Practice Problems

- Show that the area of the parallelogram spanned by two (non-zero) vectors A and B is given by $\|A\times B\|$.

- Show that for two (non-zero) vectors A and B, $A\times B=0$ if and only if one vector is a scalar multiple of the other.

- Show that for three (non-zero) vectors A, B and C, $A\cdot(B\times C)=(A\times B)\cdot C$. What is the geometric meaning of this identity? Can you prove it?

- What is $\frac{d^3A}{dt^3}$? (see lecture 3)

1.1 Given the three vectors $\mathbf{A}=A_x\hat{\mathbf{i}}+A_y\hat{\mathbf{j}}+A_z\hat{\mathbf{k}}$, $\mathbf{B}=B_x\hat{\mathbf{i}}+B_y\hat{\mathbf{j}}+B_z\hat{\mathbf{k}}$, and $\mathbf{C}=C_x\hat{\mathbf{i}}+C_y\hat{\mathbf{j}}+C_z\hat{\mathbf{k}}$, show analytically that

(a) $\mathbf{A}\cdot\mathbf{A}=A^2$

(b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$ (interchangeability of the dot and cross)

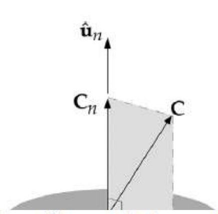
(c) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ (the bac-cab rule)

(Hint: Simply compute the expressions on each side of the = signs and demonstrate conclusively that they are the same.) Do *not* substitute numbers to "prove" your point. Use Eqs. (1.9) and (1.16).

1.2 Use just the vector identities in Problem 1.1 to show that

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

1.3 Let $\mathbf{A}=8\hat{\mathbf{i}}+9\hat{\mathbf{j}}+12\hat{\mathbf{k}}$, $\mathbf{B}=9\hat{\mathbf{i}}+6\hat{\mathbf{j}}+\hat{\mathbf{k}}$, and $\mathbf{C}=3\hat{\mathbf{i}}+5\hat{\mathbf{j}}+10\hat{\mathbf{k}}$. Calculate the (scalar) projection C_{AB} of $\mathbf{C}$ onto the plane of $\mathbf{A}$ and $\mathbf{B}$ (see illustration below).
(Hint: $C^2=C_n^2+C_{AB}^2$)
{Ans.: $C_{AB}=11.58$}



1.4 Since $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ are perpendicular and $\hat{\mathbf{u}}_t\times\hat{\mathbf{u}}_n=\hat{\mathbf{u}}_b$, use the bac-cab rule to show that $\hat{\mathbf{u}}_b\times\hat{\mathbf{u}}_t=\hat{\mathbf{u}}_n$ and $\hat{\mathbf{u}}_n\times\hat{\mathbf{u}}_b=\hat{\mathbf{u}}_t$, thereby verifying Eq. (1.29).

1.5 The x, y, and z coordinates (in meters) of a particle P as a function of time (in seconds) are $x=\sin 3t$, $y=\cos t$, and $z=\sin 2t$. At $t=3$ s, determine:

(a) The velocity $\mathbf{v}$ in Cartesian coordinates.

(b) The speed v.

(c) The unit tangent vector $\hat{\mathbf{u}}_t$.

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(d) The angles θ_x, θ_y, and θ_z that $\mathbf{v}$ makes with the x, y, and z axes.
(e) The acceleration $\mathbf{a}$ in Cartesian coordinates.
(f) The unit binormal vector $\hat{\mathbf{u}}_b$.
(g) The unit normal vector $\hat{\mathbf{u}}_n$.
(h) The angles ϕ_x, ϕ_y, and ϕ_z that $\mathbf{a}$ makes with the x, y, and z axes.
(i) The tangential component a_t of the acceleration.
(j) The normal component a_t of the acceleration. a_n
(k) The radius of curvature of the path of P.
(l) The Cartesian coordinates of the center of curvature of the path.
{Partial Ans.: (b) 2.988 m/s; (d) $\theta_x = 139.7$deg; (j) $a_n = 5.398$ m/s^2; (l) $x_C = -0.4068$ m}

