AE 242
Aerospace Measurements Laboratory

Attitude Determination

Angular displacement between two frames.

AHRS – Attitude heading reference system

Attitude Determination

By using kinematic equations: involves integration of differential equations. Prone to errors during long time estimations.

Using algebraic methods: Uses measurement of a vector fixed in inertial space in body frame and obtains body attitude with respect to the vector. To find body's attitude with respect to inertial frame, components of the vector fixed in inertial frame must be known. Such combination will give only two attitude angles, it is an under determined problem. Two vector measurements can be used for finding the attitude. When two vectors are used it is an over determined problem.

Can we use accelerometers to determine attitude?

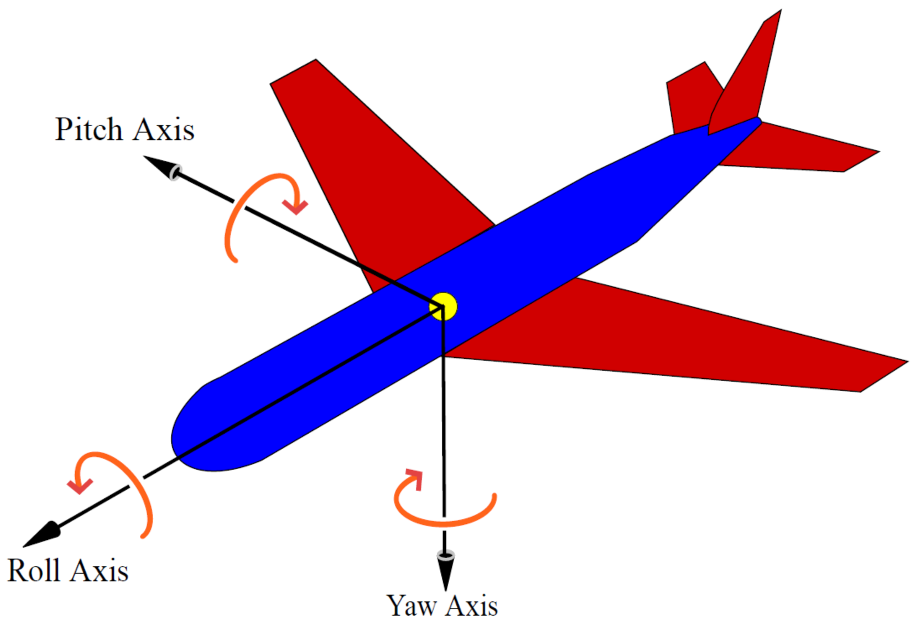
What are its limitations?

Roll - Pitch – Yaw

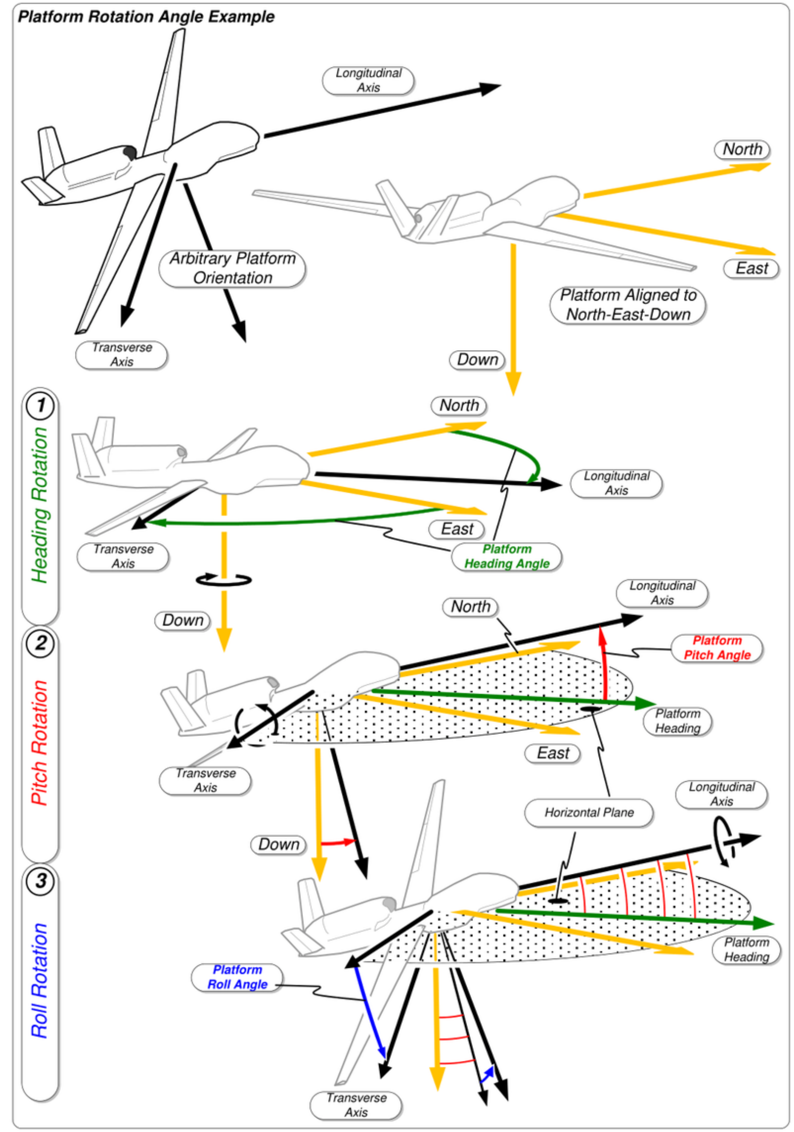
Roll – Rotation about x-axis

Pitch – Rotation about y-axis

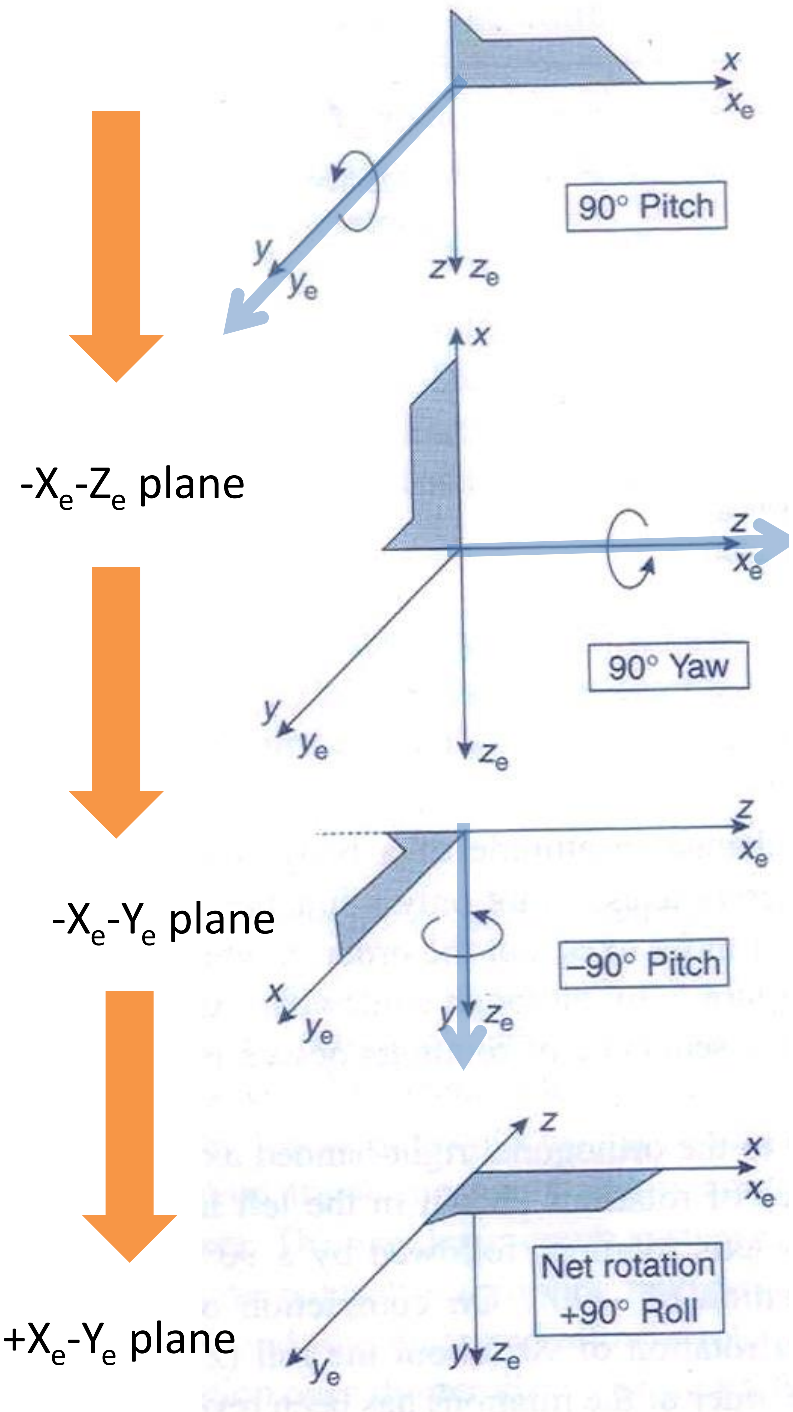
Yaw – Rotation about z axis



By Yaw_Axis.svg: Auawisederivative work: Jrvz (talk) - Yaw_Axis.svg, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=9441238



