Buckling: structural Instability

Consider the pinned (pin-ended) column AB. We assume that it is in the displaced state of neutral equilibrium associated with buckling so that the compressive load P has attained a critical value P_{cr}. From bending theory we have:

$$EI \frac{d^2w}{dx^2} = -M$$

$$\Rightarrow EI \frac{d^2w}{dx^2} = -P_{cr} w \quad @ \text{ any } x \text{ from } A$$

$$\Rightarrow \frac{d^2w}{dx^2} + \frac{P_{cr}}{EI} w = 0$$

$$\Rightarrow \frac{d^2w}{dx^2} + k^2 w = 0 \qquad \text{where, } k = \sqrt{\frac{P_{cr}}{EI}}$$

Solution to this equation is

$$w = C_1 \cos kx + C_2 \sin kx$$

Boundary conditions:

$w = 0 \quad @ \; x = 0$ and $x = l$

$$\Rightarrow \quad C_1 = 0$$

$$C_2 \sin kl = 0$$

$$\Rightarrow kl = n\pi \qquad \text{where } n = 1, 2, 3 \ldots$$

$$\Rightarrow k^2 l^2 = n^2 \pi^2$$

$$\Rightarrow \frac{P_{cr} l^2}{EI} = n^2 \pi^2$$

$$P_{cr} = \frac{n^2 \pi^2 EI}{l^2}$$

The smallest value of buckling load, which can maintain the column in a neutral equilibrium state, is obtained for $n = 1$

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

Other integer values of n correspond to higher modes.

$P_{cr} = \frac{4\pi^2 EI}{l^2}$ P_{cr}

$P_{cr} = 9\pi^2 EI/l^2$ $P_{cr} = 9\pi^2 EI/l^2$

now I can be written in terms of radius of gyration as

$$I = Ar^2$$

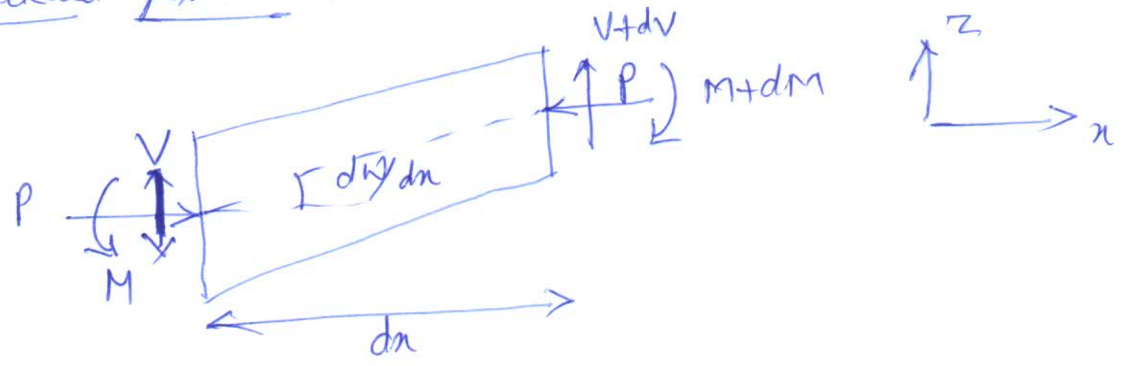
$$\therefore \quad P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EAr^2}{l^2}$$

$$\Rightarrow \frac{P_{cr}}{A} = \sigma_{cr} = \frac{\pi^2 E}{(l/r)^2}$$

$l/r \rightarrow$ slenderness ratio

A more general formulation

Buckled position:



$$\sum F_z = 0 \Rightarrow (V+dV) - V = 0$$

$$\Rightarrow \frac{dV}{dx} = 0$$

$$\sum M = 0 \Rightarrow (V+dV)\,dx + P\frac{dw}{dx}\,dx = dM$$

or $V = \frac{dM}{dx} - \frac{P\,dw}{dx}$

$$\Rightarrow \frac{dV}{dx} = 0 \Rightarrow \frac{d^2M}{dx^2} - \frac{P\,d^2w}{dx^2} = 0.$$

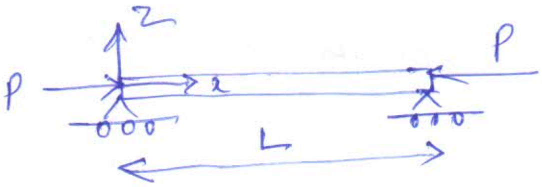
NOW, $M = -EI\frac{d^2w}{dx^2}$

$\therefore$ We have,

$$\frac{d^4w}{dx^4} + k^2\frac{d^2w}{dx^2} = 0 \qquad k = \sqrt{\frac{P}{EI}}$$

$$\therefore w = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4$$

Different cases:



(1) Pinned - Pinned bar:

@ $x = 0$: $W = 0$; $M = 0$

@ $x = L$: $W = 0$; $M = 0$

$$\Rightarrow \quad C_2 + C_4 = 0$$
$$C_2 = 0$$

$$C_1 \sin kL + C_2 \cos kL + C_3 L + C_4 = 0$$
$$C_1 k^2 \sin kL + C_2 k^2 \cos kL = 0$$

