– Given the position and velocity vectors of a spacecraft in the geocentric equitorial frame, how do we obtain the orbital elements?

1. Calculate the distance, $r=\sqrt{\mathbf{r}\cdot\mathbf{r}}=\sqrt{X^2+Y^2+Z^2}$.
2. Calculate the speed, $v=\sqrt{\mathbf{v}\cdot\mathbf{v}}=\sqrt{v_X^2+v_Y^2+v_Z^2}$.
3. Calculate the radial velocity, $v_r=\mathbf{r}\cdot\mathbf{v}/r=(Xv_X+Yv_Y+Zv_Z)/r$.
   Note that if $v_r>0$, the spacecraft is flying away from perigee. If $v_r<0$, it is flying toward perigee.
4. Calculate the specific angular momentum,

$$\mathbf{h}=\mathbf{r}\times\mathbf{v}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ X & Y & Z\\ v_X & v_Y & v_Z\end{vmatrix}$$

5. Calculate the magnitude of the specific angular momentum, $h=\sqrt{\mathbf{h}\cdot\mathbf{h}}$.
   This is the first orbital element.
6. Calculate the inclination,

$$i=\cos^{-1}(h_Z/h) \tag{4.7}$$

   This is the second orbital element. Recall that i must lie between 0° and 180°, which is precisely the range (principal values) of the arccosine function. Hence, there is no quadrant ambiguity to contend with here. If $90°<i\le 180°$, the angular momentum $\mathbf{h}$ points in a southerly direction. In that case, the orbit is retrograde, which means that the motion of the satellite around the earth is opposite to earth's rotation.
7. Calculate

$$\mathbf{N}=\hat{\mathbf{K}}\times\mathbf{h}=\begin{vmatrix}\hat{\mathbf{I}} & \hat{\mathbf{J}} & \hat{\mathbf{K}}\\ 0 & 0 & 1\\ h_X & h_Y & h_Z\end{vmatrix} \tag{4.8}$$

   This vector defines the node line.
8. Calculate the magnitude of $\mathbf{N}$, $N=\sqrt{\mathbf{N}\cdot\mathbf{N}}$.
9. Calculate the right ascension of the ascending node, $\Omega=\cos^{-1}(N_X/N)$. This is the third orbital element. If $N_X>0$, then Ω lies in either the first or fourth quadrant. If $N_X<0$, then Ω lies in either the second or third quadrant. To place Ω in the proper quadrant, observe that the ascending node lies on the positive side of the vertical XZ plane ($0\le\Omega<180°$) if $N_Y>0$. On the other hand, the ascending node lies on the negative side of the XZ plane ($180°\le\Omega<360°$) if $N_Y<0$. Therefore, $N_Y>0$ implies that $0\le\Omega<180°$, whereas $N_Y<0$ implies that $180°\le\Omega<360°$. In summary,

$$\Omega=\begin{cases}\cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y\ge 0)\\ 360°-\cos^{-1}\left(\dfrac{N_X}{N}\right) & (N_Y<0)\end{cases} \tag{4.9}$$

10. Calculate the eccentricity vector. Starting with Eq. (2.40),

$$\mathbf{e}=\frac{1}{\mu}\left[\mathbf{v}\times\mathbf{h}-\mu\frac{\mathbf{r}}{r}\right]=\frac{1}{\mu}\left[\mathbf{v}\times(\mathbf{r}\times\mathbf{v})-\mu\frac{\mathbf{r}}{r}\right]=\frac{1}{\mu}\left[\overbrace{\mathbf{r}v^2-\mathbf{v}(\mathbf{r}\cdot\mathbf{v})}^{bac-cab\ rule}-\mu\frac{\mathbf{r}}{r}\right]$$

   so that

$$\mathbf{e}=\frac{1}{\mu}\left[\left(v^2-\frac{\mu}{r}\right)\mathbf{r}-rv_r\mathbf{v}\right] \tag{4.10}$$

