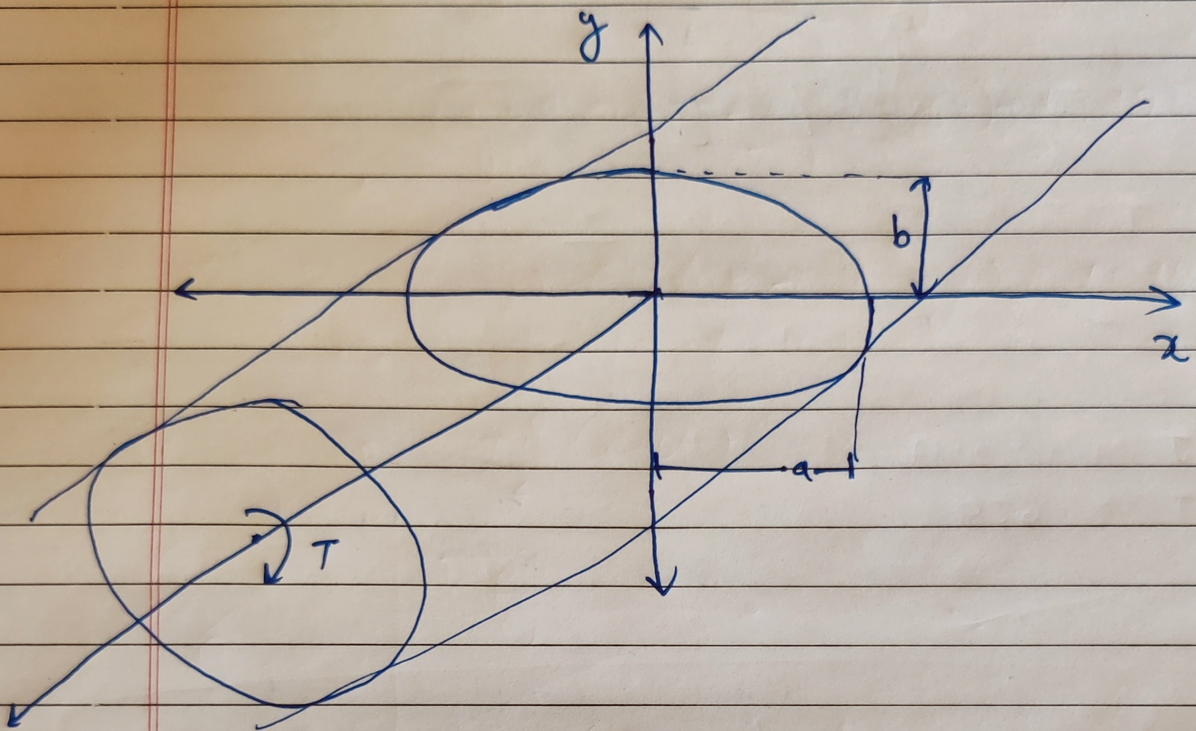
CHIRAG S AE238 Assignment 4.

20D180012



a) Since the lateral surface is traction free we know that $\phi(x,y) = 0$. at lateral surface.

Then $\phi(x,y) = C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$, c is constant

We know, compatability equation,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

$$2C\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta$$

$$c = \frac{-G\theta}{\frac{1}{a^2}+\frac{1}{b^2}}$$

$$\phi(x,y) = -\frac{G\theta}{\frac{1}{a^2}+\frac{1}{b^2}}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

b) Polar moment of Inertia.

$$J = \iint_A r^2 dA = \iint_A (x^2+y^2)\, dA$$

$$A = \left\{ (x,y) \mid \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1,\ |x| \le a, |y| \le b \right\}.$$

