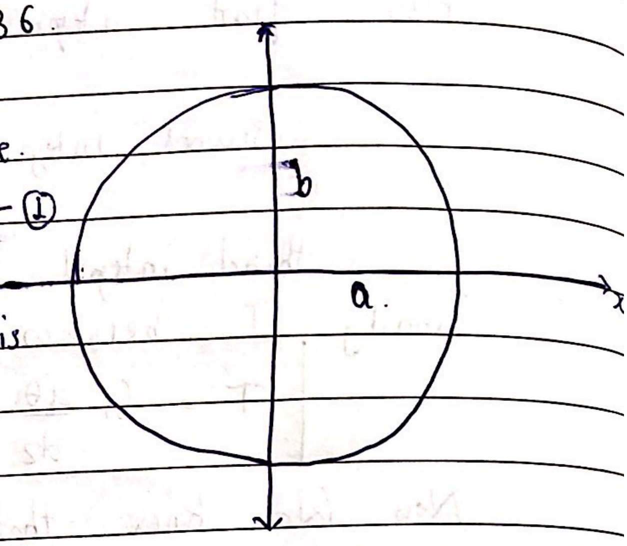
Assignment #3.

16001 0036.

Let the Prandtl's stress function be.



$$\phi = C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \quad \text{—(1)}$$

since ellipse of eq. is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

so B.C i.e.

$\phi = 0$; at boudary is already satisfie

and C is const.

Since we know that

Compatibility eq. ← $\nabla^2\phi = F = -2G\frac{d\theta}{dz}$.

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\frac{d\theta}{dz}$$

$$\Rightarrow 2C\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\frac{d\theta}{dz}$$

$$\Rightarrow C = -G\frac{d\theta}{dz}\frac{a^2b^2}{(a^2+b^2)}$$

putting it back in eq. (1)

$$\boxed{\phi = -G\frac{d\theta}{dz}\frac{a^2b^2}{(a^2+b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)} \quad \text{—(2)}$$

Now we have ϕ. so.

$$T = 2\int\phi\, dx\, dy$$

$$\Rightarrow T = -2G\frac{d\theta}{dz}\frac{a^2b^2}{(a^2+b^2)}\left(\frac{1}{a^2}\iint x^2 dx\,dy + \frac{1}{b^2}\iint y^2 dx\,dy - \int dx\,dy\right)$$

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Here first integral $= \frac{\pi a^3 b}{4} = I_{yy}$

second integral $= \frac{\pi a b^3}{4} = I_{xx}$

third integral $A = \pi ab$.

Finally T becomes

$$\boxed{T = G \frac{d\theta}{dz} \frac{\pi a^3 b^3}{(a^2 + b^2)}} \quad - ③$$

Now. We know that

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} \quad ; \quad \tau_{xz} = \frac{\partial \phi}{\partial y}$$

Using eq. ③ putting $G\frac{d\theta}{dz}$ in eq. ② &

than diff. according to above relation

$$\tau_{yz} = + \frac{T(a^2 + b^2)}{\pi a^3 b^3} \times \frac{a^2 b^2}{(a^2 + b^2)} \frac{2x}{a^2}.$$

$$\Rightarrow \boxed{\tau_{yz} = \frac{2Tx}{\pi a^3 b}}$$

Similarly we can solve for τ_{xz}

$$\Rightarrow \boxed{\tau_{zx} = -\frac{2Ty}{\pi a b^3}}$$

Now for warping displacement. since we know

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = -\theta \frac{d\theta}{dz} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}.$$

and $$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

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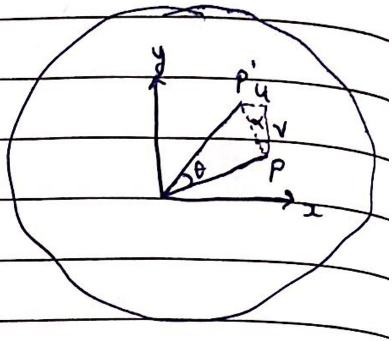
Here $u = -r\theta \sin\alpha$

