

# AE 236 : Compressible Fluid Mechanics

(Module III: Supersonic Flow Turning)

- a. Oblique shock waves
- b. Expansion waves

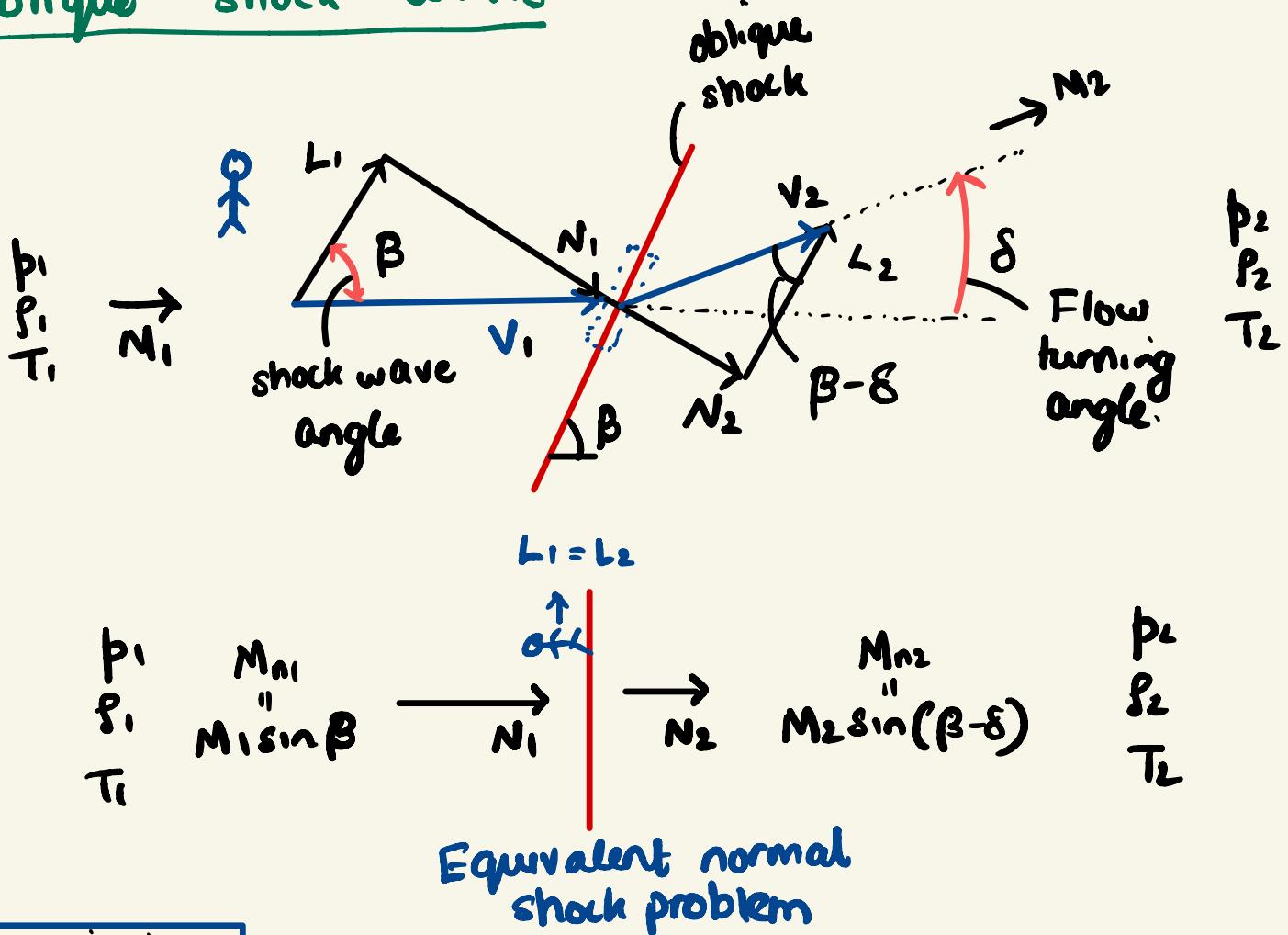
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## Oblique shock waves



Continuity

$$p_1 N_1 = p_2 N_2 \quad \text{--- ①}$$

Momentum

$$p_1 - p_2 = p_2 N_2^2 - p_1 N_1^2 \quad \text{--- ②}$$

Energy

$$\frac{2\gamma}{\gamma-1} \frac{p_1}{\rho_1} + v_1^2 = \frac{2\gamma}{\gamma-1} \frac{p_2}{\rho_2} + v_2^2$$

$$\Rightarrow \left( \frac{2\gamma}{\gamma-1} \right) \left[ \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right] = N_1^2 - N_2^2 \quad \text{--- ③}$$

$$\frac{P_2}{P_1} = \frac{\left(\frac{r+1}{r-1}\right) \frac{P_1}{M_1^2} + 1}{\left(\frac{r+1}{r-1}\right) + \frac{P_1}{M_1^2}} = \frac{N_1}{N_2}$$

$$\frac{T_2}{T_1} = \frac{\left(\frac{r+1}{r-1}\right) \frac{P_2}{P_1}}{\left(\frac{r+1}{r-1}\right) + \frac{P_1}{P_2}}$$

Rankine - Hugoniot relations for oblique shock waves

$$N_1 = V_1 \sin \beta$$

$$N_2 = V_2 \sin (\beta - \delta)$$

⇒ Instead of  $M_1$  and  $M_2$ , we can substitute  
 $M_{n1} = M_1 \sin \beta$  and  $M_{n2} = M_2 \sin (\beta - \delta)$  in the normal  
shock relationships

$$\frac{P_2}{P_1} = \frac{2r M_{n1}^2 \sin^2 \beta - (r-1)}{r+1}$$

$$\frac{P_2}{P_1} = \frac{(r+1) M_{n1}^2 \sin^2 \beta}{2 + (r-1) M_{n1}^2 \sin^2 \beta}$$

$$\frac{T_2}{T_1} = \frac{[2 + (r-1) M_{n1}^2 \sin^2 \beta] [2r M_{n1}^2 \sin^2 \beta - (r-1)]}{(r+1)^2 M_{n1}^2 \sin^2 \beta}$$

$$M_{n2}^2 \sin^2 (\beta - \delta) = \frac{M_{n1}^2 \sin^2 \beta + 2/(r-1)}{2r M_{n1}^2 \sin^2 \beta / (r-1) - 1}$$

From consideration of normal shock waves

$$M_1 \sin \beta \geq 1$$

For a given  $M_1$ ,  $\beta_{\max} = 90^\circ$  Normal shock

$$\beta_{\min} = \sin^{-1} \frac{1}{M_1}$$
 Mach wave

$$\sin^{-1} \frac{1}{M_1} \leq \beta \leq 90^\circ$$

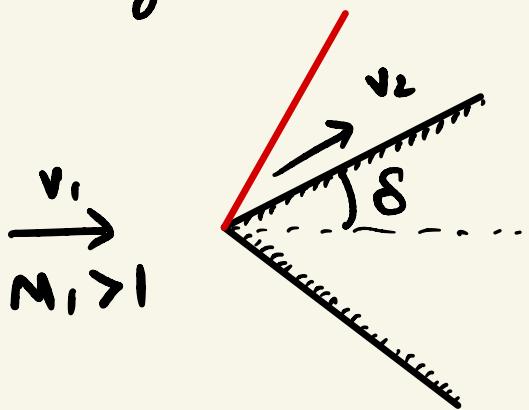
Also, from normal shock wave analysis

$$M_2 \sin(\beta - \delta) \leq 1$$

$M_2$  can be supersonic for oblique shock waves

Oblique shocks are one of two ways in which flow turning is achieved in supersonic flows.

For a given  $M_1$  &  $\delta$ ,  $\beta$  is usually unknown



Flow has to remain parallel to the wall which is ensured by an oblique shock of appropriate strength/inclination

$\Rightarrow$  We need a relation connecting  $M_1$ ,  $\delta$  and  $\beta$

We have

$$\tan \beta = \frac{N_1}{L_1}, \quad \tan(\beta - \delta) = \frac{N_2}{L_2}$$

$$\Rightarrow \frac{\tan(\beta - \delta)}{\tan \beta} = \frac{N_2}{N_1} = \frac{P_1}{P_2} = \frac{2 + (r-1) M_1^2 \sin^2 \beta}{(r+1) M_1^2 \sin^2 \beta} = X$$

From l.h.s. and r.h.s., we get

$$\left( \frac{\tan \beta - \tan \delta}{1 + \tan \beta \tan \delta} \right) / \tan \beta = X$$

$$\Rightarrow \tan \delta = \frac{\tan \beta (1-X)}{1 + X \tan^2 \beta}$$

Substituting for  $X$ , we have

$$\tan \delta = \frac{2 \cot \beta (M_1^2 \sin^2 \beta - 1)}{2 + M_1^2 (r + \cos 2\beta)}$$

- ⚡

Two cases where  $\delta = 0$

$$(i) \cot \beta = 0 \Rightarrow \beta = 90^\circ \text{ Normal shock}$$

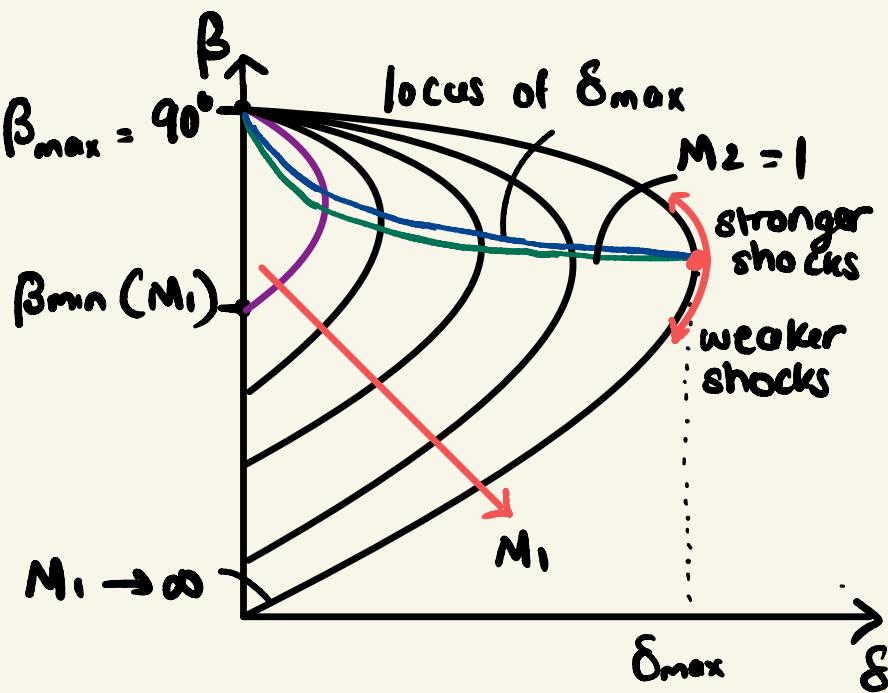
$$(ii) M_1^2 \sin^2 \beta = 1 \Rightarrow \beta = \sin^{-1} \frac{1}{M_1} \text{ Mach wave}$$

