SC 618: An introduction to manifolds

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Outline

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Engineering applications

- Articulated mechanisms, rolling mechanisms, robotic manipulators and underwater vehicles - configurations spaces are not necessarily the real line OR copies of the real line OR Euclidean space.
- Issues: Develop kinematic level and dynamic level modelling framework in non-Euclidean settings. Backdrop for path planning, feedback control and stabilization.
- Other reasons: Interconnected mechanisms and more complex mechanisms- satellite with tethers, mobile robots with internal driving mechanisms - need for a better theoretical framework.
- All these issues motivate better mathematical tools. We explore the geometric framework OR differential geometry

A few problems

- The acrobot

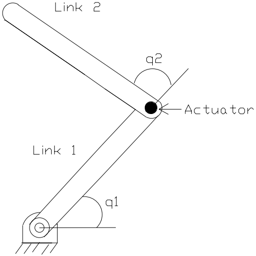

Figure: Two link manipulator

Wheeled mobile robots

- Wheeled mobile robots, fingers handling an object (rolling contact)
- Objective - move from one configuration to another
- Constraints - pure rolling (no sliding or slipping)

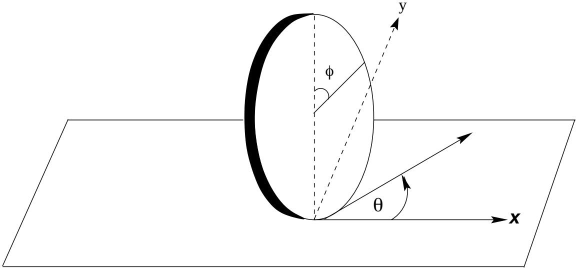

Figure: Vertical coin on a plane

Underwater vehicles

- *Kinematics*

$$\begin{aligned}\dot{x} &= v_x \cos\theta - v_y \sin\theta \\ \dot{y} &= v_x \sin\theta + v_y \cos\theta \\ \dot{\theta} &= \omega_z\end{aligned}$$

- *Dynamics*

$$\begin{aligned}m_{11}\dot{v}_x - m_{22}v_y\omega_z + d_{11}v_x &= F_x \\ m_{22}\dot{v}_y + m_{11}v_x\omega_z + d_{22}v_y &= 0 \\ m_{33}\dot{\omega}_z + (m_{22} - m_{11})v_x v_y + d_{33}\omega_z &= \tau_z\end{aligned}$$

Classification of constraints in mechanical systems

Holonomic constraints

Restrict the allowable configurations of the system

Nonintegrable constraints

Do not restrict the allowable configurations of the system but restrict instantaneous velocities/accelerations

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Velocity level constraints - parking of a car, wheeled mobile robots, rolling contacts in robotic applications.

Classification of constraints in mechanical systems

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Do not restrict the allowable configurations of the system but restrict instantaneous velocities/accelerations
Velocity level constraints - parking of a car, wheeled mobile robots, rolling contacts in robotic applications.
Acceleration level - fuel slosh in spacecrafts/launch vehicles, underwater vehicles, underactuated mechanisms (on purpose or loss of actuator) systems - serial link manipulators.

Velocity level constraints

- *Rolling coin* : Configuration variables

$$\boldsymbol{q} = (x, y, \theta, \phi) \in \mathcal{M} = \mathbb{R}^1 \times \mathbb{R}^1 \times S^1 \times S^1$$

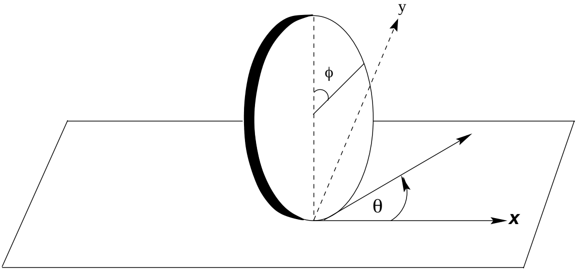


Figure: Rolling coin

- Constraints - No slip and no sliding

Velocity level constraints

- Constraints of motion are expressed as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \qquad \text{No lateral motion}$$

$$\dot{x}\cos\theta + \dot{y}\sin\theta = r\dot{\phi} \qquad \text{Pure rolling}$$

- Constraints in a matrix form

$$\begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -r \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0$$

