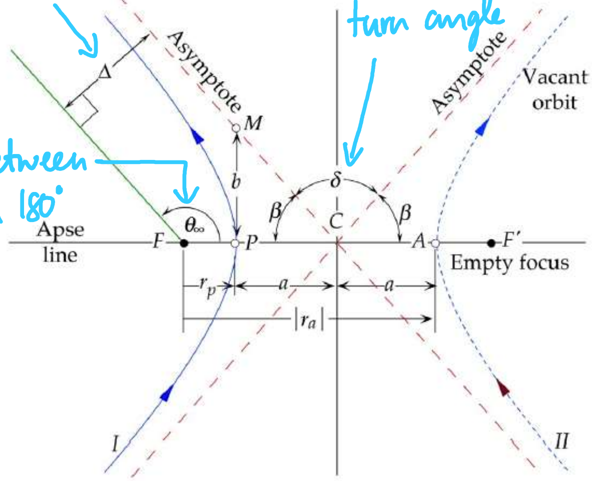
Hyperbolic Trajectories $(\|e\|>1)$

aiming radius

lies between 90° and 180°

turn angle



physical trajectory $(-\theta_\infty < \theta < \theta_\infty)$

- $\|r\| = \dfrac{\|h\|^2}{\mu} \dfrac{1}{1+\|e\|\cos\theta}$

- $1+\|e\|\cos\theta = 0 \Rightarrow \theta_\infty = \cos^{-1}\left(\dfrac{-1}{\|e\|}\right)$

- $\sin\theta_\infty = \sqrt{\dfrac{\|e\|^2-1}{\|e\|^2}}$

- $\beta = 180^\circ - \theta_\infty \Rightarrow \cos\beta \quad -\cos\theta_\infty$

- $\beta = \cos^{-1}\left(\dfrac{1}{\|e\|}\right)$

- $\delta = 180^\circ - 2\beta \Rightarrow \sin\delta/2 = \sin(90-\beta)$
  $= \cos\beta$
  $= \dfrac{1}{\|e\|}$

$\|e\|$

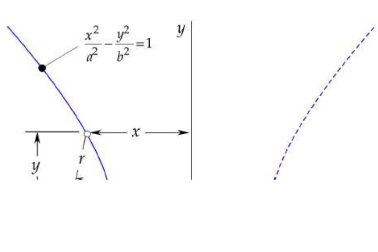
- $$2a = |r_a| - r_p = -r_a - r_p = \frac{-\|h\|^2}{\mu}\left(\frac{1}{1-\|e\|} + \frac{1}{1+\|e\|}\right) = \frac{\|h\|^2}{\mu}\frac{1}{\|e\|^2-1}$$

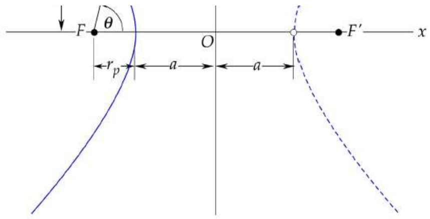
- $$\|r\| = a\,\frac{\|e\|^2 - 1}{1 + \|e\|\cos\theta}$$

- $r_p = a(\|e\| - 1)$, $r_a = -a(\|e\| + 1)$

- $b = a\tan\beta = -a\tan\theta_\infty = a\sqrt{\|e\|^2 - 1}$

- $$\Delta = (r_p + a)\sin\beta = a\|e\|\sin\beta = a\|e\|\sin\theta_\infty = a\|e\|\sqrt{1 - \frac{1}{\|e\|^2}} = a\sqrt{\|e\|^2 - 1}$$





- $x = -a - r_p + \|r\| \cos\theta$, $y = \|r\| \sin\theta$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

$$= \left(\frac{\|e\| + \cos\theta}{1 + \|e\| \cos\theta}\right)^2 - \left(\frac{\sqrt{\|e\|^2 - 1}\,\sin\theta}{1 + \|e\| \cos\theta}\right)^2$$

$$= 1$$

- $\varepsilon = \underbrace{\frac{\mu}{2a}}$ (why?)

