AE 240/338/713 Quiz No. 1 Tuesday 24th January, 2017 Weightage: 10%
Time: 1900 – 2025 Hrs. Venue: LC 001, LT 001, LT 002
Note: Examination is Closed Notes. Only calculator is permitted.
Model Solutions

Q.1 (a) Derive the expressions for the burnout velocity and altitude for a single stage rocket undergoing vertical rectilinear motion under constant sea level gravity and no atmosphere assumptions, starting from the simplified equations applicable for this case, as given below. (2)

$$\boxed{m(t)\frac{dV(t)}{dt} = -\dot{m}g_0 I_{sp} - m(t)g_0(t)}$$

Here, m_0 is lift-off mass, m_p is total propellant mass having specific impulse I_{sp}, which is being consumed at the constant burn rate of β. Using the above relations, answer the following additional questions.

$$\frac{dV}{dt} = -\frac{\dot{m}}{m}g_0 I_{sp} - g_0; \quad V(t) = g_0 I_{sp}\ln\frac{m_0}{m(t)} - g_0 t; \quad V_b = g_0 I_{sp}\ln\frac{m_0}{m_b} - g_0 t_b$$

$$V_b = g_0 I_{sp}\ln\frac{m_0}{m_0 - m_p} - g_0 t_b; \quad m_p \to \text{Total Propellant Mass}; \quad m(t) = m_0 - \beta t;$$

$$t_b = \frac{m_p}{\beta}; \quad V_b(t) = g_0 I_{sp}\ln\frac{m_0}{(m_0 - m_p)} - g_0\left(\frac{m_p}{\beta}\right)$$

$$h(t) = \int V(t)dt = \int\left(g_0 I_{sp}\ln\frac{m_0}{m(t)} - g_0 t\right)dt; \quad h(t) = g_0 I_{sp}\int \ln\frac{m_0}{m_0 - \beta t}dt - \frac{1}{2}g_0 t^2;$$

$$h(t) = \frac{m_0 g_0 I_{sp}}{\beta}\left[\left(1-\frac{\beta}{m_0}t\right)\ln\left(1-\frac{\beta}{m_0}t\right) - \left(1-\frac{\beta}{m_0}t\right)\right] - \frac{1}{2}g_0 t^2 + C; \quad h = h_0 \text{ at } t = 0$$

$$C = h_0 + 1; \quad h_b = \frac{m_0 g_0 I_{sp}}{\beta}\left[(1-\Lambda)\ln(1-\Lambda) + \Lambda\right] - \frac{1}{2}g_0\left(m_p/\beta\right)^2 + h_0; \quad t_b = \frac{m_p}{\beta}; \quad \Lambda = m_p/m_0$$

(b) Calculate the burnout velocity and altitude reached by the above rocket with m_o of 10,000 kg, m_p of 7,500 kg, I_{sp} of 200 seconds and β of 250 kg/s. Assume $g_0 = 9.81$ m/sec^2. (2)

$$V_b(t) = 9.81\times 200\times\ln\frac{10000}{2500} - 9.81\times\frac{7500}{250} = 2425.6 m/s$$

$$\Lambda = 0.75, \quad h_b = \frac{10000\times 9.81\times 200}{250}\left[0.25\times\ln 0.25 + 0.75\right] - \frac{1}{2}\times 9.81\times 900 = 27246.4 m$$

(c) If it is known that peak drag acceleration occurs at the end of the burnout, determine the value of the average drag acceleration applicable for this trajectory, using standard triangular approximation. (S_r = 0.1 m^2, C_{D0} = 0.8, Air Density as per the table below). (1)

$$\rho = 0.041 - \frac{0.041 - 0.018}{25 - 30}(25 - 27.246) = 0.030 kg/m^3$$

$$Q = 0.5\times 0.03\times(2425.6)^2 = 90219.3 N/m^2; \quad D = 90219.3\times 0.8\times 0.1 = 7217.5 N$$

$$a_{DPeak} = \frac{7217.5}{2500} = 2.887 m/s^2; \quad a_{DAvg} = 1.443 m/s^2$$

Q.2 Obtain the relation between time (t) and inclination (θ) for a constant specific thrust gravity turn trajectory ($T/m = n_0 g_0$), for the case when 'n_0' is 1. (Hint: Use following basic relations, as applicable.). (1)

$$\boxed{\dot{\theta} = \frac{g_0 \sin^2\theta}{k\left\{\tan\left(\theta/2\right)\right\}^{n_0}}; \quad k = V_0 \sin(\theta_0)\left|\cot\left(\theta_0/2\right)\right|^{n_0}}$$

$$\frac{d\theta}{dt} = \frac{g_0 \sin\theta}{V} \to \int dt = \int \frac{V d\theta}{g_0 \sin\theta} \to t = \int \frac{k \tan\left(\theta/2\right)}{g_0 \sin^2\theta} d\theta = \frac{k}{4g_0}\int \frac{\sec^2\left(\theta/2\right) d\theta}{\sin\left(\theta/2\right)\cos\left(\theta/2\right)}$$

$$t = \frac{k}{4g_0}\int\left[\frac{2}{\sin\theta} d\theta + \sec^2\left(\theta/2\right)\tan\left(\theta/2\right) d\theta\right] = \frac{k}{4g_0}\left[2\ln\tan\left(\theta/2\right) + \sec^2\left(\theta/2\right)\right]$$

