MA 214 Tutorial-1 Solutions

Problem 1
Find the 4^{th} Taylor Polynomial $P_4(x)$ for the function $f(x) = xe^{x^2}$ at $x = 0$

Solution
We note that that function $f(x) = e^{x^2}$ is differentiable infinitely many times and hence by Taylor's formula the n^{th} Taylor polynomial is given by

$$P_n(x) = \Sigma_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k.$$

$$f(x) = xe^{x^2}, f(0) = 0$$
$$f'(x) = e^{x^2} + 2x^2e^{x^2}, f'(0) = 1$$
$$f''(x) = (4x^3 + 6x)e^{x^2}, f''(0) = 1$$
$$f^3(x) = (8x^4 + 24x^2 + 6)e^{x^2}, f^3(0) = 6$$
$$f^(4) = (16x^4 + 80x^3 + 60x)e^{x^2}, f^4(0) = 0$$

Using the Taylor formula with $a = 0$ we get that

$$P_4(x) = x + \frac{6}{3!}x^3 = x + x^3$$

Problem 2
Let $f(x) = (1-x)^{-1}$. Find the n-th Taylor polynomial $P_n(x)$ for $f(x)$ about $x = 0$.

Solution
Recall Taylor's Theorem from MA 105. The given function $f(x)$ is at least n times differentiable at $x = 0$. Thus, $\exists c \in [0, x]$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}$$

where

$$f^{(k)}(x) = \begin{cases} f(x) & k = 0 \\ \frac{d^k f(x)}{dx^k} & k \neq 0 \end{cases}$$

We can separate the polynomial term $P_n(x)$ and the remainder term $R_n(x)$ as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$
$$R_n(x) = \frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}$$

Now, consider our function $f(x) = (1-x)^{-1}$. Using standard differentiation rules,

$$
\begin{aligned}
f^{(0)}(x) &= (1-x)^{-1} \;\Rightarrow f^{(0)}(0) = 1 \\
f^{(1)}(x) &= (1-x)^{-2} \;\Rightarrow f^{(1)}(0) = 1 \\
f^{(2)}(x) &= 2(1-x)^{-3} \Rightarrow f^{(2)}(0) = 2 \\
f^{(3)}(x) &= 6(1-x)^{-4} \Rightarrow f^{(3)}(0) = 6
\end{aligned}
$$

and so on. We begin to see a pattern here. Let us assume

$$f^{(k)}(x) = k!(1-x)^{-(k+1)}$$

We use induction to prove this. Differentiation of the above equation yields

$$f^{(k+1)}(x) = k!(1-x)^{-(k+2)} \times (-(k+1)) \times (-1) = (k+1)!(1-x)^{-(k+2)}$$

which complies with the form

$$f^{(k)}(x) = k!(1-x)^{-(k+1)}$$

Using this,

$$f^{(k)}(0) = k!(1-0)^{-(k+1)} = k!$$

Substituting into the equation for $P_n(x)$, we get

$$
\begin{aligned}
P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\
&= \sum_{k=0}^{n} \frac{k!}{k!} x^k \\
&= \sum_{k=0}^{n} x^k
\end{aligned}
$$

Thus, $\boxed{P_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n}$

Problem 3 For $f(x)$ and $P_n(x)$ as in the above problem, find a value of n such that $P_n(x)$ approximates $f(x)$ to within 10^{-6} on $[0, 0.5]$.

Solution
From, previous solution we have that $P_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$. We have to find n such that $|f(x) - P_n(x)| < 10^{-6}\ \forall x \in [0, 0.5]$ ie

$$|\frac{1}{1-x} - (1 + x + x^2 + \cdots + x^n)| < 10^{-6}\ \forall x \in [0, 0.5]$$

$$|\frac{1}{1-x} - \frac{1-x^{n+1}}{1-x}| < 10^{-6}\ \forall x \in [0, 0.5]$$

$$\frac{x^{n+1}}{1-x}| < 10^{-6}\ \forall x \in [0, 0.5]$$

$\frac{x^{n+1}}{1-x}$ is an increasing function on the given interval so we have that $0.5^n < 10^{-6}$ which gives that $2^n > 10^6$ and thus $n > 6log_2(10) = 19.93$. So $n \geqslant 20$ satisfies the given condition.

Problem 4
If we use k digits and the chopping method to approximate a real number $y \neq 0$ then prove that the relative error is $\leqslant 10^{-k+1}$.

Solution
W.L.O.G Let $y = 0.d_1d_2d_3...d_kd_{k+1}... \times 10^n$ where $1 \leqslant d_1 \leqslant 9$ and $0 \leqslant d_i \leqslant 9$ for i=2,3,... and let the approximation be y^*. In chopping method if we use only k digits then in y all the digits from d_{k+1} will be chopped off and this is taken as y^*. So,

$$y^* = 0.d_1d_2d_3...d_k \times 10^n$$

Now , consider the absolute error $|y - y^*|$ which turns out to be

$$A.E = |y - y^*| = 0.000...0d_{k+1}d_{k+2}... \times 10^n$$

And,

$$R.E = |\frac{y - y^*}{y}|$$

$$R.E = |\frac{A.E}{y}| \Rightarrow R.E = \frac{0.000...0d_{k+1}d_{k+2}... \times 10^n}{0.d_1d_2d_3...d_kd_{k+1}... \times 10^n}$$

$$\Rightarrow R.E = \frac{0.d_{k+1}d_{k+2}... \times 10^{-k}}{0.d_1d_2d_3...d_kd_{k+1}...}$$

$$\Rightarrow R.E \leqslant \frac{0.d_{k+1}d_{k+2}... \times 10^{-k}}{0.1} \leqslant 10^{-k+1}$$

Thus, $\boxed{R.E \leqslant 10^{-k+1}}$

Problem 5
If we use k digits and the rounding method to approximate a real number $y \neq 0$ then prove that the relative error is $\leqslant 0.5 \times 10^{-k+1}$

Solution
Let $y = (0.d_1d_2...d_kd_{k+1}...) \times 10^n$ where $1 \leqslant d_1 \leqslant 9$ and let approximation be y^*

We know that y^* is obtained after adding $5 \times 10^{n-(k+1)}$ to y and chopping till k digits. So,

$$y^* \leqslant y + 5 \times 10^{n-(k+1)}$$

Also, when $d_{k+1} < 5$, we get $y^* = (0.d_1d_2...d_k) \times 10^n$. In this case, $y - 5 \times 10^{n-(k+1)} \leqslant y^*$.
When $d_{k+1} \geqslant 5$, as we will be adding 5 to d_{k+1}, y^* will have digit in k^{th} place 1 more than d_k or it will be carried, so $y \leqslant y^*$ which implies $y - 5 \times 10^{n-(k+1)} \leqslant y^*$. Therefore, in both cases.

$$y - 5 \times 10^{n-(k+1)} \leqslant y^*$$

Combining above inequalities, we get

$$y - 5 \times 10^{n-(k+1)} \leqslant y^* \leqslant y + 5 \times 10^{n-(k+1)}$$
$$- 5 \times 10^{n-(k+1)} \leqslant y^* - y \leqslant 5 \times 10^{n-(k+1)}$$

Hence, we get

$$|y^* - y| \leqslant 5 \times 10^{n-(k+1)}$$

We know that relative error $= \dfrac{|y^* - y|}{|y|}$. Therefore,

$$R.E \leqslant \frac{5 \times 10^{n-(k+1)}}{|y|}$$

$$R.E \leqslant \frac{5 \times 10^{n-(k+1)}}{(0.d_1 d_2 ... \times 10^n}$$

Since, $1 \leqslant d_1 \leqslant 9$,

$$R.E \leqslant \frac{5 \times 10^{n-(k+1)}}{0.1 \times 10^n}$$

$$R.E \leqslant \frac{5 \times 10^{-(k+1)}}{0.1}$$

$$R.E \leqslant 0.5 \times 10^{-k+1}$$

Problem 6
Suppose $x = \frac{5}{7}$ and $y = \frac{1}{3}$. Use five-digit chopping to compute $x \oplus y, x \ominus y$, $x \otimes y$ and $x \oplus y$. Compute the absolute and the relative errors in the above 4 operations.

Solution
To get floating point representation of $x = 0.d_1 d_2 \cdots d_k d_{k+1} \cdots \times 10^n$ we chop the part starting from d_{k+1} and get $x = 0.d_1 d_2 \cdots d_k \times 10^n$. So, $x = 0.71428 \times 10^0$ and $y = 0.33333 \times 10^0$.

a)$(x \oplus y)$
$(x \oplus y) = (0.71428 + 0.33333) \times 10^0 = 1.04761 = 0.10476 \times 10^1$. For a real number p with approximation p^* we have that
absolute error $= |p - p^*|$
relative error $= \frac{|p-p^*|}{p}$.
With $p = x + y = \frac{5}{7} + \frac{1}{3} = \frac{22}{21}$, we get
absolute error$= |\frac{22}{21} - 0.10476 \times 10^1| = 0.19047 \times 10^{-4}$.
Relative error$= \frac{|\frac{22}{21} - 0.10476 \times 10^1|}{\frac{22}{21}} = 0.18181 \times 10^{-4}$.

b)$(x \ominus y)$
$(x \ominus y) = (0.71828 - 0.33333) = 0.38095$. With $p = x - y = \frac{5}{7} - \frac{1}{3} = \frac{8}{21}$, we get
absolute error$= |\frac{8}{21} - 0.38095| = 0.23089 \times 10^{-5}$.
Relative error$= \frac{|\frac{8}{21} - 0.38095|}{\frac{8}{21}} = 0.62499 \times 10^{-5}$.

c)$(x \otimes y)$
$(x \otimes y) = 0.71428 \times 0.33333 = 0.23809$. With $p = x \times y = \frac{5}{7} \times \frac{1}{3} = \frac{5}{21}$, we get
absolute error$= |\frac{5}{21} - 0.23809| = 0.42856 \times 10^{-5}$.

Relative error$= \frac{|\frac{5}{21}-0.23809|}{\frac{5}{21}} = 0.17999 \times 10^{-4}$.

d)$(x \oplus y)$
$(x \oplus y) = (0.71428/0.33333) = 0.21428 \times 10^1$. With $p = x/y = \frac{15}{7}$, we get
absolute error$= |\frac{15}{7} - 0.21428 \times 10^1| = 0.57182 \times 10^{-4}$.
Relative error$= \frac{|\frac{15}{7}-0.21428|}{\frac{15}{7}} = 0.26666 \times 10^{-4}$.

Problem 7
Let $p = 0.546217$ and $q = 0.546201$. Use five-digit arithmetic to compute $p \ominus q$ and determine the absolute and the relative errors using the methods of chopping and rounding. Compute the number of significant digits in both these methods for the result.

Solution
$p = 0.546217$ and $q = 0.546201$ so $z = p - q = 1.6 \times 10^{-5}$

a)Chopping
$p^* = 0.54621$ and $q^* = 0.54620$ So $z^* = p \ominus q = p^* - q^* = 1 \times 10^{-5}$.
Absolute error$= |z - z^*| = |1.6 \times 10^{-5} - 10^{-5}| = 6 \times 10^{-6}$.
Relative error$= \frac{|z-z^*|}{z} = \frac{6\times10^{-6}}{1.6\times10^{-5}} = 0.375 < 0.5 = 5 \times 10^{-1}$. So, z^* approximates z to one significant digit.

b)Rounding
$p^* = 0.54622$ and $q^* = 0.54620$ So $z^* = p \ominus q = p^* - q^* = 2 \times 10^{-5}$.
Absolute error$= |z - z^*| = |1.6 \times 10^{-5} - 2 \times 10^{-5}| = 4 \times 10^{-6}$.
Relative error$= \frac{|z-z^*|}{z} = \frac{4\times10^{-6}}{1.6\times10^{-5}} = 0.25 < 0.5 = 5 \times 10^{-1}$. So, z^* approximates z to one significant digit.

Problem 8
Consider the quadratic equation $x^2 + 62.10x + 1 = 0$ whose roots are (approximately)$x = -0.01610723$ and $x = -62.08390$. Use the four-digit rounding arithmetic to compute the roots using the formula
$x_1 = \frac{-b+\sqrt{b62-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b62-4ac}}{2a}$. Compute the absolute and the relative errors.

Solution
From the equation $x^2 + 62.10x + 1 = 0$ we have that $a = 1, b = 62.10$ and $c = 1$. $b^2 = 3856.41 \simeq 3856$, so $b^-4ac = 3852$ and $\sqrt{b^2 - 4ac} = 62.06$ Using this and the given formula we get $x_1^* = \frac{-62.10+62.06}{2} = -0.02$ and $x_2^* = \frac{-62.10-62.06}{2} = -62.1$.

Absolute error$(x_1) = |x_1 - x_1^*| = 3.89 \times 10^{-3}$.
Relative error$(x_1) = \frac{|x_1-x_1^*|}{x_1} = 2.42 \times 10^{-1}$.

Absolute error$(x_2) = |x_2 - x_2^*| = 1.61 \times 10^{-2}$.
Relative error$(x_2) = \frac{|x_2-x_2^*|}{x_2} = 2.59 \times 10^{-4}$.

Problem 9
Evaluate $f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$ at $x = 4.71$ using three-digit arithmetic in both the chopping and the rounding methods. Compute the absolute and the relative errors.
Solution
$f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$
a)Chopping

First term: $3.2 * x = 3.2 * 4.71 = 15.072$ chopped to 15.0

Second term: $6.1 * (x * x) = 6.1 * (22.1) = 134$
Third Term: $x * x * x = 4.71 * 22.1 = 104$
$f(4.71) = (104 - 134) + (15 + 1.5) = (104 - 134) + 16.5 = -30.0 + 16.5 = -13.5$
Exact value : $f(4.71) = 104.487111 - 135.32301 + 15.072 + 1.5 = -14.2638999$
Absolute error: $|-13.5 - (-14.2638999)| = 0.7638999$
Relative error=$0.7638999/14.2638999 = 0.05355$.

b)Rounding off

We shall add $0.0005 * 10^n$
First Term: $3.2 * 4.71 = 15.072$ rounded to 15.1
Second term: $6.1 * (x * x) = 6.1 * (22.1841)$ rounded to $6.1 * 22.2 = 135.42$ rounded to 135
Third Term: $x * x * x = 4.71 * 22.2 = 104.562$ rounded to 105
$f(4.71) = 105 - 135 + 15.1 + 1.5 = -30.0 + 16.6 = -13.4$
Absolute error: $|-13.4 - (-14.2638999)| = 0.8638999$
Relative error=$0.8638999/14.2638999 = 0.06056$
Note: Order of addition or subtraction in the final computation of f will change the answer because of the rounding/chopping

MA214 TUTORIAL-2 SOLUTION

January 17, 2022

Problem 1): Use the Bisection method to find the approximations $p_1, p_2, \cdots, p_5$ for the root of $f(x) = \sqrt{x} - cos(x) = 0$ on $[0, 1]$.

solution): We have $f(0) = -1; f(1) = \sqrt{1} - cos(1) = .4597 \implies f(0)f(1) < 0$

$$p_1 = \frac{0+1}{2} = .5$$

Now $f(p_1) = \sqrt{.5} - cos(.5) = -.1705 \implies f(p_1)f(1) < 0$

$$p_2 = \frac{.5+1}{2} = .75$$

Now $f(p_2) = \sqrt{.75} - cos(.75) = .1343 \implies f(p_2)f(p_1) < 0$

$$p_3 = \frac{.75+.5}{2} = .625$$

Now $f(p_3) = \sqrt{.625} - cos(.625) = -.0204 \implies f(p_3)f(p_2) < 0$

$$p_4 = \frac{.625+.75}{2} = .6875$$

Now $f(p_4) = \sqrt{.6875} - cos(.6875) = .0563 \implies f(p_4)f(p_3) < 0$

$$p_5 = \frac{.6875+.625}{2} = .65625$$

Problem 2): Use the Bisection method to find solutions accurate to within 10^{-2} for the root of $f(x) = x^3 - 7x^2 + 14x - 6 = 0$ on $[1, 3.2]$.

solution): By trail and error method we can guess $x = 3$ is a root of f, Hence we can factorize $f(x)$ as:

$$f(x) = x^3 - 7x^2 + 14x - 6 = (x-3)(x^2 - 4x + 2)$$

And the root 3 lies in the interval $[1, 3.2]$.
Let, $a_0 = 1; b_0 = 3.2$ then,

$$f(a_0) = f(1) = 1^3 - 7 * 1^2 + 14 * 1 - 6 = 2$$

$$f(b_0) = f(3.2) = (3.2)^3 - 7 * (3.2)^2 + 14 * (3.2) - 6 = -.112$$

$$\implies \qquad f(a_0) * f(b_0) < 0$$

Therefore by intermediate value theorem there exists a $c \in (a_0, b_0)$ such that $f(c) = 0$. For accuracy within 10^{-2}, the root c should lie in the interval $[a_n, b_n]$ such that $[a_n, b_n] < 10^{-2}$. In the bisection method, the size of the range interval reduces by a factor of 2 in every iteration.

$$d_n = \frac{d_0}{2^n} = \frac{3.2 - 1}{2^n} = \frac{2.2}{2^n}$$

To get accuracy within 10^{-2}, we have the relation $d_n < 10^{-2}$.

$$\implies \qquad \frac{2.2}{2^n} < 10^{-2}$$

$$\implies \qquad 220 < 2^n$$

As $2^7 = 128$ and $2^8 = 256$ we clearly have $n > 7$ and n should be at-most 8, as n is an integer. So we need to perform our iteration at max 8th times, because at 8th iteration it has been guaranteed from our above discussion that we will reach the given accuracy level in finding the root.
So we will denote a_i, b_i be our interval end points such that $f(a_i)f(b_i) < 0$, $p_i = \frac{a_i+b_i}{2}$. So for $i = 1, 2, \cdots, 8$ we have the following table:

i	a_i	b_i	p_i	$f(a_i)$	$f(b_i)$	$f(p_i)$
0	1	3.2	2.1	2	-.112	1.791
1	2.1	3.2	2.65	.791	-.112	.552125
2	2.65	3.2	2.925	.552125	-.112	.08582
3	2.925	3.2	3.0625	.08582	-.112	-.05444
4	2.925	3.0625	2.99375	.08582	-.05444	.00632
5	2.99375	3.0625	3.028125	.00632	-.05444	-.02652
6	2.99375	3.028125	3.0109375	.00632	-.02652	-.01069
7	2.99375	3.0109375	3.00234375	.00632	-.01069	-.00233
8	2.99375	3.00234375	2.998046875	.00632	-.00233	.00196

So our required root is $c = p_8 = 2.998046875$

Problem 3): Let $f(x) = (x+2)(x+1)^2 x(x-1)^3(x-2)$. To which zero of f does the Bisection method converge when applied on the interval $[-1.5, 2.5]$?

solution): Notice $f(-1.5) = -10.254; f(2.5) = 232.5586 \implies f(-1.5) * f(2.5) < 0$, hence the interval $[-1.5, 2.5]$ contains at-least a root of $f(x)$. Let us denote $a_0 = -1.5; b_0 = 2.5$ and for $i = 1, 2, 3...$ we denote the iterated interval by $[a_i, b_i]$, with $f(a_i) * f(b_i) < 0$ and $p_i = \frac{a_i + b_i}{2}$. We have the following table:

i	a_i	b_i	p_i	$f(a_i)$	$f(b_i)$	$f(p_i)$
0	-1.5	2.5	.5	-10.254	232.5586	.52734
1	-1.5	.5	-.5	-10.254	.52734	-6.3281
2	-.5	.5	0	-6.3281	.52734	0

Check the second iteration $p_2 = 0$ we have $f(p_2) = 0$. Which means we have arrived at a root of $f(x)$. So our Bisection method converges to the root '0' of $f(x)$.

Problem 4): Show that the fixed point theorem does not ensure a unique fixed point of $f(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exists.

solution): Notice that $f(x)$ is continuous and differentiable over $[0, 1]$.

$$f'(x) = -3^{-x} \ln 3 < 0 \qquad \forall\, x \in [0, 1]$$

Which implies $f(x)$ is a strictly decreasing function for $x \in [0, 1]$. So the maximum value of f obtained at the point $x = 0$ and minimum value of f obtained at the point $x = 1$.

$$f(0) = 1,\ f(1) = \frac{1}{3}$$

Thus, the image of f is contained in $[\frac{1}{3}, 1] \subseteq [0, 1]$. Hence by fixed point theorem f has a unique fixed point in $[0, 1]$.

Now observe $|f'(\log_3 \ln 2.9)| = \frac{\ln 3}{\ln 2.9} > 1$, where $\log_3 \ln 2.9 \in (0, 1)$ as $e < 2.9 < e^3 \implies 1 < \ln 2.9 < 3 \implies 0 < \log_3 \ln 2.9 < 1$. Thus the second condition of fixed point theorem does not hold. So fixed point theorem does not ensure the fixed point is unique for $f(x)$ in the interval $[0, 1]$.

We now show that $f(x)$ indeed has a unique fixed point on the interval $[0, 1]$.
So let's assume the opposite, i.e there exists two different fixed points x_1, x_2 of $f(x)$ on the interval $[0, 1]$, i.e:

$$x_1 = f(x_1) \text{ and } x_2 = f(x_2) \qquad \text{where } x_1 \neq x_2$$

Without loss of generality, we take $x_1 < x_2$.
We already have $f(x)$ is monotonically strictly decreasing on $[0, 1]$, so using this;

$$f(x_1) > f(x_2)$$

But $x_1 = f(x_1)$ and $x_2 = f(x_2)$, which implies:

$$x_1 > x_2 \qquad \text{CONTRADICTION}$$

So $f(x)$ has a unique fixed point on the interval $[0,1]$.