1.11 $\mathbf{F}$ is a force vector of fixed magnitude embedded on a rigid body in plane motion (in the xy plane). At a given instant, $\boldsymbol{\omega} = 2\hat{\mathbf{k}}$ rad/s, $\dot{\boldsymbol{\omega}} = -5\hat{\mathbf{k}}$ rad/s^2, $\ddot{\boldsymbol{\omega}} = 3\hat{\mathbf{k}}$ rad/s^3, and $\mathbf{F} = \cancel{(15+10)}$ (N). At that instant, calculate $\dddot{\mathbf{F}}$.
$15\hat{i}+10\hat{j}$
{Ans.: $\ddot{\mathbf{F}} = 500\hat{\mathbf{i}} + 225\hat{\mathbf{j}}$ (N/s^3)}

1.12 The absolute position, velocity, and acceleration of O are

$$\mathbf{r}_O = -16\hat{\mathbf{I}} + 84\hat{\mathbf{J}} + 59\hat{\mathbf{K}}\,(\text{m})$$
$$\mathbf{v}_O = 7\hat{\mathbf{I}} + 9\hat{\mathbf{J}} + 4\hat{\mathbf{K}}\,(\text{m/s})$$
$$\mathbf{a}_O = 3\hat{\mathbf{I}} - 7\hat{\mathbf{J}} + 4\hat{\mathbf{K}}\,(\text{m/s}^2)$$

The angular velocity and acceleration of the moving frame are

$$\boldsymbol{\Omega} = -0.8\hat{\mathbf{I}} + 0.7\hat{\mathbf{J}} + 0.4\hat{\mathbf{K}}\,(\text{rad/s}) \quad \dot{\boldsymbol{\Omega}} = -0.4\hat{\mathbf{I}} + 0.9\hat{\mathbf{J}} - 1.0\hat{\mathbf{K}}\,(\text{rad/s}^2)$$

The unit vectors of the moving frame are

$$\hat{\mathbf{i}} = -0.15670\hat{\mathbf{I}} - 0.31235\hat{\mathbf{J}} + 0.93704\hat{\mathbf{K}}$$
$$\hat{\mathbf{j}} = -0.12940\hat{\mathbf{I}} + 0.94698\hat{\mathbf{J}} + 0.29409\hat{\mathbf{K}}$$
$$\hat{\mathbf{k}} = -0.97922\hat{\mathbf{I}} - 0.075324\hat{\mathbf{J}} - 0.18831\hat{\mathbf{K}}$$

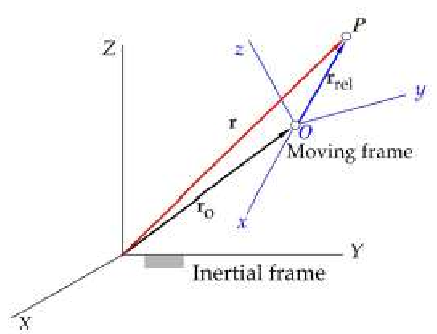
The absolute position of P is

$$\mathbf{r} = 51\hat{\mathbf{I}} - 45\hat{\mathbf{J}} + 36\hat{\mathbf{K}}\,(\text{m})$$

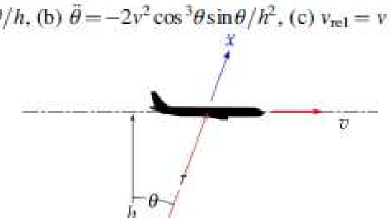
The velocity and acceleration of P relative to the moving frame are

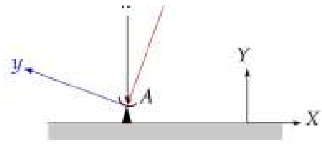
$$\mathbf{v}_{\text{rel}} = 31\hat{\mathbf{i}} - 68\hat{\mathbf{j}} - 77\hat{\mathbf{k}}\,(\text{m/s}) \quad \mathbf{a}_{\text{rel}} = 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}\,(\text{m/s}^2)$$

