Problem 1

Find the 4^{th} Taylor Polynomial $P_4(x)$ for the function $f(x) = xe^{x^2}$ at $x = 0$

Solution

We note that that function $f(x) = e^{x^2}$ is differentiable infinitely many times and hence by Taylor's formula the n^{th} Taylor polynomial is given by

$$P_n(x) = \Sigma_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k.$$

$$f(x) = xe^{x^2}, f(0) = 0$$

$$f'(x) = e^{x^2} + 2x^2e^{x^2}, f'(0) = 1$$

$$f''(x) = (4x^3 + 6x)e^{x^2}, f''(0) = 1$$

$$f^3(x) = (8x^4 + 24x^2 + 6)e^{x^2}, f^3(0) = 6$$

$$f^{(}4) = (16x^4 + 80x^3 + 60x)e^{x^2}, f^4(0) = 0$$

Using the Taylor formula with $a = 0$ we get that

$$P_4(x) = x + \frac{6}{3!}x^3 = x + x^3$$

Problem 2
Let $f(x) = (1-x)^{-1}$. Find the n-th Taylor polynomial $P_n(x)$ for $f(x)$ about $x = 0$.

Solution
Recall Taylor's Theorem from MA 105. The given function $f(x)$ is at least n times differentiable at $x = 0$. Thus, $\exists c \in [0, x]$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1}$$

where

$$f^{(k)}(x) = \begin{cases} f(x) & k = 0 \\ \frac{d^k f(x)}{dx^k} & k \neq 0 \end{cases}$$

We can separate the polynomial term $P_n(x)$ and the remainder term $R_n(x)$ as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$$R_n(x) = \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1}$$

Now, consider our function $f(x) = (1-x)^{-1}$. Using standard differentiation rules,

$$\begin{aligned} f^{(0)}(x) &= (1-x)^{-1} \Rightarrow f^{(0)}(0) = 1 \\ f^{(1)}(x) &= (1-x)^{-2} \Rightarrow f^{(1)}(0) = 1 \\ f^{(2)}(x) &= 2(1-x)^{-3} \Rightarrow f^{(2)}(0) = 2 \\ f^{(3)}(x) &= 6(1-x)^{-4} \Rightarrow f^{(3)}(0) = 6 \end{aligned}$$

and so on. We begin to see a pattern here. Let us assume

$$f^{(k)}(x) = k!(1-x)^{-(k+1)}$$

We use induction to prove this. Differentiation of the above equation yields

$$f^{(k+1)}(x) = k!(1-x)^{-(k+2)} \times (-(k+1)) \times (-1) = (k+1)!(1-x)^{-(k+2)}$$

which complies with the form

$$
\begin{aligned}
f^{(1)}(x) &= (1-x)^{-2} \ \Rightarrow f^{(1)}(0) = 1 \\
f^{(2)}(x) &= 2(1-x)^{-3} \Rightarrow f^{(2)}(0) = 2 \\
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Using this,

$$f^{(k)}(0) = k!(1-0)^{-(k+1)} = k!$$

Substituting into the equation for $P_n(x)$, we get

$$P_n(x) = \sum^{n} \frac{f^{(k)}(0)}{k!} x^k$$

$$f^{(k)}(x) = k!(1-x)^{-(k+1)}$$

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Using this,

$$f^{(k)}(0) = k!(1-0)^{-(k+1)} = k!$$

Substituting into the equation for $P_n(x)$, we get

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{k!}{k!} x^k \\ &= \sum_{k=0}^{n} x^k \end{aligned}$$

Thus, $\boxed{P_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n}$

Problem 3 For $f(x)$ and $P_n(x)$ as in the above problem, find a value of n such that $P_n(x)$ approximates $f(x)$ to within 10^{-6} on $[0, 0.5]$.

Solution

From, previous solution we have that $P_n(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$. We have to find n such that $|f(x) - P_n(x)| < 10^{-6}\ \forall x \in [0, 0.5]$ ie

$$|\frac{1}{1-x} - (1 + x + x^2 + \cdots + x^n)| < 10^{-6}\ \forall x \in [0, 0.5]$$

$$|\frac{1}{1-x} - \frac{1-x^{n+1}}{1-x}| < 10^{-6}\ \forall x \in [0, 0.5]$$

$$\frac{x^{n+1}}{1-x}| < 10^{-6}\ \forall x \in [0, 0.5]$$

$\frac{x^{n+1}}{1-x}$ is an increasing function on the given interval so we have that $0.5^n < 10^{-6}$ which gives that $2^n > 10^6$ and thus $n > 6log_2(10) = 19.93$. So $n \geqslant 20$ satisfies the given condition.

