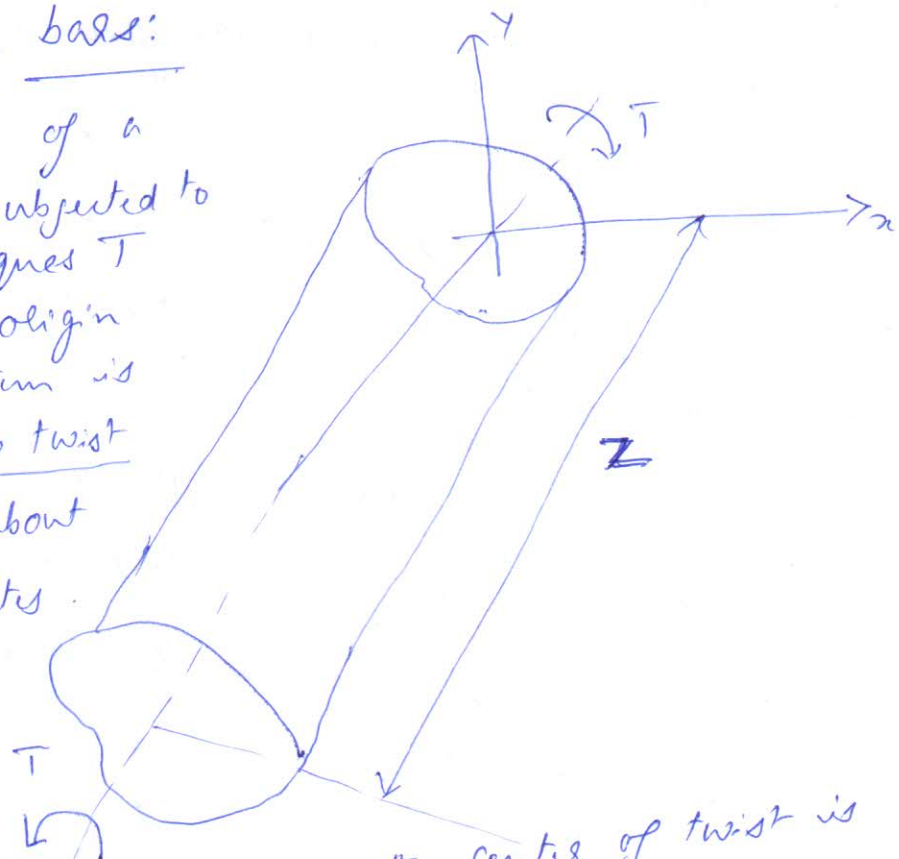
TORSION

Torsion of uniform bars:

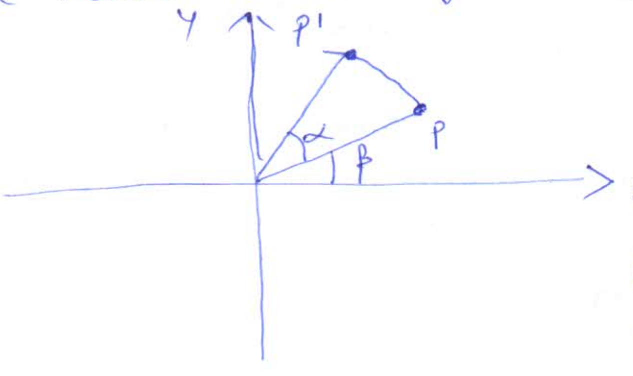


Consider a straight bar of a constant cross-section subjected to equal and opposite torques T at each end. The origin of the coordinate system is located at the center of twist of the cross-section, about which cross-section rotates during twisting.

This implies in-plane displacements 'u' & 'v' vanish along the Z-axis. Location of the center of twist is function of shape of the cross-section. Let α denote the total angle of rotation (twist angle) at Z relative to the end at $Z = 0$. The twist angle per unit length is denoted by

$$\theta = \alpha / Z$$

St. Venant assumed that during torsional deformation the in-plane sections warp, but the projections on the x-y plane rotate as a rigid body.



