Control of Nonlinear Dynamical Systems

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Outline

1. Lipschitz continuity
2. Flow of a vector field
3. Autonomous systems
4. 2nd order systems
5. Lyapunov theorem

References

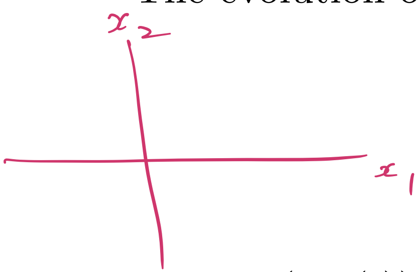
Textbooks

- *Nonlinear Systems*, by H. K. Khalil, Prentice Hall, 2002.
- *Stability of motion*, by Wolfgang Hahn, Springer-Verlag, 1967.
- *Stability by Liapunov's Direct Method* - J. La Salle and S. Lefschetz, Academic Press, 1961.
- *Ordinary Differential Equations* - V. Arnold, Springer, 1992.
- *Elementary Topics in Differential Geometry*, by J. A. Thorpe, Springer, 1979.
- *Dynamic models in Biology* - S. P. Ellner and J. Guckenheimer, Princeton University Press.

Most of the slides for this course are composed from these references.

Dynamical systems as differential equations

- Different paradigms exist to model the behaviour of dynamical systems - two prevalent ones are deterministic differential equations and stochastic differential equations.
- These slides focus on "deterministic ordinary differential equations." The evolution of the systems is taken forward in time - $t \geq 0$.
- A dynamical system at any instant of time t, in our context, is described by a set of variables called states $x_1(t), , x_n(t)$, where n denotes the order of the dynamical system. The state vector is denoted as $x(t)$.
- The evolution of the state vector of the system is given by



$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \rightarrow \frac{dx}{dt} = f(t, x)$$

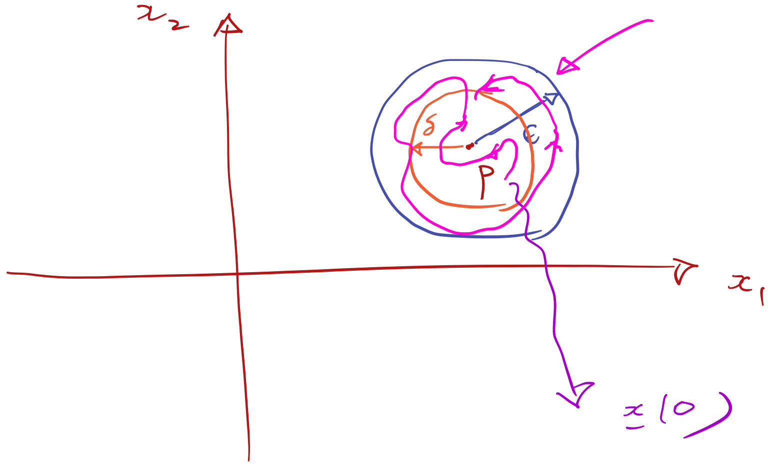
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

where $f(t, x(t))$ is called the vector field describing the dynamical system.

Autonomous and non-autonomous systems

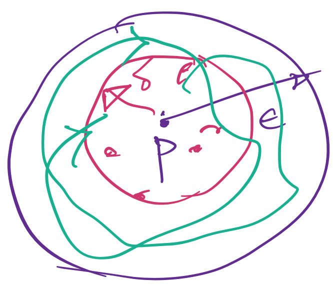
- If the vector field $f(t,x)$ is dependent (independent) on time t, then we have a non-autonomous (autonomous) system.
- For the autonomous system $f(x)$, let $\alpha > 0$, and consider two different initial conditions $x(0) = x_0$ and $x(\alpha) = x_0$ and the corresponding solutions $x(t)$ and $x_\alpha(t)$. Then

$$x_\alpha(t) = x(t-\alpha) \quad \forall \quad t \geq \alpha$$



$\epsilon > 0$

$\delta > 0$



Outline

Local existence and uniqueness of the solution of an ODE

It is of interest whether for an initial condition $x(0) = x_0$, the solution of the ODE exists for a small interval $[0, t_1]$ (however small) and further, if this solution is unique ? A sufficient condition to guarantee this is that the right hand side of the ODE, $f(t, x)$, satisfy a local Lipschitz condition.

