Order Of Convergence Of An Iterative Scheme

Let the sequence of iterative values $\{ x_n \}_{n=0}^{\infty}$ converges to 's'. Also let $\varepsilon_n = s - x_n$ and $\varepsilon_{n+1} = s - x_{n+1}$ for $n \geq 0$ are the errors at nth and (n+1)th iterations respectively. If two positive constants $A \neq 0$ and $R > 0$ exist, and

$$\lim_{n \to \infty} \frac{|s - x_{n+1}|}{|s - x_n|^R} = \lim_{n \to \infty} \frac{|\varepsilon_{n+1}|}{|\varepsilon_n|^R} = A$$

then the sequence is said to converge to 's' with order of convergence R. The number A is called the asymptotic error constant.

This can be derived from Taylor series as follows

Let $x_{i+1} = g(x_i)$ define an iterative method, and let 's' and x_n respectively are the exact and approximate solutions of $x = g(x)$. Then $\mathbf{x_n = s + \varepsilon_n}$, where ε_n is the error in x_n. Suppose that g is differentiable any number of times, then from Taylor's formula

$$\mathbf{x_{n+1} = g(x_n) = g(s + \varepsilon_n)}$$

$$\mathbf{= g(s) + \varepsilon_n g'(s) + \frac{1}{2}\varepsilon_n^2 g''(s) + \ldots}$$

The exponent of ε_n in the first non-vanishing term after g(s) is called the order of the iteration process defined by g. Since $x_{n+1} - s = \varepsilon_{n+1}$ (and $g(s) = s$) the above equation can now written as

$$\mathbf{\varepsilon_{n+1} = \varepsilon_n g'(s) + \frac{1}{2}\varepsilon_n^2 g''(s) + \ldots}$$

that is the error at (n+1)th iteration can be written as a function of error at the previous iteration. In the case of convergence ε_n is small for large n and hence the order is a measure for the speed of convergence. For example if a scheme is second order, that is

$$\mathbf{\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n^2 g''(s)}$$

then the number of significant digits are approximately doubled in each step.

Order of Newton's Method: Since for Newton's method

$$\mathbf{g(x) = x - \frac{f(x)}{f'(x)}}$$

$$\mathbf{g'(x) = \frac{f(x) * f''(x)}{}}$$

$$(f'(x))^2$$

at x = s, g' = 0 since f(s) = 0 and

$$g''(x) = \frac{2f''(x)}{f'(x)}$$

at x = s, g''(s) need not be zero, hence Newton-Raphson method is of order two. That is for each iteration the scheme converges approximately to two significant digits.

Order of Fixed Point Iteration method : Since the convergence of this scheme depends on the choice of g(x) and the only information available about g'(x) is |g'(x)| must be lessthan 1 in some interval which brackets the root. Hence g'(x) at x = s may or may not be zero. That is the order of fixed point iterative scheme is only one.

Order of Secant method : The g(x) for secant method is

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1}) * f(x_i)}{f(x_i) - f(x_{i-1})}$$

If $x_{i+1} = s + \varepsilon_{i+1}$, $x_i = s + \varepsilon_i$ and $x_{i-1} = s + \varepsilon_{i-1}$ then

$$s + \varepsilon_{i+1} = (s + \varepsilon_i) - \frac{(\varepsilon_i - \varepsilon_{i-1}) * f(s + \varepsilon_i)}{f(s + \varepsilon_i) - f(s + \varepsilon_{i-1})}$$

$$\rightarrow \varepsilon_{i+1} = \varepsilon_i - \frac{(\varepsilon_i - \varepsilon_{i-1}) * f(s + \varepsilon_i)}{f(s + \varepsilon_i) - f(s + \varepsilon_{i-1})}$$

$$= \varepsilon_i - \frac{(\varepsilon_i - \varepsilon_{i-1}) * (\varepsilon_i f'(s) + (½)\varepsilon_i^2 f''(s) + \ldots)}{(\varepsilon_i f'(s) + (½)\varepsilon_i^2 f''(s) + \ldots) - (\varepsilon_{i-1} f'(s) + (½)\varepsilon_{i-1}^2 f''(s) + \ldots)}$$

$$= \varepsilon_i - \frac{(\varepsilon_i f'(s) + (½)\varepsilon_i^2 f''(s) + \ldots)}{(f'(s) + (½)(\varepsilon_i + \varepsilon_{i-1})f''(s) + \ldots)}$$

$$= \varepsilon_i - (\varepsilon_i + \frac{1}{2}\varepsilon_i^2 \frac{f''(s)}{f'(s)} + \ldots)(1 + \frac{1}{2}(\varepsilon_{i-1} + \varepsilon_i)\frac{f''(s)}{f'(s)} + \ldots)^{-1}$$

$$\varepsilon_{i+1} = \frac{1}{2} \varepsilon_i \varepsilon_{i-1} \frac{f''(s)}{f'(s)} + 0(\varepsilon_i^2 \varepsilon_{i-1} + \varepsilon_i \varepsilon_{i-1}^2)$$

$$\varepsilon_{i+1} = C \varepsilon_i \varepsilon_{i-1} \quad \text{where } C = \frac{1 f''(s)}{2 f'(s)}$$

and higher order terms are neglected.

since $\varepsilon_i = A\varepsilon^p_{i-1}$

$$\varepsilon_{i-1} = A^{-1/p} \varepsilon_i^{1/p}$$

=> $\varepsilon_{i+1} = A\varepsilon_i(A^{-1/p}\varepsilon_i^{1/p}) = A^{1-1/p}\varepsilon_i^{1+1/p}$

=> order of the scheme **p = 1 + 1/p**

$$p = \frac{1}{2}(1 \pm 5^{1/2}) = 1.618$$

and **A** =**C** $^{p/(p+1)}$

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