Mid-semester Examination *AE 240/338/713* *Monday February 20, 2017*
Weightage: 20% *Time: 1730 – 1930 Hrs.* *Venue: LA101*
Examination is closed Notes/Books. *Only Calculator is permitted.*

Model Solutions

Q.1 (a) Derive the expressions for m_b, t_b, h_b & x_b, starting $t_0 = 0$, $h_0 = 0$, $x_0 = 0$, in respect of constant speed ($V = V_0$, zero drag) gravity turn trajectory, using the following equations of motion. (2)

$$\dot{V} = -\frac{\dot{m} g_0 I_{sp}}{m(t)} - \tilde{g}\cos\theta = 0 \rightarrow \frac{dm}{m} = -\frac{1}{I_{sp}}\cos\theta dt; \quad \dot{\theta} = \frac{g_0 \sin\theta}{V_0};$$

$$\frac{dm}{m} = -\frac{1}{I_{sp}}\frac{\cos\theta}{\dot{\theta}}d\theta = -\frac{V_0}{g_0 I_{sp}}\frac{\cos\theta}{\sin\theta}d\theta; \quad \int\frac{dm}{m} = -\frac{V_0}{g_0 I_{sp}}\int\frac{\cos\theta}{\sin\theta}d\theta$$

$$\ln m = -\frac{V_0}{g_0 I_{sp}}\ln(\sin\theta) + C; \quad m(\theta) = k(\sin\theta)^{-\frac{V_0}{g_0 I_{sp}}}; \quad \frac{m_b}{m_0} = \left(\frac{\sin\theta_0}{\sin\theta_b}\right)^{\frac{V_0}{g_0 I_{sp}}}$$

$$\dot{\theta} = \frac{g_0 \sin\theta}{V_0}; \quad \frac{d\theta}{dt} = \frac{g_0\sin\theta}{V_0} \rightarrow dt = \frac{V_0}{g_0}\frac{d\theta}{\sin\theta}; \quad \int dt = \frac{V_0}{g_0}\int\frac{d\theta}{\sin\theta}$$

$$t = \frac{V_0}{g_0}\ln\tan\frac{\theta}{2} + C; \quad t_b = \frac{V_0}{g_0}\left(\ln\tan\frac{\theta_b}{2} - \ln\tan\frac{\theta_0}{2}\right) = \frac{V_0}{g_0}\ln\left(\frac{\tan\frac{\theta_b}{2}}{\tan\frac{\theta_0}{2}}\right)$$

$$\frac{dh}{dt} = V_0\cos\theta \rightarrow \int dh = \frac{V_0^2}{g_0}\int\frac{\cos\theta}{\sin\theta}d\theta; \quad h = \frac{V_0^2}{g_0}\ln\sin\theta + C$$

$$h_b = \frac{V_0^2}{g_0}\left(\ln\sin\theta_b - \ln\sin\theta_0\right); \quad \frac{dx}{dt} = V_0\sin\theta \rightarrow \int dx = \frac{V_0^2}{g_0}\int d\theta$$

$$x = \frac{V_0^2}{g_0}\theta + C \rightarrow x_b = \frac{V_0^2}{g_0}\left(\theta_b - \theta_0\right)$$

(b) A rocket having $m_0 = 25$ T starts gravity turn trajectory from earth's surface in an impulsive launch by burning instantaneously, $m_p = 10$ T having I_{sp} of 250s. If the rocket immediately also switches on a speed-hold autopilot and after travelling for some time and burning another 10T of propellant of same I_{sp}, attains $\theta_b = 60^o$, determine (a) the extent of initial 'pitch kick' (θ_0) required, (b) time taken to complete the manoeuvre and (c) horizontal (x) and vertical (h) distances travelled. ($g_0 = 9.81$ m/s^2). (2)

$$V_0 = g_0 I_{sp}\ln\frac{m_0}{m_{b1}} = 9.81\times 250\times\ln\frac{25}{15} = 1252.8 m/s \ (\beta{=}\infty)$$

