- $v = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$
  $= v_x'\hat{i}' + v_y'\hat{j}' + v_z'\hat{k}'$

- $v_x\hat{i} + v_y\hat{j} + v_z\hat{k} = v_x'\hat{i}' + v_y'\hat{j}' + v_z'\hat{k}'$

$$v_x'\hat{\mathbf{i}}' + v_y'\hat{\mathbf{j}}' + v_z'\hat{\mathbf{k}}' = \left(Q_{11}\hat{\mathbf{i}}' + Q_{21}\hat{\mathbf{j}}' + Q_{31}\hat{\mathbf{k}}'\right)v_x + \left(Q_{12}\hat{\mathbf{i}}' + Q_{22}\hat{\mathbf{j}}' + Q_{32}\hat{\mathbf{k}}'\right)v_y + \left(Q_{13}\hat{\mathbf{i}}' + Q_{23}\hat{\mathbf{j}}' + Q_{33}\hat{\mathbf{k}}'\right)v_z$$

$$v_x'\hat{\mathbf{i}}' + v_y'\hat{\mathbf{j}}' + v_z'\hat{\mathbf{k}}' = \left(Q_{11}v_x + Q_{12}v_y + Q_{13}v_z\right)\hat{\mathbf{i}}' + \left(Q_{21}v_x + Q_{22}v_y + Q_{23}v_z\right)\hat{\mathbf{j}}' + \left(Q_{31}v_x + Q_{32}v_y + Q_{33}v_z\right)\hat{\mathbf{k}}'$$

- $v_x' = Q_{11}v_x + Q_{12}v_y + Q_{13}v_z$
  $v_y' = Q_{21}v_x + Q_{22}v_y + Q_{23}v_z$
  $v_z' = Q_{31}v_x + Q_{32}v_y + Q_{33}v_z$

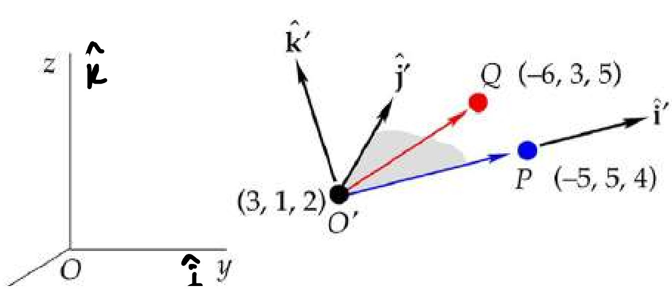
- $v' = Qv$

- $Q^Tv' = Q^TQv = v$

Example

In Fig. 4.10, the x' axis is defined by the line segment $O'P$. The $x'y'$ plane is defined by the intersecting line segments $O'P$ and $O'Q$. The z' axis is normal to the plane of $O'P$ and $O'Q$ and obtained by rotating $O'P$ toward $O'Q$ and using the right-hand rule.

(a) Find the direction cosine matrix $[\mathbf{Q}]$.
(b) If $\{\mathbf{v}\} = \lfloor 2 \ \ 4 \ \ 6 \rfloor^T$, find $\{\mathbf{v}'\}$.
(c) If $\{\mathbf{v}'\} = \lfloor 2 \ \ 4 \ \ 6 \rfloor^T$, find $\{\mathbf{v}\}$.



x $\hat{i}$

Details

(a) $\overrightarrow{O'P} = (-5-3)\hat{i} + (5-1)\hat{j} + (4-2)\hat{k}$

$\overrightarrow{O'Q} = (-6-3)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k}$

