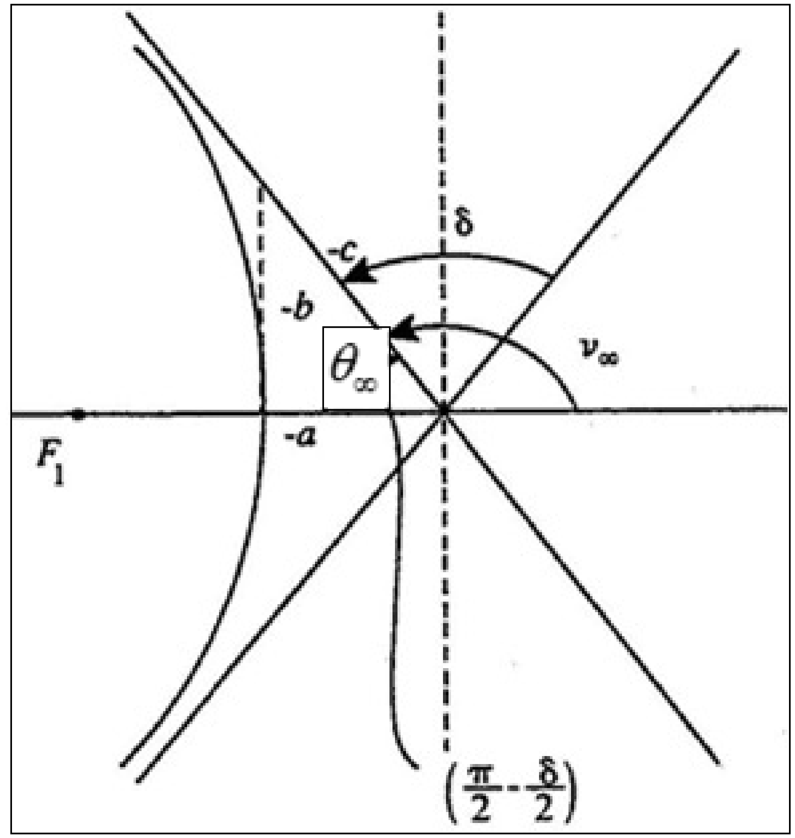
This can be done by **identifying** an applicable auxiliary **geometry** that can be **used** to set up 'Δt' **relations.**

Hyperbolic Trajectory Features

Applicable **auxiliary geometry** is arrived at by considering hyperbolic **features** as shown alongside.

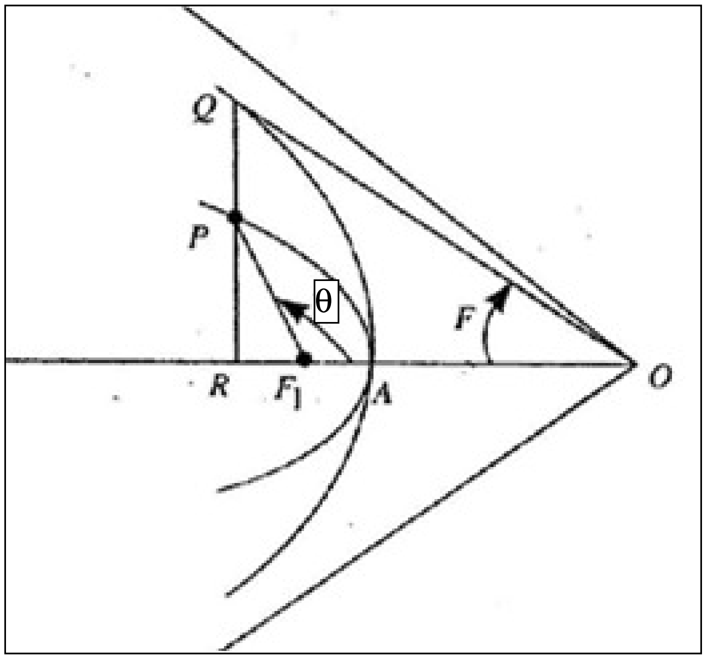
Here, 'δ' is angle **between** asymptotes at **r = ±∞**, and is related to '**e**', as follows.