Let $x = au$, $y = bv$ and by change of variables,

$$J = \iint_{A'} (a^2u^2+b^2v^2)(ab)(du\,dv)$$

$$A' = \left\{ (u,v) \mid u^2+v^2 \le 1,\ |u| \le 1,\ |v| < 1 \right\}$$

$$J = \iint_{A'} (a^3bu^2 + ab^3v^2)\, du\,dv.$$

~~(A~~Changing u-v cartesian system to (ρ,θ) polar co-ordinate system

Date

Page

$$u = \rho\cos\theta \quad , v = \rho\sin\theta$$

$$0 \le \rho \le 1 \quad , \quad 0 \le \theta \le 2\pi.$$

$$J = \iint_{A''} (a^3 b \rho^2 \cos^2\theta + ab^3 \rho^2 \sin^2\theta)\, \rho\, d\rho\, d\theta$$

$$J = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} a^3 b\cos^2\theta + ab^3 \sin^2\theta)\, \rho^3\, d\rho\, d\theta$$

$$J = \frac{\pi ab}{4}(a^2+b^2)$$

(c) $$\tau_{xz} = \frac{\partial\phi}{\partial y}$$

$$\tau_{xz} = -\left(\frac{2(G\theta)}{1+\frac{b^2}{a^2}}\right) y.$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x}$$

$$\tau_{yz} = \left(\frac{2G\theta}{1+\frac{a^2}{b^2}}\right) x$$

d) $$\gamma_{xz} = \frac{\tau_{xz}}{G} = -\frac{2\theta a^2 y}{a^2+b^2}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{-2\theta a^2 y}{a^2+b^2}$$

$$-\theta y + \theta \frac{\partial \psi}{\partial x} = \frac{-2\theta a^2 y}{a^2+b^2}$$

$$\frac{\partial \psi}{\partial x} = \left(\frac{(b^2-a^2)}{b^2+a^2}\right) y$$

$$\psi(x,y) = \frac{(b^2-a^2)}{(b^2+a^2)} xy + f(y)$$

also, $$\frac{\partial \psi}{\partial y} = \left(\frac{b^2-a^2}{a^2+b^2}\right) x + f'(y)$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{2\theta b^2 x}{a^2+b^2}$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{2\theta b^2 x}{a^2+b^2}$$

$$\frac{\partial \psi}{\partial y} = \left(\frac{b^2-a^2}{a^2+b^2}\right) x.$$

$$\therefore f'(y) = 0 \Rightarrow f(y) = K$$

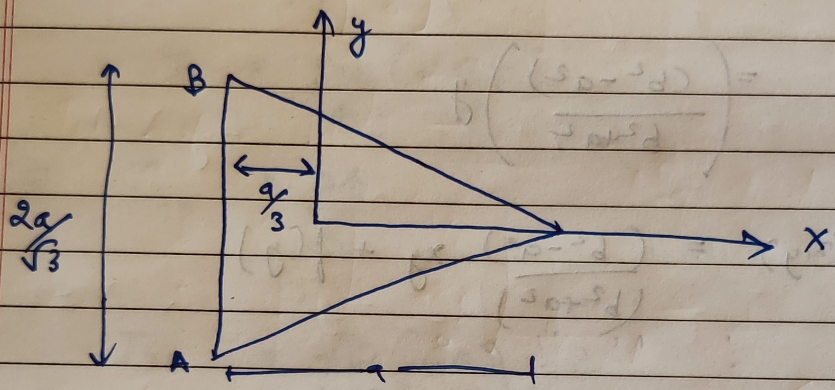
So, warping function.

$$\psi(x,y) = \frac{(b^2-a^2)}{(b^2+a^2)} xy + k$$ where k is a constant

$(0,0) \Rightarrow \psi(0,0) = 0 \Rightarrow k = 0.$

$$\psi(x,y) = \frac{(b^2-a^2)}{(b^2+a^2)} xy.$$

(2)



$$\phi = -G\theta\left[\frac{1}{2}(x^2+y^2) - \frac{1}{2a}(x^3 - 3xy^2) - \frac{2}{27}a^2\right]$$

a) $$\frac{\partial\phi}{\partial x} = -G\theta\left(x - \frac{3}{2a}(x^2-y^2)\right)$$

$$\frac{\partial^2\phi}{\partial x^2} = -G\theta\left(1 - \frac{3x}{a}\right)$$

$$\frac{\partial \phi}{\partial y} = -G\theta\left(y + \frac{3xy}{a}\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = -G\theta\left(1 + \frac{3x}{4}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -G\theta\left(1 - \frac{3x}{4}\right) - G\theta\left(1 + \frac{3x}{4}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta.$$