Consider an arbitrary point P on the c/s at Z that moves through a rotation of a small angle α to P' after torque is applied.

Assume that the c/s at $z = 0$ remains stationary.
If rotation angle is small, then the displacement components at point P are given by,

$$u = -r\alpha \sin\beta = -\alpha y = -\theta z y \longrightarrow (1)$$

$$v = r\alpha \cos\beta = \alpha x = \theta z x \longrightarrow (2)$$

The displacement w in the z-direction is assumed to be independent of z

$$w(x,y) = \theta\, \psi(x,y) \longrightarrow (3)$$

where, $\psi(x,y)$ is the warping function

From (1) - (3) we get the following strain terms, which are zero.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0 \;;\quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \;;\quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \longrightarrow (4)$$

(4) implies, from constitutive relations,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = 0$$

Thus τ_{xz} and τ_{yz} are the only two nonvanishing stress components. In the absence of body forces, the equations of equillibrium reduce to,

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \longrightarrow (5)$$

Prandtl introduced a stress function $\phi(x,y)$ such that

$$\tau_{xz} = \frac{\partial \phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x}$$

the form of τ_{xz} & τ_{yz} using $\phi(x,y)$ satisfies eqn (5).

The strain- displacement relations from (1) - (3) are

$$\left.\begin{aligned} \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} - \theta y \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} + \theta x \end{aligned}\right\} \quad (6)$$

From (6) we can see that

$$\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} = 2\theta \longrightarrow (7)$$

This is the compatibility eqn for torsion.

Using stress-strain relationships, we have

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

From (7) we now get

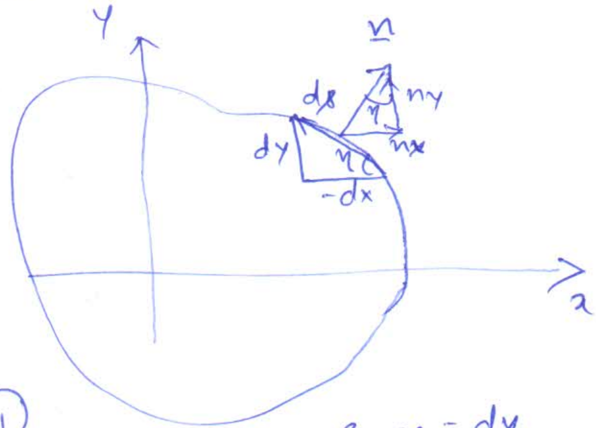
$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = 2G\theta \longrightarrow (8)$$

(8). can be written in terms of $\phi(x,y)$ as,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \longrightarrow (9)$$

The torsion problem reduces to finding stress function ϕ and requiring that stresses derived from its stress function satisfy the boundary conditions.

On the lateral surface of the bar, no loads are applied. Thus the traction $\underline{t}$ must vanish



$$\{t\} = [\sigma]\{n\}$$

$$\Rightarrow \quad \{t\} = [\sigma]\begin{Bmatrix} n_x \\ n_y \\ 0 \end{Bmatrix}$$

$$\Rightarrow \left. \begin{aligned} & t_x = 0 \quad t_y = 0 \\ & t_z = \tau_{xz} n_x + \tau_{yz} n_y \end{aligned} \right\} \quad (11)$$

$$= \frac{\partial \phi}{\partial y} n_x - \frac{\partial \phi}{\partial x} n_y$$

$$(12) \quad \begin{cases} n_x = \sin\eta = \dfrac{dy}{ds} \\ n_y = \cos\eta = -\dfrac{dx}{ds} \end{cases}$$

$\therefore$ We can now write

$$t_z = \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} \longrightarrow (13)$$

