Orbital Time Solutions

Mean Motion Concept for Time in Orbit

Circle – Ellipse Equivalence

Mean Orbital Velocity Solution

Kepler's Equation for Time of Flight

TOF for Elliptic Orbits

Time and Angle Solutions

In order to fix time (or θ), we need to know **angular velocity**, which **is neither known nor a constant**. What is actually **constant and known** is the areal velocity, in terms of the **angular momentum.**

In a general sense, **following integral equation** describes the relation between the **time and the angular position.**

$$\left|\vec{h}\right| = h = \left|\vec{r}\times\vec{v}\right| = rv_p = r\frac{rd\theta}{dt}; \quad dt = \frac{1}{h}r^2 d\theta \rightarrow t = \frac{1}{h}\int r^2 d\theta$$

$$t_B - t_A = \frac{h^3}{\mu^2}\int_{\theta_A}^{\theta_B}\frac{d\theta}{(1+e\cos\theta)^2}; \quad A, B: \text{ Two points on conic}$$

Time of Flight Concept

It can be seen that **time of flight** between two points ‘A’ and ‘B’ **can be obtained** in terms of the initial **angular momentum**, orbit eccentricity and angles **corresponding to the two positions.**

However, it is also seen that the **above integral can not be evaluated in the closed form** as ‘e’ is a real number, except for the **case of circle** when e = 0, or **for the case of parabola** when e = 1.

Therefore, **we need a methodology** that gives orbital time **solution that does not require** explicit integration of the **time of flight equation.**

Mean Angular Velocity Concept

Concept of **mean angular velocity** has been evolved, in order to arrive at **time solutions without explicitly integrating** the time **equation.**

It is to be noted here that, while **working with elliptic orbits**, Kepler found the corresponding **circle more convenient** to work with, and for the first time, introduced the concept of **mean** angular motion for **an elliptic orbit.**

In order to do this, Kepler **defined an auxiliary circle**, whose projected f**orm** was the **elliptic orbit**.

Mean Angular Velocity Concept

In this regard, it is known that the **orbital angular velocity** is a maximum for perigee at **$\theta \approx 0^o$**, and is a minimum for apogee at **$\theta = 180^o$**.

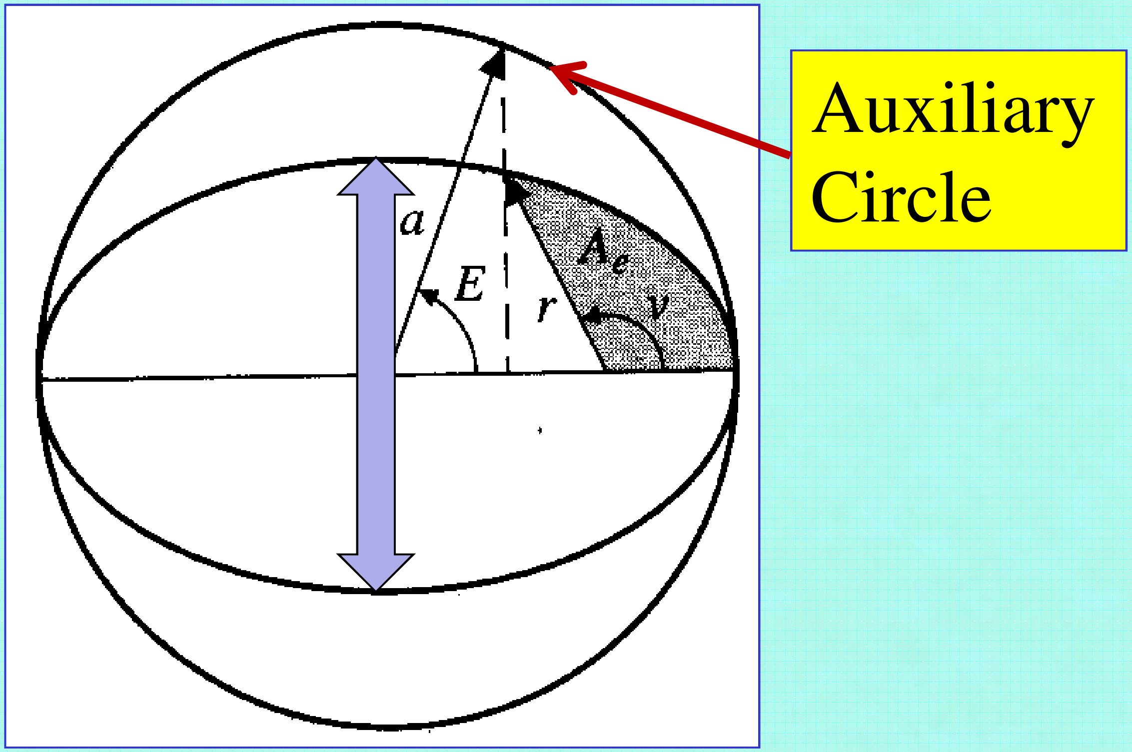
Mean angular velocity is defined as the **average** of angular velocity function, taken over **one complete cycle**.