$Z' = \overrightarrow{O'P} \times \overrightarrow{O'Q}$

$Y' = Z' \times \overrightarrow{O'P}$

$$\hat{i}' = \frac{\overrightarrow{O'P}}{\|\overrightarrow{O'P}\|}$$

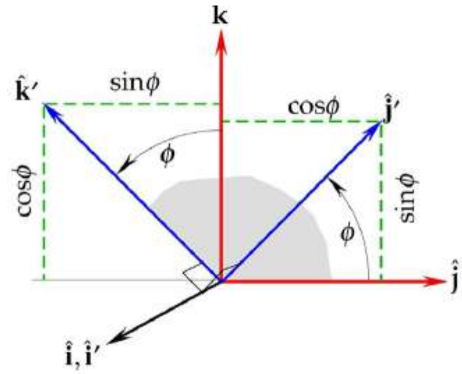
$$\hat{j}' = \frac{Y'}{\|Y'\|}$$

$$\hat{k}' = \frac{Z'}{\|Z'\|}$$

(b) $v' = Qv$

(c) $v = Q^T v$

$$- \begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{i}'\cdot\hat{i} & \hat{i}'\cdot\hat{j} & \hat{i}'\cdot\hat{k} \\ \hat{j}'\cdot\hat{i} & \hat{j}'\cdot\hat{j} & \hat{j}'\cdot\hat{k} \\ \hat{k}'\cdot\hat{i} & \hat{k}'\cdot\hat{j} & \hat{k}'\cdot\hat{k} \end{bmatrix}}_{\downarrow \; Q} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

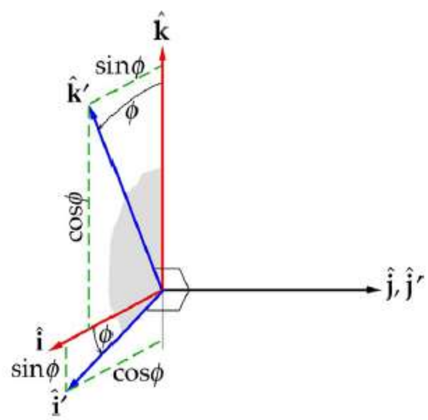


- $\hat{i}' = \hat{i}$

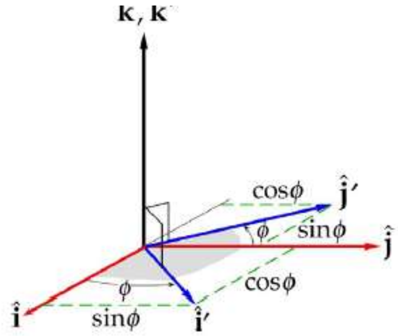
  $\hat{j}' = (\hat{j}' \cdot \hat{j})\hat{j} + (\hat{j}' \cdot \hat{k})\hat{k} = \cos\phi\,\hat{j} + \sin\phi\,\hat{k}$

  $\hat{k}' = (\hat{k}' \cdot \hat{j})\hat{j} + (\hat{k}' \cdot \hat{k})\hat{k} = -\sin\phi\,\hat{j} + \cos\phi\,\hat{k}$

- $$\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{R_1(\phi)} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$



- $$\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}}_{R_2(\phi)} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$



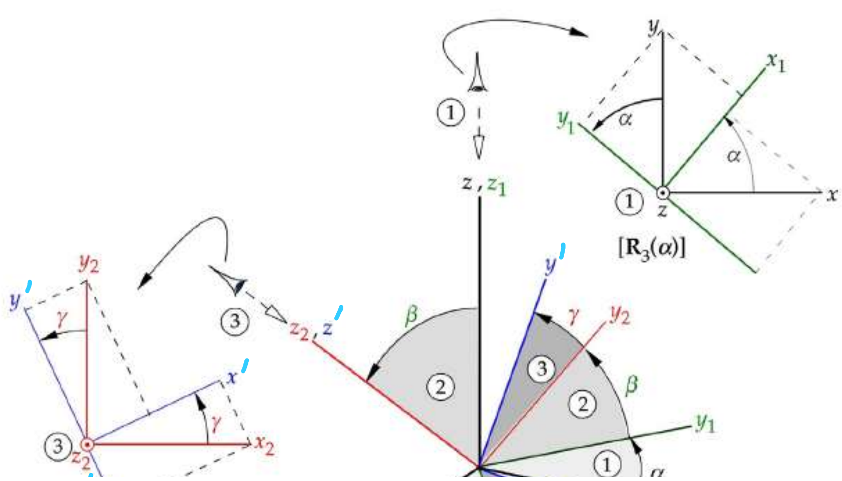
- $$\begin{bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_3(\phi)} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