Problem 5): The following two methods are proposed to compute $21^{\frac{1}{3}}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$:

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}; \qquad p_n = \Big(\frac{21}{p_{n-1}}\Big)^{\frac{1}{2}}$$

solution):

CASE 1: Let take first method: $p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$. we have $p_0 = 1,\ p_1 = \frac{23}{3}$
Now we have $21^2 = 441$ and $(\frac{23}{3})^3 = (7+\frac{2}{3})^3 = 7^3+3*7^2*\frac{2}{3}+3*7*\frac{4}{9}+\frac{8}{27} = 343+98+.. > 441$

$$\implies 21^2 < (\frac{23}{3})^3 \implies 21^{\frac{2}{3}} < \frac{23}{3} = p_1$$

$$p_n - 21^{\frac{1}{3}} = p_{n-1} - 21^{\frac{1}{3}} - \frac{p_{n-1}^3 - 21}{3p_{n-1}}$$

$\implies$

$$p_n - 21^{\frac{1}{3}} = p_{n-1} - 21^{\frac{1}{3}} - (p_{n-1} - 21^{\frac{1}{3}})\frac{p_{n-1}^2 + p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}}}{3p_{n-1}^2}$$

$\implies$

$$\frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}} = 1 - \frac{1}{3} - \Big(\frac{p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}}}{3p_{n-1}^2}\Big)$$

Now let's use induction, we already have $p_1 > 21^{\frac{2}{3}}$. So if $p_{n-1} > 21^{\frac{2}{3}}$ we have $p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}} > p_{n-1}p_{n-1} + p_{n-1}^2 \implies \frac{p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}}}{3p_{n-1}^2} > \frac{2}{3}$. $\implies$

$$\frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}} > 1 - \frac{1}{3} - \frac{2}{3} = 0$$

$\implies$

$$p_n > 21^{\frac{1}{3}}$$

So by induction we get $p_n > 21^{\frac{1}{3}}\ \ \forall\ n = 1, 2, ...$
Now we have:

$$0 < \frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}} < 1 - \frac{1}{3} - \Big(\frac{p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}}}{3p_{n-1}^2}\Big) \leq \frac{2}{3}$$

$\implies$

$$\Big|\frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}}\Big| \leq \frac{2}{3}$$

$\implies$

$$0 \leq |p_n - 21^{\frac{1}{3}}| \leq \frac{2}{3}|p_{n-1} - 21^{\frac{1}{3}}| \leq \cdots \leq \Big(\frac{2}{3}\Big)^{n-1}|p_1 - 21^{\frac{1}{3}}|$$

As $\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^{n-1}|p_1 - 21^{\frac{1}{3}}| = 0$ by sandwich theorem we get,

$$\lim_{n\to\infty}|p_n - 21^{\frac{1}{3}}| = 0$$

So p_n converges to $21^{\frac{1}{3}}$, again we have:

$$0 < \frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}} < 1 - \frac{1}{3} - \Big(\frac{p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}}}{3p_{n-1}^2}\Big)$$

$\implies$

$$\Big|\frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}}\Big| < \Big|1 - \frac{1}{3} - \Big(\frac{p_{n-1}21^{\frac{1}{3}} + 21^{\frac{2}{3}}}{3p_{n-1}^2}\Big)\Big|$$

Using $\lim\limits_{n\to\infty}|p_n - 21^{\frac{1}{3}}| = 0$ we get:

$$\lim_{n\to\infty}\Big|\frac{p_n - 21^{\frac{1}{3}}}{p_{n-1} - 21^{\frac{1}{3}}}\Big| = 0$$

So p_n converges to $21^{\frac{1}{3}}$ with order of convergence 1.

CASE 2: Let's take second method: $p_n = \left(\frac{21}{p_{n-1}}\right)^{\frac{1}{2}}$, $\quad p_0 = 1;\ p_1 = 21^{\frac{1}{2}}$

Claim: $p_n = 21^{\frac{2^n - 2^{n-1} + \cdots + (-1)^n 2^0}{2^{n+1}}}$

We will prove this via induction. Clearly it is true for $n = 1$. Let it be true for p_{n-1}. So we have the following:

$$p_n = \Big(\frac{21}{p_{n-1}}\Big)^{\frac{1}{2}} = \Bigg(\frac{21}{21^{\frac{2^{n-1} - 2^{n-2} + \cdots + (-1)^{n-1}2^0}{2^n}}}\Bigg)^{\frac{1}{2}}$$

$\implies$

$$p_n = \Big(21^{1 - \frac{2^{n-1} - 2^{n-2} + \cdots + (-1)^{n-1}2^0}{2^n}}\Big)^{\frac{1}{2}}$$

$\implies$

$$p_n = 21^{\frac{2^n - 2^{n-1} + \cdots + (-1)^n 2^0}{2^{n+1}}}$$

Now we compute the sum $2^n - 2^{n-1} + \cdots + (-1)^n 2^0$. It is a geometric series with common ratio (-2). It will take two different expression for odd or even n.

$$2^n - 2^{n-1} + \cdots + (-1)^n 2^0 = \begin{cases} \frac{1+(2)^{n+1}}{3} & \text{if } n \text{ even} \\ \frac{-1+(2)^{n+1}}{3} & \text{if } n \text{ odd} \end{cases}$$

Taking $n \to \infty$ in both the even and odd cases we get:

$$\lim_{n\to\infty} \frac{2^n - 2^{n-1} + \cdots + (-1)^n 2^0}{2^{n+1}} = \frac{1}{3}$$

$\implies$

$$\lim_{n\to\infty} 21^{\frac{2^n - 2^{n-1} + \cdots + (-1)^n 2^0}{2^{n+1}}} = 21^{\frac{1}{3}}$$

$$\text{Hence } \lim_{n\to\infty} p_n = 21^{\frac{1}{3}}$$

Our next claim: $p_n > 21^{\frac{1}{3}} \quad \forall\ n = 1, 2, 3...;$. Clearly we have $p_1 = 21^{\frac{1}{2}} > 21^{\frac{1}{3}}$. Let the relation holds true for p_{n-1}, $\implies$

$$p_n = \left(\frac{21}{p_{n-1}}\right)^{\frac{1}{2}} > \left(\frac{21}{21^{\frac{1}{3}}}\right)^{\frac{1}{2}} = 21^{\frac{1}{3}}$$

Therefore by induction: $p_n > 21^{\frac{1}{3}} \quad \forall\ n = 1, 2, 3...;$.
Now,

$$p_n - 21^{\frac{1}{3}} = p_{n-1} - 21^{\frac{1}{3}} - p_n + \left(\frac{21}{p_{n-1}}\right)^{\frac{1}{2}}$$

$\implies$

$$p_n - 21^{\frac{1}{3}} = p_{n-1} - 21^{\frac{1}{3}} - \frac{p_{n-1}^{\frac{3}{2}} - 21^{\frac{3}{6}}}{p_{n-1}^{\frac{1}{2}}}$$

$$0 \leq \frac{|p_n - 21^{\frac{1}{3}}|}{|p_{n-1} - 21^{\frac{1}{3}}|^q} \leq \frac{|p_{n-1} - 21^{\frac{1}{3}}|}{|p_{n-1} - 21^{\frac{1}{3}}|^q} + \frac{|p_{n-1}^{\frac{3}{2}} - 21^{\frac{3}{6}}|}{p_{n-1}^{\frac{1}{2}}\big(|p_{n-1} - 21^{\frac{1}{3}}|^q\big)}$$

$\implies$

$$0 \leq \frac{|p_n - 21^{\frac{1}{3}}|}{|p_{n-1} - 21^{\frac{1}{3}}|^q} \leq |p_{n-1} - 21^{\frac{1}{3}}|^{1-q} + \frac{|p_{n-1}^{\frac{1}{2}} - 21^{\frac{1}{6}}||p_{n-1} + p_{n-1}^{\frac{1}{2}} 21^{\frac{1}{6}} + 21^{\frac{1}{3}}|}{p_{n-1}^{\frac{1}{2}}\big(|p_{n-1} - 21^{\frac{1}{3}}|^q\big)}$$

$\implies$

$$0 \leq \frac{|p_n - 21^{\frac{1}{3}}|}{|p_{n-1} - 21^{\frac{1}{3}}|^q} \leq |p_{n-1} - 21^{\frac{1}{3}}|^{1-q} + \frac{|p_{n-1} - 21^{\frac{1}{3}}||p_{n-1} + p_{n-1}^{\frac{1}{2}} 21^{\frac{1}{6}} + 21^{\frac{1}{3}}|}{p_{n-1}^{\frac{1}{2}}\big(|p_{n-1}^{\frac{1}{2}} + 21^{\frac{1}{6}}|\big)\big(|p_{n-1} - 21^{\frac{1}{3}}|^q\big)}$$

$\Longrightarrow$

$$0 \leq \frac{|p_n - 21^{\frac{1}{3}}|}{|p_{n-1} - 21^{\frac{1}{3}}|^q} \leq |p_{n-1} - 21^{\frac{1}{3}}|^{1-q} + \frac{|p_{n-1} - 21^{\frac{1}{3}}|^{1-q}|p_{n-1} + p_{n-1}^{\frac{1}{2}}21^{\frac{1}{6}} + 21^{\frac{1}{3}}|}{p_{n-1}^{\frac{1}{2}}\left(|p_{n-1}^{\frac{1}{2}} + 21^{\frac{1}{6}}|\right)}$$

Using $\lim_{n\to\infty} p_n = 21^{\frac{1}{3}}$ we have:(for any $q < 1$)

$$\lim_{n\to\infty}\left(|p_{n-1} - 21^{\frac{1}{3}}|^{1-q} + \frac{|p_{n-1} - 21^{\frac{1}{3}}|^{1-q}|p_{n-1} + p_{n-1}^{\frac{1}{2}}21^{\frac{1}{6}} + 21^{\frac{1}{3}}|}{p_{n-1}^{\frac{1}{2}}\left(|p_{n-1}^{\frac{1}{2}} + 21^{\frac{1}{6}}|\right)}\right) = 0 + \frac{0 * 3 * 21^{\frac{1}{3}}}{21^{\frac{1}{6}} * 2 * 21^{\frac{1}{6}}} = 0$$

So by sandwich theorem we can conclude:

$$\lim_{n\to\infty} \frac{|p_n - 21^{\frac{1}{3}}|}{|p_{n-1} - 21^{\frac{1}{3}}|^q} = 0$$

So p_n converges to $21^{\frac{1}{3}}$ with order of convergence $q < 1$.

So the first method method will converge faster as it's order of convergence is higher, so rankwise it supersedes the second method.

Problem 6): Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_3.

solution): $f(x) = x^2 - 6, \qquad p_0 = 1$. Newton's method of iteration is (p_n be the nth iteration)

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{1-6}{2} = 3.5$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 3.5 - \frac{3.5^2 - 6}{2 * 3.5} = 2.6071$$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)} = 2.6071 - \frac{2.6071^2 - 6}{2 * 2.6071} = 2.4543$$

Problem 7): Let $f(x) = x^2 - 6$. Use the Secant method to find p_4, with $p_0 = 3$ and $p_1 = 2$.

solution): $f(x) = x^2 - 6, \qquad p_0 = 3;\ p_1 = 2$. Secant method of iteration is (p_n be the nth iteration)

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

We have $f(p_0) = 3; f(p_1) = -2$

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{-2(2-3)}{(-2-3)} = 2.4$$

We have $f(p_1) = -2; f(p_2) = -.24$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = 2.4 - \frac{-.24(2.4-2)}{(-.24+2)} = 2.4545$$

We have $f(p_2) = -.24; f(p_3) = .0248$

$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_2)}{f(p_3) - f(p_2)} = 2.4545 - \frac{.0248(2.4545-2.4)}{(.0248+.24)} = 2.4494$$

Problem 8): Let $f(x) = x^2 - 6$. Use the method of false position to find p_4, with $p_0 = 3$ and $p_1 = 2$.

solution): $f(x) = x^2 - 6, \qquad p_0 = 3;\ p_1 = 2$. method of false position of iteration is (p_n be the nth iteration) exactly the Secant method and only difference here is we will check the interval where at the end points given function has opposite sign. We have:

$$f(p_0) = f(3) = 3;\ f(p_1) = f(2) = -2 \implies f(p_0) * f(p_1) < 0$$

Now we use secant method in the interval $[p_1, p_0]$. So using first iteration in false position method we get,

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{-2(2-3)}{(-2-3)} = 2.4$$

$$f(p_2) = f(2.4) = -.24 \implies f(p_2) * f(p_1) > 0;\ f(p_2) * f(p_0) < 0$$

Now we use secant method in the interval $[p_2, p_0]$. So using first iteration in false position method we get,

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_0)}{f(p_2) - f(p_0)} = 2.4 - \frac{-2.4(2.4-3)}{(-.24-3)} = \frac{22}{9}$$

$$f(p_3) = f(2.444) = -\frac{2}{81} = -.0247 \implies f(p_3) * f(p_2) > 0;\ f(p_3) * f(p_0) < 0$$

Now we use secant method in the interval $[p_3, p_0]$. So using first iteration in false position method we get,

$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_0)}{f(p_3) - f(p_0)} = \frac{22}{9} - \frac{-\frac{2}{81}(\frac{22}{9} - 3)}{(-\frac{2}{81} - 3)} = \frac{120}{49} = 2.449$$

Problem 9): An object falling vertically through air is faces various resistance as well as gravitational force.Assume that an object with mass a is dropped from a height d_0 and that the height of the object after t seconds is

$$d_t = d_0 - \frac{ag}{k}t + \frac{a^2 g}{k^2}\left(1 - e^{-\frac{kt}{a}}\right)$$

Where $g = 9.80665\ m/s^2$ and $k = .24\ kg/m$ represents the coefficient of air resistance.If an object of mass $.25\ kg$ is dropped from a height of $300\ meters$ then find, to within $.01$ error, the time it takes this object to hit the ground.

solution): We need to determine t such that $d_t = 0$.

$$\text{Consider } f(t) = d_0 - \frac{ag}{k}t + \frac{a^2 g}{k^2}\left(1 - e^{-\frac{kt}{a}}\right)$$

We need to find t' such that $f(t') = 0$. The first derivative of $f(t)$ gives us $f'(t) = \frac{ag}{k}\left(e^{-\frac{kt}{a}} - 1\right) \implies$ for $t > 0$, $f'(t) < 0 \implies f(t)$ is strictly decreasing $\forall\ t \in [0, \infty)$ so if it has a solution it will be the only solution. Now $f(0) = d_0 > 0$ and $f\left(\frac{d_0 + 3\frac{a^2 g}{k^2}}{\frac{ag}{k}}\right) = \frac{a^2 g}{k^2}\left(e^{-\frac{kt}{a}} - 2\right) \le -\frac{a^2 g}{k^2} < 0$

$$\text{As } \quad e^{-\frac{kt}{a}} \le 1\ \ \forall\ t \ge 0$$

So by Intermediate value theorem $f(t)$ attains a root in the interval $\left[0, \frac{d_0 + 3\frac{a^2 g}{k^2}}{\frac{ag}{k}}\right]$, and the root is unique(as $f(t)$ is strictly decreasing). First we will solve $f(t) = 0$ ignoring the exponential term. It gives:

$$d_0 - \frac{ag}{k}t + \frac{a^2 g}{k^2} = 0 \implies 300 + \frac{.25^2 * 9.80665}{.24^2} = \frac{.25 * 9.80665}{.24}t \implies t = 30.4095$$

Consider the interval $[30.40, 30.41]$. We have, $f(30.40) = .096$; $f(30.41) = -.0052 \implies f(30.40)f(30.41) < 0$. So our required root will be in the interval $[30.40, 30.41]$.

Using Bisection method we can choose

$$p_1 = \frac{30.40 + 30.41}{2} = 30.405$$

Now our root(t') either lies in the interval $[30.40, 30.405]$, or in the interval $[30.405, 30.41]$. In either case as length of both the two intervals is $.005$ and $p_1 = 30.405$, we can conclude

$$|t' - p_1| \leq .005 < .01$$

So $p_1 = 30.405$ is our required root, within $.01$ error.

MA 214

Spring-2022

Tutorial 3

The accuracy in problem (1) - (4) is expected within 10^{-2}.

Problem 1: *Use the Newton-Raphson method with* $p_0 = -1.5$ *to solve*

$$cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0$$

Solution: We have

$$f(x) = cos(x + \sqrt{2}) + x(x/2 + \sqrt{2})$$

On differentiating the above function, we get

$$f'(x) = -sin(x + \sqrt{2}) + x + \sqrt{2}$$

Based on the Newton-Raphson method, we can write

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Calculations:

i	p_i	$f(p_i)$	$f'(p_i)$	$\lvert p_i - p_{i-1}\rvert$
0	-1.5	0.00000226	-0.00010518	-
1	-1.47855076	0.00000071	-0.00004438	0.02144924
2	-1.46246535	0.00000023	-0.00001872	0.01608541
3	-1.45040194	0.00000007	-0.00000790	0.01206342
4	-1.44135464	0.00000002	-0.00000333	0.00904729

It is clear that after the 4^{th} iteration $|p_n - p_{n-1}| < 10^{-2}$.
Hence, our solution to the above equation would be

$$p_4 = -1.44135464$$

We can also note that the exact solution of the above quation is $-\sqrt{2} = -1.41421356$.

Problem 2: *Use the Newton-Raphson method with $p_0 = -0.5$ to solve*

$$e^{6x} + 3(ln2)^2 e^{2x} - (ln8)e^{4x} - (ln2)^3 = 0$$

Solution: Note that we have $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$

$$\Rightarrow f'(x) = 6e^{6x} + 6(ln2)^2 e^{2x} - 4(ln8)e^{4x}$$

First note that

$$f(x) = (e^{2x} - ln2)^3$$

so that our only solution is

$$x = \frac{ln(ln2)}{2} \approx -0.1833$$

Now we start applying Newton-Raphson method with $p_0 = -0.5$ and $p = \frac{ln(ln2)}{2} \approx -0.1833$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

i	p_i	$f(p_i)$	$f'(p_i)$	p_{i+1}	$e_i = p_{i+1} - p$
0	-0.5	-0.03441303	0.23352789	-0.35263844	0.31674354
1	-0.35263844	-0.00790145	0.11757765	-0.28543642	0.16938198
2	-0.28543642	-0.00210282	0.05564489	-0.24764649	0.10217996
3	-0.24764649	-0.00058753	0.02564886	-0.22473983	0.06439003
4	-0.22473983	-0.00016808	0.01165791	-0.21032222	0.04148337
5	-0.21032222	-0.00004872	0.00525552	-0.20105165	0.02706576
6	-0.20105165	-0.00001424	0.00235736	-0.1950131	0.01779519
7	-0.1950131	-0.00000418	0.00105402	-0.19104778	0.01175664
8	-0.19104778	-0.00000123	0.00047031	-0.18843034	0.00779132
9	-0.18843034	-0.00000036	0.00020957	-0.18669676	0.00517388
10	-0.18669676	-0.00000011	0.0000933	-0.18554604	0.0034403

We reach our desired accuracy at the iteration $i = 8$ and hence we can stop there. Thus $x \approx -0.19$ is the solution by Newton-Raphson method.

Problem 3: *Use the modified Newton-Raphson method in problem (1) above.*

Solution: We have

$$f(x) = cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) \tag{01}$$

And $$f(-\sqrt{2}) - cos(0) = 0$$

On differentiating the equation (01), we get

$$f'(x) = -sin(x + \sqrt{2}) + x + \sqrt{2} \tag{02}$$

And $$f'(-\sqrt{2}) = 0$$

On differentiating the equation (02), we get

$$f''(x) = -cos(x + \sqrt{2}) + 1 \tag{03}$$

And $$f''(-\sqrt{2}) = 0$$

On differentiating the equation (03), we get

$$f'''(x) = sin(x + \sqrt{2}) \tag{04}$$

And $$f'''(-\sqrt{2}) = 0$$

On differentiating the equation (04), we get

$$f''''(x) = cos(x + \sqrt{2}) \tag{05}$$

And $f''''(-\sqrt{2}) = 1 \neq 0$ Hence $-\sqrt{2}$ is a zero of f. Using modified Newton-Raphson method, we get

$$g(x) = x - \frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)}$$

$$p_1 = p_0 - \frac{f(p_0)f'(p_0)}{f'(p_0)^2 - f(p_0)f''(p_0)}$$

Hence we get the following

i	p_i	$\|p_i - p_{i-1}\|$
0	-1.5	-
1	-1.4142346	0.0857654
2	-1.4142416	0.0000070

After 2^{nd} iteration $|p_i - p_{i-1}| < 10^{-2}$ and $p_2 = -1.414216$.

Problem 4: *Use the modified Newton-Raphson method in problem (2) above.*

Solution: We have

$$f(x) = e^{6x} + 3(ln2)^2e^{2x} - 3(ln2)e^{4x} - (ln2)^3$$

By observation, we get a root of $f(x)$ as

$$x = ln(ln2)^{1/2} = -0.18325646$$

Also $$f'(x) = 6(e^{6x} + (ln)^2e^{2x} - 2(ln2)e^{4x})$$
and $$f''(x) = 12(3e^{6x} + (ln2)^2e^{2x} - 4(ln2)e^{4x})$$
We have

$$p_0 = -0.5 \text{ and } p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{f'(P_n)^2 - f(p_n)f''(p_n)}$$

Hence we get the following

i	p_i	$\|p_i - p_{i-1}\|$
0	-0.5	-
1	-0.26536892	0.23463107
2	-0.18964449	0.07572442
3	-0.18329709	0.00634739

where p_3 would be the root.