$$M_1 \rightarrow | \rightarrow \delta = 0$$

No flow turning for normal shocks and Mach waves

$$M_1 \rightarrow \cancel{|} \rightarrow \delta \neq 0$$

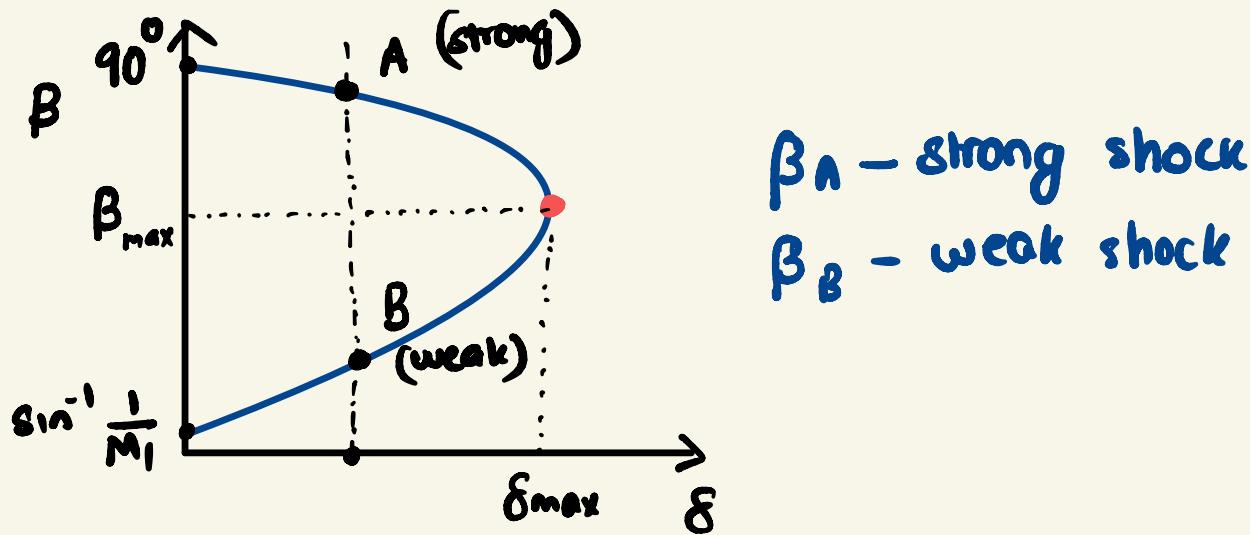
$$M_1 \rightarrow \cancel{|} \rightarrow \delta = 0 \quad \downarrow \beta \text{ decreasing}$$



As  $\beta$  increases for fixed  $M_1$ ,  $M_{n1} = M_1 \sin \beta$  increases,  $p_2/p_1$  increases and shocks become stronger

$\beta > \delta_{max}$  (strong shock solution) Always subsonic  
 $\beta < \delta_{max}$  (weak shock solution) Subsonic / supersonic

Given  $\delta, M_1$ , two possible solutions for  $\beta$



External flows — weak shock solution as the required pressure gradients cannot be maintained

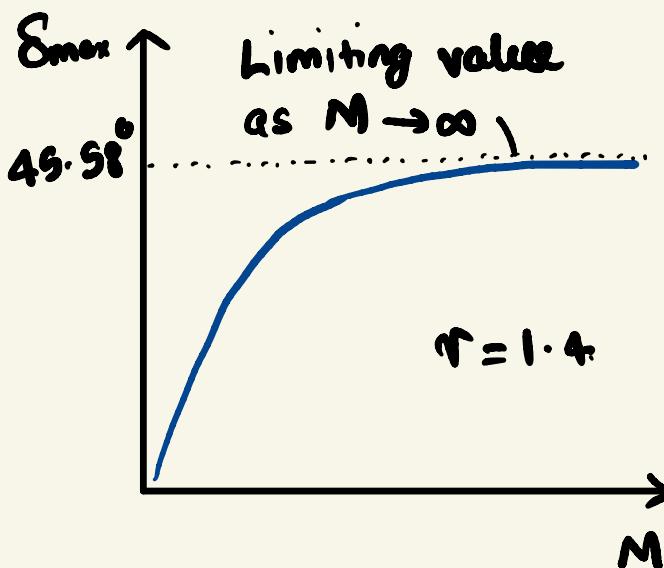
Internal flows — strong shock solution as required pressure gradients are easier to maintain

## Finding $\delta_{max}$

Set  $\frac{d\delta}{d\beta} = 0$  in  $\textcircled{2}$

$$\Rightarrow \sin^2 \beta_{max} = \frac{\gamma+1}{4\gamma} - \frac{1}{\gamma M_2^2} \left[ 1 - \sqrt{(\gamma+1) \left( 1 + \frac{\gamma-1}{2} M_1^2 + \frac{\gamma+1}{16} M_1^4 \right)} \right]$$

Substitute  $\beta_{max}$  in  $\textcircled{2}$  to get  $\delta_{max}$



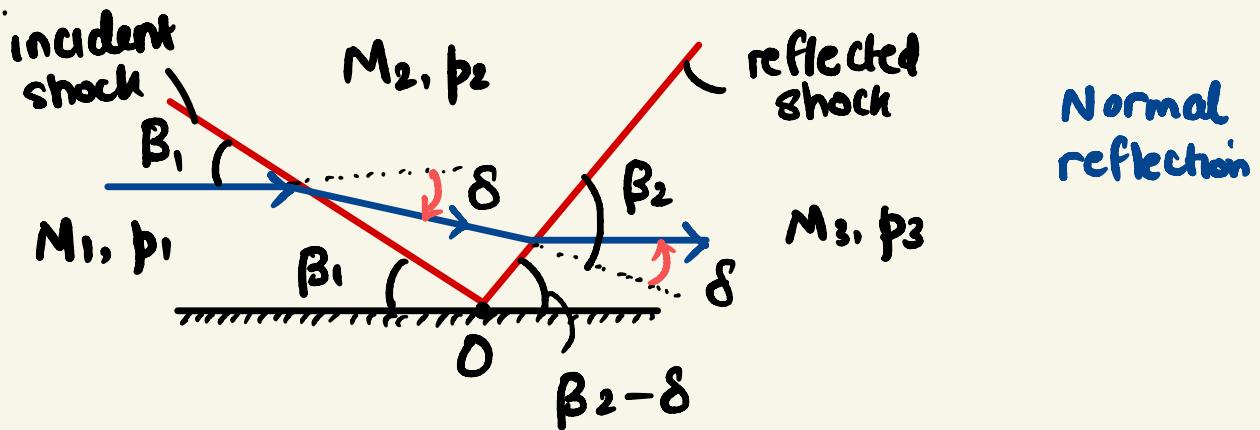
Tells us maximum achievable flow turning at a given  $M$   
Maximum possible turning by an oblique shock =  $45.58^\circ$



For  $\delta > \delta_{max}$ , we have a **detached curved shock wave** which consists of all possible solutions at the  $M_1$  ( $\sin^{-1} \frac{1}{M_1} \leq \beta \leq 90^\circ$ )

Decreasing  $M_1$  at a given  $\delta$  shifts the solution from an oblique shock to a detached curved shock.

## Reflections of oblique shocks

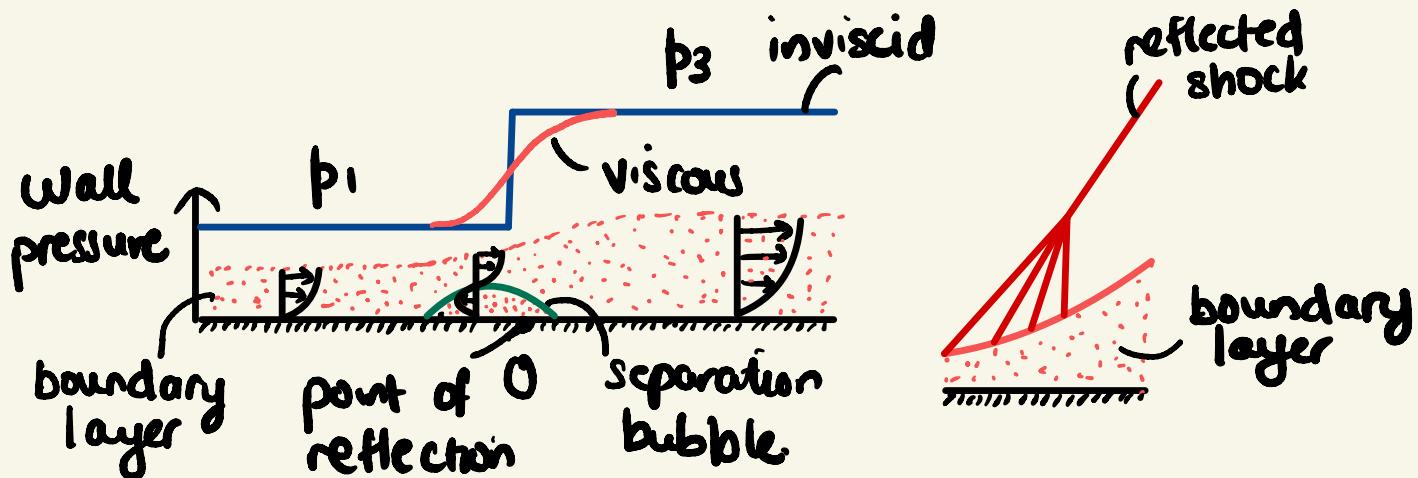


Normal reflection

- (1) For the given  $M_1, \delta$ , determine  $\beta_1, M_2, p_2/p_1$
- (2) For  $M_2$  and known  $\delta$ , find  $\beta_2, M_3, p_3/p_2$
- (3) Then overall pressure ratio

$$\frac{p_3}{p_1} = \frac{p_3}{p_2} \cdot \frac{p_2}{p_1}$$

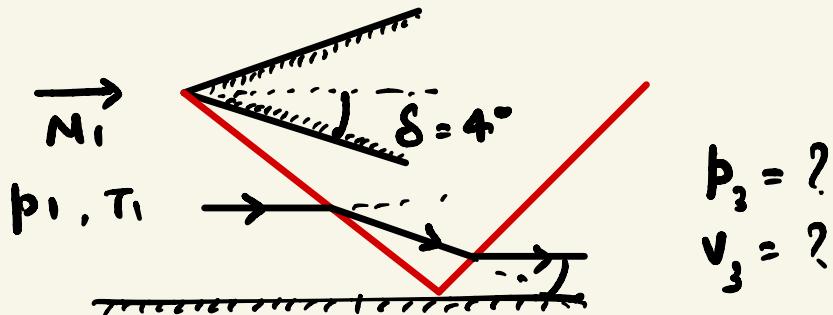
- (4) Find  $\beta_2 - \delta$  which gives angle of reflected shock