$= -\theta y$

$v = \theta x$



So.

$$\frac{\partial w}{\partial y} = \frac{\tau_{xz}}{G} - \frac{\partial u}{\partial z}$$

$$\& \quad \frac{\partial w}{\partial x} = \frac{\tau_{yz}}{G} - \frac{\partial v}{\partial z}$$

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{-2Ty}{\pi a b^3 G} + y\frac{d\theta}{dz} \quad \text{— I}$$

$$\& \quad \frac{\partial w}{\partial y} = \frac{2Tx}{\pi a^3 b G} - x\frac{d\theta}{dz} \quad \text{— II}$$

using value of $\frac{d\theta}{dz} = \frac{T}{G}\frac{(a^2+b^2)}{\pi a^3 b^3}$ from previous..

and integrating both I & II we get.

$$w = \frac{T(b^2-a^2)}{\pi a^3 b^3 G} xy + f_1(y)$$

$$\& \quad w = \frac{T(b^2-a^2)}{\pi a^3 b^3 G} + f_2(x)$$

Since both w are same so

$f_1(y)$ & $f_2(x)$ should be zero so we get

$$\boxed{w = \frac{T(b^2-a^2)}{\pi a^3 b^3 G} xy}$$

Now,

Case 1, So $a = b$ & let $a^4 = b^4 = c$.

$$\text{So} \quad \tau_{yz} = \frac{2Tx}{\pi c}$$

$$\& \quad \tau_{xz} = \frac{-2Ty}{\pi c}$$

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Case 2. $\frac{a}{b} = 46 \Rightarrow a = 46b.$

and ~~b~~ taking $b = 1$.

We know that for $a = b = 1$. the eq. will be.

$$x^2 + y^2 = r^2$$

$$\Rightarrow \tau_{yz}^2 + \tau_{xz}^2 = r^2 = 1$$

For ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\tau_{yz}^2 \frac{\pi^2}{4T^2} \frac{a^6 b^2}{a^2} + \frac{\tau_{xz}^2 \pi^2 b^6 a^2}{4T^2 b^2} = 1$$

$$\Rightarrow \frac{a^3 b^3 \pi^2}{4T^2} \left[\tau_{yz}^2 \left(\frac{a}{b}\right) + \tau_{xz}^2 \left(\frac{b}{a}\right) \right] = 1.$$

Now for $\frac{b}{a} = 1$.

Normal. Eq will be. $\tau_{yz}^2 + \tau_{xz}^2 = 1.$
that will be a circle.

& otherwise $\frac{b}{a} = 46$

$$\Rightarrow \frac{\tau_{yz}^2}{46} + \tau_{zx}^2 \times 46 = 1.$$

$$\Rightarrow \boxed{\tau_{yz}^2 = 46 - 46^2 \tau_{zx}^2}$$

(d) Maximum shear stress ~~occurs~~ at

$$\tau_{max} = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}$$

$$= 2G\frac{d\theta}{dz} \frac{1}{(a^2+b^2)} \sqrt{b^4 x^2 + a^4 y^2}$$

$$= 2G\frac{d\theta}{dz} \sqrt{\frac{b^4 x^2 + a^4 y^2}{(a^2+b^2)^2}}$$

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$$\rightarrow \tau_{net} = \frac{2G\,d\theta}{dz} \sqrt{\frac{r^4 x^2 + y^2}{(1+r^2)^2}} \quad \text{when } r = \frac{b}{a}.$$

i) if $r = 1$ $\quad \tau_{net} = \frac{2G\,d\theta}{dz} \sqrt{x^2 + y^2}$

ii) if $r = 46$ $\quad \tau_{net} = \frac{2G\,d\theta}{dz} = \sqrt{\frac{(46)^4 x^2 + y^2}{(1 + 46^2)^2}}$

So We can see that max shear stress will occur at boundaries.