https://en.wikipedia.org/wiki/Aircraft_principal_axes

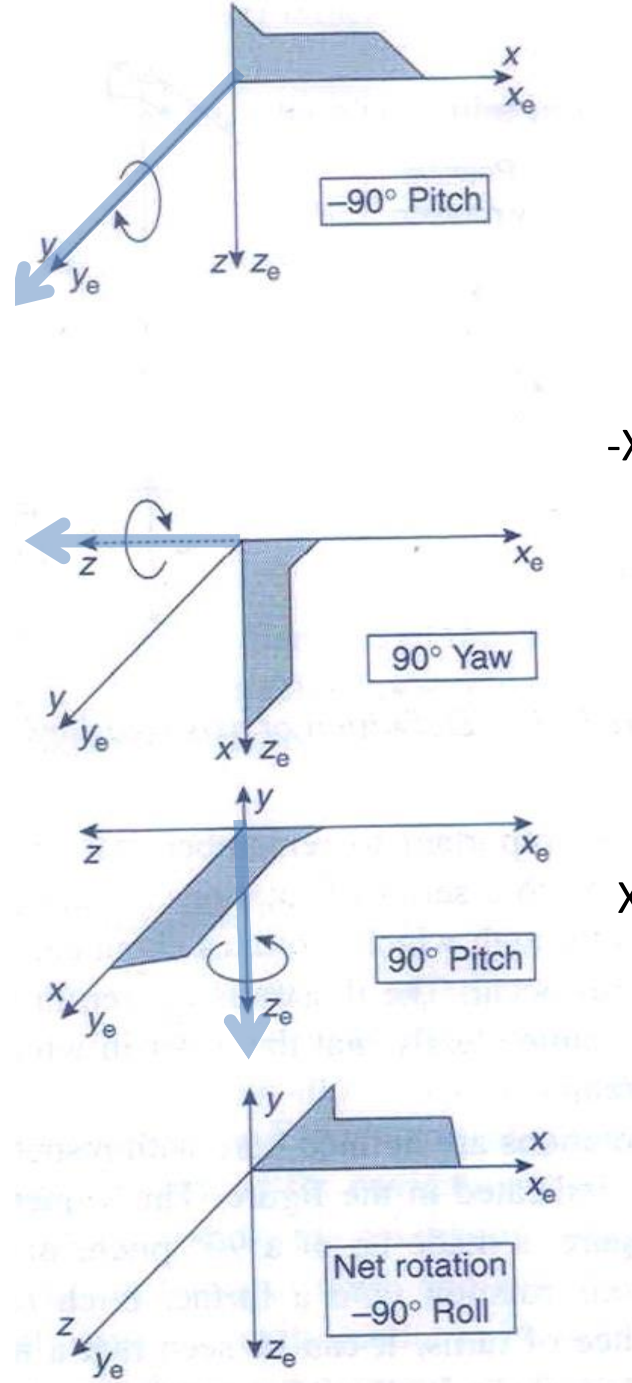


Pitch = 90 degree

Yaw = 90 degree

Pitch = -90 degree

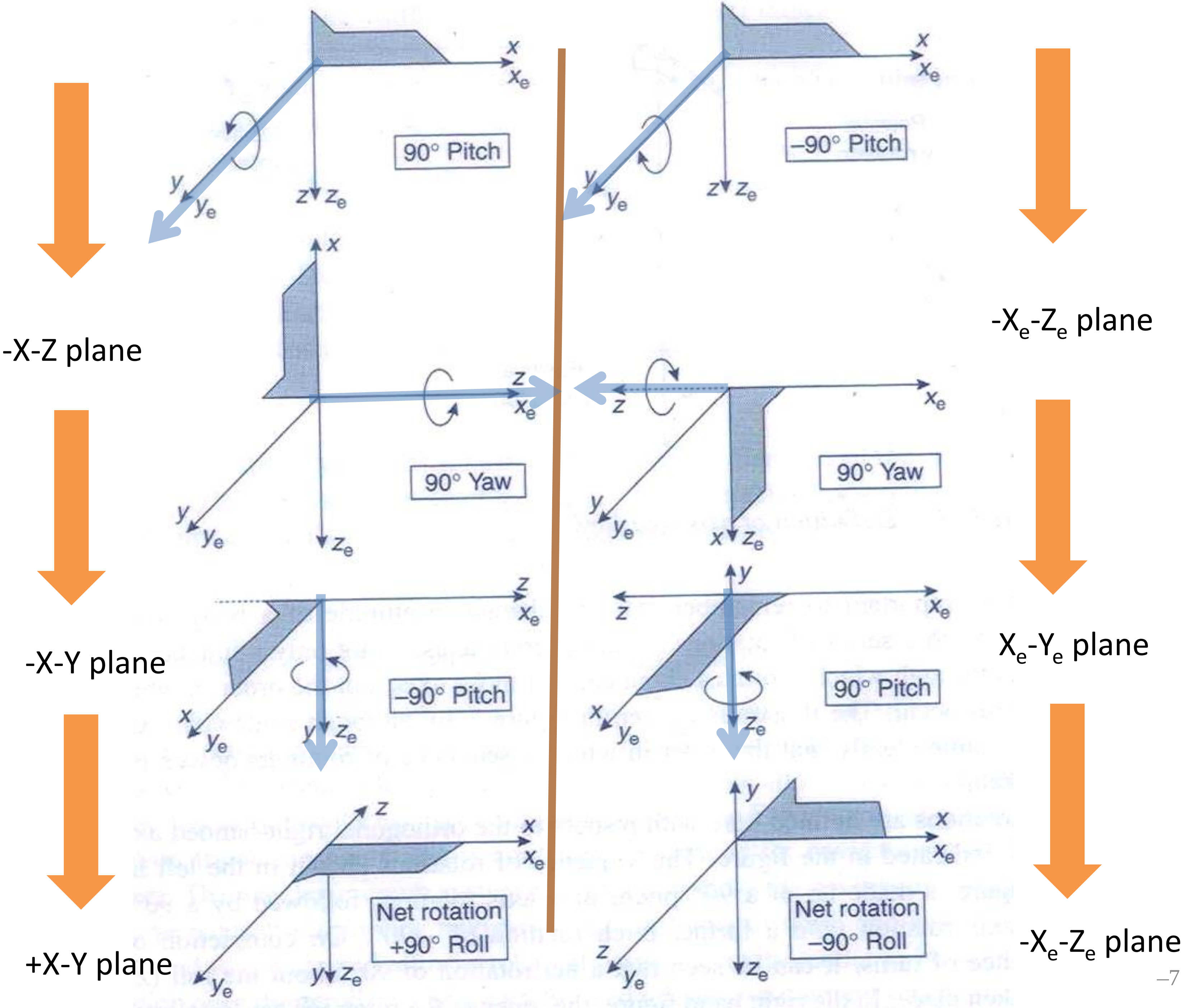
Pitch = -90 degree
Yaw = 90 degree
Pitch = 90 degree



-X-Z plane

X-Y plane

-X-Z plane


-X-Z plane
-X-Y plane
+X-Y plane
$-X_e$-Z_e plane
X_e-Y_e plane
$-X_e$-Z_e plane
90° Pitch
–90° Pitch
90° Yaw
90° Yaw
–90° Pitch
90° Pitch
Net rotation +90° Roll
Net rotation –90° Roll
x
x_e
y
y_e
z
z_e

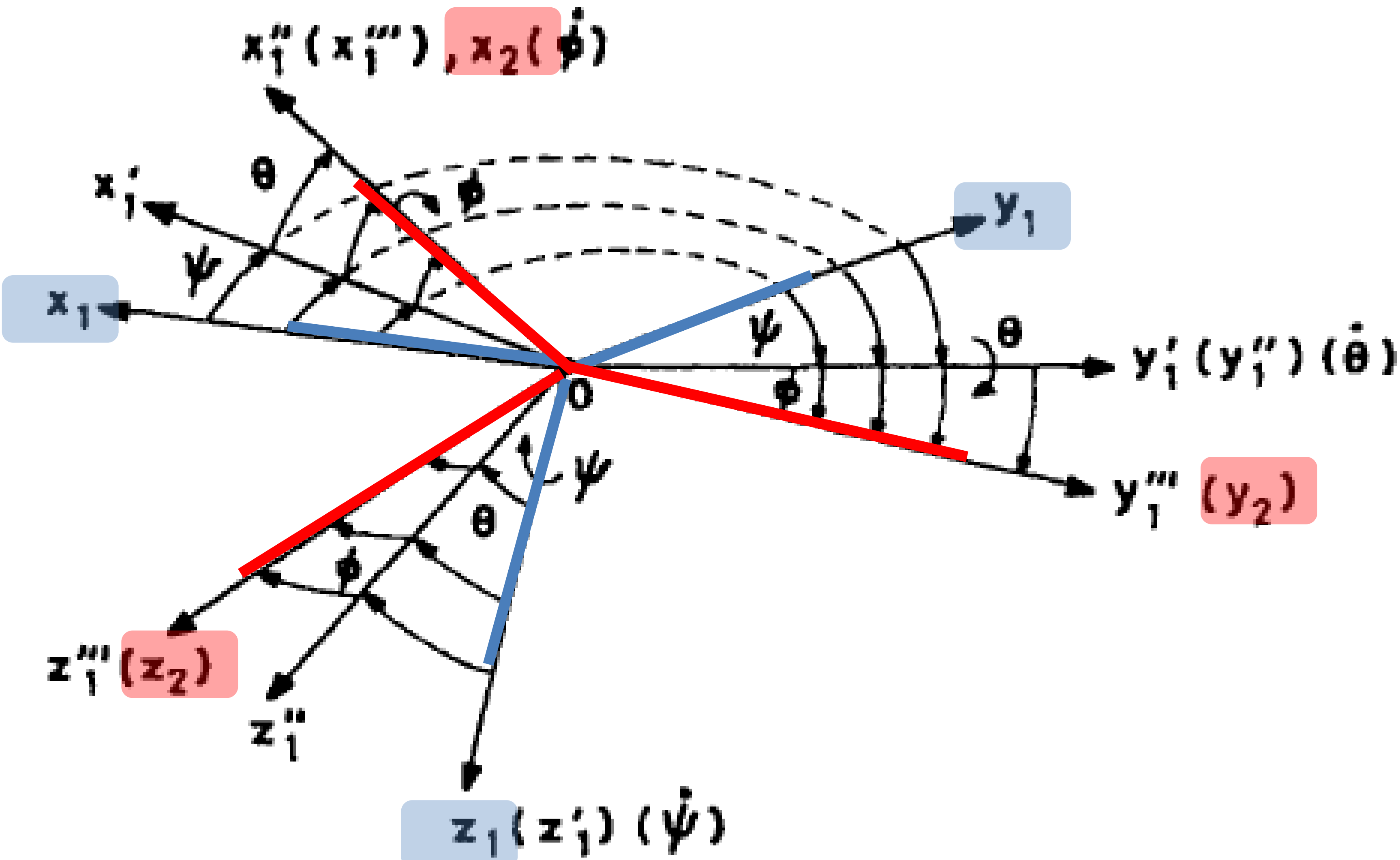
Euler Angles

Euler angles are also used to describe the orientation of a frame of reference relative to another. They are typically denoted as φ, θ, ψ.

–**Three parameterization.**

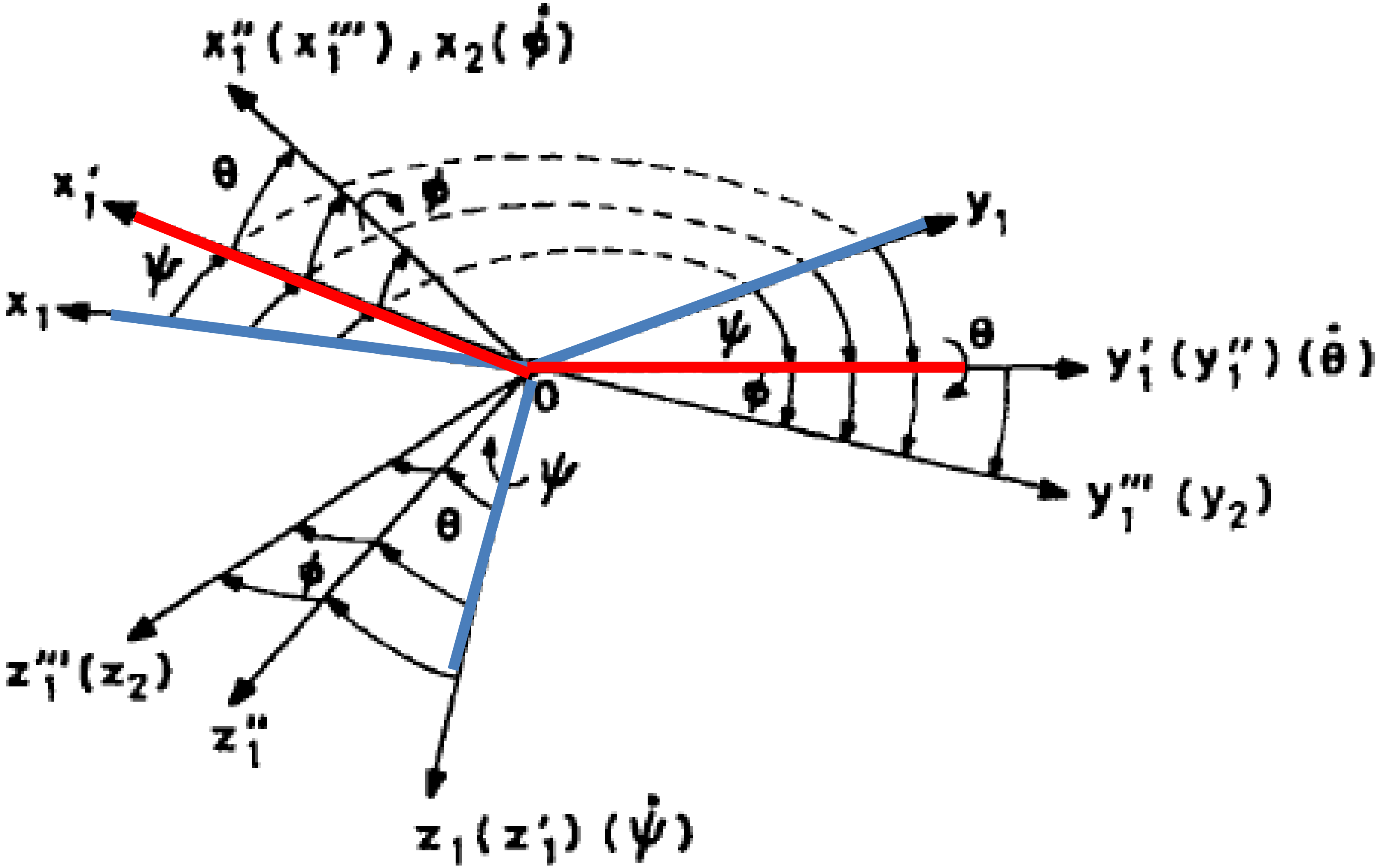
Euler angles represent a sequence of three elemental rotations, i.e. rotations about the axes of a coordinate system