Problem 4

If we use k digits and the chopping method to approximate a real number $y \neq 0$ then prove that the relative error is $\leqslant 10^{-k+1}$.

Solution

W.L.O.G Let $y = 0.d_1d_2d_3...d_kd_{k+1}... \times 10^n$ where $1 \leqslant d_1 \leqslant 9$ and $0 \leqslant d_i \leqslant 9$ for i=2,3,... and let the approximation be y^*. In chopping method if we use only k digits then in y all the digits from d_{k+1} will be chopped off and this is taken as y^*. So,

$$y^* = 0.d_1d_2d_3...d_k \times 10^n$$

Now , consider the absolute error $|y - y^*|$ which turns out to be

$$A.E = |y - y^*| = 0.000...0d_{k+1}d_{k+2}... \times 10^n$$

And,

$$R.E = |\frac{y - y^*}{y}|$$

$$R.E = |\frac{A.E}{y}| \Rightarrow R.E = \frac{0.000...0d_{k+1}d_{k+2}... \times 10^n}{0.d_1d_2d_3...d_kd_{k+1}... \times 10^n}$$

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$$\Rightarrow R.E = \frac{0.d_{k+1}d_{k+2}... \times 10^{-k}}{0.d_1d_2d_3...d_kd_{k+1}...}$$

$$\Rightarrow R.E \leqslant \frac{0.d_{k+1}d_{k+2}... \times 10^{-k}}{0.1} \leqslant 10^{-k+1}$$

Thus, $\boxed{R.E \leqslant 10^{-k+1}}$

Problem 5

If we use k digits and the rounding method to approximate a real number $y \neq 0$ then prove that the relative

Problem 5

If we use k digits and the rounding method to approximate a real number $y \neq 0$ then prove that the relative error is $\leqslant 0.5 \times 10^{-k+1}$

Solution

Let $y = (0.d_1 d_2 ... d_k d_{k+1} ...) \times 10^n$ where $1 \leqslant d_1 \leqslant 9$ and let approximation be y^*

We know that y^* is obtained after adding $5 \times 10^{n-(k+1)}$ to y and chopping till k digits. So,

$$y^* \leqslant y + 5 \times 10^{n-(k+1)}$$

Also, when $d_{k+1} < 5$, we get $y^* = (0.d_1 d_2 ... d_k) \times 10^n$. In this case, $y - 5 \times 10^{n-(k+1)} \leqslant y^*$.
When $d_{k+1} \geqslant 5$, as we will be adding 5 to d_{k+1}, y^* will have digit in k^{th} place 1 more than d_k or it will be carried, so $y \leqslant y^*$ which implies $y - 5 \times 10^{n-(k+1)} \leqslant y^*$. Therefore, in both cases.

$$y - 5 \times 10^{n-(k+1)} \leqslant y^*$$

Combining above inequalities, we get

$$y - 5 \times 10^{n-(k+1)} \leqslant y^* \leqslant y + 5 \times 10^{n-(k+1)}$$
$$-5 \times 10^{n-(k+1)} \leqslant y^* - y \leqslant 5 \times 10^{n-(k+1)}$$

Hence, we get

$$|y^* - y| \leqslant 5 \times 10^{n-(k+1)}$$

We know that relative error $= \dfrac{|y^* - y|}{|y|}$. Therefore,

$$R.E \leqslant \frac{5 \times 10^{n-(k+1)}}{|y|}$$

$$R.E \leqslant \frac{5 \times 10^{n-(k+1)}}{(0.d_1 d_2 ... \times 10^n}$$

Since, $1 \leqslant d_1 \leqslant 9$,

$$R.E \leqslant \frac{5 \times 10^{n-(k+1)}}{0.1 \times 10^n}$$

$$R.E \leqslant \frac{5 \times 10^{-(k+1)}}{0.1}$$

We know that relative error $= \frac{|y - \hat{y}|}{|y|}$. Therefore,

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$$R.E \leqslant 0.5 \times 10^{-k+1}$$

Problem 6

Suppose $x = \frac{5}{...}$ and $y = \frac{1}{...}$. Use five-digit chopping to compute $x \oplus y$, $x \ominus y$, $x \otimes y$ and $x \oslash y$. Compute the

Problem 6

Suppose $x = \frac{5}{7}$ and $y = \frac{1}{3}$. Use five-digit chopping to compute $x \oplus y, x \ominus y$, $x \otimes y$ and $x \oplus y$. Compute the absolute and the relative errors in the above 4 operations.