Problem 5: *For* $p_0 = 0.5$ *and* $p_n = \dfrac{2 - e^{p_{n-1}} + p_{n-1}^2}{3}$, *generate first five terms of the sequence* $\{\hat{p}_n\}$ *using the Aitken's* Δ^2*-method.*

Solution: Given that $p_0 = 0.5$ and $p_n = \dfrac{2 - e^{p_{n-1}} + p_{n-1}^2}{3}$.
Using Aitken's method, we have

$$\hat{p}_n = p_n \frac{(p_{n+1} - p_n)^2}{p_{n+2} - 2p_{n+1} + p_n}$$

So we get the following

i	p_i	$\hat{p}_i$
0	0.5	0.25868
1	0.20043	0.25761
2	0.27275	0.25753
3	0.25361	0.25753
4	0.25855	0.25753
5	0.25727	0.25753
6	0.25760	0.25753

Clearly, the first five terms of the sequence $\{\hat{p}_n\}$ using Aitken's Δ^2-method are 0.25868, 0.25761, 0.25753, 0.25753, 0.25753. Here the sequence $\{\hat{p}_n\}$ using Aitken's method converged at $n = 2$ itself as compared to the given sequemce which converges at $n = 8$.

Problem 6: *Find appropriate polynomials of degree at most one and at most two interpolating* $f(x) = \cos x$ *on* $x_0 = 0, x_1 = 0.6, x_2 = 0.9$ *to approximate* $cos(0.45)$. *Find the absolute errors.*

Solution: We have $cos(0.95) = 0.900$ and

$$y_0 = f(x_0) = cos(0) = 1$$
$$y_1 = f(x_1) = cos(0.6) = 0.825$$
$$y_2 = f(x_2) = cos(0.9) = 0.622$$

The Lagrange interpolating polynomials of degree at most one will be as follows

$$P_1 = \frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1$$

Putting known values we get

$$P_1(x) = 1 - 0.292x \text{ and } P_1(0.45) = 0.869$$

$$\therefore \text{Absolute error} = |0.900 - 0.869| = 0.031$$

The Lagrange interpolating polynomials of degree at most two will be as follows

$$P_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

Putting known values we get

$$P_2(x) = -0428x^2 - 0.035x + 1 \text{ and } P_2(0.45) = 0.898$$

$$\therefore \text{Absolute error} = |0.900 - 0.898| = 0.002$$

Problem 7: *Repeat the above problem for* $f(x) = \sqrt{1 + x}$.

Solution: We have $f(x) = \sqrt{1 + x}$, $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$. Hence we get

$$y_0 = f(x_0) = 1$$
$$y_1 = f(x_1) = 1.264911$$
$$y_2 = f(x_2) = 1.378405$$

The Lagrange interpolating polynomial of degree at most one will be as follows

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x + 1 - x_0) y_1}$$

For $x \in [0, 0.6]$ we get $P_1(x) = 0.441518x + 1$ and $P_1(0.45) = 1.196863$.

$$\therefore \text{Absolute error} = |1.196863 - 1.204159| = 0.007296$$

The Lagrange interpolating polynomial of degree at most two will be as follows

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

Putting values of x_0, x_1, x_2 we get the value of $P_2(x)$ at $x = 0.45$ as $P_2(0.45) = 1.203424$.

$$\therefore \text{Absolute error} = |1.203424 - 1.204159| = 0.000735$$

Problem 8: *Use aprrropriate Lagrange polynomials of degrees one, two and three to find $f(8.4)$ with the following data:*

$$\begin{aligned} f(8.1) &= 16.94410 \\ f(8.3) &= 17.56492 \\ f(8.6) &= 18.50515 \\ f(8.7) &= 18.82091 \end{aligned}$$

Solution: Let

$$\begin{aligned} y_0 &= f(x_0) = f(8.1) = 16.94410 \\ y_1 &= f(x_1) = f(8.3) = 17.56492 \\ y_2 &= f(x_2) = f(8.6) = 18.50515 \\ y_3 &= f(x_3) = f(8.7) = 18.82091 \end{aligned}$$

The Lagrange interpolating polynomial of degree one will be as follows

$$P_1(x) = \frac{(x-x_2)}{(x_1-x_2)}y_1 + \frac{(x-x_1)}{(x_2-x_1)}y_2$$

Putting values of x_1, x_2 we get the value of $P_1(x)$ at $x = 8.4$ as

$$f(x) = P_1(8.4) = 17.87833$$

The Lagrange interpolating polynomials of degree two will be as follows

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

Putting values of x_0, x_1, x_2 we get the value of $P_2(x)$ at $x = 8.4$ as

$$f(x) = P_2(8.4) = 17.87713$$

And

$$Q_2(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}y_3$$

Putting values of x_1, x_2, x_3 we get the value of $Q_2(x)$ at $x = 8.4$ as

$$f(x) = Q_2(8.4) = 17.877155$$

The Lagrange interpolating polynomials of degree three will be as follows :

$$P_3(x) = y_0L_0(x) + y_1L_1(x) + y_2L_2(x)$$

where

$$L_i(x) = \frac{(x-x_0)\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_n)}$$

Putting values of x_0, x_1, x_2, x_3 we get the value of $P_3(x)$ at $x = 8.4$ as

$$f(x) = P_3(8.4) = 17.8771425$$

Problem 9: *Use appropriate Lagrange polynomials of degree one, two and three to find $f(0.25)$ with the following data:*

$$\begin{aligned} f(0.1) &= 0.29004986 \\ f(0.2) &= -0.56079734 \\ f(0.3) &= -0.81401972 \\ f(0.4) &= -1.0526302 \end{aligned}$$

Solution: Note that $0.1 < 0.2 < 0.25 < 0.3 < 0.4$

For degree 1, we use $f(0.2) = -0.56079734, f(0.3) = -0.81401972$

$$L_0(x) = (x - 0.3)/(0.2 - 0.3)$$
$$L_1(x) = (x - 0.2)/(0.3 - 0.2)$$
$$P_1(x) = f(0.2)L_0(x) + f(0.3)L_1(x)$$

So $$P_1(0.25) = -0.68740853$$

For degree two, using $0.1, 0.2$ and 0.3

$$\begin{aligned}
L_0(0.25) &= \frac{(0.25-0.2)(0.25-0.3)}{(0.1-0.2)(0.1-0.3)} = -0.125 \\
L_1(0.25) &= \frac{(0.25-0.1)(0.25-0.3)}{(0.2-0.3)(0.2-0.3)} = 0.75 \\
L_2(0.25) &= \frac{(0.25-0.1)(0.25-0.2)}{(0.3-0.1)(0.3-0.2)} = -0.375
\end{aligned}$$

So $$P_2(0.25) = -0.0790843775$$

For degree three, use $0.1, 0.2, 0.3$ and 0.4

$$\begin{aligned}
L_0(0.25) &= \frac{(0.25-0.2)(0.25-0.3)(0.25-0.4)}{(0.1-0.2)(0.1-0.3)(0.1-0.4)} = -0.5625 \\
L_1(0.25) &= \frac{(0.25-0.1)(0.25-0.3)(0.25-0.4)}{(0.2-0.3)(0.2-0.3)(0.2-0.4)} = 0.5625 \\
L_2(0.25) &= \frac{(0.25-0.1)(0.25-0.2)(0.25-0.4)}{(0.3-0.1)(0.3-0.2)(0.3-0.4)} = 0.5625 \\
L_3(0.25) &= \frac{(0.25-0.1)(0.25-0.2)(0.25-0.3)}{(0.4-0.1)(0.4-0.2)(0.4-0.3)} = -0.0625
\end{aligned}$$

So $$P_3(0.25) = -0.5443921625$$

MA214 TUTORIAL-4 SOLUTION

February 7, 2022

Problem 1): Let $f : [0,1] \to \mathbf{R}$ be continously differentiable and define

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

prove that $\lim_{n\to\infty} B_n(x) = f(x)$ for each $x \in [0,1]$.

solution):

We will prove this results using four intermediary results. Let define

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

RESULT 1: $\sum_{k=0}^{n} b_{n,k} = 1 \qquad \forall x \in [0,1]$

By binomial theorem,

$$((1-x)+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \qquad \forall x \in [0,1]$$

$$\implies \qquad \sum_{k=0}^{n} b_{n,k} = 1^n = 1$$

end of the proof of result 1.

RESULT 2: $\sum_{k=0}^{n} \left(\frac{k}{n}\right) b_{n,k}(x) = x \qquad \forall\ x \in [0,1]$

By binomial theorem,

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

Differentiating both sides with respect to p:

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = n(p+q)^{n-1}$$

$\implies$

$$\sum_{k=0}^{n}\binom{n}{k}kp^k q^{n-k} = n(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n}\Big(\frac{k}{n}\Big)b_{n,k}(x) = x \qquad \forall\ x \in [0,1]$$

end of the proof of result 2.

RESULT 3: $\sum_{k=0}^{n}\Big(\frac{k^2}{n^2}\Big)b_{n,k}(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x \qquad \forall\ x \in [0,1]$

By binomial theorem,

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$$

Differentiating both sides with respect to p:

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = n(p+q)^{n-1}$$

$\implies$

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k}{n}\Big)p^k q^{n-k} = (p+q)^{n-1}p$$

Again differentiating both sides with respect to p, and multiplied by p we have:

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k^2}{n^2}\Big)p^k q^{n-k} = \frac{n-1}{n}(p+q)^{n-2}p^2 + \frac{1}{n}(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n}\binom{n}{k}\Big(\frac{k^2}{n^2}\Big)p^k q^{n-k} = \frac{n-1}{n}x^2 + \frac{1}{n}x$$

end of the proof of result 3.

RESULT 4: $\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_{n,k}(x)=\frac{x(1-x)}{n} \qquad \forall\ x\in[0,1]$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_{n,k}(x)=\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)b_{n,k}(x)-2x\ \sum_{k=0}^{n}\left(\frac{k}{n}\right)b_{n,k}(x)+x^2\ \sum_{k=0}^{n}b_{n,k}$$

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 b_{n,k}(x)=\frac{n-1}{n}x^2+\frac{1}{n}x-2x^2+x^2=\frac{x(1-x)}{n} \qquad \forall\ x\in[0,1]$$

end of the proof of result 4

.

Finally we use the fact that f is continuous on $[0,1]$. i.e,

$$\exists\ M>0 \text{ such that } |f(x)|\leq M \qquad \forall x\in[0,1]$$

For a given point x and $\epsilon>0$,

$$\exists\ \delta>0 \text{ such that } |f(x)-f(y)|\leq\epsilon \qquad |x-y|\leq\delta$$

$$\begin{aligned}
|B_n(x)-f(x)| &= \\
&\leq \Big|\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}-f(x)\Big| \\
&\leq \Big|\sum_{k=0}^{n}\binom{n}{k}\left(f\left(\frac{k}{n}\right)-f(x)\right)x^k(1-x)^{n-k}\Big| \\
&\text{(Using Result 1)} \\
&\leq \sum_{k=0}^{n}\binom{n}{k}\Big|\left(f\left(\frac{k}{n}\right)-f(x)\right)\Big|x^k(1-x)^{n-k}
\end{aligned}$$

$$\leq \sum_{\substack{k=0\\ |\frac{k}{n}-x|\leq\delta}}^{n}\binom{n}{k}\Big|\left(f\left(\frac{k}{n}\right)-f(x)\right)\Big|x^k(1-x)^{n-k}+\sum_{\substack{k=0\\ |\frac{k}{n}-x|>\delta}}^{n}\binom{n}{k}\Big|\left(f\left(\frac{k}{n}\right)-f(x)\right)\Big|x^k(1-x)^{n-k}$$

$$\leq \epsilon + \frac{2M}{\delta^2} \sum_{\substack{k=0 \\ |\frac{k}{n}-x|>\delta}}^{n} \binom{n}{k} \Big(\Big(\frac{k}{n}\Big) - x\Big)^2 x^k (1-x)^{n-k}$$

$$\leq \epsilon + \frac{2M}{\delta^2} \frac{x(1-x)}{n} \leq \epsilon + \frac{2M}{n\delta^2}$$

Taking $n \geq \left[\frac{2M}{\epsilon\delta^2}\right] + 1$, we have:

$$|B_n(x) - f(x)| \leq 2\epsilon$$

$\Longrightarrow$

$$\lim_{n\to\infty} B_n(x) = f(x) \text{ for each } x \in [0,1]$$

Problem 2): If $f(x) = x^2$ then show that $B_n(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x$.

solution):

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{k}{n}\Big) x^k (1-x)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \Big(\frac{k^2}{n^2}\Big) x^k (1-x)^{n-k}$$

Again we divide this into three intermediary results. Let define

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

RESULT 1: $\sum_{k=0}^{n} b_{n,k} = 1 \qquad \forall x \in [0,1]$

By binomial theorem,

$$((1-x)+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \qquad \forall x \in [0,1]$$

$$\Longrightarrow \qquad \sum_{k=0}^{n} b_{n,k} = 1^n = 1$$

end of the proof of result 1.

RESULT 2: $\sum_{k=0}^{n} \Big(\frac{k}{n}\Big) b_{n,k}(x) = x \qquad \forall\ x \in [0,1]$

By binomial theorem,

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

Differentiating both sides with respect to p:

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = n(p+q)^{n-1}$$

$\Longrightarrow$

$$\sum_{k=0}^{n}\binom{n}{k}kp^k q^{n-k} = n(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n}\left(\frac{k}{n}\right)b_{n,k}(x) = x \qquad \forall\ x \in [0,1]$$

end of the proof of result 2.

RESULT 3: $\sum_{k=0}^{n}\left(\frac{k^2}{n^2}\right)b_{n,k}(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x \qquad \forall\ x \in [0,1]$

By binomial theorem,

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}$$

Differentiating both sides with respect to p:

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = n(p+q)^{n-1}$$

$\Longrightarrow$

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}\right)p^k q^{n-k} = (p+q)^{n-1}p$$

Again differentiating both sides with respect to p, and multiplied by p we have:

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k^2}{n^2}\right)p^k q^{n-k} = \frac{n-1}{n}(p+q)^{n-2}p^2 + \frac{1}{n}(p+q)^{n-1}p$$

Taking $p = x; q = 1-x$ we have,

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k^2}{n^2}\right)p^k q^{n-k} = \frac{n-1}{n}x^2 + \frac{1}{n}x$$

end of the proof of result 3.
As, $B_n(x) = \sum_{k=0}^{n} \left(\frac{k^2}{n^2}\right) b_{n,k}(x)$; we have: $B_n(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x$.

Problem 3): Use the above $B_n(x)$ to determine n such that $|B_n(x) - x^2| < 10^{-2}$ for all $x \in [0,1]$.

solution):

$$|B_n(x) - x^2| \leq \left|\frac{n-1}{n}x^2 + \frac{1}{n}x - x^2\right| \leq \frac{1}{n}\left|x - x^2\right|$$

Now $x - x^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$, is maximize at $x = 2$, and the maximum value is $\frac{1}{4}$. Using this,

$$|B_n(x) - x^2| \leq \frac{1}{4n}$$

So, it is enough to solve for the n, for which, $\frac{1}{4n} \leq 10^{-2}$, i.e $n > 25 \implies n = 26$.

Problem 4): Use Neville's method to approximate $\sqrt{3}$ with $f(x) = 3^x$ and $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1$ and $x_4 = 2$. Find the absolute and relative errors.

solution): The nth order polynomial in Neville's method:

$$P_{0,1,2,\cdots,n}(x) = \frac{(x - x_0)P_{1,2,\cdots,n} - (x - x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

We have the following table for $x = \frac{1}{2}$:

i	x_i	$P_i(x)$	$P_{i-1,i}(x)$	$P_{i-2,i-1,i}(x)$	$P_{i-3,i-2,i-1,i}(x)$	$P_{i-4,i-3,i-2,i-1,i}(x)$
0	-2	$\frac{1}{9}$				
1	-1	$\frac{1}{3}$	$\frac{2}{3}$			
2	0	1	$\frac{4}{3}$	$\frac{3}{2}$		
3	1	3	2	$\frac{11}{6}$	$\frac{16}{9}$	
4	2	9	0	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{41}{24}$

$$\text{Absolute error} = \left|\sqrt{3} - \frac{41}{24}\right| = 0.023717$$

$$\text{Relative error} = \frac{\left|\sqrt{3} - \frac{41}{24}\right|}{\sqrt{3}} = 0.013693$$

Problem 5): Use Neville's method to approximate $\sqrt{3}$ with $f(x) = \sqrt{3}$ and $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4$ and $x_4 = 5$. Find the absolute and relative errors.

solution): The nth order polynomial in Neville's method:

$$P_{0,1,2,\cdots,n}(x) = \frac{(x-x_0)P_{1,2,\cdots,n} - (x-x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

We have the following table for $x = 3$:

i	x_i	$P_i(x)$	$P_{i-1,i}(x)$	$P_{i-2,i-1,i}(x)$	$P_{i-3,i-2,i-1,i}(x)$	$P_{i-4,i-3,i-2,i-1,i}(x)$
0	0	0				
1	1	1	3			
2	2	1.4142	1.4244	1.2426		
3	4	2	1.7071	1.7475	1.6213	
4	5	2.2341	1.7439	1.7260	1.7368	1.6906

$$\text{Absolute error} = \left|\sqrt{3} - 1.6906\right| = |1.73205 - 1.6906 = 0.04145$$

$$\text{Relative error} = \frac{\left|\sqrt{3} - \frac{41}{24}\right|}{\sqrt{3}} = \frac{|1.73205 - 1.6906|}{1.73205} = 0.02393$$

One thing to note: the approximation is better if we choose $f(x) = 3^x$ instead of $f(x) = \sqrt{3}$, as in problem 4; as both the absolute and relative error is smaller in comparison to problem 5.

Problem 6): If $P_3(x)$ is the interpolating polynomial for the following data then we use Neville's method to find y if $P_3(1.5) = 0$.

x	0	.5	1	2
$f(x)$	0	y	3	2

solution): Here we define $P_{0,1,2,3}$ as the Neville's P_3 polynomial. So we have: $P_{0,1,2,3}(1.5) = 0$.

$$P_0 = f(0) = 0; P_1 = f(.5) = y; P_2 = f(1) = 3; P_3 = f(2) = .2$$

The nth order polynomial in Neville's method:

$$P_{0,1,2,\cdots,n}(x) = \frac{(x-x_0)P_{1,2,\cdots,n} - (x-x_n)P_{0,1,\cdots,n-1}}{x_n - x_0}$$

Using this:

$$P_{0,1}(1.5) = \frac{(1.5-0)y - (1.5-.5)\times 0}{.5} = 3y$$

$$P_{1,2}(1.5) = \frac{(1.5-0.5)\times 3 - (1.5-1)\times y}{.5} = 6-y$$

$$P_{2,3}(1.5) = \frac{(1.5-1)\times 2 - (1.5-2)\times 3}{1} = 2.5$$

$$P_{0,1,2}(1.5) = \frac{(1.5-0)\times(6-y) - (1.5-1)\times 3y}{1} = 9-3y$$

$$P_{1,2,3}(1.5) = \frac{(1.5-.5)\times 2.5 - (1.5-2)\times(6-y)}{1.5} = \frac{11-y}{3}$$

$$P_{0,1,2,3}(1.5) = \frac{(1.5-0)\times\frac{(11-y)}{3} - (1.5-2)\times(9-3y)}{2} = 5-y = 0$$

$$\Longrightarrow \qquad y = 5$$

Problem 7): Use the forward difference formula to construct interpolating polynomials of degree one,two, and three for the following data and approximate $f(-\frac{1}{3})$.

x	-0.75	-0.5	-0.25	0
$f(x)$	-0.07181250	-0.02475000	-0.33493750	1.10100000

solution): Note that $-.75 < -.5 < -\frac{1}{3} < -.25 < 0.$
From the given data data we have the given divided difference table, where divided difference formula is give by ,

$$f[x_i] = f(x_i); \text{ and } f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}:$$

x	$f(x)$			
$x_0 = -.75$	$-.0718125$			
		$.18825$		
$x_1 = -.5$	$-.02475$		2.501	
		1.43875		1
$x_2 = -.25$	$.3349375$		3.251	
		3.06425		
$x_3 = 0$	1.101			

For polynomial of degree 1, we use the subtable corresponding to $[x_1, x_2]$.

$$P_1(x) = -.02475 + 1.43875(x + .5)$$

$$P_1(-\frac{1}{3}) = -.02475 + 1.43875(-\frac{1}{3} + .5) = .215041$$

For polynomial of degree 2 we use subtable corresponding to $[x_0, x_2]$ and denoted it by $P_2^1(x)$, and the subtable corresponding to $[x_1, x_2]$, denoting it by $P_2^2(x)$.

$$P_2^1(x) = -.0718125 + .18825(x + .75) + 2.501(x + .75)(x + .5)$$

$$P_2^1(-\frac{1}{3}) = -.0718125 + .18825(-\frac{1}{3} + .75) + 2.501(-\frac{1}{3} + .75)(-\frac{1}{3} + .5) = .180306$$

Similarly:

$$P_2^2(x) = -.02475 + 1.43875(x + .5) + 3.251(x + .5)(x + .25)$$

$$P_2^2(-\frac{1}{3}) = -.02475 + 1.43875(-\frac{1}{3} + .5) + 3.251(-\frac{1}{3} + .5)(-\frac{1}{3} + .25) = .169895$$

For polynomial of degree 3, we will use the entire table corresponding to the interval $[x_0, x_3]$.

$$P_3(x) = P_2^1(x) + (x + .75)(x + .5)(x + .25)$$

$$P_3(-\frac{1}{3}) = P_2^1(-\frac{1}{3}) + (-\frac{1}{3} + .75)(-\frac{1}{3} + .5)(-\frac{1}{3} + .25) = .174519$$