Actual form of pressure distribution depends on the type of boundary layer flow (laminar/turbulent), the thickness of the boundary layer and shock strength

## Problems

①  $M_1 = 2.5, p_1 = 60 \text{ kPa}, T = -20^\circ\text{C}$



for  $M_1 = 2.5, \delta = 4^\circ, \beta_1 = 26.6^\circ$  (weak solution)

$$M_{n1} = M_1 \sin \beta_1 = 1.12$$

From normal shock tables at  $M = 1.12$

$$M_{n2} = 0.897, \frac{p_2}{p_1} = 1.336, \frac{T_2}{T_1} = 1.087$$

$$M_2 = \frac{0.897}{\sin(26.6^\circ - 4^\circ)} = 2.334$$

For  $M_2 = 2.334, \delta = 4^\circ, \beta_2 = 28.5^\circ$

$$M_{n2} = M_2 \sin \beta_2 = 1.113$$

From normal shock tables at  $M = 1.113$

$$M_{n3} = 0.9018, \frac{p_3}{p_2} = 1.297, \frac{T_3}{T_2} = 1.078$$

$$M_3 = \frac{0.9018}{\sin(28.5^\circ - 4^\circ)} = 2.17$$

$$p_3 = \frac{p_3}{p_2} \cdot \frac{p_2}{p_1} \cdot p_1 = 104.0 \text{ kPa}$$

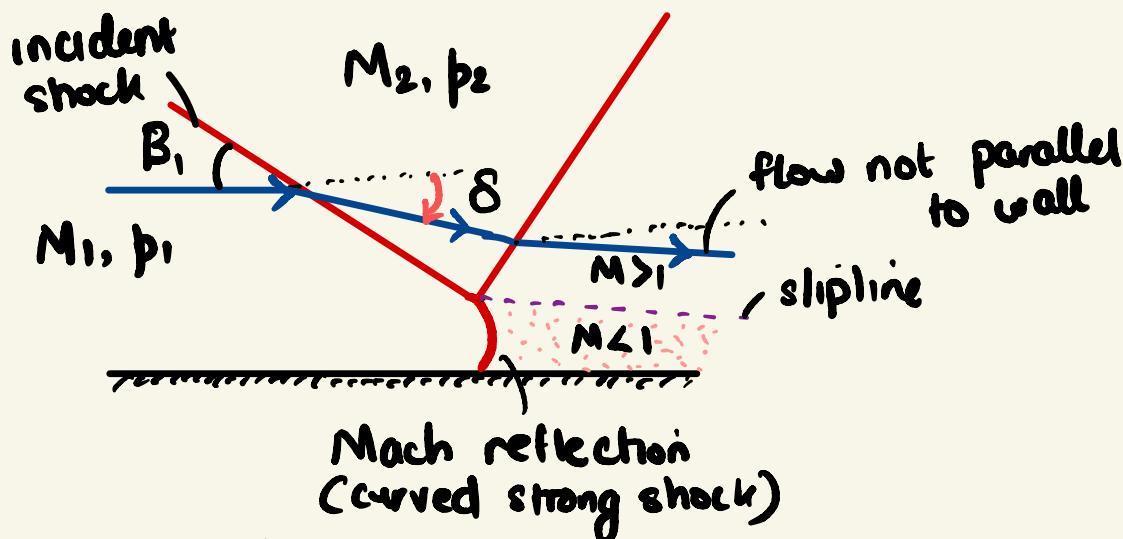
$$T_3 = \frac{T_3}{T_2} \cdot \frac{T_2}{T_1} \cdot T_1 = 296 \text{ K}$$

$$q_3 = \sqrt{\gamma R T} = 345 \text{ m/s} \Rightarrow v_3 = M_3 v_3 = 749 \text{ m/s}$$

## Special cases

① What happens if  $\delta > \delta_{\max}(M_2)$  ?

We have **Mach reflection** with formation of **sliplines**

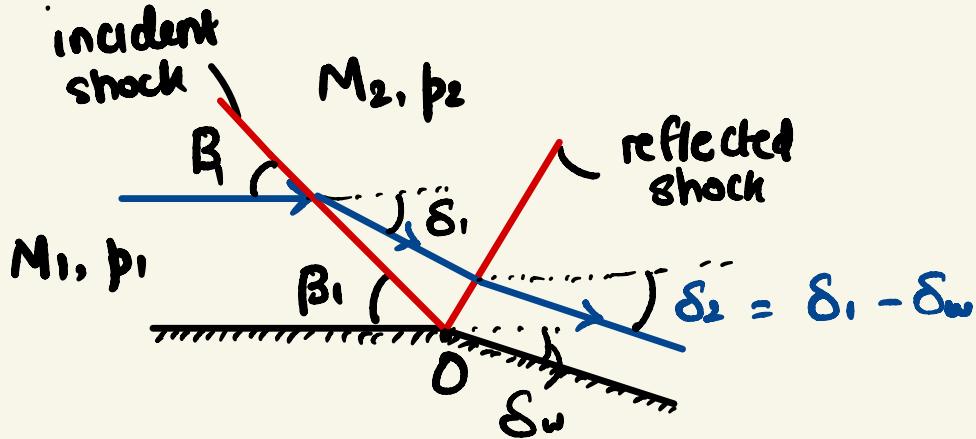


Properties of sliplines:

- ① Pressure is continuous across a slipline
- ② Velocity should have same direction either side of a slipline
- ③ P, T, V, S can have different values across a slipline

In the previous problem with  $M_1 = 2.5$ , if  $\delta > 17.9^\circ$  then we have Mach reflection

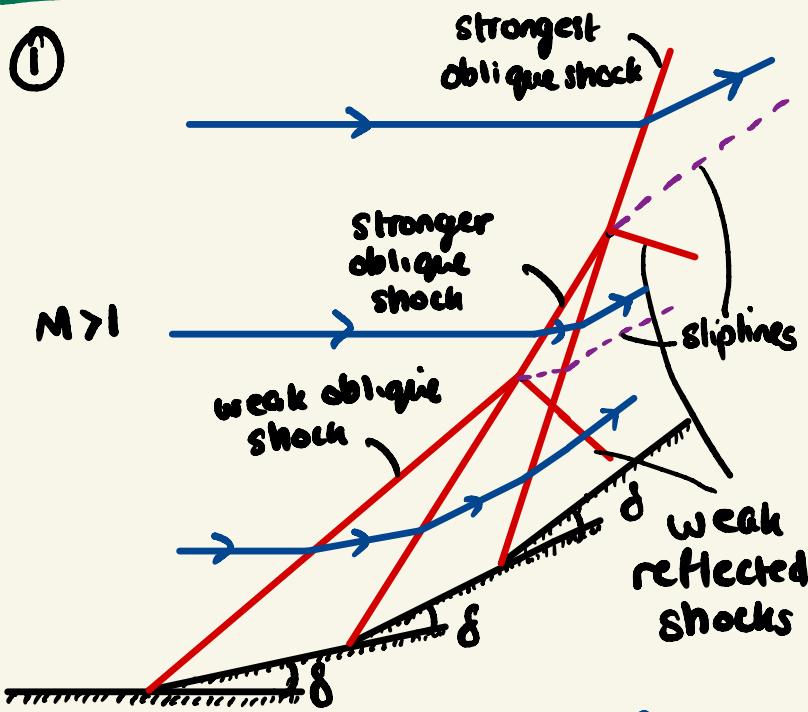
② What happens if wall is inclined? ( $\delta_w$ )



If  $\delta_w > \delta_1 \Rightarrow \delta_2 > 0 \Rightarrow$  No reflected shock

## Interaction of oblique shock waves

①

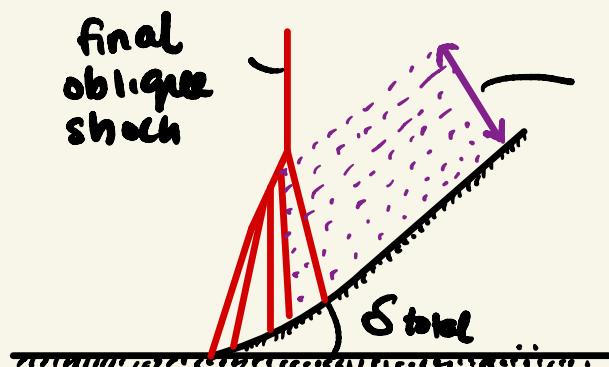


The oblique shock waves generated at each step will converge and coalesce into a single oblique shock that is stronger than any of the initial waves

Oblique shocks converge as  $\beta$  increases with decreasing  $M$

Reflected shocks are formed as the weaker oblique shocks can't produce the same change as the stronger shock

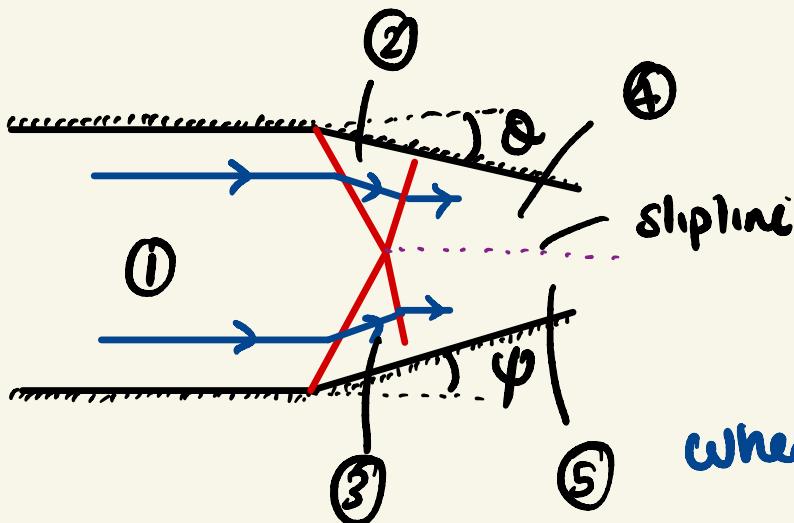
② What if we have a smooth ramp?