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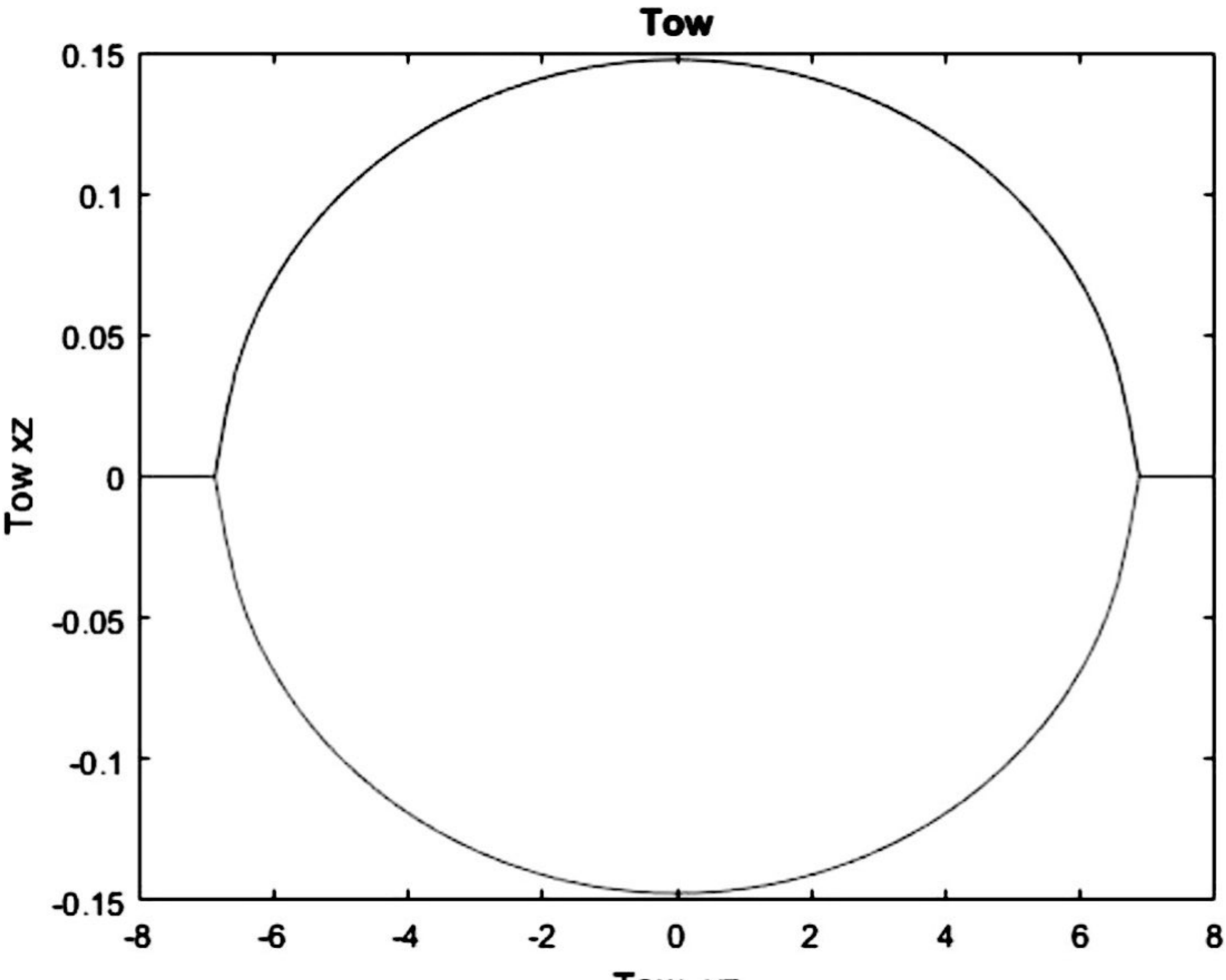