Calculate the absolute velocity $\mathbf{v}_P$ and acceleration $\mathbf{a}_P$ of P.
{Ans.: $\mathbf{v}_P = 156.4\hat{\mathbf{u}}_v\,(\text{m/s})$ $\quad \hat{\mathbf{u}}_v = 0.7790\hat{\mathbf{I}} - 0.3252\hat{\mathbf{J}} + 0.5360\hat{\mathbf{K}}$
$\mathbf{a}_P = 85.13\hat{\mathbf{u}}_a\,(\text{m/s}^2)$ $\quad \hat{\mathbf{u}}_a = -0.3229\hat{\mathbf{I}} + 0.8284\hat{\mathbf{J}} - 0.4576\hat{\mathbf{K}}$}



1.13 An airplane in level flight at an altitude h and a uniform speed v passes directly over a radar tracking station A. Calculate the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$ of the radar antenna as well as the rate $\dot{r}$ at which the airplane is moving away from the antenna. Use the equations of this chapter (rather than polar coordinates, which you can use to check your work). Attach the inertial frame of reference to the ground and assume a nonrotating earth. Attach the moving frame to the antenna, with the x axis pointing always from the antenna toward the airplane.
{Ans.: (a) $\dot{\theta} = v\cos^2\theta/h$, (b) $\ddot{\theta} = -2v^2\cos^3\theta\sin\theta/h^2$, (c) $v_{\text{rel}} = v\sin\theta$}





Assume, μ= 398,600km^3/sec^2, r_{Earth}= 6378km

2.1 Two particles of identical mass m are acted on only by the gravitational force of one upon the other. If the distance d between the particles is constant, what is the angular velocity of the line joining them? Use Newton's second law with the center of mass of the system as the origin of the inertial frame.

{Ans.: $\omega = \sqrt{2Gm/d^3}$}

2.2 Three particles of identical mass m are acted on only by their mutual gravitational attraction. They are located at the vertices of an equilateral triangle with sides of length d. Consider the motion of any one of the particles about the system center of mass G and, using G as the origin of the inertial frame, employ Newton's second law to determine the angular velocity ω required for d to remain constant.

{Ans.: $\omega = \sqrt{3Gm/d^3}$}

2.3 Consider the two-body problem illustrated in Fig. 2.1. If a force $\mathbf{T}$ (such as rocket thrust) acts on m_2 in addition to the mutual force of gravitation $\mathbf{F}_{21}$, show that

(a) Eq. (2.22) becomes

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{T}}{m_2}$$

(b) If the thrust vector $\mathbf{T}$ has a magnitude T and is aligned with the velocity vector $\mathbf{v}$, then

$$\mathbf{T} = T\frac{\mathbf{v}}{v}$$

2.6 If $\mathbf{r}$, in meters, is given by $\mathbf{r} = t\sin t\hat{\mathbf{I}} + t^2\cos t\hat{\mathbf{J}} + t^3\sin^2 t\hat{\mathbf{K}}$, where t is the time in seconds, calculate (a) $\dot{r}$ (where $r = \|\dot{\mathbf{r}}\|$) and (b) $\|\dot{\mathbf{r}}\|$ at $t = 2$ s.

{Ans.: (a) $\dot{r} = 4.894$ m/s; (b) $\|\dot{\mathbf{r}}\| = 6.563$ m/s}

$\widehat{u} \to \widehat{u}_r$

2.7 Starting with Eq. (2.35a), prove that $\dot{r} = \mathbf{v}\cdot\hat{\mathbf{u}}$ and interpret this result.

2.8 Show that $\hat{\mathbf{u}}_r \cdot d\hat{\mathbf{u}}_r/dt = 0$, where $\hat{\mathbf{u}}_r = \mathbf{r}/r$. Use only the fact that $\hat{\mathbf{u}}_r$ is a unit vector. Interpret this result.