$$\frac{m_b}{m_0} = \left(\frac{\sin\theta_0}{\sin\theta_b}\right)^{\frac{V_0}{g_0 I_{sp}}} \rightarrow \left(\frac{\sin\theta_0}{\sin 60^o}\right)^{0.5108} = \frac{5}{15} = 0.333 \rightarrow \theta_0 = 5.78^o$$

$$t_b = \frac{V_0}{g_0}\ln\left(\frac{\tan\frac{\theta_b}{2}}{\tan\frac{\theta_0}{2}}\right) = \frac{1252.8}{9.81}\times\ln\left(\frac{\tan 30^o}{\tan 2.89^o}\right) = 311.2s$$

$$h_b = \frac{V_0^2}{g_0}\left(\ln\sin\theta_b - \ln\sin\theta_0\right) = \frac{1252.8^2}{9.81}\times(\ln\sin 60 - \ln\sin 5.78) = 344.25km$$

$$x_b = \frac{V_0^2}{g_0}\left(\theta_b - \theta_0\right) = \frac{1252.8^2}{9.81}\times(1.0473 - 0.1009) = 151.42km$$

Q.2 (a) Formulate the objective function for optimizing the burnout velocity under the mission payload constraint and obtain the expressions for π and V_*, for an n-stage rocket under the assumption that all stages have same (ε) and I_{sp}, using the Lagrange multiplier approach. (Hint: Use the following as required). (2)

$$V_* = \sum_{i=1}^{n} \Delta V_i = -\sum_{i=1}^{n} g_0 I_{spi} \ln \frac{m_{0i} - m_{pi}}{m_{0i}} = -g_0 \sum_{i=0}^{n} I_{spi} \ln\left(\varepsilon_i + \pi_i \times (1-\varepsilon_i)\right)$$

$$\pi_* = \frac{m_*}{m_0} = \prod_{i=1}^{n} \pi_i \rightarrow \ln \pi_* = \sum_{i=1}^{n} \ln \pi_i$$

$$H_V(\lambda, \pi_i) = V_* + \lambda(\text{Error}) = -g_0 \sum_{i=1}^{n} I_{sp_i} \ln\left[\varepsilon_i + (1-\varepsilon_i)\pi_i\right] + \lambda\left(\ln \pi_* - \sum_{i=1}^{n} \ln \pi_i\right)$$

$$H_V(\lambda, \pi_i) = -g_0 I_{sp} \sum_{i=1}^{n} \ln\left[\varepsilon + (1-\varepsilon)\pi_i\right] + \lambda\left(\ln \pi_* - \sum_{i=1}^{n} \ln \pi_i\right)$$

$$\frac{\partial H_V}{\partial \pi_i} = \frac{g_0 I_{sp}(1-\varepsilon)}{\varepsilon + (1-\varepsilon)\pi_i} + \frac{\lambda}{\pi_i} = 0; \quad \pi_i = \frac{-\lambda\varepsilon}{(1-\varepsilon)\left(\lambda + g_0 I_{sp}\right)} = \sqrt[n]{\pi_*}$$

$$\lambda = -\frac{g_0 I_{sp}(1-\varepsilon) \times \sqrt[n]{\pi_*}}{\varepsilon + (1-\varepsilon) \times \sqrt[n]{\pi_*}}; \quad V_{*-\text{optim}} = -g_0 n I_{sp} \ln\left[\varepsilon + (1-\varepsilon) \times \sqrt[n]{\pi_*}\right]$$