Euler Angles



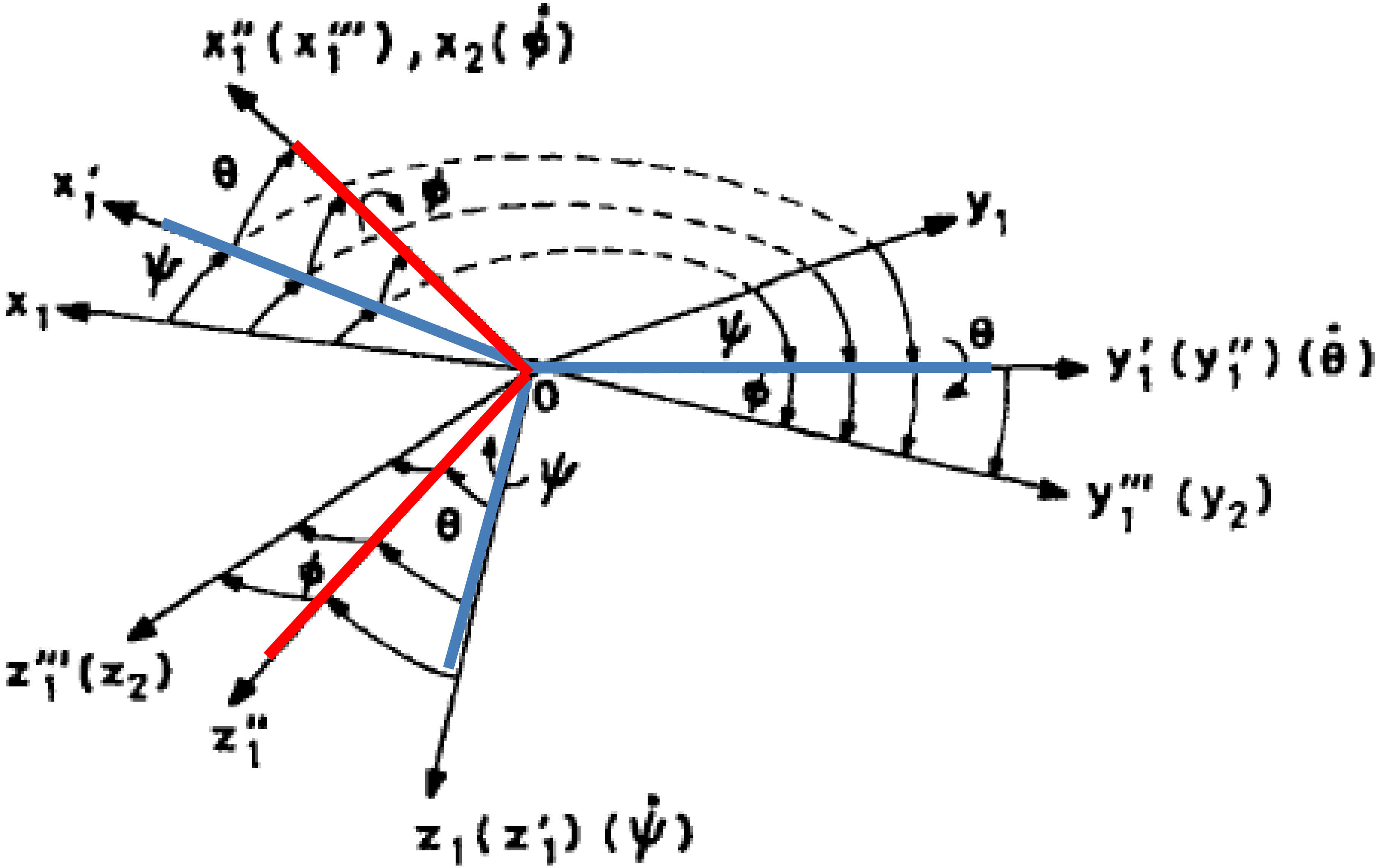
Φ, θ and ψ are Euler angles

Euler Angles



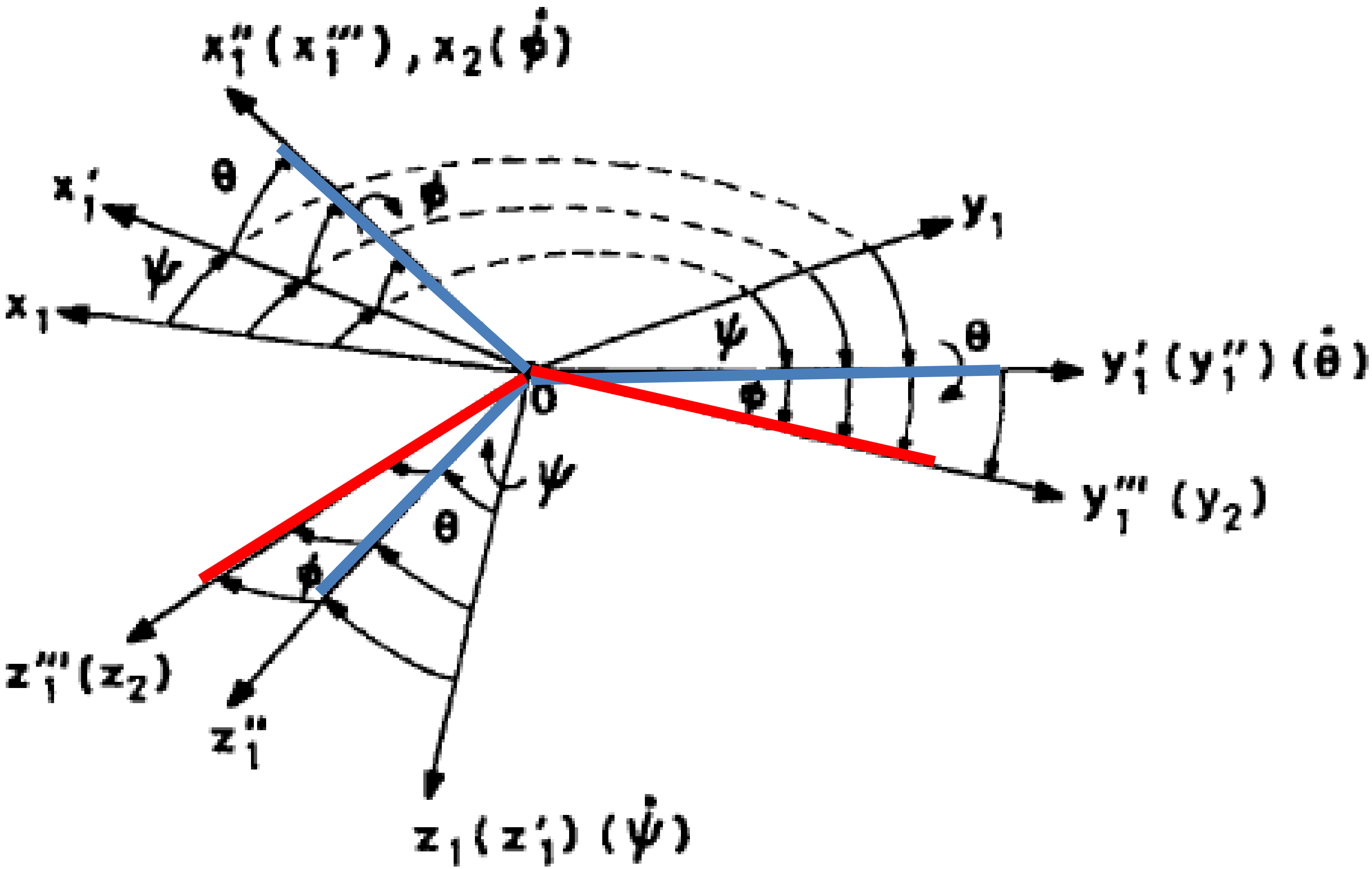
Rotation about z_1

Euler Angles



Rotation about y_1'

Euler Angles



Rotation about x_1''

Euler Angle Rates

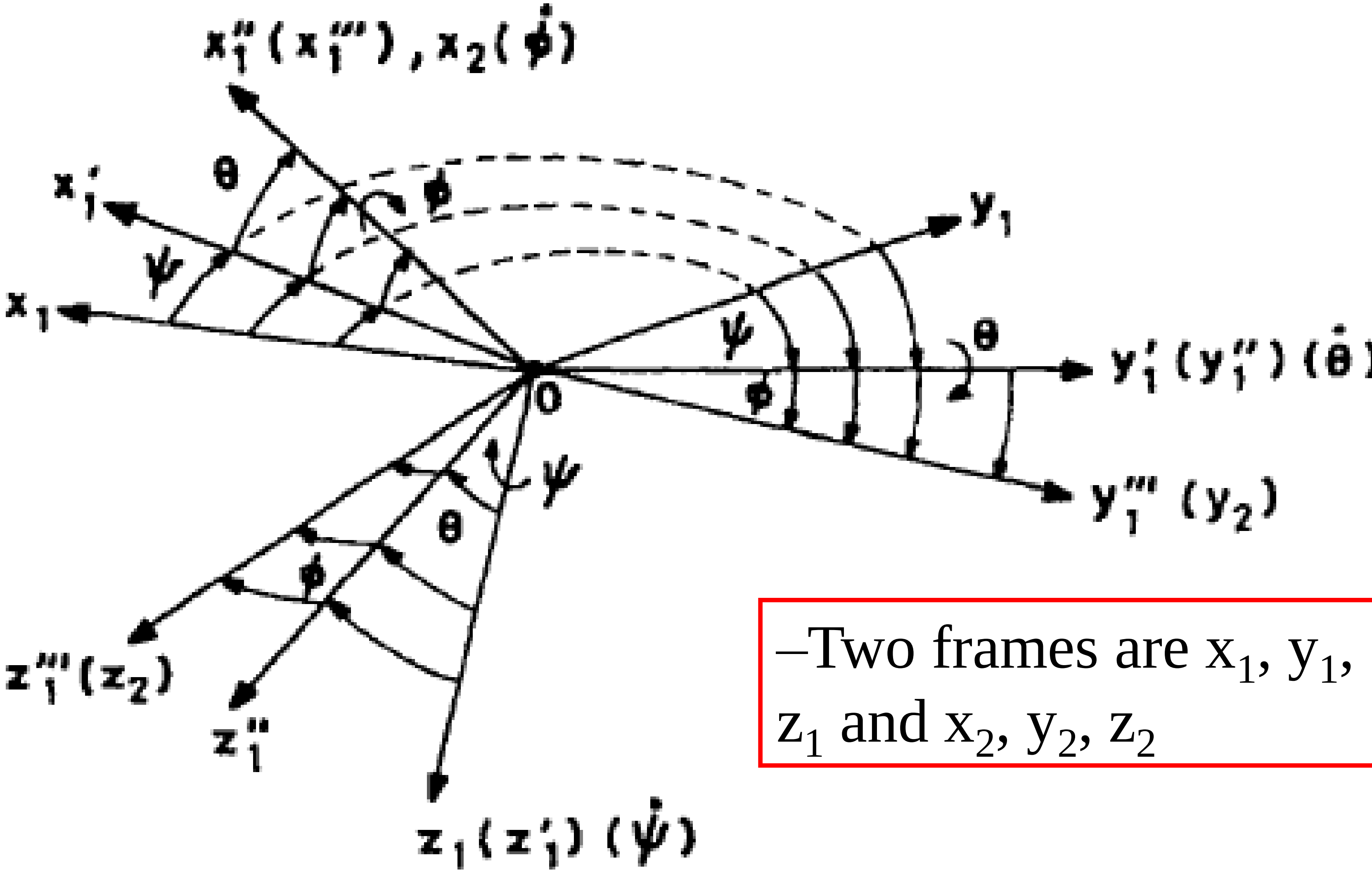
Euler angle rates are not directly measured. Angular velocity components p, q, r, which are the body axes components of angular velocity of vehicle with respect to inertial frame are available. These are available from onboard gyros.

