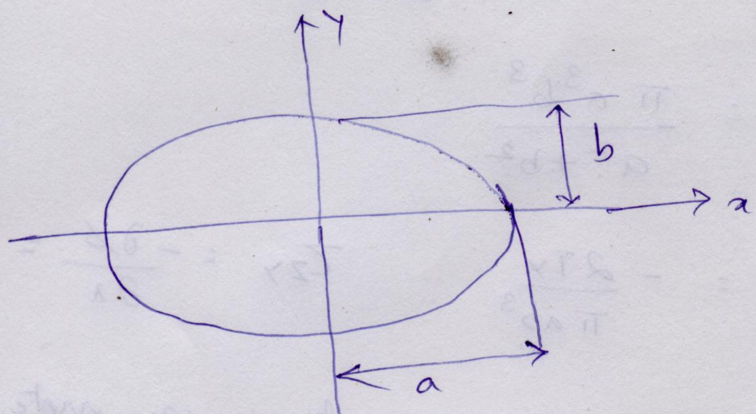
Torsion of elliptical c/s cylindrical bar



Equation of boundary is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we choose a stress function of the form
$$\phi = c\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$
then the boundary condition $\phi = 0$ is satisfied at every point on the boundary and the constant C is chosen such that
$$\nabla^2 \phi = -2G\theta$$
$$\Rightarrow 2c\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\theta$$
$$\therefore c = -\frac{G\theta\, a^2 b^2}{(a^2+b^2)}$$

$$\therefore \phi = -\frac{G\theta\, a^2 b^2}{(a^2+b^2)}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$\therefore$ Torque, $T = \iint 2\phi \, dx\, dy$

$$T = -2G\theta \frac{a^2b^2}{a^2+b^2}\left(\frac{1}{a^2}\iint x^2\, dx\, dy + \frac{1}{b^2}\iint y^2\, dx\, dy - \iint dx\, dy\right)$$

$$\Rightarrow T = -2G\theta \frac{a^2b^2}{a^2+b^2}\left(\frac{1}{a^2}\frac{\pi a^3 b}{4} + \frac{1}{b^2}\frac{\pi a b^3}{4} - \pi ab\right)$$

$$\cancel{= -2G}\quad T = G\theta \frac{\pi a^3 b^3}{(a^2+b^2)}$$

$$\therefore \quad T = GJ\theta = G\theta \frac{\pi a^3 b^3}{a^2+b^2}$$

$$\Rightarrow J = \frac{\pi a^3 b^3}{a^2+b^2}$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -\frac{2Ty}{\pi a b^3} \qquad \tau_{zy} = -\frac{\partial \phi}{\partial x} = \frac{2Tx}{\pi a^3 b}$$

To determine warping function, we note that

$$\frac{\partial w}{\partial x} = -\frac{2Ty}{\pi a b^3 G} + \frac{T}{G}\frac{(a^2+b^2)}{\pi a^3 b^3} y$$

$$\frac{\partial w}{\partial y} = \frac{2Tx}{\pi a^3 b G} - \frac{T}{G}\frac{a^2+b^2}{\pi a^3 b^3} x$$

By integrating the two eqns we get,

$$w = \frac{T(b^2-a^2)}{\pi a^3 b^3 G} xy + f_1(y) \quad ; \quad w = \frac{T(b^2-a^2)}{\pi a^3 b^3 G} xy + f_2(x)$$

at same (x, y) w from the two eqns should be same $\Rightarrow f_1(y) = f_2(x) = 0$.

$$\therefore \quad w = \frac{T(b^2-a^2)}{\pi a^3 b^3 G} xy$$

Show that warping function $\psi = kxy$, in which k is an unknown constant maybe used to solve the torsion problem for the elliptical section.

Boundary condition on the surface is

$$\tau_{xz}\frac{dy}{ds} + \tau_{yz}\frac{dx}{ds} = 0$$

$$\tau_{xz} = G\gamma_{xz} \qquad \tau_{yz} = G\gamma_{yz}$$

$$\& \quad \gamma_{xz} = \frac{\partial w}{\partial x} - \theta y \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \theta x$$

$$w = \theta\psi(x,y) = \theta kxy.$$

$\therefore$ we have

$$y(k-1)\frac{dy}{ds} - x(k+1)\frac{dx}{ds} = 0$$

or $$\frac{d}{ds}\left[-\frac{x^2}{2}(k+1) + \frac{y^2}{2}(k-1)\right] = 0$$

$$\Rightarrow \quad -\frac{x^2}{2}(k+1) + \frac{y^2}{2}(k-1) = \text{constant} \rightarrow (1)$$

