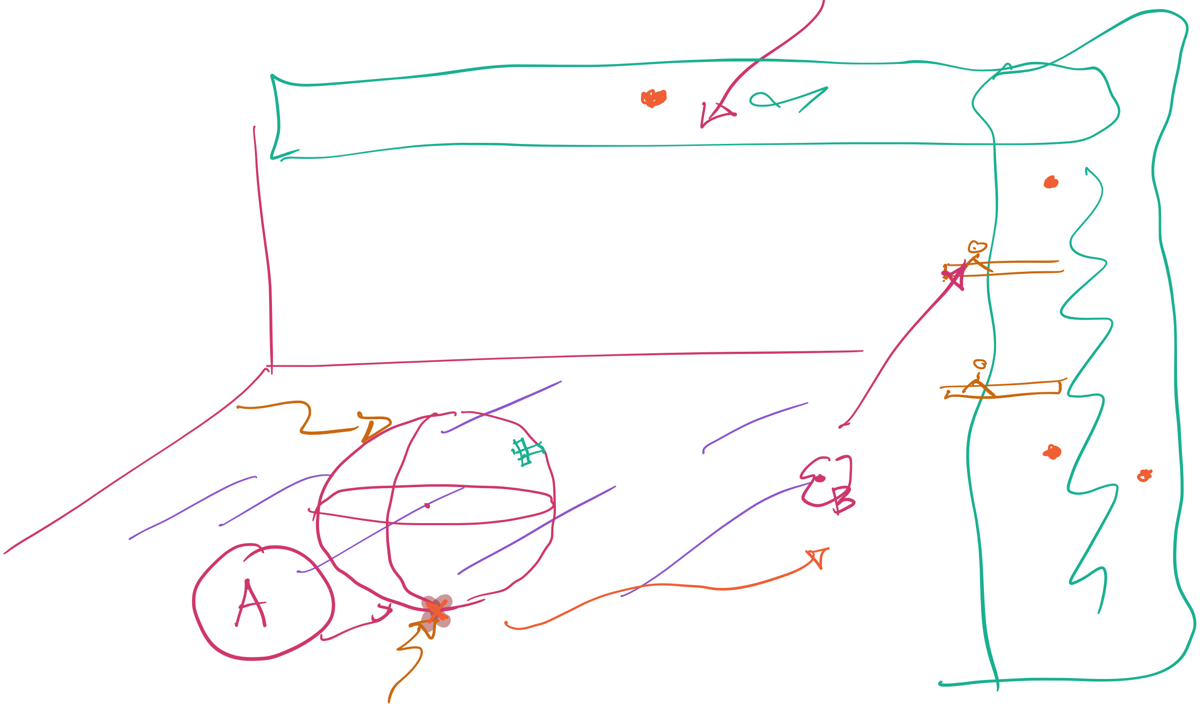
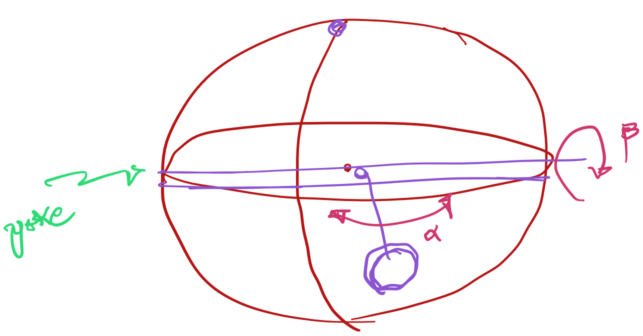
* Quadrotors, multiple air vehicles, ← satellite clusters

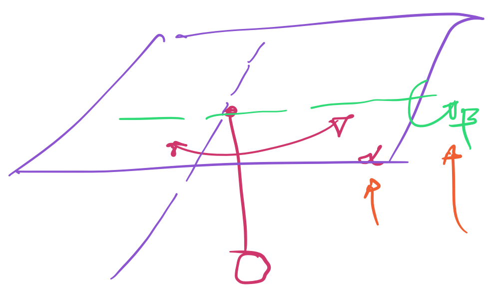
Rolling robots, Spheres

Mobile robotics

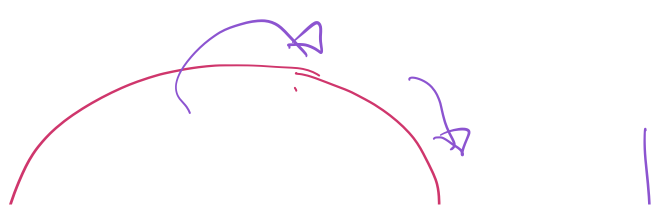


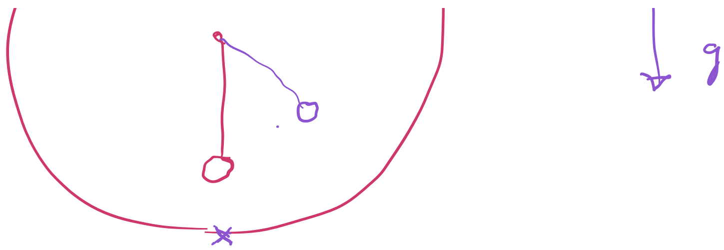
Robot location?





τ_x, τ_y



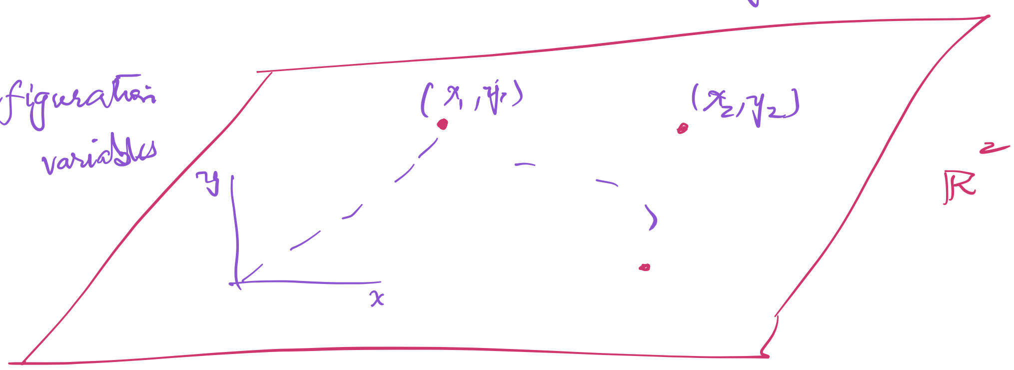


1 mechanism for locomotion

2 Dynamic model for the sphere

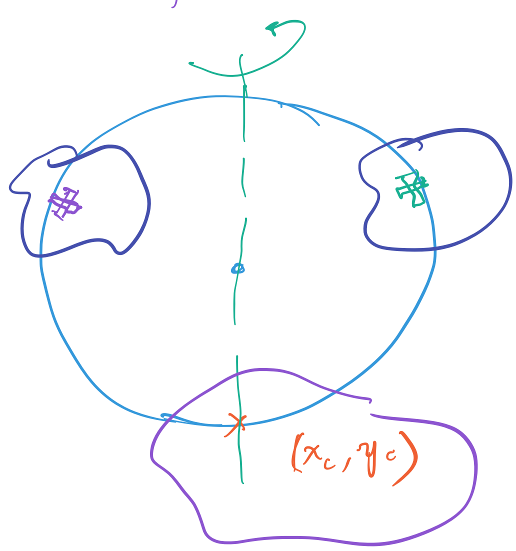
→ Problem defn.

Configuration variables



v_1, v_2, ...

Rigid body



"orientation"

$\mathbb{R}^2 \times ?$

$\mathbb{R}^2 \times SO(3)$

3 Lagrangian model

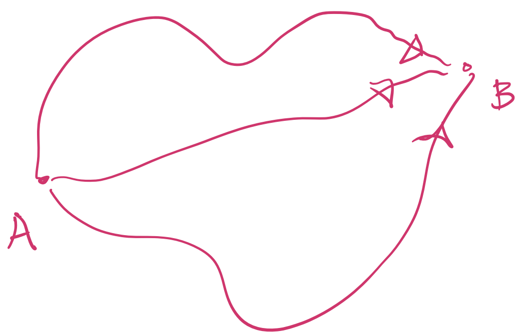
T , V

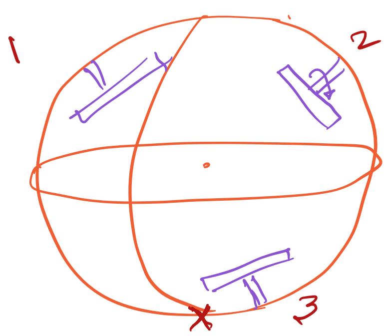
$\mathcal{L} = T - V$

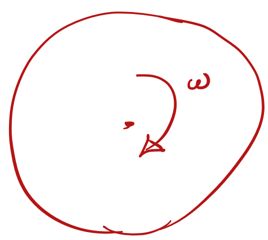
Dynamics

4 Problem defn. Measurement strategy

5 Analysis, path planning and Control







$\omega_1, \omega_2, \omega_3$

"Conservation of angular momentum"

