Lecture 11

Jacobi Constant

$$\dot x\left(\ddot x - 2\Omega\dot y - \Omega^2 x\right) = \left(-\frac{\mu_1}{\|r_1\|^3}(x+\pi_2 r_{12}) - \frac{\mu_2}{\|r_2\|^3}(x-\pi_1 r_{12})\right)\dot x$$

$$\dot y\left(\ddot y + 2\Omega\dot x - \Omega^2 y\right) = \left(-\frac{\mu_1}{\|r_1\|^3}y - \frac{\mu_2}{\|r_2\|^3}y\right)\dot y$$

$$\dot z\left(\ddot z = -\frac{\mu_1}{\|r_1\|^3}z - \frac{\mu_2}{\|r_2\|^3}z\right)\dot z$$

$$\mu = G(m_1+m_2)$$

$$\pi_1 = \frac{m_1}{m_1+m_2}, \quad \pi_2 = \frac{m_2}{m_1+m_2} \qquad \mu_1 = Gm_1, \quad \mu_2 = Gm_2$$

$$\ddot x\dot x + \ddot y\dot y + \ddot z\dot z - \Omega^2(x\dot x + y\dot y) = -\left(\frac{\mu_1}{\|r_1\|^3} + \frac{\mu_2}{\|r_2\|^3}\right)(\dot x x + \dot y y + \dot z z) + r_{12}\left(\frac{\pi_1\mu_2}{\|r_2\|^3} - \frac{\pi_2\mu_1}{\|r_1\|^3}\right)\dot x$$

$$\underbrace{\ddot x\dot x + \ddot y\dot y + \ddot z\dot z}\; - \Omega^2\underbrace{(x\dot x + y\dot y)} = \underbrace{-\frac{\mu_1}{\|r_1\|^3}(\dot x x + \dot y y + \dot z z + \pi_2 r_{12}\dot x)}$$

$$\underbrace{-\frac{\mu_2}{\|r_2\|^3}(\dot x x + \dot y y + \dot z z - \pi_1 r_{12}\dot x)}$$

$$\ddot x\dot x + \ddot y\dot y + \ddot z\dot z = \frac{1}{2}\frac{d}{dt}\underbrace{(\dot x^2 + \dot y^2 + \dot z^2)} = \frac{1}{2}\|v\|^2$$

$$\dot x x + \dot y y = \frac{1}{2}\frac{d}{dt}(x^2+y^2)$$

$$\|r_1\|^2 = (x+\pi_2 r_{12})^2 + y^2 + z^2$$

$$2\|r_1\|\,\dot{\|r_1\|} = 2(x+\pi_2 r_{12})\dot x + 2y\dot y + 2z\dot z$$

$$\text{or, } \dot{\|r_1\|} = \underbrace{\frac{1}{\|r_1\|}\left(\pi_2 r_{12}\dot x + x\dot x + y\dot y + z\dot z\right)}$$

It follows, $\displaystyle \frac{d}{dt}\frac{1}{\|r_1\|} = -\frac{1}{\|r_1\|^2}\dot{\|r_1\|} = -\frac{1}{\|r_1\|^3}(\dot x x + \dot y y + \dot z z + \pi_2 r_{12}\dot x)$

$$\dot{\|r_2\|} = -1 \quad (\dot x x + \dot y y + \dot z z - \pi_1 r_{12}\dot x)$$

Similarly, $\frac{d}{dt}\frac{1}{\|r_2\|} = -\frac{1}{\|r_2\|^2}\dot{\overline{\|r_2\|}} = -\frac{1}{\|r_2\|^3}\left(\dot{x}x+\dot{y}y+\dot{z}z-\pi_1 r_{12}\dot{x}\right)$

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \frac{1}{2}\Omega^2\frac{d}{dt}(x^2+y^2) = \mu_1\frac{d}{dt}\frac{1}{\|r_1\|} + \mu_2\frac{d}{dt}\frac{1}{\|r_2\|}$$

$$\frac{d}{dt}\left[\frac{1}{2}\|v\|^2 - \frac{1}{2}\Omega^2(x^2+y^2) - \frac{\mu_1}{\|r_1\|} - \frac{\mu_2}{\|r_2\|}\right] = 0$$

$$\Rightarrow \frac{1}{2}\|v\|^2 - \frac{1}{2}\Omega^2(x^2+y^2) - \frac{\mu_1}{\|r_1\|} - \frac{\mu_2}{\|r_2\|} = \underbrace{C}_{\text{Jacobi Constant}} \qquad -C/2$$

- Total energy of the secondary mass relative to the rotating frame.