  ↓

  does not depend on the eccentricity

- $\dfrac{\|v\|^2}{2} - \dfrac{\mu}{\|r\|} = \dfrac{\mu}{2a}$

- $V_\infty = \sqrt{\dfrac{\mu}{a}}$ (hyperbolic excess speed)

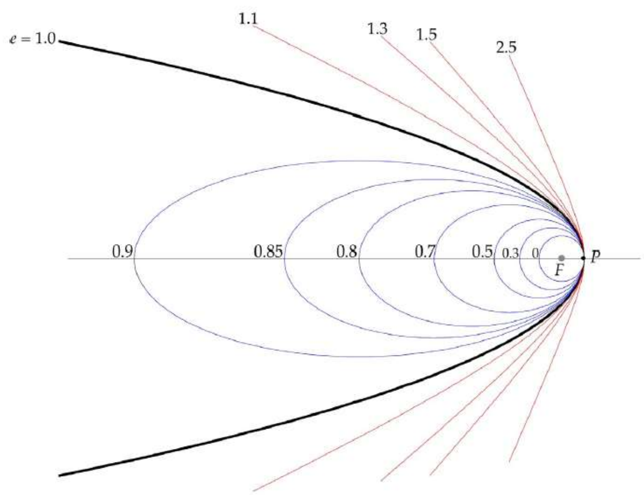
- $\dfrac{\|v\|^2}{2} - \dfrac{\mu}{\|r\|} = \dfrac{V_\infty^2}{2}$

- $\|v\|^2 = V_{esc}^2 + \underbrace{V_\infty^2}_{C_3}$

- V_∞ represents the excess K.E. over that which is required to simply escape from the centre of attraction.

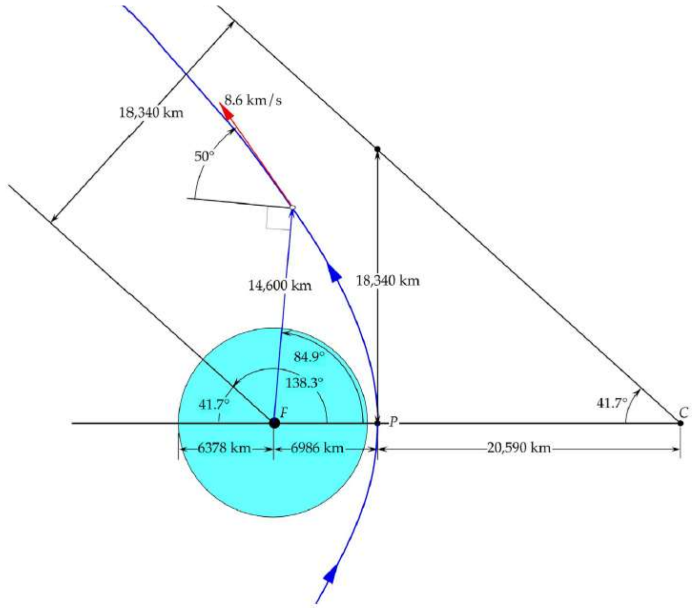
- C_3 is a measure of energy required for an inter-planetary mission. It is also a measure of the maximum energy a launch vehicle can impart to a spacecraft of a given mass.

- $V_\infty = \frac{\mu}{\|h\|} \|e\| \sin\theta_\infty = \frac{\mu}{\|h\|} \sqrt{\|e\|^2 - 1}$



Example

At a given point of a spacecraft's geocentric trajectory, the radius is 14,600 km, the speed is 8.6 km/s, and the flight path angle is 50°. Show that the path is a hyperbola and calculate the following:
(a) angular momentum
(b) eccentricity
(c) true anomaly
(d) radius of the perigee
(e) semimajor axis
(f) C_3
(g) turn angle
(h) aiming radius



