- The angular momentum of body m_2 relative to m_1 is

$$H_{2/1} = r \times m_2 \dot{r}$$

- Let $h = \frac{H_{2/1}}{m_2}$, so that

$$h = r \times \dot{r}$$

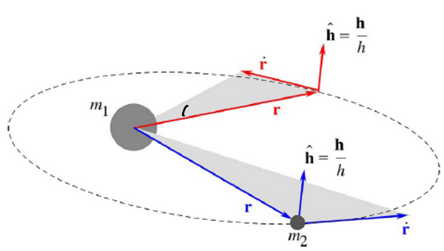
relative angular momentum of m_2 per unit mass (i.e., the specific relative angular momentum)

- $$\frac{dh}{dt} = \underbrace{\dot{r} \times \dot{r}}_{0} + r \times \ddot{r}$$
$$= r \times \ddot{r}$$
$$= r \times \left(\underbrace{-\frac{\mu}{\|r\|^3}}\, r \right)$$
$$= 0$$

- Therefore, $\underbrace{\frac{dh}{dt} = 0}_{\text{conservation of angular momentum}}$, or, $r \times \dot{r} =$ constant.

- If $r, \dot{r}$ are parallel, i.e., $r \times \dot{r} = 0$, then $r \times \dot{r}$ remains zero at all points of the trajectory.

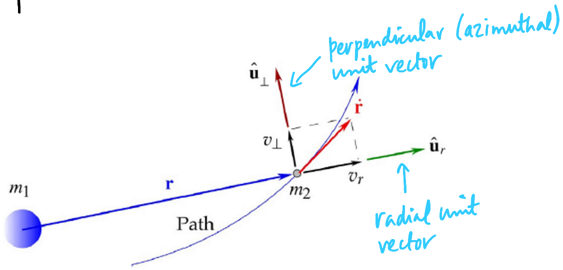
- Zero angular momentum characterises rectilinear trajectories, where m_2 moves towards or away from m_1 in a straight line.



- Since $h = r \times \dot{r}$, $\hat{h} = \frac{h}{\|h\|}$

- Note that $\hat{h}$ is constant along all points of the trajectory.

- Path of m_2 around m_1 lies in a single plane.

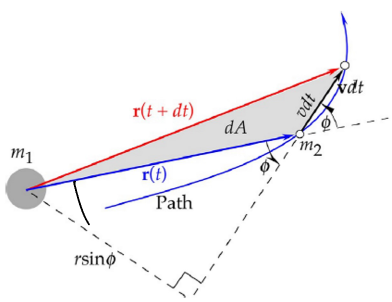


- $V = V_r \hat{u}_r + V_\perp \hat{u}_\perp$

- Then, $h = r \times \dot{r}$

$$\begin{aligned} &= r \times V \\ &= \|r\| \hat{u}_r \times (V_r \hat{u}_r + V_\perp \hat{u}_\perp) \\ &= \|r\| V_\perp \hat{h} \end{aligned}$$

- $\|h\| = \|r\| |v_\perp|$



- In differential time dt, the position vector r sweeps out an area:

$$\begin{aligned} dA &= \frac{1}{2} \cdot \|v\| dt \cdot \|r\| \sin\phi \\ &= \frac{1}{2} \cdot \|r\| \cdot \|v\| \sin\phi \, dt \\ &= \frac{1}{2} \|r\| |v_\perp| \, dt \end{aligned}$$

Alternatively, $dA = \frac{1}{2} \|r \times v\, dt\|$ (why?)

$$\begin{aligned} &= \frac{1}{2} \|r \times \dot{r}\| \, dt \\ &= \frac{\|h\|}{2} \, dt \end{aligned}$$

- $\dfrac{dA}{dt} = \dfrac{\|h\|}{2} = \text{constant}$

– Kepler's second law: The position vector sweeps out equal areas in equal intervals of time.

