MA 214: Introduction to Numerical Analysis

Department of Mathematics, Indian Institute of Technology Bombay

Quiz 2

Time: **1 hour** | Marks: **15**

Instructor: **S. Baskar** | Date: **15/03/2017**

Instructions:

(1) Write your **Name, Roll Number, and Tutorial Batch** clearly on your answer book as well as every supplement you may use. A **penalty of -1 mark will be awarded** for failing to do so.

(2) Number the pages of your answer book and make a question-page index on the front page. A **penalty of -1 mark will be awarded** for failing to do so.

(3) The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.

(4) Only scientific calculators are allowed. Any kind of programing device is not allowed.

(5) Formulas used need not be proved but **needs to be stated clearly**.

(6) The question paper contains 5 questions each carries 3 marks. **Answer all the questions**.

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(1) Let the bisection method be used to obtain an approximate value of the smallest positive root r of the nonlinear equation

$$e^x + x^2 - 2 = 0.$$

After choosing an appropriate initial interval $[a_0, b_0]$, find the number of iterations (with a mathematical justification and without performing the actual iterations) needed to obtain an approximate value of the root r so that the absolute relative error is less than e^{-10}.

[**Note:** You may use any number of decimal places as precision in your calculation]

(2) Consider the equation $x^2 - 6x + 5 = 0$. Take the initial interval as $[a_0, b_0] = [0, 4.5]$ and generate the first 3 iterations using regula-falsi method and find the approximate value of the root after the third iteration.

[**Note:** Write all steps as per the algorithm of the method in each iteration. You may use any number of decimal places as precision in your calculation]

(3) Let x_0 and x_1 be initial guesses in secant method for the equation $f(x) = 0$, where

$$f(x) = x^2 + 2x - 3.$$

For $x_0 = 0$, if x_1 is such that $x_2 (\neq x_1)$ exists but x_3 does not exist, then obtain a nonlinear equation $g(x) = 0$ for which x_1 is a root.

(4) Using graphical illustrations (only), explain three different reasons where Newton-Raphson method fail although the nonlinear equation has a root.

(5) Consider the fixed-point iteration $x_{n+1} = g(x_n)$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function, which has a unique fixed point $r \in \mathbb{R}$. Let the initial guess $x_0 \in \mathbb{R}$ be such that

$$g(x_0) < x_0 \text{ and } r < x_0.$$

Show that $x_{n+1} < x_n$ for each $n = 0, 1, 2, \cdots$. Does the fixed-point iteration sequence converges to r in this case, even if the function g is not a contraction map in $\mathbb{R}$? Justify your answer.

— **End of the Question Paper** —

MA 214 Quiz-2 Spring 2016-17

Marking Scheme

(1) The formula to be used is

$$n \geq \frac{\log(b_0 - a_0) - \log \varepsilon - \log a_0}{\log 2}, \qquad \text{——— } 1$$

where $a_0 > 0$ and $b_0 > a_0$ are to be chosen, and $\varepsilon = e^{-10}$ is given.

Note:- One may choose any $a_0 > 0$ and any $b_0 > a_0$ such that there is a root in the interval $[a_0, b_0]$

Any correct choice as mentioned above → 1

For instance choose $a_0 = 0.5$ and $b_0 = 1$. Then

$$\underbrace{n \geq \frac{\log 0.5 - \log(e^{-10}) - \log 0.5}{\log 2} \approx 14.42}_{} \rightarrow \frac{1}{2}$$

Therefore the required number of iterations $n = 15$.

Choosing the nearest integer larger than the computed real number → $\frac{1}{2}$

Note: If a student does calculation mistake while computing the value out of the formula, no mark is deducted.

(2) Formula to be used is

$$x_{n+1} = a_n - f(a_n)\frac{b_n - a_n}{f(b_n) - f(a_n)} \quad \textbf{or} \quad x_{n+1} = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}$$ or $\frac{1}{2}$

where $f(x) = x^2 - 6x + 5$.

Iteration 1:– Take $[a_0, b_0] = [0, 4.5]$

$$x_1 = 0 - f(0)\frac{4.5 - 0}{f(4.5) - f(0)} \quad \textbf{or} \quad x_1 = \frac{0 \times f(4.5) - 4.5 \times f(0)}{f(4.5) - f(0)}$$ $\frac{1}{2}$

$= 3.3333\ldots$ $\qquad = 3.3333\ldots$

$f(a_0) = f(0) = 5$, $f(x_1) = f(3.333\ldots) \approx -3.888\ldots$, $f(b_0) = f(4.5) = -1.75$

$\Rightarrow f(a_0) f(x_1) < 0$ and $f(x_1) f(b_0) > 0$ $\frac{1}{2}$

This step has to be written atleast once. Otherwise this mark is cut.

