MA 214: Introduction to numerical analysis
Lecture 13

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Fixed points and roots

A fixed point of $f : [a, b] \to \mathbb{R}$ is $p \in [a, b]$ such that $f(p) = p$.

Note that $p \in [a, b]$ is a root of the equation $f(x) = 0$ if and only if p is a fixed point of $g(x) = f(x) - x$. WRONG!

The sign of x should be positive, thank you Aryaman.

Fixed points of various functions are studied well in Mathematics. There are many nice results guaranteeing the existence of fixed points.

Cumulative calculation of interpolating polynomials

The polynomials P_0, Q_0 and P_1 interpolating f on $\{x_0\}$, $\{x_1\}$ and $\{x_0, x_1\}$, respectively, are related by

$$P_1(x) = \frac{(x - x_1)P_0(x) - (x - x_0)Q_0(x)}{(x_0 - x_1)}.$$

Further, P_1, Q_1 and P_2 interpolating f on $\{x_0, x_1\}$, $\{x_1, x_2\}$ and $\{x_0, x_1, x_2\}$, respectively, are related by

$$P_2(x) = \frac{(x - x_2)P_1(x) - (x - x_0)Q_1(x)}{(x_0 - x_2)}.$$

If P_2 interpolates f on $\{x_0, x_1, x_2\}$, Q_2 on $\{x_1, x_2, x_3\}$ and P_3 on $\{x_0, x_1, x_2, x_3\}$ then do we get

$$P_3(x) = \frac{(x - x_3)P_2(x) - (x - x_0)Q_2(x)}{(x_0 - x_3)}?$$

Neville's formula

Let f be defined on $\{x_0, x_1, \ldots, x_n\}$.

Choose two distinct nodes x_i and x_j.

Let Q_i be the polynomial interpolating f on all nodes except x_i and let Q_j be the one interpolating f on all nodes except x_j.

If P denotes the polynomial interpolating f on all nodes then

$$P(x) = \frac{(x - x_j)Q_j(x) - (x - x_i)Q_i(x)}{x_i - x_j}.$$

Proof: Just verify that $P(x_k) = f(x_k)$ for all $0 \leqslant k \leqslant n$.

Neville's formula

In Neville's formula you can get the interpolating for higher degree from any two polynomials for two subsets of nodes which are obtained by removing a single node.

Let $P_{m_1,m_2,\dots,m_k}$ denote the polynomial interpolating the given function on $x_{m_1}, x_{m_2}, \dots, x_{m_k}$ then

$$P_{0,1} = \frac{(x - x_0)P_1 - (x - x_1)P_0}{x_1 - x_0},$$

$$P_{1,2} = \frac{(x - x_1)P_2 - (x - x_2)P_1}{x_2 - x_1},$$

$$P_{0,1,2} = \frac{(x - x_0)P_{1,2} - (x - x_2)P_{0,1}}{x_2 - x_0} = \frac{(x - x_1)P_{0,2} - (x - x_0)P_{1,2}}{x_0 - x_1}$$

and so on.

Neville's formula

We then have the following table:

$$\begin{array}{llllll}
x_0 & P_0 & & & & \\
x_1 & P_1 & P_{0,1} & & & \\
x_2 & P_2 & P_{1,2} & P_{0,1,2} & & \\
x_3 & P_3 & P_{2,3} & P_{1,2,3} & P_{0,1,2,3} & \\
x_4 & P_4 & P_{3,4} & P_{2,3,4} & P_{1,2,3,4} & P_{0,1,2,3,4}
\end{array}$$

Assume that we are given a function f on $(n+1)$-nodes and that we want to compute $f(x)$ for some x.

We then go on computing various interpolating polynomials in the order P_0, P_1, $P_{0,1}$, P_2, $P_{1,2}$, $P_{0,1,2}$, $\ldots$ until a sufficient number of digits of the values of the interpolating polynomials of the two highest degrees at x agree.

In this case, they are $P_{0,1,2,3}$, $P_{1,2,3,4}$ and $P_{0,1,2,3,4}$.