Definition

Lipschitz continuity

A function $f(t, x(t))$ is said to be Lipschitz continuous with respect to x at x_0 if there exists a neighbourhood V containing x_0, a time interval $[0, c]$, and a constant $L > 0$ such that

$$\|[f(t, x_1) - f(t, x_2)]\| \leq L \|(x_1 - x_2)\| \quad \forall x_1, x_2 \in V \quad \text{and} \quad \forall t \in [0, c]$$

Lipschitz condition

Definition

Locally Lipschitz on an open set $U \subset \mathbb{R}^n$

Given a neighbourhood $U \subset \mathbb{R}^n$, the function $f(t,x)$ is said to be locally Lipschitz in the set U if for every $x_0 \in U$, there exists a neighbourhood V_{x_0} and a constant L_{x_0} such that

$$\|[f(t,x_1) - f(t,x_2)]\| \leq L_{x_0} \|(x_1 - x_2)\| \quad \forall x_1, x_2 \in V_{x_0} \quad \text{and} \quad \forall t \in [0,c]$$

Definition

Lipschitz on a set $U \subset \mathbb{R}^n$

Given a set $U \subset \mathbb{R}^n$, the function $f(t,x)$ is said to be Lipschitz on the set U if for every $x_0 \in U$, there exists a neighbourhood V_{x_0} and a fixed constant L such that

$$\|[f(t,x_1) - f(t,x_2)]\| \leq L \|(x_1 - x_2)\| \quad \forall x_1, x_2 \in V_{x_0} \quad \text{and} \quad \forall t \in [0,c]$$

Lipschitz condition

Definition

Globally Lipschitz The function $f(t, x)$ is said to be globally Lipschitz if it is Lipschitz on $\mathbb{R}^n$.

Remark

A sufficient condition for the Lipschitz condition to be satisfied is that the function has bounded partial derivatives with respect to the x_is.

Theorem

Suppose $f(t, x)$ is globally Lipschitz on the interval $[0, \infty)$. Then the solution of the ODE

$$\dot{x} = f(t, x) \quad x(0) = x_0$$

exists and is unique for all $t \in [0, \infty)$.

Equlibrium of a dynamical system

Definition

A point x_{eq} is said to be an equilibrium of the dynamical system $\dot{x} = f(t, x)$ if

$$f(t, x_{eq}) = 0 \;\; \forall t \geq 0.$$

Towards a notion of stability

A topic of interest is the following: If the dynamical system is perturbed slightly away from the equilibrium, does the evolving trajectory stay close to the equilibrium x_{eq} for all future time ?

With more refinement, we ask: Is it possible, given a region W around x_{eq} (think of ϵ-balls), to specify a region V around x_{eq} in which the initial conditions must lie (think of δ-balls), such that a trajectory originating in V, stays within the region W for all time ($t \geq 0$) ?

Lyapunov stability

Definition

Lyapunov stability The equilibrium 0 of the dynamical system $\dot{x} = f(t, x)$ is called *stable in the sense of Lyapunov* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|x(t)\| < \epsilon \quad \forall t \geq 0$$

for all $\|x(0)\| < \delta$, where $\|\cdot\|$ is the chosen norm on $\mathbb{R}^n$.

Outline

A linear system

A linear system with an input

$$\dot{x} = Ax + Bu \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in $\mathbb{R}^n$, a vector space.

A linear autonomous system

$$\dot{x} = Ax, \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in $\mathbb{R}^n$, a vector space.

A linear system

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A linear autonomous system

$$\dot{x} = Ax, \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in $\mathbb{R}^n$, a vector space.
The right-hand side of the differential equation is termed a vector field. For the linear system, it is a linear vector field.
Linearity

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$$

Solution and flow

Solution to the autonomous linear system

$$x(t) = e^{At}x_0, \quad x(0) = x_0, \quad e^{At} \triangleq I + A + A^2/2! + \ldots$$

The term $e^{At}x_0$ is termed the flow associated with the linear vector field Ax.

A nonlinear system

A nonlinear system with an input

$$\dot{x} = f(x) + g(x)u \quad x(t) \in M$$

$f(\cdot), g(\cdot)$ are smooth functions, $x(t)$ lives in M, a smooth manifold.

A nonlinear autonomous system

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in M, a smooth manifold.

A nonlinear system

A nonlinear system with an input

$$\dot{x} = f(x) + g(x)u \quad x(t) \in M$$

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A nonlinear autonomous system

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in M, a smooth manifold.
The right-hand side of the differential equation is a nonlinear vector field.

A nonlinear system

A nonlinear system with an input

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A nonlinear autonomous system

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in M, a smooth manifold.
The right-hand side of the differential equation is a nonlinear vector field.
Linearity does not hold.

$$f(\alpha_1 x_1 + \alpha_2 x_2) \neq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

Solution and flow

Solution to the autonomous nonlinear system

$$x(t) = \Phi(t, x_0) \quad x(0) = x_0$$

The term $\Phi(t, x_0)$ is termed the flow associated with the nonlinear vector field $f(x)$.