On Lateral Surfaces

$$[t] = [\sigma][4]$$

$$\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{zy} & \tau_{yz} & 0 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ 0 \end{bmatrix}$$

$$t_z = \tau_{xz} h_x + \tau_{yz} h_y$$

For surface AB : $\rightarrow$ $x = -a/3 \ \forall y$

$$\tau_{xz} = -G\theta y\left(1 + \frac{3x}{a}\right) = 0.$$

$$\tau_{yz} = G\theta\left(x - \frac{3}{2a}(x^2 - y^2)\right)$$

$$= G\theta\left(\frac{3y^2}{2a} - \frac{a}{2}\right)$$

$$n = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$t_{z)AB} = (0)(-1) + G\theta\left(\frac{3y^2}{2a} - \frac{a}{2}\right)(0)$$

$$= 0$$

$$[t] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For surface BC $\rightarrow$

$$n = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix}$$

Equation of line BC, $y = -\frac{1}{\sqrt{3}}\left(x - \frac{2a}{3}\right)$

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = \frac{G\theta}{\sqrt{3}}\left(\frac{3x^2}{a} - x - \frac{2a}{3}\right)$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = G\theta\left(\frac{-x^2}{a} + \frac{x}{3} + \frac{2a}{9}\right)$$

$$t_z = \frac{G\theta}{2\sqrt{3}}\left(\frac{3x^2}{a} - x - \frac{2a}{3}\right)$$

$$+ \frac{\sqrt{3}G\theta}{2}\left(-\frac{x^2}{a} + \frac{x}{3} + \frac{2a}{9}\right)$$

$$= \frac{\sqrt{3}G\theta}{2}\left(\frac{x^2}{a} - \frac{x}{3} - \frac{2a}{9}\right) + \frac{\sqrt{3}G\theta}{2}\left(-\frac{x^2}{a} + \frac{x}{3} + \frac{2a}{9}\right)$$

$t_z = 0$

$[t] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

For surface AC :→

$n = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \\ 0 \end{bmatrix} \qquad y = \frac{1}{\sqrt{3}}\left(x - \frac{2a}{3}\right)$

$\tau_{xz} = -\sqrt{3}\,G\theta\left(\frac{x^2}{a} - \frac{x}{3} - \frac{2a}{9}\right)$

$\tau_{yz} = G\theta\left(-\frac{x^2}{a^2} + \frac{x}{3} + \frac{2a}{9}\right)$

$t_z = 0$

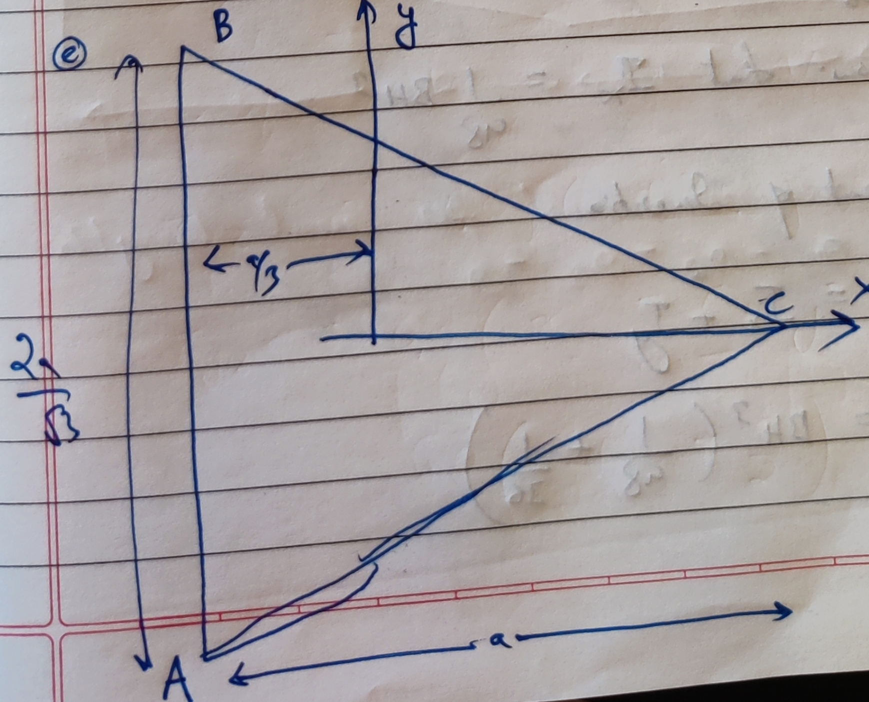
$[t] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