Problem 8): Use the backward difference formula to construct interpolating polynomials of degree one,two, and three for the following data and approximate $f(0.25)$.

x	0.1	0.2	0.3	0.4
$f(x)$	-0.62049958	-0.28398668	0.00660095	0.24842440

solution): Note that $.1 < .2 < .25 < .3 < .4.$

From the given data data we have the given divided difference table, where divided difference formula is give by ,

$$f[x_i] = f(x_i);\ \text{and}\ f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}:$$

x	$f(x)$			
$x_0 = .1$	$-.62049958$			
		3.651286		
$x_1 = .2$	$-.28398668$		-2.2962615	
		2.9058763		$-.4731583$
$x_2 = .3$	.00660095		-2.438209	
		2.4182345		
$x_3 = .4$	.24842440			

For polynomial of degree 1, we use the subtable corresponding to $[x_1, x_2]$.

$$P_1(x) = .006095 + 2.9058763(x - .3)$$

$$P_1(.25) = .006095 + 2.9058763(.25 - .3) = -.138692365$$

For polynomial of degree 2 we use subtable corresponding to $[x_1, x_3]$ and denoted it by $P_2^1(x)$, and the subtable corresponding to $[x_0, x_2]$, denoting it by $P_2^2(x)$.

$$P_2^1(x) = .24842440 + 2.4182345(x - .4) - 2.438209(x - .4)(x - .3)$$

$$P_2^1(.25) = .24842440 + 2.4182345(.25 - .4) - 2.438209(.25 - .4)(.25 - .3) = -.13279734$$

$$P_2^2(x) = .00660095 + 2.9058763(x - .3) - 2.2962615(x - .3)(x - .2)$$

$$P_2^2(.25) = .00660095 + 2.9058763(.25 - .3) - 2.2962615(.25 - .3)(.25 - .2) = -.132952$$

For polynomial of degree 3, we will use the entire table corresponding to the interval $[x_0, x_3]$.

$$P_3(x) = P_2^1(x) - .4731583(x - .4)(x - .3)(x - .2)$$

$$P_3(.25) = P_2^1(.25) - .4731583(.25 - .4)(.25 - .3)(.25 - .2) = -.13297478$$

Problem 9): A fourth degree polynomial $P(x)$ satisfies $\Delta^4 P(0) = 24, \Delta^3 P(0) = 6,$ and $\Delta^2 P(0) = 0,$ where $\Delta P(x) = P(x+1) - P(x)$. Compute $\Delta^2 P(10)$.

solution): We have the following divided difference table:

x	$f(x)$				B
$x_0 = 0$	$P(0)$				
		$\Delta P(0)$			
$x_1 = 1$	$P(1)$		$\Delta^2 \frac{P(0)}{2} = 0$		
		$\Delta P(1)$		$\Delta^3 \frac{P(0)}{6} = 1$	
$x_2 = 2$	$P(2)$		$\Delta^2 \frac{P(1)}{2}$		$\Delta^4 \frac{P(0)}{24} = 1$
		$\Delta P(2)$		$\Delta^3 \frac{P(1)}{6}$	
$x_3 = 3$	$P(3)$		$\Delta^2 \frac{P(2)}{2}$		$\Delta^4 \frac{P(1)}{24}$
		$\Delta P(3)$		$\Delta^2 \frac{P(2)}{2}$	
$x_4 = 4$	$P(4)$				

So our required polynomial will be:

$$P_5(x) = P(0) + \Delta P(0)(x - 0) + \Delta^2 \frac{P(0)}{2}(x - 0)(x - 1) + \Delta^3 \frac{P(0)}{6}(x - 0)(x - 1)(x - 2) + \Delta^4 \frac{P(1)}{24}(x - 0)(x - 1)(x - 2)(x - 3)$$

$\Longrightarrow$

$$P_5(x) = P(0) + \Delta P(0)x + x(x-1)(x-2) + x(x-1)(x-2)(x-3)$$

$$\Delta^2 P(10) = \Delta P(11) - \Delta P(10) = P(12) - 2P(11) - P(10)$$

$$= 12\times11\times10+12\times11\times10\times9+10\times9\times8+10\times9\times8\times7-2\times11\times10\times9-2\times11\times10\times9\times8$$

$$= 1140$$

MA214 Tutorial-5 Solutions

february 12,2022

Problem 1

Let $L_0, L_1, L_2 and L_3$ be the Lagrange polynomial for distinct nodes x_0, x_1, x_2 and x_3. Find all $j \geq 0$ such that

$$x_0^j L_0(x) + x_1^j L_1(x) + x_2^j L_2(x) + x_3^j L_3(x) = x^j$$

Solution

If $P(x)$ is the interpolating polynomial of the function $f(x)$, we have:

$$P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

For the given equation,

$$L.H.S = \sum_{i=0}^{3} x_i^j L_i(x)$$

By comparing the above two equations, we can see that the $L.H.S$ of the given equation is the interpolating polynomial of $f(x) = x^j$.

We can also note that,

$$R.H.S = x^j = f(x)$$

So we need all the values of $j \geq 0$ such that $P(x) = f(x)$.

It is obvious that we can have such a result only when we have $j \leq 3$ because we are interpolating a polynomial function, $f(x) = x^j$, using only 4 distinct interpolation points.

Let us prove the result using the error formula

$$f(x) = P(x) + \frac{f^{(n+1)}(x)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

$$\text{Error term} = \frac{f^{(n+1)}(x)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Here we have $f(x) = x^j$ and $n = 3$

$$f^{(n+1)}(x) = f^{(4)}(x) = \begin{cases} 0 & j \leq 3 \\ j(j-1)(j-2)(j-3)x^{j-4} & \text{otherwise} \end{cases}$$

For $j \leq 3$, Error term $= 0 \implies P(x) = f(x)$

For $j > 3$, there exists some value of x such that Error term $\neq 0$.

Hence the required values of j are $0, 1, 2$ and 3.

Problem 2

Let $x_0, x_1, \cdots, x_k$ be distinct nodes and define $g(x) := [x_0, x_1, \cdots, x_k, x]$. Prove that $g[y_0, y_1, \cdots, y_n] = f[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n]$.

Solution

First, we note that the divided difference is symmetric in the nodes.

Lemma 1. *The divided difference is a symmetric function of its arguments, that is, if $z_0, z_1, \cdots, z_k$ is a permutation of $x_0, x_1, \cdots, x_k$, then*

$$f[x_0, x_1, \cdots, x_k] = f[z_0, z_1, \cdots, z_k] \tag{1}$$

Proof. $z_0, z_1, \cdots, z_k$ is a permutation of $x_0, x_1, \cdots, x_k$, which means that the nodes $x_0, x_1, \cdots, x_k$ have only been re-labelled as $z_0, z_1, \cdots, z_k$, and hence the polynomial interpolating the function f at both these sets of nodes is the same. By definition, $f[x_0, x_1, \cdots, x_k]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $x_0, x_1, \cdots, x_k$, and $f[z_0, z_1, \cdots, z_k]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $z_0, z_1, \cdots, z_k$. Since both the interpolating polynomials are equal, so are the coefficients of x^n in them. This completes the proof. □

We also note that the formula for divided difference.

Lemma 2. *The divided difference satisfies the recurrence relation*

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0} \tag{2}$$

Proof. Refer to the lectures. □

Using lemmas 1 and 2, we show that

$$
\begin{aligned}
g\left[y_i, y_{i+1}\right] &= \frac{g\left[y_{i+1}\right]-g\left[y_i\right]}{y_{i+1}-yi} && (3)\\
&= \frac{g\left(y_{i+1}\right)-g\left(y_i\right)}{y_{i+1}-yi} && (4)\\
&= \frac{f\left[x_0, x_1, \cdots, x_k, y_{i+1}\right]-f\left[x_0, x_1, \cdots, x_k, y_i\right]}{y_{i+1}-yi} && (5)\\
&= \frac{f\left[x_0, x_1, \cdots, x_k, y_{i+1}\right]-f\left[y_i, x_0, x_1, \cdots, x_k\right]}{y_{i+1}-yi} \quad \text{(Using lemma 1)} && (6)\\
&= f\left[x_0, x_1, \cdots, x_k, y_i, y_{i+1}\right] \quad \text{(Using lemma 2)} && (7)\\
& && (8)
\end{aligned}
$$

Likewise, we can show that

$$
g\left[y_i, y_{i+1}, y_{i+2}\right]=f\left[x_0, x_1, \cdots, x_k, y_i, y_{i+1}, y_{i+2}\right] \tag{9}
$$

We use induction to prove that

$$
g\left[y_0, y_1, \cdots, y_n\right]=f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n\right] \tag{10}
$$

Proof. Let us assume that $g\left[y_0, y_1, \cdots, y_n\right]=f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n\right]$. Then, using lemmas 1 and 2,

$$
\begin{aligned}
g\left[y_0, y_1, \cdots, y_n, y_{n+1}\right] &= \frac{g\left[y_0, y_1, \cdots, y_{n-1}, y_{n+1}\right]-g\left[y_0, y_1, \cdots, y_{n-1}, y_n,\right]}{y_{n+1}-y_n} && (11)\\
&= \frac{f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_{n-1}, y_{n+1}\right]-f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_{n-1}, y_n\right]}{y_{n+1}-y_n} && (12)\\
&= f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n, y_{n+1}\right] && (13)
\end{aligned}
$$

This completes the proof. □

If for some i, j, $y_i=y_j$, we take y_j as a distinct node and perform $\lim y_i-y_j \rightarrow 0$ in the last step.

Problem 3

If $f(x)=g(x)h(x)$ then find a formula for the divided difference for f in terms of those of g and h.

Solution

$f(x)=g(x)h(x)$
Aim: To get the divided difference for f in terms of those of g and h

Let us consider the nodes $x_0, x_1, \ldots, x_n$ between $[a, b]$ $f[x_0, x_1, \ldots, x_n]$ is the coefficient of x^n in polynomial which interpolates $f(x)$ using the nodes.

Let us consider the interpolating polynomial for $g(x)$ using nodes$x_0, x_1, \ldots, x_n$ be $P_g(x)$ and $h(x)$ using nodes $x_0, x_1, \ldots, x_n$ be $P_h(x)$
Using forward divided difference method for $g(x)$

$$P_g(x) = g[x_0] + g[x_0, x_1](x - x_0) + \cdots + g[x_0, x_1 \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

Using backward divided difference method for $h(x)$

$$P_h(x) = h[x_n] + h[x_{n-1}, x_n](x - x_n) + \cdots + h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_1)$$

$P(x) = P_g(x)P_h(x)$

$$\begin{aligned} &= \big(g[x_0] + g[x_0, x_1](x - x_0) + \cdots + g[x_0, x_1 \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1})\big) \\ &\qquad \big(h[x_n] + h[x_{n-1}, x_n](x - x_n) + \cdots + h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_1)\big) \end{aligned}$$

$$\begin{aligned} &= g[x_0]h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_1) + \ldots \\ &\qquad + g[x_0, x_1]h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_0) + \ldots \\ &\qquad\qquad + g[x_0, x_1 \ldots, x_n]h[x_0, x_1 \ldots, x_n](x - x_n)^2 \ldots (x - x_0)^2 \end{aligned}$$

$$\begin{aligned} P(x) = P_n(x) &+ \big(g[x_0, x_1]h[x_0, x_1 \ldots, x_n] + \cdots + g[x_0, x_1 \ldots, x_n]h[x_{n-1}, x_n]\big) \\ &(x - x_0)(x - x_1) \ldots (x - x_n) + \cdots + g[x_0, x_1 \ldots, x_n]h[x_0, x_1 \ldots, x_n](x - x_n)^2 \ldots (x - x_0)^2 \end{aligned}$$

Since the higher order terms have a factor of $(x - x_0)(x - x_1) \ldots (x - x_n)$
consider $P(x) = p_n(x) + q(x)(x - x_0)(x - x_1) \ldots (x - x_{n-1})$
we observe that $p_n(x)$ would also interpolate $f(x)$ and now has degree$\leq n$.
the coefficients of x^n is
$g[x_0, x_1]h[x_0, x_1 \ldots, x_n] + \cdots + g[x_0, x_1 \ldots, x_n]h[x_{n-1}, x_n]$
Hence $f[x_0, x_1 \ldots, x_n] = \sum_{r=0}^{n} g[x_0, x_1 \ldots, x_r] \quad h[x_r, x_{r+1} \ldots, x_n]$

Problem 4

Construct a Hermite polynomial $H_3(x)$ for the following data for $(x, f(x), f'(x))$: (8.3,17.56492,3.116256) and (8.6,18.50515,3.151762)
If the function here is $f(x) = xlnx$ then compute $f(8.4)$ and the errors

Solution

x	$f(x)$	$f'(x)$
8.3	17.56492	3.116256
8.6	18.50515	3.151762

Given $x_0, x_1, \ldots, x_n$ points we construct a hermite polynomial $H(x)$ of degree atmost $2n+1$ where

$$H(x) = \sum_{i=0}^{1} f(x_i)H_i(x) + \sum_{i=0}^{1} f'(x_i)\hat{H}_i(x)$$
$$H_i(x) = [1 - 2(x - x_i)L_i'(x_i)](L_i(x))^2$$
$$\hat{H}_i(x) = (x - x_i)(L_i(x))^2$$

let us calculate Lagrange polynomial

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$
$$L_0'(x) = \frac{1}{x_0 - x_1} \qquad L_1'(x) = \frac{1}{x_1 - x_0}$$

$$L_0(8.4) = 0.66666 \quad L_1(8.4) = 0.33333 \quad L_0'(x) = -3.3333333333 \quad L_1'(x) = 3.3333333333$$

$$H_0(x) = [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) = [1 - 2(x - 8.3)(-3.3333333333)]\left(\frac{x - 8.6}{-0.3}\right)^2$$

$$H_1(x) = [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x) = [1 - 2(x - 8.6)(3.3333333333)]\left(\frac{x - 8.3}{0.3}\right)^2$$

$$\hat{H}_0(x) = (x - x_0)L_0^2(x) = (x - 8.3)\left(\frac{x - 8.6}{-0.3}\right)^2$$

$$\hat{H}_1(x) = (x - x_1)L_1^2(x) = (x - 8.6)\left(\frac{x - 8.3}{0.3}\right)^2$$

$$H(x) = f(x_0)H_0(x) + f(x_1)H_1(x) + f'(x_0)\hat{H}_0(x) + f'(x_1)\hat{H}_1(x)$$

on substituting the values we get the Hermite polynomial as

$$H(x) = (-0.0020222222)x^3 + (0.11044)x^2 + (1.7008846667)x - 3.0043539553$$

$$H(8.4) = 17.8771444582 \quad f(8.4) = 17.8771463291$$

Absolute error $=|H(8.4) - f(8.4)| = 1.8709 \times 10^{-6}$ relative error$=\frac{Abs.err}{|f(8.4)|} = 1.047 \times 10^{-7}$

Problem 5

Construct a Hermite polynomial $H_3(x)$ for the following data for $(x, f(x), f'(x))$:
(0.8,0.22363362,2.1691753) and (1,0.65809197,2.0466965).If the function here is
$f(x) = sin(e^x - 2)$ then compute $f(0.9)$ and the errors

Solution

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1	0.65809197	2.0466965

$x_0 = 0.8 \quad x_1 = 1$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-1}{-0.2} = 5(1-x)$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-0.8}{0.2} = 5(x-0.8) = 5x - 4$$

$$L_0'(x) = -5 \quad L_1'(x) = 5$$

$$\begin{aligned} H_0(x) &= [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) \\ &= [1 - 2\big(x - \frac{4}{5}\big)(-5)]5^2(1-x)^2 \\ &= [1 + 2(5x-4)]25(1-x)^2 \\ &= (10x-7)25(x-1)^2 \end{aligned}$$

$$\begin{aligned} H_1(x) &= [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x) \\ &= [1 - 2(x-1)5](5x-4)^2 \\ &= (11-10x)(5x-4)^2 \end{aligned}$$

and

$$\begin{aligned} \hat{H}_0(x) &= (x - x_0)L_0^2(x) \\ &= \big(x - \frac{4}{5}\big)5^2(1-x)^2 \\ &= (5x-4)5(1-x)^2 \end{aligned}$$

$$\begin{aligned} \hat{H}_1(x) &= (x - x_1)L_1^2(x) \\ &= (5x-4)^2(x-1) \end{aligned}$$

hence

$$\begin{aligned} H_3(x) &= f(x_0)H_0(x) + f(x_1)H_1(x) + f'(x_0)\hat{H}_0(x) + f'(x_1)\hat{H}_1(x) \\ &= \Big\{(0.22363362)(10x-7)25(x-1)^2 + (0.65809197)(11-10x)(5x-4)^2 + (2.1691753) \\ &(5x-4)5(x-1)^2 + (2.0466965)(x-1)(5x-4)^2\Big\} \\ &= \Big\{5(x-1)^2[5(10x-7)(0.22363362) + (5x-4)(2.1691753)] \\ &+ (5x-4)^2[(11-10x)(0.65809197) + (x-1)(2.0466965)]\Big\} \\ &= 25[x^3(-0.1287117) + x^2(0.33527371) + x(-0.20254446) + (0.02230613)] \end{aligned}$$

$H_3(0.9)$=0.443924795
Given $f(x) = sin(e^x - 2)$ $f(0.9) = 0.4435924388$.
Hence,
absolute error= $0.0003323562 = 3.323562 \times 10^{-4}$
Relative error=$0.00074923775 = 7.4923775 \times 10^{-4}$.

Problem 6

Use five digit rounding arithmetic and compute the table for the values of $sin(x)$ and its derivative $cos(x)$ at 0.30,0.32 and 0.35.Obtain the corresponding Hermite polynomial $H(x)$ and compute H(0.34).Compute the actual error and the one predicted by the error formula.

Solution

$$H(x) = \sum_i f(x_i)H_i(x) + \sum_i f'(x_i)\hat{H}_i(x)$$

$$H_i(x) = [1 - 2(x - x_i)L_i(x_i)](L_i(x))^2$$

$$L_i^n(x) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

$$\hat{H}_i(x) = (x - x_i)(L_i(x))^2$$

given data

x	$sin(x)$	$cos(x)$
$x_0 = 0.30$	0.29552	0.95534
$x_1 = 0.32$	0.31457	0.94924
$x_2 = 0.35$	0.34290	0.93937

$$L_0(x) = \frac{(x - 0.32)(x - 0.35)}{0.02 \times 0.05} = 1000(x - 0.32)(x - 0.35)$$

$$L_1(x) = \frac{(x - 0.30)(x - 0.35)}{-0.02 \times 0.03} = \frac{1000}{6}(x - 0.30)(x - 0.35)$$

$$L_2(x) = \frac{(x - 0.30)(x - 0.32)}{0.02 \times 0.05} = -1000(x - 0.30)(x - 0.32)$$

$$L_0'(x) = 1000(2x - 0.67) \implies L_0'(x - 0) = -70.00$$

$$L_1'(x) = \frac{-10000}{6}(2x - 0.65) \implies L_1'(x_1) = 16.6666$$

$$L_2'(x) = 10000(2x - 0.62) \implies L_2'(x_2) = 80.00$$

$$
\begin{aligned}
H_0(x) &= [1-2(x-0.30)(-70)](x-0.32)^2(x-0.35)^2 10^6 \\
&= [140x-41](x-0.32)^2(x-0.35)^2 10^6
\end{aligned}
$$

$$
\begin{aligned}
H_1(x) &= [1-2(x-0.32)(16.66667)](x-0.30)^2(x-0.35)^2 \frac{10^8}{36} \\
&= [-33.33334x+11.66667](x-0.30)^2(x-0.35)^2 \frac{10^8}{36}
\end{aligned}
$$

$$
\begin{aligned}
H_2(x) &= [1-2(x-0.35)(80.00)](x-0.30)^2(x-0.32)^2 10^6 \\
&= [-160x+57.00](x-0.30)^2(x-0.32)^2 10^6
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}_0(x) &= (x-0.30)(x-0.32)^2(x-0.35)^2 10^6 \\
\hat{H}_1(x) &= (x-0.32)(x-0.30)^2(x-0.35)^2 \frac{10^8}{36} \\
\hat{H}_2(x) &= (x-0.35)(x-0.30)^2(x-0.32)^2 10^6
\end{aligned}
$$

Hermite polynomial for the given function $f(x) = sin(x)$ using the formula

$H(x) = 0.29552H_0(x)+0.31457H_1(x)+0.34290H_2(x)+0.95534\hat{H}_0(x)+0.94924\hat{H}_1(x)+0.93937\hat{H}_2(x)$
$H(0.34) = 0.33719$ whereas $sin(0.34) = 0.33349$

therefore Actual error$= -0.00370$
using error formula we have : $f(x) = sin(x)$ $f^{(6)}(x) = -sin(x)$
Maximum value of $f^{(6)}$ in 0.30 to 0.35 is sin(0.35) = 0.34290 since sin is an increasing function in 0 to $\pi/2$

$$
error = \frac{(0.34-0.30)^2(0.34-0.32)^2(0.34-0.35)^2}{6!} \times 0.34290
$$

$$
\approx 3.048 \times 10^{-14} <<< \text{Actual error}
$$

this means that we should use many ore decimal digits to get accurate Hermite interpolation.