Bead in a slot

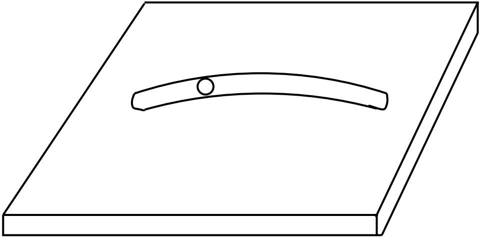


Figure: A bead in a slot

- Configuration variables of the bead are (x, y)
- Restriction to stay in the slot gives rise to an algebraic constraint

$$g(x, y) = 0$$

- g is the equation of the curve describing the slot
- Slot restricts the possible configurations that the bead can assume

Acceleration level constraints

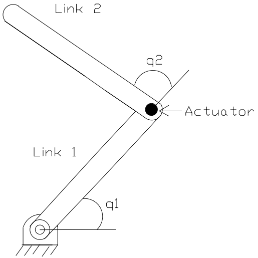


Figure: A two-link manipulator

- Two link manipulator moving in a vertical plane with one actuator at its second joint (acrobot)

Acrobot

-

$$
\begin{aligned}
d_{11}(q)\ddot{q}_1 + d_{12}(q)\ddot{q}_2 + h_1(q,\dot{q}) + \psi_1(q) &= 0 \\
d_{21}(q)\ddot{q}_1 + d_{22}(q)\ddot{q}_2 + h_2(q,\dot{q}) + \psi_2(q) &= \tau_2
\end{aligned}
\tag{1}
$$

- First equation (with the right hand side being zero) denotes the lack of actuation at the first joint
- The acrobot can assume any configuration but cannot assume arbitrary accelerations.

Fuel slosh in a launch vehicle

- A launch vehicle with liquid fuel in its tank. The motion of the fluid and the outer rigid body are coupled.
- Unactuated pivoted pendulum model. The motion of the pendulum is solely affected by the motion of the outer rigid body

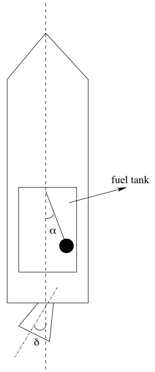


Figure: Fuel slosh phenomenon

- The equations of motion are of the form

$$\begin{aligned} \ddot{q}_u + f_1(q, \dot{q}, \ddot{q}) &= 0 \\ \ddot{q}_a + f_2(q, \dot{q}, \ddot{q}) &= F \end{aligned} \tag{2}$$

  where $q \triangleq (q_u, q_a)$, q_u corresponds to the configuration variable of the pendulum, q_a corresponds to the configuration variable of the outer rigid body and F is the external force.
- While the acceleration level constraint in the acrobot arises due to purpose of design or loss of actuation, in the case of the launch vehicle it is the inability to directly actuate the fluid dynamics.

Mechanics, geometry and control

The thumb experiment

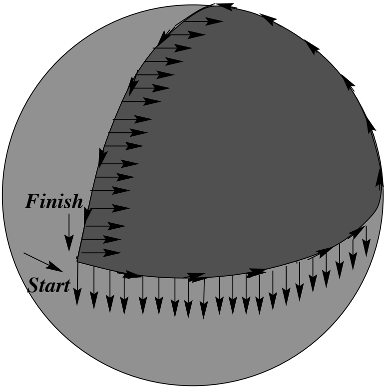


Figure: Motion on a sphere (Courtesy: Marsden and Ostrowski)

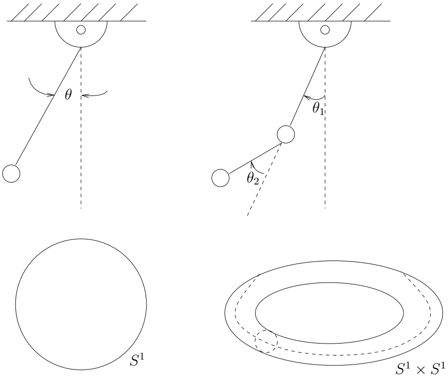
Mechanics, geometry and control

- Phenomenon similar to the thumb experiment is also common to a large class of systems which are termed blue.
- Variables describing the motion of these systems can be classified into two sets called the blueshape variables and the group variables.
- Cyclic motion in the shape variables produces motion in the group (fiber) variables. Biological systems (fishes, snakes, paramecia) and robotic mechanisms, the falling cat, the steering of a car, the motion of underactuated systems of linkages.
- The net changes in position due to changes in the shape variables is explicable either due to an interaction with the environment or some conservation law.