Flow of a vector field

Flow of a vector field $X(x)$

The flow of the vector field $X(x)$, denoted by $\Phi(t, x_0)$, is a mapping from $(-a, a) \times U \to \mathbb{R}^n$ (where $a(> 0) \in \mathbb{R}$ and U is an open region in the state-space) and satisfies the differential equation

$$\frac{d\Phi(t, x_0)}{dt} = X(\Phi(t, x_0)) \quad \forall t \in (-a, a), x(0) = x_0 \in U.$$

over the interval $(-a, a)$ and with initial conditions starting in the region U.

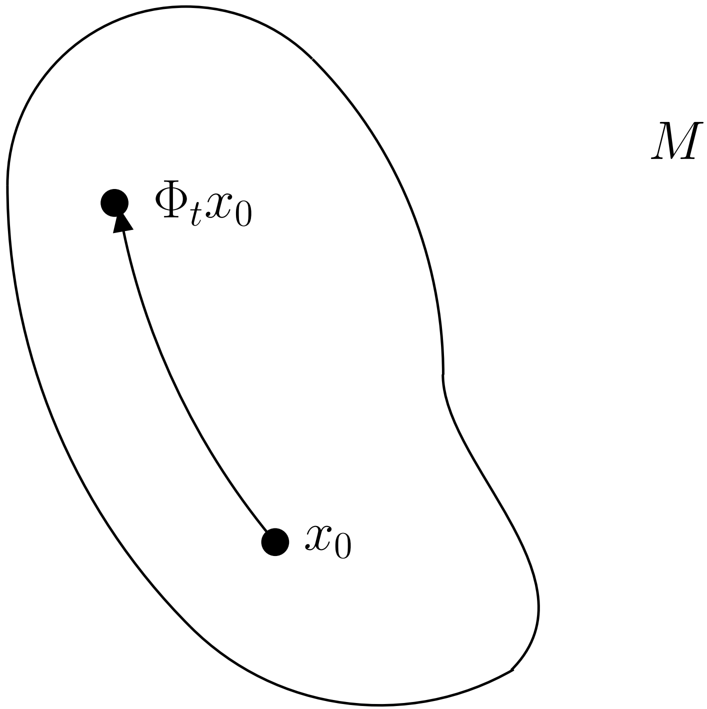

Figure: Flow of a vector field

Properties of flows

The group structure

Denote

$$\Phi_t(x_0) \triangleq \Phi(t, x_0)$$

The set of transformations $\{\Phi_t\} : U \to \mathbb{R}^n$ satisfies the following properties.

- $\Phi_{t+s}x_0 = \Phi_t \circ \Phi_s x_0 \quad \forall t, s, t+s \in (-a, a)$ (the group binary operation.)
- $\Phi_0 x_0 = x_0$ (the group identity.)
- For a fixed $t \in (-a, a)$ we have $\Phi_t \Phi_{-t} x_0 = x_0 \Rightarrow [\Phi_t]^{-1} = \Phi_{-t}$ (existence of an inverse.)

The flow of a linear system

The group property

Remark

The three properties mentioned above impart a group structure to the set $\{\Phi_t\}$. This set is called a one-parameter (time) group of diffeomorphisms (Φ_t and its inverse are smooth mappings).

The flow of a linear system

The group property

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Linear flow

Remark

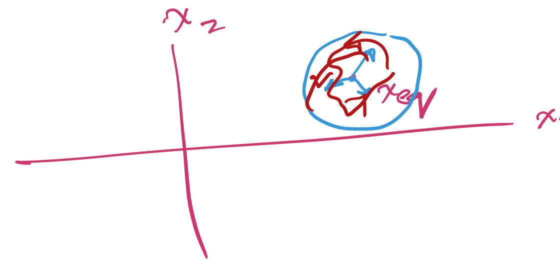
For a linear system described by

$$\dot{x} = Ax \qquad A \in \mathbb{R}^{n\times n}$$

the flow $\Phi_t x_0 = e^{At}x_0$ where $\{e^{At} : t \in (-\infty, \infty)\}$ constitutes the one-parameter group of diffeomorphisms.

Outline

Equilibria



Equilibrium point

Given an autonomous system of the form

$$\dot{x} = f(x) \tag{1}$$

$\dot{x}\Big|_{x_{eq}} = 0$

a state x_{eq} that satisfies $f(x_{eq}) = 0$ is called an equilibrium state (point) of the dynamical system.

Lyapunov stability

An equilibrium point p of (1) is said to be *Lyapunov stable* if for every $\epsilon > 0$ there exist a corresponding $\delta > 0$ such that
$x(0) \in \mathcal{B}_\delta(p) \Rightarrow x(t) \in \mathcal{B}_\epsilon(p) \forall t \geq 0$.