Mean motion of an elliptic orbit helps in determining either '$\Delta\theta$' from given 'Δt' or '**Δt**' from given '$\Delta\theta$'.

As **mean motion represents an exact solution** for circular orbit, we **can use this idea,** along with **circle – ellipse equivalence**, to mathematically **formulate** the problem of **time solution.**

Circle – Ellipse Equivalence

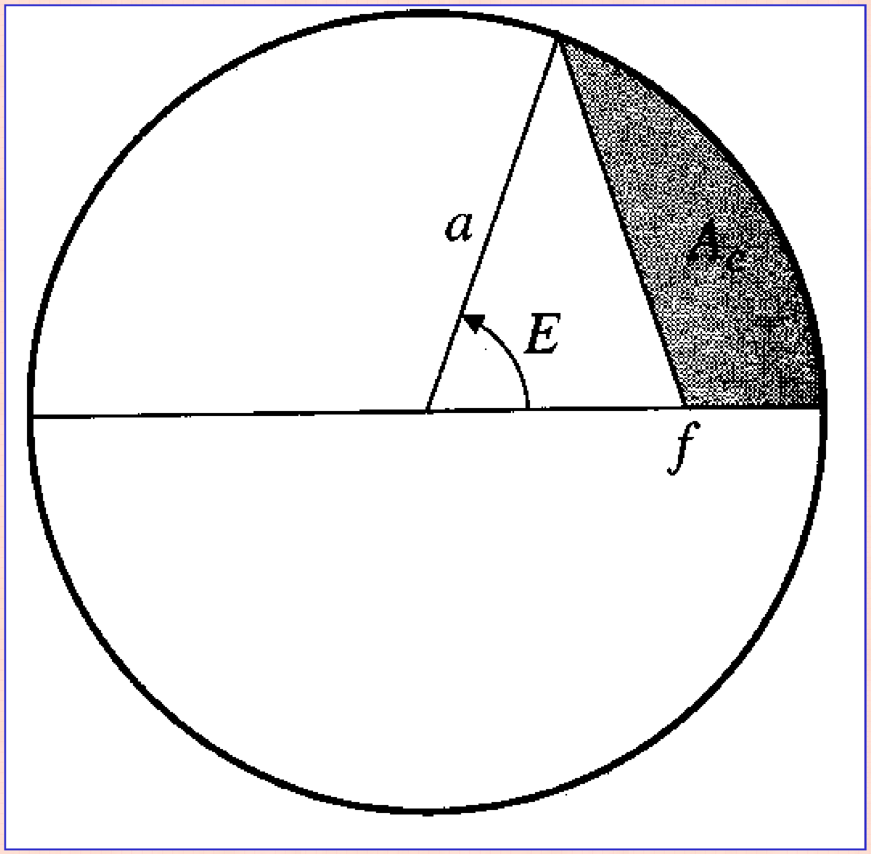
Consider the **following figure**, showing the elliptic orbit, along with its **projected equivalent circle.**



Here, **E** is the angle made by **projected tip** of the radius vector for the **hypothetical circle.**

Circle – Ellipse Mapping

Area segment 'A_e' in ellipse **projects** to the **area segment 'A_c'** in the auxiliary circle as **shown below.**



It can be shown that **this mapping** is based on **'a/b'.**

Area Swept by Radius Vector

It can be seen that if **we rotate a circle** about its axis, we can **visualize** the resulting ellipse, indicating that **areas scale as per 'b' contraction,** mentioned earlier.

As a result, **shaded area in ellipse A_e**, will project to the corresponding **shaded area in circle A_c**, through the same expansion 'a/b'.

