- Recall that orbit equation describes conic sections:

  circular ($\|e\| = 0$)
  elliptical ($0 < \|e\| < 1$)
  parabolic ($\|e\| = 1$)
  hyperbolic ($\|e\| > 1$)

- The component of the velocity normal to the position vector:

$$V_\perp = \|r\| \dot{\theta}$$

- $\|h\| = \|r\| V_\perp = \|r\|^2 \dot{\theta}$

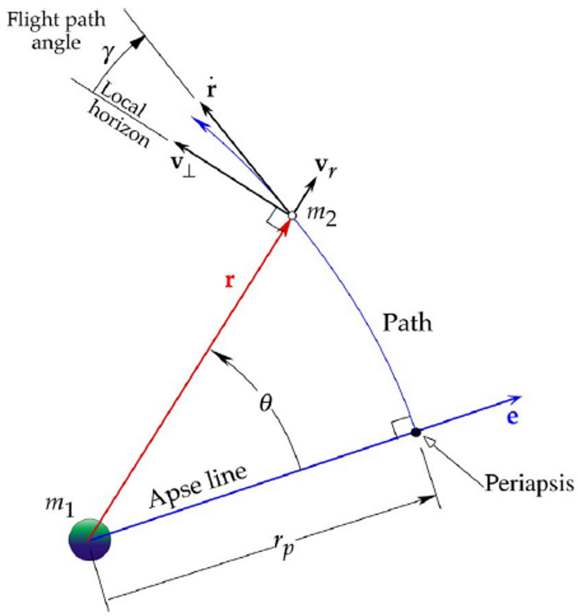
- $V_\perp = \dfrac{\|h\|}{\|r\|} = \dfrac{\mu}{\|h\|} (1 + \|e\| \cos\theta)$

- $$V_r = \dot{\overline{\|r\|}} = \frac{d}{dt}\left[\frac{\|h\|^2}{\mu} \frac{1}{1 + \|e\| \cos\theta}\right]$$

$$= \frac{\|h\|^2}{\mu}\left[\frac{\|e\| \sin\theta\, \dot{\theta}}{(1 + \|e\| \cos\theta)^2}\right]$$

$$= \frac{\|h\|^2}{\mu} \frac{\|e\| \sin\theta}{(1 + \|e\| \cos\theta)^2} \frac{\|h\|}{\|r\|^2}$$

$$= \frac{\mu}{\|h\|} \|e\| \sin\theta$$



$(V_r = 0)$

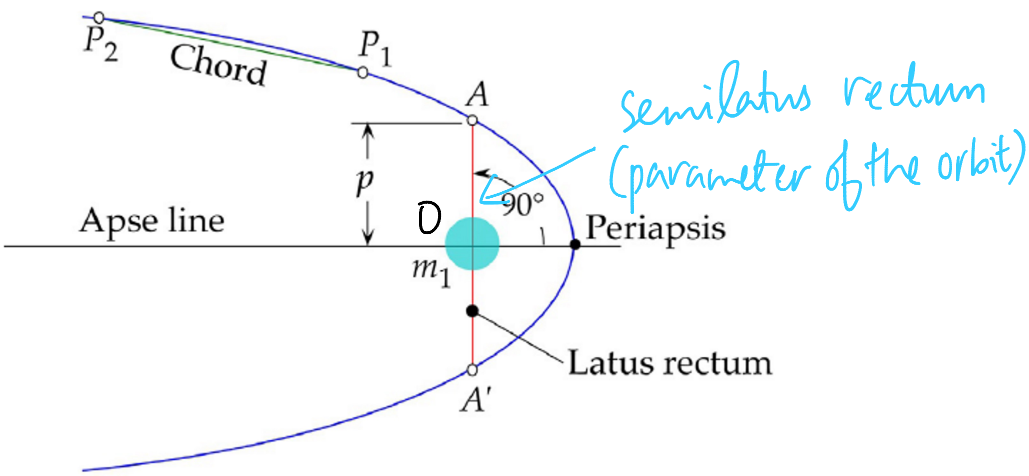
$V_r > 0, \quad 0° < \theta < 180°$

$V_r < 0, \quad 180° < \theta < 360°$

- m_2 is closest to m_1, i.e., $\|r\|$ is the smallest, when $\theta = 0°$ (unless $\|e\| = 0$, in which case the distance between m_1 and m_2 is constant).

- $$r_p = \frac{\|h\|^2}{\mu} \frac{1}{1+\|e\|}$$

- $$\tan\gamma = \frac{V_r}{V_\perp} = \frac{\|e\| \sin\theta}{1+\|e\|\cos\theta}$$



- $p = \dfrac{\|h\|^2}{\mu}$ (why?)

- $\ddot{r} \cdot \dot{r} = -\dfrac{\mu}{\|r\|^3}\, r \cdot \dot{r}$ (taking dot product with the relative linear momentum per unit mass)

- $\ddot{r} \cdot \dot{r} = \dfrac{1}{2}\, \dot{\overline{(\dot{r} \cdot \dot{r})}} = \dfrac{1}{2}\, \dot{\overline{(v \cdot v)}} = \dfrac{1}{2}\, \dot{\overline{\|v\|^2}}$

- $r \cdot \dot{r} = \|r\|\, \dot{\overline{\|r\|}}$ and $\dot{\overline{\|r\|^{-1}}} = -\|r\|^{-2}\, \dot{\overline{\|r\|}}$