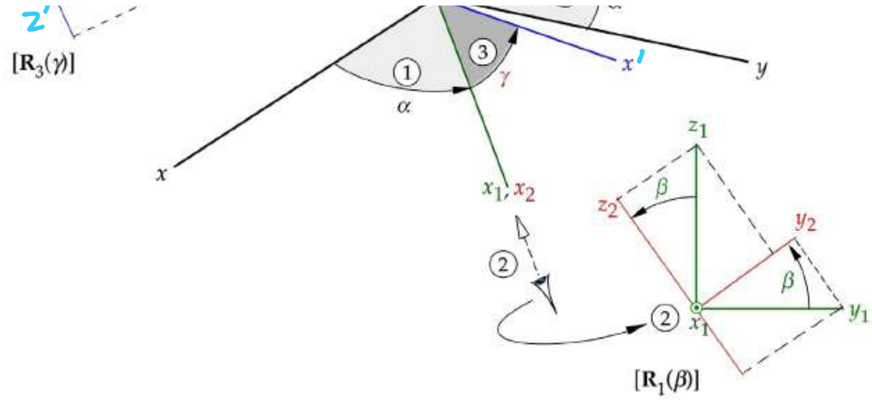
- $xyz \xrightarrow{\alpha} x_1y_1z_1 \xrightarrow{\beta} x_2y_2z_2 \xrightarrow{\gamma} x'y'z'$ (Euler angle sequence)

$$\begin{array}{ll} [\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] & [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] \\ [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] & [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ \underline{[\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)]} & [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)] \end{array}$$ (symmetric Euler sequences)

$$\begin{array}{ll} \underline{[\mathbf{R}_1(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_3(\alpha)]} & [\mathbf{R}_1(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_2(\alpha)] \\ [\mathbf{R}_2(\gamma)][\mathbf{R}_3(\beta)][\mathbf{R}_1(\alpha)] & [\mathbf{R}_2(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_3(\alpha)] \\ [\mathbf{R}_3(\gamma)][\mathbf{R}_1(\beta)][\mathbf{R}_2(\alpha)] & [\mathbf{R}_3(\gamma)][\mathbf{R}_2(\beta)][\mathbf{R}_1(\alpha)] \end{array}$$ (asymmetric Euler sequences)

- $Q = R_3(\gamma)R_1(\beta)R_3(\alpha)$ ($0^\circ \le \alpha < 360^\circ$, $0^\circ \le \beta \le 180^\circ$, $0^\circ \le \gamma < 360^\circ$)





- $b = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

- $Qb = R_3(\gamma)\,R_1(\beta)\,R_3(\alpha)\,b = R_3(\gamma)\,R_1(\beta)\,b|_{x_1y_1z_1} = R_3(\gamma)\,b|_{x_2y_2z_2} = b|_{x'y'z'}$

$$[\mathbf{Q}] = \begin{bmatrix} -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\sin\alpha\cos\beta\sin\gamma - \cos\alpha\cos\gamma & \cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix}$$

Given the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

find the angles $\alpha\beta\gamma$ of the classical Euler rotation sequence. This algorithm is implemented by the MATLAB function *dcm_to_euler.m* in Appendix D.20.

1. $\alpha = \tan^{-1}(-Q_{31}/Q_{32}) \quad (0 \le \alpha < 360°)$
2. $\beta = \cos^{-1} Q_{33} \quad (0 \le \beta \le 180°)$
3. $\gamma = \tan^{-1}(Q_{13}/Q_{23}) \quad (0 \le \gamma < 360°)$