Region of uniform pressure and flow direction but changing  $V, T, P, S$

Final shock turns the flow by  $\delta_{\text{total}}$

③ Internal flows



Flow should be parallel to the centerline as flow from ② and ③ approach it, which creates reflected shocks & sliplines

When  $\theta = \psi$ , sliplines vanish

## What if $\delta_w > \delta_1$ ?

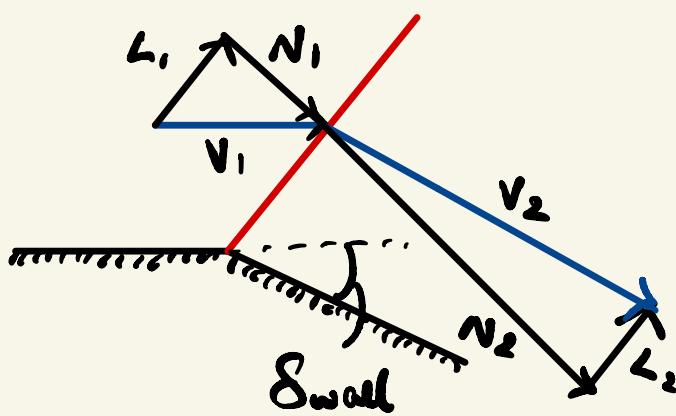
Supersonic flow

- ↳ turning into itself — compression (oblique shock)
- ↳ turning away from itself — expansion?

Happens when

- ① the wall is convex
- ② duct flows when pressure just at the exit is greater than outside pressure outside

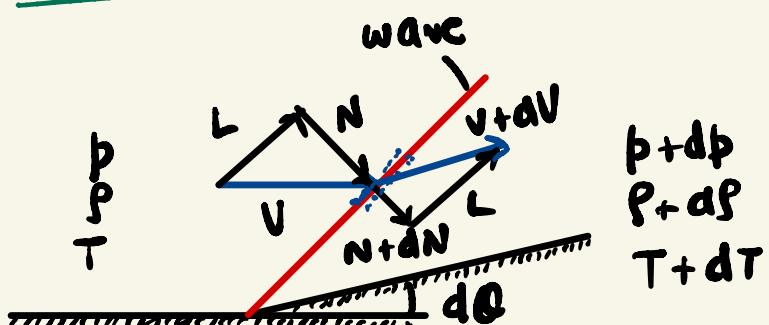
Can oblique shocks achieve this turn?



$N_2 > N_1$ , which is unphysical because 2nd law of thermodynamics is violated (Expansion shocks)

Oblique shocks cannot make the flow turn this way

Solution procedure: consider an infinitesimal turn  $d\theta$



$d\theta$  is infinitesimal. Also  $dp, dp, dT, dV$  small  
 $\Rightarrow$  flow  $\approx$  isentropic

Since flow is isentropic  $d\theta$  can be +ve or -ve

- ↳ +ve for "compression", -ve for "expansion"

## Continuity

$$p_N = (p + dp)(N + dN)$$

$$\Rightarrow pdN + Ndp = 0 \quad \text{--- ①}$$

## Momentum

$$p - (p + dp) = p_N (N + dN - 1)$$

$$\Rightarrow -dp = p_N dN \quad \text{--- ②} \quad \text{Euler's equation}$$

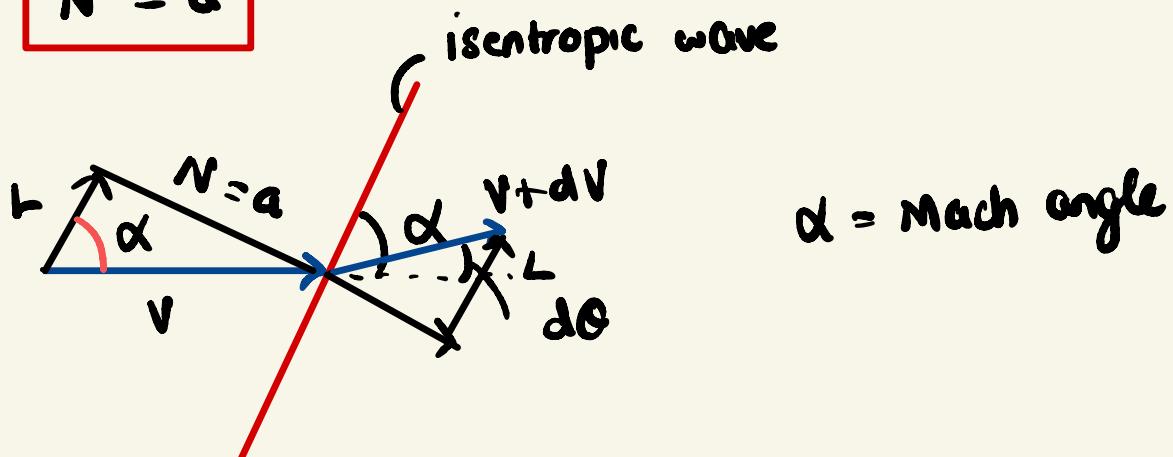
Substitute for  $dN$  from ① in ②

$$+dp = p_N \left( + \frac{Ndp}{p} \right)$$

$$\Rightarrow \frac{dp}{dp} = N^2$$

Process nearly isentropic  $\left(\frac{dp}{dp}\right)_s = a^2 = \frac{rP}{P}$

$$\Rightarrow N = a$$



$$L = V \cos \alpha = (V + dV) \cos (\alpha - d\theta)$$

$$\Rightarrow (V + dV) [\cos \alpha \cos d\theta + \sin \alpha \sin d\theta] \approx V \cos \alpha$$

$$\Rightarrow V \cos \alpha + V \sin \alpha d\theta + dV \cos \alpha = V \cos \alpha$$

$$\Rightarrow \frac{dV}{V} = -\tan \alpha d\theta$$

$$\tan \alpha = \frac{1}{\sqrt{M^2-1}}$$

$$\Rightarrow \boxed{\frac{dV}{V} = \frac{-d\theta}{\sqrt{M^2-1}}} - \star$$

## Energy

$$\left(\frac{2r}{r-1}\right)\left(\frac{p}{s}\right) + v^2 = \left(\frac{2r}{r-1}\right)\left(\frac{p+dp}{s+ds}\right) + (v+dv)^2$$

$$\cancel{\left(\frac{2r}{r-1}\right)\left(\frac{p}{s}\right) + v^2} = \left(\frac{2r}{r-1}\right)\left(\frac{p}{s}\right) \left[ 1 + \frac{dp}{p} - \frac{ds}{s} \right] + \cancel{v^2} + 2vdv$$

$$\Rightarrow \left(\frac{2r}{r-1}\right)\left(\frac{p}{s}\right) \left[ \frac{dp}{p} - \frac{1}{r} \frac{dp}{p} \cdot \frac{rs}{s} \cdot \frac{ds}{dp} \right] = -2vdv$$

$$\Rightarrow \left(\frac{a^2}{r-1}\right) \frac{dp}{p} \left[ 1 - \frac{1}{r} \right] = -vdv$$

$$\Rightarrow \frac{dp}{p} = -\frac{r v dv}{a^2} = -\frac{r v^2}{a^2} \frac{dv}{V} = -r M^2 \frac{dv}{V}$$

Substitute  $\star$

$$\boxed{\frac{dp}{p} = \frac{r M^2}{\sqrt{M^2-1}} d\theta} \rightarrow \star\star$$

$$\frac{ds}{p} = \frac{dp}{dp} \cdot \frac{dp}{p} \cdot \frac{p}{s} = \frac{1}{a^2} \frac{dp}{p} \cdot \frac{a^2}{r} = \frac{1}{r} \frac{dp}{p}$$

$$\Rightarrow \boxed{\frac{ds}{p} = \frac{M^2}{\sqrt{M^2-1}} d\theta} \rightarrow \star\star\star$$

## Entropy

$$\frac{ds}{R} = \underbrace{\frac{1}{\gamma-1} \ln \left[ 1 + \frac{dp}{p} \right]}_{C_v/R} - \underbrace{\frac{\gamma}{\gamma-1} \ln \left[ 1 + \frac{df}{f} \right]}_{C_p/R}$$

$$\approx \frac{1}{\gamma-1} \frac{dp}{p} - \frac{\gamma}{\gamma-1} \frac{df}{f} = 0 \quad \text{from } \star \star \text{ and } \star \star \star$$

Flow process is isentropic

what is the fractional change in Mach number?

$$M^2 = \frac{V^2}{a^2} = \frac{V^2 p}{\gamma p}$$

$$\begin{aligned} \Rightarrow 2 \frac{dM}{M} &= 2 \frac{dV}{V} + \frac{df}{f} - \frac{dp}{p} \\ &= [2 - M^2 + \gamma M^2] \frac{-d\theta}{\sqrt{M^2 - 1}} \end{aligned}$$

$$\Rightarrow \boxed{\frac{dM}{M} = \left[ 1 + \left( \frac{\gamma-1}{2} \right) M^2 \right] \frac{(-d\theta)}{\sqrt{M^2 - 1}}} - \star \star \star$$

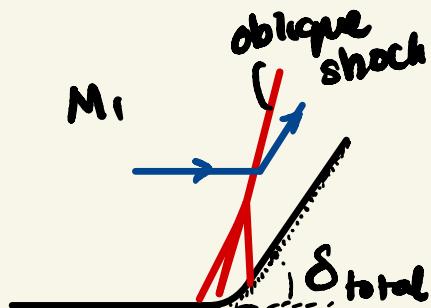
For a differential isentropic ( $ds = 0$ ) flow turn ( $d\theta$ )

$$\begin{aligned} dV &\propto -d\theta \\ dp &\propto d\theta \\ df &\propto d\theta \\ dM &\propto -d\theta \end{aligned}$$

All property changes directly proportional to  $d\theta$

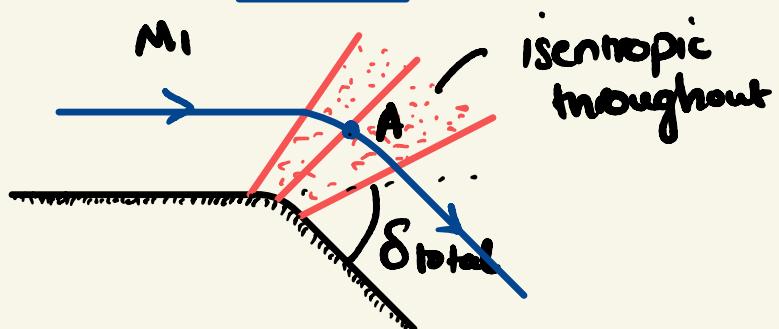
If results seem counterintuitive to intuition, that is because our intuition is SUBSONIC