- $$\overset{\cdot}{\overline{\frac{1}{2}\|v\|^2}} = -\frac{\mu}{\|r\|^3}\|r\|\,\overset{\cdot}{\overline{\|r\|}} = -\mu\|r\|^{-2}\,\overset{\cdot}{\overline{\|r\|}}$$

$$= \overset{\cdot}{\overline{\mu\|r\|^{-1}}}$$

- $$\frac{d}{dt}\left(\frac{\|v\|^2}{2} - \frac{\mu}{\|r\|}\right) = 0, \text{ or,}$$

$$\underbrace{\frac{\|v\|^2}{2}}_{\substack{\text{relative K.E.} \\ \text{per unit mass}}} - \underbrace{\frac{\mu}{\|r\|}}_{\substack{\text{P.E. per unit mass} \\ \text{of the body } m_2 \text{ in the} \\ \text{gravitational field of} \\ m_1}} = \underbrace{\varepsilon}_{\substack{\text{total mechanical} \\ \text{energy per unit} \\ \text{mass}}} \text{ (constant)}$$

- Conservation of energy, namely, the specific mechanical energy is the same at all points of the trajectory.

- At periapsis, i.e., $\theta = 0°$:

$$\varepsilon = \varepsilon_p = \frac{\|v_p\|^2}{2} - \frac{\mu}{r_p}$$

$$= \frac{|v_\perp|}{2} - \frac{\mu}{r_p}$$

$$= \frac{\|h\|^2}{2r_p^2} - \frac{\mu}{r_p}$$

$$= \frac{-\mu^2}{2\|h\|^2}\left(1 - \|e\|^2\right)$$

- Energy $\mathcal{E}$ of a spacecraft of mass m :

$$\mathcal{E} = m\varepsilon$$

<u>Circular Orbits ($\|e\| = 0$)</u>

- $\|r\| = \frac{\|h\|^2}{\mu} = \text{constant}$

- $\|h\| = \|r\| \, |v_\perp| = \|r\| \|v\|$ (why?)

- $v_{circular} = \sqrt{\frac{\mu}{\|r\|}}$

- Period of a circular orbit (time required for one orbit):

$$T = \frac{2\pi \|r\|}{v_{circular}} = \frac{2\pi}{\sqrt{\mu}} \|r\|^{\frac{3}{2}}$$

- $$\mathcal{E} = \frac{-\mu^2}{2\|h\|^2} = \frac{-\mu}{2\|r\|}$$

- As the radius goes up, the energy becomes less negative (i.e., it increases). In other

- As the ... less negative (i.e., it increases). In other words, the larger the orbit is, the greater is its energy.

- To launch a satellite from the surface of the earth into a circular orbit requires increasing $\mathcal{E}$

- A propulsion system that can place a large mass in a low earth orbit, can place a smaller mass in a high earth orbit. altitude lies between 150 Km and 2000 Km

- If a satellite remains always above the same point on the earth's equator, then it is in a circular geostationary orbit (GEO).

Example

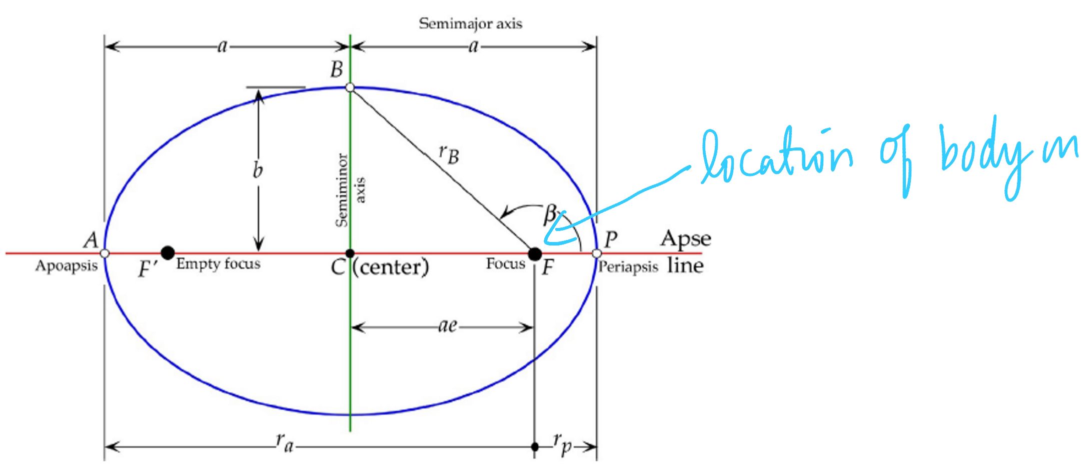
Calculate the altitude z_{GEO} and speed v_{GEO} of a geostationary earth satellite.

Details

$$v_{GEO} = \sqrt{\frac{\mu}{r_{GEO}}}, \quad v_{GEO} = r_{GEO}\,\omega_E$$

$$z_{GEO} = r_{GEO} - r_{Earth}$$

Elliptical Orbits $(0 < \|e\| < 1)$



- $r_a = \frac{\|h\|^2}{\mu} \frac{1}{1-\|e\|}$

- $2a = r_a + r_p$

  $$a = \frac{\|h\|^2}{\mu} \frac{1}{1-\|e\|^2}$$

- $$\|r\| = a \frac{1-\|e\|^2}{1+\|e\| \cos\theta}$$

- $CF = a - FP = a - r_p = a\|e\|$

- $$r_B = a \frac{1-\|e\|^2}{1+\|e\| \cos\beta}$$

- $a\|e\| = r_B \cos(180^\circ - \beta) \Rightarrow \|e\| = -\cos\beta$

- $r_B = a$

- $b^2 = r_B^2 - (a\|e\|)^2 \Rightarrow b = a\sqrt{1-\|e\|^2}$