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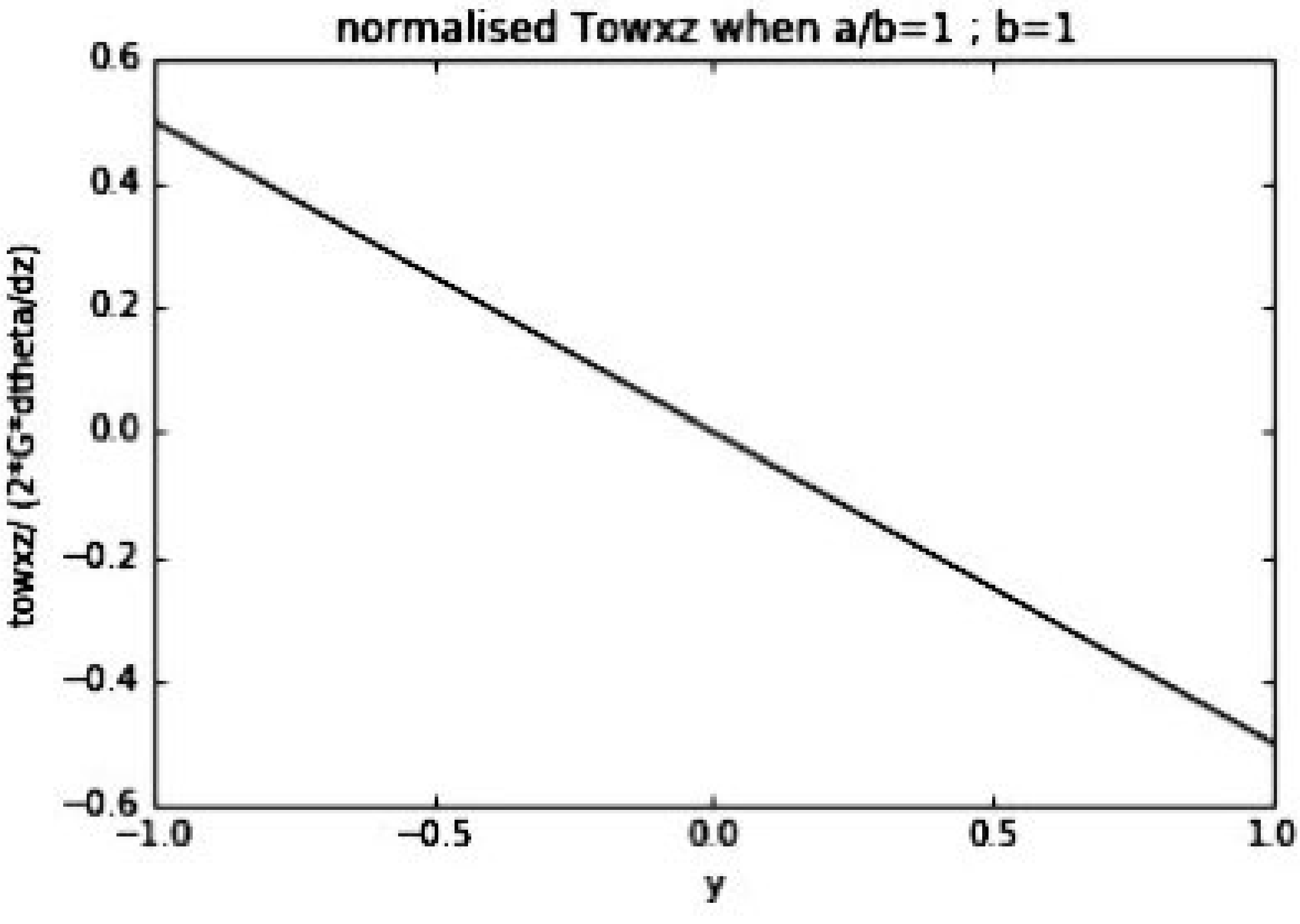