ϕ satisfies traction free boundary condition.

(c) $\tau_{xz} = \frac{\partial \phi}{\partial y} = -G\theta y\left(1 + \frac{3x}{a}\right)$

$\tau_{yz} = -\frac{\partial \phi}{\partial x} = G\theta\left(x - \frac{3}{2a}(x^2 - y^2)\right)$

(e)



$$J = \iint_A x^2 dA$$

$$= \iint_A (x^2 + y^2)\, dx\,dy$$

$$= \iint_A x^2 dA + \iint_A y^2 dA$$

$$J = I_y + I_x$$

where I_x, I_y are area moment of Inertia

$I_{AB} = \frac{1}{12} BH^3$, $B \to$ base, $H \to$ height.

$$I_{AB} = I_y + Ad^2 = I_y + \frac{1}{2}(BH)\left(\frac{H}{3}\right)^2$$

$$I_y = \frac{1}{12} BH^3 - \frac{1}{18} BH^3$$

$$= \frac{1}{36} BH^3.$$

We also know that $I_x = \frac{1}{48} BH^3$

Polar moment of Inertia.

$$J = I_x + I_y$$

$$= BH^3\left(\frac{1}{48} + \frac{1}{36}\right)$$

Date ____
Page ____

$$J = \frac{7BH^3}{144}$$

$$B = \frac{2a}{\sqrt{3}}, \quad H = 9$$

$$J = \frac{7}{72\sqrt{3}} a^4$$

(8) $$\gamma_{xz} = \frac{\tau_{xz}}{G} = -\theta y\left(1 + \frac{3x}{9}\right)$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\theta y\left(1 + \frac{3x}{9}\right)$$

$$-\theta y + \theta \frac{\partial \psi}{\partial x} = \theta y\left(1 + \frac{3x}{9}\right)$$

$$\frac{\partial \psi}{\partial x} = \frac{-3xy}{9}$$

$$\psi(x,y) = \frac{-3yx^2}{2a} + f(y)$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \theta\left(x - \frac{3}{2a}(x^2 - y^2)\right)$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \theta\left(x - \frac{3}{2a}(x^2 - y^2)\right)$$