2.9 Show that $v = (\mu/h)\sqrt{1 + 2e\cos\theta + e^2}$ for any orbit.

2.10 Relative to a nonrotating, earth-centered Cartesian coordinate system, the position and velocity vectors of a spacecraft are $\mathbf{r} = 7000\hat{\mathbf{i}} - 2000\hat{\mathbf{j}} - 4000\hat{\mathbf{k}}$ (km) and $\mathbf{v} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (km/s). Calculate the orbit's (a) eccentricity vector and (b) the true anomaly.

{Ans.: (a) $\mathbf{e} = 0.2888\hat{\mathbf{i}} + 0.08523\hat{\mathbf{j}} - 0.3840\hat{\mathbf{k}}$; (b) $\theta = 33.32°$}

2.11 Show that the eccentricity is 1 for rectilinear orbits ($\mathbf{h} = \mathbf{0}$).

2.12 Relative to a nonrotating, earth-centered Cartesian coordinate system, the velocity of a spacecraft is $\mathbf{v} = -4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ (km/s) and the unit vector in the direction of the radius $\mathbf{r}$ is $\hat{\mathbf{u}}_r = 0.26726\hat{\mathbf{i}} + 0.53452\hat{\mathbf{j}} + 0.80178\hat{\mathbf{k}}$. Calculate (a) the radial component of velocity v_r, (b) the azimuth component of velocity $v_\perp$, and (c) the flight path angle γ.

{Ans.: (a) −3.474 km/s; (b) 6.159 km/s; (c) −29.43°}

2.13 If the specific energy ε of the two-body problem is negative, show that m_2 cannot move outside a sphere of radius $\mu/|\varepsilon|$ centered at m_1.

2.14 Relative to a nonrotating Cartesian coordinate frame with the origin at the center O of the earth, a spacecraft in a rectilinear trajectory has the velocity $\mathbf{v}=2\hat{\mathbf{i}}+3\hat{\mathbf{j}}+4\hat{\mathbf{k}}$ (km/s) when its distance from O is 10,000 km. Find the position vector $\mathbf{r}$ when the spacecraft comes to rest.

{Ans.: $\mathbf{r}=5837.4\hat{\mathbf{i}}+8756.1\hat{\mathbf{j}}+11{,}675\hat{\mathbf{k}}$ (km)}

2.15 The specific angular momentum of a satellite in circular earth orbit is 60,000 km^2/s. Calculate the period.

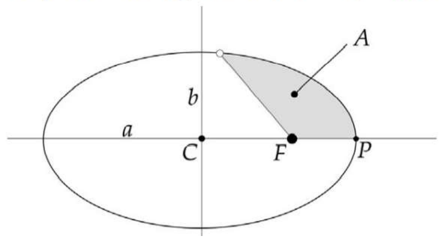
{Ans.: 2.372 h}

2.16 A spacecraft is in a circular orbit of Mars at an altitude of 200 km. Calculate its speed and its period.

{Ans.: 3.451 km/s; 1 h 49 min}

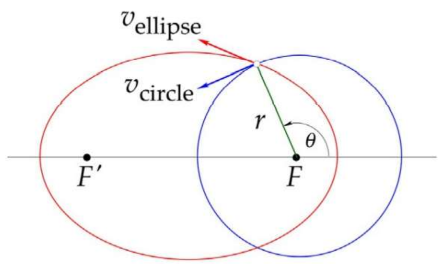
2.17 Calculate the area A swept out during the time $t = T/4$ since periapsis, where T is the period of the elliptical orbit.

{Ans.: $0.7854ab$}



2.18 Determine the true anomaly θ of the point(s) on an elliptical orbit at which the speed equals the speed of a circular orbit with the same radius (i.e., $v_{\text{ellipse}} = v_{\text{circle}}$).