Problem7

Compute the natural cubic spline for the following data:

x	-0.5	-0.25	0
$f(x)$	-0.0247500	0.3349375	1.1010000

Solution

$$x_0 = -0.5 \qquad x_1 = -0.25 \qquad x_2 = 0$$
$$f_0 = -0.0247500 \qquad f_1 = 0.3349375 \qquad f_2 = 1.101000$$

$S_i(x)$ is the cubic polynomial on $[x_i, x_i + 1] \quad i = 0, 1$
and $h = |x_{i+1} - x_i| = 0.25$

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & x_0 \le x \le x_1 \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & x_1 \le x \le x_2 \end{cases}$$

$$a_0 = S_0(x_0) = f_0 = -0.0247500$$
$$a_1 = S_1(x_1) = f_1 = 0.3349375$$

By the above relations we have

$$\begin{aligned} S_0''(x_0) = S_1''(x_2) &= 0 \\ 6d_0(x_0 - x_0) + 2c_0 &= 0 \\ c_0 &= 0 \end{aligned}$$

$$\begin{aligned} 6d_1(x_2 - x_1) + 2c_1 &= 0 \\ 6d_1 h + 2c_1 &= 0 \\ d_1 &= \frac{-c_1}{3h} \end{aligned}$$

$$\begin{aligned} S_0''(x_1) &= S_1''(x_1) \\ 6d_0(x_1 - x_0) + 2c_0 &= 2c_1 \\ 6d_0 h + 2c_0 &= 2c_1 \\ d_0 &= \frac{c_1}{3h} \end{aligned}$$

$$\begin{aligned} f_1 &= S_0(x_1) \\ f_1 &= a_0 + b_0 h + c_0 h^2 + d_0 h^3 \\ \frac{f_1 - f_0}{h} &= b_0 + c_0 h + d_0 h^2 \\ b_0 &= \frac{f_1 - f_0}{h} - \frac{c_1 h}{3} \end{aligned}$$

$$\begin{aligned} f_2 &= S_1(x_2) \\ f_2 &= a_1 + b_1 h + c_1 h^2 + d_1 h^3 \\ \frac{f_2 - f_1}{h} &= b_1 + c_1 h - \frac{c_1 h}{3} \\ b_1 &= \frac{f_2 - f_1}{h} - \frac{2c_1 h}{3} \end{aligned}$$

$$\begin{aligned} S_0'(x_1) &= S_1'(x_1) \\ S_0'(x) &= b_0 + 2c_0(x - x_0) + 3d_0(x - x_0)^2 \\ S_1'(x) &= b_1 + 2c_1(x - x_1) + 3d_1(x - x_1)^2 \end{aligned}$$

$$\begin{aligned} b_1 &= 3d_0 h^2 + 2c_0 h + b_0 \\ \frac{f_2 - f_1}{h} - \frac{2c_1 h}{3} &= c_1 h + \frac{f_1 - f_0}{h} - \frac{c_1 h}{3} \\ \frac{f_2 + f_0 - 2f_1}{h^2} &= \frac{4c_1 h}{3} \\ c_1 &= \frac{f_2 + f_0 - 2f_1}{h^2} \times \frac{3}{4} \end{aligned}$$

$c_1 = \frac{1.101 - 0.02475 + 2(0.3349375)}{(0.25)^2} \times \frac{3}{4} = 4.8765$

on substituting the values we get

$$b_0 = 1.032375 \qquad b_1 = 2.2515 \qquad d_0 = 6.502 \qquad d_1 = -6.502$$

$S_0(x) = 6.502x^3 + 9.573x^2 + 5.908875x + 1.3041875$
$S_1(x) = -6.502x^3 + 3.470625x + 1.101$
Hence

$$S(x) = \begin{cases} 6.502x^3 + 9.573x^2 + 5.908875x + 1.3041875 & -0.5 \le x \le -0.25 \\ -6.502x^3 + 3.470625x + 1.101 & -0.25 \le x \le 0 \end{cases}$$

Problem 8

Compute the natural cubic spline for the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	-0.062049958	-0.28398668	0.00660095	0.24842440

Solution

$$x_0 = 0.1 \qquad x_1 = 0.2 \qquad x_2 = 0.3 \qquad x_3 = 0.4$$
$$y_0 = -0.62049958 \qquad y_1 = -0.28398668 \qquad y_2 = 0.00660095 \qquad y_3 = 0.24842440$$

$$S(x) = \begin{cases} S_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0 & x_0 \le x \le x_1 \\ S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1 & x_1 \le x \le x_2 \\ S_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2 & x_2 \le x \le x_3 \end{cases}$$

for $i = 1, 2$ we have

$$S_{i-1}(x_i) = S_i(x_i) \qquad S'_{i-1}(x_i) = S'_i(x_i) \qquad S''_{i-1}(x_i) = S''_i(x_i)$$

and $\quad S(x_i) = y_i \quad 0 \le i \le 3 \quad S''(x_0) = S''(x_3) = 0.$

$$S_0(x_0) = y_0 \qquad S_0(x_1) = S_1(x_1) = y_1 \quad S_1(x_2) = S_2(x_2) = y_2 \quad S_2(x_3) = y_3$$

on substituting the values we have

$$a_0(0.1)^3 + b_0(0.1)^2 + c_0(0.1) + d_0 = -0.62049958$$
$$a_0(0.2)^3 + b_0(0.2)^2 + c_0(0.2) + d_0 = -0.28398668$$
$$a_1(0.2)^3 + b_1(0.2)^2 + c_1(0.2) + d_1 = -0.28398668$$
$$a_1(0.3)^3 + b_1(0.3)^2 + c_1(0.3) + d_1 = 0.00660095$$

$$a_2(0.3)^3 + b_2(0.3)^2 + c_2(0.3) + d_2 = 0.00660095$$

$$a_3(0.4)^3 + b_3(0.4)^2 + c_3(0.4) + d_3 = 0.24842440$$

substituting the values in the first derivative equations we have

$$S_0'(x_1) = S_1'(x_1) \quad S_1'(x_2) = S_2'(x_2)$$

$$3a_0(0.2)^2 + 2b_0(0.2) + c_0 = 3a_1(0.2)^2 + 2b_1(0.2) + c_1$$

$$3a_1(0.3)^2 + 2b_1(0.3) + c_1 = 3a_2(0.3)^2 + 2b_2(0.3) + c_2$$

for the second derivative equations

$$S_1''(x_0) = 0 = S_2''(x_3) \quad S_0''(x_1) = S_1''(x_1) \quad S_1''(x_2) = S_2''(x_2)$$

$$6a_0(0.1) + 2b_0 = 0 = 6a_1(0.4) + 2b_2$$

$$6a_0(0.2) + 2b_0 = 6a_1(0.2) + 2b_1$$

$$6a_1(0.3) + 2b_1 = 6a_2(0.3) + 2b_2$$

$$\begin{bmatrix} 0.001 & 0.01 & 0.1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.008 & 0.04 & 0.2 & 1 & -0.008 & -0.04 & -0.2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & -0.027 & -0.09 & -0.3 & -1 \\ 0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.064 & 0.16 & 0.4 & 1 \\ 0.6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & 2 & 0 & 0 \\ 1.2 & 2 & 0 & 0 & -1.2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.8 & 2 & 0 & 0 & -1.8 & -2 & 0 & 0 \\ 0.12 & 0.4 & 1 & 0 & -0.12 & -0.4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.24 & 0.6 & 1 & 0 & -0.27 & -0.6 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \begin{bmatrix} y_0 \\ 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

on solving we get

$$a_0 = -8.99579832 \qquad a_1 = -0.94630872 \qquad a_2 = 9.94210526$$

$$b_0 = 2.69874477 \qquad b_1 = -2.13095238 \qquad b_2 = -11.93051380$$

$$c_0 = 3.18521341 \qquad c_1 = 4.15115115 \qquad c_2 = 7.09102091$$

$$d_0 = -0.95701357 \qquad d_1 = -1.02140673 \qquad d_2 = -1.31539611$$

Problem 9

Compute the cubic spline for the data in the above problem and $f'(0.1) = 3.58502082$ and $f'(0.4) = 2.16529366$.

Solution

$$S_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0$$
$$S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$$
$$S_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$$

So we get Let $x_0 = 0.1, x_1 = 0.2$ and so on The continuity equations equations are:

$$a_0(0.1)^3 + b_0(0.1)^2 + c_0(0.1) + d_0 = -0.62049958$$
$$a_0(0.2)^3 + b_0(0.2)^2 + c_0(0.2) + d_0 = -0.28398668$$
$$a_1(0.2)^3 + b_1(0.2)^2 + c_1(0.2) + d_1 = -0.28398668$$
$$a_1(0.3)^3 + b_1(0.3)^2 + c_1(0.3) + d_1 = 0.00660095$$
$$a_2(0.3)^3 + b_2(0.3)^2 + c_2(0.3) + d_2 = 0.00660095$$
$$a_3(0.4)^3 + b_3(0.4)^2 + c_(0.4) + d_3 = 0.24842440$$

The first derivative equations are

$$3a_0(0.1)^2 + 2b_0(0.1) + c_0 = 3.58502082$$
$$3a_0(0.2)^2 + 2b_0(0.2) + c_0 = 3a_1(0.2)^2 + 2b_1(0.2) + c_1$$
$$3a_1(0.3)^2 + 2b_1(0.3) + c_1 = 3a_2(0.3)^2 + 2b_2(0.3) + c_2$$
$$3a_2(0.4)^2 + 2b_2(0.4) + c_2 = 2.16529366$$

The second derivative equations are:

$$6a_0(0.2) + 2b_0 = 6a_1(0.2) + 2b_1$$
$$6a_1(0.3) + 2b_1 = 6a_2(0.3) + 2b_2$$

As a matrix :

$$\begin{bmatrix} 0.001 & 0.01 & 0.1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.064 & 0.16 & 0.4 & 1 \\ 0.03 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.12 & 0.4 & 1 & 0 & -0.12 & -0.4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.27 & 0.6 & 1 & 0 & -0.27 & -0.6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.64 & 0.8 & 1 & 0 \\ 1.2 & 2 & 0 & 0 & -1.2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.8 & 2 & 0 & -1.8 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} -0.6204995 \\ -0.28398668 \\ -0.28398668 \\ 0.00660095 \\ 0.00660095 \\ 0.24842440 \\ 3.5850208 \\ 0 \\ 0 \\ 2.16529366 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above equations we get the coefficients

$$[-5.4278927], [-0.02776892], [3.75341136], [-0.99013505], [14.33700811], [-11.88670941],$$
$$[6.12519946], [-1.14825426], [1.13523922], [-5.59455431], [5.914384], [-1.29485582$$

MA214 Tutorial 6

5 Mar 2022

Problem1

Use the forward-difference and backward-difference formulae to determine each missing entry in the following table:

x	0.5	0.6	0.7
$f(x)$	0.4794255386	0.5646424734	0.6442176872
$f'(x)$	?	?	?

Solution:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

If $h > 0$, it is Forward difference,
If $h < 0$, it is Backward difference.

for $x = 0.5$ We use Forward difference

$$f'(0.5) \approx \frac{f(0.5+0.1) - f(0.5)}{0.1} = \frac{0.5646424739 - 0.4794255386}{0.1} = 0.85216935$$

for $x = 0.7$ we use backward difference

$$f'(0.7) \approx \frac{f(0.7+0.1) - f(0.7)}{-0.1} = 0.79575214$$

for $x = 0.6$ we use either forward or backward difference.
using forward difference

$$f'(0.6) \approx \frac{f(0.7) - f(0.6)}{0.1} = 0.79575214$$

using backward difference

$$f'(0.6) \approx \frac{f(0.6-0.1) - f(0.6)}{-0.1} = 0.85216935$$

We thus see that we could uniquely calculate the values of f only the end points. We can approximate

$$f'(0.6) \approx \frac{f'(0.6)_{fwd} + f'(0.6)_{bwd}}{2}$$
$$f'(0.6) \approx \frac{f(0.6+0.1) - f(0.6-0.1)}{2 \times 0.1}$$

x	0.5	0.6	0.7
$f(x)$	0.4794255386	0.564642473	0.6442176872
$f'(x)$	0.85216935	0.823960745	0.79575214

Problem 2

Use the forward-difference and backward difference formula to determine each missing entry in the following table:

x	0.0	0.2	0.4
$f(x)$	0.0	0.7414027582	1.3718246976
$f'(x)$	?	?	?

solution:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

If $h > 0$, it is Forward difference,
If $h < 0$, it is Backward difference. for x=0: We use Forward difference

$$f'(0) \approx \frac{f(0.2) - f(0)}{0.2} = 3.7070137951.$$

for $x = 0.2$ we use Forward difference

$$f'(0.2) \approx \frac{f(0.4) - f(0.2)}{0.2} = 3.152109697$$

for $x = 0.2$ we use backward difference.
using Backward difference

$$f'(0.2) \approx \frac{f(0) - f(0.2)}{-0.2} = 3.7070137915.$$

for $x = 0.4$ using Backward difference

$$f'(0.4) \approx \frac{f(0.2) - f(0.4)}{-0.2} = 3.152109697.$$

Problem 3

The data in the above problems were taken from the following functions.Compute the actual errors and find error bounds using the error formulae:

$$(1)f(x) = Sinx \qquad (2)f(x) = e^x - 2x^2 + 3x - 1$$

Solution:

x	0.5	0.6	0.7
$f'_{app}(x)$ $f'_{ac}(x)$	0.4794255386 $cos(0.5) = 0.87758$	0.564624734 $cos(0.6) = 0.8253365$	0.6442176872 $cos(0.7) = 0.76484$
Actual error	0.25888046	0.029406068	0.033455963

$$|f'_{app}(x) - f'_{ac}(x)| \leq |\frac{Mh}{2}| \leq |\frac{f''(\xi)h}{2}|$$

$f''(x) = -sinx$ Error bound=$|\frac{hsinx}{2}| = E$

x	0.5	0.6	0.7
E	$\lvert\frac{0.1sin(0.6)}{2}\rvert$ =0.0282321236	$\lvert\frac{0.1sin(0.7)}{2}\rvert$ =0.0322108844	$\lvert\frac{0.1sin(0.6)}{2}\rvert$ =0.0358678046

$$2)f(x) = e^x - 2x^2 + 3x - 1$$

x	0.0	0.2	0.4
$f(x)$ $f'_{app}(x)$ $f'_{ac}(x)$	0.0 3.707013791 $e^0 + 3 = 4$	0.7414027582 3.152109697 $e^{0.2} - 0.8 + 3 = 3.42140275816$	1.3718246976 2.65147051395 $e^{0.4} + 1.4 = 2.891324887$
Error	0.292986209	0.26929306116	0.24035418305

Error Bound =$|\frac{f''(\xi)h}{2}|$=E

x	0.0	0.2	0.4
$f'(\xi)$ E	$\lvert 1-4\rvert = 3$ 0.3	$\lvert 1.221402 - 4\rvert$ 0.2778598	$\lvert 1.491824697 - 4\rvert$ 0.250817532

$$\begin{aligned} f(0.6) &= e^(0.6) - 2(0.36) + 1.8 - 1 \\ &= 1.82211880039 - 0.72 + 0.8 \\ &= 1.90211880039 \end{aligned}$$

Problem 4

If $f(x) : [a,b] \to \mathbb{R}$ is continuously differentiable and $c_i \geq 0, \theta_i \in (a,b)$ for $i = 0, 1, \ldots, n$ such that $\sum_i c_i f'(\theta_i) = (\sum_i c_i) f'(\theta)$.

Solution:

To show that there is a $\theta \in (a,b)$ such that

$$\sum_i c_i f'(\theta_i) = \left(\sum_i c_i\right) f'(\theta)$$

We will use induction over the value of M to show that above statement is true.

For $n = 0$: $c_0 f'(\theta_0) = c_0 f'(\theta)$ for $\theta = \theta_0$ therefore for the base case the above statement is true.
Suppose the given statement holds for $n = k$ i.e., $\exists \theta' \in (a,b)$ such that

$$\sum_i^k c_i f'(\theta_i) = \left(\sum_i^k c_i\right) f'(\theta^{'})$$

Given that the above statement holds, we now need to prove that $\exists \theta = \theta^{''} \in (a,b)$ such that

$$\sum_i^{k+1} c_i f'(\theta_i) = \left(\sum_i^{k+1} c_i\right) f'(\theta^{''})$$

$$L.H.S = \sum_i^{k+1} c_i f'(\theta_i) = \sum_i^{k} c_i f'(\theta_i) + c_{k+1} f'(\theta^{''})$$

L.H.S=$\sum_{i=1}^{k+1} c_i f'(\theta_i) = \sum_{i=1}^{k} c_i f'(\theta_i) + c_{k+1} f'(\theta_{k+1}) = (\sum_{i=1}^{k} c_i) f'(\theta^{'}) + c_{k+1} f'(\theta_{k+1})$

Let $\sum_{i=1}^{k} c_i = A$
since $c_i \geq 0$ for $i = 1, 2, \ldots, n \implies A \geq 0$

Case-1: $A + c_{k+1} = 0 \implies A = 0$ and $c_{k+1} = 0$
then $c_i = 0$ for $i = 0, 1, \ldots, k+1$.
Case-2: $A + c_{k+1} \neq 0$ i.e.,$A + c_{k+1} > 0$
Step:1

$$Af'(\theta') + c_{k+1} f'(\theta_{k+1}) = (A + c_{k+1}) \left(\frac{Af'(\theta') + c_{k+1} f'(\theta_{k+1})}{A + c_{k+1}} \right)$$

let

$$B = \frac{Af'(\theta') + c_{k+1} f'(\theta_{k+1})}{A + c_{k+1}}$$

WLOG let $f'(\theta') \leq f'(\theta_{k+1})$, then we know B is the weighted average of$f'(\theta')$ and $f'(\theta_{k+1})$
Hence $B \in [f'(\theta'), f'(\theta_{k+1})]$.

Step:2 Given f is continuously differentiable over [a,b] then f' is continuous over (a,b).Since $\theta', \theta_{k+1} \in (a,b)$ By intermediate value theorem there exist $\theta^{'''} \in (a,b)$ such that $f'(\theta''') = B$. Thus the L.H.S becomes

$$A + C_{k+1}f'(\theta''') = (\sum_{i=1}^{k} c_i)f'(\theta''')$$

Hence Proved.

Problem 5

Assume that for any sufficiently continuously differentiable function f,we have

$$f''(t) \approx Af(t+h) + Bf(t) + Cf(t-h)$$

where A,B,C are Constants,depending on h,to be determined.Replace $f(t+h)$ by the Taylor expansions.Ignoring the terms involving h^3 or higher powers of h, Solve for A,B,C. Write the approximate formula for $f''(t)$ obtained thus.

Solution:

$$f''(t) \approx Af(t+h) + Bf(t) + Cf(t-h)$$

$$f(t+h) \approx f(t) + hf'(t) + \frac{h^2}{2}f''(t)$$

$$f(t-h) \approx f(t) - hf'(t) + \frac{h^2}{2}f''(t)$$

Comparing coefficients of $f(t), f'(t)$ *and* $f''(t)$ respectively :

$$A + B + C = 0$$

$$hA - hC = 0 \leftrightarrow A = C$$

$$\frac{h^2}{2}A + \frac{h^2}{2}C = 1$$

This gives us :

$$A = C = \frac{1}{h^2}$$

and

$$B = -\frac{2}{h^2}$$

Thus :

$$f''(t) \approx \frac{1}{h^2} f(t+h) - \frac{2}{h^2} f(t) + \frac{1}{h^2} f(t-h)$$

Problem 6

Derive Simpson's $\frac{1}{3}$rd rule with error term by using

$$\int_{x_0}^{x_2} f(x)dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi)$$

Find a_0, a_1 and a_2 from the fact that the rule is exact for $f(x) = x^n$ when $n = 1, 2$ and 3.Then find k by applying the integration formula with $f(x) = x^4$.