Asymptotic stability

Invariant sets

Definition

(*Invariant set*) A set $M \subset U$ is said to be invariant with respect to the flow of f if

$$x(s) \in M \Rightarrow x(t+s) \in M \quad \forall t \in (-\infty, \infty)(\text{where } s \in \mathbb{R} \text{ is fixed})$$

Definition

(*Positively (negatively) invariant set*) A set $M \subset U$ is said to be positively (negatively) invariant with respect to the flow of f if

$$x(s) \in M \Rightarrow x(t+s) \in M \quad \forall t \in (0, \infty)(\forall t \in (-\infty, 0))$$

Definition

(*Distance to a set*) The distance of a point $p \in U$ to a set $M \subset U$ is defined as

$$\text{dist}(p, M) = \inf_{x \in M} \|(p - x)\|$$

Limit sets

Definition

(Positive limit set) Given a trajectory (solution) $\Phi(t, x_0)$ of the system (1), a point p is said to be a positive limit point of the trajectory $x(t)$ if there exists an increasing sequence of times $\{t_k\}$ with $t_k \to \infty$ (as $k \to \infty$) satisfying

$$\lim_{t_k \to \infty} \Phi(t_k, x_0) = p$$

The set of all positive limit points of a trajectory $\Phi(t, x_0)$ forms the positive limit set of *the trajectory* $\Phi(t, x_0)$.

Remark

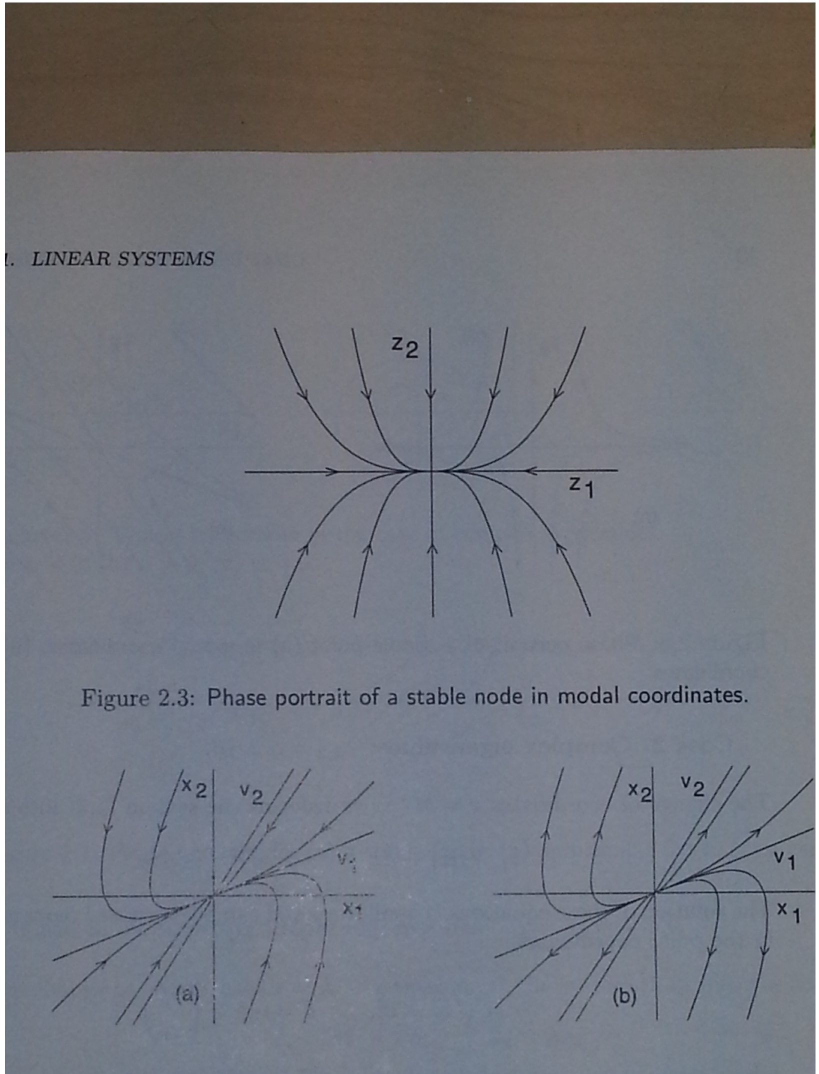
Take the example of spring-mass system (with and without damper) or a LC circuit (with and without a resistor) and comprehend these new definitions and notions.