{Ans.: $\theta = \cos^{-1}(-e)$, where e is the eccentricity of the ellipse}



2.19 Calculate the flight path angle at the locations found in Problem ~~2.19~~ 2.18.

{Ans.: $\gamma = \tan^{-1}\left(e/\sqrt{1-e^2}\right)$}

2.20 An unmanned satellite orbits the earth with a perigee radius of 10,000 km and an apogee radius of 100,000 km. Calculate:

(a) the eccentricity of the orbit;
(b) the semimajor axis of the orbit (km);
(c) the period of the orbit (h);
(d) the specific energy of the orbit (km^2/s^2);
(e) the true anomaly (degrees) at which the altitude is 10,000 km;
(f) v_r and $v_\perp$ (km/s) at the points found in part (e);
(g) the speed at perigee and apogee (km/s).

{Partial Ans.: (c) 35.66 h; (e) 82.26°; (g) 8.513 km/s, 0.8513 km/s}

2.21 A spacecraft is in a 400-km-by-600-km low earth orbit. How long (in minutes) does it take to coast from the perigee to the apogee?

{Ans.: 48.34 min}

2.22 The altitude of a satellite in an elliptical orbit around the earth is 2000 km at apogee and 500 km at perigee. Determine:

(a) the eccentricity of the orbit;
(b) the orbital speeds at perigee and apogee;
(c) the period of the orbit.

{Ans.: (a) 0.09832; (b) $v_p = 7.978$ km/s, $v_a = 6.550$ km/s; (c) $T = 110.5$ min}

2.23 A satellite is placed into an earth orbit at perigee at an altitude of 500 km with a speed of 10 km/s. Calculate the flight path angle γ and the altitude of the satellite at a true anomaly of 120°.

{Ans.: $\gamma = 44.60°$, $z = 12,247$ km}

2.24 A satellite is launched into earth orbit at an altitude of 1000 km with a speed of 10 km/s and a flight path angle of 15°. Calculate the true anomaly of the launch point and the period of the orbit.

{Ans.: $\theta = 32.48°$; $T = 30.45$ h}

2.25 A satellite has perigee and apogee altitudes of 500 and 21,000 km. Calculate the orbit period, eccentricity, and the maximum speed.

{Ans.: 6.20 h, 0.5984, 9.625 km/s}

2.26 A satellite is launched parallel to the earth's surface with a speed of 7.6 km/s at an altitude of 500 km. Calculate the period.

{Ans.: 1.61 h}

2.27 A satellite in orbit around the earth has a speed of 8 km/s at a given point of its orbit. If the period is 2 h, what is the altitude at that point?

{Ans.: 648 km}

2.28 A satellite in polar orbit around the earth comes within 200 km of the north pole at its point of closest approach. If the satellite passes over the pole once every 100 min, calculate the eccentricity of its orbit.

{Ans.: 0.07828}

2.29 For an earth orbiter, the altitude is 1000 km at a true anomaly of 40° and 2000 km at a true anomaly of 150°. Calculate

(a) the eccentricity;
(b) the perigee altitude (km);
(c) the semimajor axis (km).

{Partial Ans.: (c) 7863 km}

2.30 An earth satellite has a speed of 7.5 km/s and a flight path angle of 10° when its radius is 8000 km. Calculate

(a) the true anomaly (degrees);
(b) the eccentricity of the orbit.

{Ans.: (a) 63.82°; (b) 0.2151}

2.31 For an earth satellite, the specific angular momentum is $70,000$ km^2/s and the specific energy is -10 km^2/s^2. Calculate the apogee and perigee altitudes.

{Ans.: 25,889 and 1214.9 km}

2.32 A rocket launched from the surface of the earth has a speed of 7 km/s when the powered flight ends at an altitude of 1000 km. The flight path angle at this time is 10°. Determine the eccentricity and the period of the orbit.

{Ans.: 0.1963 and 92.0 min}

2.33 If the perigee velocity is c times the apogee velocity, calculate the eccentricity of the orbit in terms of c.

{Ans.: $e = (c - 1)/(c + 1)$}

2.34 At what true anomaly does the speed on a parabolic trajectory equal α times the speed at the periapsis, where $\alpha \leq 1$?
{Ans.: $\cos^{-1}(2\alpha^2 - 1)$}

increase

2.35 What velocity, ~~relative to the earth~~, is required to escape the solar system on a parabolic path from earth's orbit?