Lecture 1 : Rotations

Motivation → satellite

spherical robot

Rotation matrix ⇝ Difference from a vector space

Distinguishing features

A parametrized curve ~

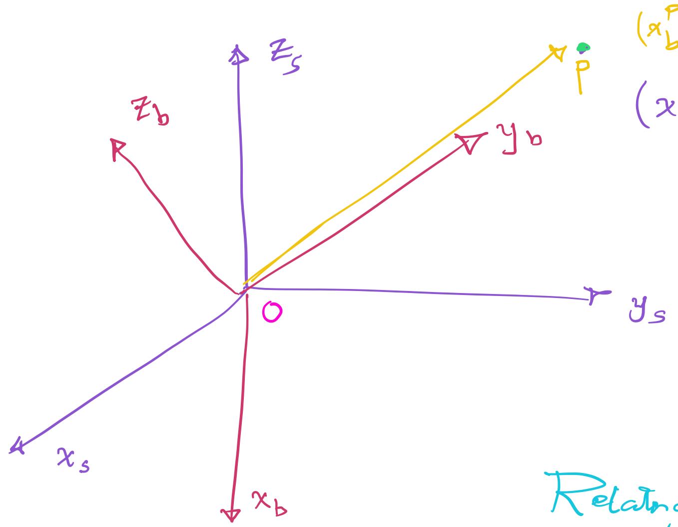
Representations of rotations → Euler angles

→ Quaternions

Rotations

(x_b^p, y_b^p, z_b^p) — B

(x_s^p, y_s^p, z_s^p) — S



Relating the two coordinates

$(e_{1_b}, e_{2_b}, e_{3_b})$

$(e_{1_s}, e_{2_s}, e_{3_s})$

$\underline{\langle \underbrace{x_b^p e_{1_b}} , e_{1_s} \rangle e_{1_s}}$ $\quad$ $\underline{\langle x_b^p e_{1_b}, e_{2_s} \rangle e_{2_s}}$ $\quad$ $\underline{\langle x_b^p e_{1_b}, e_{3_s} \rangle e_{3_s}}$

$\underbrace{\langle e_{1_b}, e_{1_s} \rangle} e_{1_s} x_b^p$ $\quad$ $\underbrace{\langle e_{1_b}, e_{2_s} \rangle} e_{2_s} x_b^p$ $\quad$ $\underbrace{\langle e_{1_b}, e_{3_s} \rangle} e_{3_s} x_b^p$

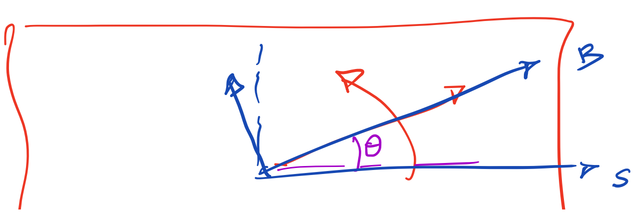
$$\begin{bmatrix} \langle e_{1_b}, e_{1_s}\rangle & \langle e_{2_b}, e_{1_s}\rangle & \langle e_{3_b}, e_{1_s}\rangle \\ \langle e_{1_b}, e_{2_s}\rangle & \langle e_{2_b}, e_{2_s}\rangle & \langle e_{3_b}, e_{2_s}\rangle \\ \langle e_{1_b}, e_{3_s}\rangle & \langle e_{2_b}, e_{3_s}\rangle & \langle e_{3_b}, e_{3_s}\rangle \end{bmatrix}$$

$$e_{1_s} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} x_b^p \\ y_b^p \\ z_b^p \end{pmatrix}$$

$$e_{1_s} x_b^p = \begin{pmatrix} x_b^p \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_s^p \\ y_s^p \\ z_s^p \end{pmatrix} \longleftarrow$$

$$\underbrace{\begin{pmatrix} \ \\ \ \\ \ \end{pmatrix}}_{S} = \underbrace{\begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}}_{\text{Rotation matrix}} \underbrace{\begin{pmatrix} \ \\ \ \\ \ \end{pmatrix}}_{B}$$



$$\underbrace{\begin{pmatrix} \\ \\ \end{pmatrix}}_{S} = \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} \underbrace{\begin{pmatrix} \\ \\ \end{pmatrix}}_{B}$$

The set of rotation matrices do not form a vector space. $\rightarrow$ $RR^T = I$, $\det R = 1$ $\quad R \in \mathbb{R}^{3\times 3}$

Group $\rightarrow$ $R_1 \times R_2$ $\rightarrow$ Rotation

But

$R_1 \rightarrow R_1^T$

Identity

$R_1 R_1^T = I \rightarrow$ identity

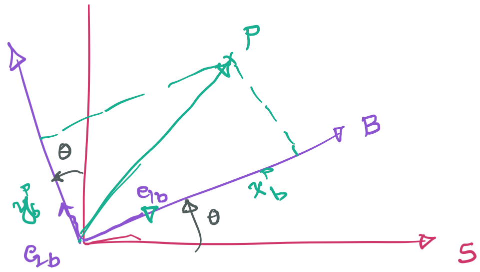
$$R_1 \times (R_2 \times R_3) = (R_1 \times R_2) \times R_3$$

Associativity

SO (3) Special orthogonal group in 3 dim.

O(3) $\quad R^T R = I$

SO(3) $\quad \det R = 1$



$$e_{1b} = c\theta \; e_{1s} + s\theta \, e_{2s}$$

$$e_{2b} = (s\theta)(-e_{1s}) + (c\theta)\, e_{2s}$$

$$\langle e_{1b}, e_{1s} \rangle = c\theta$$

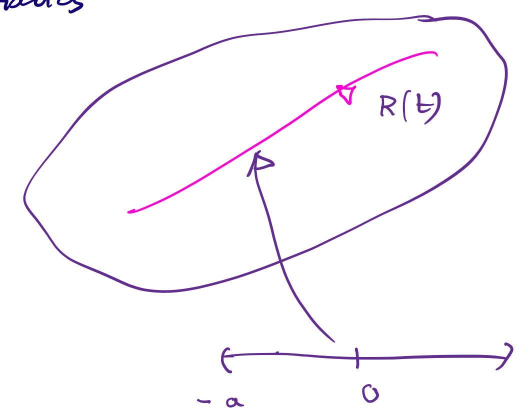
$$\langle e_{1b}, e_{2s} \rangle = s\theta$$

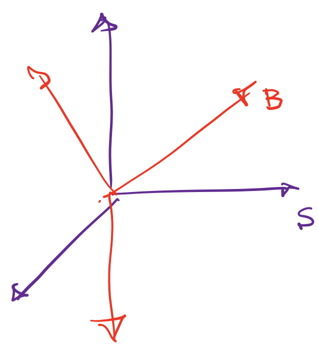
$$\begin{pmatrix} x_s^P \\ y_s^P \end{pmatrix} = \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} \begin{pmatrix} x_b^P \\ y_b^P \end{pmatrix}$$