Geometry

- Provides better insight, better framework for solution (control law) and an elegant setting for these problems
- Essential tools - Lie groups and differentiable manifolds

Dynamical systems and geometry



A glimpse of differential geometry

- Familiar with calculus (differentiation, integration) on the real line ($\mathbb{R}$) or $\mathbb{R}^n$. Extend this calculus to surfaces - say on a circle, a sphere, a torus and more complicated surfaces
- But why ? From a dynamical systems and control theory point of view, systems need not always evolve on the real line ($\mathbb{R}$) or multiple real lines ($\mathbb{R}^n$). They may evolve on *non-Euclidean* surfaces. On such surfaces, we wish to talk of "rate of change (velocity) $\Rightarrow$ differentiation, to talk of "rate of rate of change (acceleration), to talk of accumulation $\Rightarrow$ integration
- Can these surfaces which are *not Euclidean* be *locally represented* as Euclidean ? This would allow us to employ the calculus that we are familiar with for these surfaces

Outline

Level Sets in $\mathbb{R}^{n+1}$

- Consider a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and consider the set

$$S = \{x \in \mathbb{R}^{n+1} : x = f^{-1}(c)\} \quad \text{for some fixed} \quad c \in \mathbb{R}$$

- Examples: a circle - $x_1^2 + x_2^2 = c(c \neq 0)$is a 1-surface $\in \mathbb{R}^2$, a sphere $x_1^2 + x_2^2 + x_3^2 = c(c \neq 0)$ is a 2-surface $\in \mathbb{R}^3$
- The gradient of f is defined as

$$\bigtriangledown f(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

- We restrict our study to those level sets such that

$$\bigtriangledown f(x) \neq 0 \quad \forall x \in S$$

We shall also call these level sets as n-surfaces.

Vector field

- A vector field $\mathbf{X}$ on $U \subset \mathbb{R}^{n+1}$ is an assignment of a vector at each point of U
- Our interest centres on *smooth vector fields* - here the assignment is in terms of a smooth function
- *Gradient vector field* - It is the smooth vector field associated with each smooth function $f : U \to \mathbb{R}^{n+1}$, and is given by

$$(\nabla f)(p) = (p, \frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_{n+1}}(p))$$

A parametrized curve in $\mathbb{R}^{n+1}$ and an integral curve

- A *parametrized curve* in $\mathbb{R}^{n+1}$ is smooth mapping from an open interval of the real-line ($\mathbb{R}$) to $\mathbb{R}^{n+1}$
- $\alpha(\cdot) : (a, b) \to \mathbb{R}^{n+1}$ is a parametrized curve implies that each of the $\alpha_i(\cdot)$s ($i = 1, \ldots, n+1$) is a smooth function on (a, b)
- Associated with a parametrized curve is the notion of a *velocity vector*. At any point p on the parametrized curve, the associated velocity vector is
$$v_\alpha(p) \triangleq \dot{\alpha}(t)|_{t=p}$$
- A parametrized curve $\alpha(\cdot)$ on an open interval (α, β) is said to be the integral curve of a smooth vector field $\mathbf{X}$ if
$$v_\alpha(p) = \mathbf{X}(p) \quad \forall p \in (a, b)$$

Integral curves and solutions of differential equations

- In engineering we are often interested in solving a system of equations of the following form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_{n+1}) \\ &\vdots \\ \dot{x}_{n+1} &= f_{n+1}(x_1, \ldots, x_{n+1}) \end{aligned}$$

  where the f_is are smooth functions on an open interval (a, b) and with an initial condition as $x(p) = (x_1(p), \ldots, x_{n+1}(p))$. Let us now try to understand what does it mean *in a geometric sense* to seek a solution of this equation.

- A solution to this set of equations describes an integral curve of the vector field $f(\cdot)$.