(b) Using the results of (a), determine π_i and V_* of a 2-stage rocket with same structural ratio of 0.1 and same I_{sp} of 400s for both the stages, so that it achieves the mission payload fraction of 0.07. (2)

$$\pi_1 = \pi_2 = \frac{-\lambda\varepsilon}{(1-\varepsilon)\left(\lambda + g_0 I_{sp}\right)} = \sqrt[n]{\pi_*} = 0.2646$$

$$\lambda = -\frac{g_0 I_{sp}(1-\varepsilon) \times \sqrt[n]{\pi_*}}{\varepsilon + (1-\varepsilon) \times \sqrt[n]{\pi_*}} = -\frac{9.81 \times 400 \times 0.9 \times 0.2646}{0.1 + 0.9 \times 0.2646} = -2763.5$$

$$V_{*-\text{optim}} = -g_0 n I_{sp} \ln\left[\varepsilon + (1-\varepsilon) \times \sqrt[n]{\pi_*}\right] = 9.81 \times 2 \times 400 \times \ln 0.3381 = 8509.5 m/s$$

(c) Next, obtain π_i & V_* through the approximate optimization methodology for the rocket described in (b), by ignoring the partial derivative with respect to π_2 and comment on the differences or lack of them. (Hint: Use general relations given in (a) to formulate and solve the problem). (3)

$$\pi_1 \times \pi_2 = 0.07; \quad V_* = -9.81 \times 400 \times \ln\left\{\left[0.1 + 0.9\pi_1\right] \times \left[0.1 + 0.9 \times \frac{0.07}{\pi_1}\right]\right\}$$

$$\frac{\partial V_*}{\partial \pi_1} = -9.81 \times 400 \times \frac{\left[0.1 + 0.9\pi_1\right] \times \left[-\frac{0.063}{\pi_1^2}\right] + 0.9 \times \left[0.1 + 0.9 \times \frac{0.07}{\pi_1}\right]}{\left[0.1 + 0.9\pi_1\right] \times \left[0.1 + 0.9 \times \frac{0.07}{\pi_1}\right]} = 0$$

$$0.09 + \frac{0.0567}{\pi_1} = \frac{0.0063}{\pi_1^2} + \frac{0.0567}{\pi_1} \rightarrow 0.09\pi_1^2 = 0.0063 \rightarrow \pi_1^2 = 0.07$$

$$\pi_1 = 0.2646, \quad \pi_2 = 0.2646$$

Q.3 (a) Show that starting from first principles, the structural trade-off ratios for a 2-stage rocket is given by the following expressions. (Hint: Use velocity invariance & basic equations given in Q.2, as necessary). (2)

$$\frac{\partial m_*}{\partial m_{s1}} = \frac{-I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right)}{I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right) + I_{sp2}\left(\frac{1}{m_{02}} - \frac{1}{m_{f2}}\right)}; \quad \frac{\partial m_*}{\partial m_{s2}} = -1$$

$$m_{01} = m_* + m_{s1} + m_{p1} + m_{s2} + m_{p2}; \quad m_{f1} = m_* + m_{s1} + m_{s2} + m_{p2};$$

$$m_{02} = m_* + m_{s2} + m_{p2}; \quad m_{f2} = m_* + m_{s2}; \quad \frac{V_b}{g_0} = I_{sp1} \ln \frac{m_{01}}{m_{f1}} + I_{sp2} \frac{m_{02}}{m_{f2}}$$

$$\frac{dV_b}{g_0}(\text{for } m_*) = \begin{bmatrix} I_{sp1} \times \left(\frac{1}{m_{01}}\frac{\partial m_{01}}{\partial m_*} - \frac{1}{m_{f1}}\frac{\partial m_{f1}}{\partial m_*}\right) + \\ I_{sp2} \times \left(\frac{1}{m_{02}}\frac{\partial m_{02}}{\partial m_*} - \frac{1}{m_{f2}}\frac{\partial m_{f2}}{\partial m_*}\right) \end{bmatrix} \delta m_* = \sum_{i=1}^{2} I_{spi} \times \left(\frac{1}{m_{0i}} - \frac{1}{m_{fi}}\right)\delta m_*$$