— Lecture 5 content —

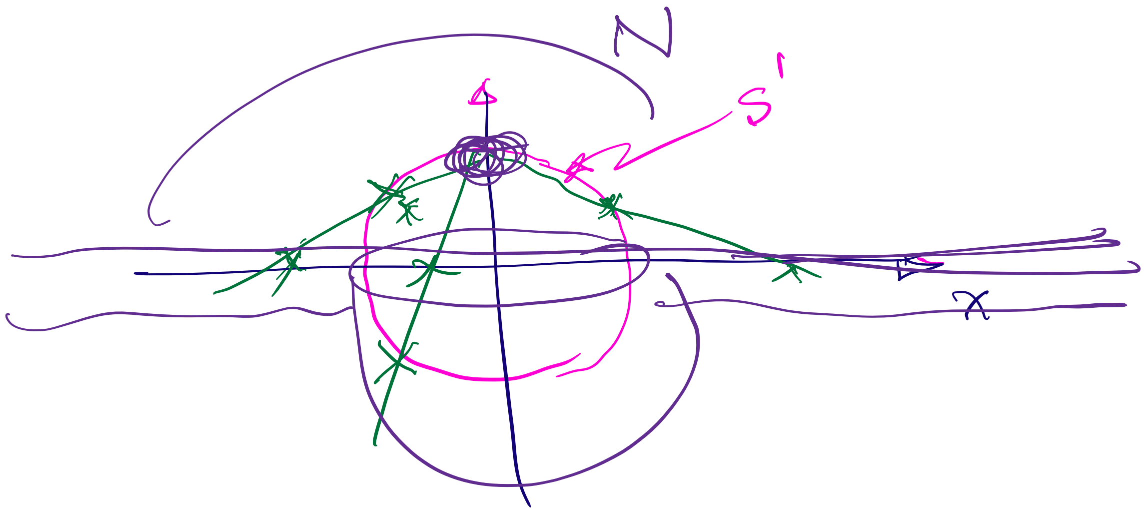
Kinematics

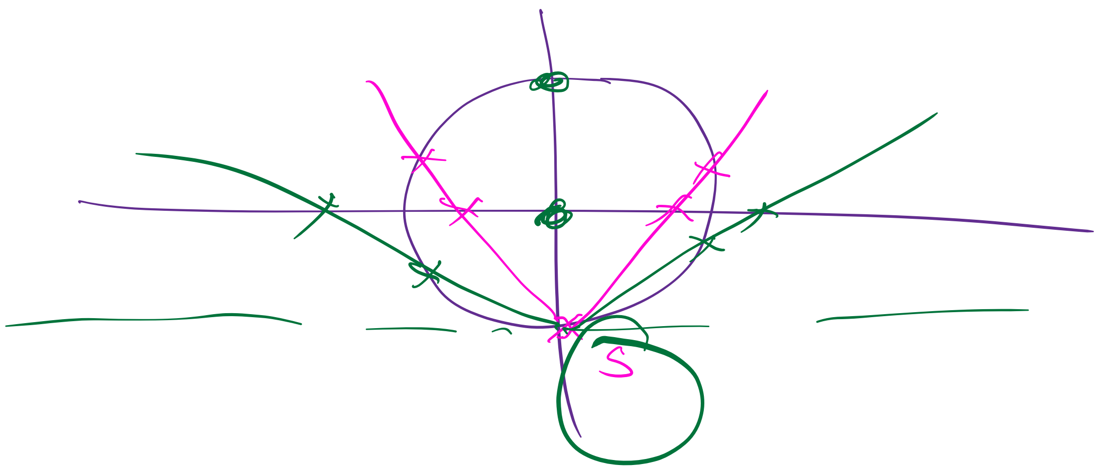
$SO(3)$





$R(t)$ is a smooth curve


N
S'
x


S

Lecture 2 : Continuing with rotations

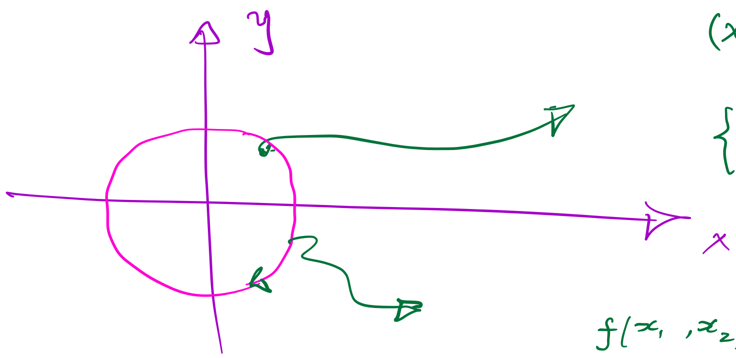
$R \in \mathbb{R}^{3\times 3}$

$$\boxed{\begin{aligned} R^T R &= I \\ \det R &= 1 \end{aligned}}$$

$\underbrace{\text{Euler angles}}$ $\qquad$ $\underbrace{\text{Quaternions}}$

Lecture 3 : Quaternions

Lecture 4 : Rotational motion on a surface



(x, y)

$\{ (x, y) \mid x^2 + y^2 = 1 \}$

$f(x_1, x_2) = x_1^2 + x_2^2 - 1$

$$S^1 = \{ (x_1, x_2) : f(x_1, x_2) = 0 \}$$

$R \in \mathbb{R}^{3\times 3}$ $\qquad$ $R^T R = I$

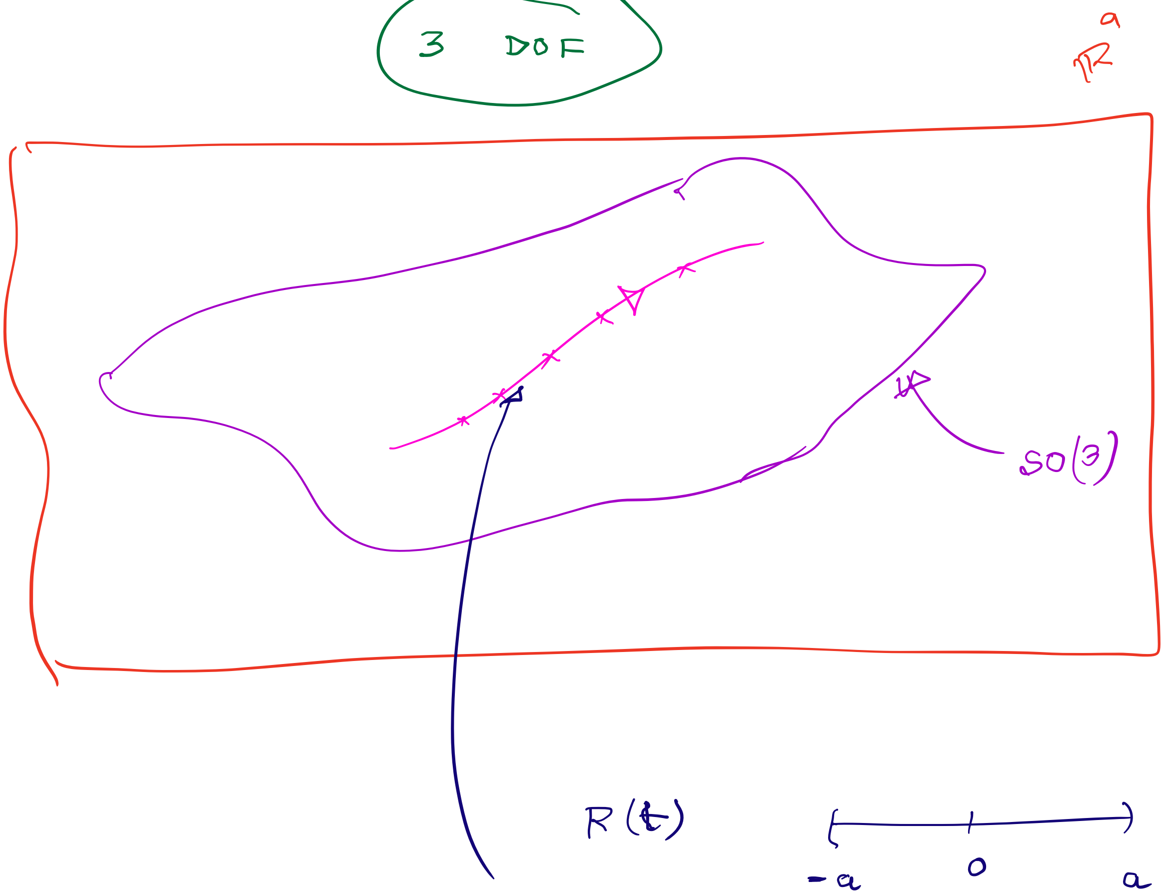
$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \qquad \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$r_{11}^2 + r_{12}^2 + r_{13}^2 - 1 = 0$$

$$\det R = 1$$

9 DOF , 6 eqns

3 DOF



$R(t)$

$-a \quad 0 \quad a$

$t \in (-a, a)$

$0 \quad b$

$$R(0) = I \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\dot{R}(t)$

$$\boxed{\begin{aligned} R^T(t)\, R(t) &= I \\ \det(R(t)) &= 1 \end{aligned}}$$

$$R^T R = I$$

Lecture 5

$$\dot{R}^T R + R^T \dot{R} = 0 \implies (R^T \dot{R}) = -(R^T \dot{R})^T$$

$R^T \dot{R}$ is skew - symmetric.

$$A = -A^T$$

$$(\dot{R} R^T) = -(\dot{R} R^T)^T$$

$RR^T = I$

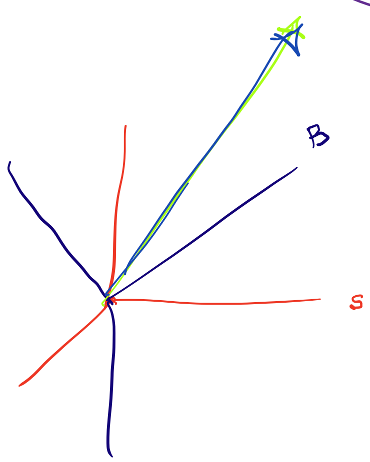
$$(\dot{R} R^T) = -(\dot{R} R^T)^T$$

$$\dot{R} R^T \longleftrightarrow R^T \dot{R}$$

$$\omega \in \mathbb{R}^3 \longleftrightarrow \hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \longleftrightarrow$$

$$\dot{R} R^T = \hat{\omega}_s \qquad R^T \dot{R} = \hat{\omega}_b$$



$$R^T \hat{\omega}_s R = \hat{\omega}_b$$

$$A^{-1} B A = C$$

$$R \hat{\omega}_b R^T = \hat{\omega}_s$$

Lecture

Lecture 6

$R(t)$ $\quad t=0 \quad R(0) = I$

$$\dot{R} = R\,\hat{\omega}_b \qquad \dot{R} = \hat{\omega}\,R$$

$$\begin{pmatrix} a^s \\ b^s \\ c^s \end{pmatrix} = R \begin{pmatrix} a^b \\ b^b \\ c^b \end{pmatrix}$$

$(x_1, \dots, x_n)$ $\quad x_1(t) \in \mathbb{R}, \dots$

$x_n(t) \in \mathbb{R}$

ODEs

$$\dot{x}_1 = a_1(x_1, \dots, x_n)$$
$$\vdots$$
$$\dot{x}_n = a_n(x_1, \dots, x_n)$$