{Ans.: 12.34 km/s}

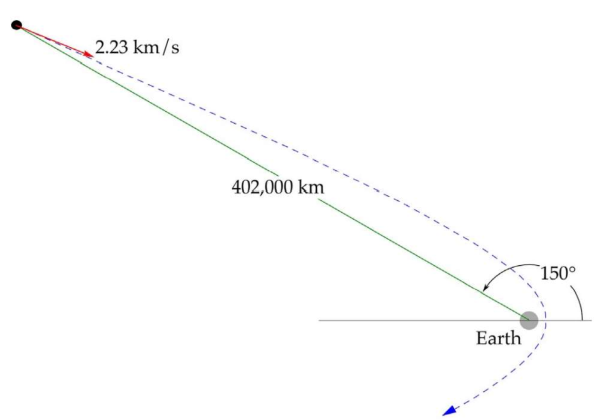
for the earth

2.36 A hyperbolic earth departure trajectory has a perigee altitude of 250 km and a perigee speed of 11 km/s. Calculate:
(a) the hyperbolic excess speed (km/s);
(b) the radius (km) when the true anomaly is 100°;
(c) v_r and $v_\perp$ (km/s) when the true anomaly is 100°.

{Partial Ans.: (b) 16,179 km}

2.37 A meteoroid is first observed approaching the earth when it is 402,000 km from the center of the earth with a true anomaly of 150°. If the speed of the meteoroid at that time is 2.23 km/s, calculate:
(a) the eccentricity of the trajectory;
(b) the altitude at closest approach;
(c) the speed at the closest approach.

{Ans.: (a) 1.086; (b) 5088 km; (c) 8.516 km/s}



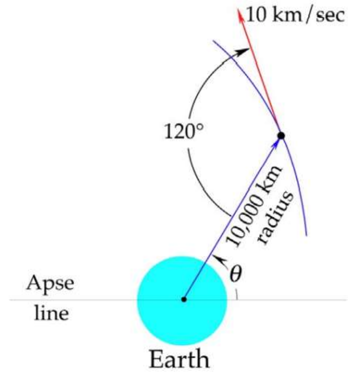
2.38 If α is a number between 1 and $\sqrt{(1+e)/(1-e)}$, calculate the true anomaly at which the speed on a hyperbolic trajectory is α times the hyperbolic excess speed.
$\left\{\text{Ans.: } \cos^{-1}\left[\frac{(\alpha^2-1)(e^2-1)-2}{2e}\right]\right\}$

2.39 For a hyperbolic orbit, find the eccentricity in terms of the radius at periapsis r_p and the hyperbolic excess speed v_∞.

{Ans.: $e = 1 + r_p v_\infty^2/\mu$}

2.40 A space vehicle has a velocity of 10 km/s in the direction shown when it is 10,000 km from the center of the earth. Calculate its true anomaly.

{Ans.: 51°}



2.41 A spacecraft at a radius r has a speed v and a flight path angle γ. Find an expression for the eccentricity of its orbit in terms of r, v, and γ.

{Ans.: $e = \sqrt{1 + \sigma(\sigma - 2)\cos^2\gamma}$, where $\sigma = rv^2/\mu$}

2.42 For an orbiting spacecraft, $r = r_1$ when $\theta = \theta_1$, and $r = r_2$ when $\theta = \theta_2$. What is the eccentricity?

{Ans.: $e = (r_1 - r_2)/(r_2 \cos\theta_2 - r_1 \cos\theta_1)$}

2.46 For the sun–earth system, find the distance of the collinear Lagrange points L_1, L_2, and L_3 from the barycenter.

{Ans.: $x_1 = 148.108(10^6)$ km, $x_2 = 151.101(10^6)$ km, and $x_3 = -149.600(10^6)$ km (opposite side of the sun)}