$\therefore$ stress free condition is given as $t_z = 0$

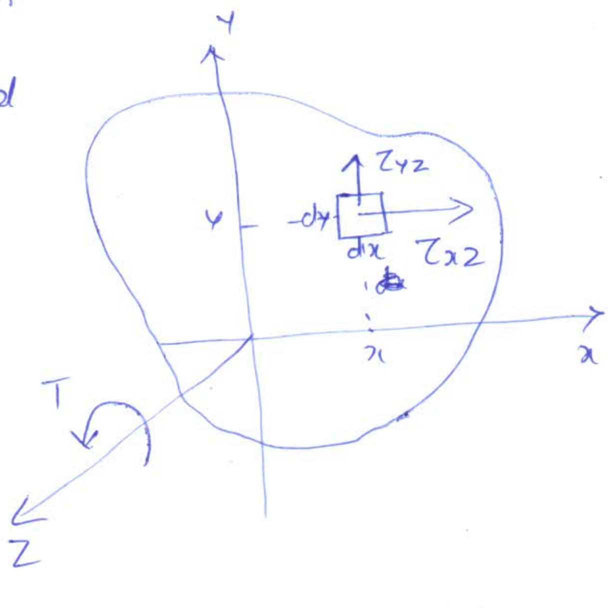
$$\Rightarrow \frac{d\phi}{ds} = 0 \longrightarrow (14) \quad \text{or} \quad \phi = \text{constant on the lateral surface.}$$

For solid sections with a single contour boundary, this constant is arbitrary and can be chosen to be zero.

$\therefore$ Boundary condition is expressed as

$\phi = 0$ on the lateral surface of the bar

Consider a differential area $dA = dx\,dy$. The torque produced by stresses in this area is



$$dT = x\tau_{yz}\,dA - y\tau_{xz}\,dA$$

$$= \left[-x\frac{\partial \phi}{\partial x} - y\frac{\partial \phi}{\partial y}\right] dA$$

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The total Resultant torque is obtained by integrating dT over the entire c/s

$$T = -\iint \left[x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right] dx\,dy$$

$$= -\iint \left[\frac{\partial}{\partial x}(x\phi) - \phi \right] dx\,dy - \iint \left[\frac{\partial}{\partial y}(y\phi) - \phi \right] dx\,dy$$

$$= 2\iint \phi\, dx\,dy - \int [x\phi]_{x_1}^{x_2} dy - \int [y\phi]_{y_1}^{y_2} dx$$

where, x_1, x_2, y_1, y_2 are integration limits on the boundary, the boundary. Since ϕ vanishes on the boundary, the last two terms in the above equation vanishes

$$\therefore\ T = 2\iint \phi\, dx\,dy \longrightarrow (15)$$

The above derivation clearly indicates that the solution of the torsion problem lies in finding ϕ that vanishes on the boundary of the bar (lateral boundary).

The induced warping can be found by using eqn (6) integrating $\partial w/\partial x$ and $\partial w/\partial y$

Bars with circular c/s

Consider a uniform bar of circular c/s. If the origin of the coordinates is chosen to coincide with the centre of the c/s, the boundary contour is given by the equation

$$x^2 + y^2 = a^2 \,, \quad a \rightarrow \text{radius of the circular boundary}$$

Assume the stress function as

$$\phi = c\left(\frac{x^2}{a^2} + \frac{y^2}{a^2} - 1\right) \longrightarrow (16)$$

This form of ϕ satisfies the boundary condition eqn (14).

Using compatibility eqn, we have.

$$\nabla^2\phi = -2G\theta \,,$$

$$c = -\frac{1}{2}a^2 G\theta$$

We can now determine torque as

$$T = 2c \iint \left(\frac{x^2}{a^2} + \frac{y^2}{a^2} - 1\right) dx\,dy$$

$$= 2c \iint \left(\frac{r^2}{a^2} - 1\right) dA = 2c\left(\frac{J}{a^2} - A\right)$$

note: $r^2 = x^2 + y^2$

where, $J = \iint r^2\, dA = \frac{1}{2}\pi a^4$ is the polar moment of inertia of the c/s. $A = \pi a^2 \Rightarrow a^2 A = 2J$