$d\theta > 0$



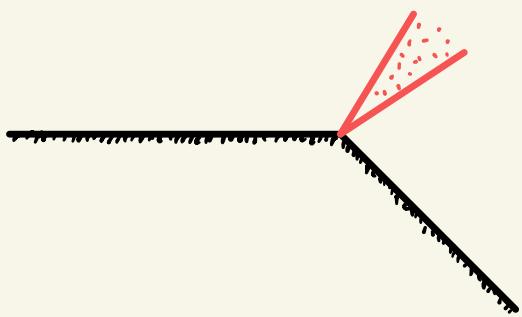
Mach waves converge

$d\theta < 0$

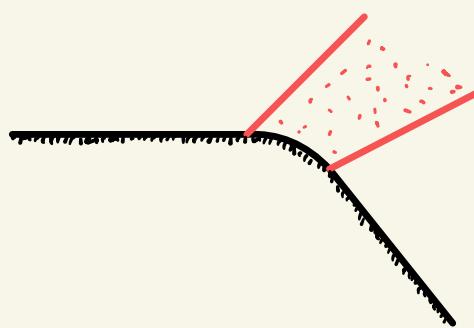


Mach waves diverge

$d\theta < 0$  produces Prandtl-Meyer flows / expansion waves



centered expansion fan



noncentered expansion fan

$$\int_{\theta_1}^{\theta_2} -d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M}$$

At  $M=1, \theta=0, V=a=a^*$  Reference condition

$$\int_0^{\theta_1} d\theta = \int_1^{M_1} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM} - \int_1^{M_2} \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}$$

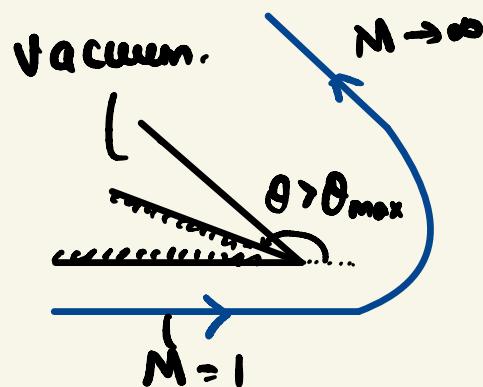
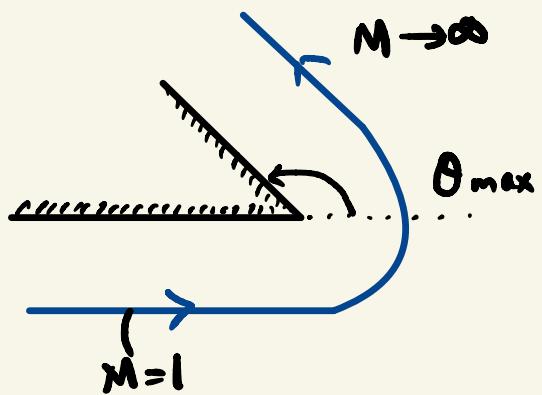
$$\Rightarrow \theta = \int_1^M \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \quad \text{function only of } M$$

$$\theta = \sqrt{\frac{r+1}{r-1}} \tan^{-1} \sqrt{\frac{r-1}{r+1} (M^2 - 1)} - \tan^{-1} \sqrt{M^2 - 1}$$

tabulated in isentropic tables

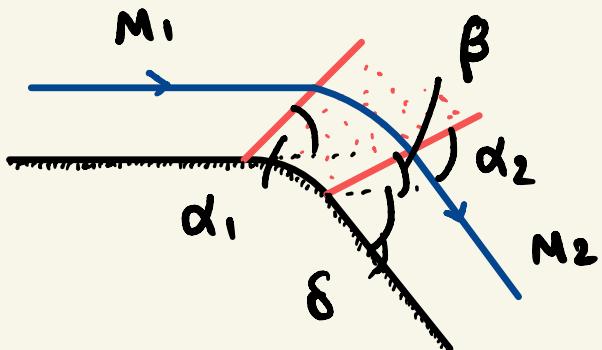
$$\text{As } M \rightarrow \infty, \theta_{\max} = \sqrt{\frac{r+1}{r-1}} \frac{\pi}{2} - \frac{\pi}{2} = 130.5^\circ \text{ for } r = 1.4$$

If we turn a  $M=1$  flow by  $130.5^\circ$  we achieve  $M \rightarrow \infty$  downstream.



$p, P, T \rightarrow 0$  downstream

Not realized physically as continuum violated,  
ideal gas assumption violated, liquefaction etc.



$$\alpha_1 = \sin^{-1} \frac{1}{M_1}$$

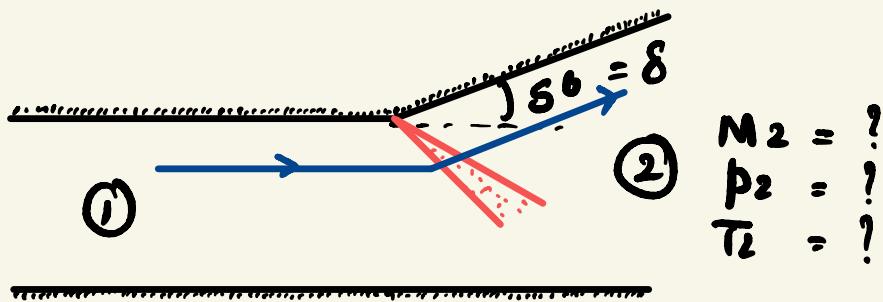
$$\alpha_2 = \sin^{-1} \frac{1}{M_2}$$

$$\beta = \alpha_2 - \delta$$

## Problems

②

$$M_1 = 1.8 \\ T_1 = 15^\circ C \\ p_1 = 90 \text{ kPa}$$



From isentropic tables corresponding to  $M_1 = 1.8$

$$\theta_1 = 20.73^\circ, \frac{p_{01}}{p_1} = 5.746, \frac{T_{01}}{T_1} = 1.648$$

$$\Rightarrow \theta_2 = \theta_1 + \delta \\ = 25.73^\circ$$

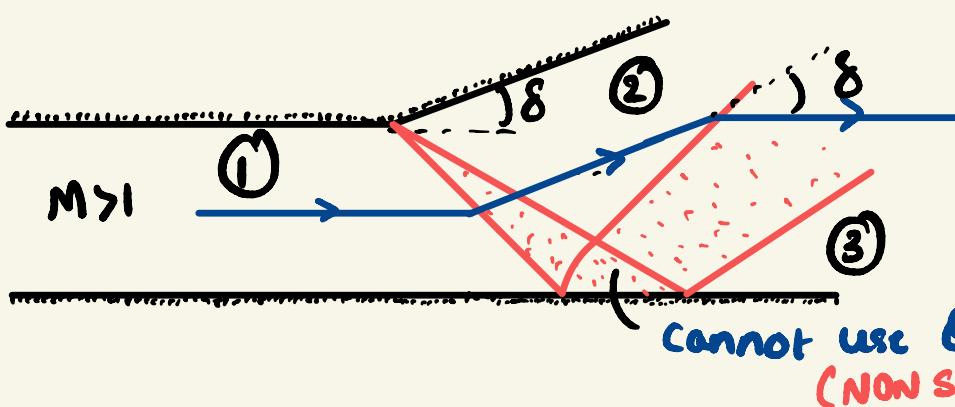
From isentropic tables corresponding to  $\theta_2$

$$M_2 = 1.98, \quad \frac{p_{02}}{p_2} = 7.585, \quad \frac{T_{02}}{T_2} = 1.784$$

$$T_2 = \frac{T_2}{T_{02}} \cdot \frac{T_{02}}{T_{01}} \cdot \frac{T_{01}}{T_1} \\ = \frac{1.648}{1.784} \cdot (288) = 266 \text{ K} = -7^\circ C$$

$$p_2 = \frac{p_2}{p_{02}} \cdot \frac{p_{02}}{p_{01}} \cdot \frac{p_{01}}{p_1} \\ = \frac{5.746}{7.585} \cdot 90 = 68.2 \text{ kPa}$$

## Reflection of expansion wave



$$\theta_2 = \theta_1 + \delta$$

$$\theta_3 = \theta_2 + \delta = \theta_1 + 2\delta$$

### Problems

③  $M_1 = 2.0$ ,  $\delta = 10^\circ$ ,  $p_1 = 90 \text{ kPa}$ . Find  $M_3, p_3$

From isentropic tables at  $M_1 = 2.0$

$$\theta_1 = 26.38^\circ, \frac{p_{01}}{p_1} = 7.83$$

From the figure,

$$\theta_3 = \theta_1 + 2\delta = 46.38^\circ$$

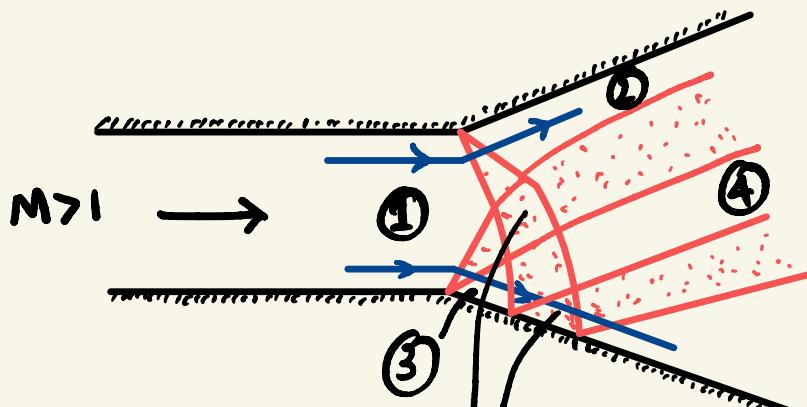
Using isentropic tables at  $\theta_3$

$$\Rightarrow M_3 = 2.83, \frac{p_{03}}{p_3} = 28.41$$

$$p_3 = \frac{p_3}{p_{03}} \cdot \frac{p_{03}}{p_{01}} \cdot \frac{p_{01}}{p_1}$$

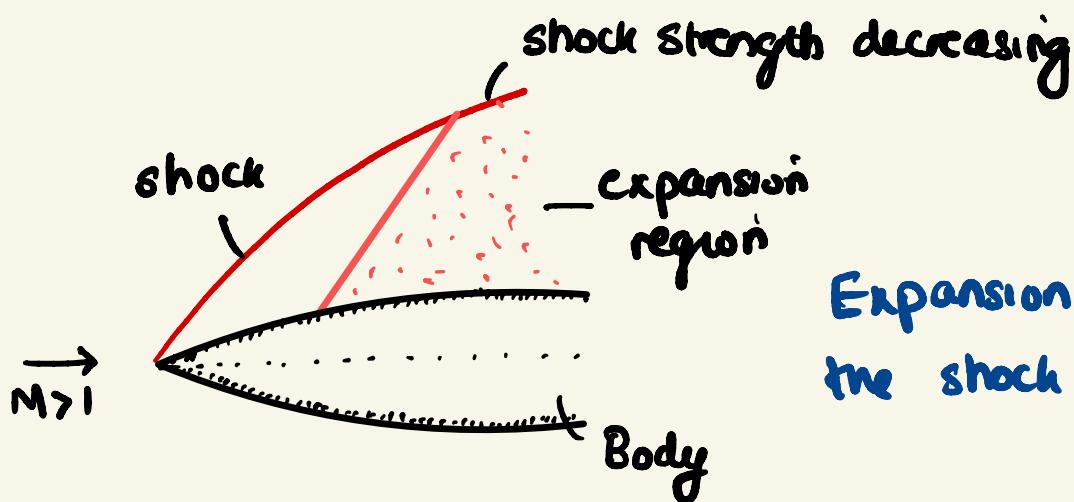
$$= \frac{7.83}{28.41} \cdot 90 = 24.81 \text{ kPa}$$