Theorem

If a trajectory $\Phi(t, x_0)$ of (1) is bounded and belongs to U for all $t \geq 0$, then its positive limit set L is non-empty, compact and invariant and $\Phi(t, x_0)$ approaches L as $t \to \infty$.

Outline

Stable node


LINEAR SYSTEMS
z_2
z_1
Figure 2.3: Phase portrait of a stable node in modal coordinates.
x_2
v_2
v_1
x_1
(a)
(b)

Saddle point

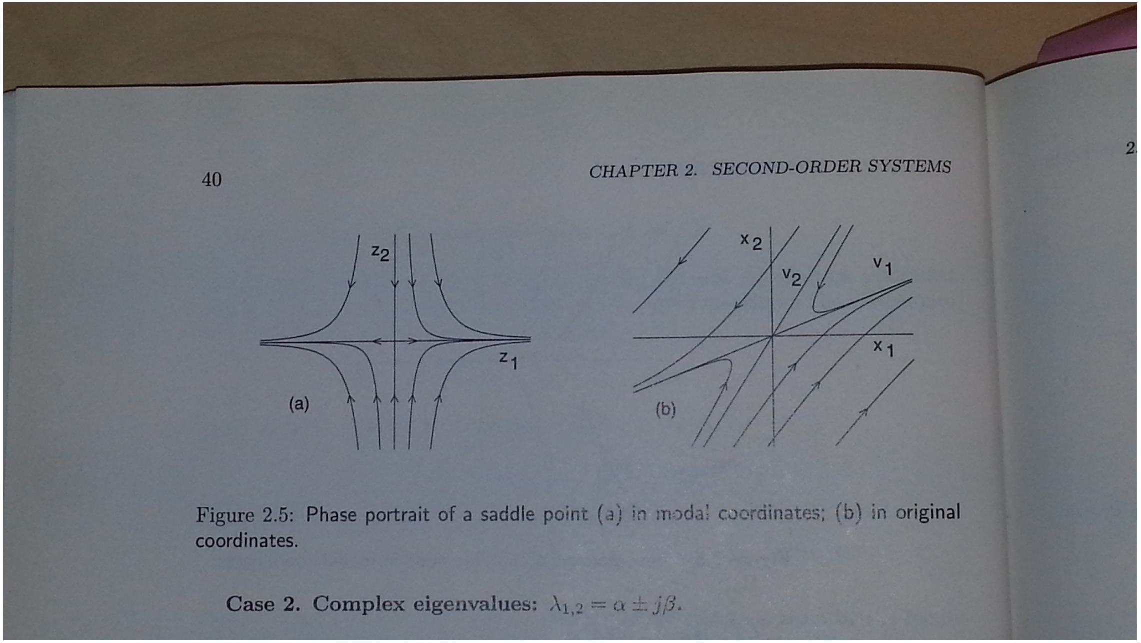

40 CHAPTER 2. SECOND-ORDER SYSTEMS

Figure 2.5: Phase portrait of a saddle point (a) in modal coordinates; (b) in original coordinates.

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$.