Euler Angle Rates

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sec\theta\sin\phi & \sec\theta\cos\phi \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Euler angle rates expressed as function of body angular rate

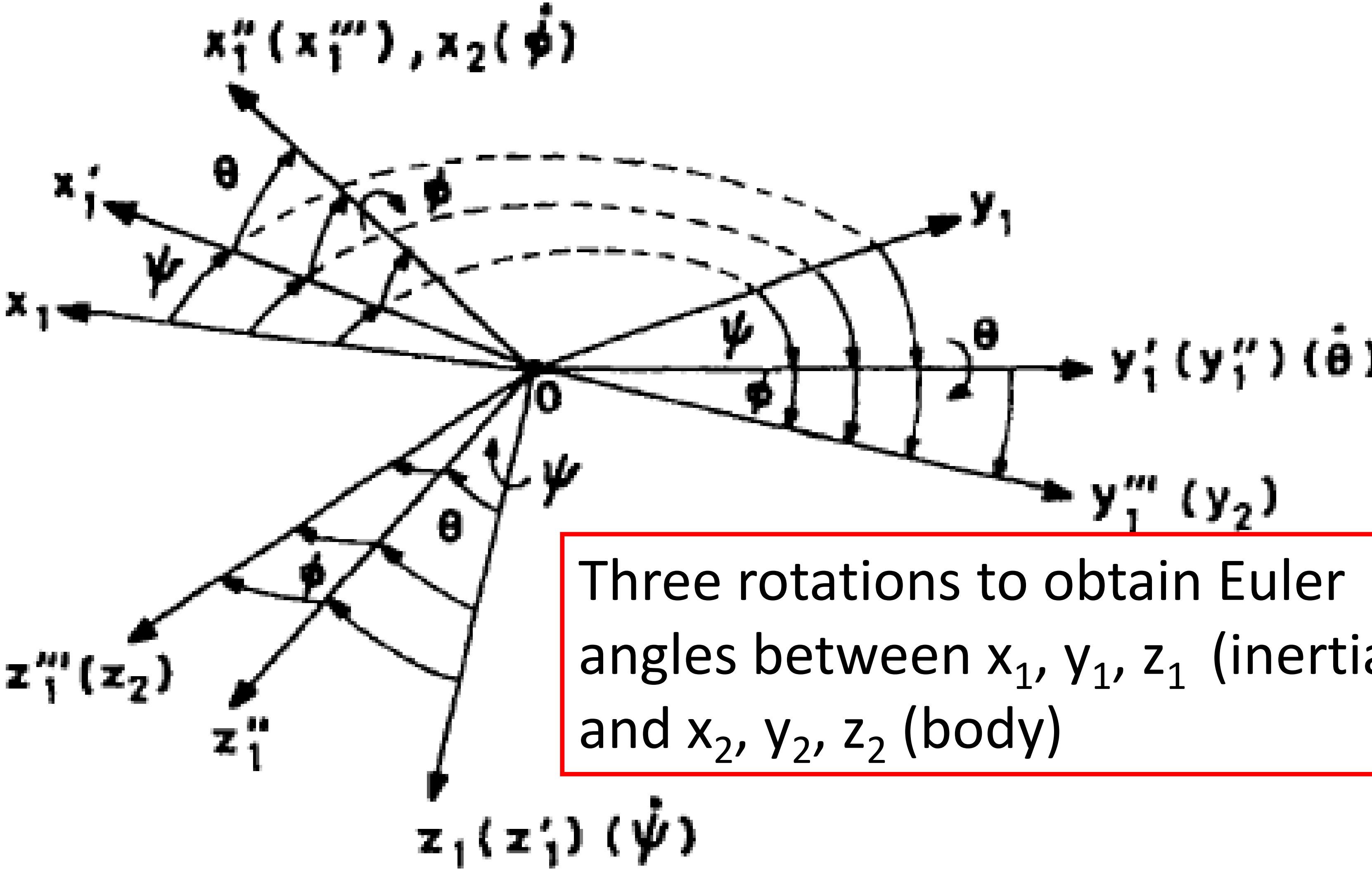
–Euler Angles



–Two frames are x_1, y_1, z_1 and x_2, y_2, z_2

Fig. 4.3 Euler angles.

Axes Transformation



Three rotations to obtain Euler angles between x_1, y_1, z_1 (inertial) and x_2, y_2, z_2 (body)

Fig. 4.3 Euler angles.

Euler Angle Rates

Consider $\dot{\psi}$ vector. It has to be transformed from $Ox_i^{'}y_i^{'}z_i^{'}$ system to $Ox_b y_b z_b$ (x_2, y_2, z_2)

$$\dot{\psi}_b = AB\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

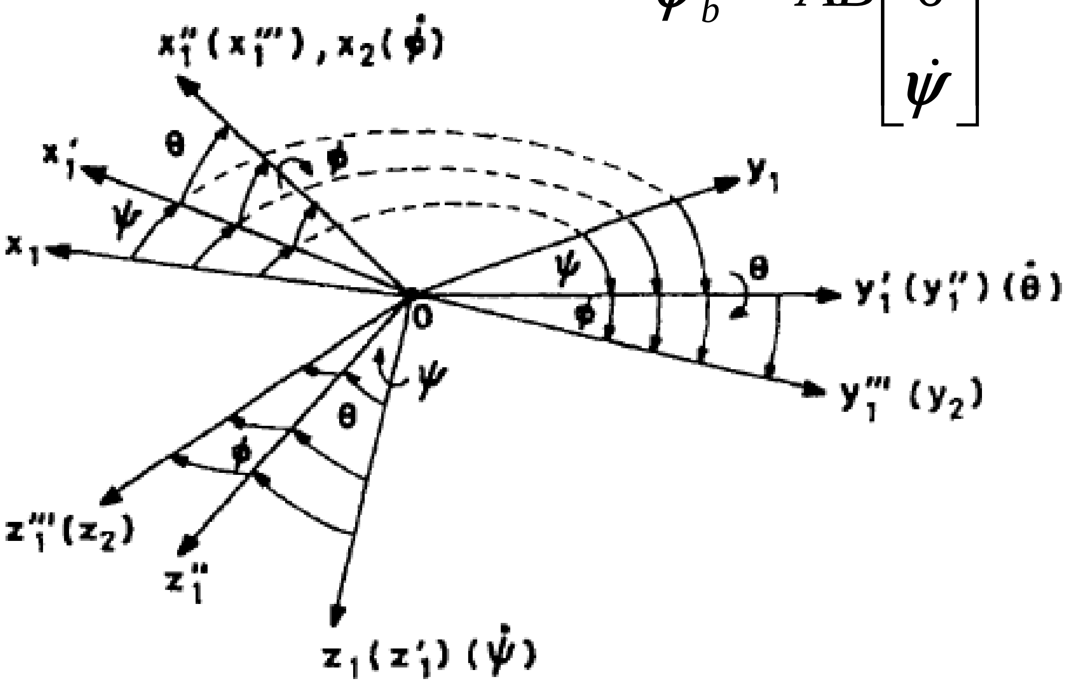


Fig. 4.3 Euler angles.

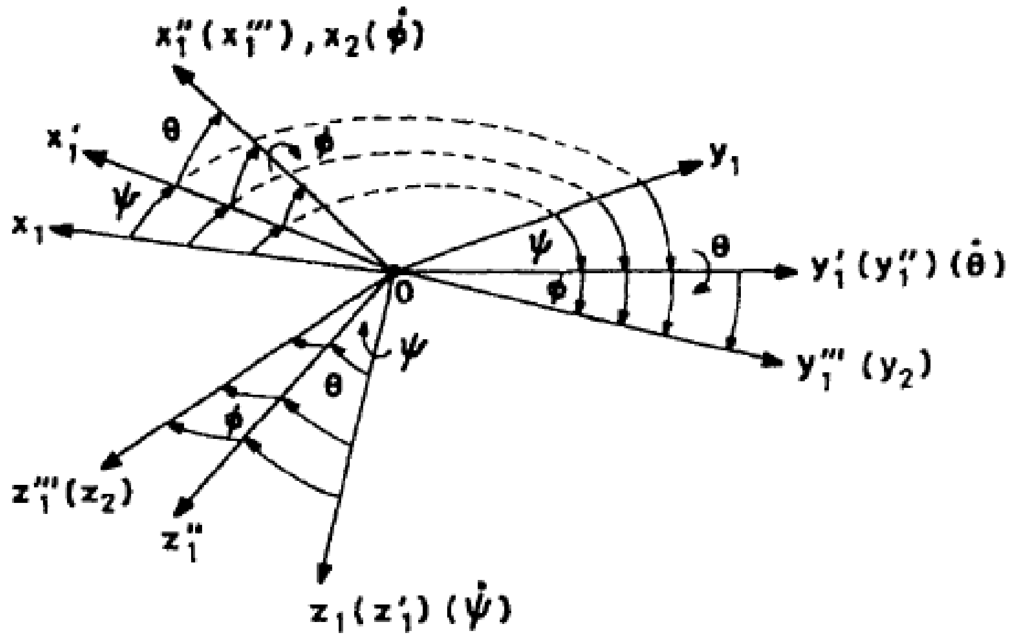


Fig. 4.3 Euler angles.

Consider $\dot{\theta}$ vector. It has to be transformed from $Ox_i''y_i''z_i''$ system to $Ox_b y_b z_b$ (x_2, y_2, z_2)

$$\dot{\theta}_b = A\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

Euler Angle Rates

Consider $\dot{\phi}$ vector. It has to be transformed from $Ox_b y_b z_b$ system to $Ox_b y_b z_b$ (x_2, y_2, z_2)

$$\dot{\phi}_b = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

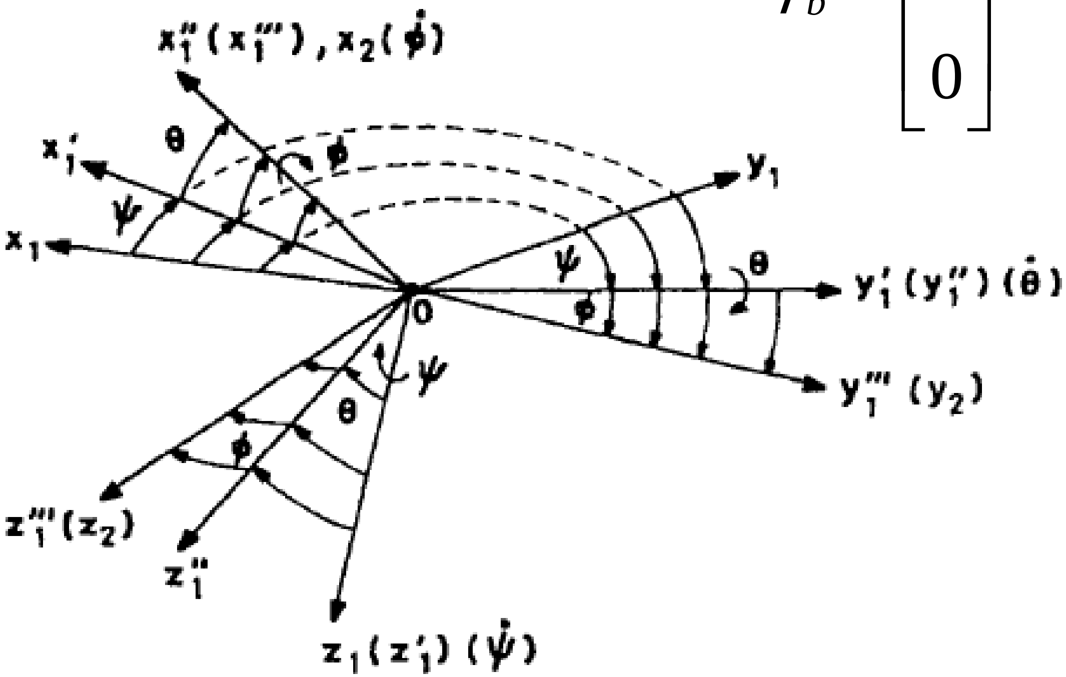


Fig. 4.3 Euler angles.

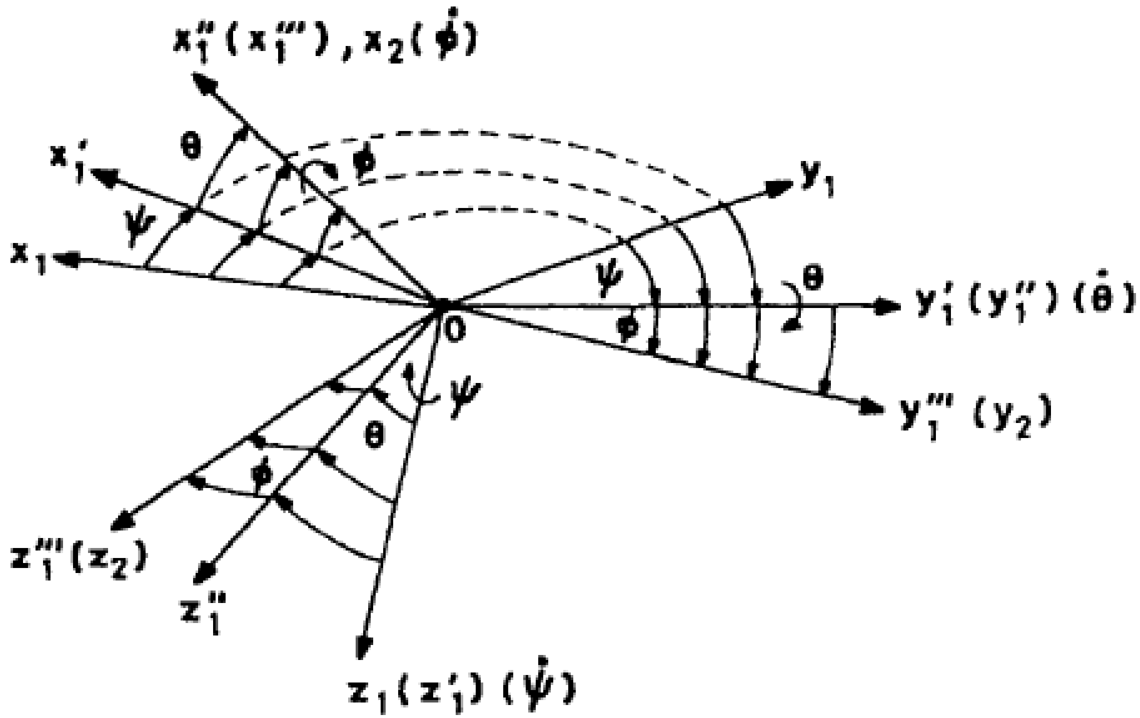


Fig. 4.3 Euler angles.