## Interaction of expansion waves



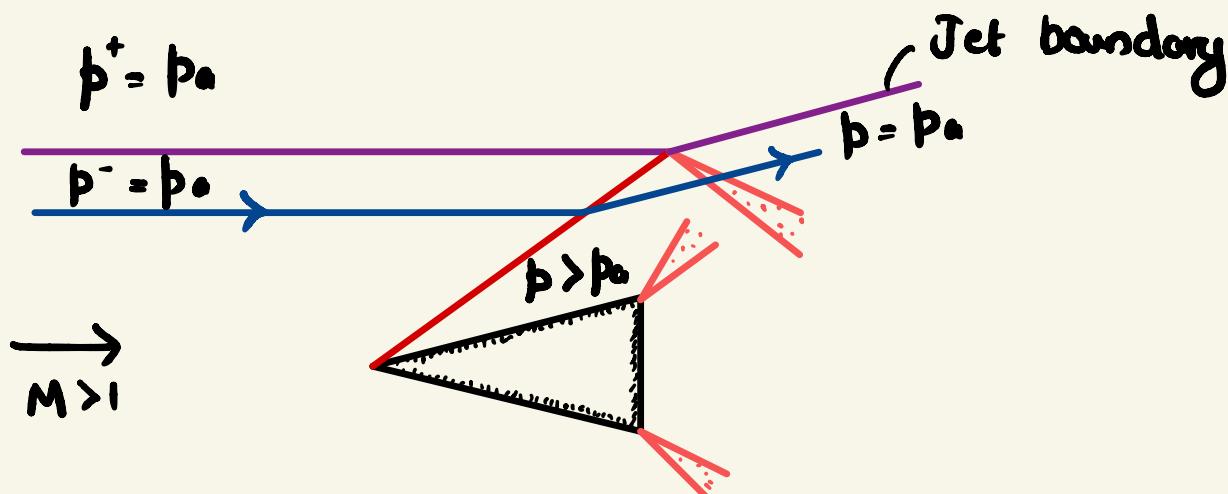
Region ④ has uniform properties with no slipstreams as the entire flow is isentropic

## Interaction of expansion waves with shock



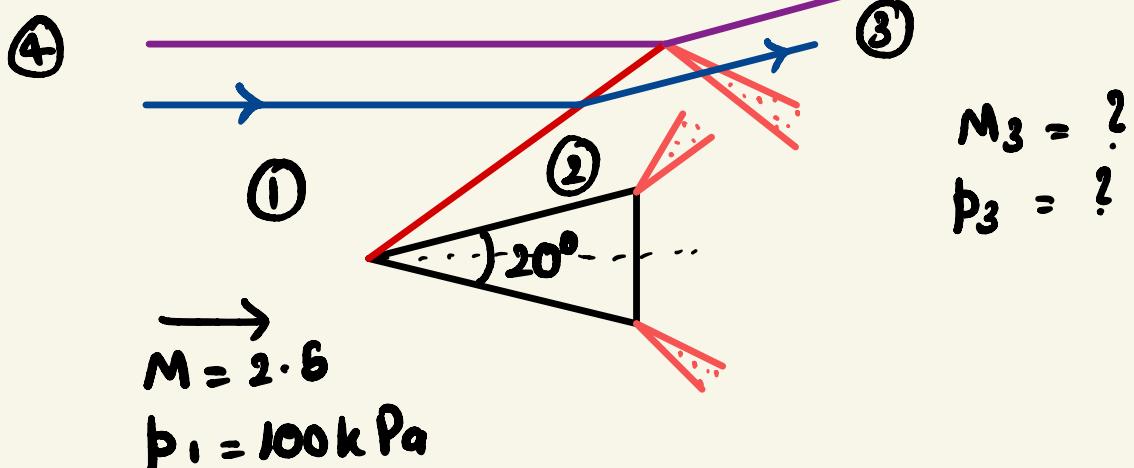
Expansion fans weaken the shock on interaction

## Interaction of an oblique shock with a jet boundary



Oblique shock interacting with a jet boundary produces an expansion wave and vice versa.

## Problems



At  $M_1 = 2.5$ ,  $\delta = 10^\circ$ , using oblique shock tables

$$M_2 = 2.086, \frac{p_2}{p_1} = 1.866$$

Isentropic tables at  $M_2 = 2.086$  give

$$\frac{p_{02}}{p_2} = 8.945$$

We need  $p_3 = p_1$

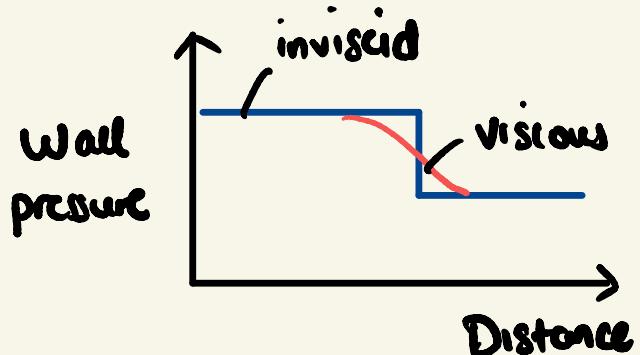
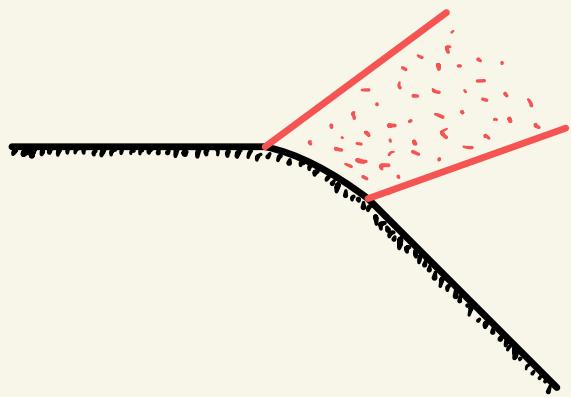
$$\begin{aligned} \frac{p_{03}}{p_3} &= \underbrace{\frac{p_{03}}{p_{02}}}_{\substack{1 \\ \text{from}}} \cdot \frac{p_{02}}{p_2} \cdot \frac{p_2}{p_3} \\ &= 1.8945 \cdot 1.866 \\ &= 16.70 \end{aligned}$$

Using isentropic tables corresponding to  $\frac{p_0}{p_3}$

$$M_3 = 2.485$$

$$p_3 = p_1 = 100 \text{ kPa}$$

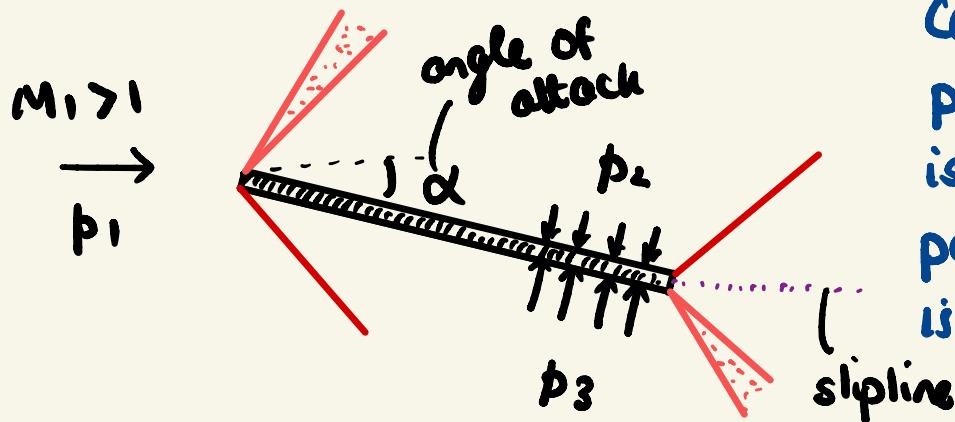
## Boundary layer effects



Pressure gradient depends on the type of boundary layer (laminar/turbulent) and its thickness

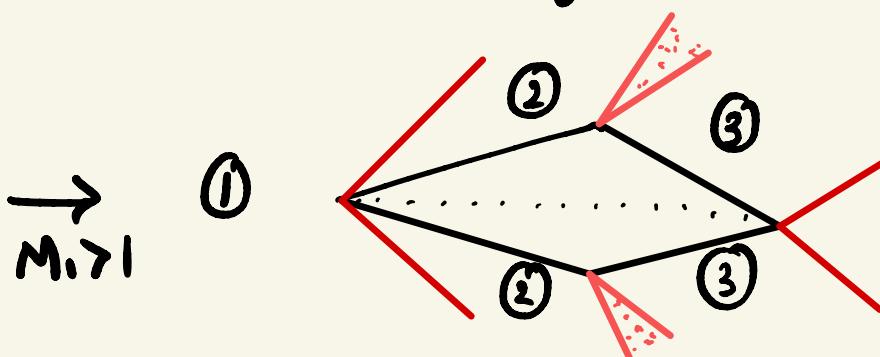
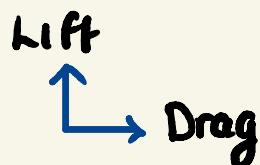
## Shock-expansion theory

### Flat plate



Component of force perpendicular to flow is **LIFT** and component parallel to the flow is **DRAG (WAVE DRAG)**

### Airfoil



Diamond airfoil

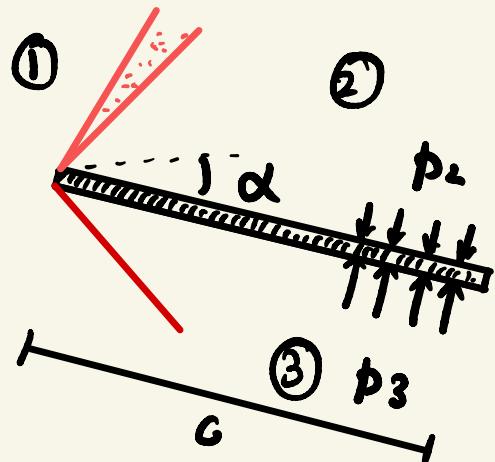
## Problems

⑥ Wing modelled as flat plate of width 0.25 m :  $\alpha = 3^\circ$ ,  $M_1 = 2.5$ ,  $p_1 = 60 \text{ kPa}$ . Estimate lift and drag / span.