Parametrized curves on a level set

- Consider a smooth curve $\alpha(\cdot) : I \to S$ where $0 \in I \subset \mathbb{R}$, S is a level set of a smooth function f and let $\alpha(0) = p$. Then

$$v_p \triangleq \frac{d(f \circ \alpha)(t)}{dt}|_{t=0}$$

  is a tangent vector to the level set S at p.
- The set of all tangent vectors at a point p to a level set S forms a vector space and is denoted by S_p
- It can be shown that the S_p is orthogonal to $\bigtriangledown f(p)$

Tangent vector field

- A tangent vector field on a surface S is the assignment of a tangent vector at each point $p \in S$. If this assignment is smooth, we have a smooth tangent vector field on the surface
- Associated with a surface S are two smooth unit normal vector fields defined as

$$\frac{\bigtriangledown f(p)}{\|\bigtriangledown f(p)\|} \quad \text{and} \quad -\frac{\bigtriangledown f(p)}{\|\bigtriangledown f(p)\|}$$

Smooth functions: $f : X \subset \mathbb{R}^m \to Y \subset \mathbb{R}^n$

- Differentiable (or smooth functions) help us to characterize the derivative maps.
- From linear algebra, recall that the rank of an $n \times m$ matrix A is defined in 3 equivalent ways:
  1. The dimension of the subspace $V(\subset \mathbb{R}^m)$ spanned by the n rows of A.
  2. The dimension of the subspace $W(\subset \mathbb{R}^n)$ spanned by the m columns of A.
  3. The maximum order of any non vanishing minor determinant.
- The rank of $Df(x)$ (the the $n \times m$ Jacobian) is referred to the rank of f at x. And since f is smooth, and the value of the determinant is a continuous function of its entries, the rank remains constant in some neighborhood of x.
- Diffeomorphisms have non-singular Jacobians.
- So rank of f is the same as the rank of $h \circ f$, where h is a diffeomorphism.

Submersion

Let $f : X \subset \mathbb{R}^m \to Y \subset \mathbb{R}^n$ be smooth. Then

- If the Jacobian $Df(p)$ is *onto* for all $p \in X$, then f is called a submersion. (this definition makes sense only when $n \leq m$.)

Submersion

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- Note that Jacobian maps vectors from $\mathbb{R}^m$ to $\mathbb{R}^n$, and in coordinates is expressed as

$$[Df(p)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}_{x=p}$$

Immersion

Let $f : X \subset \mathbb{R}^m \to Y \subset \mathbb{R}^n$ be smooth, Then

- If the Jacobian $Df(p)$ is *one-to-one (injective)* for all $p \in X$, then f is called a immersion. (this definition makes sense only in the case when $n \geq m$.)

Immersion

Let $f : X \subset \mathbb{R}^m \to Y \subset \mathbb{R}^n$ be smooth, Then

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- Once again the Jacobian maps vectors from $\mathbb{R}^m$ to $\mathbb{R}^n$, and in coordinates is expressed as

$$[Df(p)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_p} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}_{x=p}$$

Outline

1. Motivation
2. Mathematical preliminaries
3. Submanifolds of $\mathbb{R}^n$
4. Optional material

A submanifold of $\mathbb{R}^n$

Three equivalent definitions

Each of these definitions is equivalent to the other and makes $M(\subset \mathbb{R}^n)$ an m-dimensional submanifold of $\mathbb{R}^n$.

- For every $x \in M \subset \mathbb{R}^n$, there exists a neighbourhood U containing x and a smooth submersion $f : U \to \mathbb{R}^{n-m}$ such that $U \cap M$ is a level set of f.
- For every $x \in M \subset \mathbb{R}^n$, there exists a neighbourhood U containing x and a smooth function $g : V \subset \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that $U \cap M$ is a graph of g.
- For every $x \in M \subset \mathbb{R}^n$, there exists a neighbourhood U containing x and an smooth embedding $\psi : V \subset \mathbb{R}^m \to \mathbb{R}^n$ such that $\psi(V) = M \cap U$.

Examples seen : $S^1, O(2), SO(3), O(n), S^n$

S^1

The unit circle - S^1

- Consider the function $f(x, y) = x^2 + y^2 : U \to \mathbb{R}^1$. Note that f is a smooth submersion on U and $U \cap S^1$ is a level set of f.
- Consider the function $g(x) = \sqrt{1 - x^2}$ and note that $U \cap S^1 = (x, g(x))$ - the graph of g.
- Consider the function $\psi(x) = (x, \sqrt{1 - x^2})$ in a neighbourhood $V (\subset R^1)$ which satisfies $U \cap S^1 = \psi(V)$, is an immersion and $\psi^{-1}(a, b) = a$ is continuous in $U \cap S^1$.