Q.3 (a) Derive the expressions for the burnout velocity (V_b) and burnout mass fraction (m_b/m_0), in respect of constant specific thrust ($T/m = n_0 g_0$, neglecting drag effect) gravity turn trajectory, using the following equations of motion. (2)

$$\dot{V} = -\frac{\dot{m} g_0 I_{sp}}{m} - g_0 \cos\theta; \quad \dot{\theta} = \frac{g_0 \sin\theta}{V};$$

Where, all the quantities have their standard meaning. (V is the velocity along the vehicle axis and thrust direction, g_0 is sea level gravitational acceleration, and θ is the angle measured from local vertical). Answer part (b) using results obtained in part (a).

$$\dot{V} = -\frac{\dot{m} g_0 I_{sp}}{m(t)} - \tilde{g}\cos\theta = -\tilde{g}\cos\theta + n_0 \tilde{g}; \quad \dot{\theta} = \frac{\tilde{g}\sin\theta}{V};$$

$$\frac{\dot{V}}{\dot{\theta}} = \frac{dV}{d\theta} = V\left(-\cot\theta + n_0 \operatorname{cosec}\theta\right); \quad \int \frac{dV}{V} = -\int \frac{\cos\theta}{\sin\theta} d\theta + n_0 \int \frac{1}{\sin\theta} d\theta$$

$$\int \frac{dV}{V} = -\int \frac{d(\sin\theta)}{\sin\theta} + n_0 \int \frac{1}{2}\frac{\sec^2 \theta/2}{\tan \theta/2} d\theta; \quad \ln V = \ln \operatorname{cosec}\theta + n_0 \ln\left|\tan(\theta/2)\right| + C$$

$$V = k\frac{|\tan(\theta/2)|^{n_0}}{\sin\theta}; \quad k = V_0 \frac{\sin(\theta_0)}{|\tan(\theta_0/2)|^{n_0}}; \quad V_b = V_0 \frac{\sin\theta_0}{\sin\theta_b}\frac{|\tan(\theta_b/2)|^{n_0}}{|\tan(\theta_0/2)|^{n_0}}$$

$$\frac{T}{m} = -\frac{\dot{m} g_0 I_{sp}}{m} = n_0 g_0 \to \frac{dm}{m} = -\frac{n_0}{I_{sp}} dt \to \ln m = -\frac{n_0}{I_{sp}} t + C; \quad \frac{m_0}{m} = e^{\frac{n_0}{I_{sp}}(t - t_0)} \to t_b = \frac{I_{sp}}{n_0}\ln\left(\frac{m_0}{m_0 - m_p}\right)$$

(b) A single stage rocket is directly launched into a gravity turn trajectory (constant specific thrust) in vacuum with n_0 of 1.1 & θ_0 of 0.15°. If at the end of trajectory lasting for 250 s, the inclination from vertical is 90°, determine the initial velocity at the start of the gravity turn (Hint: Use time expression given below). Also, what is the final velocity achieved, and the value of the propellant mass fraction ($\Lambda = m_p/m_0$) needed to complete the mission, assuming constant seal level gravity and flat earth ($g_0 = 9.81$ m/s^2, $I_{sp} = 275$ s)? (2)

$$250 = \frac{V_0 \times 0.0026}{2 \times 9.81 \times (0.0013)^{1.1}}\left[\frac{1}{2.1} + \frac{1}{0.1} - \frac{(0.0013)^{2.1}}{2.1} - \frac{(0.0013)^{0.1}}{0.1}\right] = 0.198 V_0 \times 5.331$$

$$V_0 = \frac{250}{0.198 \times 5.331} = 236.8 m/s; \quad V_b = k = 236.8 \times \frac{0.0026}{|0.0013|^{1.1}} = 9206 m/s; \quad \theta_b = 90^o$$

$$250 = \frac{275}{1.1} \times \ln\left(\frac{m_0}{m_0 - m_p}\right) \to \frac{m_0}{m_0 - m_p} = 2.718; \quad 1 - \frac{m_p}{m_0} = 0.368 \to \frac{m_p}{m_0} = 0.632$$