Example

Compute $f(2.1)$ using Neville's method on the following data:

x	$f(x)$
2.0	0.6931
2.2	0.7885
2.3	0.8329

The computations give

x_i	$P_i(x)$	$P_{i,i+1}(x)$	$P_{i,i+1,i+2}(x)$
2.0	0.6931		
2.2	0.7885	0.7410	
2.3	0.8329	0.7441	0.7420

Neville's method gives the values of interpolating polynomials at a specific point, without having to compute the polynomials themselves.

Divided differences

We will see another method to construct the interpolating polynomials.

Given the function f on distinct $(n+1)$-nodes, $x_0, \ldots, x_n$, there is a unique polynomial P_n interpolating f on these nodes.

We define $f[x_0, \ldots, x_n]$ to be the coefficient of x^n in P_n.

Now, it follows readily that the value of $f[x_0, \ldots, x_n]$ does not depend on the ordering of the nodes x_i.

We will get a recurrence formula for the coefficients $f[x_0, \ldots, x_n]$.

Divided differences

Let P_{n-1} and Q_{n-1} be the polynomials interpolating f on the nodes $x_0, \ldots, x_{n-1}$ and $x_1, \ldots, x_n$, respectively:

$$f(x_0) = P_{n-1}(x_0), f(x_1) = P_{n-1}(x_1), \ldots, f(x_{n-1}) = P_{n-1}(x_{n-1}),$$

and

$$f(x_1) = Q_{n-1}(x_1), f(x_2) = Q_{n-1}(x_2), \ldots, f(x_n) = Q_{n-1}(x_n).$$

By Neville's method,

$$P_n(x) = \frac{(x - x_0)Q_{n-1}(x) - (x - x_n)P_{n-1}(x)}{x_n - x_0}.$$

Divided differences

The coefficient of x^n in P_n is then

$$\frac{(\text{the coefficient of } x^{n-1} \; in \; Q_{n-1}) - (\text{the coefficient of } x^{n-1} \; in \; P_{n-1})}{x_n - x_0}$$

$$= \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Let us now see whether we can get the recurrence relation for the polynomials P_n in terms of the divided differences.

Recurrence relation for P_n

We note that for $i < n$, $P_n(x_i) = P_{n-1}(x_i)$.

In other words, $P_n - P_{n-1}$ has a zero at each of the points $x_0, x_1, \ldots, x_{n-1}$. Hence

$$P_n - P_{n-1} = \alpha(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where α is a real number.

Now, α has to be the coefficient of the monomial x^n in P_n, as the degree of P_{n-1} is $\leqslant n - 1$.

Hence $f[x_0, x_1, \ldots, x_n] = \alpha$ and we have

$$P_n = P_{n-1} + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_0, x_1, \ldots, x_n].$$

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Divided differences

We are studying divided differences, introduced by Newton, to construct the interpolating polynomials recursively.

For the polynomial P_n interpolating a given function f on nodes $x_0, \ldots, x_n$ we define

$$f[x_0, \ldots, x_n]$$

to be the coefficient of x^n in P_n.

If P_{n-1} interpolates f on the nodes $x_0, \ldots, x_{n-1}$ then

$$P_n - P_{n-1} = f[x_0, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}).$$

We also have

$$f[x_0, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

Properties of the divided differences

The divided differences can be computed in the following way

$$
\begin{array}{cccccc}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & \ddots & \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \ldots, x_n] \\
 & & \vdots & & \iddots & \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}
$$

Since everything is independent of the order of the points, we can construct the polynomial p_n in the forward way as well as in the backward way.

The forward formula

$$
\begin{array}{cccccc}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & \ddots & \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \ldots, x_n] \\
 & & \vdots & & \cdot^{\cdot^{\cdot}} & \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}
$$

$$P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +$$

$$\cdots + f[x_0, x_1, \ldots, x_n](x - x_0) \cdots (x - x_{n-1}).$$

The backward formula

$$
\begin{array}{ccccc}
x_0 & f(x_0) & & & \\
 & & f[x_0, x_1] & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & \\
 & & f[x_1, x_2] & & \ddots \\
x_2 & f(x_2) & & \vdots & & f[x_0, x_1, \ldots, x_n] \\
 & & \vdots & & \iddots \\
\vdots & \vdots & & f[x_{n-2}, x_{n-1}, x_n] & \\
 & & f[x_{n-1}, x_n] & & \\
x_n & f(x_n) & & &
\end{array}
$$

$$
P_n(x) = f(x_n) + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1})
$$

$$
+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_n) \cdots (x - x_1).
$$