Figure: Saddle point

Focus and center

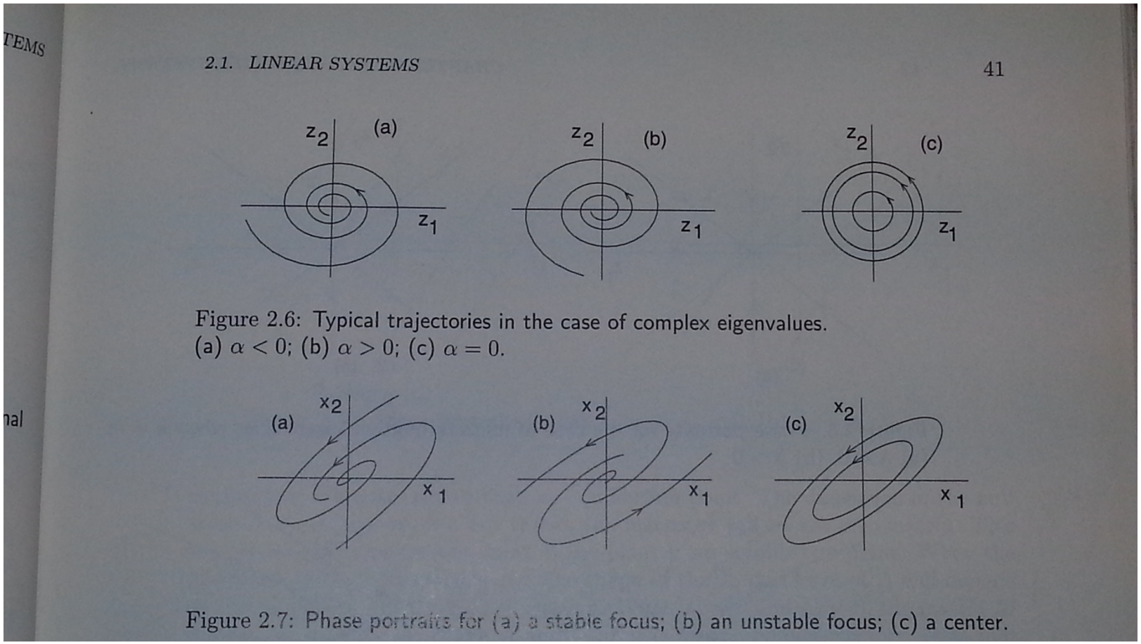


Figure: Focus and center

Repeated eigen values

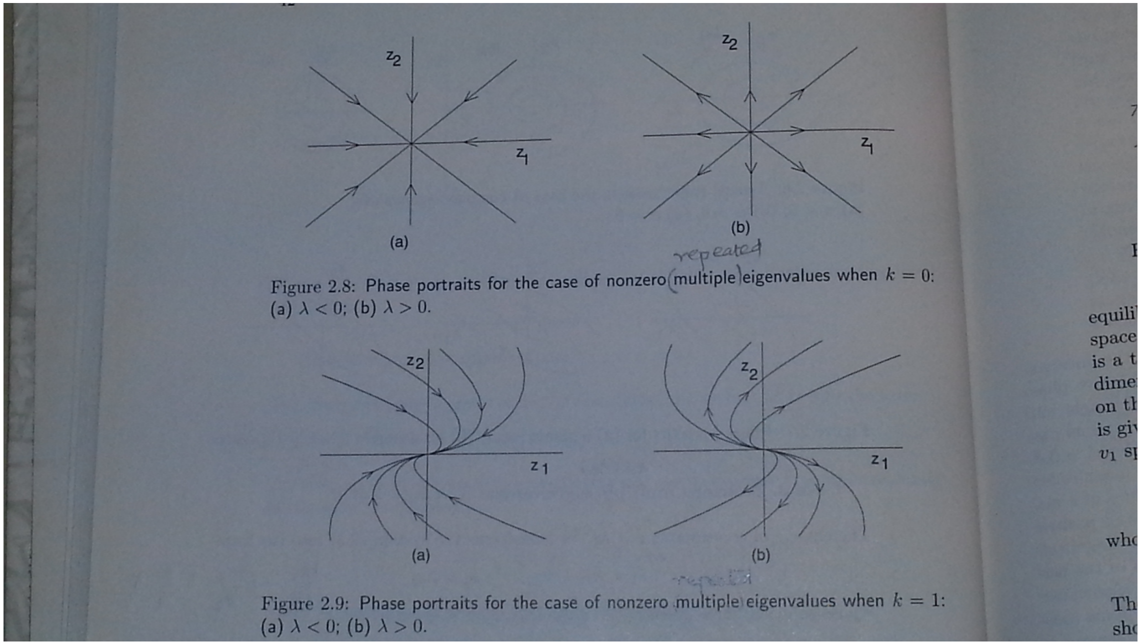


Figure: Repeated eigen values

One zero eigen value