Solution:

Simpson's$\frac{1}{3}$rd rule:

$$\int_{x_0}^{x_2} f(x)dx = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi)$$

where $x_0 < \xi < x_2$
Given formula is exact for $f(x) = x, x^2, x^3$
$f(x) = x$

$$\int_{x_0}^{x_2} x dx = a_0 x_0 + a_1 x_1 + a_2 x_2 = \frac{x_2^2}{2} - \frac{x_0^2}{2}$$

$f(x) = x^2$

$$\int_{x_0}^{x_2} x^2 dx = a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = \frac{x_2^3}{3} - \frac{x_0^3}{3}$$

$f(x) = x^3$

$$\int_{x_0}^{x_2} x^3 dx = a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3 = \frac{x_2^4}{4} - \frac{x_0^4}{4}$$

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_0^2 & x_1^2 & x_2^2 \\ x_0^3 & x_1^3 & x_2^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2^2}{2} - \frac{x_0^2}{2} \\ \frac{x_2^3}{3} - \frac{x_0^3}{3} \\ \frac{x_2^4}{4} - \frac{x_0^4}{4} \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{(h+x_0)(2h+x_0)}{2h^2x} & \frac{-3}{2hx_0} - \frac{1}{h^2} & \frac{1}{2h^2x_0} \\ \frac{-x(2h+x_0)}{h^2(h+x_0)} & \frac{2}{h^2} & \frac{1}{h^2(h+x_0)} \\ \frac{x(h+x_0)}{2h^2(2h+x_0)} & \frac{-h^2+x_0}{h^2(2h+x_0)} & \frac{1}{2h^2(2h+x_0)} \end{bmatrix} \begin{bmatrix} \frac{x_2^2}{2} - \frac{x_0^2}{2} \\ \frac{x_2^3}{3} - \frac{x_0^3}{3} \\ \frac{x_2^4}{4} - \frac{x_0^4}{4} \end{bmatrix}$$

To make Calculations easier, consider
$x_0 = x_0$ for a_0
$x_0 = x_1 - h, x_2 = x_1 + h$ for a_1
$x_0 = x_2 - 2h, x_1 = x_2 - h$ for a_2

$$a_0 = \frac{h}{3} \quad a_1 = \frac{4h}{3} \quad a_2 = \frac{4}{3}$$

For k,consider $f(x) = x^4; f^N(\xi) = 4! = 24$

$$\frac{x_2^5}{5} - \frac{x_0^5}{5} = \int_{x_0}^{x_2} x^4 dx = \frac{h}{3}x_0^4 + \frac{4h}{3}x_1^4 + \frac{h}{3}x_2^4 + 24k$$

gives

$$\begin{aligned} 24k &= 0.2\Big[(x_1+h)^5 - (x_1-h)^5\Big] - \frac{h}{3}(x_1-h)^4 - \frac{4h}{3}x_1^4 - \frac{h}{3}(x_1+h)^4 \\ &= \frac{-8}{30}h^5 \\ k &= \frac{-h^5}{90} \end{aligned}$$

Problem 7

Approximate the following using Simpson's$\frac{1}{3}$rd rule:
Compute the actual error and compare it with the error given by the error formulae. $\int_{0.5}^{1} x^4 dx$

Solution

Given, $f(x) = x^4$, we are required to compute $\int_{0.5}^{1} f(x)dx$ using the trapezoidal and Simpson's rule
To calculate the true value,

$$\int_a^b f(x)dx = \left.\frac{x^5}{5}\right|_{0.5}^{1} = \frac{1 - 0.03125}{5} = 0.19375$$

Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12} f''(\xi)$$

where $x_0 = a$, $x_1 = b$. For our case, $f(x_0) = 0.0625$, $f(x_1) = 1$ and $h = 0.5$. Substituting values, we get,

$$\int_a^b f(x)dx \approx \frac{0.5}{2}(0.0625 + 1) = 0.265625$$

True error $= 0.265625 - 0.19375 = 0.071875$
Error from formula $= \frac{0.5^3}{12} f''(\xi)$. $f''(x) = 12x^2$. In the given range the max value of $f''(x) = 12$.
So the max error possible from the formula $= 0.125$

Simpson's rule

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) - \frac{h^5}{90} f^{(4)}(\xi)$$

where $x_1 = \frac{x_0+x_1}{2}$, For our case, $x_0 = 0.5$, $x_1 = 0.75$, $x_2 = 1$, $f(x_0) = 0.0625$, $f(x_1) = 0.3164$, $f(x_2) = 1$ and h=0.25. Substituting these values, we get.

$$\int_a^b f(x)dx \approx \frac{0.25}{3}(0.0625 + 1.2656 + 1) = 0.19401$$

True error $= 0.19401 - 0.19375 = 0.00026$
Error from formula $= \frac{h^5}{90} f^{(4)}(\xi)$. $f^{(4)}(x) = 24$. So, the max error possible from the formula is $= 24\frac{0.25^5}{90} = 0.00026$

Problem 8

Approximate the following using the Trapezoidal and Simpson's $\frac{1}{3}$rd rule. Compute the actual error and compare it with the error given by the error formulae.$\int_0^{0.5} \frac{2}{x-4} dx$

Solution:

$f(x) = \frac{2}{x-4} \quad a = 0 \quad b = 0.5$
Trapezoidal rule:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a) + f(b))$$

by Trapezoidal rule we have

$$f(0) = -0.5 \qquad\qquad f(0.5) = -\frac{2}{3.5} = -0.5714$$

$\int_a^b f(x)dx \approx -\frac{0.5}{2}(0.5 + 0.5714) = -0.26785$

Simpson's $\frac{1}{3}$rd rule:

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = \frac{0.5}{6}\left[f(0) + 4f(0.25) + f(0.5)\right]$$
$$= \frac{0.5}{6}\left[0.5 + 4 \times 0.5333 + 0.5714\right] = -0.26706$$

Actual integral value:
$\int_0^{0.5} \frac{2}{x-4} dx = -2ln(|x-4|)\Big|_0^{0.5} = -0.26706$
error:
Trapezoidal-Actual error=-0.26706+0.26785=0.00079

$$
\begin{aligned}
E &= \frac{(b-a)^3}{12}\|f''\| \\
&= \frac{(0.5)^3}{12} \sup_{x\in[0,0.5]} \left\{ \left| \frac{4}{(x-4)^3} \right| \right\} \\
&= \frac{(0.5)^3}{12}(0.09329) = 0.00097
\end{aligned}
$$

Simpson's-Actual error= -0.26706+0.26706 = 0 (upto 5 de)

$$
\begin{aligned}
E &= \frac{(b-a)^5}{90}||f^k|| \\
&= \frac{(0.5)^5}{90} \sup_{x\in[0,0.5]} \left\{ \left| \frac{48}{(x-4)^5} \right| \right\} \\
&= \frac{(0.5)^5}{90}(0.09139) = 0.0000317.
\end{aligned}
$$

We are approximating the $f'''(\xi)$ and $f^4(\xi)$ in the error formula with corresponding max norms. Since that gives the Upper bound on the error.

Problem 9

The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value 4, and the Simpson's$\frac{1}{3}$rd rule gives the value 2.What is $f(1)$?
The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value 5, and the Mid-point rule gives the value 4.What value does is $f(1)$ Simpson's$\frac{1}{3}$rd rule give ?

Solution

Trapezoidal rule:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a)+f(b))$$

by Trapezoidal rule we have

$$\int_0^2 f(x)dx = 4 \approx -\frac{0.5}{2}(0.5 + 0.5714) = -0.26785$$

Simpson's $\frac{1}{3}$rd rule:

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right]$$

$$\int_0^2 f(x)dx = 2 \approx \frac{2-0}{2\times 3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$6 = f(x_0) + 4f(x_1) + f(x_2)$$

$$f(x_1) = \frac{1}{2}$$

From Midpoint rule we have

$$\int_a^b f(x)dx \approx 2hf(a)\left(\frac{a+b}{2}\right) \;\; where \;\; h = \frac{b-a}{2}$$

$$\int_0^2 f(x)dx = 4 \approx 2\times f(1)\left(\frac{2-0}{2}\right) \implies f(1) = 2$$

Using Trapezoidal rule:

$$\int_0^2 f(x)dx = 5 \approx \frac{2-0}{2}(f(0) + f(2)) \implies f(0) + f(2) = 5$$

By Simpson's$\frac{1}{3}$rd rule:

$$\begin{aligned}\int_0^2 f(x)dx &= \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]\\ &= \frac{2-0}{6}\left[f(0) + 4f(x_1) + f(2)\right]\\ &= \frac{1}{3}[5+8] = \frac{13}{3}\end{aligned}$$

Tut-07

16 March 2022

1 Question 1

Use the composite trapezoidal rule with $n = 6$ to approximate the following integral:

$$1) \int_0^2 \frac{2}{x^2+4}\,dx \qquad\qquad 2) \int_0^{\pi} x^2 cos dx$$

Solution

Given $n = 6$, so we will have 6 partitions,
Now for every partition, $h = \frac{1}{3}$

$$\begin{aligned}\mathbf{I} = &\frac{1}{2}\times\frac{1}{3}\big[f(0)+f(\frac{1}{3})\big]+\frac{1}{2}\times\frac{1}{3}\big[f(\frac{1}{3})+f(\frac{2}{3})\big]+\frac{1}{2}\times\frac{1}{3}\big[f(\frac{2}{3})+f(1)\big]\\ &+\frac{1}{2}\times\frac{1}{3}\big[f(1)+f(\frac{4}{3})\big]+\frac{1}{2}\times\frac{1}{3}\big[f(\frac{4}{3})+f(\frac{5}{3})\big]+\frac{1}{2}\times\frac{1}{3}\big[f(\frac{5}{3})+f(2)\big]\end{aligned}$$

1) $\displaystyle\int_0^2 \frac{2}{x^2+4}\,dx$

$$f(0) = \frac{2}{0^2+4} = \frac{1}{2} \qquad f(\frac{1}{3}) = \frac{2}{(\frac{1}{3})^2+4} = \frac{18}{37} \qquad f(\frac{2}{3}) = \frac{9}{20}$$

$$f(1) = \frac{2}{5} \qquad f(\frac{4}{3}) = \frac{9}{26} \qquad f(\frac{5}{3}) = \frac{18}{61} \qquad f(2) = \frac{1}{4}$$

On substituting values in the formula

$$\mathbf{I} \approx 0.7842$$

2) $\int_0^{\pi} x^2 \cos x dx$

As above $n = 6$ partitions i.e. $h = \frac{\pi}{6}$,

$$\mathbf{I} = \frac{1}{2}\times\frac{\pi}{6}\times\big[f(0)+2f(\frac{\pi}{6})+2f(\frac{\pi}{3})+2f(\frac{\pi}{2})+2f(\frac{2\pi}{3})+2f(\frac{5\pi}{6})+f(\pi)\big]$$

We have
$f(0) = 0,\ f(\frac{\pi}{6}) = 0.2374,\ f(\frac{\pi}{3}) = 0.5483,\ f(\frac{\pi}{2}) = 0,\ f(\frac{2\pi}{3}) = -2.1932,\ f(\frac{5\pi}{6}) = -5.9356,\ f(\pi) = -9.8696$
Hence

$$\mathbf{I} \approx -6.4287$$

2 Question 2

Use the composite simpson's rule to approximate the integrals:

$$1) f(x) = \frac{2}{x^2+4} \qquad\qquad 2) f(x) = x^2 \cos x$$

solution

1) $f(x) = \frac{2}{x^2+4}$

$$\begin{aligned}
\int_0^2 f(x)\,dx &\approx \sum_{j=0}^{5} \int_{\frac{j}{6}}^{\frac{j+1}{6}} f(x)\,dx \\
&= \frac{2}{6} \times \frac{1}{6}\Big[f(0) + 4f(\frac{1}{6}) + f(\frac{2}{6})\Big] + \frac{2}{6} \times \frac{1}{6}\big[f(\frac{2}{6}) + 4f(\frac{3}{6}) + f(\frac{4}{6})\big] + \cdots \\
&= \frac{2}{36}\big[f(0) + 4f(\frac{1}{6}) + 2f(\frac{2}{6}) + 4f(\frac{3}{6}) + 2f(\frac{4}{6}) + 4f(\frac{5}{6}) + 2f(1) + 4f(\frac{7}{6}) + \\
&\qquad 2f(\frac{8}{6}) + 4f(\frac{9}{6}) + 2f(\frac{10}{6}) + 4f(\frac{11}{6}) + f(2)\big] \\
&= 0.785398160076 \\
Error &= 0.785398163397 - 0.785398160076 = 0.000000007
\end{aligned}$$

2) $f(x) = x^2 \cos x$

$$\begin{aligned}
\int_0^{\pi} f(x)\,dx &\approx \frac{\pi}{6} \times \frac{1}{6}\big[f(0) + 4f(\frac{\pi}{12}) + 2f(\frac{2\pi}{12}) + \cdots + 2f(\frac{10\pi}{12}) + 4f(\frac{11\pi}{12}) + f(\pi)\big] \\
&= -5.400681848
\end{aligned}$$

3 Question 3

Suppose that $f(0) = 1, f(0.5) = 2.5, f(1) = 2, f(0.25) = f(0.75) = \alpha$. Find α if the composite trapezoidal rule with $n = 4$ gives the value 1.75 for $\int_0^1 f(x)dx$.

solution

$f(0) = 1$; $f(0.5) = 2.5$; $f(1) = 2$; $f(0.25) = f(0.75) = \alpha$
We will calculate $\int_0^1 f(x)\,dx$ by trapezoidal rule with $n = 4$

$$\begin{aligned}
\mathbf{I} &= \frac{0.25}{2}\big[f(0) + f(0.25)\big] + \frac{0.25}{2}\big[f(0.25) + f(0.5)\big] \\
&\qquad + \frac{0.25}{2}\big[f(0.5) + f(0.75)\big] + \frac{0.25}{2}\big[f(0.75) + f(1)\big] \\
1.75 &= \frac{0.25}{2}\big[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\big] \\
1.75 &= \frac{0.25}{2}\big[1 + 2\alpha + 2 \times 2.5 + 2\alpha + 2\big] \\
14 &= 1 + 4\alpha + 5 + 2 \\
14 &= 8 + 4\alpha \\
6 &= 4\alpha \\
\alpha &= 1.5
\end{aligned}$$

4 Question 4

Use adaptive quadrature to compute the following integral with accuracy within 10^{-2}

$$f(x) = e^{2x} sin(3x)$$

solution

Given $f(x) = e^{2x} sin(3x)$

$$I = \int_1^3 f(x)dx$$

$$S(a, b) = \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$$

$$F(a, b) = \big|S(a, b) - S(a, \frac{a+b}{2}) + S(\frac{a+b}{2}, b)\big|$$

Now if $F(1, 3) = 67.0899164 > 15\epsilon$
so we divide $(1, 3)$ into further as $(1, 2)$ and $(2, 3)$
$F(1, 2) = 0.8008379 > \frac{15\epsilon}{2} = 0.075$
$F(1, 1.5) = 0.0240668 < \frac{15\epsilon}{4} = 0.0375$
$F(1.5, 2) = 0.011841 < \frac{15\epsilon}{4} = 0.0375$

For(2,3)
$F(2, 3) = 6.5656500 > \frac{15\epsilon}{2} = 0.0375$ so we divide $(2, 3)$ into $(2, 2.5)$ and $(2.5, 3)$
$F(2, 2.5) = 0.1466767 > \frac{15\epsilon}{4} = 0.0375$
so we divide $(2, 2.5)$ into $(2, 2.25)$ and $(2.25, 2.5)$
$F(2, 2.25) = 0.0029153 < \frac{15\epsilon}{8} = 0.01875$

For $(2.5, 3)$
$F(2.5, 3) = 0.0.1439066 > \frac{15\epsilon}{4} = 0.0375$
so we divide $(2.5, 3)$ into $(2.5, 2.75)$ and $(2.75, 3)$
$F(2.5, 2.75) = 0.0069736 < \frac{15\epsilon}{8} = 0.01875$
$F(2.75, 3) = 0.0000401 < \frac{15\epsilon}{8} = 0.01875$
So the sub intervals that we should use to approximate are :
$(1, 1.5), (1.5, 2), (2, 2.25), (2.25, 2.5), (2.5, 2.75), (2.75, 3)$

$S(1, 1.5) + S(1.5, 2) + S(2, 2.25) + S(2.25, 2.5) + S(2.5, 2.75) + S(2.75, 3) = 108.5722885$
and

$$I = \int_1^3 f(x)dx = 108.5722885$$

so we get error as $0.006 < 0.01$.

5 Question 5

Use adaptive quadrature to compute the following integral with accuracy within 10^{-3}:

$$\int_0^5 2x\cos 2x - (x-2)^2\, dx$$

solution

Approximate $\int_0^5 2x\cos 2x - (x-2)^2\, dx$ using adaptive quadrature with $\epsilon = 10^{-3}$
First we check for $[0, 5]$ tolerance, we use simpson's rule

$$\begin{aligned}
\mathbf{I}_1 = S(a,b) = S(0,5) &= \frac{5}{6}\big[f(0) + 4f(2.5) + f(5)\big] \\
&= -13.93123 \\
\mathbf{I}_2 = S(0,2.5) + S(2.5,5) &= \frac{5}{12}\big[f(0) + 4f(1.25) + 2f(2.5) + 4f(3.75) + f(5)\big] \\
&= -12.98603.
\end{aligned}$$

$|\mathbf{I}_2 - \mathbf{I}(f)| \approx \frac{1}{15}|\mathbf{I}_1 - \mathbf{I}_2| = 0.063 > 10^{-3}$
So we check for $[0.2, 5]$ and $[2.5, 5]$ independently with tolerance , $\frac{\epsilon}{2}$

check for $[0, 2.5]$

$$\mathbf{I}_1 = S(0, 2.5) = -5.45547$$
$$\mathbf{I}_2 = S(0, 1.25) + S(1.25, 2.5) = -5.48316$$

$|\mathbf{I}_2 - \mathbf{I}(f)| \approx \frac{1}{15}|\mathbf{I}_2 - \mathbf{I}_1| = 0.0018 > 0.0005$
So we apply Simpson's rule for interval $[0, 1.25]$ and $[1.25, 2.5]$. If we keep on going in this way we can confirm that for subinterval $[0, 0.625]$ with corresponding tolerance, $\frac{\epsilon}{8} = 0.000125$ satisfies for $[0, 0.625]$. We get

$$\mathbf{I}_1 = S(0, 0.625) = -1.54788$$

$$\mathbf{I}_2 = S(0, 0.3125) + S(0.3125, 0.625) = -1.54926$$

Clearly $\frac{|\mathbf{I}_1 - \mathbf{I}_2|}{15} = 1.213 \times 10^{-5} < 0.000125$.
Similarly we can also confirm that for sub-intervals
$[0.625, 1.25]$;$[1.25, 1.875]$; $[1.875, 2.1875]$; $[2.1875, 2.5]$; $[2..5, 3.125]$; $[3.125, 3.4375]$; $[3.4375, 3.75]$; $[3.75, 4.375]$; $[4.375, 4.6875]$; and $[4.6875, 5]$ satisfy for their corresponding tolerances.
So,
$$\int_0^5 f(x)\,dx = \int_0^5 \left[2x\cos 2x - (x-2)^2\right] dx \text{ is } \approx \text{ sum of } \mathbf{I}'s$$
of above subintervals which satisfy their corresponding tolerances such that occuracy of our approximation is within 10^{-3}.
So,

$$\begin{aligned}\int_0^5 f(x)\,dx &\approx -1.54926 + (-1.12907) + (-1.96954) + (-0.75058) + (-0.06526) \\ &\quad + 2.21864 + 1.4210604 + 0.56188 + (-5.40321) \\ &= -15.306116\end{aligned}$$

Then the error is
$\int_0^5 f(x)\,dx = -15.3063 - 7 < 10^{-3}$.

6 Question 6

Let $T(a, b), T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)$ be the single and double applications of the trapezoidal rule to$\int_a^b f(x)\,dx$. Derive the relationship between

$$T(a, b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)$$

and
$$\int_a^b f(x)\,dx - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)$$

solution

From the composite trapezoidal rule, we have

$$\int_a^b f(x)\,dx = \frac{h}{2}\big[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\big] - \frac{b-a}{12}h^2 f''(\mu)$$

for some $\mu \in (a,b)$ and $h = \frac{b-a}{n}$
In this question we have $[a, \frac{a+b}{2}]$, $[\frac{a+b}{2}, b] \implies n = 2$.
Since

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{1}{2}\big(\frac{b-a}{2}\big)\big[f(a) + 2f(\frac{a+b}{2}) + f(b)\big] - \frac{1}{2^2}\frac{(b-a)^3}{12}f''(\mu)\\ &= T(a, \frac{a+b}{2}) + T(\frac{a+b}{2}, b) - \frac{1}{4}\frac{(b-a)^3}{12}f''(\mu)\end{aligned}$$

$$\Big|\int_a^b f(x)\,dx - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)\Big| = \frac{1}{4}\frac{(b-a)^3}{12}|f''(\mu)| \tag{1}$$

Now let us go to adaptive quadrature method method we know that by putting $n = 1$ in composite Trapezoidal rule, we get

$$\begin{aligned}\int_a^b f(x)\,dx &= \frac{h}{2}\big[f(a) + f(b)\big] - \frac{(b-a)^3}{12}f''(\mu')\\ &= T(a,b) - \frac{(b-a)^3}{12}f''(\mu')\end{aligned}$$

Thus we have

$$T(a,b) - \frac{(b-a)^3}{12}f''(\mu') = T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b) - \frac{1}{4}\frac{(b-a)^3}{12}f''(\mu)$$

Take the opproximation $f''(\mu') \approx f''(\mu)$ the strength of our result depends on the validity of this approximation, we get

$$T(a,b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b) = \frac{3}{4}\frac{(b-a)^3}{12}f''(\mu) \tag{2}$$

Thus from equation (1) and (2) we get

$$\big|T(a,b) - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)\big| \approx 3\Big|\int_a^b f(x)\,dx - T(a, \frac{a+b}{2}) - T(\frac{a+b}{2}, b)\Big|$$

7 Question 7

Approximate the integrals using Gaussian quadrature with ($n = 2$) and compare from results to the exact values of the integrals.

$$1) \int_1^{1.5} x^2 \log x\,dx \qquad\qquad 2) \int_0^1 x^2 e^{-x}\,dx$$

solution

Gaussian quadrature rule for $n = 2$,

$$\int_{-1}^{1} f(x)\,dx \approx f(\frac{-\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3}) \tag{1}$$

First we need to transform our integral $\int_a^b f(x)\,dx$ into an integral defined over $[-1, 1]$.Let

$$x = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)t$$

$$dx = \frac{(b-a)}{2}dt$$

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{-1}^{1} f\big(\frac{1}{2}(a+b) + \frac{1}{2}(b-a)t\big)\big(\frac{b-a}{2}\big)\,dt \\
\int_1^{1.5} f(x)\,dx &= \int_{-1}^{1} f\big(\frac{1}{2}(1.5+1) + \frac{1}{2}(1.5-1)t\big)\big(\frac{1.5-1}{2}\big)\,dt \\
&= \frac{1}{4}\int_{-1}^{1} f\big(\frac{t+5}{4}\big)\,dt \\
&= \frac{1}{4}\int_{-1}^{1} \big(\frac{t+5}{4}\big)^2 \log\big(\frac{t+5}{4}\big)\,dt \\
&Now using (1) \\
&= \frac{1}{4}\big[\big(\frac{\frac{-\sqrt{3}}{3}+5}{4}\big)^2 \log\big(\frac{\frac{-\sqrt{3}}{3}+5}{4}\big)\big] + \big[\big(\frac{\frac{\sqrt{3}}{3}+5}{4}\big)^2 \log\big(\frac{\frac{\sqrt{3}}{3}+5}{4}\big)\big] \\
&= 0.1922687
\end{aligned}$$

Exact value of the integral

$$\int_1^{1.5} x^2 \log x\,dx = 0.1922594$$

$$|\text{estimate value - exact value}| = 9.3 \times 10^{-6}$$

2) $\int_0^1 x^2 e^{-x}\,dx$, using the similar process as in (I) we get

$$\text{estimate value} = 0.1593326$$

and

$$\text{Exact value } = 0.0012702$$

$$|\text{estimate value - exact value}| = 0.0012702.$$

8 Question 8

Approximate the integrals using Gaussian quadrature with $n = 4$ and compare from results to the exact values of the integrals.
I) $\int_1^{1.5} x^2 \log x \, dx$

solution

1) For interval $[-1, 1]$, $n = 4$, by Gaussian quadrature

$x_i,$	$\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$	$-\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$	$\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$	$-\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$
$C_i,$	$\frac{18 + \sqrt{30}}{36}$	$\frac{18 + \sqrt{30}}{36}$	$\frac{18 - \sqrt{30}}{36}$	$\frac{18 - \sqrt{30}}{36}$

1)

$$\int_1^{1.5} x^2 \log x \, dx = \int_{-1}^{1} (\frac{t+5}{4})^2 \log (\frac{t+5}{4}) \, dx$$

By sustituting $x = \frac{t+5}{4}$.
Actual value of the integral calculated using $\int_1^{1-5} f(x)g(x) \, dx$ is $\frac{x^3}{9}(3 \log x - 1) + c$. Actual ans from 1 to 1.5 = 0.1922593577
Using Gaussian quadrature in interval $[-1, 1]$ for transformed integral

$$\text{approx value} = \sum_{i=1}^{4} \frac{c_i}{4} \left(\frac{x_i+5}{4}\right)^2 \log \left(\frac{x_i + 5}{4}\right) = 0.1922593578$$

2)

$$\int_0^1 x^2 e^{-x} \, dx = \int (\frac{t+1}{2})^2 e^{-\frac{t+1}{2}} \frac{dt}{2}$$

by substituting $x = \frac{t+1}{2}$,

Actual value of integral calculated using $\int f(x)g(x) \, dx$ is $-(x^2+2x+2)e^{-x}+c$
Actual answer from 0 to 1 = 0.160602794
Using Gaussian quadrature in interval $[-1, 1]$

$$\sum_{i=1}^{4} \frac{c_i}{2} (\frac{x_i + 1}{2})^2 e^{-\left(\frac{x_i+1}{2}\right)} = 0.160602777$$

Note:- we could have also solved for $\{x_i\}_{i=1}^4$ $\{c_i\}_{i=1}^4$ for both functions.