Example

Find the polynomial interpolating f on $\{0,1,2\}$ with $f(0) = 1$, $f(1) = 4$ and $f(2) = 15$. The forward table is

$$\begin{array}{cccc} 0 & 1 & & \\ & & 3 & \\ 1 & 4 & & 4 \\ & & 11 & \\ 2 & 15 & & \end{array}$$

hence $p_2(x) = 1 + 3x + 4x(x-1)$
$= 4x^2 - x + 1.$

The backward table is

$$\begin{array}{cccc} 0 & 1 & & \\ & & 3 & \\ 1 & 4 & & 4 \\ & & 11 & \\ 2 & 15 & & \end{array}$$

then $p_2(x) = 15 + 11(x-2) + 4(x-2)(x-1)$
$= 4x^2 - x + 1.$

Nested form of the interpolating polynomial

The forward formula is

$$P_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) +$$

$$\cdots + f[x_0, x_1, \ldots, x_n](x - x_0)\cdots(x - x_{n-1}).$$

This polynomial can be expressed in a nested form as follows:

$$P_n(x) = f(x_0) + (x - x_0)\Big[f[x_0, x_1] + (x - x_1)\big[f[x_0, x_1, x_2] + \cdots$$

$$+ (x - x_{n-1}) f[x_0, x_1, \ldots, x_n] \cdots \big]\Big].$$

Nested form of the interpolating polynomial

In particular, we have

$$P_2(x) = f(x_0) + (x - x_0)\Big[f[x_0, x_1] + (x - x_1) f[x_0, x_1, x_2]\Big]$$

$$\begin{aligned} P_3(x) = f(x_0) \quad & + \quad (x - x_0)\Bigg[f[x_0, x_1] \\ & + \quad (x - x_1)\Big[f[x_0, x_1, x_2] \\ & + \quad (x - x_2) f[x_0, x_1, x_2, x_3]\Big]\Bigg]. \end{aligned}$$

This nested form of the interpolating polynomial is useful for computing the polynomials P_n effectively.

Divided differences as a function

In the definition of $f[x_0, \ldots, x_n]$, we need that the nodes x_i be all distinct.

We now give the definition of the divided differences when some of the nodes may be equal to each other.

By the Mean Value Theorem, $f[x_0, x_1] = f'(\xi)$ for some ξ between x_0 and x_1. In fact, we also have the following theorem:

Theorem

If f is n-times continuously differentiable on $[a, b]$ then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}$$

for some $\xi \in [a, b]$.

Divided differences as a function

Since $f[x_0, x_1] = f'(\xi)$ for some ξ between x_0 and x_1, we define

$$f[x_0, x_0] = f'(x_0).$$

This gives

$$f[x_0, x_0] = \lim_{x_1 \to x_0} f[x_0, x_1].$$

We define $f[x_0, \ldots, x_n]$ in a similar way when the nodes are not necessarily distinct, by taking limits. For instance,

$$f[x_0, x_1, x_0] = f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

$$= \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}.$$

And $f[x_0, x_0, x_0] = \dfrac{f^{(2)}(x_0)}{2}$.

Divided differences as a function

We have thus defined $f[x_0, \dots, x_n]$ in general.

Now, by letting the last x_n as a variable x, we get a function of x:

$$f[x_0, x_1, \dots, x_{n-1}, x].$$

This function is continuous. Indeed,

$$f[x_0, x] = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & x \neq x_0 \\ f'(x_0) & x = x_0 \end{cases}$$

which implies continuity of $f[x_0, x]$.

Divided differences as a function

In general,

$$f[x_0, x_1, \ldots, x_{n-2}, x_{n-1}, x] = f[x_0, x_1, \ldots, x_{n-2}, x, x_{n-1}]$$

$$= \frac{f[x_1, \ldots, x_{n-2}, x, x_{n-1}] - f[x_0, \ldots, x_{n-2}, x]}{x_{n-1} - x_0}$$

which gives continuity by induction.

We need to take care when there are equalities among the nodes.