$$x + \frac{\partial \psi}{\partial y} = x - \frac{3}{2a}(x^2 - y^2)$$

$$-\frac{3x^2}{2a^2} + f'(y) = -\frac{3x^2}{2a} + \frac{3y^2}{2a}$$

$$f'(y) = \frac{3y^2}{2a}$$

$$f(y) = \frac{y^3}{2a} + K$$

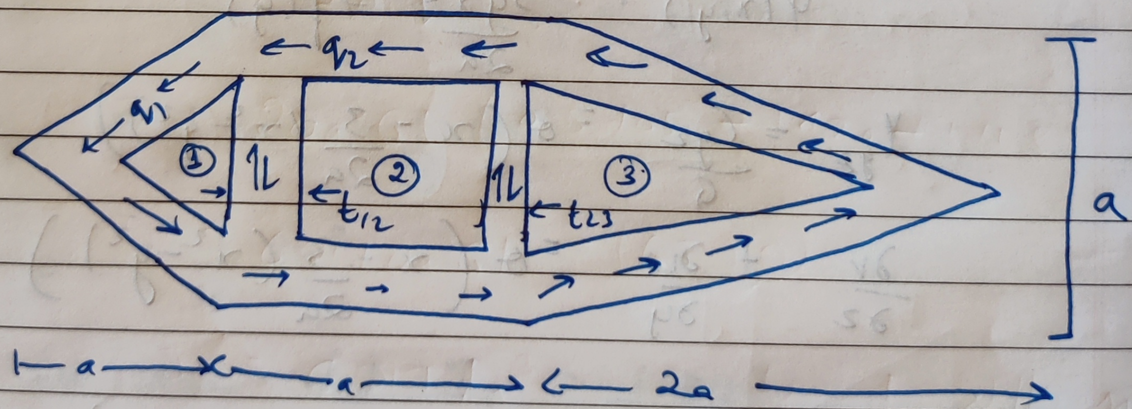
$$\psi(x,y) = \frac{-3yx^2}{2a} + \frac{y^3}{2a} + K$$

At $(0,0)$, $\psi(0,0) = 0 \Rightarrow K = 0$

$$\psi(x,y) = \frac{-3yx^2}{2a} + \frac{y^3}{2a}$$

$$\psi(x,y) = \frac{-3yx^2}{2a} + \frac{y^3}{2a}.$$

③.



Let us assume twist per unit length is given 4 $\theta_1 = \theta_2 = \theta_3 = \theta$.

Date
Page

$t_1 = t_2 = t_3 = 2t$
$t_{12} = t_{23} = t$

$\bar{A}_1 = \frac{1}{2}(a)(a) = a^2/2$

$\bar{A}_2 = a^2$, $\bar{A}_3 = \frac{1}{2}(a)(2a) = a^2$

$$\theta_1 = \frac{1}{2\bar{A}_1 G} \oint q \frac{ds}{t} = \frac{1}{2\bar{A}_1 G}\left(\frac{q_1}{2t}\left(\frac{\sqrt{5}a}{2} + \frac{\sqrt{5}a}{2}\right) + \frac{(q_1 - q_2)a}{t}\right)$$

$$\theta_2 = \frac{1}{2\bar{A}_1 G} \oint q \frac{ds}{t} = \frac{1}{2\bar{A}_2 G}\left(\frac{q_2}{2t}(2a) + \frac{(q_2 - q_3)a}{t} + \frac{(q_2 - q_1)a}{t}\right)$$

$$\theta_3 = \frac{1}{2\bar{A}_3 G} \oint q \frac{ds}{dt} = \frac{1}{2\bar{A}_3 G}\left(\frac{q_3}{2t}\left(\frac{\sqrt{17}a}{2} + \frac{\sqrt{17}a}{2}\right) + \frac{(q_3 - q_2)(-a)}{t}\right)$$

Solving these:

$$\left(\frac{\sqrt{5}}{2} + 1\right) q_1 - q_2 + (0) q_3 = \theta a G t$$

$$-q_1 + 3q_2 - q_3 = 2\theta a G t$$

$$0(q_1) - q_2 + \left(\frac{\sqrt{17}}{2} + 1\right) q_3 = 2\theta a G t$$

$$2.12q_1 - q_2 + (0)q_3 = \theta a G t$$

$$-q_1 + 3q_2 - q_3 = 2\theta a G t$$

$$(0)q_1 - q_2 + 3.06q_3 = 2\theta a G t$$

$$\begin{bmatrix} 2.12 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3.06 \end{bmatrix} \begin{bmatrix} q_1 \\ -q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \theta a G t \\ 2\theta a G t \\ 2\theta a G t \end{bmatrix}$$

Solving the above equations we get

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1.141\,\theta a G t \\ 1.428\,\theta a G t \\ 1.118\,\theta a G t \end{bmatrix}$$

$$T = \Sigma\, 2\bar{A}_i q_i$$

$$T = 2\theta a^3 G t\,(0.5705 + 1.428 + 1.118)$$

$$T = 6.217\,\theta a^3 G t = (GJ)(\theta)$$

$$J = 6.217\,t a^3$$ torsional stiffness