Rearranging we have,

$$x^2 + \left(\frac{1-k}{1+k}\right)y^2 = \text{constant}$$

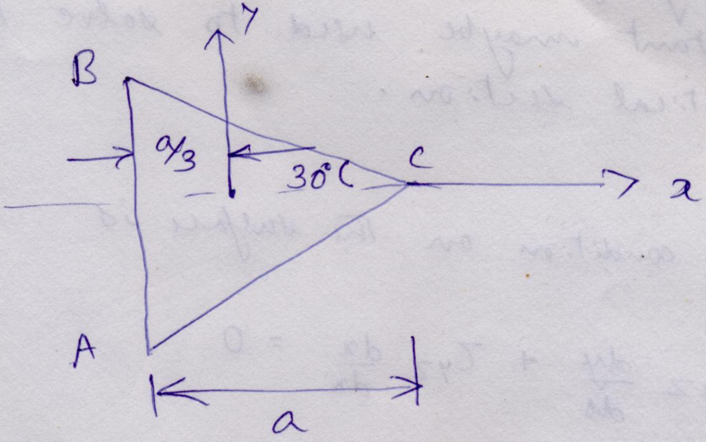
eqn of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

or $$x^2 + \frac{a^2}{b^2}y^2 = a^2 \rightarrow (2)$$

Comparing (1) & (2) we have

$$\frac{a^2}{b^2} = \left(\frac{1-k}{1+k}\right) \quad \therefore \psi = \frac{b^2-a^2}{a^2+b^2}xy$$

Bar having a c/s in the form of triangular section (equilateral)



$$\phi = -G\theta\left[\frac{1}{2}(x^2+y^2) - \frac{1}{2a}(x^3 - 3xy^2) - \frac{2}{27}a^2\right]$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} = -G\theta\left(1 - \frac{3x}{a}\right)$$

$$\frac{\partial^2\phi}{\partial y^2} = -G\theta\left(1 + \frac{3x}{a}\right)$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta$$ (condition satisfied)

On AB $x = -a/3$; $y = y$

BC $y = \frac{-x}{\sqrt{3}} + \frac{2a}{3\sqrt{3}}$

AC $y = \frac{x}{\sqrt{3}} - \frac{2a}{3\sqrt{3}}$

substituting the above expressions in ϕ

$$\Rightarrow \phi_{AB} = \phi_{BC} = \phi_{AC} = 0$$

Now,

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -G\theta\left(y + \frac{3xy}{a}\right)$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} = G\theta\left(x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right)$$

@ each corner of the triangle $\tau_{xz} = \tau_{yz} = 0$

Max value of τ_{zy} occurs at $y = 0$

$$(\tau_{zy})_{max} = -\frac{Ga\theta}{2}$$

$T = \iint 2\phi \, dx\, dy$

Eqn of the side AC of the triangle is $y = (x - 2a/3)/\sqrt{3}$

and BC $y = -(x - 2a/3)/\sqrt{3}$

$$\therefore T = \int_{-a/3}^{2a/3} \int_{(x-2a/3)/\sqrt{3}}^{-(x-2a/3)/\sqrt{3}} 2\phi \, dx\, dy$$

$$\Rightarrow T = \frac{Ga^4 \theta}{15\sqrt{3}}$$

For warping

$$\frac{\partial w}{\partial x} = -\theta \left(y + \frac{3xy}{\cancel{a}} - y \right) \qquad \left(\frac{\partial w}{\partial x} = \gamma_{xz} + \theta y \right)$$

$$w = -\frac{3x^2 y}{2a}\theta + f(y) \quad \rightarrow (a)$$

Similarly $\dfrac{\partial w}{\partial y} = \gamma_{yz} - \theta x$

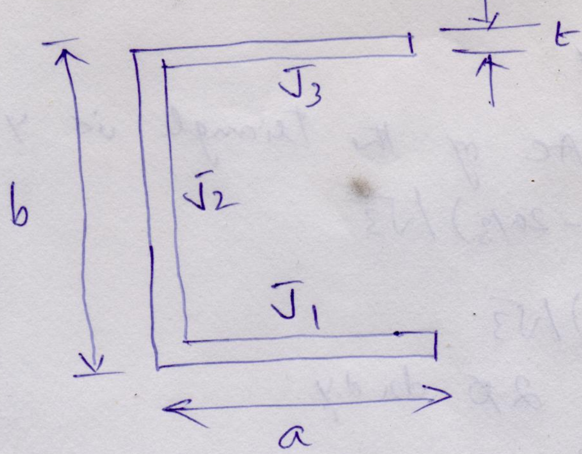
$$\Rightarrow w = \int (\gamma_{yz} - \theta x)\, dy$$

$$w = -\frac{3x^2 y \theta}{2a} + \frac{y^3}{2a}\theta + f_1(x) \quad \rightarrow (b)$$

Comparing (a) & (b)

$f(x) = 0$ & $f(y) = \dfrac{y^3}{2a}\theta$

$$\therefore w = \frac{1}{2a}\theta (y^3 - 3x^2 y)$$



Determine the max shear stress in this section.

$$J = (J_1 + J_3) + J_2$$
$$= \frac{2at^3}{3} + \frac{bt^3}{3}$$

$$J = \frac{2at^3}{3} + \frac{bt^3}{3}$$

$$\therefore \theta = \frac{T}{GJ} = \frac{3T}{G(2a+b)t^3}$$

$$\tau_{max} = \pm G\theta t = \pm \frac{3T}{(2a+b)t^2}$$