$\therefore$ we get : $C_2 = C_4 = 0$

$$C_1 \sin kL + C_3 L = 0$$
$$C_1 \sin kL = 0$$

$$\Rightarrow \quad C_3 = 0 \text{ and } C_1 \sin kL = 0.$$

Since $C_1 \neq 0$ $\sin kL = 0$

$\therefore kL = n\pi$ $\quad n = 1, 2, 3 \ldots$

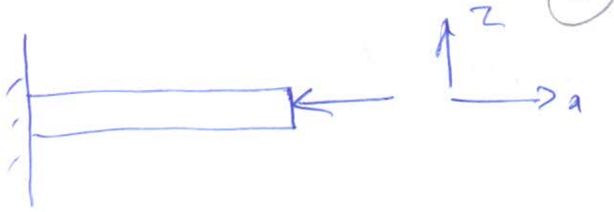
$$\therefore \quad P_{cr} = \frac{n^2 \pi^2 EI}{L^2}$$

Deflection for each critical load.

$$W^{(n)}(x) = C_1 \sin k^{(n)} x$$

for $n = 1, 2, 3 \ldots$
n for different modes.

(2) Clamped - Free bar



Boundary conditions:

@ $x=0$: $w = 0$

$\frac{dw}{dx} = 0$

@ $x = L$: $M = 0 \quad - EI\frac{d^2w}{dx^2} = 0$

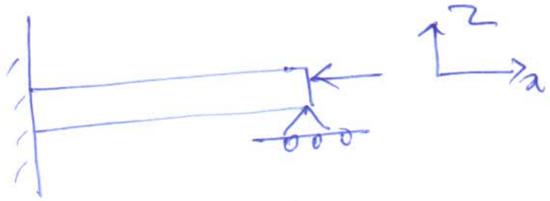
$V = 0 \quad \frac{dM}{dx} - P\frac{dw}{dx} = 0$

After solving the general equation for the above conditions:

$$w(x) = C_2 \cos kx + C_4 = C_2(\cos kx - 1)$$

$$P_{cr} = \frac{n^2\pi^2 EI}{4L^2} \qquad n = 1,3,5 \ldots$$

(3) Clamped - Pinned bar



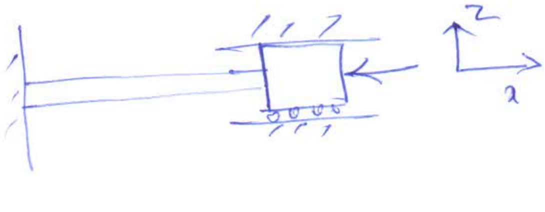
@ $x=0$: $w = 0$; $\frac{dw}{dx} = 0$

@ $x = L$: $w = 0$; $-\frac{d^2w}{dx^2}EI = M = 0$

Solution gives

$$P_{cr} = \frac{20.16EI}{L^2}$$

(4) Clamped - Clamped bar



@ $x=0$ & @ $x=L$

$w = 0$ & $\frac{dw}{dx} = 0$.

$$P_{cr} = \frac{4n^2\pi^2EI}{L^2} \qquad n = 1,2,3, \ldots$$

Effective length of buckling.

Buckling loads of all the cases considered earlier can be written in a single form:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

L_e is the effective length of buckling whose value depends on boundary conditions.

for example

$L_e = 0.5L$ for clamped-clamped condition.

Further,

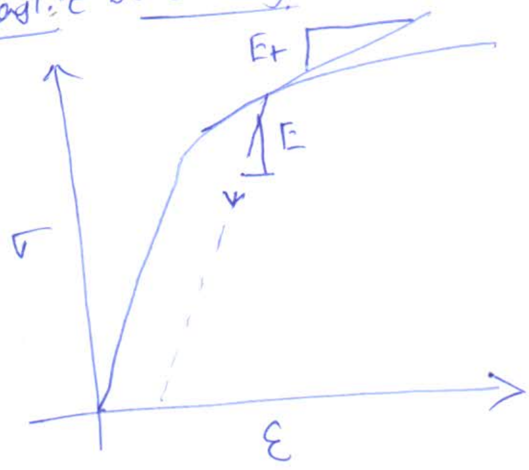
$I = Ar^2$ $\quad$ $r \rightarrow$ radius of gyration

$A \rightarrow$ cross-sectional area

$$\therefore P_{cr} = \frac{\pi^2 EA}{(L_e/r)^2}$$

$L_e/r \rightarrow$ effective slenderness ratio.

Inelastic buckling



Beyond yield, we should consider the tangent modulus E_t in the calculation of P_{cr}.

$$\therefore P_{cr} = \frac{\pi^2 E_t}{(L_e/r)^2}$$

Effect of initial imperfections

Initial imperfection: w_0

For this case, bending moment at any point x is equal to change in curvature

$$\Rightarrow \quad EI\frac{d^2w}{dx^2} - EI\frac{d^2w_0}{dx^2} = -Pw$$

$$\Rightarrow \quad \frac{d^2w}{dx^2} + k^2 w = \frac{d^2w_0}{dx^2} \qquad k = \sqrt{P/EI}$$

Solution depends on form of 'w_0'.