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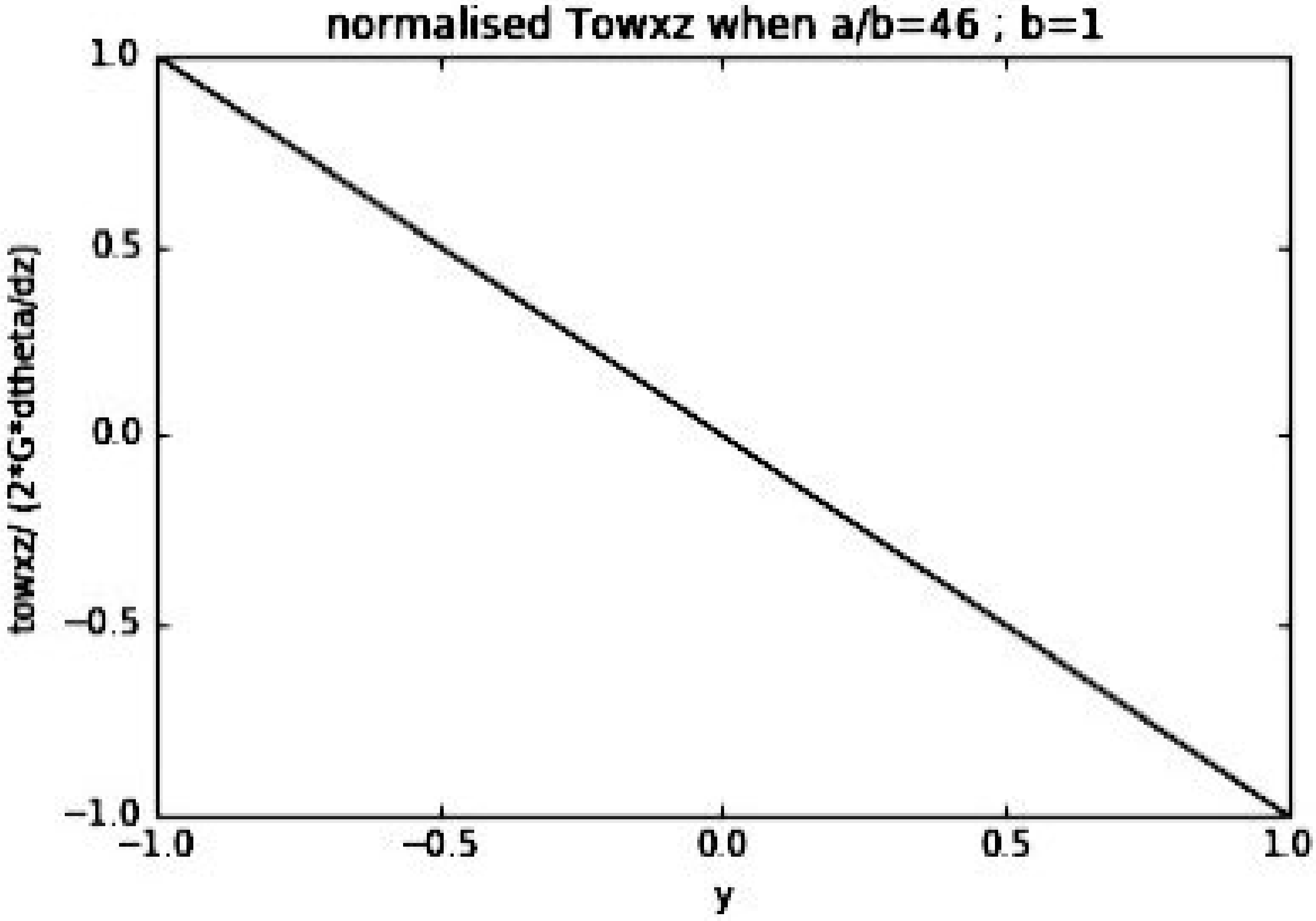