$x_1(0), \dots$

$x_n(0)$

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \dot{\underline{x}} = \underbrace{\begin{pmatrix} a_1(\underline{x}) \\ \vdots \\ a_n(\underline{x}) \end{pmatrix}}$$

$$\dot{\underline{x}} = \underset{n\times n}{A}\,\underline{x}$$

Linear system

$$\underline{x}(t) = e^{At}\,\underline{x}(0)$$

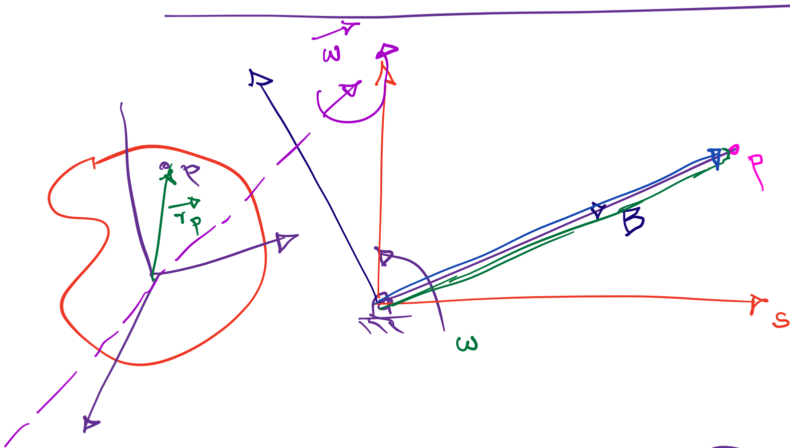
$$e^{At} \triangleq I + At + \frac{A^2 t^2}{2!} + \cdots$$

$A \in \mathbb{R}^{n \times n}$

$$\dot{R} = R\,\hat{\omega}_b \qquad R = \begin{pmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{pmatrix}$$

$$\dot{r}_1 = (R\,\hat{\omega}_b)_{1^{st}} \text{ column}$$

$$= r_1$$



$\vec{r}_p$

$$\frac{d\vec{r}_p}{dt} = \vec{\omega} \times \vec{r}_p$$

$$\vec{v}_p = \vec{\omega} \times \vec{r}_p$$

$$\omega \times x \longleftrightarrow \hat{\omega}\,x$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

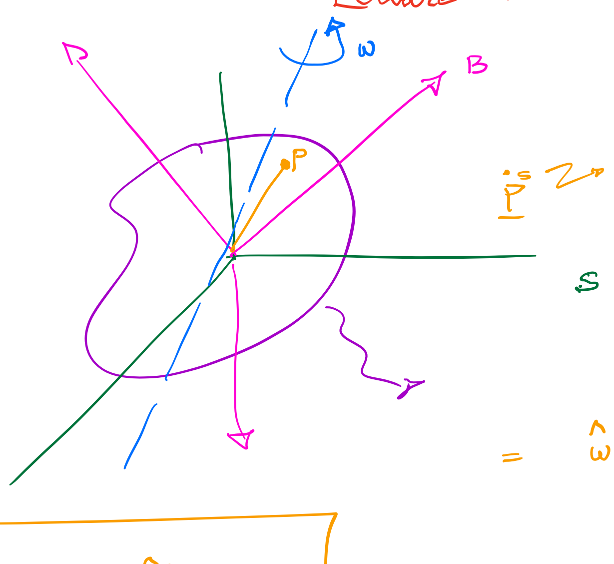
$$\begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{pmatrix} = \hat{\omega} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\underline{\dot{x}} = A \underline{x}$$

$$\underline{x}(t) = e^{\hat{\omega} t} \underline{x}(0)$$

$$\left(e^{\hat{\omega} t}\right)^T \left(e^{\hat{\omega} t}\right) = I, \qquad \det\left(e^{\hat{\omega} t}\right) = 1$$

Lecture 7 material



$$\underline{\dot{p}}^s \rightarrow \begin{pmatrix} \dot{p}_x^s \\ \dot{p}_y^s \\ \dot{p}_z^s \end{pmatrix} = \underline{\omega}^s \times \underline{p}^s$$

$$= \hat{\omega}^s \underline{p}^s$$

$$\boxed{\underline{\dot{p}}_s = \hat{\omega}^s \underline{p}^s} \rightarrow \dot{x} = Ax$$

$\mathbb{R}^3$

$$p^s(t) = \underbrace{\exp(\hat{\omega}^s t)}_{R} \; \underbrace{p^s(0)}$$

$$\boxed{\exp(\hat{\omega}^s t) \in SO(3)}$$

$$\begin{cases} 1. \quad (\exp(\hat{\omega}_s t))^T (\exp(\hat{\omega}_s t)) = I \\ 2. \quad \det(\exp(\hat{\omega}^s t)) = 1 \end{cases}$$

$R \quad , \quad \exp(\cdot)$

$$\underline{R \hat{\omega}_b R^T} = \underline{\hat{\omega}_s}$$

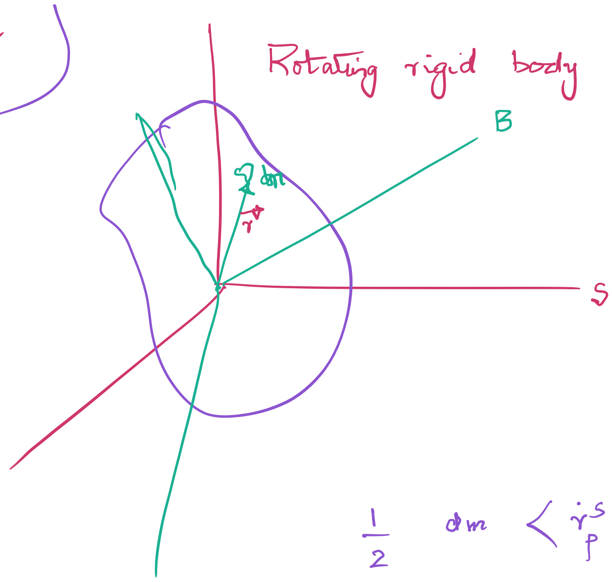
$$R\omega_b = \omega_s$$

$\mathbb{R}^3$

$SO(3)$

11/1/22

Rotating rigid body



elemental mass

dm

$$K.E = \frac{1}{2} dm \langle \dot{\vec{r}}, \dot{\vec{r}} \rangle$$

$\vec{r}$ → r_p^s

→ r_p^b

$$\frac{1}{2} dm \langle \dot{r}_p^s, \dot{r}_p^s \rangle$$

$$r^s = R r^b \qquad \dot{r} = \dot{R} r^b$$

$\boxed{r^b \text{ for an elemental mass is constant.}}$

$$\frac{1}{2} dm \langle \dot{R} r^b, \dot{R} r^b \rangle = \frac{1}{2} dm (r^b)^T \dot{R}^T \dot{R} r^b$$

$$= \frac{1}{2} dm (r^b)^T \dot{R}^T \underbrace{R R^T}_{I} \dot{R} r^b$$

$$= \frac{1}{2} dm \langle \underbrace{R^T \dot{R}}_{\hat{\omega}^b} r^b, \underbrace{R^T \dot{R}}_{\hat{\omega}^b} r^b \rangle$$

$$= \frac{1}{2} dm \langle \hat{\omega}^b r^b, \hat{\omega}^b r^b \rangle = \frac{1}{2} dm (r^b)^T (-\hat{\omega}^{b2}) r^b$$

$$= -\frac{1}{2} dm (r^b)^T \underbrace{\left(\omega^b \omega^{bT} - \|\omega^b\|^2 I_{3\times3} \right)}_{} r^b$$

T. Y. Lee's text

1.1.7 Identities Involving Vectors, Orthogonal Matrices, and Skew-Symmetric Matrices

There are a number of matrix identities that involve vectors, orthogonal matrices, and skew-symmetric matrices. A few of these identities are summarized.

For any vectors $x,\, y \in \mathbb{R}^n$:

$$x^T y = \mathsf{trace}[xy^T]. \tag{1.14}$$

For any $m \times n$ matrix A and any $n \times m$ matrix B:

$$\mathsf{trace}[AB] = \mathsf{trace}[BA]. \tag{1.15}$$

For any $x \in \mathbb{R}^3$:

$$S^T(x) = -S(x), \tag{1.16}$$

$$x \times x = S(x)x = 0, \tag{1.17}$$

$\hat{\omega}^2 = \omega\omega^T - \|\omega\|^2 \cancel{I}$

$$\boxed{S(x)^2 = xx^T - \|x\|^2 I_{3\times 3},} \tag{1.18}$$

$$S(x)^3 = -\|x\|^2 S(x). \tag{1.19}$$

For any $R \in \mathsf{SO}(3)$ and any $x,\, y \in \mathbb{R}^3$:

$$R(x \times y) = (Rx) \times (Ry), \tag{1.20}$$

$$RS(x)R^T = S(Rx). \tag{1.21}$$

For any $x,\, y,\, z \in \mathbb{R}^3$:

$$S(x \times y) = yx^T - xy^T, \tag{1.22}$$

$$S(x \times y) = S(x)S(y) - S(y)S(x), \tag{1.23}$$

$$S(x)S(y) = -x^T y I_{3\times 3} + yx^T, \tag{1.24}$$

$$(x \times y) \cdot z = x \cdot (y \times z), \tag{1.25}$$

$$x \times (y \times z) = (x \cdot z)y - (x \cdot y)z, \tag{1.26}$$

$$y \times (x \times z) + x \times (z \times y) + z \times (y \times x) = 0, \tag{1.27}$$

$$\|x \times y\|^2 = \|x\|^2 \|y\|^2 - (x^T y)^2. \tag{1.28}$$

The proofs of these identities are not given here but the proofs depend only on the definitions and properties previously introduced.