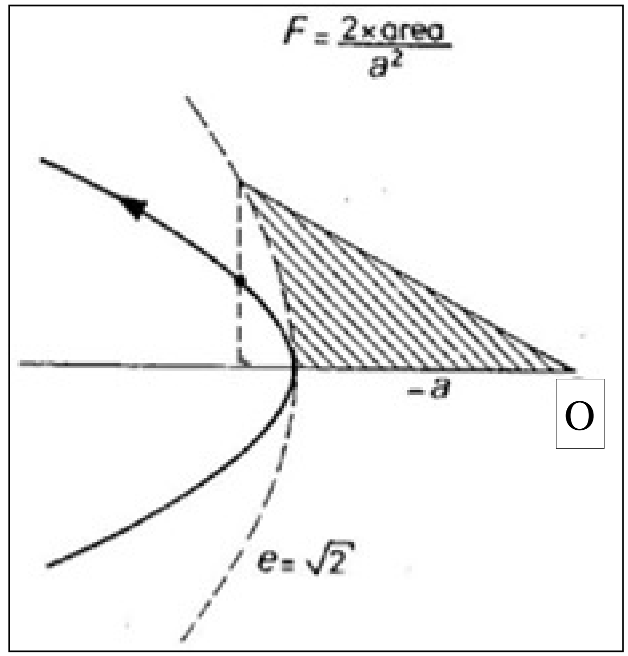
$$\delta = 2\sin^{-1}\frac{1}{e}$$



Auxiliary Hyperbola Concept

In this case, we employ an **equilateral** (rectangular) hyperbola **($e = \sqrt{2}$, $\delta = 90^o$)** , as shown below.





Hyperbola Features

We see that the **direction** of eccentricity vector **reverses**, with origin being **outside** of the hyperbola.

This is same as **$a < 0$** and $a(1 - e) < r < \infty$, which permits us to define '**F**' such that $r = a(1 – e \cosh F)$, **similar** to what is done for **elliptic** orbits ($r = a\{1\text{-}e \cos\theta\}$).

Hyperbola Features

It is seen that **minimum** value of ‘coshF’ is **1**, so that **minimum** value of r is **‘(-a)(e - 1)’,** which is $\mathbf{r_p}$.

It is to be noted that here, ‘**F**’ can take **any** value between **-∞** and **+∞.**

Hyperbolic Δt Formulation

We can **employ** ellipse – hyperbola (or **conic** section) **similarity** to arrive at **E – F** transformation, as follows.

$$\boxed{E = \pm iF,\ \mathrm{i}=\sqrt{-1};\quad 0 < E < 2\pi,\quad -\infty < F < +\infty}$$

Hyperbolic Kepler's Equation

This **permits** us to write **hyperbolic** form of Kepler's **equation**, as given below.

$$\cosh F = \frac{e+\cos\theta}{1+e\cos\theta}; \quad M = e\sinh F - F$$

The sign of '**F**' is resolved on the **basis** of sign of '**θ**'. i.e. **F < 0** for $\theta < 0$, under the **constraint** that get **M > F > θ.**

Hyperbolic Δt Solution Features

We can now **find** the explicit **expression** 'Δt' based on the **hyperbolic** relations, by using **applicable** hyperbolic and trigonometric **identities**, which is as given **below.**

$$\Delta t = \left\{ \sqrt{\frac{(-a)^3}{\mu}} \right\} M = \left\{ \sqrt{\frac{(-a)^3}{\mu}} \right\} \left\{ \frac{e\left(\sqrt{e^2-1}\right)\sin\theta}{1+e\cos\theta} - \ln \frac{\sqrt{e+1} + \sqrt{e-1}\tan\left(\theta/2\right)}{\sqrt{e+1} - \sqrt{e-1}\tan\left(\theta/2\right)} \right\}$$

It is **interesting** to note that above **expression** is also the exact time **integral** for e > 1.

Summary

In **conclusion**, we see that TOF for **hyperbolic** trajectories is obtained by a **suitable** extension of the Kepler's equation **derived** for elliptic orbits.

This is **due** to the fact that both these are the **solutions** of the same conic **section** relation.