We know that **areal velocity in ellipse is a constant**, so if we use this and find **its equivalent term for the projected circle**, we can obtain **equivalent mean motion**.

Mean Orbital Velocity Solution

From basic **triangle geometry**, we can obtain the **area of the sector in ellipse** in terms of the area of the projected **sector of the equivalent circle** as follows.

$$A_c = \frac{1}{2}a^2(E - e\sin E); \quad A_e = \frac{b}{a}A_c = \frac{ab}{2}(E - e\sin E);$$

We know that the angular **velocity of a circle is constant**, which can be obtained **from Kepler's 3rd law**, as follows.

Circular Orbital Velocity: $n = \frac{2\pi}{T} = \frac{2\pi}{\left(2\pi\sqrt{a^3}\big/\sqrt{\mu}\right)} = \sqrt{\frac{\mu}{a^3}};$

Mean Orbital Velocity Solution

As '**auxiliary**' circle completely **corresponds to the ellipse**, '**n**', for auxiliary circle is **same as that for ellipse**.

Further, as we know that **area 'A_e'** is same as the area **'A_c.(b/a)',** we can relate the mean angular position in the elliptic orbit to the angle **'E',** as follows.

$$A_c = \frac{1}{2}a^2(E - e\sin E); \quad A_e = \frac{b}{a}A_c = \frac{ab}{2}(E - e\sin E);$$

$$A_e = \frac{\pi ab \cdot \Delta t}{\left(2\pi\sqrt{a^3}\Big/\sqrt{\mu}\right)} = \frac{n \cdot \Delta t \cdot ab}{2} = \frac{ab}{2}(E - e\sin E)$$

$$\Delta\bar{\theta} = n \cdot \Delta t = (E - e\sin E) = M \text{ (Mean Angle)}$$

Kepler's Solution

The expression **(E – e sin E)** is the amount of mean **angle traversed** by the **radius vector 'r'**, in time interval **from** t_A to t_B**,** in respect of any two points 'A' and 'B'.

The equation relating **mean angle 'M' to** the **angle 'E',** is called the **Kepler's equation**, which he used, in order to provide **accurate solutions for elliptic orbits.**

Kepler's equation is one of the **earliest transcendental equations**, which cannot be solved in **closed form** and **only numerical solutions** are possible.

Concept of Anomalies

In orbital mechanics, **all angles are termed ‘anomalies’,** so that angle ‘E’ is called ‘**Eccentric Anomaly**’, while ‘M’ is called ‘**Mean Anomaly**’ and ‘θ’ is called ‘**True Anomaly**’.

This is a **hangover from earlier times** when circle was treated **as a perfect shape**, to be aimed for at all times, and **any shape that deviated** from circle, was termed ‘**anomalous**’.

Thus, even today **the above nomenclature** is used for solving **orbital mechanics problems**, though we now know **ellipse to be a valid** conic section geometry.