$$\text{Total K.E.} = \int_B dm\,(r^b)^T \left(\omega\omega^T - \|\omega\|^2 I_{3\times 3}\right) r^b$$

$$=$$

$$= \frac{1}{2}\left\langle \omega^b, \underbrace{\mathbb{I}^b}_{\text{Body moment of inertia.}} \omega^b \right\rangle$$

Lagrangian $\quad \mathcal{L}(R, \dot{R}) = \ell(\omega^b) = \frac{1}{2}\left\langle \omega^b, \mathbb{I}^b \omega^b \right\rangle$

$$\int_0^{t_f} \ell(\omega^b)\, dt \qquad\qquad \delta\left(\int_0^{t_f} \ell(\omega^b)\, dt\right) = 0$$

$$R(0) = R_0 \qquad\qquad R(t_f) = R_f$$

Hamilton's principle

- Hamilton's principle - The trajectory of a simple mechanical system moving from a fixed configuration q_a at time a to a fixed configuration q_b at time b is the stationary point of the functional

$$\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt$$

where $\mathcal{L}(q^i, \dot{q}_i)$ is the Lagrangian of the system

- The equations of motion are thus obtained by solving the equation

$$\delta(\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt) = 0$$

- The stationarity condition yields the familiar Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) - \frac{\partial \mathcal{L}}{\partial q^i} = 0 \quad i = 1, \dots, n$$

Variational calculus on a vector space V

- Denote the vector space of admissible trajectories as V. Each trajectory is differentiable in $[a, b]$ and must satisfy $q(a) = q_a$ and $q(b) = q_b$
- The functional
$$\int_a^b \mathcal{L}(q^i, \dot{q}_i) dt$$
is a mapping from V to $\mathbb{R}$.
- Characterize (parametrize) admissible trajectories with the variable ϵ ($\epsilon \in [0, 1]$) as
$$q(t, \epsilon) \quad \text{is differentiable and} \quad q(a, \epsilon) = q_a, \;\; q(b, \epsilon) = q_b$$
- The extremal trajectory corresponds to $\epsilon = 0$

Variational calculus

- The variation of q at time t is

$$\delta q(t) = \frac{d}{d\epsilon} q(t, \epsilon)|_{\epsilon=0}$$

- The first order condition for an extremum is

$$\delta\left(\int_a^b \mathcal{L}(q^i, \dot{q}_i)dt\right) = 0 \Rightarrow \frac{d}{d\epsilon}\left(\int_a^b \mathcal{L}(q^i(t, \epsilon), \dot{q}_i(t, \epsilon))dt\right)|_{\epsilon=0} = 0$$

Variational calculus on a manifold M (here $SO(3)$)

- The functional
$$\int_a^b \mathcal{L}(R, \dot{R})dt$$
is a mapping from TM to $\mathbb{R}$.
- Denote the manifold of admissible trajectories as W. Each trajectory is differentiable in $[a, b]$ and must satisfy $R(a) = R_a$ and $R(b) = R_b$
- Characterize (parametrize) admissible trajectories with the variable $\epsilon \in \mathbb{R}$ and $\eta \in \mathbb{R}^3$ as follows
$$R(t, \eta, \epsilon) = R(t) \exp(\epsilon\hat{\eta})$$
$$R(t, \eta, \epsilon) \quad \text{is differentiable and} \quad R(a, \eta, \epsilon) = R_a, \quad R(b, \eta, \epsilon) = R_b$$
- The extremal trajectory corresponds to $\epsilon = 0$

Variational calculus on a manifold M

- The variation at time t along the direction η is

$$\delta R(t,\eta) = \frac{d}{d\epsilon} R(t,\eta,\epsilon)|_{\epsilon=0} = R\hat{\eta}$$

- The first order condition for an extremum is

$$\delta(\int_a^b \mathcal{L}(R,\dot{R})dt) = 0 \Rightarrow \frac{d}{d\epsilon}(\int_a^b \mathcal{L}(R(t,\eta,\epsilon),\dot{R}(t,\eta,\epsilon))dt)|_{\epsilon=0} = 0 \quad \forall \eta \in \mathbb{R}^3$$

The Euler Poincaré equations for rotational motion

- The configuration manifold is $SO(3)$. There is no potential energy. The Lagrangian ($\mathcal{L} : T\,SO(3) \to \mathbb{R}$) is a function of the kinetic energy alone. So $\mathcal{L}(R, \dot{R}) = T(R, \dot{R})$.
- Evaluate the *kinetic energy* by integrating the kinetic energy of an elemental volume over the entire body, as

$$T = \int_V \frac{1}{2}[\rho(r_b)(dr_b)^3]\,\|v(r_b)\|^2 = \int_V \frac{1}{2}[\rho(r_b)(dr_b)^3] \ll \dot{R}r_b, \dot{R}r_b \gg =$$

$$\int_V \frac{1}{2}[\rho(r_b)(dr_b)^3] \ll R^T\dot{R}r_b, R^T\dot{R}r_b \gg = \int_V \frac{1}{2}[\rho(r_b)(dr_b)^3] \ll \hat{\Omega}_b r_b, \hat{\Omega}_b r_b \gg$$

$$= \int_V \frac{1}{2}[\rho(r_b)(dr_b)^3] \ll \hat{r}_b\Omega_b, \hat{r}_b\Omega_b \gg = \frac{1}{2}\Omega_b^T[\int_V [\rho(r_b)(dr_b)^3]\hat{r}_b^T\hat{r}_b]\Omega_b$$

$$= \frac{1}{2}\Omega_b^T\mathbb{I}_b\Omega_b$$

- $\mathbb{I}_b$ - the moment of inertia in the body frame.

Invariance with respect to R

- Notice that the kinetic energy is invariant with respect to R, the orientation.

$$\mathcal{L}(R_1 R, R_1 \dot{R}) = \mathcal{L}(R, \dot{R}) = \mathcal{L}(R^T R, R^T \dot{R}) = \mathcal{L}(I, \hat{\Omega}_b)$$

- Since the Lagrangian is now a function of Ω_b alone, and idependent of R, we say

$$l(\Omega_b) \triangleq \mathcal{L}(I, \hat{\Omega}_b)$$

- Using Hamilton's principle, between two fixed orientations, R_0 and R_1, starting at time t_0 and terminating at t_1 respectively, the trajectory folllowed by the system is a stationary trajectory of the functional

$$\int_{t_0}^{t_1} l(\Omega_b)dt \quad R(t_0) = R_0 \quad R(t_1) = R_1$$

- The necessary stationarity condition is

$$\delta(\int_{t_0}^{t_1} l(\Omega_b)dt) = 0$$

Constructing variations

- Let the variation in R be δR. Note that $\delta R(t_0) = 0$ and $\delta R(t_1) = 0$.
- From $R^T R = I$, we notice that $\hat{\Sigma} \triangleq R^T \delta R$ is skew-symmetric.
- Now from $\hat{\Omega}_b = R^T \dot{R}$, we have

$$- \hat{\Sigma}\hat{\Omega}_b + \hat{\Omega}_b \hat{\Sigma}$$

$$\delta\hat{\Omega}_b = \underbrace{-\hat{\Sigma}\hat{\Omega}_b + [\dot{\hat{\Sigma}} + \hat{\Omega}_b R^T \delta R]} = \dot{\hat{\Sigma}} + \underbrace{[\hat{\Omega}_b, \hat{\Sigma}]}_{\text{Lie bracket}}$$

- Substituting the expression for the variation of $\hat{\Omega}_b$ is the action integral, and applying integration by parts and then the scalar triple product rule $< a, b \times c >=< c, a \times b >=< b, c \times a >$, we have the Euler-Poincaré equation

$$l = \frac{1}{2}\langle \Omega_b, \mathbb{I}_b \Omega_b \rangle$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \Omega_b}\right) = \frac{\partial l}{\partial \Omega_b} \times \Omega_b$$