Angular velocity vector in the body axes system is given as

$$\boldsymbol{\omega}_{i,b}^{b} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \dot{\boldsymbol{\psi}}_b + \dot{\boldsymbol{\theta}}_b + \dot{\boldsymbol{\phi}}_b$$

Euler Angle Rates

Substituting matrices A and B

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = AB\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + A\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\dot{\psi}_b = AB\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \qquad \dot{\theta}_b = A\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \qquad \dot{\phi}_b = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

Euler Angle Rates

Substituting matrices A and B

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = AB\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + A\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Euler Angle Rates

Substituting matrices A and B

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = AB\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + A\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi & \sin\phi\cos\theta \\ \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

Euler Angle Rates

Substituting matrices A and B

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = AB\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + A\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$B = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi & \sin\phi\cos\theta \\ \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Euler Angle Rates

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = L_w \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \qquad L_w = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

$$p = \dot{\phi} - \dot{\psi}\sin\theta$$

$$q = \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta$$

$$r = \dot{\psi}\cos\phi\cos\theta - \dot{\theta}\sin\phi$$

Euler angle rates can be obtained by inverting above matrix (It is not an orthogonal matrix)

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = L_w^{-1} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad L_w^{-1} = \left(\frac{1}{\Delta(L)}\right) adj(L_w)$$

orthogonal matrix $A^T A = AA^T = 1$

Euler Angle Rates

$$\Delta(L) = \cos^2\phi\cos\theta + \sin^2\phi\cos\theta = \cos\theta$$

$$adj(L_w) = \begin{bmatrix} \cos\theta & \sin\theta\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$L_w^{-1} = \left(\frac{1}{\Delta(L)}\right) adj(L_w)$$

Euler Angle Rates

$$\Delta(L) = \cos^2\phi\cos\theta + \sin^2\phi\cos\theta = \cos\theta$$

$$adj(L_w) = \begin{bmatrix} \cos\theta & \sin\theta\sin\phi & \cos\phi\sin\theta \\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$L_w^{-1} = \left(\frac{1}{\Delta(L)}\right) adj(L_w)$$

$$L_w^{-1} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sec\theta\sin\phi & \sec\theta\cos\phi \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sec\theta\sin\phi & \sec\theta\cos\phi \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = \sec\theta(q\sin\phi + r\cos\phi)$$

Euler angle rates expressed as function of body angular rate

Euler Angle Rates

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = \sec\theta\left(q\sin\phi + r\cos\phi\right)$$

Singularity when pitch angle is 90 degree. It is for 3-**2**-1 rotation

When body angular rate and Euler angle rates will be equal?

Euler Angle Rates

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = \sec\theta(q\sin\phi + r\cos\phi)$$

When Euler angles are zero?

$$\dot{\phi} = p$$

$$\dot{\theta} = q$$

$$\dot{\psi} = r$$

When Euler angles are small?
Euler angular rates can be approximated as.

$$\dot{\phi} = p$$

$$\dot{\theta} = q$$

$$\dot{\psi} = r$$

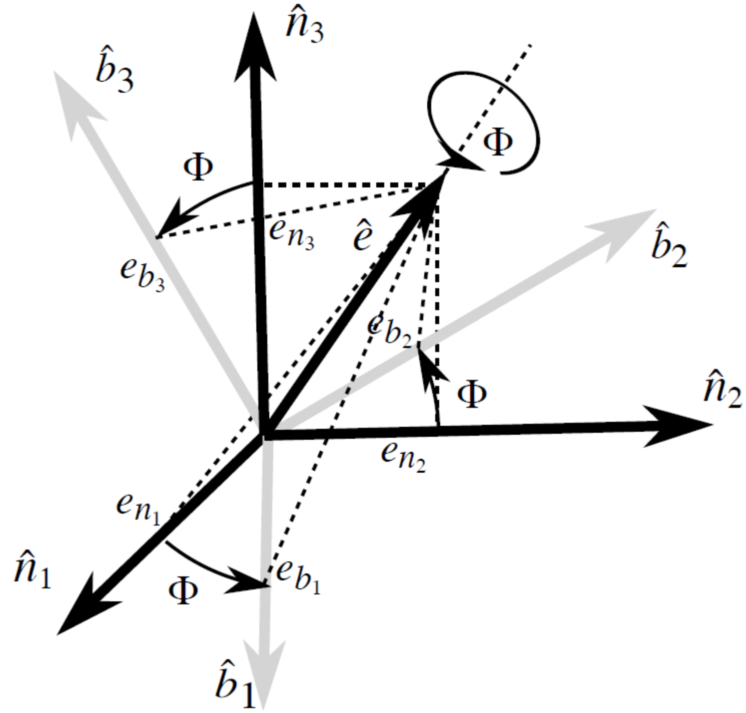
Euler Angle Rates

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$
$$\dot{\theta} = q\cos\phi - r\sin\phi$$
$$\dot{\psi} = \sec\theta(q\sin\phi + r\cos\phi)$$

Singularity when pitch angle is 90 degree. It is for 3-**2**-1 rotation

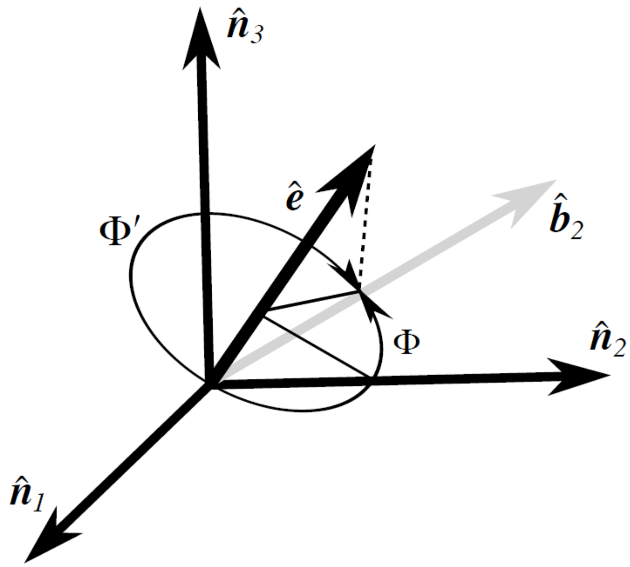
How to overcome singularity? In the framework of Euler Angles.

Quarternions



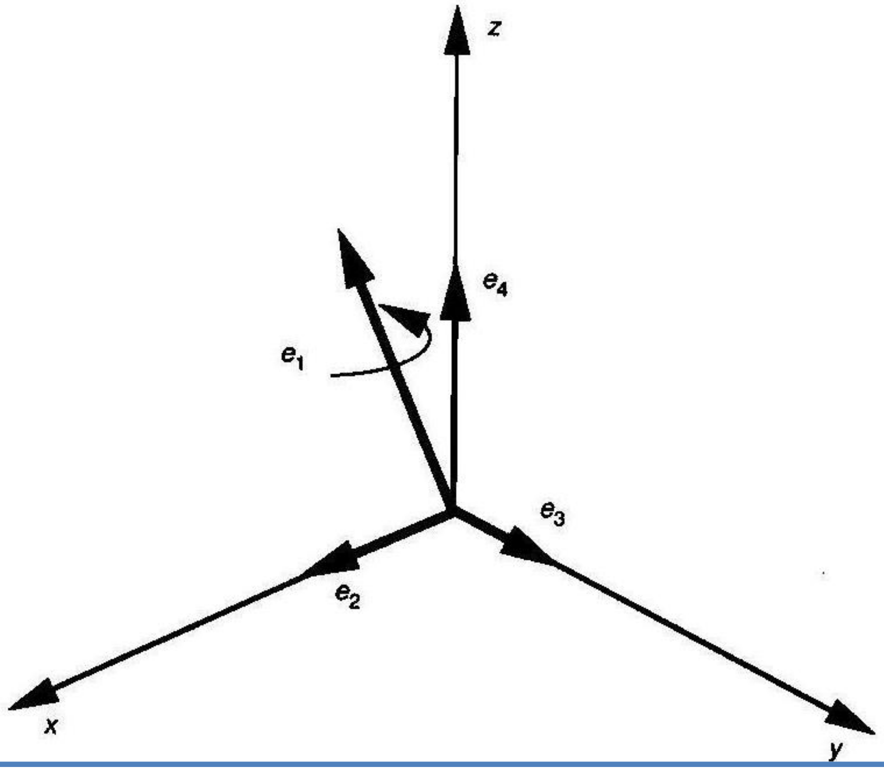
A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle " about the principal axis eˆ; the principal axis being a judicious axis fixed in both the initial and final orientation.

Quarternions



Rotation of the judiciously selected axis is not unique. Rotation can be achieved in clockwise or anticlockwise i.e. one rotation can be more than the other. It becomes tricky in some situations.

Quarternions

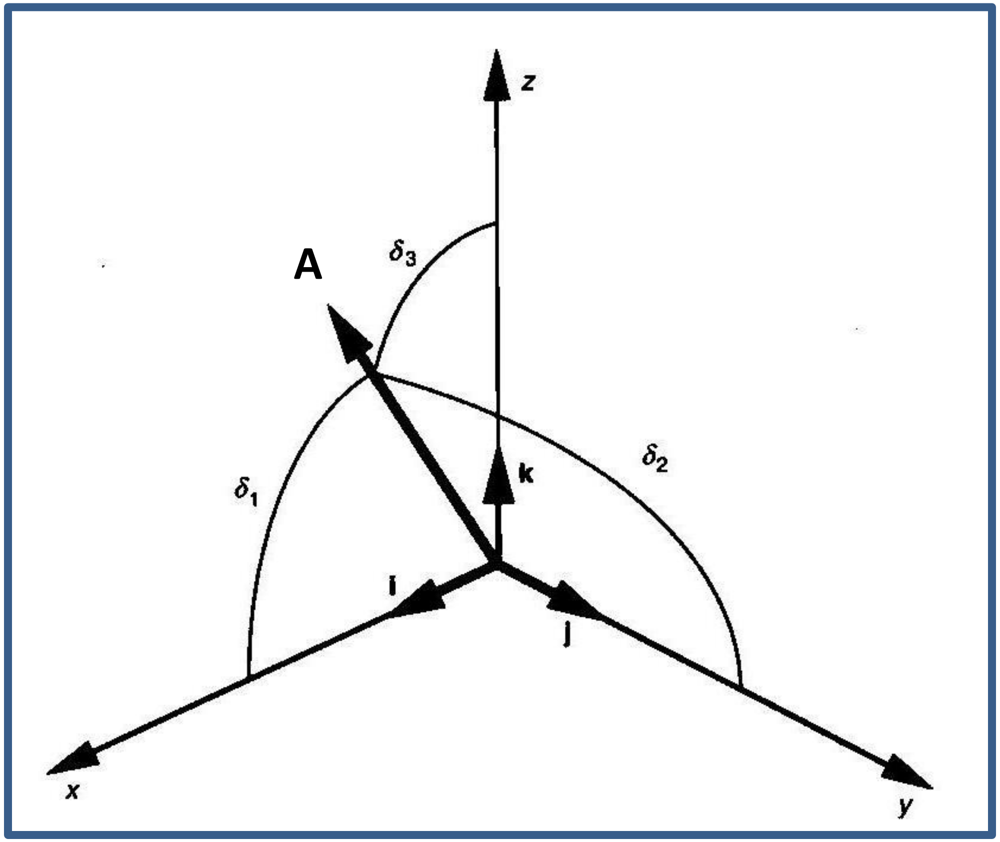


$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

$$\boxed{(e_1^2 + e_2^2 + e_3^2 + e_4^2) = 1}$$

A solid body rotation from one attitude to another, by a single rotation about some axis in reference frame. Four parameters are, three direction cosines of a unit vector aligned with rotational axis and fourth parameter is rotation angle