Iteration 2:– Take $[a_1, b_1] = [0, 3.333\ldots]$ $\frac{1}{2}$

$$x_2 = 0 - f(0)\frac{3.333\ldots - 0}{f(3.333\ldots) - f(0)} \quad \textbf{or} \quad x_2 = \frac{0 \times f(3.333\ldots) - 3.333\ldots f(0)}{f(3.333\ldots) - f(0)}$$

≈ 1.875 $\qquad \approx 1.875$

$f(a_1) = f(0) = 5$, $f(x_2) = f(1.875) \approx -2.734$, $f(b_1) = f(3.33\ldots) \approx -3.88\ldots$

$f(a_1) f(x_2) < 0$ and $f(x_2) f(b_1) > 0$ $\frac{1}{2}$

Checking the condition and choosing the interval

Iteration 3:– Take $[a_2, b_2] = [0, 1.875]$

$$x_3 = 0 - f(0)\frac{1.875 - 0}{f(1.875) - f(0)} \quad \textbf{or} \quad x_3 \approx \frac{0 \times f(1.875) - 1.875 f(0)}{f(1.875) - f(0)}$$

$\approx 1.2121\ldots$ $\qquad \approx 1.2121\ldots$

$f(a_2) = f(0) = 5$, $f(x_3) \approx -0.8035$, $f(b_2) \approx -2.734$

$f(a_2) f(x_3) < 0$ and $f(x_3) f(b_2) > 0$

Finally, we take

$$x_4 \approx 1.0443$$ or $\frac{1}{2}$

as the approximate value of the root of the given equation.

Note:– If a student makes calculation mistake but otherwise the steps are written clearly (with perhaps wrong numbers with one or more calculation mistakes) then over all only ½ mark is reduced.

(3) Given $x_0 = 0$. Thus, for any given x_1, we have

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$= x_1 - \frac{(x_1^2 + 2x_1 - 3)x_1}{(x_1^2 + 2x_1 - 3) + 3}$$

$$\Rightarrow x_2 = \frac{3}{x_1 + 2}. \longrightarrow 1$$

x_3 does not exists if $f(x_1) = f(x_2)$. $\longrightarrow$ 1

This gives

$$x_1^2 + 2x_1 - 3 = \left(\frac{3}{x_1+2}\right)^2 + \frac{6}{x_1+2} - 3.$$

Therefore, the required nonlinear equation $g(x) = 0$ is such that

$$g(x) = (x+2)^2(x^2+2x-3) - 6(x+2) + 3(x+2)^2 - 9 \longrightarrow 1$$

$$= x^4 + 6x^3 + 12x^2 + 2x - 21.$$

Aliter:—

Given $x_0 = 0$. Thus, for any given x_1, we have

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$= x_1 - \frac{(x_1^2 + 2x_1 - 3)x_1}{(x_1^2 + 2x_1 - 3) + 3}$$

$$\Rightarrow x_2 = \frac{3}{x_1 + 2}. \longrightarrow 1$$

x_3 does not exists if $f(x_1) = f(x_2)$. $\longrightarrow$ 1

$$\Rightarrow x_1^2 + 2x_1 - 3 = x_2^2 + 2x_2 - 3$$

$$\Rightarrow x_1^2 - x_2^2 + 2(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 2) = 0$$

$$\Rightarrow x_1 + x_2 + 2 = 0 \quad [\because x_1 \neq x_2]$$

$$\Rightarrow x_1 + \frac{3}{x_1+2} + 2 = 0.$$

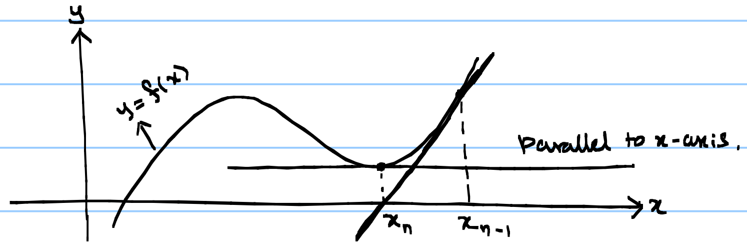
Therefore the required nonlinear equation is

$$g(x) = x(x+2) + 2(x+2) + 3 \longrightarrow 1$$

$$= x^2 + 4x + 7.$$

(4) Reason 1 :- For some $n \geq 0$, x_n is such that $f'(x_n) = 0$, then $x_{n+1} = \pm\infty$ and therefore Newton-Raphson method fails. → $\frac{1}{2}$