$$\therefore\ T = -\frac{2cJ}{a^2} = \underline{\underline{\theta G J}}$$

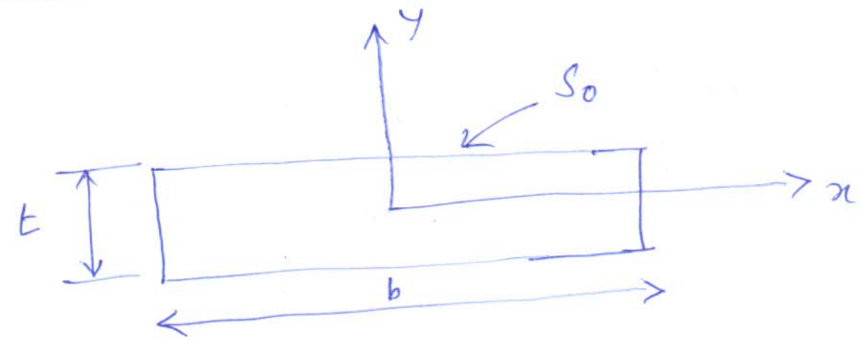
$GJ \rightarrow$ torsional stiffness / torsional rigidity

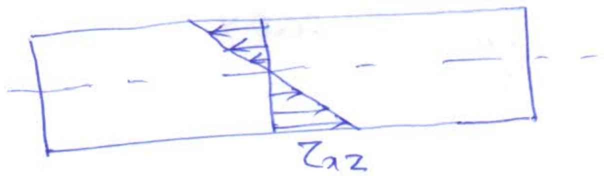
The shear stresses are,

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = 2cy/a^2 = -G\theta y$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = -2cx/a^2 = G\theta x$$

Bars with narrow rectangular c/s





Consider a bar with narrow rectangular c/s. It is assumed that $t << b$. On top and bottom faces $(y = \pm t/2)$, the boundary condition of traction free leads to

$$\underline{t} = \sigma \underline{n} = \sigma \begin{Bmatrix} 0 \\ \pm 1 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \tau_{yz} = 0$$

In terms of stress function it means $\longrightarrow$ (1)

$$-\frac{\partial \phi}{\partial x} = \tau_{yz} = 0$$

on the top and bottom surface

Since t is very small, and τ_{yz} must vanish at $y = \pm t/2$, it is unlikely that the shear stress τ_{yz} would build up across the thickness. Therefore we can assume that $\tau_{yz} \approx 0$ through the thickness. Consequently ϕ is independent of 'x'.

$\therefore$ Governing eqn reduces to

$$\frac{d^2\phi}{dy^2} = -2G\theta \quad \rightarrow (2)$$

$$\Rightarrow \phi = -G\theta y^2 + C_1 y + C_2$$

The boundary condition requires,

$$\phi = 0 \quad @ \ y = \pm t/2$$

$$\Rightarrow C_1 = 0 \quad \& \quad C_2 = G\theta t^2/4$$

$$\therefore \phi = -G\theta \left[y^2 - t^2/4 \right]$$

$$\Rightarrow \tau_{xz} = \frac{\partial \phi}{\partial y} = -2G\theta y \qquad \tau_{yz} = 0 = -\frac{\partial \phi}{\partial x}$$

$$\tau_{max} = (\tau_{xz})_{max} = \underbrace{G\theta t}_{\text{only magnitude}} \quad \left(@ \ y = \pm\frac{t}{2}\right)$$

Torque can now be calculated as,

$$T = \iint 2\phi \, dx\, dy = -2G\theta \int_{-b/2}^{b/2} \int_{-t/2}^{t/2} (y^2 - t^2/4) \, dx\, dy$$

$$T = \frac{bt^3}{3} G\theta$$

Define torsional constant $J = bt^3/3$

$$\therefore T = \underline{GJ\theta}$$

Further, $\dfrac{\partial w}{\partial x} = \gamma_{xz} + \theta y = \dfrac{\tau_{xz}}{G} + \theta y = -\theta y$

$\therefore$ warping $\omega = -xy\theta$ $\left(\because \omega = -\int \theta y \, dx\right.$

(note: integration constant set equal to zero since $\omega = 0$ @ center of twist)
$(x, y, z = 0, 0, 0)$

The results can be extended for sections composed of multiple thin-walled members.

$$G J = G (J_1 + J_2)$$