For region ① at  $M_1 = 2.5$

$$\frac{p_{01}}{p_1} = 17.09, \theta_1 = 39.13^\circ \quad M_1 > 1 \rightarrow p_1$$

$$\Rightarrow \theta_2 = \theta_1 + \alpha = 42.13^\circ$$



From isentropic tables,

$$M_2 = 2.63, \frac{p_{02}}{p_2} = 20.93$$

$$\Rightarrow p_2 = \frac{p_2}{p_{02}} \cdot \underbrace{\frac{p_{02}}{p_{01}}}_{\frac{1}{17.09}} \cdot \frac{p_{01}}{p_1} \cdot p_1 = \frac{17.09}{20.93} \cdot 60$$

$$= 49.02 \text{ kPa}$$

For region ③,  $\delta = \alpha = 3^\circ$ ,  $M_1 = 2.5$

$$\beta = 26^\circ, M_{n1} = 2.5 \sin 26^\circ = 1.09$$

From shock tables at  $M_{n1}$

$$\frac{p_3}{p_1} = 1.28 \Rightarrow p_3 = 74 \text{ kPa}$$

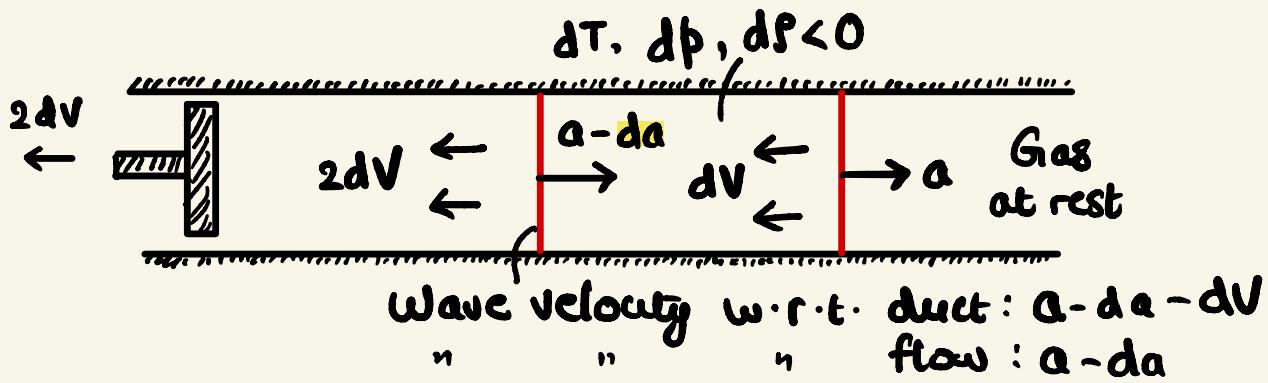
$$\text{Lift per unit span} = (p_3 - p_2) \cdot c \cdot 1 \cdot \cos \alpha$$

$$= 6.23 \text{ kN/m}$$

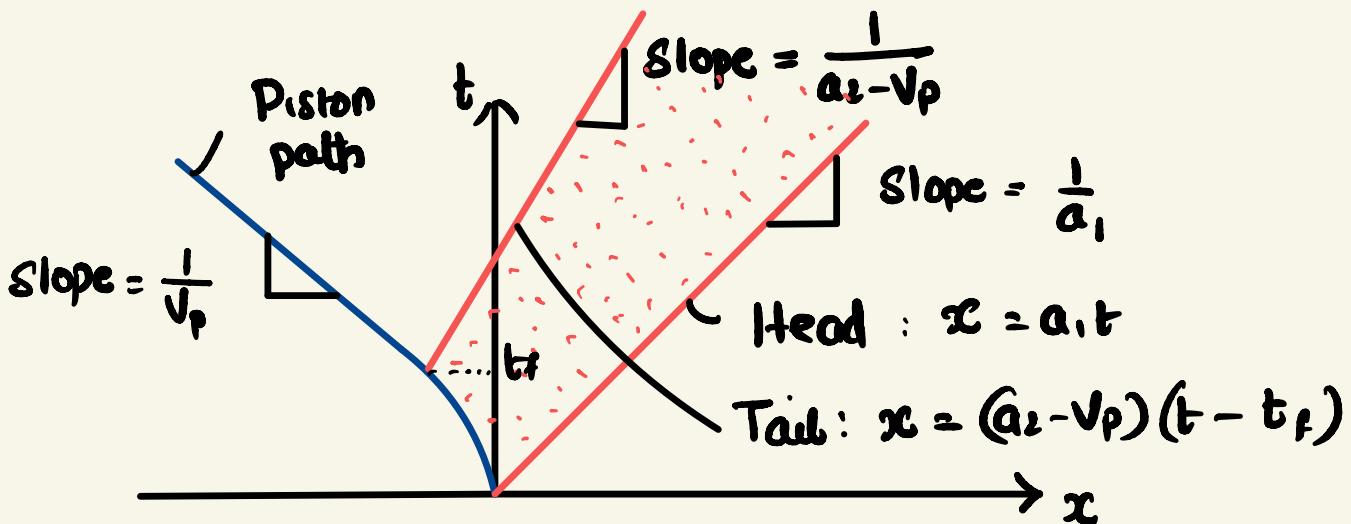
$$\text{Drag per unit span} = (p_3 - p_2) \cdot c \cdot 1 \cdot \sin \alpha$$

$$= 0.33 \text{ kN/m}$$

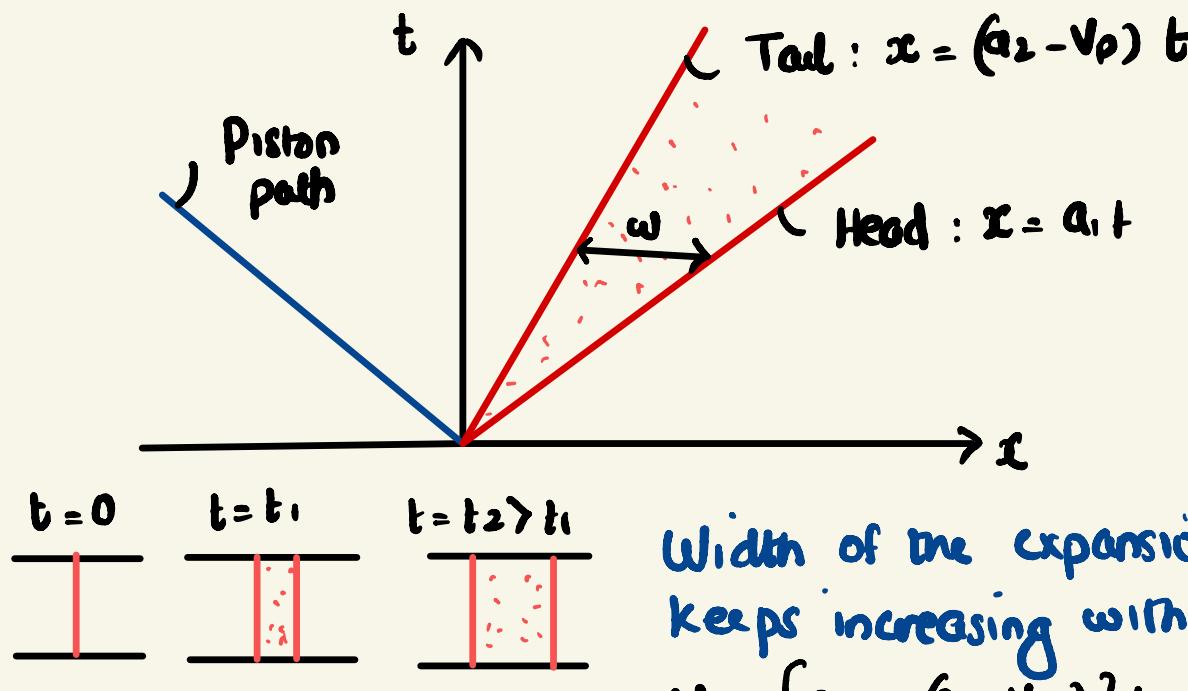
## Unsteady expansion waves



If piston velocity accelerated from 0 to  $V_p$

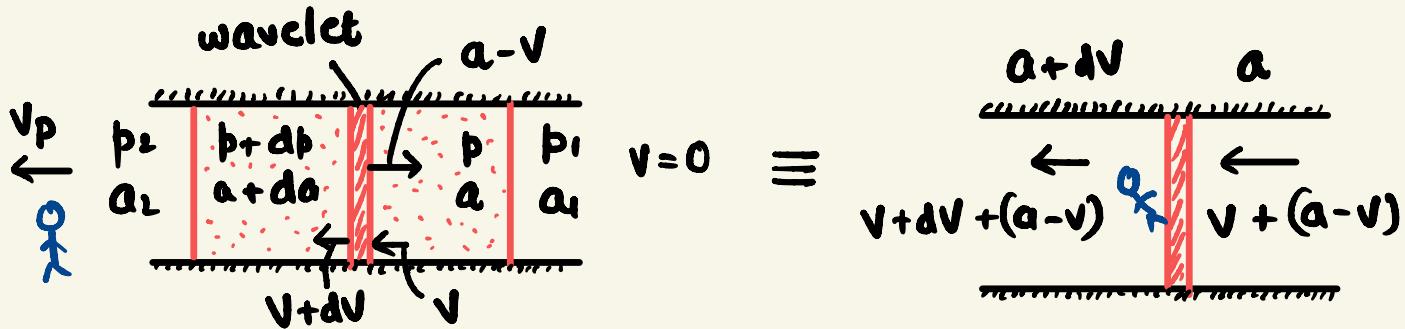


If acceleration were immediate from 0 to  $V_p$ ,



Width of the expansion wave keeps increasing with time  
 $w = [a_1 - (a_2 - V_p)] t$