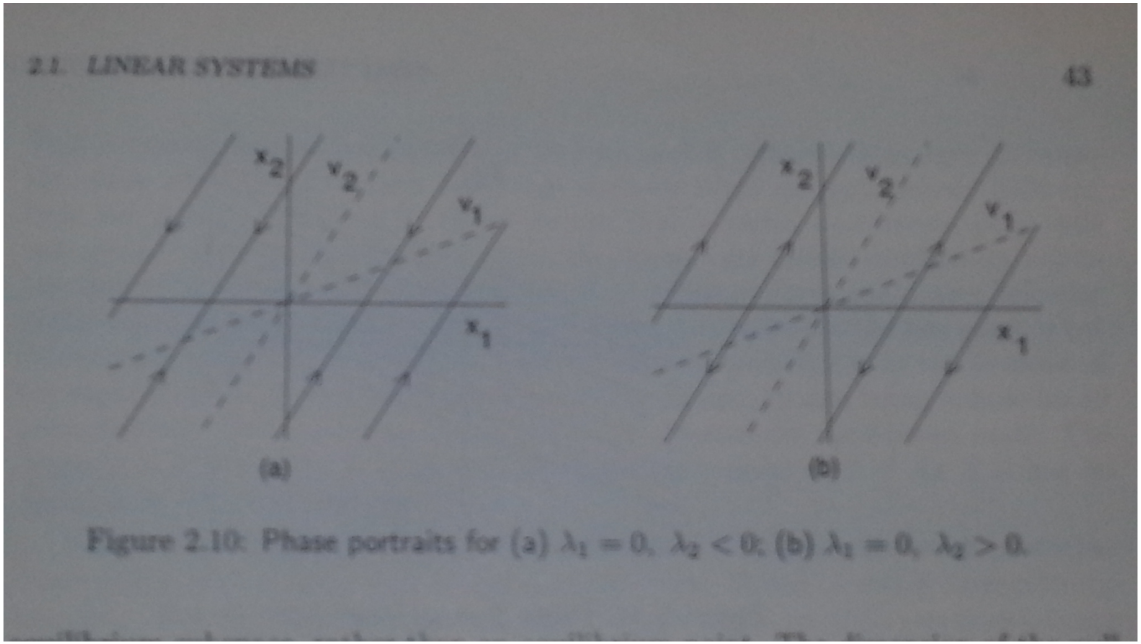
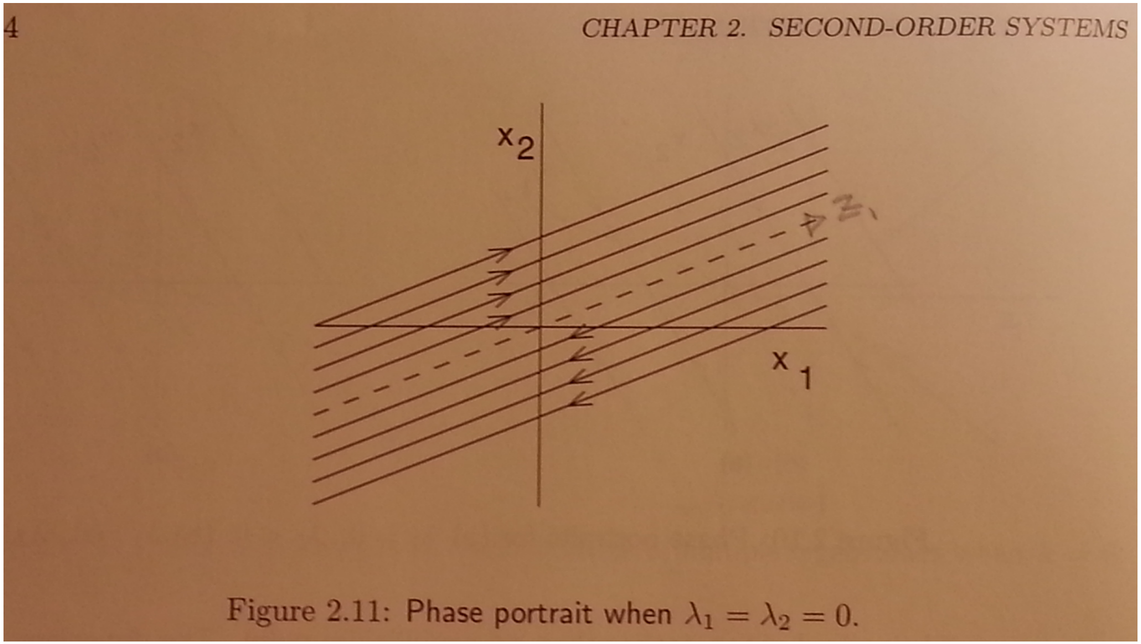


Figure: One zero eigen value

Both eigen values being zero



4 *CHAPTER 2. SECOND-ORDER SYSTEMS*

Figure 2.11: Phase portrait when $\lambda_1 = \lambda_2 = 0$.