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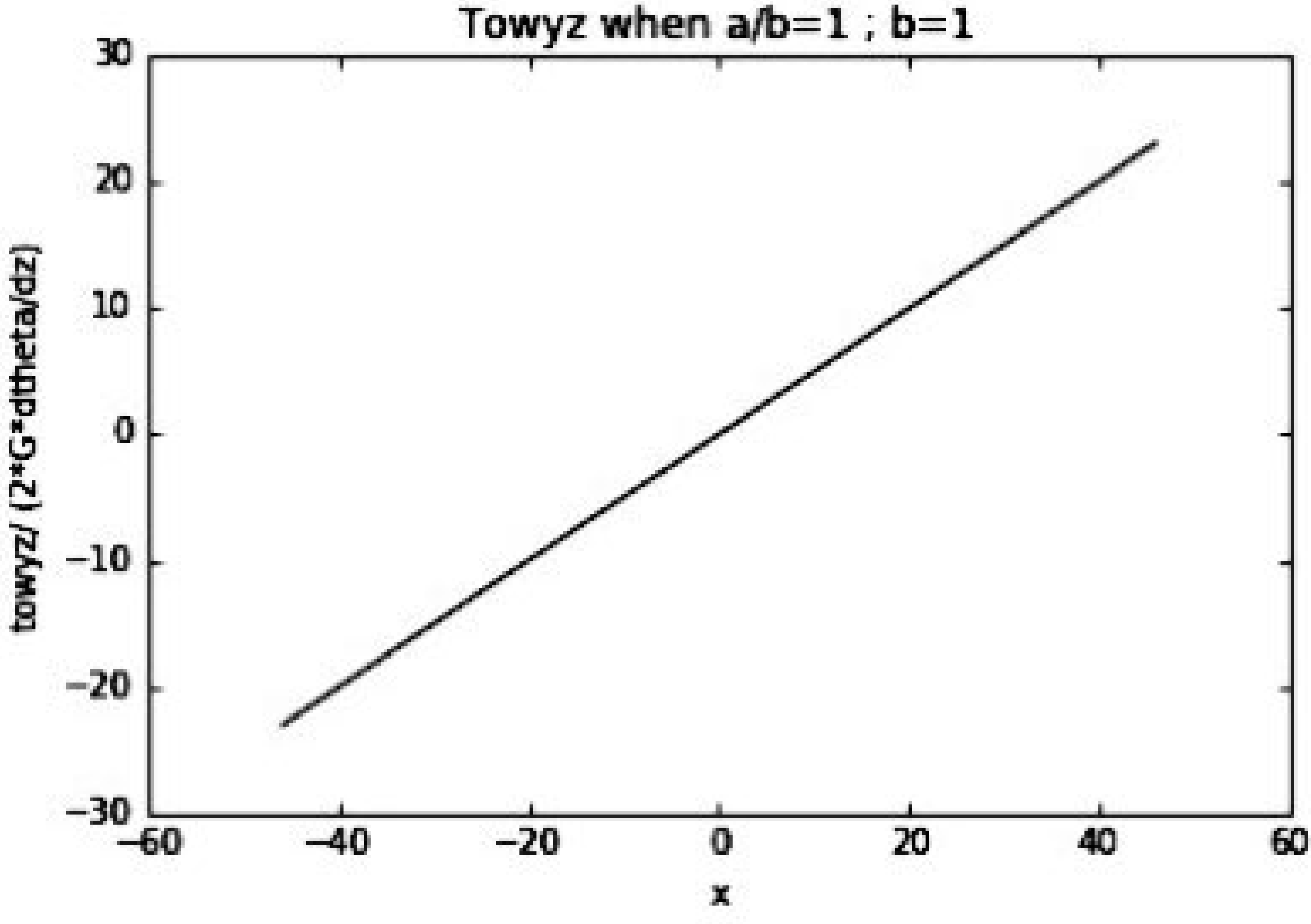