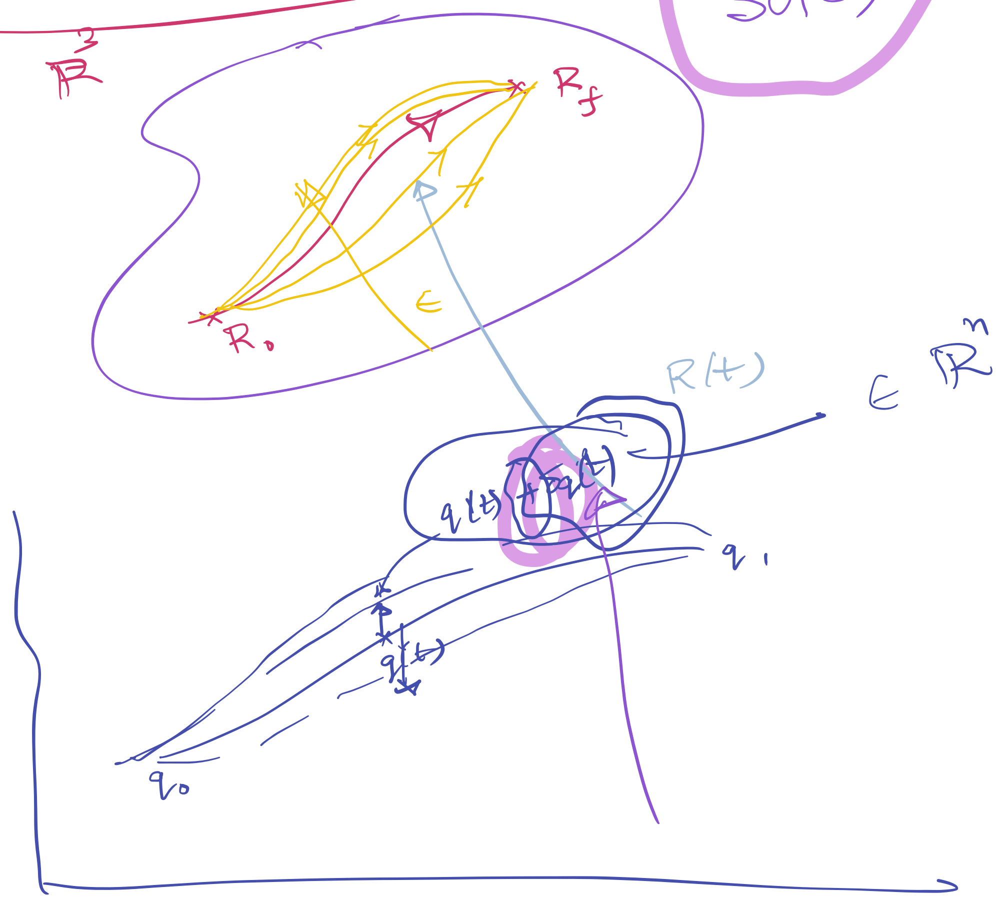
Euler eqns of motion

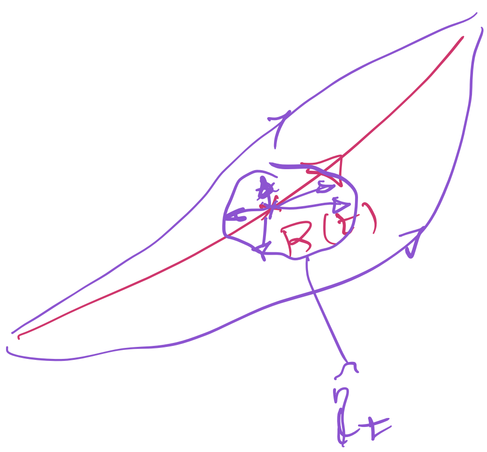
$$\mathbb{I}_b \dot{\Omega}_b = \mathbb{I}_b \Omega_b \times \Omega_b$$

$$\mathbb{I}_b \dot{\Omega}_b + \Omega_b \times \mathbb{I}\,\Omega_b = 0$$

$SO(3)$

$\Omega_b^{(t)} \in \mathbb{R}^3$





$SO(3)$

$\underline{\delta R} = \exp(\hat{\eta})$

$\dot{x} = f(x)$ $\qquad x(t) \in \mathbb{R}^n$

$f(x_{eq}) = 0$

$$\mathbb{I}_b \dot{\Omega}_b = \mathbb{I}_b \Omega_b \times \Omega_b$$

$$\mathbb{I}_b = \begin{pmatrix} I_1 & & 0 \\ 0 & I_2 & \\ & & I_3 \end{pmatrix} \quad 3\times 3$$

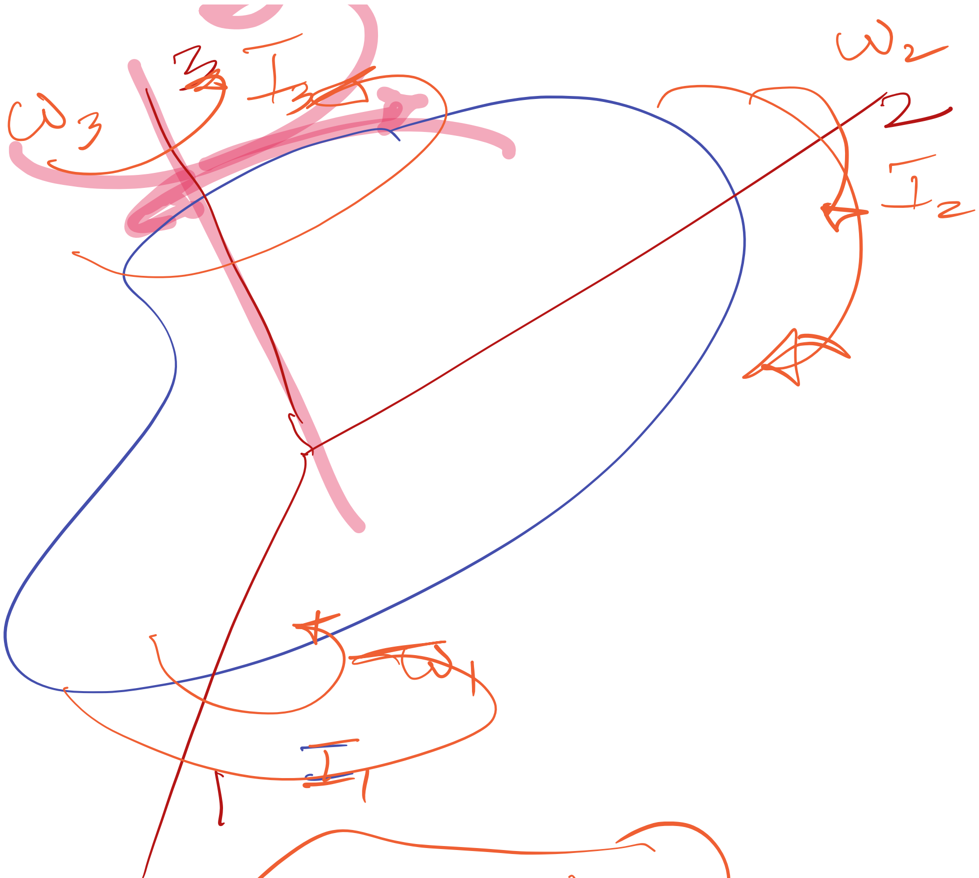
$$\begin{pmatrix} I_1 \dot{\Omega}_1 \\ I_2 \dot{\Omega}_2 \\ I_3 \dot{\Omega}_3 \end{pmatrix} = \begin{pmatrix} I_1 \Omega_1 \\ I_2 \Omega_2 \\ I_3 \Omega_3 \end{pmatrix} \times \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}$$

$$= \begin{pmatrix} (I_2 - I_3)\,\Omega_2 \Omega_3 \\ (I_3 - I_1)\,\Omega_1 \Omega_3 \\ (I_1 - I_2)\,\Omega_1 \Omega_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$I_1 > I_2 > I_3$$

$\Omega_2 \Omega_3 = 0$, $\Omega_1 \Omega_3 = 0$, $\Omega_1 \Omega_2 = 0$

$$\begin{pmatrix} \Omega_{1e} \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \Omega_{2e} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \Omega_{3e} \end{pmatrix}$$

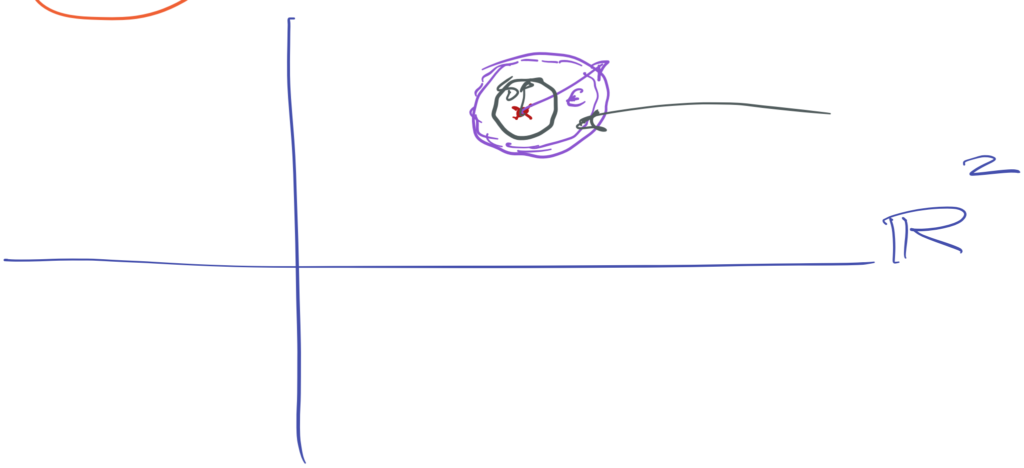


$$\dot{x} = f(x)$$

x_{eq}

stable ?

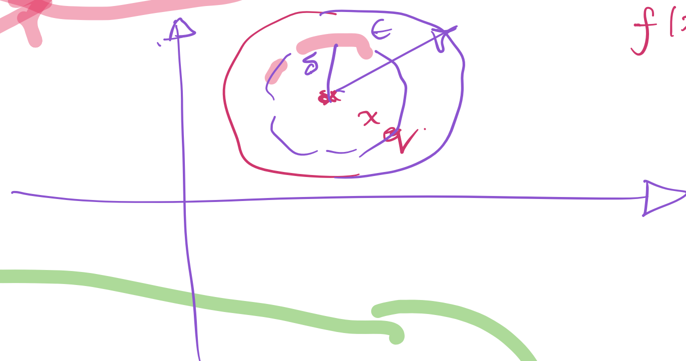
unstable ?