Kepler 1571-1630

– Recall that $r \cdot r = \|r\|^2$

– $\overset{\cdot}{\widehat{r \cdot r}} = 2\|r\| \, \overset{\cdot}{\widehat{\|r\|}}$

$$\cancel{2} r \cdot \dot{r} = \cancel{2} \|r\| \, \overset{\cdot}{\widehat{\|r\|}}$$

$$r \cdot v = \|r\| \, \overset{\cdot}{\widehat{\|r\|}}$$

– $\ddot{r} \times h = -\frac{\mu}{\|r\|^3} r \times h$ (taking cross product with h)

– $$\frac{d}{dt}(\dot{r} \times h) = \ddot{r} \times h + \dot{r} \times \underbrace{\dot{h}}_{0}$$
$$= \ddot{r} \times h$$

– $$\frac{d}{dt}(\dot{r} \times h) = -\frac{\mu}{\|r\|^3} r \times h$$

– $$\frac{1}{\|r\|^3} r \times h = \frac{1}{\|r\|^3}\left[r \times (r \times \dot{r})\right]$$

$$= \frac{1}{\ }\left[r(r \cdot \dot{r}) - \dot{r}(r \cdot r)\right] \text{ (bac-cab rule)}$$

$\|r\|^3$

$$= \frac{1}{\|r\|^3}\left[r \left(\|r\| \, \dot{\overline{\|r\|}} - \dot{r} \, \|r\|^2 \right) \right]$$

$$= \frac{r \, \dot{\overline{\|r\|}} - \dot{r} \, \|r\|}{\|r\|^2}$$

$$= -\frac{d}{dt}\left(\frac{r}{\|r\|} \right)$$

- $$\frac{d}{dt}(\dot{r} \times h) = \frac{d}{dt}\left(\mu \frac{r}{\|r\|} \right)$$

- $$\frac{d}{dt}\left(\dot{r} \times h - \mu \frac{r}{\|r\|} \right) = 0$$

- $$\dot{r} \times h - \mu \frac{r}{\|r\|} = C$$ $\rightarrow$ Laplace vector (named after Pierre-Simon Laplace)

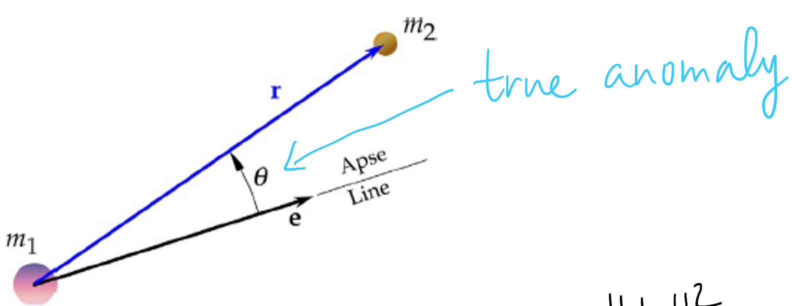
$\downarrow$

First integral of the equation of motion

- $$\underbrace{(\dot{r} \times h) \cdot h}_{0} - \mu \frac{\overbrace{r \cdot h}^{0}}{\|r\|} = C \cdot h$$ (taking dot product with h)

- $C \cdot h = 0$, i.e., C must lie in the orbital plane.

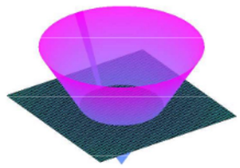
- $$\frac{r}{\|r\|} + \underbrace{\frac{C}{\mu}}_{e \text{ (eccentricity vector)}} = \frac{\dot{r} \times h}{\mu}$$



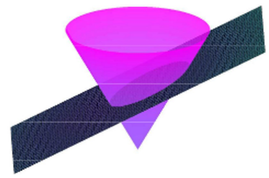
- $$\frac{r \cdot r}{\|r\|} + r \cdot e = \frac{\overbrace{r \cdot (\dot{r} \times h)}^{\|h\|^2}}{\mu}$$ (taking dot product with r)

- $$\|r\| + r \cdot e = \frac{\|h\|^2}{\mu}$$

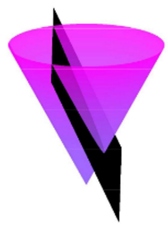
- $$\|r\| = \frac{\|h\|^2}{\mu} \frac{1}{1 + \|e\| \cos\theta}$$ (μ, $\|h\|$, $\|e\|$ are constants)



Circle



Ellipse



Parabola

Hyperbola

- Kepler's first law: Planets follow elliptical paths around the sun.

- Two-body orbits are often referred to as Keplerian orbits.