Direction Cosine Matrix



A vector A makes angle δ_1 , δ_2 and δ_3 with the axis x, y and z

Axes Transformation

Consider a vector $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$

$\delta_1, \delta_2, \delta_3$ angles with the x, y and x axis

$$\cos\delta_1 = \frac{A_x}{|A|}$$

$$\cos\delta_2 = \frac{A_y}{|A|}$$

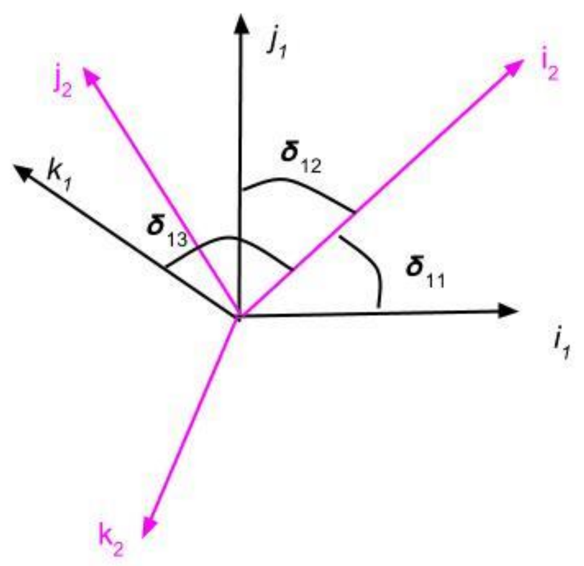
$$\cos\delta_3 = \frac{A_z}{|A|}$$

$$\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$$

$\cos\delta_1, \cos\delta_2, \cos\delta_3$ direction cosines of vector *A* with respect to x, y and z axis

Direction Cosine Transformation

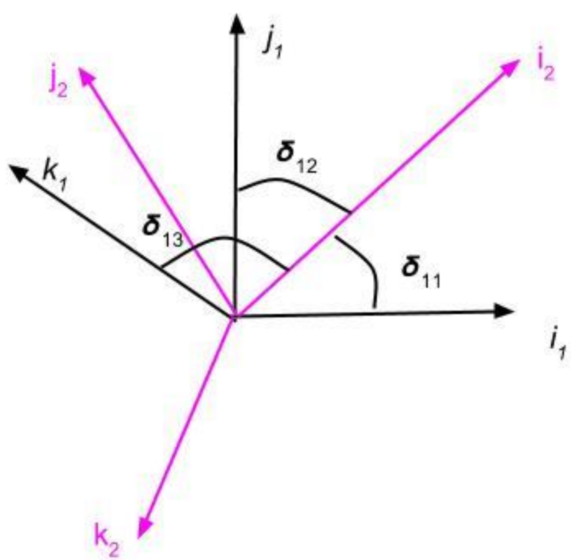
Consider the transformation of a vector from one reference frame to another reference frame. Transformation from x_1, y_1, z_1 to x_2, y_2, z_2



Direction Cosine Transformation

Consider the transformation of a vector from one reference frame to another reference frame. Transformation from x_1, y_1, z_1 to x_2, y_2, z_2

$$\hat{i}_2 = C_{11}\hat{i}_1 + C_{12}\hat{j}_1 + C_{13}\hat{k}_1$$



$$C_{11} = \cos(\delta_{11})$$
$$C_{12} = \cos(\delta_{12})$$
$$C_{13} = \cos(\delta_{13})$$

Direction Cosine Transformation

Consider the transformation of a vector from one reference frame to another reference frame. Transformation from x_1, y_1, z_1 to x_2, y_2, z_2

$$\hat{i}_2 = C_{11}\hat{i}_1 + C_{12}\hat{j}_1 + C_{13}\hat{k}_1$$

$$\hat{j}_2 = C_{21}\hat{i}_1 + C_{22}\hat{j}_1 + C_{23}\hat{k}_1$$

$$\hat{k}_2 = C_{31}\hat{i}_1 + C_{32}\hat{j}_1 + C_{33}\hat{k}_1$$

C_{11}, C_{12}, C_{13} are the direction cosines of the $\hat{i}_2$ unit vector with respect to the x_1, y_1, z_1 axis system

Direction Cosine Transformation

$$\hat{i}_2 = C_{11}\hat{i}_1 + C_{12}\hat{j}_1 + C_{13}\hat{k}_1$$

$$\hat{j}_2 = C_{21}\hat{i}_1 + C_{22}\hat{j}_1 + C_{23}\hat{k}_1$$

$$\hat{k}_2 = C_{31}\hat{i}_1 + C_{32}\hat{j}_1 + C_{33}\hat{k}_1$$

In matrix form

$$\begin{bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$

$$C_1^2 = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_2^1 = \left[C_1^2\right]^{-1} = \left[C_1^2\right]^T$$

$$C_2^1 C_1^2 = I$$

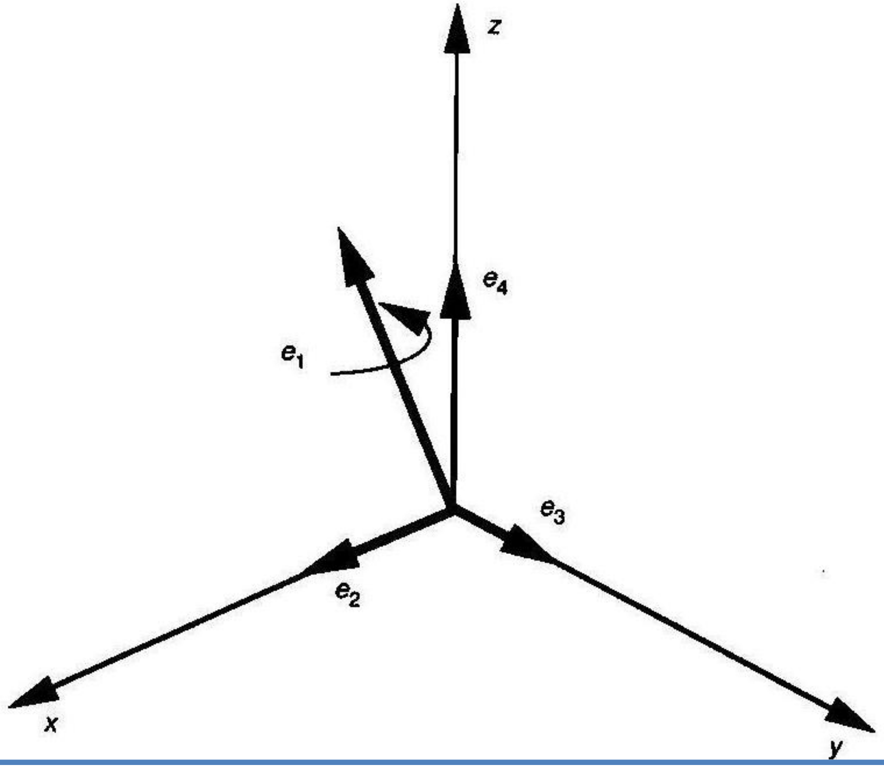
Direction Cosine Transformation

Nine equations can be obtained by matrix multiplication. DCM is an orthogonal matrix

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Out of nine equations three equations are repeated. We will have six equations relating nine parameters. There are six constraint equations and nine parameters. Only three of them are independent.

Quarternions



$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

$$\boxed{(e_1^2 + e_2^2 + e_3^2 + e_4^2) = 1}$$

A solid body rotation from one attitude to another, by a single rotation about some axis in reference frame. Four parameters are, three direction cosines of a unit vector aligned with rotational axis and fourth parameter is rotation angle

Quarternions

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad \boxed{\text{or}} \quad e = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad \boxed{\text{or}} \quad q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad \boxed{\text{or}} \quad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Some other notation in literature. Careful when converting quaternions to euler angles.

Quarternions

$$\mathcal{R}_v^b(\phi,\theta,\psi) \doteq \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

Vector transformation from *v* to *b*

$$R_v^B = \begin{bmatrix} (e_1^2 - e_2^2 - e_3^2 + e_4^2) & 2(e_1e_2 + e_3e_4) & 2(e_2e_4 - e_1e_3) \\ 2(e_3e_4 - e_1e_2) & (e_1^2 - e_2^2 + e_3^2 - e_4^2) & 2(e_2e_3 + e_1e_4) \\ 2(e_1e_3 + e_2e_4) & 2(e_2e_3 - e_1e_4) & (e_1^2 + e_2^2 - e_3^2 - e_4^2) \end{bmatrix}$$

Quarternions

$$\mathcal{R}_v^b(\phi,\theta,\psi) \doteq \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}$$

$$\boxed{R_v^B = \begin{bmatrix} (e_1^2 - e_2^2 - e_3^2 + e_4^2) & 2(e_1e_2 + e_3e_4) & 2(e_2e_4 - e_1e_3) \\ 2(e_3e_4 - e_1e_2) & (e_1^2 - e_2^2 + e_3^2 - e_4^2) & 2(e_2e_3 + e_1e_4) \\ 2(e_1e_3 + e_2e_4) & 2(e_2e_3 - e_1e_4) & (e_1^2 + e_2^2 - e_3^2 - e_4^2) \end{bmatrix}}$$

$$\boxed{R_v^B = [3 \times 3]}$$

Quarternions

$$\theta = \sin^{-1}(-2(e_4 e_2 - e_1 e_3))$$

$$\phi = \cos^{-1}\left(\frac{(e_1^2 - e_4^2 - e_3^2 + e_2^2)}{\sqrt{1-4(e_4 e_2 - e_1 e_3)^2}}\right)\mathrm{sgn}(2(e_2 e_3 + e_1 e_4))$$

$$\psi = \cos^{-1}\left(\frac{(e_1^2 + e_4^2 - e_3^2 - e_2^2)}{\sqrt{1-4(e_4 e_2 - e_1 e_3)^2}}\right)\mathrm{sgn}(2(e_4 e_3 + e_1 e_2))$$

Euler angles from Quaternions

Quarternions

$$e_1 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}$$

$$e_2 = \sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}$$

$$e_3 = \cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}$$

$$e_4 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}$$

Quaternions from euler angles

Quarternions

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Euler angles rate 3-2-1

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -r & -q & -p \\ r & 0 & -p & q \\ q & p & 0 & -r \\ p & -q & r & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

$$(e_1^2 + e_2^2 + e_3^2 + e_4^2) = 1$$

Quaternions rate

Quaternion computations

$$\dot{\mathbf{e}}_0 = -\frac{1}{2}(\mathbf{e}_1\mathbf{p} + \mathbf{e}_2\mathbf{q} + \mathbf{e}_3\mathbf{r}) + \lambda\boldsymbol{\varepsilon}\mathbf{e}_0$$

$$\dot{\mathbf{e}}_1 = \frac{1}{2}(\mathbf{e}_0\mathbf{p} + \mathbf{e}_2\mathbf{r} - \mathbf{e}_3\mathbf{q}) + \lambda\boldsymbol{\varepsilon}\mathbf{e}_1$$

$$\dot{\mathbf{e}}_2 = \frac{1}{2}(\mathbf{e}_0\mathbf{q} + \mathbf{e}_3\mathbf{p} - \mathbf{e}_1\mathbf{r}) + \lambda\boldsymbol{\varepsilon}\mathbf{e}_2$$

$$\dot{\mathbf{e}}_3 = \frac{1}{2}(\mathbf{e}_0\mathbf{r} + \mathbf{e}_1\mathbf{q} - \mathbf{e}_2\mathbf{p}) + \lambda\boldsymbol{\varepsilon}\mathbf{e}_3$$

$$\boldsymbol{\varepsilon} = 1 - (\mathbf{e}_0^2 + \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2)$$

λ = small multiple of the integration time step

DCM Rates

In matrix form

$$\begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b$$

$$\Omega_{ib}^b = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

Derivative of direction cosines can be used to find variation of direction cosines with time