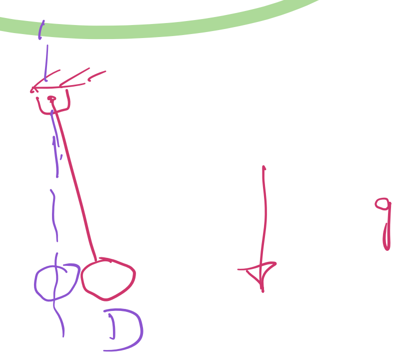
~ A primer on stability ~

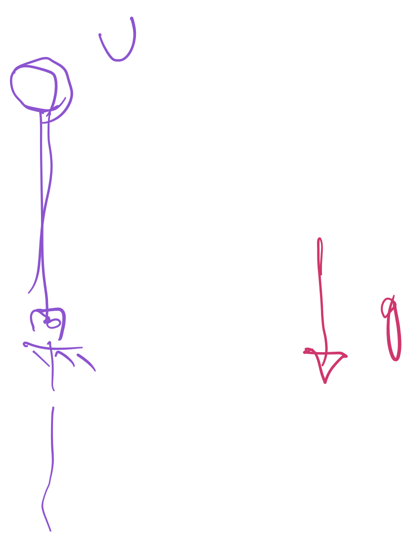
$\dot{x} = f(x)$ $\qquad x(t) \in \mathbb{R}^n$

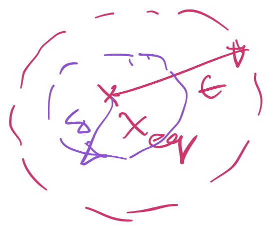
$f(x_{eq}) = 0$

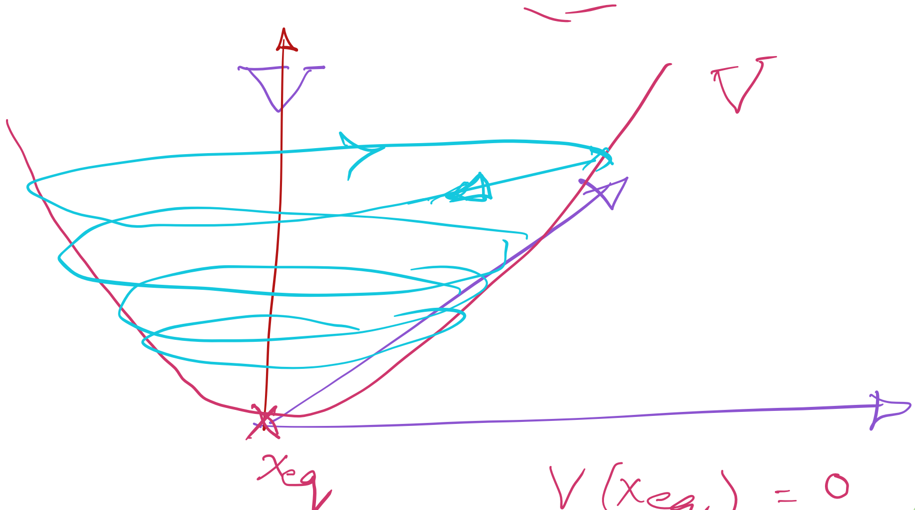


$\dot{x} = f(t, x)$ U









$V(x_{eq}) = 0$

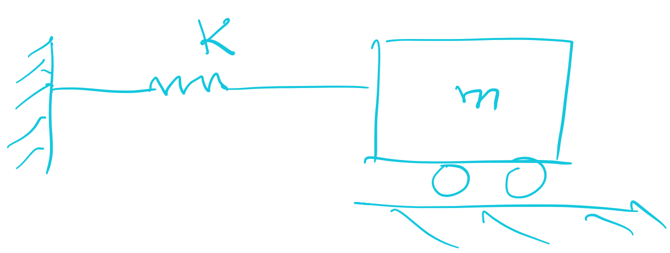
$V(x) > 0$

$x \neq x_{eq}$

x

$\frac{dV}{dt} \leq 0$

$\dot{x} = f($



$m\ddot{x} + Kx = 0$

$$\begin{pmatrix} x_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} x_2 \\ -\frac{k}{m} x_1 \end{pmatrix}}_{f(x)}$$

$$V(x) = \frac{1}{2} k x_1^2 + \frac{1}{2} m x_2^2$$

x_2

$f \quad \mathbb{R}^1 \leftarrow \mathbb{R}^2$

x_1

M

$x_2 = \sqrt{1 - x_1^2}$

$f(x_1, x_2) = x_1^2 + x_2^2 - 1$

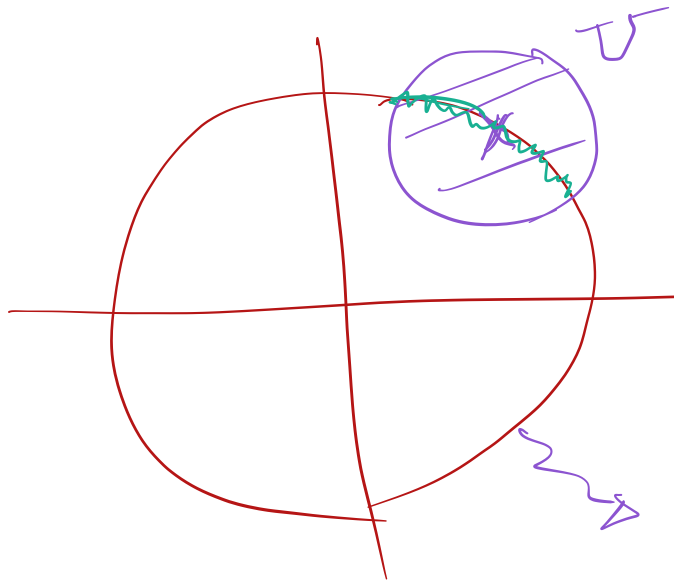
$f^{-1}(0)$

$U \cap M$

$U \longrightarrow \mathbb{R}^1$

$U \cap M$

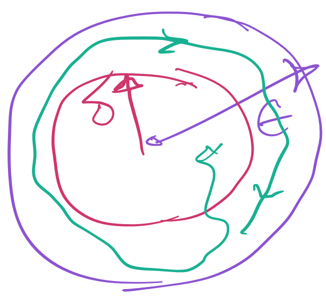
$f(U)$



$f : U \to \mathbb{R}^1$

$f(x_1, x_2) = x_1^2 + x_2^2 - 1$

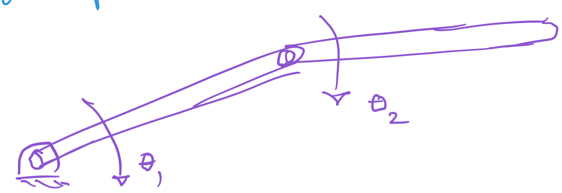
$f(U \cap M) = 0$



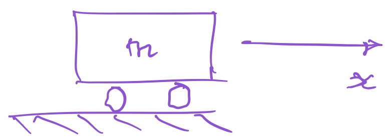
Degrees of freedom of a mechanical system