9 Question 9

Determine constants a, b, c, d that produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3.

solution

For $f(x) = 1$ we have

$$2 = a + b$$

For $f(x) = x$ we have

$$0 = -a + b + c + d$$

For $f(x) = x^2$

$$2/3 = a + b - 2c + 2d$$

For $f(x) = x^3$

$$0 = -a + b + 3c + 3d$$

We solve this system of equations to get:

$$a = 1$$

$$b = 1$$

$$c = 1/3$$

$$d = 1/3$$

MA 214

Spring-2022

Tutorial 8

Problem 1: *Use Taylor polynomial P_4 and composite Simpson's rule with $n = 6$ to approximate the improper integral* $\displaystyle\int_0^1 \frac{e^{2x}}{\sqrt[5]{x^2}}\,dx$.

Solution: Let $f(x) = \frac{e^{2x}}{\sqrt[5]{x^2}}$ has a singularity at $x = 0$, it is an indefinite integral.

We use $P_4(x)$ to approximate e^{2x}

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{e^{2x} - P_4(x)}{\sqrt[5]{x^2}}\,dx + \int_0^1 \frac{P_4(x)}{\sqrt[5]{x^2}}\,dx = I_1 + I_2$$

where $P_4(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!}$

Now computting $I_2 = \displaystyle\int_0^1 \frac{P_4(x)}{\sqrt[5]{x^2}}\,dx = 4.20119553$

We compute I_1 using Simpson's one-third rule.

Let $G(x) = \frac{e^{2x} - P_4(x)}{\sqrt[5]{x^2}}$.

Now , $\lim_{x\to 0} G(x) = 0$

$$\int_0^1 G(x)\,dx \approx \frac{1}{18}\{G(0) + 2[G(x_2) + G_(x_4)] + 4[g(x_1) + G(x_3) + G(x_5)] + G(1)\} = 0.11782813$$

$$\therefore I = I_1 + I_2 = 4.31902366$$

Problem 2: *Use Taylor polynomial P_4 and composite Simpson's rule with $n = 6$ to approximate the improper integral* $\int_0^1 \frac{cos2x}{x^{\frac{1}{3}}}\,dx$.

Solution: Let $f(x) = \frac{cos2x}{x^{\frac{1}{3}}}$, it has a singularity at $x = 0$. Since we are using Simpson's rule, we use $P_4(x)$ to approximate $cos2x$.

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{cos2x - P_4(x)}{x^{\frac{1}{3}}}\,dx + \int_0^1 \frac{P_4(x)}{x^{\frac{1}{3}}}\,dx = I_1 + I_2$$

Here $P_4(x) = 1 - 2x^2 + (2/3)x^4$
$I_2 = \int_0^1 \frac{P_4(x)}{x^{\frac{1}{3}}}\,dx = 25/28 = 0.892857...$

Let $G(x) = cos(2x) - 1 + 2x^2 - (2/3)x^4$, and $I_1[0,1] = [0,1/3] \cup [1/3,2/3] \cup [2/3,1]$
So $I_1 = \sum_{i=1}^{3} I_i^{'}$ and
$I_1^{'} = (1/18)[G(0) + 4G(1/6) + G(1/3)]$
$I_2^{'} = (1/18)[G(1/3) + 4G(1/2) + G(2/3)]$
$I_3^{'} = (1/18)[G(2/3) + 4G(5/6) + G(1)]$
This gives
$I_1 = I_1^{'} + I_2^{'} + I_3^{'} = -0.00632808$
And $I = I_1 + I_2 = 0.88652905$

Problem 3: *Approximate the valie of the improper integral* $\int_1^\infty x^{\frac{-3}{2}} sin\frac{1}{x} dx.$

Solution: Given I = $\int_1^\infty x^{\frac{-3}{2}} sin\frac{1}{x} dx$
Put $t = 1/x \implies dt = \frac{-1}{x^2}dx \implies dx = -x^2 dt = \frac{-1}{t^2}dt$
Thus $x = 1 \to t = 1, x = \inf \to t = 0$

$$\begin{aligned} I &= \int_1^0 t\sqrt{t} sin(t)(\frac{-1}{t^2})dt \\ &= \int\limits_0^1 \frac{1}{\sqrt{t}} sin\, t\ dt \\ &= \int_0^1 f(t)\, dt \end{aligned}$$

where $f(t) = \frac{g(t)}{(t-a)^p}$; thus $g(t) = sin\, t, a = 0, p = 1/2$.

Therefore

$$I = \int_0^1 f(t)\, ft = \int_0^1 \frac{(f(t) - P_4(t))}{t^{1/2}} dt + \int_0^1 \frac{P_4(t)}{t^{1/2}}$$

where

$$P_4(t) = \sum_{i=0}^{4} \frac{g(0)}{i!}(t-0)^i = t - \frac{t^3}{6}$$

$$T_2 = \int_0^1 \frac{P_4(t)}{t^{1/2}}\, dt = \frac{13}{21}$$

Define $G(t) = \begin{cases} \frac{sin\, t - t + \frac{t^3}{6}}{t^{1/2}} & 0 < t \leq 1 \\ 0 & otherwise \end{cases}$ Using the composite Simpson's Rule,

$$T_1 = \int_0^1 G(t)dt \approx \frac{h}{3}(G(0) + 4G(1/4) + 2G(1/2) + 4G(1/3) + G(4))$$

Thus

$$\begin{aligned} T_1 &= \frac{1}{12}(0 + 4 * 0.0.00001625184 + 2 * 0.00036610020 + \\ &\quad 4 * 0.00225312100 + 0.00813765147) \\ &= 0.001496 \end{aligned}$$

Thus $I = T_1 + T_2 = 13/21 + 0.001496 = 0.620543$

Problem 4: *Use Euler's method with $h = 0.25$ to approximate the solution for the intial-value problem: $y' = 1 + (t-y)^2,\ 2 \le t \le 3$ and $y(2) = 1$. Compare the results with $y(t) = t + \frac{1}{1-5}$.*

Solution: Here $f(t,y) = 1 + (t-y)^2$ wherin $t_0 = 2$ and $w_0 = 1$

$t_1 = t_0 + h = 2.25$
$w_1 = w_0 + hf(t_0, w_0) = 1 + (0.25)(2) = 1.5$

$t_2 = t_0 + 2h = 2.5$
$w_2 = w_1 + hf(t_1, w_1) = 1.8906$

$t_3 = t_0 + 3h = 2.75$
$w_3 = w_2 + hf(t_2, w_2) = 2.2334$

$t_4 = t_0 + 4h = 3$
$w_4 = w_3 + hf(t_3, w_3) = 2.5501$

Thus we have the following approximations

$$\begin{aligned} y(2.25) &\approx w_1 = 1.15 \\ y(2.5) &\approx w_2 = 1.8906 \\ y(2.75) &\approx w_3 = 2.2334 \\ y(3) &\approx w_4 = 2.5501 \end{aligned}$$

We compare the above valies with the actual values given by $y(t) = t + 1/(1-t)$

t_i	w_i	y_i	$\|y_i - w_i\|$
2	1	1	0
2.25	1.5	1.45	0.05
2.5	1.8906	1.8333	0.0573
2.75	2.2334	2.1786	0.0548
3	2.5501	2.5	0.0501

Problem 5: *Use Euler's method with $h = 0.25$ to approximate the solution for the initial value problem: $y' = cos2t + sint2t, 0 \leq t \leq 1$ and $y(0) = 1$. Compare the results with $y(t) = \frac{1}{2}sin2t - \frac{1}{2}cos\,2t + \frac{3}{2}$.*

Solution:

$$y' = cos(2t) + sin(2t), 0 <= t <= 1$$

$$y(0) = 1$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

$$w_0 = y(0) = 1$$

$$f(t, y) = cos(2t) + sin(2t)$$

$t_0 = 0$, $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, $t_4 = 1$

Using 5 digit chopping :

$$w_1 = 1 + 0.25f(0, 1) = 1 + 0.25 * 1 = 1.25$$

$$w_2 = 1.25 + 0.25f(0.25, 1.25) = 1.25 + 0.25 * 1.35700 = 1.58925$$

$$w_3 = 1.58925 + 0.25f(0.5, 1.58925) = 1.58925 + 0.25 * 1.38177 = 1.93469$$

$$w_4 = 1.93469 + 0.25f(0.75, 1.93469) = 1.93469 + 0.25 * 1.06823 = 2.20174$$

Comparing with

$$y(t) = \frac{sin(2t)}{2} - \frac{cos(2t)}{2} + \frac{3}{2}$$

$$y(0) = 1.0 \ (w_0 = 1)$$

$$y(0.25) = 1.30092 \ (w_1 = 1.25)$$

$$y(0.5) = 1.65058 \ (w_2 = 1.58925)$$

$$y(0.75) = 1.96337 \ (w_3 = 1.93469)$$

$$y(1) = 2.16272 \ (w_4 = 2.20174)$$

Relative error in approximating $y(1) = \frac{|2.20174 - 2.16272|}{2.16272} = 1.8\%$

Problem 6: *Consider the initial-value problem:* $y' = -10y, \ 0 \leq t \leq 2$, $y(0) = 1$ *with solution* $y(t) = e^{-10t}$. *What happens when Euler's method is applied to this problem with* $h = 0.1$*? Does this behavior violate the erroe bound?*

Solution: For the given initial value problem, by Euler's Method
$w_{i+1} = w_i + hf(t_i, w_i)$ which implies $w_{i+1} = w_i(1 - 10h)$
Thus $w_n = w_0(1-10h)^n$. But since $h = 0.1$, all the $w_n = 0$ for $n = 1, 2, \ldots, 20$
The given $f(t, y)$ satisfies the Lipschitz condition as $|f(t, y_2) - f(t, y_1)| = |10(y_1 - y_2)| \leq 10|y_1 - y_2| \implies L = 10$ which is our Lipschitz constant.
The error bound between $y(t_i)$ and w_i is given by

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left(e^{L(t_i-a)} - 1\right)$$

where $h = 0.1, L = 10, M = \underset{t\in[0,2]}{max}|y''(t)| = 100$

For $i = 0$ this relation holds as $|w_0 - w_0| = 0$
For $1 \leq i \leq 20$, let us look at the error inequality
$t_i = 0 + ih = 0.1i$
$y(t_i) = e^{-10t_i} = e^{-1} > 0$ for $i \in \{1, 2, \ldots, 20\}$, $\ w_i = 0$; $\ e^{L(t_i-a)} = e^i$
For $i > 1$, $e^i - 1 > 0 \implies 0.5e^i - 0.5 - e^{-i} \geq 0$ has to hold for error bound to be accurate.
Let $g(i) = \frac{e^i}{2} - e^{-i} - 0.5$
$g(1) = 0.49$ and $g'(i) > 0$ for $i = 1, 2, \ldots, 20$. Therefore $g(i) > 0$ for all $1 \leq i \leq 20$
Thus the error bound holds for all $i = 0, 1, 2, \ldots 20$.

Problem 7: *Use Taylor's Method of order 2 and 4 with* $h = 0.25$ *to approximate the solution for the intial value problem:* $y' = 1 + \frac{y}{t},\ 1 \le t \le 2, y(1) = 2$

Solution: Given $h = 0.25$, and the initial value problem $y' = 1 + \frac{y}{t}$, we are required to approximate the solution for $1 \le t \le 2$. So $f(t, y(t)) = 1 + \frac{y}{t}$ and $w_{i+1} = w_i + hT(t_i, w_i)$

Part 1: $\mathbf{n = 2}$

$$T(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$
$$f'(t, y(t)) = \frac{ty' - y}{t^2} = \frac{t\left(1 + \frac{y}{t}\right) - y}{t^2} = \frac{1}{t}$$

So, we get the following expression for $T(t_i, w_i)$,

$$T(t_i, w_i) = 1 + \frac{w_i}{t_i} + \frac{h}{2}\frac{1}{t_i}$$

So we have $w_{i+1} = 0.25 + w_i\left(1 + \frac{0.25}{t_i}\right) + \frac{0.0625}{2t_i}$ Thus, we get the following values (with $w_0 = y(1) = 2$),

t_i	w_i
1	2
1.25	2.78125
1.5	3.6125
1.75	4.4854
2	5.394

Part 2: $n = 4$

$$T(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i)$$

From previous part, $f'(t, y(t)) = \frac{1}{t}$, so we have, $f''(t, y(t)) = -\frac{1}{t^2}$ and $f'''(t, y(t)) = \frac{2}{t^3}$. Using these expressions in the above equation, we get,

$$T\left(t_i, w_i\right) = 1 + \frac{w_i}{t_i} + \frac{h}{2}\frac{1}{t_i} - \frac{h^2}{6}\frac{1}{t_i^2} + \frac{h^3}{12}\frac{1}{t_i^3}$$

So we have, $w_{i+1} = 0.25 + w_i\left(1 + \frac{0.25}{t_i}\right) + \frac{0.25^2}{2t_i} - \frac{0.25^3}{6t_i^2} + \frac{0.25^4}{12t_i^3}$ Thus we get the following values with $w_0 = y(1) = 2$.

t_i	w_i
1	2
1.25	2.77897
1.5	3.60826
1.75	4.47941
2	5.38639

Problem 8: *Use Taylor's Method of order 2 and 4 with $h = 0.25$ to approximate the solution for the intial value problem: $y' = cos2t + sin3t,\ 0 \leq t \leq 1, y(0) = 1$*

Solution: Taylor's Method of Order 2
$y(x_i + h) = y(x_0) + hy'(x_i) + \frac{h^2}{2}y''(x_i)$ where $x_0 = 0, h = 0.25,$
$x_{i+1} = x_i + h, y'' = -2sin2t + 3cos3t$

Thus $y(0.25) = y(0) + hy'(0) + \frac{h^2}{2}y''(0) = 1.34375$

Now for $x_0 = 0.25$
$y(0.5) = y(0.25) + hy'(0.25) + \frac{h^2}{2}y''(0.25) = 1.7722$

Now for $x_0 = 0.5$
$y(0.75) = y(0.5) + hy'(0.5) + \frac{h^2}{2}y''(0.5) = 2.1107$

Now for $x_0 = 0.75$
$y(1) = y(0.75) + hy'(0.75) + \frac{h^2}{2}y''(0.75) = 2.2017$

Taylor's Method of Order 2
$y(x_i + h) = \sum_{i=0}^{4} \frac{h^i}{i!}y^{(i)}(x_i)$ where $h = 0.25$, $x_{i+1} = x_i + h$ and $x_0 = 0$.
Thus we have

$$\begin{aligned} y(0.25) &= 1.3289 \\ y(0.5) &= 1.7296 \\ y(0.75) &= 2.0372 \\ y(1) &= 2.1133 \end{aligned}$$

Problem 9: *Use Taylor's method of order 2 and 4 with $h = 0.5$ to approximate the solution for the initial-value problem: $y' = te^{3t} - 2y$, $0 \leq t \leq 1$, $y(0) = 0$.*

Solution: Given $h = 0.5$, $y' = te^{3t} - 2y$, $0 \leq t \leq 1$, $y(0) = 0$
Let $w_{i+1} = w_i + hT(t_i, w_i)$
For $n = 2$,
$T(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) = t_i e^{3t_i} - 2w_i + \frac{h}{2}(e^{3t_i} + t_i e^{3t_i} + 4y)$
This gives $w_{i+1} = \frac{1}{2}t_i e^{3t_i} + \frac{1}{8}e^{3t_i} + \frac{1}{8}t_i e^{3t_i} + \frac{w_i}{2}$

t_i	w_i
0	0.125
0.5	2.02324
1	16.07572

For $n = 4$,

$$
\begin{aligned}
T(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \frac{h^2}{6}f''(t_i, w_i) + \frac{h^3}{24}f'''(t_i, w_i) \\
&= t_i e^{3t_i} - 2w_i + \frac{0.5}{2}(e^{3t_i} + t_i e^{3t_i} + t_i w_i) + \\
&\frac{(0.5)^2}{6}(4e^{3t_i} + 7t_i e^{3t_i} - 8w_i) + \frac{(0.5^3)}{24}(19e^{3t_i} + 13t_i e^{3t_i} 16w_i)
\end{aligned}
$$

t_i	w_i
0	0.765625
0.5	9.535
1	35.7840877

Answers

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April 2022

1 Questions

Question 1:-Use the Modified Euler method with $h = 0.25$to approximate the solution to the initial-value problem:

$$y^{'} = 1 + (t - y)^2, 2 \leq t \leq 3,$$

$y(2) = 1$ and compare the result with actual values $y(t) = t + (\frac{1}{1-t})$.
Solution:- we know that modified euler method is,

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, \tilde{w_{i+1}})]$$

where

$$\tilde{w_{i+1}} = w_i + hf(t_i, w_i), f(t, y) = 1 + (t - y)^2$$

here we have $h = 0.25$ and $t_{i+1} = t_i + h$
after putting these values in the given formula we get the following table-

$\mathbf{i}$	$\mathbf{t_i}$	$\mathbf{w_i}$	$\mathbf{exact(y_i)}$	**error**
0	2	1	1	0
1	2.25	1.44531	1.45	4.69×10^{-3}
2	2.5	1.82790	1.833333	5.43×10^{-3}
3	2.75	2.17345	2.17857	5.12×10^{-3}
4	3	2.49544	2.5	4.56×10^{-3}

2 Question

Use the midpoint method to solve for $y(t)$ such that $y' = 1+(t-y)^2$, $2 \leq t \leq 3$, $y(2) = 1$. Compare the solution with actual solution $y(t) = t + \frac{1}{1-t}$. Use $h = 0.25$.
Solution: We shall use the midpoint method formula

$$t_{i+1} = t_i + h \tag{1}$$

$$y_{i+1} = y_i + hft_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i) \tag{2}$$

with $t_0 = 2$, $y_0 = 1$, $h = 0.25$, $f(t, y) = 1 + (t - y)^2$. The table below gives the desired results.