$$\frac{dV_b}{g_0}(\text{for } m_{s1}) = \begin{bmatrix} I_{sp1} \times \left(\frac{1}{m_{01}}\frac{\partial m_{01}}{\partial m_{s1}} - \frac{1}{m_{f1}}\frac{\partial m_{f1}}{\partial m_{s1}}\right) \\ + I_{sp2} \times \cancel{\left(\frac{1}{m_{02}}\frac{\partial m_{02}}{\partial m_{s1}} - \frac{1}{m_{f2}}\frac{\partial m_{f2}}{\partial m_{s1}}\right)} \end{bmatrix} \delta m_{s1} = I_{sp1} \times \left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right)\delta m_{s1}$$

$$(V_b \text{ Invartiant}): \frac{dV_b}{g_0}(\text{for } m_*) + \frac{dV_b}{g_0}(\text{for } m_{s1}) = 0 \; = \sum_{i=1}^{2} I_{spi} \times \left(\frac{1}{m_{0i}} - \frac{1}{m_{fi}}\right)\delta m_* + I_{sp1} \times \left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right)\delta m_{s1}$$

$$\frac{\partial m_*}{\partial m_{s1}} = \frac{-I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right)}{I_{sp1}\left(\frac{1}{m_{01}} - \frac{1}{m_{f1}}\right) + I_{sp2}\left(\frac{1}{m_{02}} - \frac{1}{m_{f2}}\right)}$$

(b) Obtain $\frac{\partial m_*}{\partial m_{s1}}$ for rocket with $m_{p1} = 21000kg; \quad m_{s1} = 1300kg; \quad m_{p2} = 3900kg; \quad m_{s2} = 350kg;$

$m_* = 650kg; \quad I_{sp1} = 250s; \quad I_{sp2} = 300s.$

$m_{01} = 27200; \quad m_{f1} = 6200; \quad m_{02} = 4900; \quad m_{f2} = 1000$

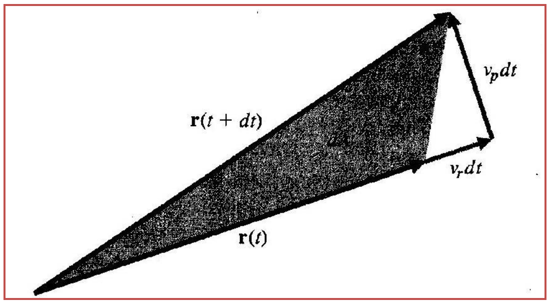
(1)

$$\frac{\partial m_*}{\partial m_{s1}} = \frac{-250\times\left(\frac{1}{27200} - \frac{1}{6200}\right)}{250\times\left(\frac{1}{27200} - \frac{1}{6200}\right) + 300\times\left(\frac{1}{4900} - \frac{1}{1000}\right)} = -\frac{0.03113}{-0.03113 - 0.2388} = -0.115$$

Q.4 Prove the Kepler's laws from conservation of angular momentum and solution of the central force motion equation given below. (2)

$$r = \frac{h^2/\mu}{1 + e\cos\theta}; \quad \vec{e}: \text{Constant Vector}; \quad \theta: \angle \text{ between } \vec{r} \;\&\; \vec{e}$$

$$\vec{r} \times \dot{\vec{r}} = \vec{r} \times \left(d\vec{r}/dt\right) = \vec{H} \rightarrow \text{A constant angular momentun vector (per unit mass)}$$



It is seen that vector 'dr' is not along the vector 'r' and thus, 'dr' and 'r', define a plane. Further, as H is constant in both magnitude & direction, it must remain normal to plane defined by vectors 'r' & 'dr', proving that motion is confined to plane. Also, solution of the central force motion equation is a conic section, of which ellipse is one of the most generic geometries. (1st Law)

Also, from previous diagram we can show that the area 'dA', swept by the radius vector 'r' in time interval 'dt' is expressed as follows.

$$dA=\frac{1}{2}(Base)\times(Alt.)=\frac{1}{2}(r)\left(v_p dt\right);\quad \frac{dA}{dt}=\frac{1}{2}H;\ \text{Areal Velocity (2nd Law)}$$