Holonomy

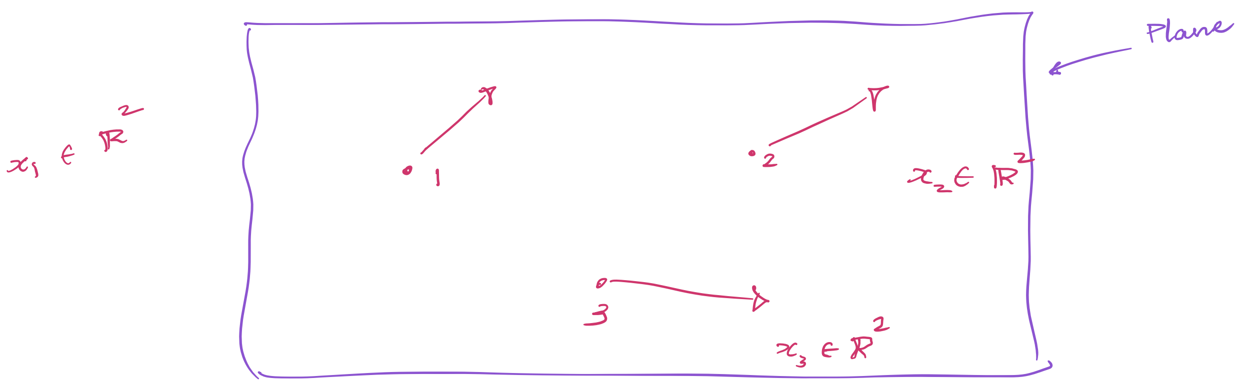
Nonholonomy



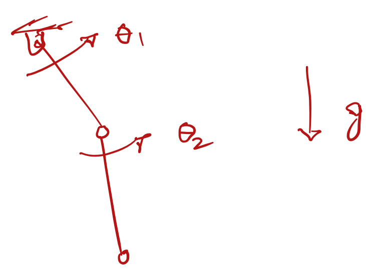
$\theta_1, \theta_2 \in S^1$

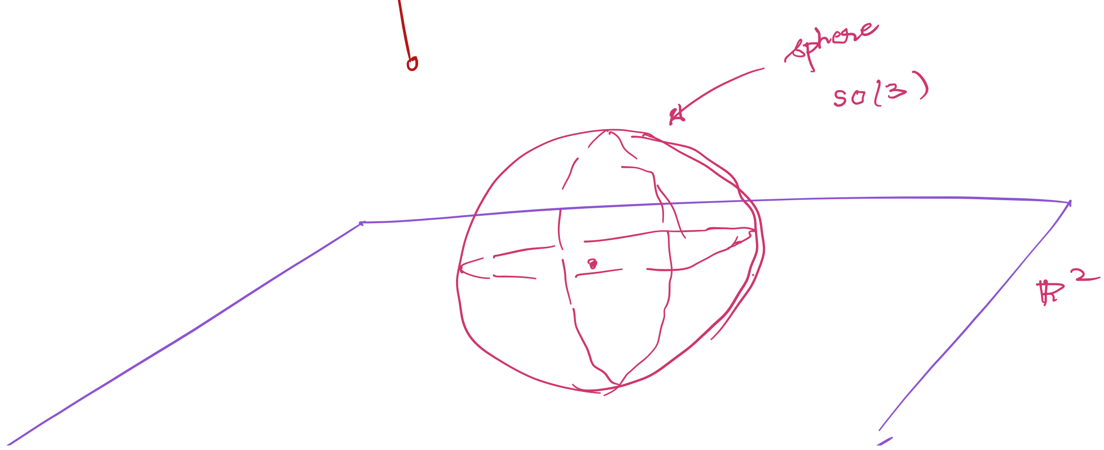


$x \in \mathbb{R}^1$



double pendulum



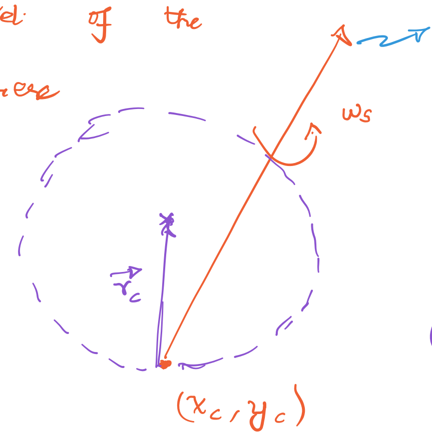


Rolling constraint

$$\text{Configuration} \;-\; \underbrace{SO(3)}_{\text{Orientation}} \times \underbrace{\mathbb{R}^2}_{\text{Point of contact.}}$$

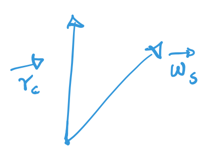
Nonholonomic

$$\boxed{\underbrace{\vec{\omega}_s}_{\text{angular vel. of the sphere}} \times \vec{r}_c = \dot{x}_c \hat{i} + \dot{y}_c \hat{j}}$$



(x_c, y_c, a) coordinates of the centre of the sphere (radius 'a').

$$\vec{\omega}_s = \omega_x \hat{i} + \omega_y \hat{j} \quad , \quad \vec{r}_c = a\hat{k}$$



$$\vec{\omega}_s \times \vec{r}_c = -a\omega_x \hat{j} + a\omega_y \hat{i}$$

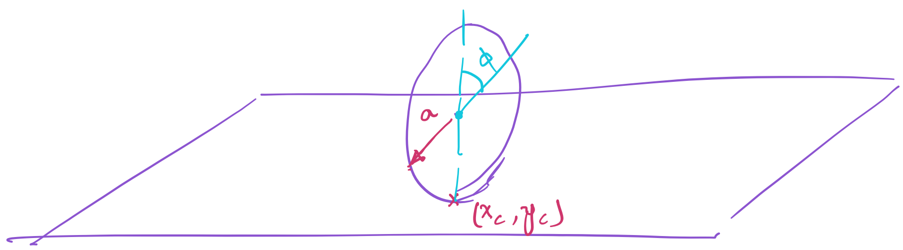
$$= \dot{x}_c \hat{i} + \dot{y}_c \hat{j}$$

$$\begin{pmatrix} \dot{x}_c \\ \dot{y}_c \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix}$$

$$\begin{pmatrix} 0 & +a & 0 & -1 & 0 \\ -a & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \\ \dot{x}_c \end{pmatrix} = 0$$

$\dot{y}_c$

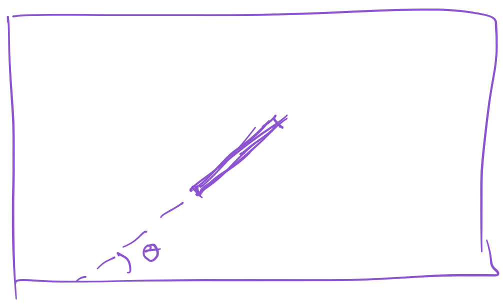
Rolling coin



$$\boxed{\begin{aligned} \dot{x}_c &= a\dot{\phi}\, c\theta \\ \dot{y}_c &= a\dot{\phi}\, s\theta \end{aligned}}$$

Config. space

$$(\underbrace{x_c, y_c}_{\mathbb{R}^2}, \underbrace{\theta}_{S^1}, \underbrace{\phi}_{S^1})$$



$$\boxed{\delta\left(\int_{t_0}^{t_f} \mathcal{L}(q, \dot{q})\, dt\right) + \int_{t_0}^{t_f} F \cdot \delta q \, dt = 0}$$

Lagrange D'Alembert Principle

$$\dot{x}_c\, c\theta + \dot{y}_c\, s\theta = a\dot{\phi}$$

$$\dot{x}_c\, s\theta - \dot{y}_c\, c\theta = 0$$

$$\begin{pmatrix} c\theta & s\theta & 0 & -a \\ s\theta & -c\theta & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = 0$$

$$\boxed{A(q)\,\dot{q} = 0}$$

$$F_{c_1} = \lambda^1 \underbrace{\begin{pmatrix} c\theta \\ s\theta \\ 0 \\ -a \end{pmatrix}}_{\xi^1} \qquad F_{c_2} = \lambda^2 \underbrace{\begin{pmatrix} s\theta \\ -c\theta \\ 0 \\ 0 \end{pmatrix}}_{\xi^2}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q^i} = F^i$$

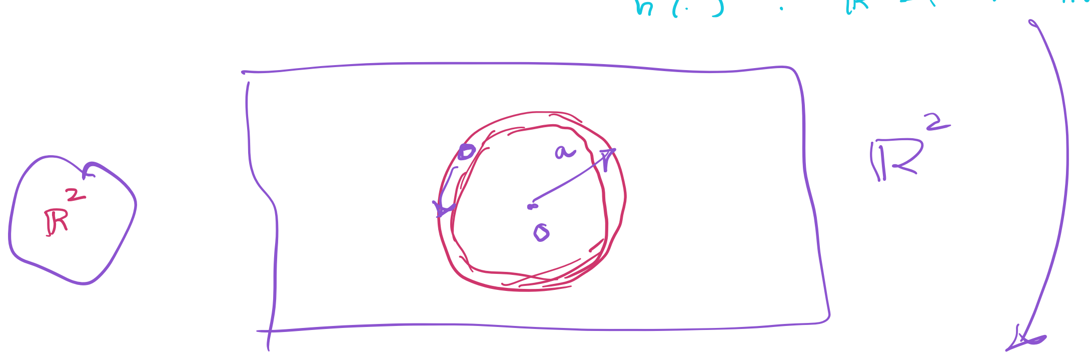
$$= \lambda^1_i \xi^1_i + \lambda^2_i \xi^2_i$$

Nonholonomic constraint $A(q)\,\dot{q} = 0$

Holonomic constraints : $h(q) = 0$

$q \in \mathbb{R}^n$

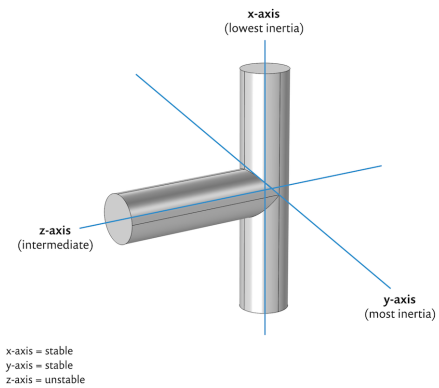
$h(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$



$$h(\underbrace{q}_{x,y}) = x^2 + y^2 - a^2 = 0$$

Supplementary lecture on Stability of spinning bodies — Feb 2, 2022

Dedicated to Radha Lahoti, Akshay Wakode, Kriti Verma and Akhilesh Katdare



Energy-momentum picture

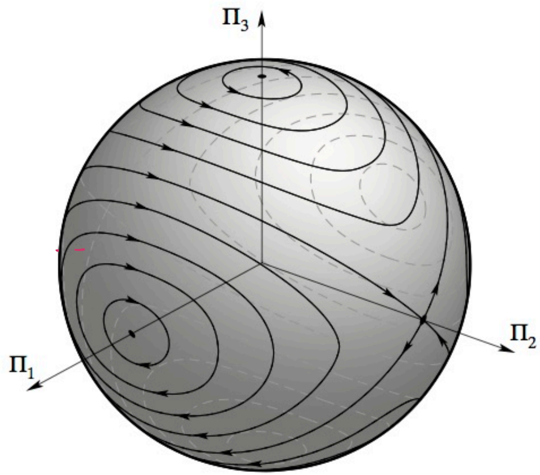


FIGURE 15.3.1. Rigid body flow on the angular momentum spheres for the case $I_1 < I_2 < I_3$.

Figure : Flow lines of the Euler equations - intersection of the momentum sphere and the kinetic energy ellipsoid (Courtesy: Introduction to Mechanics and Symmetry, J. E. Marsden and T. Ratiu)