MA 214: Introduction to Numerical Analysis

Department of Mathematics, Indian Institute of Technology Bombay

End-Semester Examination

Time: 3 hours | Marks: 40

Instructor: S. Baskar and S. Sivaji Ganesh

Date: 26/4/2019

Instructions:

(1) Write your **Name, Roll Number, and Tutorial Batch** clearly on your answer book as well as every supplement you may use. Also write your name and roll number on this question paper. A **penalty of -1 mark will be awarded** for failing to do so.

(2) Number the pages of your answer book and make a question-page index on the front page. A **penalty of -1 mark will be awarded** for failing to do so.

(3) The answer to each question should start on a new page. If the answer for a question is split into two parts and written at two different places, the first part alone will be evaluated.

(4) Only scientific calculators are allowed. Any kind of programing device is not allowed.

(5) Formulas used need not be proved but **needs to be stated clearly.**

(6) There are **13 questions** in this question paper. **Answer all the questions.**

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(1) Find $\alpha, \beta \in \mathbb{R}$ such that the following two conditions hold: [3 Marks]

(a) bisection method hypotheses hold for approximating a root of the equation $f(x) = 0$ on the interval $[0, 1]$, where $f(x) = (x - \alpha)(x - \beta)$, and

(b) the errors e_1, e_2, e_3 satisfy $|e_1| > |e_2|$, and $|e_2| < |e_3|$.

Justify your answer by showing the computation of the errors in each of the three iterations. Note that, $e_k := r - x_k$, where $\{x_k\}$ denotes the sequence generated by the bisection method.

(2) Draw the graph of a function $f : [0, 1] \to \mathbb{R}$ such that $f(0)f(1) < 0$ and the first three terms of the sequence $\{x_n\}$ in the regula-falsi method are given by $x_1 = 0.25$, $x_2 = 0.75$, and $x_3 = 0.5$. Justify your answer graphically. [3 Marks]
[No need to give formula for such an f and no need to give any analytical justification]

(3) Using the concepts of the fixed-point iteration method, prove that the sequence $\{x_n\}$ defined recursively as follows is convergent. [3 Marks]

$$x_0 = 1, \quad x_{n+1} = 3 - \frac{1}{2}|x_n|, \quad (n = 0, 1, 2, \ldots).$$

(4) Let $\alpha \in \mathbb{R}$. Construct the divided difference table for the following data: [3 Marks]

x	-2	-1	0	1	2
y	α	3	1	-1	3

Let $p(x)$ denote the interpolating polynomial for the above data. Find the value of α such that the coefficient of x^4 in $p(x)$ is equal to 0. Also determine the interpolating polynomial $p(x)$.

(5) Derive the piecewise quadratic interpolating function for

$$f(x) = \sin x$$

on the interval $[0, 2\pi]$ with the partition $\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$. [3 Marks]

(6) Derive the trapezoidal rule using the method of undetermined coefficients. [3 Marks]

(7) Let $f : [a, b] \to \mathbb{R}$ be a twice continuously differentiable function. Derive an expression for the mathematical error involved in approximating the integral $\int_a^b f(x)\,dx$ using the mid-point rule. [3 Marks]

(8) Obtain the expression for the arithmetic error in approximating the value of the integral $\int_a^b f(x)dx$ using Simpson's rule. Further if $a = -1$, $b = 1$, and $f(x) = e^{-x^2}$, then find an upper bound for the absolute value of the arithmetic error, given that the function values are approximated to 6 significant digits. [3 Marks]

(9) State the central difference formula for approximating $f'(x)$. Assume that $f \in C^3(\mathbb{R})$ with $|f'''(x)| \leq 3$ for all $x \in \mathbb{R}$ and the function values (used in the central difference formula) are approximated with an absolute error which is at most 10^{-3}. Derive an upper bound for the total error involved in this approximation and find the optimal h that minimizes the upper bound. [3 Marks]

[**Note:** You may directly use the formula for Mathematical Error $= -\frac{h^2}{6} f'''(\xi)$ for some ξ.]

(10) Obtain an expression for the mathematical error in the approximation

$$f'(x) \approx \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h}.$$

Find an upper bound for the absolute value of the mathematical error. What is the order of accuracy of the approximation? [3 Marks]

(11) Consider the following initial value problem (IVP)

$$y' = xe^{-5x} - 5y, \quad y(0) = 0.$$

(a) Let $N \in \mathbb{N}$. Give the formula of Euler method for approximating the solution of the IVP on the interval $[0, 1]$ with step size $h = \frac{1}{N}$. Show that [2 Marks]

$$y_N = \frac{1}{N^2} e^{-5/N} \left(1 - \frac{5}{N}\right)^{N-2} \sum_{k=0}^{N-2} (k+1) \left(\frac{e^{-5/N}}{1 - (5/N)}\right)^k.$$

(b) Show that $\lim_{N\to\infty} y_N = \frac{e^{-5}}{2}$. [**Hint:** $e^x = \lim_{N\to\infty} \left(1 + \frac{x}{N}\right)^N$.] [2 Marks]

(12) Derive the Euler's trapezoidal method and use it to obtain an approximate value of $y(0.4)$ with $h = 0.2$, where $y(x)$ is the solution of the initial value problem [3 Marks]

$$y' = xy, \; y(0) = 1.$$

(13) Using Runge-Kutta method of order 2 with $h = 0.5$, obtain approximate values of $y(0.5)$ and $y(1)$, where $y(x)$ is the solution of the initial value problem [3 Marks]

$$y' = x^2 + y^2; \quad y(0) = 1.$$

— **End of Question Paper** —