Consider the flow relative to a wavelet of the expansion wave



### Momentum

$$p - (p + dp) = \rho a [(a + dv) - a]$$

$$-dp = \rho a dv$$

Since flow is isentropic  $\frac{dp}{ds} = a^2$

$$\Rightarrow -\frac{dp}{s} = \frac{dv}{a}$$

Integrating across the wave

$$\int_0^{V_p} dv = - \int_{p_1}^{p_2} a \frac{dp}{s}$$

$$\frac{a}{a_1} = \left(\frac{p}{p_1}\right)^{\frac{r-1}{2}}$$

$$V_p = - \int_{p_1}^p a_1 \left(\frac{p}{p_1}\right)^{\frac{r-1}{2}} \frac{dp}{s}$$

$$= - \left[ \left(\frac{2}{r-1}\right) \frac{a_1}{p_1^{\frac{r-1}{2}}} \left( p_2^{\frac{r-1}{2}} - p_1^{\frac{r-1}{2}} \right) \right]$$

$$= - \left(\frac{2}{r-1}\right) a_1 \left[ \left(\frac{a_2}{a_1}\right) - 1 \right]$$

$$V_p = \frac{2a_1}{r-1} - \frac{2a_2}{r-1}$$

$$\frac{a_2}{a_1} = 1 - \left(\frac{r-1}{2}\right) \frac{v_0}{a_1}$$

$$\frac{p_2}{p_1} = \left(\frac{a_2}{a_1}\right)^{\frac{2r}{r-1}}$$

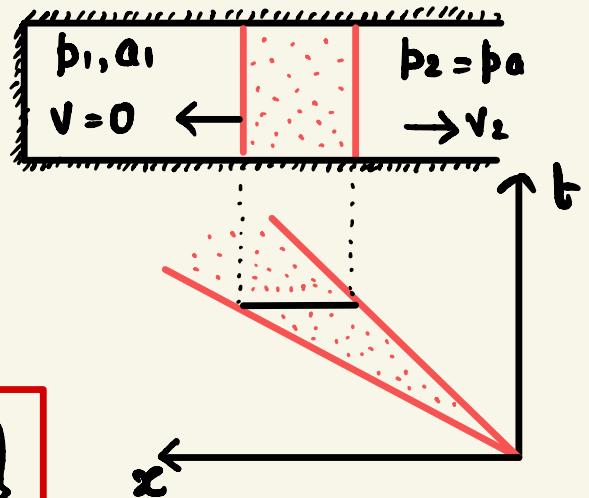
$$\frac{s_2}{s_1} = \left(\frac{a_2}{a_1}\right)^{\frac{2}{r-1}}$$

## Practical scenarios

- ① Tube sealed at both ends, contains gas at  $p_1 > p_a$ , one end suddenly opened

$$\frac{p_a}{p_1} = 1 - \left(\frac{r-1}{2}\right) \frac{v_2}{a_1}$$

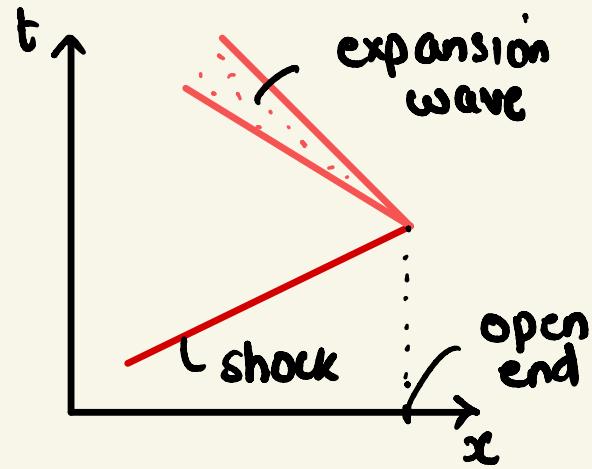
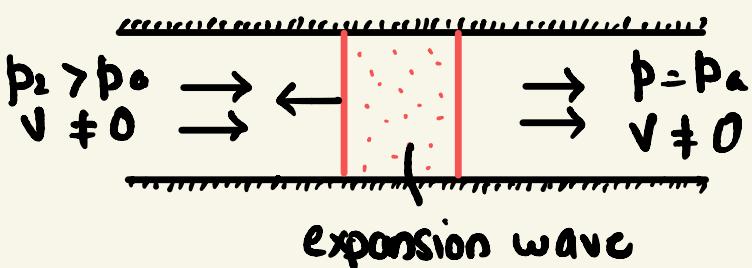
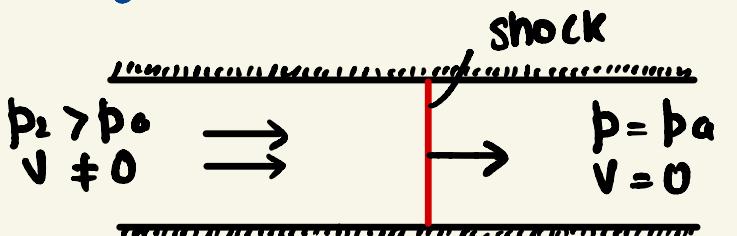
$$v_2 = \left(\frac{2a_1}{r-1}\right) \left\{ 1 - \left(\frac{p_a}{p_1}\right)^{\frac{r-1}{2r}} \right\}$$



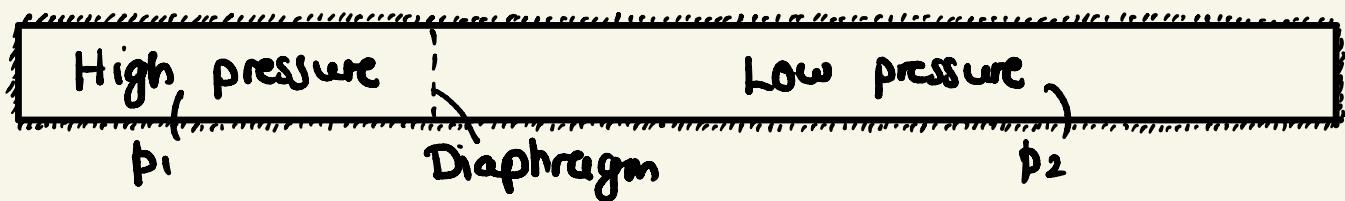
Maximum velocity that can be generated by an unsteady expansion wave propagating into a gas at rest

$$v_{2,\max} = \frac{2a_1}{r-1}$$

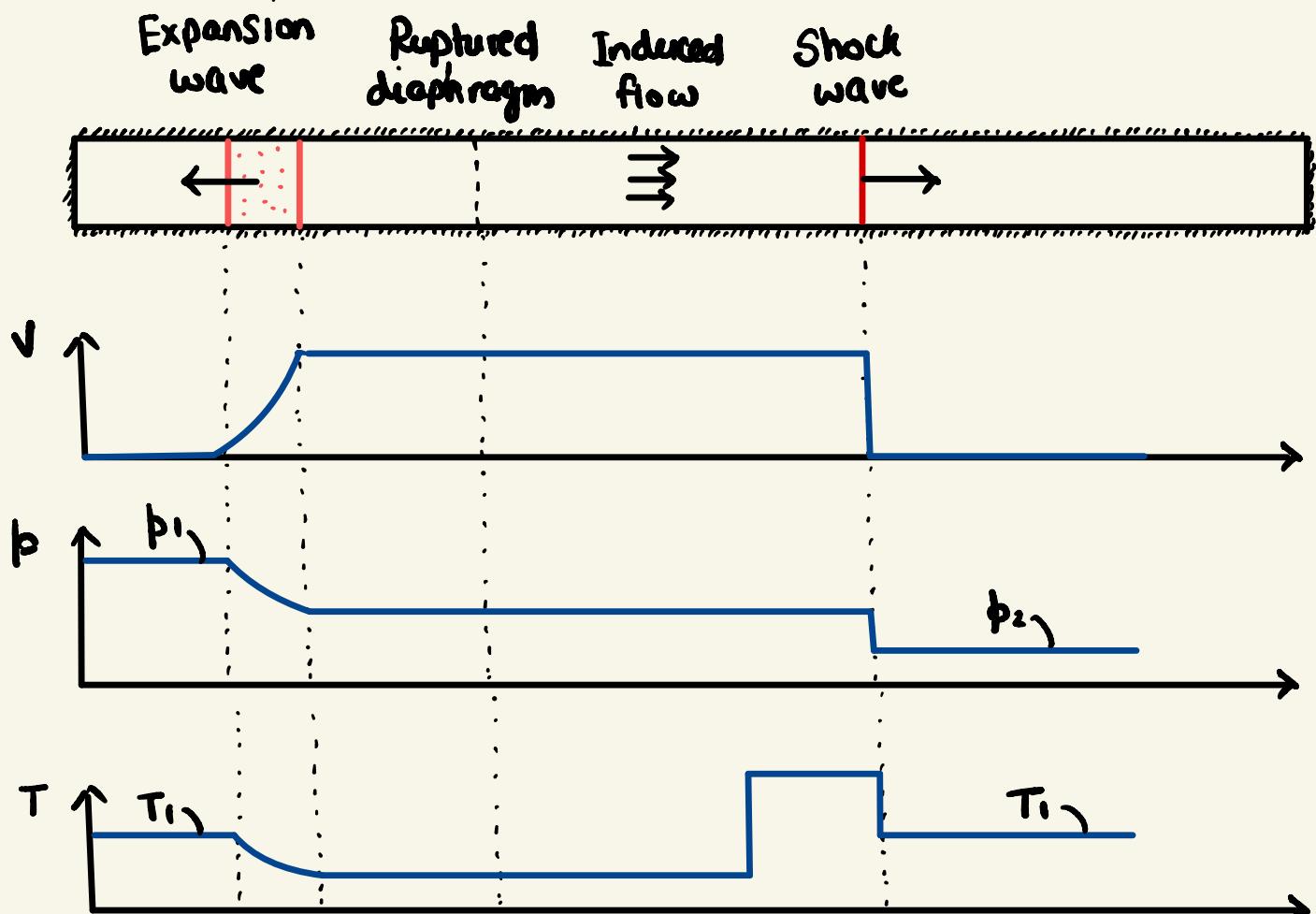
- ② Moving shock reaching the end of an open duct generates an expansion wave



③ Shock tube - A long tube of constant area divided into two sections: high pressure gas on one side and low pressure gas on the other



When diaphragm breaks, a shock propagates into low-pressure section and an expansion wave propagates into high-pressure section generating a region of uniform velocity between them



Induced velocity is determined by noting that the velocity and pressure behind the shock wave must be equal to the velocity and pressure behind the expansion wave

## Problems

- ⑥ Pipe 4m long,  $\rho = 200 \text{ kPa}$ ,  $T = 30^\circ\text{C}$  ruptures at end.  
Find velocity of air if  $\rho_a = 103 \text{ kPa}$ . Also find speed of  
tail of expansion wave and time taken for the head of  
the wave to reach pipe end.

$$V_p = \left( \frac{2\alpha_1}{\gamma-1} \right) \left[ 1 - \left( \frac{\rho_a}{\rho_1} \right)^{\frac{\gamma-1}{2\gamma}} \right]$$

$$\alpha_1 = \sqrt{\gamma R T} = \sqrt{1.4 \cdot 287 \cdot 303} = 348.9 \text{ m/s}$$

$$\Rightarrow V_p = 157.8 \text{ m/s}$$

$$\frac{\alpha_2}{\alpha_1} = 1 - \left( \frac{\gamma-1}{2} \right) \frac{V_2}{\alpha_1} = 0.91$$

$$\Rightarrow \alpha_2 = 317.5 \text{ m/s}$$

$$\text{Speed of the tail} = \alpha_2 - V_2 = 159.7 \text{ m/s}$$

Time taken for head to reach pipe end

$$t = \frac{4}{\alpha_1} = 11.5 \text{ ms}$$