Details

$$V_{esc} = \sqrt{\frac{2\mu}{\|r\|}} < \|v\|$$

(a) $$\tan \gamma = \frac{V_r}{V_\perp}$$

$$\|v\| = \sqrt{V_r^2 + V_\perp^2}$$

$$\|h\| = \|r\| V_\perp$$

(b) $$V_r = \frac{\mu}{\|h\|} \|e\| \sin\theta$$

$$\|r\| = \frac{\|h\|^2}{\mu} \frac{1}{1 + \|e\| \cos\theta}$$

(c) $$\|r\| = \frac{\|h\|^2}{\mu} \frac{1}{1 + \|e\| \cos\theta}$$

(d) $r_p = \frac{\|h\|^2}{\mu} \frac{1}{1+\|e\|}$

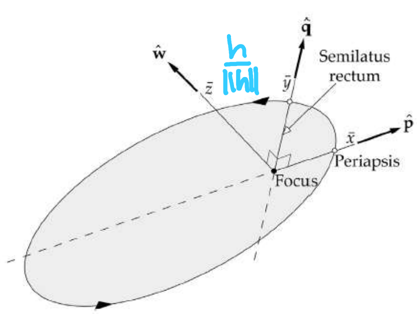
(e) $a = \frac{\|h\|^2}{\mu} \frac{1}{\|e\|^2-1}$

(f) $v_\infty^2 = \|v\|^2 - v_{esc}^2$

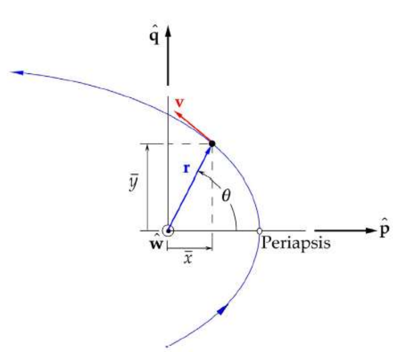
(g) $\delta = 2\sin^{-1}\left(\frac{1}{\|e\|}\right)$

(h) $\Delta = a\sqrt{\|e\|^2-1}$

Perifocal Frame



- It is the "natural frame" for an orbit.



- $r = \bar{x}\hat{p} + \bar{y}\hat{q}$, $\bar{x} = \|r\|\cos\theta$, $\bar{y} = \|r\|\sin\theta$

- $r = \dfrac{\|h\|^2}{\mu}\dfrac{1}{1+\|e\|\cos\theta}(\cos\theta\,\hat{p} + \sin\theta\,\hat{q})$

- $V = \dot{r} = \dot{\bar{x}}\hat{p} + \dot{\bar{y}}\hat{q}$

- $\dot{\bar{x}} = \dot{\widehat{\|r\|}}\cos\theta - \|r\|\dot{\theta}\sin\theta$

- $\dot{\bar{y}} = \dot{\widehat{\|r\|}}\sin\theta + \|r\|\dot{\theta}\cos\theta$

- $\dot{\widehat{\|r\|}} = \dfrac{\mu}{\|h\|}\|e\|\sin\theta$

- $\|r\|\dot{\theta} = \dfrac{\mu}{\|h\|}(1+\|e\|\cos\theta)$

- $\dot{\bar{x}} = -\dfrac{\mu}{\|h\|}\sin\theta$, $\dot{\bar{y}} = \dfrac{\mu}{\|h\|}(\|e\|+\cos\theta)$

Example

An earth orbit has an eccentricity of 0.3, an angular momentum of 60, 000km^2/s, and a true anomaly of 120°. What are the position vector r and velocity vector v in the perifocal frame of reference?

Details

$$r = \frac{\|h\|^2}{\mu}\frac{1}{1+\|e\|\cos\theta}(\cos\theta\,\hat{p} + \sin\theta\,\hat{q})$$

$$v = \frac{\mu}{\|h\|}\left[-\sin\theta\,\hat{p} + (\|e\| + \cos\theta)\,\hat{q}\right]$$

Example

An earth satellite has the following position and velocity vectors at a given instant:

$$\mathbf{r} = 7000\hat{\mathbf{p}} + 9000\hat{\mathbf{q}} \text{ (km)}$$
$$\mathbf{v} = -3.3472\hat{\mathbf{p}} + 9.1251\hat{\mathbf{q}} \text{ (km/s)}$$

Calculate the specific angular momentum h, the true anomaly θ, and the eccentricity e.

Details

$$h = r \times v$$

$$r \cdot \hat{p} = \|r\| \cos\theta$$

$$\|r\| = \frac{\|h\|^2}{\mu} \frac{1}{1 + \|e\| \cos\theta}$$