11. Calculate the eccentricity, $e=\sqrt{\mathbf{e}\cdot\mathbf{e}}$, which is the fourth orbital element. Substituting Eq. (4.10) leads to a form depending only on the scalars obtained thus far,

$$e=\sqrt{1+\frac{h^2}{\mu^2}\left(v^2-\frac{2\mu}{r}\right)} \tag{4.11}$$

12. Calculate the argument of perigee,

$$\omega=\cos^{-1}\left(\frac{\mathbf{N}}{N}\cdot\frac{\mathbf{e}}{e}\right)$$

   This is the fifth orbital element. If $\mathbf{N}\cdot\mathbf{e}>0$, then ω lies in either the first or fourth quadrant. If $\mathbf{N}\cdot\mathbf{e}<0$, then ω lies in either the second or third quadrant. To place ω in the proper quadrant, observe that perigee lies above the equatorial plane ($0°\le\omega<180°$) if $\mathbf{e}$ points up (in the positive Z direction) and that perigee lies below the plane ($180°\le\omega<360°$) if $\mathbf{e}$ points down. Therefore, $e_Z\ge 0$ implies that $0°<\omega<180°$, whereas $e_Z<0$ implies that $180°<\omega<360°$. To summarize,

$$\omega=\begin{cases}\cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z\ge 0)\\ 360°-\cos^{-1}\left(\dfrac{\mathbf{N}\cdot\mathbf{e}}{Ne}\right) & (e_Z<0)\end{cases} \tag{4.12}$$

13. Calculate the true anomaly,

$$\theta = \cos^{-1}\left(\frac{\mathbf{e}}{e}\cdot\frac{\mathbf{r}}{r}\right)$$

This is the sixth and final orbital element. If $\mathbf{e}\cdot\mathbf{r} > 0$, then θ lies in the first or fourth quadrant. If $\mathbf{e}\cdot\mathbf{r} < 0$, then θ lies in the second or third quadrant. To place θ in the proper quadrant, note that if the satellite is flying away from perigee ($\mathbf{r}\cdot\mathbf{v} \geq 0$), then $0 \leq \theta < 180°$, whereas if the satellite is flying toward perigee ($\mathbf{r}\cdot\mathbf{v} < 0$), then $180° \leq \theta < 360°$. Therefore, using the results of Step 3 above

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{\mathbf{e}}{e}\cdot\dfrac{\mathbf{r}}{r}\right) & (v_r \geq 0) \\ 360° - \cos^{-1}\left(\dfrac{\mathbf{e}}{e}\cdot\dfrac{\mathbf{r}}{r}\right) & (v_r < 0) \end{cases} \tag{4.13a}$$

Substituting Eq. (4.10) yields an alternative form of this expression,

$$\theta = \begin{cases} \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r \geq 0) \\ 360° - \cos^{-1}\left[\dfrac{1}{e}\left(\dfrac{h^2}{\mu r} - 1\right)\right] & (v_r < 0) \end{cases} \tag{4.13b}$$

Example

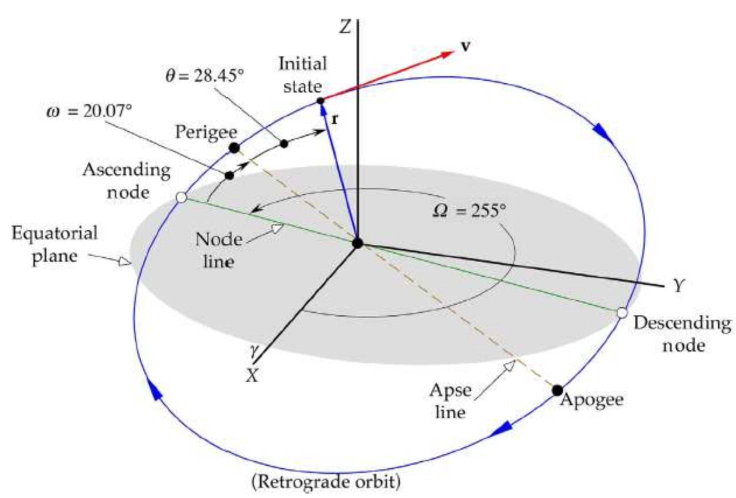
Given the state vector,

$$\mathbf{r} = -6045\hat{\mathbf{I}} - 3490\hat{\mathbf{J}} + 2500\hat{\mathbf{K}}\ (\text{km})$$
$$\mathbf{v} = -3.457\hat{\mathbf{I}} + 6.618\hat{\mathbf{J}} + 2.533\hat{\mathbf{K}}\ (\text{km/s})$$

find the orbital elements h, i, Ω, e, ω, and θ using Algorithm 4.2.