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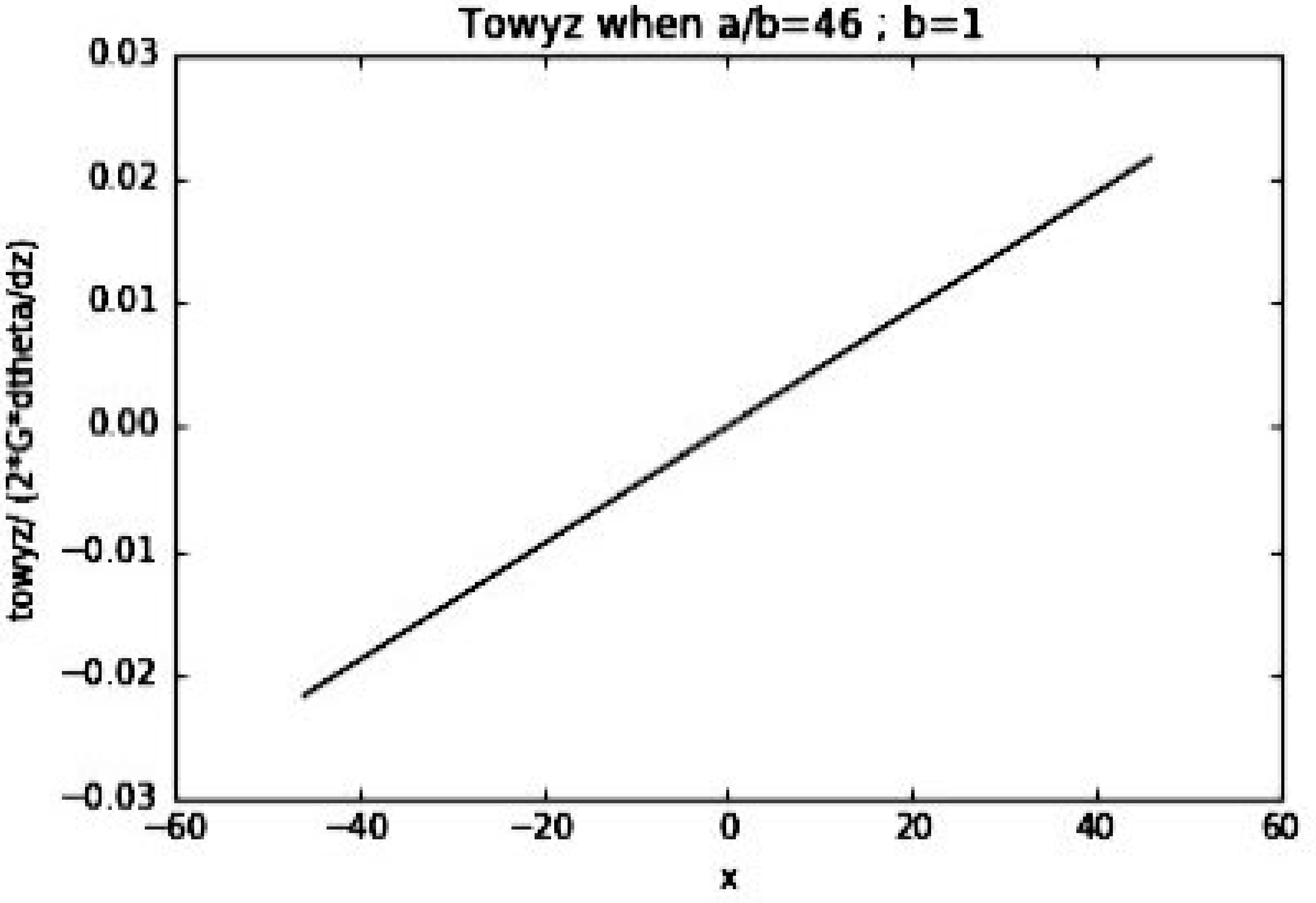

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