Jacobi Constant
Carl Gustav Jacobi (1804-1851)

1836

$$\underbrace{\|v\|^2}_{\geq 0} = \Omega^2(x^2+y^2) + \frac{2\mu_1}{\|r_1\|} + \frac{2\mu_2}{\|r_2\|} + 2C$$

$\|r\|^2 = x^2+y^2$ $\qquad\qquad \|r_1\| =$

$$\|r_1\| = \sqrt{(x+\pi_2 r_{12})^2 + y^2}$$

$$\|r_2\| = \sqrt{(x-\pi_1 r_{12})^2 + y^2}$$

Since, $\|v\|^2$ cannot be negative, it must be true that

$$\underbrace{\Omega^2(x^2+y^2)}_{} + \underbrace{\frac{2\mu_1}{\|r_1\|}}_{} + \underbrace{\frac{2\mu_2}{\|r_2\|}}_{} + 2C \geq 0$$

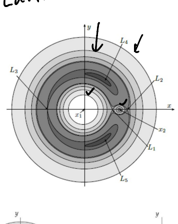
$$\|v\|^2 = 0$$

$$\Omega^2\underbrace{(x^2+y^2)}_{} + \frac{2\mu_1}{\underbrace{\|r_1\|}_{}} + \frac{2\mu_2}{\underbrace{\|r_2\|}_{}} + 2C = 0$$

Zero velocity curves correspond to negative values of the Jacobi constant.

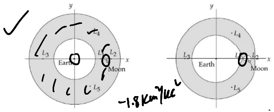
$$\Omega = \sqrt{\frac{G(m_1+m_2)}{r_{12}^3}} = 2.66538 \times 10^{-6} \text{ rad/sec}$$

Earth - Moon System



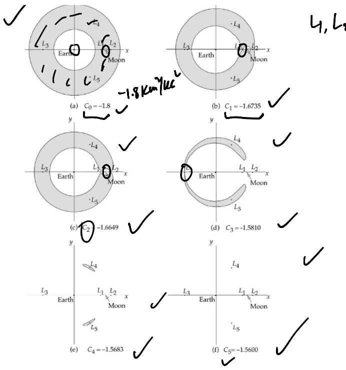
$$\mu_1 = Gm_1 = 398{,}620 \text{ Km}^3/\text{sec}^2$$

$$\mu_2 = Gm_2 = 4903.02 \text{ Km}^3/\text{sec}^2$$



-1.8 Km²/sec²

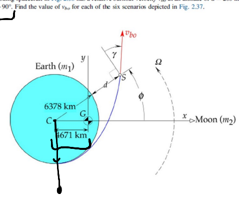
L_1, L_2, L_3



L_1, L_2, L_3

Example

The earth-orbiting spacecraft in Fig. 2.38 has a relative burnout velocity v_{bo} at an altitude of $d = 200$ km on a radial for which $\phi = -90°$. Find the value of v_{bo} for each of the six scenarios depicted in Fig. 2.37.



Details

$$\pi_1 = \frac{m_1}{m_1 + m_2} = 0.9878$$

$$\pi_2 = 1 - \pi_1 = 0.01215$$

$$x_1 = -\pi_1 r_{12} = -4670.6\ km$$

$$\phi = -90°, \quad x = -4670.6\ km, \quad y = -6578\ km$$

$$C = -1.8\ km^2/sec^2, \quad V_{bo} = 10.84518\ km/sec$$

$$C = -1.6735\ km^2/sec^2, \quad V_{bo} = 10.85683\ km/sec$$

$\vdots$

$$C = -1.56\ km^2/sec^2, \quad V_{bo} = 10.86728\ km/sec$$

$$V_{esc} = \sqrt{\frac{2\mu}{r}} = 11.01\ km/sec$$