Example

If the direction cosine matrix for the transformation from xyz to $x'y'z'$ is

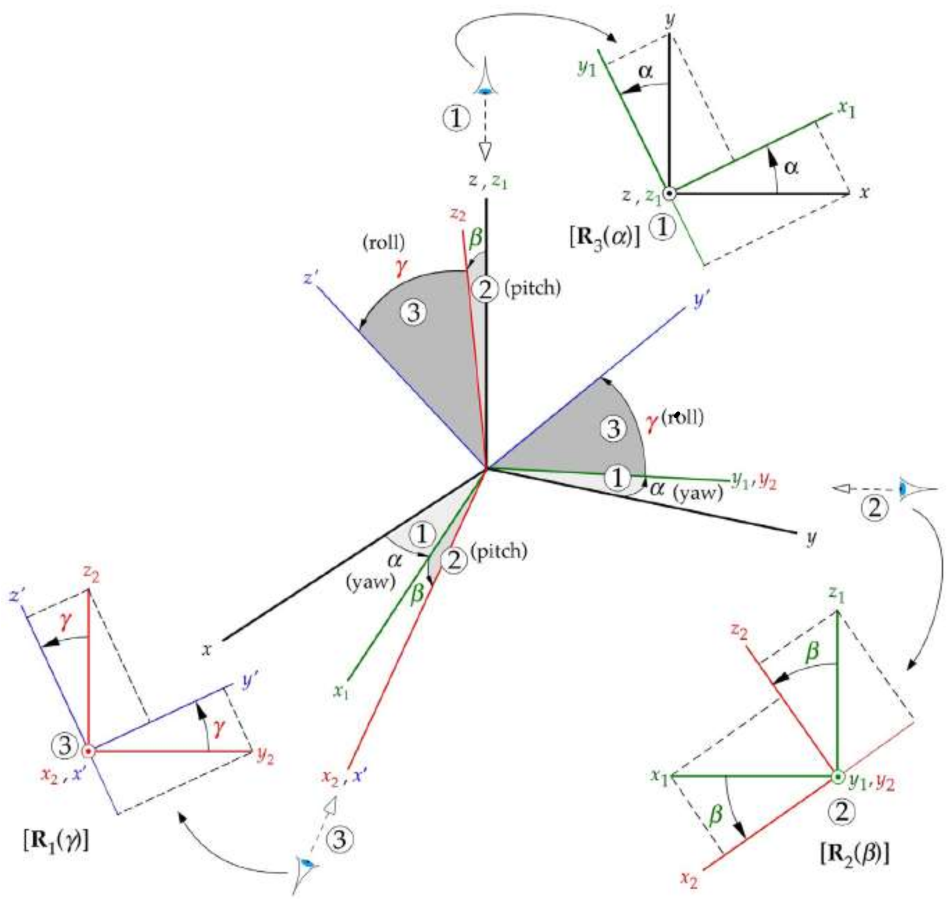
$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75309 & -0.15038 \\ 0.76737 & -0.63530 & 0.086823 \\ -0.30152 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles α, β, and γ of the classical Euler sequence.

Details

Follow the above algorithm.

- $Q = R_1(\gamma)\,R_2(\beta)\,R_3(\alpha)$ $(0^\circ \le \alpha < 360^\circ,\ -90^\circ < \beta < 90^\circ,\ 0 \le \gamma < 360^\circ)$



$$[\mathbf{Q}] = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & -\sin\beta \\ \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Given the direction cosine matrix

$$[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

Find the angles $\alpha\beta\gamma$ of the yaw, pitch, and roll sequence. This algorithm is implemented by the MATLAB function *dcm_to_ypr.m* in Appendix D.21.

1. $\alpha = \tan^{-1}(Q_{12}/Q_{11})$ $(0 \le \alpha < 360^\circ)$
2. $\beta = \sin^{-1}(-Q_{13})$ $(-90^\circ < \beta < 90^\circ)$
3. $\gamma = \tan^{-1}(Q_{23}/Q_{33})$ $(0 \le \gamma < 360^\circ)$

Example

If the direction cosine matrix for the transformation from xyz to $x'y'z'$ is the same as it was in Example 4.5,

$$[\mathbf{Q}] = \begin{bmatrix} 0.64050 & 0.75309 & -0.15038 \\ 0.76737 & -0.63530 & 0.086823 \\ -0.30152 & -0.17101 & -0.98481 \end{bmatrix}$$

find the angles α, β, and γ of the yaw, pitch, and roll sequence.

Details

Follow the above algorithm.