Solution

To get floating point representation of $x = 0.d_1 d_2 \cdots d_k d_{k+1} \cdots \times 10^n$ we chop the part starting from d_{k+1} and get $x = 0.d_1 d_2 \cdots d_k \times 10^n$. So, $x = 0.71428 \times 10^0$ and $y = 0.33333 \times 10^0$.

a)$(x \oplus y)$

$(x \oplus y) = (0.71428 + 0.33333) \times 10^0 = 1.04761 = 0.10476 \times 10^1$. For a real number p with approximation p^* we have that

absolute error $= |p - p^*|$

relative error $= \frac{|p-p^*|}{p}$.

With $p = x + y = \frac{5}{7} + \frac{1}{3} = \frac{22}{21}$, we get

absolute error$= |\frac{22}{21} - 0.10476 \times 10^1| = 0.19047 \times 10^{-4}$.

Relative error$= \frac{|\frac{22}{21} - 0.10476 \times 10^1|}{\frac{22}{21}} = 0.18181 \times 10^{-4}$.

b)$(x \ominus y)$

$(x \ominus y) = (0.71828 - 0.33333) = 0.38095$. With $p = x - y = \frac{5}{7} - \frac{1}{3} = \frac{8}{21}$, we get

absolute error$= |\frac{8}{21} - 0.38095| = 0.23089 \times 10^{-5}$.

Relative error$= \frac{|\frac{8}{21} - 0.38095|}{\frac{8}{21}} = 0.62499 \times 10^{-5}$.

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c)$(x \otimes y)$

$(x \otimes y) = 0.71428 \times 0.33333 = 0.23809$. With $p = x \times y = \frac{5}{7} \times \frac{1}{3} = \frac{5}{21}$, we get

absolute error$= |\frac{5}{21} - 0.23809| = 0.42856 \times 10^{-5}$.

$(x \otimes y) = 0.71428 \times 0.33333 = 0.23809$. With $p = x \times y = \frac{5}{7} \times \frac{1}{3} = \frac{5}{21}$, we get absolute error$= |\frac{5}{21} - 0.23809| = 0.42856 \times 10^{-5}$.

Relative error$= \frac{|\frac{5}{21} - 0.23809|}{\frac{5}{21}} = 0.17999 \times 10^{-4}$.

d)$(x \oplus y)$

$(x \oplus y) = (0.71428/0.33333) = 0.21428 \times 10^{1}$. With $p = x/y = \frac{15}{7}$, we get absolute error$= |\frac{15}{7} - 0.21428 \times 10^{1}| = 0.57182 \times 10^{-4}$.
Relative error$= \frac{|\frac{15}{7} - 0.21428|}{\frac{15}{7}} = 0.26666 \times 10^{-4}$.

Problem 7

Relative error$= \frac{\quad}{\frac{5}{21}} = 0.17999 \times 10$.

d)$(x \oplus y)$
$(x \oplus y) = (0.71428/0.33333) = 0.21428 \times 10^1$. With $p = x/y = \frac{15}{7}$, we get
absolute error$= |\frac{15}{7} - 0.21428 \times 10^1| = 0.57182 \times 10^{-4}$.
Relative error$= \frac{|\frac{15}{7} - 0.21428|}{\frac{15}{7}} = 0.26666 \times 10^{-4}$.

Problem 7
Let $p = 0.546217$ and $q = 0.546201$. Use five-digit arithmetic to compute $p \ominus q$ and determine the absolute and the relative errors using the methods of chopping and rounding. Compute the number of significant digits in both these methods for the result.

Solution
$p = 0.546217$ and $q = 0.546201$ so $z = p - q = 1.6 \times 10^{-5}$

a)Chopping
$p^* = 0.54621$ and $q^* = 0.54620$ So $z^* = p \ominus q = p^* - q^* = 1 \times 10^{-5}$.
Absolute error$= |z - z^*| = |1.6 \times 10^{-5} - 10^{-5}| = 6 \times 10^{-6}$.
Relative error$= \frac{|z - z^*|}{z} = \frac{6 \times 10^{-6}}{1.6 \times 10^{-5}} = 0.375 < 0.5 = 5 \times 10^{-1}$. So, z^* approximates z to one significant digit.

b)Rounding
$p^* = 0.54622$ and $q^* = 0.54620$ So $z^* = p \ominus q = p^* - q^* = 2 \times 10^{-5}$.
Absolute error$= |z - z^*| = |1.6 \times 10^{-5} - 2 \times 10^{-5}| = 4 \times 10^{-6}$

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Let $p = 0.546217$ and $q = 0.546201$. Use five-digit arithmetic to compute $p \ominus q$ and determine the absolute and the relative errors using the methods of chopping and rounding. Compute the number of significant digits in both these methods for the result.