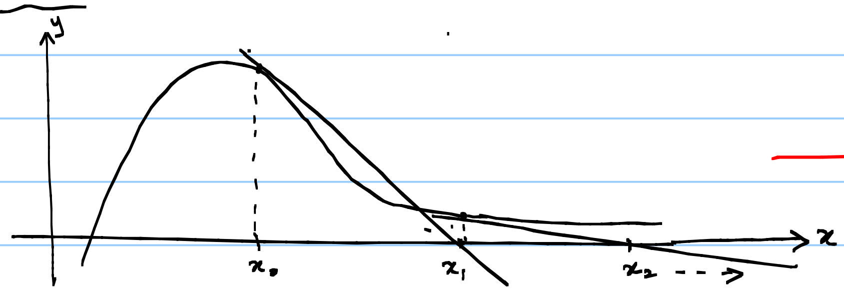
Graphical illustration :-



→ $\frac{1}{2}$

Reason 2 :- The Newton-Raphson sequence may diverge. → $\frac{1}{2}$

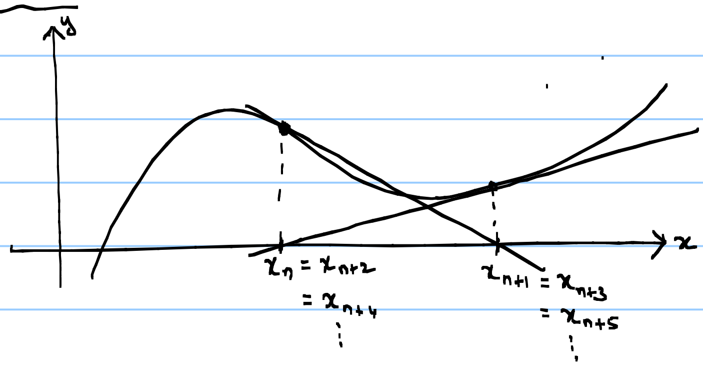
Graphical illustration :-



→ $\frac{1}{2}$

Reason 3 :- The Newton-Raphson sequence may stuck in a cycle. → $\frac{1}{2}$

Graphical illustration :-



→ $\frac{1}{2}$

(5) To prove that $x_{n+1} < x_n$, we use the mathematical induction.

First we claim that $x_1 < x_0$. By definition of the fixed-point iteration, we have

$$x_1 = g(x_0) < x_0 \text{ . (by given condition)} \longrightarrow \frac{1}{2}$$

Now, we assume that for $n = k$, for some integer $k > 0$, we have $x_{k+1} < x_k$ and show that it is true for $n = k+1$, ie., to show $x_{k+2} < x_{k+1}$. We have

$$x_{k+2} = g(x_{k+1}) < g(x_k) \quad [\because g \text{ is a strictly increasing function, by the given condition}]$$
$$= x_{k+1} \longrightarrow \frac{1}{2}$$

Note:- If a student does not use mathematical induction, but shows how $x_2 < x_1$ using the condition that g is strictly increasing, and then mentions 'Similarly, we can prove $x_{n+1} < x_n$', then the mark can be given. Same holds for the arguement below.

The fixed-point iteration converges in the present case even if g is not a contraction map. To see this, we again use mathematical induction.

First we see that,

$$\left.\begin{array}{l}\text{given } g \text{ is strictly} \\ \text{increasing and } r < x_0\end{array}\right\} \Rightarrow \quad r = g(r) < g(x_0) = x_1 \longrightarrow \frac{1}{2}$$

Now, we assume that $r < x_k$, for some integer $k > 0$, and show that $r < x_{k+1}$. Since g is strictly increasing and since $r < x_k$, we have

$$r = g(r) < g(x_k) = x_{k+1}.$$

Thus, we have shown that

$$r < x_n, \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow \{x_n\} \text{ bounded below} \longrightarrow \frac{1}{2}$$

Thus, we have shown that the sequence $\{x_n\}$ is a strictly decreasing sequence and it is bounded below. Therefore the sequence converges, say, $x_n \to \alpha$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$ $\rightarrow \frac{1}{2}$

Further since g is continuous, the limit α is a fixed point. Also, since r is the unique fixed point of g, we must have $r = \alpha$ $\rightarrow \frac{1}{2}$

Time Allotment Plan

Q.No	expected Time taken to write the answer
1	3 minutes (familiar concept)
2	1 (formula) + 3 (iteration 1) + 3 (iteration 2) + 3 (iteration 3) = 10 minutes
3	10 minutes (may need more time to think)
4	2 (Reason 1) + 2 (Reason 2) + 4 (Reason 3) = 8 minutes (Note: Reason 1 & 2 are drawn in the class whereas Reason 3 is only mentioned)
5	20 minutes (may need time to think and write the answer)
Total Required Time	51 minutes
Given Time	60 minutes