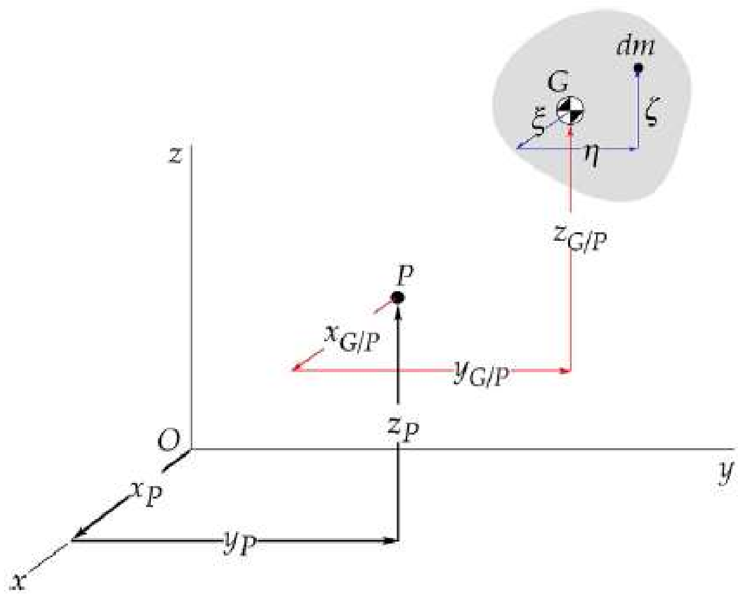
Lecture 27

Parallel Axis Theorem



$- H_P)_{rel} = I_P \omega$

$- x = x_{G/P} + \xi, \; y = y_{G/P} + \eta, \; z = z_{G/P} + \zeta$

$- H_P)_{rel} = H_G + \underbrace{r_{G/P} \times m v_{G/P}}_{H_P^{(m)})_{rel}}$

$- H_P^{(m)})_{rel} = I_P^{(m)} \omega$

$$\begin{bmatrix} m\left(y_{G/P}^2 + z_{G/P}^2\right) & -m x_{G/P} y_{G/P} & -m x_{G/P} z_{G/P} \end{bmatrix}$$

$$\left[\mathbf{I}_P^{(m)}\right] = \begin{bmatrix} \cdots & \cdots & \cdots \\ -mx_{G/P}y_{G/P} & m\left(x_{G/P}^2 + z_{G/P}^2\right) & -my_{G/P}z_{G/P} \\ -mx_{G/P}z_{G/P} & -my_{G/P}z_{G/P} & m\left(x_{G/P}^2 + y_{G/P}^2\right) \end{bmatrix}$$

$- H_G = I_G \omega$

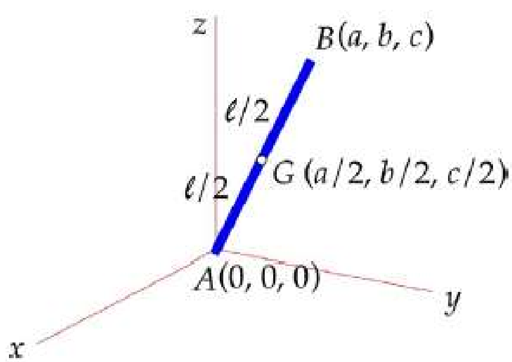
$- I_p = I_G + I_p^{(m)}$

$$I_{P_x} = I_{G_x} + m\left(y_{G/P}^2 + z_{G/P}^2\right) \quad I_{P_y} = I_{G_y} + m\left(y_{G/P}^2 + x_{G/P}^2\right) \quad I_{P_z} = I_{G_z} + m\left(x_{G/P}^2 + y_{G/P}^2\right)$$

$$I_{P_{xy}} = I_{G_{xy}} - mx_{G/P}y_{G/P} \quad I_{P_{xz}} = I_{G_{xz}} - mx_{G/P}z_{G/P} \quad I_{P_{yz}} = I_{G_{yz}} - my_{G/P}z_{G/P}$$

Example

Find the moments of inertia of the rod in Example 11.5 (Fig. 11.15) about its center of mass G.