Figure: Both eigen values being zero

Limit cycle

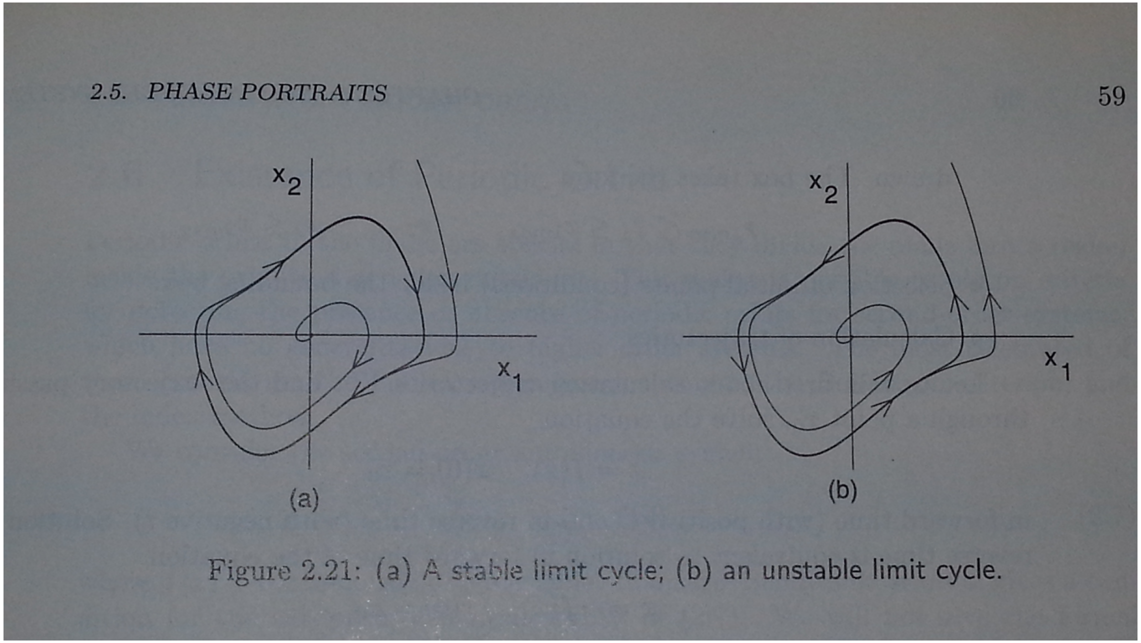


Figure: Limit cycle

Outline

Lyapunov theorem

Theorem

(Lyapunov theorem) Suppose $x = 0$ is an equilibrium point of (1). Let $V(\cdot) : U \to \mathbb{R}$ be a continuously differentiable function that satisfies

$$V(x) > 0 \quad \forall x \in U(x \neq 0) \quad \textit{and} \quad V(0) = 0$$

and

$$\frac{dV}{dt} \leq 0 \quad \forall x \in U$$

Then the equilibrium 0 is Lyapunov stable. Further, if $\frac{dV}{dt} < 0 \quad \forall x \in U(x \neq 0)$, then the equilibrium 0 is asymptotically stable

Casimir

$V(\cdot) + C(\cdot)$

La Salle's Invariance Principle

Theorem

Let $\Omega \subset U$ be a compact and positively invariant set. Let $V : U \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let

$$E = \{x \in \Omega : \dot{V}(x) = 0\}$$

Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Remark

Note that La Salle's invariance result does not impose any positive definite requirement on V.

Lyapunov's first theorem on instability

Theorem

(Lyapunov's first theorem on instability) Suppose $x = 0$ *is an equilibrium point of (1). Let* $V(\cdot) : U \to \mathbb{R}$ *be a continuously differentiable function that satisfies*

$$V(0) = 0 \quad \textit{and} \quad \frac{dV}{dt} > 0 \;\; \forall x \in U$$

and further, V *is able to assume positive values arbitrarily close to the origin. Then the origin is an unstable equilibrium point.*

Lyapunov's second theorem on instability

Theorem

(Lyapunov's second theorem on instability) Suppose $x = 0$ is an equilibrium point of (1). Let $V(\cdot) : U \to \mathbb{R}$ be a continuously differentiable function that satisfies

$$V(0) = 0 \quad \textit{and} \quad \frac{dV}{dt} = \lambda V + W \quad \forall x \in U$$

where $\lambda > 0$ and $W(\cdot) \geq 0$ in U. Then the origin is an unstable equilibrium point.

Asymptotic stability

The system described by (1) is said to be asymptotically stable if it is Lyapunov stable and

$$\lim_{t \to \infty} \|(x(t) - x_{eq})\| = 0 \quad \forall x(0) \in \mathcal{B}_\delta(x_{eq})$$

Exponential stability

The system described by (1) is said to be exponentially stable at x_{eq} if it is asymptotically stable and there exist positive constants c_1, c_2 such that

$$\|(x(t) - x_{eq})\| \leq c_1 e^{-c_2 t} \|(x(0) - x_{eq})\| \quad \forall t \geq 0 \text{ and } \forall x(0) \in \mathcal{B}_\delta(x_{eq})$$

Global asymptotic stability

The system described by (1) is said to be globally asymptotically stable at x_{eq} if it is Lyapunov stable and

$$\lim_{t\to\infty} \|(x(t) - x_{eq})\| = 0 \quad \forall x(0) \in \mathbb{R}^n$$

Global exponential stability

The system described by (1) is said to be exponentially stable at x_{eq} if it is asymptotically stable and there exist positive constants c_1, c_2 such that

$$\|(x(t) - x_{eq})\| \leq c_1 e^{-c_2 t} \|(x(0) - x_{eq})\| \quad \forall t \geq 0 \ \text{ and } \ \forall x(0) \in \mathbb{R}^n$$