$\mathbf{i}$	$\mathbf{t_i}$	$\mathbf{y_i}$	$\mathbf{f(t_i, y_i)}$	$\mathbf{ft_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i)}$	$\mathbf{y_{i+1}}$	$\mathbf{y_{actual} = t + \frac{1}{1-t}}$
0	2	1	2	1.875	1.46875	1
1	2.25	1.46875	1.610352	1.704956	1.894989	1.45
2	2.5	1.894989	1.366038	1.559256	2.284803	1.833333
3	2.75	2.284803	1.216408	1.438146	2.64434	2.178571
4	3	2.64434				2.5

3 Question 3

Repeat the above problem using the Runge-Kutta method of order four.
Solution: we know that Runge -Kutta method of order 4 is-

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(t_i, w_i)$$
$$k_2 = hf(t_i + \frac{h}{2}, w_i + \frac{k_1}{2})$$
$$k_3 = hf(t_i + \frac{h}{2}, w_i + \frac{k_2}{2})$$
$$k_4 = hf(t_i + h, w_i + k_3)$$

let us calculate w_1, we know that

$$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

so we have to calculte k_1, k_2, k_3 and k_4 Now for $i = 0$ $t_0 = 2$,$h = 0.25$,$w_0 = y(2) = 1$ $f(t, y) = 1 + (t - y)^2$.
therefore for k_1 we have

$$\begin{aligned} k_1 = hf(t_0, w_0) &= h[1 + (t_0 - w_0)^2] \\ &= 0.25[1 + (2 - 1)^2] \\ &= 0.5 \end{aligned}$$

for k_2 we have

$$\begin{aligned} k_2 = hf(t_0 + \frac{h}{2}, w_0 + \frac{k_1}{2}) &= h[1 + ((t_0 + \frac{h}{2}) - (w_0 + \frac{k_1}{2}))^2] \\ &= 0.25[1 + ((2 + \frac{0.25}{2}) - (1 + \frac{0.5}{2}))^2] \\ &= 0.25[1 + 0765625] \\ &= 0.4414062 \end{aligned}$$

for k_3 we have

$$k_3 = hf(t_0 + \frac{h}{2}, w_0 + \frac{k_2}{2} = 0.25f(2.125, 1.22070315)$$
$$= 0.25 \times 1.8177528381$$
$$= 4544382095$$

for $k4$ we have

$$k_4 = hf(t_0 + h, w_0 + k_3) = 0.25f(2.25, 1.4544382095)$$
$$= 0.25 \times 1.6329185624$$
$$= 0.4082296406$$

Now we can calculate w_1 as we know that

$$w_1 = w_0 + \frac{h}{6}(k_1 + 2K_2 + 2k_3 + k_4)$$

therefor we get

$$w_1 = 1 + \frac{0.25}{6}(0.5 + 2 \times 0.44140625 + 2 \times 0.4544382095 + 0.4082296406 = 1.4499864266$$

similarly we proceed for w_2,here is table for other numerical value of w_i

$\mathbf{i}$	$\mathbf{t_i}$	$\mathbf{w_i byR-K}$	$\mathbf{exact(y_i)}$	$\mathbf{error \times 10^{-5}}$
0	2	1	1	0
1	2.25	1.44999	1.45	1
2	2.5	1.83332	1.833333	1
3	2.75	2.17856	2.17857	1
4	3	1.49999	2.5	1

Note:here we can see ,we have more accuracy than previous both method.

4 Question 4

Given the linear system:

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

(a) find the values of α for which the system has no solution.
(b)find the values of $\alpha$for which there are infinitely many solutions.
(c) assuming that a unique solution exists for a given α, find the solution in terms of α.
Solution: Augmented matrix related to the given linear system is following.

$$A_1 = \begin{bmatrix} 2 & -6\alpha & 3 \\ 3\alpha & -1 & \frac{3}{2} \end{bmatrix}$$

changing $R_2 \Longrightarrow R_2 - \frac{3\alpha}{2} R_1$ in A ,we get following matrix

$$A_2 = \begin{bmatrix} 2 & -6\alpha & 3 \\ 0 & -1+9\alpha^2 & \frac{3}{2} - \frac{9}{2}\alpha \end{bmatrix}$$

Now,whether given system has no solution or infinite solution ,it depends on determinant

$$\Delta = \begin{pmatrix} 2 & -6\alpha \\ 3\alpha & -1 \end{pmatrix}$$

has determinant 0 or not.
since $\Delta = -2 + 18\alpha^2$,if $\Delta = 0$ we get $\alpha = \pm\frac{1}{3}$.
So at $\alpha = \pm\frac{1}{3}$ we either get infinite solution or no solution.
(a) Take $\alpha = -\frac{1}{3}$.Then matrix A_2 become

$$A_2 = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Here we get equation $2x_1 + 2x_2 = 3$ and $0 = 3$ which is not possible.
Hence we find that at $\alpha = -\frac{1}{3}$ there is no solution.
(b) Take $\alpha = \frac{1}{3}$.Then A_2 become-

$$A_2 = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have equation $2x_1 - 2x_2 = 3$,only one equation having two variable,putting any value for first variable we can get some value of other variable satisfying the equation ,Hence it has infinitely many solution.
(c) if $\Delta \neq 0$ then given system has unique solution that is if $\alpha \neq \pm\frac{1}{3}$.
After solving the given system we get-

$$x_1 = \frac{3}{2(1+3\alpha)}$$

and

$$x_2 = -\frac{3}{2(1+3\alpha)}$$

.

5 Question 5

(5) Use Gaußian elimination and three-digit chopping arithmetic to solve the linear system:

$$0.03x_1 + 58.9x_2 = 59.2$$

$$5.31x_1 - 6.10x_2 = 47.0$$

and compare the approximations to the actual solution $(10, 1)$.
Solution: we have the system of equation -

$$\begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 59.2 \\ 47.0 \end{bmatrix}$$

Here first pivot is $a_{11} = 0.03$ and $m_{21} = 177.0$
even after 3- digit chopping,$m_{21} = 177.0$
Now performing $(E_2 - m_2 1 E_1) \longrightarrow E_2$ with three digit chopping,we get

$$\begin{bmatrix} 0.03 & 58.9 \\ 0 & -10400 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} 59.2 \\ -10400 \end{bmatrix}$$

Now solving the above system we get $x_2 \approx 1$.
and so

$$x_1 \approx \frac{59.2 - 58.9 \times 1}{0.03} = 10.00$$

Hence the solution after three -digit chopping is the actual one, that is (10,1).

6 Question 6

(6) Solve the above linear system(5) using Gaußian elimination with partial pivoting and three-digit rounding arithmetic.
solution: Augmented matrix of the given system is.

$$\begin{bmatrix} 0.03 & 58.9 & 59.2 \\ 5.31 & -6.10 & 47.0 \end{bmatrix}$$

partially pivoting we get,

$$\begin{bmatrix} 5.31 & -6.10 & 47.0 \\ 0.03 & 58.9 & 59.2 \end{bmatrix}$$

Now performing $R_2 \longrightarrow R_2 - (\frac{0.03}{5.31})R_1$ we get,

$$\begin{bmatrix} 5.31 & -6.10 & 47.0 \\ 0.00 & 58.9 & 58.9 \end{bmatrix}$$

After solving the above equation we get $x_1 = 1$ and $x_2 = 10$,which are the actual solution.

7 Question 7

(7) For the matrix A given below, find a permutation matrix P such that PA has an LU factorization and find the LU factorization of PA:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution: Given matrix -

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

performing $(E_2 - 2E_1) \longrightarrow E_2$ we get matrix,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

interchanging E_1 and E_2 we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Now -permutation matrix associated to the row interchanging is given by,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

so

$$PA = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 4 & 0 \end{bmatrix}$$

Now we can perform $E_3 - 2E_1 \longrightarrow E_3$ to get an upper triangular matrix;

$$PA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

8 Question 8

For the matrix A given below, find a permutation matrix P such that PA has an LU factorization and find the LU factorization of PA:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Solution: in the given matrix $a_{11} = 0$,therefore interchanging the first and second row we found that
permutation matrix is ,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so ,

$$PA = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

now performing the operation $(E_2 - (\frac{0}{1})E_1) \longrightarrow E_2$and $(E_3 - E_1) \longrightarrow E_3$ we get,

$$M^{(1)}PA = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, L^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Again performing $(E_3 - 1 \times E_2) \longrightarrow E_3$ we get,

$$M^{(2)}M^{(1)}PA = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, L^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Hence we get LU factorisation pf PA where-

$$U = M^{(2)}M^{(1)}PA = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$L = L_1L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

9 Question 9

For the matrix A given below, find a permutation matrix P such that PA has an LU factorization and find the LU factorization of PA:

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution: interchanging the first and second row ,we find permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

therefore,

$$PA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

performing $(E_3 + 0.5 \times E_2) \longrightarrow E_3$ we get,

$$M^{(1)}PA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 2.5 \end{bmatrix}, L^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}$$

Hence we find PA hasLU factorisation where ,

$$U = M^{(1)}PA = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 2.5 \end{bmatrix}$$

$$L = L^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}$$

MA 214: Introduction to Numerical Analysis

Solutions to Tutorial 10

April 12, 2022

Solution to Q-1

In slides, we have derived $l_{11}^2 = a_{11}$, $l_{11}l_{21} = a_{21}$, $l_{11}l_{31} = a_{31}$, $l_{21}^2 + l_{22}^2 = a_{22}$, $l_{21}l_{31} + l_{22}l_{32} = a_{32}$, $l_{31}^2 + l_{32}^2 + l_{33}^2 = a_{33}$, and so on, where $A = LL^t$ with

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}.$$

This gives $l_{11} = 2$, $l_{21} = l_{31} = l_{41} = 1/2$. Again from $l_{21}^2 + l_{22}^2 = 3$, it follows that $l_{22} = \sqrt{11}/2$. This way, we obtain,

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0.5 & 1.6583 & 0 & 0 \\ 0.5 & -0.7538 & 1.0871 & 0 \\ 0.5 & 0.4523 & 0.0836 & 1.2403 \end{pmatrix}.$$

To get $A = \widetilde{L}D\widetilde{L}^t$, divide first coloumn of L by 2, second coloumn by 1.6583, third coloumn by 1.0871, fourth coloumn by 1.2403. This finally gives the desired decomposition with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 \\ 0.25 & -0.4546 & 1 & 0 \\ 0.25 & 0.2727 & 0.0769 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2.7500 & 0 & 0 \\ 0 & 0 & 1.1818 & 0 \\ 0 & 0 & 0 & 1.5383 \end{pmatrix}.$$

Solution to Q-2

Let $A = \begin{pmatrix} 4 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 5 & 2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$. Write $A = LDL^t$ with

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix}, \quad D = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}.$$

Then

$$LDL^t = \begin{pmatrix} x_1 & ax_1 & bx_1 & dx_1 \\ ax_1 & a^2x_1 + x_2 & abx_1 + cx_2 & adx_1 + ex_2 \\ bx_1 & abx_1 + cx_2 & b^2x_1 + c^2x_2 + x_3 & bdx_1 + cex_2 + fx_3 \\ dx_1 & adx_1 + ex_2 & bdx_1 + cex_2 + fx_3 & d^2x_1 + e^2x_2 + f^2x_3 + x_4 \end{pmatrix}.$$

Comparing A with LDL^t and solving for the unknowns, we get

$$x_1 = 4, \ x_2 = 11/4, \ x_3 = 50/11, \ x_4 = 78/25,$$

and

$$a = 1/4, \ b = -1/4, \ c = -3/11, \ d = e = 0, \ f = 11/25.$$

Solution to Q-3

Given, $A_3 = \begin{pmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix}$.

We know, a matrix A is positive definite if and only if each leading principal submatrix (denoted by $LPS(A)$) of A has positive determinant.

$$LPS_1(A_3) = \begin{pmatrix} 6 \end{pmatrix} \text{ and indeed } 6 > 0; \tag{1}$$

$$LPS_2(A_8) = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \text{ and indeed } \det(LPS_2(A_3)) = 20 > 0; \tag{2}$$

$$LPS_3(A_8) = \begin{pmatrix} 6 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix} \text{ and indeed } \det(LPS_3(A_3)) = 74 > 0; \tag{3}$$

$$LPS_4(A_3) = \begin{pmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} = A_3 \text{ and indeed } \det(A_3) = 191 > 0. \tag{4}$$

Thus, from (1), (2), (3) and (4), it follows that A_3 is positive definite.

Let us obtain the LDL^t factorization of the matrix A_3.

Let $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix}$ and $D = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}$ such that $A_3 = LDL^t$.

$$A_3 = \begin{pmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{pmatrix} \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} 1 & l_{21} & l_{31} & l_{41} \\ 0 & 1 & l_{32} & l_{42} \\ 0 & 0 & 1 & l_{43} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} x_1 & x_1 l_{21} & x_1 l_{31} & x_1 l_{41} \\ x_1 l_{21} & x_1 l_{21}^2 + x_2 & x_1 l_{21} l_{31} + x_2 l_{32} & x_1 l_{21} l_{41} + x_2 l_{42} \\ x_1 l_{31} & x_1 l_{21} l_{31} + x_2 l_{32} & x_1 l_{31}^2 + x_2 l_{32}^2 + x_3 & x_1 l_{31} l_{41} + x_2 l_{32} l_{42} + x_3 l_{43} \\ x_1 l_{41} & x_1 l_{21} l_{41} + x_2 l_{42} & x_1 l_{31} l_{41} + x_2 l_{32} l_{42} + x_3 l_{43} & x_1 l_{41}^2 + x_2 l_{42}^2 + x_3 l_{43}^2 + x_4 \end{pmatrix}$$

Solving for l_{ij} and x_k we get:

$$\begin{aligned}
a_{11}: &\quad & 6 &= x_1 \\
a_{21}: &\quad & 2 &= x_1 l_{21} \implies l_{21} = 1/3 \\
a_{31}: &\quad & 1 &= x_1 l_{31} \implies l_{31} = 1/6 \\
a_{41}: &\quad & -1 &= x_1 l_{41} \implies l_{41} = -1/6 \\
a_{22}: &\quad & 4 &= x_1 l_{21}^2 + x_2 \implies x_2 = 10/3 \\
a_{32}: &\quad & 1 &= x_1 l_{21} l_{31} + x_2 l_{32} \implies l_{32} = 1/5 \\
a_{42}: &\quad & 0 &= x_1 l_{21} l_{41} + x_2 l_{42} \implies l_{42} = 1/10 \\
a_{33}: &\quad & 4 &= x_1 l_{31}^2 + x_2 l_{32}^2 + x_3 \implies x_3 = 61/18 \\
a_{43}: &\quad & -1 &= x_1 l_{31} l_{41} + x_2 l_{32} l_{42} + x_3 l_{43} \implies l_{43} = -(16)^2/610 = -128/305 \\
a_{44}: &\quad & 3 &= x_1 l_{41}^2 + x_2 l_{42}^2 + x_3 l_{43}^2 + x_4 \implies x_4 = 64/25
\end{aligned}$$

Thus,

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 1/6 & 1/5 & 1 & 0 \\ -1/6 & 1/10 & -128/305 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 10/3 & 0 & 0 \\ 0 & 0 & 61/18 & 0 \\ 0 & 0 & 0 & 64/25 \end{pmatrix}.$$

Solution to Q-4

Given,

$$A_4 = \begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 3 & -1 & 1 \\ 2 & -1 & 6 & 3 \\ 1 & 1 & 3 & 8 \end{pmatrix}.$$

Let $L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}$, such that $A_4 = LL^t$.

$$A_4 = \begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 3 & -1 & 1 \\ 2 & -1 & 6 & 3 \\ 1 & 1 & 3 & 8 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{pmatrix}$$

$$\implies \begin{pmatrix} 4 & 0 & 2 & 1 \\ 0 & 3 & -1 & 1 \\ 2 & -1 & 6 & 3 \\ 1 & 1 & 3 & 8 \end{pmatrix} = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & l_{21}l_{41} + l_{22}l_{42} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \\ l_{11}l_{41} & l_{21}l_{41} + l_{22}l_{42} & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \end{pmatrix}$$

Solving for l_{ij} we get:

$$\begin{aligned}
a_{11} : &\quad 4 = l_{11}^2 \implies l_{11} = 2 \\
a_{21} : &\quad 0 = l_{11}l_{21} \implies l_{21} = 0 \\
a_{31} : &\quad 2 = l_{11}l_{31} \implies l_{31} = 1 \\
a_{41} : &\quad 1 = l_{11}l_{41} \implies l_{41} = 0.5 \\
a_{22} : &\quad 3 = l_{21}^2 + l_{22}^2 \implies l_{22} = 1.7320508 \\
a_{32} : &\quad -1 = l_{21}l_{31} + l_{22}l_{32} \implies l_{32} = -0.57735027 \\
a_{42} : &\quad 1 = l_{21}l_{41} + l_{22}l_{42} \implies l_{42} = 0.57735027 \\
a_{33} : &\quad 6 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = 2.1602469 \\
a_{43} : &\quad 3 = l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \implies l_{43} = 1.3115784 \\
a_{44} : &\quad 8 = l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \implies l_{44} = 2.3867192
\end{aligned}$$

Thus,

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1.7320508 & 0 & 0 \\ 1 & -0.57735027 & 2.1602469 & 0 \\ 0.5 & 0.57735027 & 1.3115784 & 2.3867192 \end{pmatrix}.$$

Solution to Q-5

Given,

$$A_5 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix}.$$

Let $L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}$, such that $A_5 = LL^t$

$$A_5 = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{pmatrix}$$

$$\implies \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 3 & 0 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & l_{21}l_{41} + l_{22}l_{42} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \\ l_{11}l_{41} & l_{21}l_{41} + l_{22}l_{42} & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \end{pmatrix}$$

Solving for l_{ij} we get:

$$\begin{aligned}
a_{11} :&\quad 4 = l_{11}^2 \implies l_{11} = 2 \\
a_{21} :&\quad 1 = l_{11}l_{21} \implies l_{21} = 1/2 \\
a_{31} :&\quad 1 = l_{11}l_{31} \implies l_{31} = 1/2 \\
a_{41} :&\quad 1 = l_{11}l_{41} \implies l_{41} = 1/2 \\
a_{22} :&\quad 3 = l_{21}^2 + l_{22}^2 \implies l_{22} = \sqrt{11}/2 \\
a_{32} :&\quad 0 = l_{21}l_{31} + l_{22}l_{32} \implies l_{32} = -1/2\sqrt{11} \\
a_{42} :&\quad -1 = l_{21}l_{41} + l_{22}l_{42} \implies l_{42} = -5/2\sqrt{11} \\
a_{33} :&\quad 2 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = \sqrt{19}/\sqrt{11} \\
a_{43} :&\quad 1 = l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43} \implies l_{43} = 7/\sqrt{11}\sqrt{19} \\
a_{44} :&\quad 4 = l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2 \implies l_{44} = \sqrt{56}/\sqrt{19}.
\end{aligned}$$

Solution to Q-6

Given a system of linear equations

$$\begin{aligned} 2x_1 - x_2 &= 3 \\ -x_1 + 2x_2 - x_3 &= -3 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

So the coefficient matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Write $A = LDL^t$ with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad D = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}.$$

Then

$$LDL^t = \begin{pmatrix} x_1 & ax_1 & bx_1 \\ ax_1 & a^2x_1 + x_2 & abx_1 + cx_2 \\ bx_1 & abx_1 + cx_2 & b^2x_1 + c^2x_2 + x_3 \end{pmatrix}.$$

Comparing A with LDL^t and solving for the unknowns, we get

$$x_1 = 2,\ x_2 = 3/2,\ x_3 = 4/3,$$

and

$$a = -1/2,\ b = 0,\ c = -2/3.$$

Now consider $LDL^t\mathbf{x} = \mathbf{b}$, with L and D as above and $\mathbf{b} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$.

The equation $L\mathbf{y} = \mathbf{b}$ gives $\mathbf{y} = \begin{pmatrix} 3 \\ -3/2 \\ 0 \end{pmatrix}$. Again, $D\mathbf{z} = \mathbf{y}$ gives $\mathbf{z} = \begin{pmatrix} 3/2 \\ -1 \\ 0 \end{pmatrix}$. Now finally, $L^t\mathbf{x} = \mathbf{z}$ gives $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Solution to Q-7

Given a system of linear equations $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}$.

Let $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$.

So $A = LL^t = \begin{pmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{31}l_{21} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{pmatrix}$.
Comparing values, we get the following

- $l_{11}^2 = 2 \implies l_{11} = \sqrt{2}$
- $l_{11}l_{21} = -1 \implies l_{21} = -\frac{1}{\sqrt{2}}$
- $l_{11}l_{31} = 1 \implies l_{31} = 0$
- $l_{21}^2 + l_{22}^2 = 2 \implies l_{22} = \sqrt{\frac{3}{2}}$
- $l_{22}l_{32} = -1 \implies l_{32} = -\sqrt{\frac{2}{3}}$
- $l_{32}^2 + l_{33}^2 = 2 \implies l_{33} = \sqrt{\frac{4}{3}}$

So $L = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{2}{3}} & \sqrt{\frac{4}{3}} \end{pmatrix}$. Let $L^t\mathbf{x} = \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. We have $L\mathbf{c} = \mathbf{b}$.

Using substitution, we get $\mathbf{c} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}} \\ 0 \end{pmatrix}$. Also we have $L^T\mathbf{x} = \mathbf{c}$. Again using substitution we get $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Solution to Q-8

Given, $A_8 = \begin{pmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix}$.

We know, a matrix A is positive definite if and only if each leading principal submatrix (denoted by $LPS(A)$) of A has positive determinant.

$$LPS_1(A_8) = \begin{pmatrix} 2 \end{pmatrix} \text{ and indeed } 2 > 0; \tag{5}$$

$$LPS_2(A_8) = \begin{pmatrix} 2 & \alpha \\ \alpha & 2 \end{pmatrix} \implies 4 - \alpha^2 > 0 \implies -2 < \alpha < 2; \tag{6}$$

$$LPS_3(A_8) = \begin{pmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \implies \det(A_8) = -2(2\alpha - 3)(\alpha + 2) > 0 \implies -2 < \alpha < 3/2. \tag{7}$$

Thus, from (1), (2) and (3), we get $-2 < \alpha < 3/2$.

Solution to Q-9

Given, $A_9 = \begin{pmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix}$.

We know, a matrix A is positive definite if and only if each leading principal submatrix (denoted by $LPS(A)$) of A has positive determinant.

$$LPS_1(A_9) = \begin{pmatrix} \alpha \end{pmatrix} \implies \alpha > 0; \tag{8}$$

$$LPS_2(A_9) = \begin{pmatrix} \alpha & 1 \\ 1 & 2 \end{pmatrix} \implies 2\alpha - 1 > 0 \implies \alpha > 1/2; \tag{9}$$

$$LPS_3(A_9) = \begin{pmatrix} \alpha & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 4 \end{pmatrix} \implies 7\alpha - 5 - 3 > 0 \implies \alpha > 8/7. \tag{10}$$

Thus, from (1), (2) and (3), we get $\alpha > 8/7$.