Details

$$[\mathbf{I}_A] = \begin{bmatrix} \frac{1}{3}m\left(b^2 + c^2\right) & -\frac{1}{3}mab & -\frac{1}{3}mac \\ -\frac{1}{3}mab & \frac{1}{3}m\left(a^2 + c^2\right) & -\frac{1}{3}mbc \\ -\frac{1}{3}mac & -\frac{1}{3}mbc & \frac{1}{3}m\left(a^2 + b^2\right) \end{bmatrix}$$

$$[\mathbf{I}_G] = \begin{bmatrix} \frac{1}{12}m(b^2+c^2) & -\frac{1}{12}mab & -\frac{1}{12}mac \\ -\frac{1}{12}mab & \frac{1}{12}m(a^2+c^2) & -\frac{1}{12}mbc \\ -\frac{1}{12}mac & -\frac{1}{12}mbc & \frac{1}{12}m(a^2+b^2) \end{bmatrix}$$

Euler Equations

- $M_{net} = \dot{H}$

- $H = H_x \hat{i} + H_y \hat{j} + H_z \hat{k}$

- For simplicity, assume:
  (a) The moving xyz axes are the principal axes of inertia;
  (b) The moments of inertia relative to the xyz axes are constant in time.

- $H = A\omega_x \hat{i} + B\omega_y \hat{j} + C\omega_z \hat{k}$

- $M_{net} = \dot{H})_{rel} + \Omega \times H$

- $\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}, \quad \Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$

- $$\begin{aligned} \alpha &= \dot{\omega}_x \hat{i} + \dot{\omega}_y \hat{j} + \dot{\omega}_z \hat{k} + \Omega \times \omega \\ &= (\dot{\omega}_x + \Omega_y \omega_z - \Omega_z \omega_y)\hat{i} + (\dot{\omega}_y + \Omega_z \omega_x - \Omega_x \omega_z)\hat{j} \\ &\quad + (\dot{\omega}_z + \Omega_x \omega_y - \Omega_y \omega_x)\hat{k} \end{aligned}$$

- $\dot{H})_{rel} = \frac{d}{dt}(A\omega_x)\hat{i} + \frac{d}{dt}(B\omega_y)\hat{j} + \frac{d}{dt}(C\omega_z)\hat{k}$

$$= A\dot{\omega}_x\hat{i} + B\dot{\omega}_y\hat{j} + C\dot{\omega}_z\hat{k}$$

- $M_x)_{net} = A\dot{\omega}_x + C\Omega_y\omega_z - B\Omega_z\omega_y$

  $M_y)_{net} = B\dot{\omega}_y + A\Omega_z\omega_x - C\Omega_x\omega_z$

  $M_z)_{net} = C\dot{\omega}_z + B\Omega_x\omega_y - A\Omega_y\omega_x$

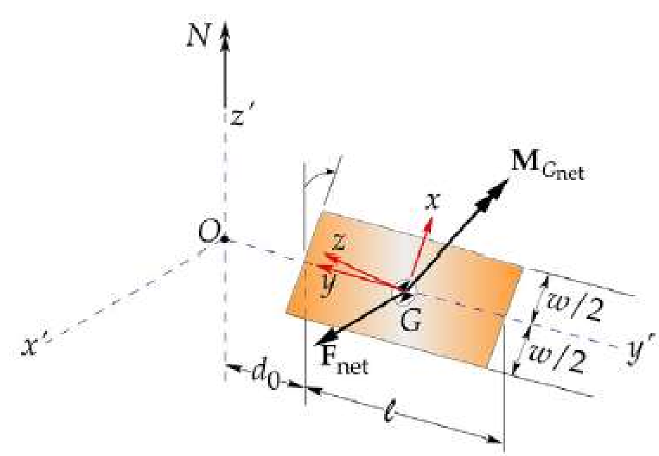
- $M_x)_{net} = A\dot{\omega}_x + (C-B)\omega_y\omega_z$

  $M_y)_{net} = B\dot{\omega}_y + (A-C)\omega_z\omega_x$

  $M_z)_{net} = C\dot{\omega}_z + (B-A)\omega_x\omega_y$

Example

Calculate the net moment on the solar panel of Examples 11.2 and 11.8 (Fig. 11.17).