Direction Cosine Transformation

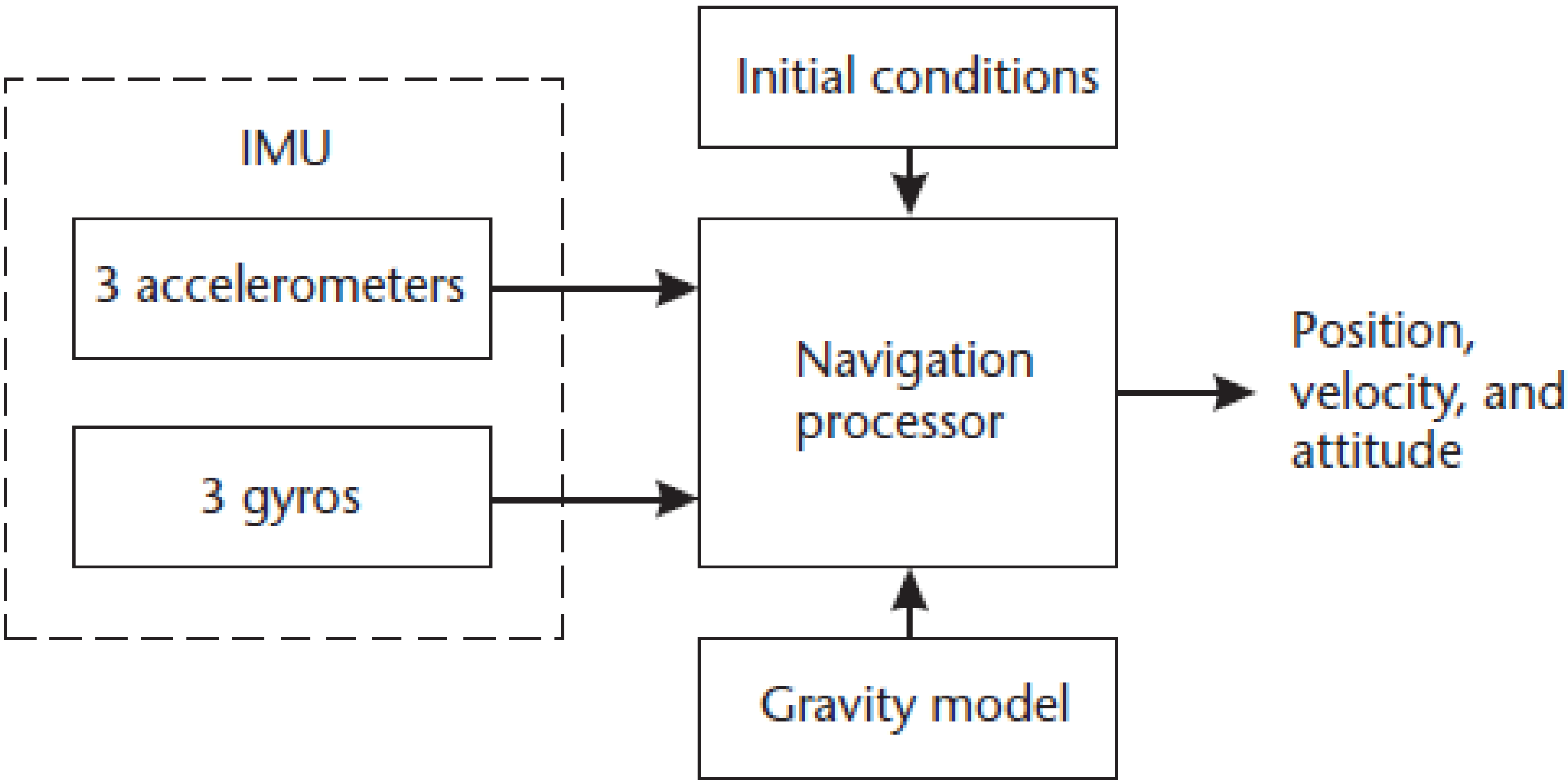
$$\begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\theta\sin\phi\cos\psi-\sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi+\cos\psi\cos\phi & \sin\phi\cos\theta \\ \sin\theta\cos\phi\cos\psi+\sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi-\cos\psi\sin\phi & \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{11} = \cos\theta\cos\psi$$
$$C_{12} = \cos\theta\sin\psi$$
$$C_{13} = -\sin\theta$$

$$C_{21} = \sin\theta\sin\phi\cos\psi - \sin\psi\cos\phi$$
$$C_{22} = \sin\theta\sin\phi\sin\psi + \cos\psi\cos\phi$$
$$C_{23} = \sin\phi\cos\theta$$

$$C_{31} = \sin\theta\cos\phi\cos\psi + \sin\psi\sin\phi$$
$$C_{32} = \sin\theta\cos\phi\sin\psi - \cos\psi\sin\phi$$
$$C_{33} = \cos\phi\cos\theta$$

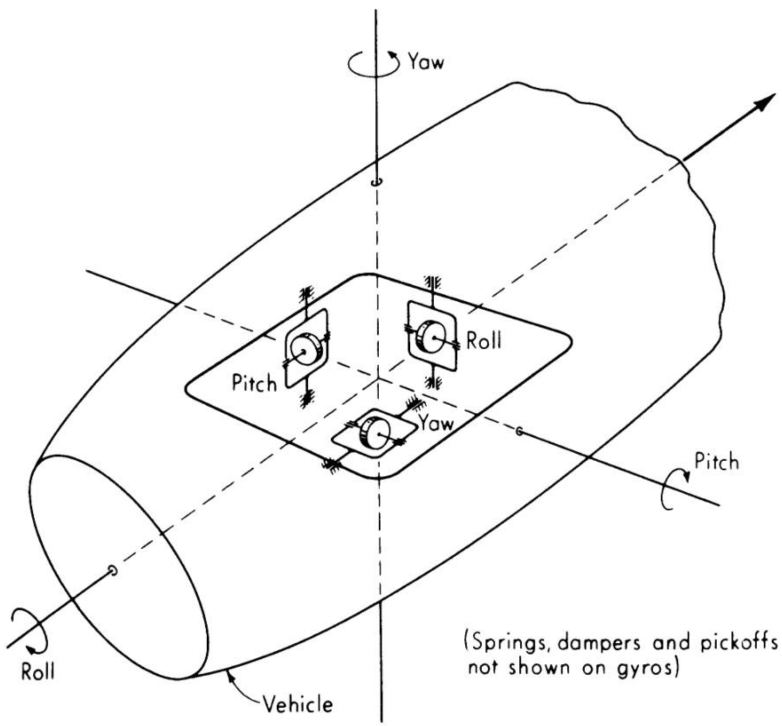
$$\theta = \sin^{-1}\left(-C_{13}\right)$$
$$\phi = \sin^{-1}\left(\frac{C_{23}}{\sqrt{1-C_{13}^2}}\right)$$
$$\psi = \sin^{-1}\left(\frac{C_{12}}{\sqrt{1-C_{13}^2}}\right)$$



Schematic of Inertial Navigation System

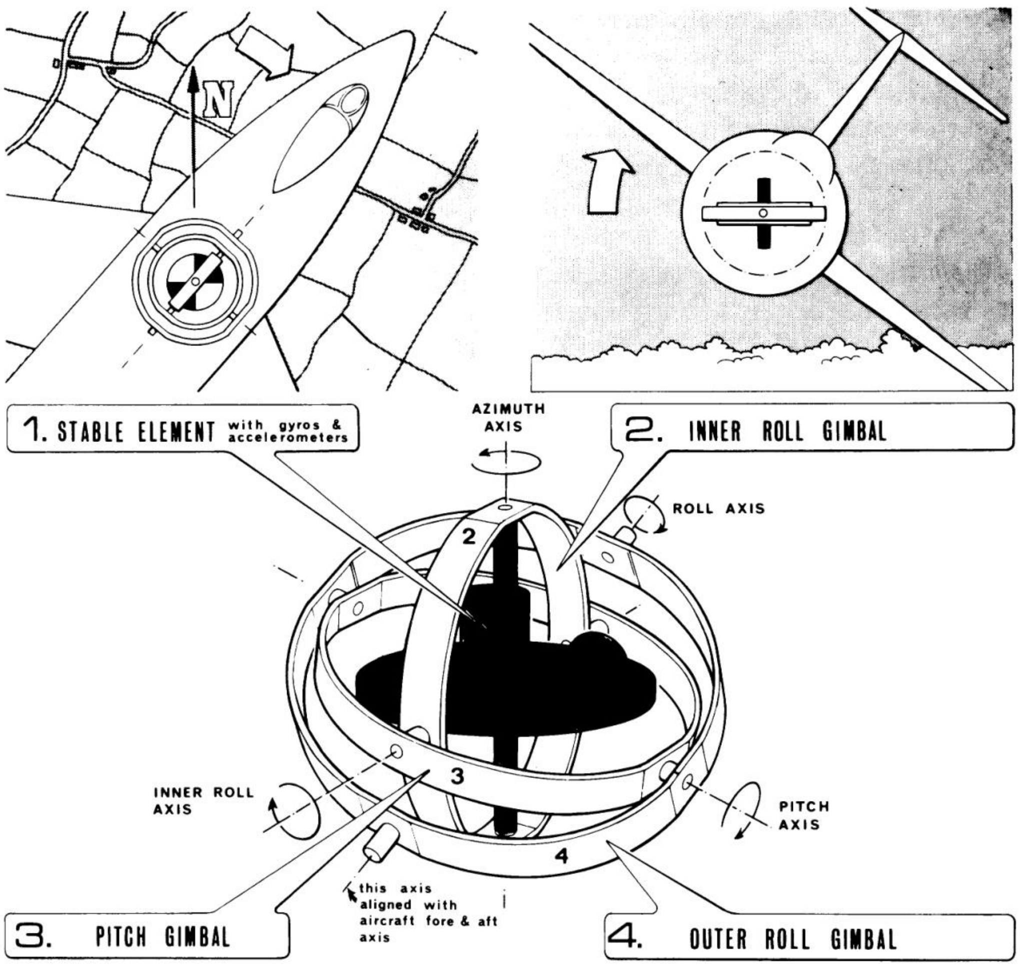
Gyroscope

Gyroscope measures body angular rates in their respective axis. Three single axis gyroscopes can be used to find three angular rates or rotations. Angular rates can be used to obtain the attitude of the aircraft.



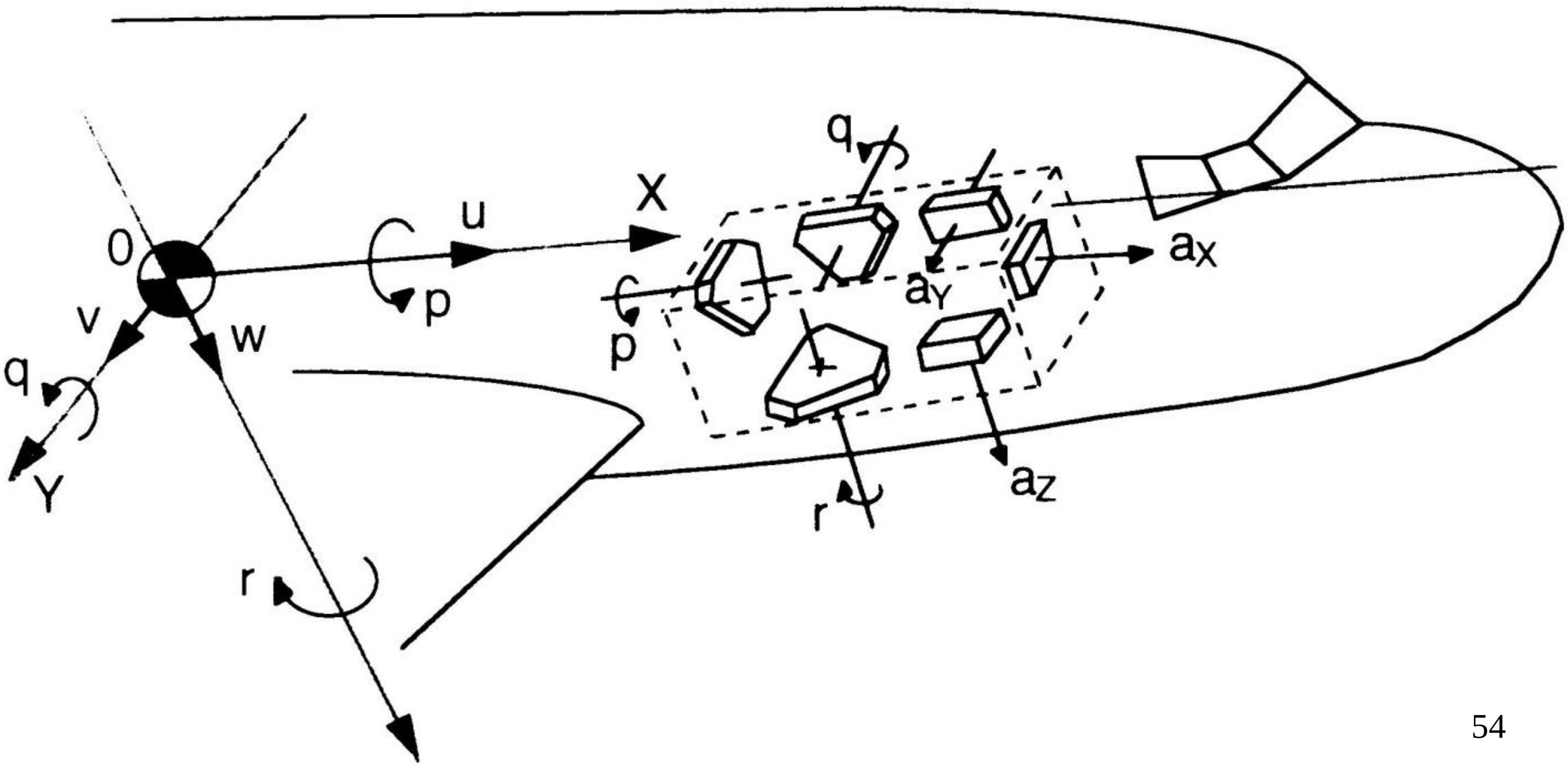
Attitude/heading reference systems

Stable platform: Rate gyros and accelerometers are suspended on a set of gimbals. Angular rotation can be obtained by the position of gimbal. The gimbals maintain the orientation and it remain constant.