The time period of the orbit can be obtained assuming an elliptic geometry, as follows.

$$\frac{dA}{dt}=\frac{h}{2}=\frac{1}{2}\sqrt{\mu a(1-e^2)};\quad T=\frac{\text{Area of ellipse}}{\text{Areal Velocity}}=\frac{\pi ab}{\left(dA/dt\right)}$$

$$T=\frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}}=\frac{2\pi}{\sqrt{\mu}}a^{(3/2)}\rightarrow T^2=\frac{4\pi^2}{\mu}a^3\rightarrow \text{Keppler's 3rd Law}$$

Q.5 (a) Derive relations between ellipse parameter 'a', 'e' & 'h', 'ε'. (Hint: Use following expressions). (1)

$$h=r_p v_p=r_a v_a;\quad \varepsilon=\frac{1}{2}v^2-\frac{\mu}{r};\quad r_p=a(1-e);\quad r_a=a(1+e);$$

$$\frac{h^2}{\mu}=p\rightarrow h=\sqrt{\mu p}=\sqrt{\mu a\left(1-e^2\right)};\quad \varepsilon=\frac{1}{2}v_p^2-\frac{\mu}{r_p}=\frac{1}{2}\frac{\mu a\left(1-e^2\right)}{a^2(1-e)^2}-\frac{\mu}{a(1-e)}$$

$$\varepsilon=\frac{1}{2}\frac{\mu(1+e)}{a(1-e)}-\frac{\mu}{a(1-e)}=\frac{\mu}{a(1-e)}\left(\frac{1}{2}(1+e)-1\right)=\frac{\mu}{a(1-e)}\left(-\frac{1}{2}(1-e)\right)=-\frac{\mu}{2a}$$

$$a=-\frac{\mu}{2\varepsilon};\quad h^2=\mu a\left(1-e^2\right)=-\frac{\mu^2}{2\varepsilon}\left(1-e^2\right)\rightarrow 1-e^2=-\frac{2\varepsilon h^2}{\mu^2}\rightarrow e=\sqrt{1+\frac{2\varepsilon h^2}{\mu^2}}$$

(b) A spacecraft is expected to form a circular orbit at 180 km altitude above earth. Determine the velocity with which it should be injected parallel to local horizon. Further, determine the maximum injection angle error to prevent impact on earth. (Hint: Use results of (a) as applicable. R_E = 6,378 km, $\mu = 3.986\times10^{14}$) (3)

$$r_p=180+6378=6558=a;\quad e=0\rightarrow\frac{2\varepsilon h^2}{\mu^2}=-1\rightarrow h^2=r_p^2v_p^2=a\mu$$

$$v_p=\sqrt{\frac{\mu}{a}}=\sqrt{\frac{3.986\times10^{14}}{6.558\times10^6}}=7796.2m/s;\quad \text{Let }\delta\theta\text{ be the allowable injection error.}$$

$$\varepsilon=\frac{v^2}{2}-\frac{\mu}{r}=\frac{7796.2^2}{2}-\frac{3.986\times10^{14}}{6.558\times10^6}=-3.039\times10^7;$$

$$h=rv\cos\delta\theta=5.113\times10^{10}\cos\delta\theta;\quad \text{Min } r=r_p=6.378\times10^6=6.558\times10^6(1-e)$$

$$e=0.0274=\sqrt{1+\frac{2\varepsilon h^2}{\mu^2}}\rightarrow\frac{2\varepsilon h^2}{\mu^2}=-0.99925=-\frac{2\times3.039\times10^7\times\left(5.113\times10^{10}\cos\delta\theta\right)^2}{\left(3.986\times10^{14}\right)^2}$$

$$\cos^2\delta\theta=0.99916\rightarrow\cos\delta\theta=0.99958\rightarrow\delta\theta=\pm1.657^o$$

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