Details

$$M_G)_{net} = \dot{H}_G)_{rel} + \omega \times H_G$$

$$\dot{H}_G)_{rel} = A\dot{\omega}_x \hat{i} + B\dot{\omega}_y \hat{j} + C\dot{\omega}_z \hat{k}$$

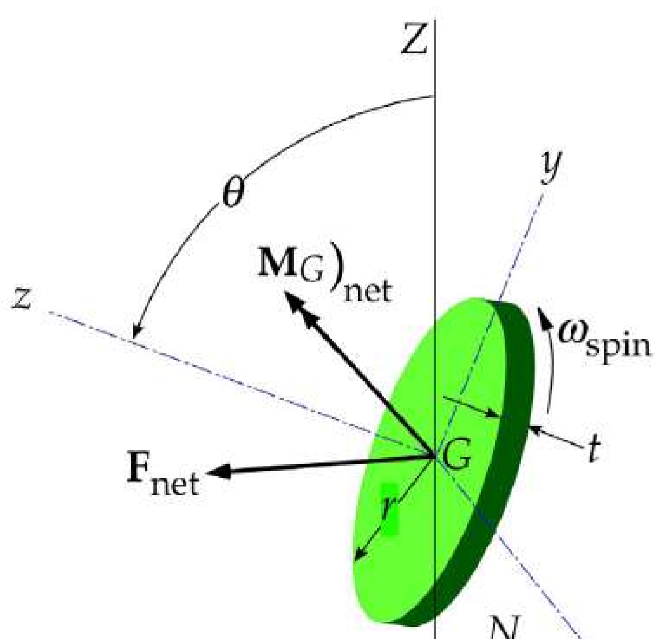
$$\omega = -\dot{\theta}\hat{j}' + N\hat{k}'$$

$$Q = \begin{bmatrix} -\sin\theta & 0 & \cos\theta \\ 0 & -1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix}$$

$$\omega|_{xyz} = Q^T \omega|_{x'y'z'}$$

Example

Calculate the net moment on the gyro rotor of Examples 11.3 and 11.6.



Details

$$M_G)_{net} = \dot{H}_G)_{rel} + \Omega \times H_G$$

$$H_G = A\omega_x \hat{i} + B\omega_y \hat{j} + C\omega_z \hat{k}$$

$$\omega_x = \dot{\theta}, \ \omega_y = N\sin\theta, \ \omega_z = \omega_{spin} + N\cos\theta$$

$$\Omega_x = \dot{\theta}, \ \Omega_y = N\sin\theta, \ \Omega_z = N\cos\theta$$

Kinetic Energy

$$-T = \int_m \frac{1}{2} v \cdot v \, dm$$

$$\begin{aligned} -v &= \dot{R} \\ &= \dot{R}_G + \dot{\rho} \\ &= v_G + \omega \times \rho \end{aligned}$$

$$-v \cdot v = \|v_G\|^2 + 2v_G \cdot (\omega \times \rho) + \omega \cdot [\rho \times (\omega \times \rho)]$$

$$-T = \int \frac{1}{2} \|v_G\|^2 dm + v_G \cdot \left(\omega \times \int \rho \, dm\right) + \frac{1}{2}\omega \cdot \int \rho \times (\omega \times \rho) \, dm$$

$$\underbrace{\quad m \quad}_{0} \qquad \underbrace{\qquad m \qquad}_{H_G}$$

$$= \frac{1}{2} m \|V_G\|^2 + \frac{1}{2} \omega \cdot H_G$$

$$-V_G = V_P + \omega \times r_{G/P}$$

$$= \omega \times r_{G/P}$$

$$-\|V_G\|^2 = (\omega \times r_{G/P}) \cdot (\omega \times r_{G/P})$$

$$= \omega \cdot (r_{G/P} \times V_G)$$

$$-T = \frac{1}{2} \omega \cdot (H_G + r_{G/P} \times m V_G)$$

$$= \frac{1}{2} \omega \cdot H_P$$

$$T_R = \frac{1}{2}(\omega_x H_x + \omega_y H_y + \omega_y H_z) = \frac{1}{2} \lfloor \omega_x \quad \omega_y \quad \omega_z \rfloor \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$T_R = \frac{1}{2} I_x \omega_x{}^2 + \frac{1}{2} I_y \omega_y{}^2 + \frac{1}{2} I_z \omega^2{}_z + I_{xy} \omega_x \omega_y + I_{xz} \omega_x \omega_z + I_{yz} \omega_y \omega_z$$

$$T_R = \frac{1}{2} A \omega_x{}^2 + \frac{1}{2} B \omega_y{}^2 + \frac{1}{2} C \omega_z{}^2$$