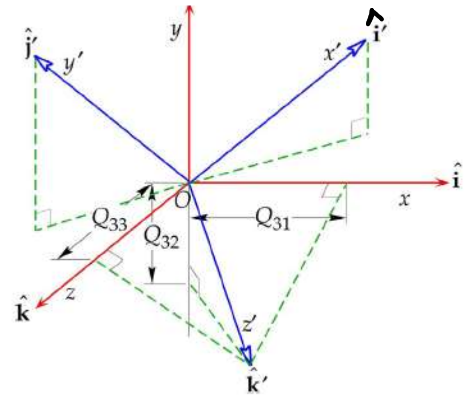
Details

Follow the above algorithm.



Coordinate Transformation

$\hat{\mathbf{j}}$



- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
  $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$

- $\hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1$
  $\hat{i}'\cdot\hat{j}' = \hat{i}'\cdot\hat{k}' = \hat{j}'\cdot\hat{k}' = 0$

- $\hat{i}' = Q_{11}\hat{i} + Q_{12}\hat{j} + Q_{13}\hat{k}$
  $\hat{j}' = Q_{21}\hat{i} + Q_{22}\hat{j} + Q_{23}\hat{k}$
  $\hat{k}' = Q_{31}\hat{i} + Q_{32}\hat{j} + Q_{33}\hat{k}$

- $\hat{i} = Q_{11}'\hat{i}' + Q_{12}'\hat{j}' + Q_{13}'\hat{k}'$
  $\hat{j} = Q_{21}'\hat{i}' + Q_{22}'\hat{j}' + Q_{23}'\hat{k}'$
  $\hat{k} = Q_{31}'\hat{i}' + Q_{32}'\hat{j}' + Q_{33}'\hat{k}'$

- $\hat{i}'\cdot\hat{i} = \hat{i}\cdot\hat{i}' \Rightarrow Q_{11} = Q_{11}'$
  $\hat{i}'\cdot\hat{j} = \hat{j}\cdot\hat{i}' \Rightarrow Q_{12} = Q_{21}'$
  $\vdots$

- $\hat{i} = Q_{11}\hat{i}' + Q_{21}\hat{j}' + Q_{31}\hat{k}'$
  $\hat{j} = Q_{12}\hat{i}' + Q_{22}\hat{j}' + Q_{32}\hat{k}'$
  $\hat{k} = Q_{13}\hat{i}' + Q_{23}\hat{j}' + Q_{33}\hat{k}'$

$\hat{i}\cdot\hat{i} = 1 \Rightarrow Q^2 + Q^2 + Q^2 = 1$

- $i \cdot i = 1 \Rightarrow Q_{11}^2 + Q_{21}^2 + Q_{31}^2 = 1$
  $\hat{j} \cdot \hat{j} = 1 \Rightarrow Q_{12}^2 + Q_{22}^2 + Q_{32}^2 = 1$
  $\hat{k} \cdot \hat{k} = 1 \Rightarrow Q_{13}^2 + Q_{23}^2 + Q_{33}^2 = 1$

- $\hat{i} \cdot \hat{j} = 0 \Rightarrow Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} = 0$
  $\hat{i} \cdot \hat{k} = 0 \Rightarrow Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} = 0$
  $\hat{j} \cdot \hat{k} = 0 \Rightarrow Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} = 0$

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{j}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{j}}' \cdot \hat{\mathbf{k}} \\ \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{j}} & \hat{\mathbf{k}}' \cdot \hat{\mathbf{k}} \end{bmatrix}$$

(Direction Cosine Matrix)

$$[\mathbf{Q}]^T = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{i}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{j}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{j}} \cdot \hat{\mathbf{k}}' \\ \hat{\mathbf{k}} \cdot \hat{\mathbf{i}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{j}}' & \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' \end{bmatrix}$$

$$[\mathbf{Q}]^T[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{11}{}^2 + Q_{21}{}^2 + Q_{31}{}^2 & Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} & Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} \\ Q_{12}Q_{11} + Q_{22}Q_{21} + Q_{32}Q_{31} & Q_{12}{}^2 + Q_{22}{}^2 + Q_{31}{}^2 & Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} \\ Q_{13}Q_{11} + Q_{23}Q_{21} + Q_{33}Q_{31} & Q_{13}Q_{12} + Q_{23}Q_{22} + Q_{33}Q_{32} & Q_{13}{}^2 + Q_{23}{}^2 + Q_{33}{}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Q is an orthogonal matrix.