Overall Solution Process

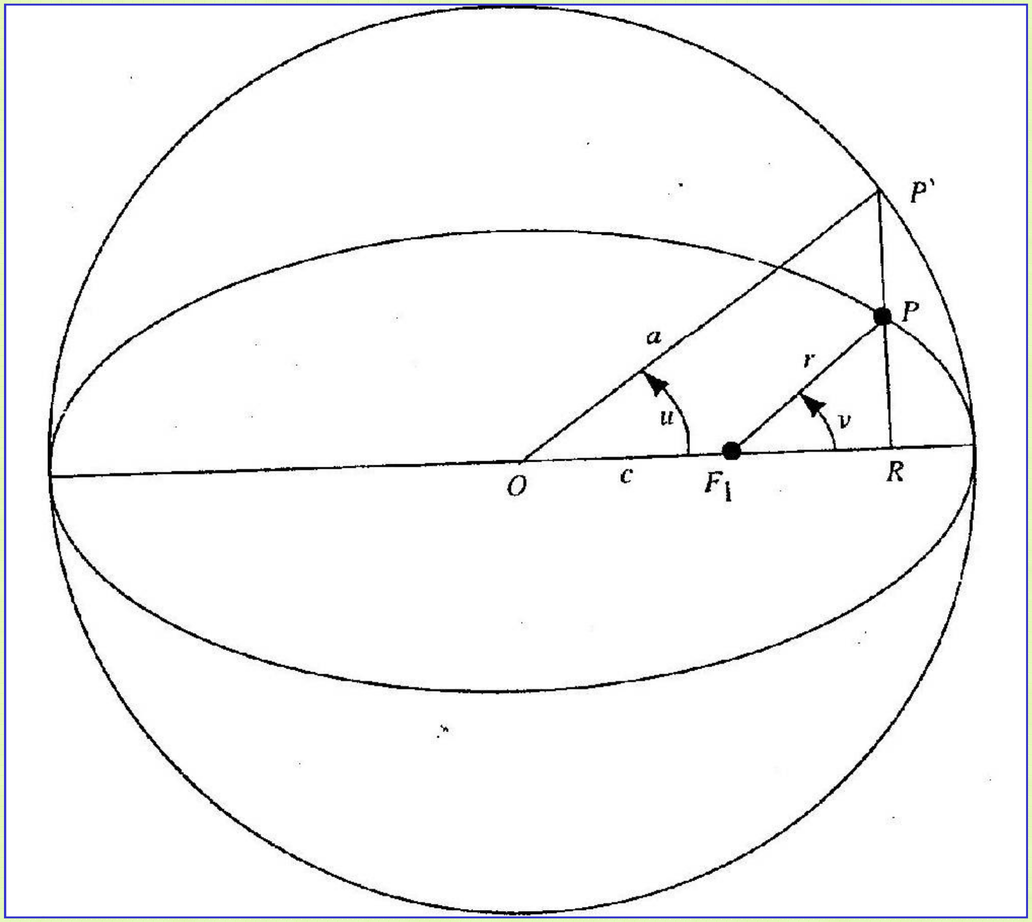
'n' is used **to predict accurately**, the angular position of a satellite **at any future time**, even if orbit **is elliptic** in nature. However, this requires **conversion of 'E' into 'θ'.**

To do this, we first **calculate 'M'** for each time interval, based on **circular nature of mean motion.** We can then **convert 'mean angle' M** into 'eccentric angle' **E**, using the relation $\mathbf{M = E - e \sin E}$.

However, as we need to obtain the '**actual or true angle**' θ, to **uniquely determine the position** of spacecraft in orbit, it is necessary to obtain the **expression for 'θ'** in terms of **'E' or 'M'**, which also fixes **TOF** for ellipse.

Eccentric – True Anomaly Mapping

Consider the **figure below.**



E - θ Relation

We can obtain **true anomaly 'θ' from** eccentric anomaly 'E', from **circle – ellipse analogy** as follows.

$$u = E; \quad \cos E = \frac{OR}{OP'} = \frac{OF_1 + FR}{a}; \quad \cos E = \frac{c + r\cos\theta(\nu)}{a}$$

$$\cos E = \frac{ae + r\cos\theta}{a}; \quad r = \frac{p}{1 + e\cos\theta} = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

$$\cos E = \frac{ae(1 + e\cos\theta) + a\left(1 - e^2\right)\cos\theta}{a(1 + e\cos\theta)}$$

$$CosE = \frac{e + \cancel{e^2\cos\theta} + \cos\theta - \cancel{e^2\cos\theta}}{1 + e\cos\theta}$$

$$\cos E = \frac{e + \cos\theta}{1 + e\cos\theta} \rightarrow \tan\frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \cdot \tan\frac{\theta}{2}$$

Solution Methodology

Following **steps are employed** for solving for ‘t’, ‘θ’.

<u>For ‘t’ Calculations:</u>

Determine ‘E_A’ & ‘E_B’ from relation: $$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$$

Evaluate ‘M’ from relation: $M = E - e \sin E$

Obtain ‘Δt’ from relation: $\Delta t = M / n$

<u>For ‘θ’ Calculations:</u>

Determine ‘M’ from relation: $n \,.\, \Delta t = M$

Evaluate ‘E’ from relation: $M = E - e \sin E$

Obtain ‘θ’ from relation: $$\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

Summary

Mean orbital motion is an important concept in arriving at time of flight **(TOF)** solutions.

Equivalence between **ellipse and circle** is used to define **mean orbital angular velocity**, which is used to arrive at some of the **orbital parameters**.

Time of flight is an important **parameter** that can be used to arrive at **orbit characteristics.**