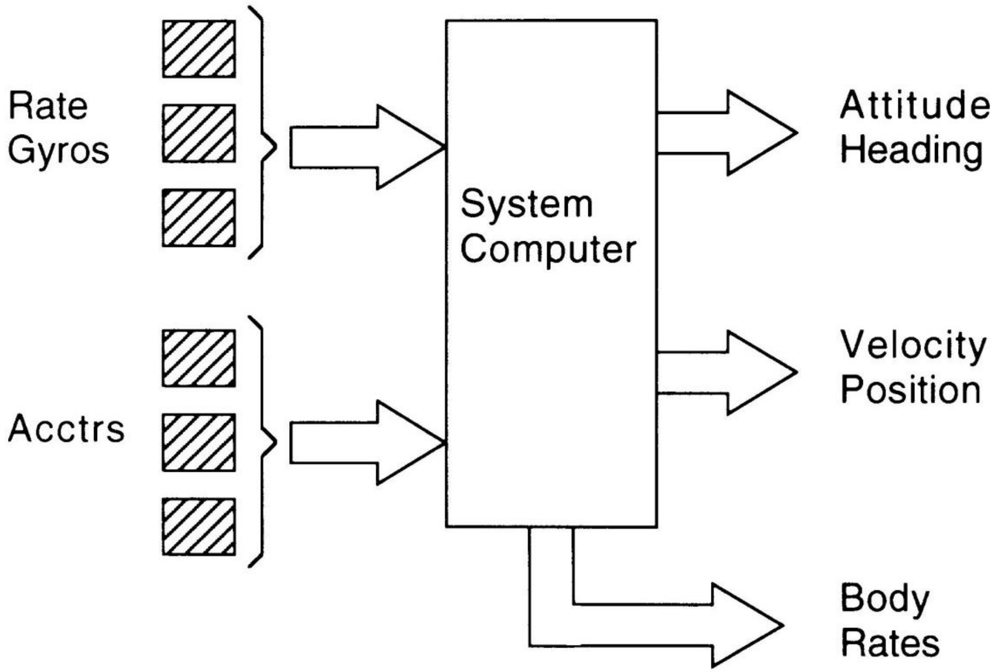


Attitude/heading reference systems

Strap down system: Rate gyros, accelerometers are strapped to the body of aircraft or all the sensor are fixed with respect to the body of aircraft. All the quantities are measured with respect to the body axis. Computation required are extensive.



Euler angles are computed by the system computer and it is equal to the gimbal angles. In case of stable platform Euler angles are computed mechanically.



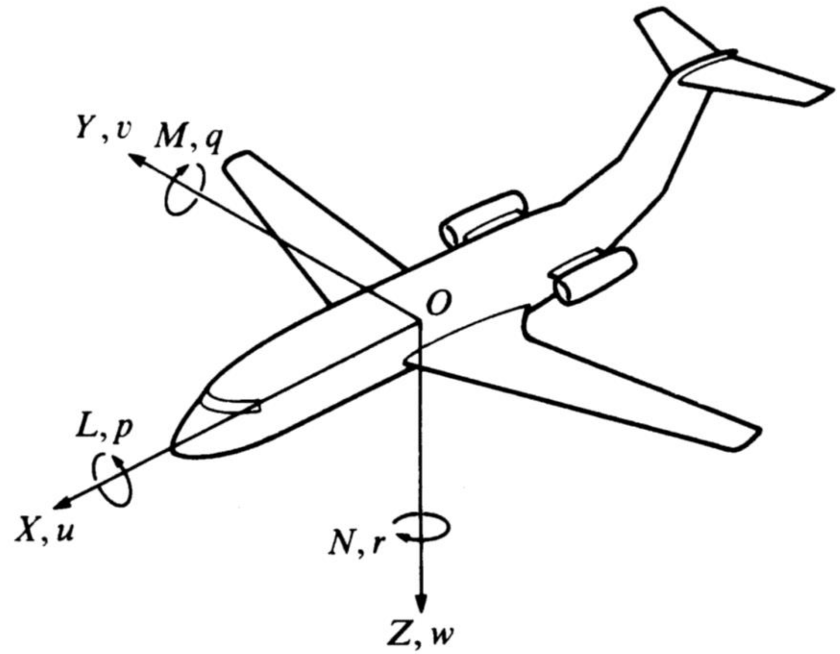
Accuracy requirement for AHRS

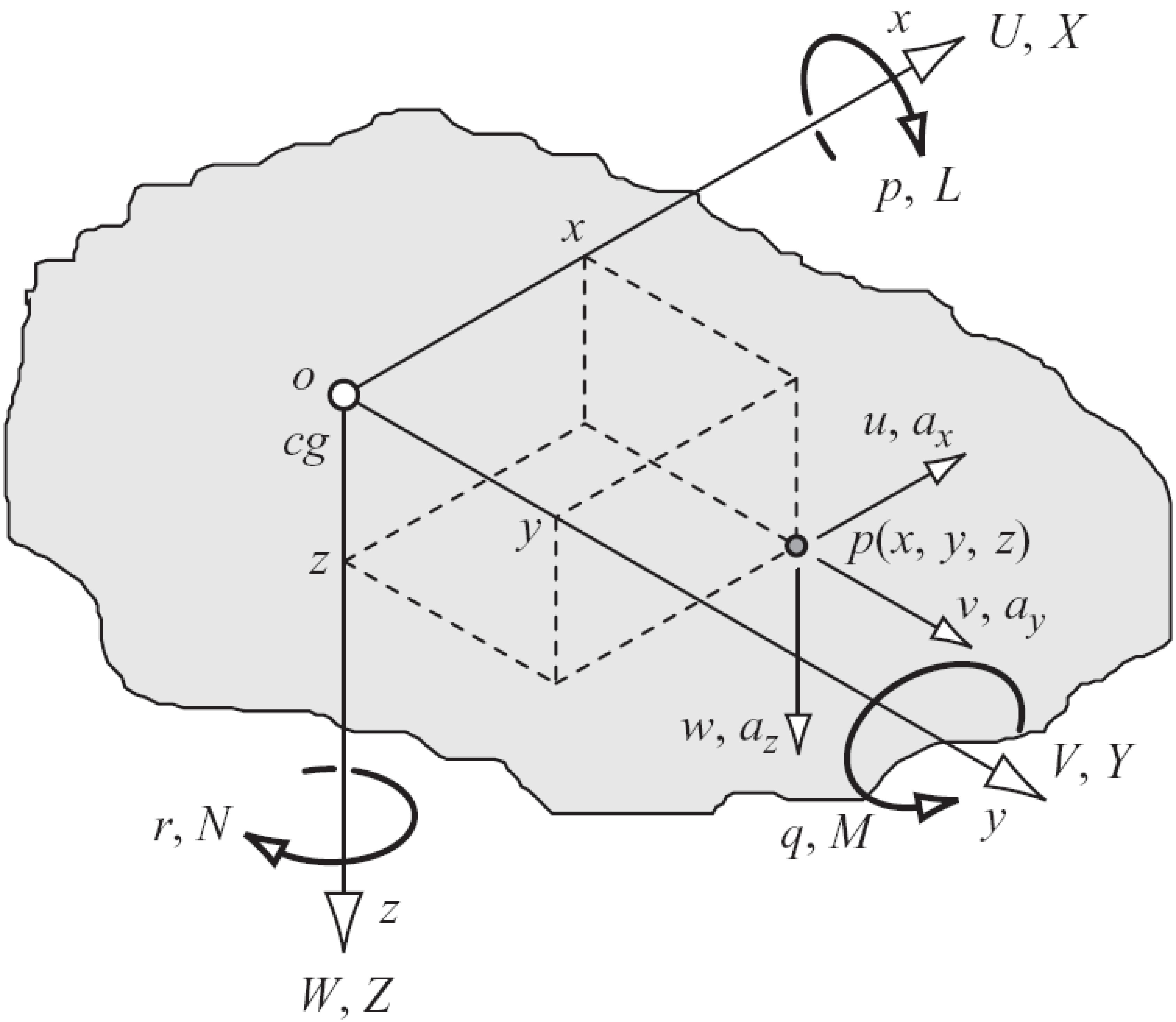
		FCS	Strap down INS
Gyro	Scale factor	0.5%	0.001 % (10 ppm)
	Zero offset	1^{o} /min	0.01^{o} /hour
Acctr	Scale factor	0.5 %	0.01% (100 ppm)
	Zero offset	5×10^{-3} g	5×10^{-5} g (50 μg)

Cost of INS is mainly due to high accuracy and reliability requirement. Moving part inertial sensors are getting replaced by solid state inertial sensors.

Inertial Measurement

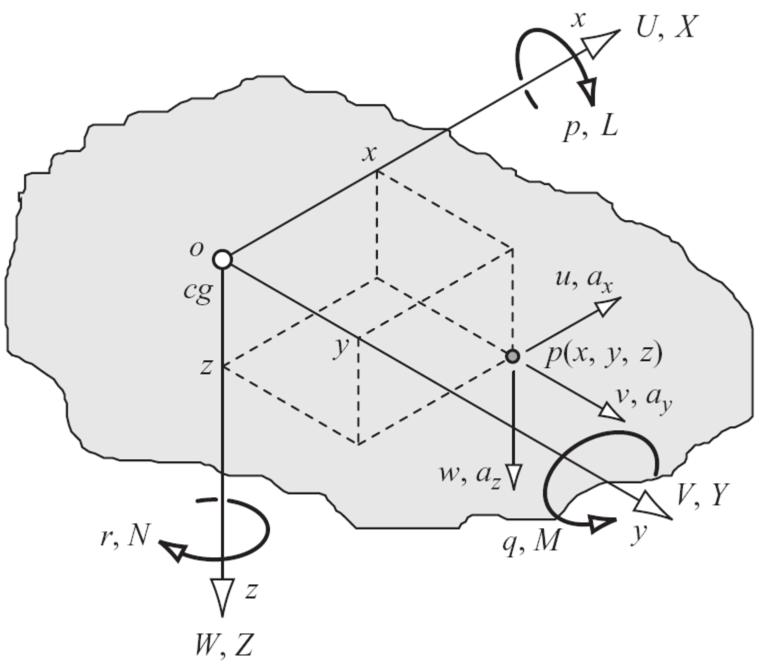
Forces on aircraft will result in to accelerations and moments in to angular accelerations. Accelerations are measured by accelerometers and angular rates by gyros. Modern IMUs are strap down i.e. sensors are strapped to body.





x
U, X
p, L
x
o
cg
u, a_x
y
z
p(x, y, z)
v, a_y
w, a_z
V, Y
r, N
q, M
y
z
W, Z

Euler Angles



$$a'_x = \dot{U} - rV + qW - x(q^2 + r^2) + y(pq - \dot{r}) + z(pr + \dot{q})$$
$$a'_y = \dot{V} - pW + rU + x(pq + \dot{r}) - y(p^2 + r^2) + z(qr - \dot{p})$$
$$a'_z = \dot{W} - qU + pV + x(pr - \dot{q}) + y(qr + \dot{p}) - z(p^2 + q^2)$$

Michael Cook, "Flight Dynamics Principles" Elsevier Publications, 2007

The Equations of Motion for INS

$$\dot{u} = a_x - qw + rv - g\sin\theta$$

$$\dot{v} = a_y - ru + pw + g\cos\theta\sin\phi$$

$$\dot{w} = a_z - pv + qu + g\cos\theta\cos\phi$$

$$\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = \sec\theta(q\sin\phi + r\cos\phi)$$

Euler angle rates

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Vector Transformation Inertial-Body

$$R_I^B = \begin{bmatrix} C\theta C\psi & C\theta S\psi & -S\theta \\ S\phi S\theta C\psi - C\phi S\psi & S\phi S\theta S\psi + C\phi C\psi & S\phi C\theta \\ C\phi S\theta C\psi + S\phi S\psi & C\phi S\theta S\psi - S\phi C\psi & C\phi C\theta \end{bmatrix}$$

$$\left[R_B^I\right] = \left[R_I^B\right]^{-1} = \left[R_I^B\right]^T$$

The Navigation equations

$$[v]_I = R_B^I [v]_B$$

$$\dot{x} = u\cos\theta\cos\psi + v(\sin\varphi\sin\theta\cos\psi - \cos\phi\sin\psi) +$$
$$w(\cos\phi\sin\theta\cos\psi + \sin\varphi\sin\psi)$$

$$\dot{y} = u\cos\theta\sin\psi + v(\sin\theta\sin\phi\sin\psi + \cos\phi\cos\psi)$$
$$+ w(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)$$

$$\dot{z} = -u\sin\theta + v\sin\phi\cos\theta + w\cos\phi\cos\theta$$