Solution

$p = 0.546217$ and $q = 0.546201$ so $z = p - q = 1.6 \times 10^{-5}$

a)Chopping

$p^* = 0.54621$ and $q^* = 0.54620$ So $z^* = p \ominus q = p^* - q^* = 1 \times 10^{-5}$.

Absolute error= $|z - z^*| = |1.6 \times 10^{-5} - 10^{-5}| = 6 \times 10^{-6}$.

Relative error= $\frac{|z-z^*|}{z} = \frac{6\times10^{-6}}{1.6\times10^{-5}} = 0.375 < 0.5 = 5 \times 10^{-1}$. So, z^* approximates z to one significant digit.

b)Rounding

$p^* = 0.54622$ and $q^* = 0.54620$ So $z^* = p \ominus q = p^* - q^* = 2 \times 10^{-5}$.

Absolute error= $|z - z^*| = |1.6 \times 10^{-5} - 2 \times 10^{-5}| = 4 \times 10^{-6}$.

Relative error= $\frac{|z-z^*|}{z} = \frac{4\times10^{-6}}{1.6\times10^{-5}} = 0.25 < 0.5 = 5 \times 10^{-1}$. So, z^* approximates z to one significant digit.

Problem 8

Consider the quadratic equation $x^2 + 62.10x + 1 = 0$ whose roots are (approximately)$x = -0.01610723$ and $x = -62.08390$. Use the four-digit rounding arithmetic to compute the roots using the formula $x_1 = \frac{-b+\sqrt{b62-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b62-4ac}}{2a}$. Compute the absolute and the relative errors.

Absolute error$= |z - z^*| = |1.6 \times 10^{-5} - 2 \times 10^{-5}| = 4 \times 10^{-6}$.
Relative error$= \frac{|z-z^*|}{z} = \frac{4\times10^{-6}}{1.6\times10^{-5}} = 0.25 < 0.5 = 5 \times 10^{-1}$. So, z^* approximates z to one significant digit.

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Solution

From the equation $x^2 + 62.10x + 1 = 0$ we have that $a = 1, b = 62.10$ and $c = 1$. $b^2 = 3856.41 \simeq 3856$, so $b^- 4ac = 3852$ and $\sqrt{b^2 - 4ac} = 62.06$ Using this and the given formula we get $x_1^* = \frac{-62.10+62.06}{2} = -0.02$ and $x_2^* = \frac{-62.10-62.06}{2} = -62.1$.

Absolute error$(x_1) = |x_1 - x_1^*| = 3.89 \times 10^{-3}$.
Relative error$(x_1) = \frac{|x_1-x_1^*|}{x_1} = 2.42 \times 10^{-1}$.

Absolute error$(x_2) = |x_2 - x_2^*| = 1.61 \times 10^{-2}$.
Relative error$(x_2) = \frac{|x_2-x_2^*|}{x_2} = 2.59 \times 10^{-4}$.

Problem 9

Evaluate $f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$ at $x = 4.71$ using three-digit arithmetic in both the chopping and the rounding methods. Compute the absolute and the relative errors.

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Solution

$f(x) = x^3 - 6.1x^2 + 3.2x + 1.5$

a)Chopping

First term: $3.2 * x = 3.2 * 4.71 = 15.072$ chopped to 15.0

Second term: $6.1 * (x * x) = 6.1 * (22.1) = 134$

Second term: $6.1 * (x * x) = 6.1 * (22.1) = 134$
Third Term: $x * x * x = 4.71 * 22.1 = 104$
$f(4.71) = (104 - 134) + (15 + 1.5) = (104 - 134) + 16.5 = -30.0 + 16.5 = -13.5$
Exact value : $f(4.71) = 104.487111 - 135.32301 + 15.072 + 1.5 = -14.2638999$
Absolute error: $|-13.5 - (-14.2638999)| = 0.7638999$
Relative error$=0.7638999/14.2638999 = 0.05355$.

b)Rounding off

We shall add $0.0005 * 10^n$
First Term: $3.2 * 4.71 = 15.072$ rounded to 15.1
Second term: $6.1 * (x * x) = 6.1 * (22.1841)$ rounded to $6.1 * 22.2 = 135.42$ rounded to 135
Third Term: $x * x * x = 4.71 * 22.2 = 104.562$ rounded to 105
$f(4.71) = 105 - 135 + 15.1 + 1.5 = -30.0 + 16.6 = -13.4$
Absolute error: $|-13.4 - (-14.2638999)| = 0.8638999$
Relative error$=0.8638999/14.2638999 = 0.06056$
Note: Order of addition or subtraction in the final computation of f will change the answer because of the rounding/chopping