$O(2)$

2×2 real, orthogonal matrices - $O(2)$

$$O(2) = \{A \in \mathbb{R}^{2\times 2} | A^T A = I\}$$

- Parameterize A as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a^2 + c^2 = 1, ab + cd, = 0, b^2 + d^2 = 1.$
- Let $f \triangleq (f_1, f_2, f_3)$ where

$$f_1(\cdot) = a^2 + c^2, \quad f_2(\cdot) = ab + cd, \quad f_3(\cdot) = b^2 + d^2, \quad U \to \mathbb{R}^1$$

This is a smooth submersion and $U \cap O(2)$ is a level set of f.

$$U \cap O(2) = f^{-1}(1, 0, 1)$$

$O(2)$ (contd.)

- The function f_1 allows us to characterize the 4-tuple as $(a, b, c, g_1(\cdot))$. Similarly, the next two functions further reduce this characterization to the form $(a, g_3(\cdot), g_2(\cdot), g_1(\cdot))$. Consider the function $g(x) \triangleq (g_1, g_2, g_3)$ and note that the set $U \cap O(2) = (x, g(x))$ - the graph of g.
- Consider the function $\psi(x) = (x, g(x))$ in a neighbourhood $V(\subset \mathbb{R}^1)$ which satisfies $U \cap O(2) = \psi(V)$, is an immersion and $\psi^{-1}(a, b, c, d) = a$ is continuous in $U \cap O(2)$.

Immersed submanifold

Let $f : X \subset \mathbb{R}^m \to Y \subset \mathbb{R}^n$ be smooth, Then

If the Jacobian $Df(p)$ is *one-to-one (injective)* for all $p \in X$, and f is one-to-one on X then $f(X)$ is called an immersed submanifold. (once again, this definition makes sense only in the case when $n \geq m$.)

Examples

- Helix $f : \mathbb{R} \to \mathbb{R}^3, f(t) = (\cos(2\pi t), \sin(2\pi t), t)$.
- Circle $f : \mathbb{R} \to \mathbb{R}^2, f(t) = (\cos(2\pi t), \sin(2\pi t))$.
- Spiralling curve $f : (1, \infty) \to \mathbb{R}^2, f(t) = (\frac{1}{t}\cos(2\pi t), \frac{1}{t}\sin(2\pi t))$.
- Spirals to a circle $f : (1, \infty) \to \mathbb{R}^2, f(t) = (\frac{t+1}{2t}\cos(2\pi t), \frac{t+1}{2t}\sin(2\pi t))$.
- A sleeping figure eight
  $f : (1, \infty) \to \mathbb{R}^2, f(t) = (2\cos(t - 0.5\pi), \sin 2(t - 0.5\pi))$.

Coordinates for a submanifold

- Choosing a basis for $\mathbb{R}^m$, one could assign coordinates to these n-tuples
- So we introduce the notion of coordinate functions $x \triangleq (x^1, \ldots, x^m)$.
- The ith coordinate of a point p on the manifold is $x^i(p) : M \to R$
- Coordinate function $x^i : U \to R$
- We could choose another set of coordinates as well - say y

Change of coordinates

- How do we move from one coordinate system to another ?
- Consider a smooth function $f(\cdot) : M \to \mathbb{R}$ and consider two coordinate systems x and y.
$$f \circ x^{-1} = f \circ y^{-1} \circ y \circ x^{-1}$$
- Then the partial derivative of $f(\cdot)$ with respect to x^j (the jth partial in the x coordinate system) is given by
$$D_j(f \circ x^{-1}) = D_j(f \circ y^{-1} \circ y \circ x^{-1}) = D_k(f \circ y^{-1}) \cdot D_j^k(y \circ x^{-1})$$
- Note
$$f \circ y^{-1} : \mathbb{R}^n \to \mathbb{R} \qquad\qquad y \circ x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

Example 1: $\mathbb{R}^2$

- Think of the Cartesian plane and two coordinate systems (x, y) and (r, θ).
- Now $x = r\cos\theta$ and $y = r\sin\theta$
- Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ with the chain rule mentioned in the last slide.

Example 2: Sphere -S^2

Sphere

Choose spherical coordinates as $(\theta, \phi) \in ((-\pi, \pi), (0, 2\pi))$ or

Example 2: Sphere -S^2

Sphere

Choose spherical coordinates as $(\theta, \phi) \in ((-\pi, \pi), (0, 2\pi))$ or

Cartesian as
$(x, y, z = \pm\sqrt{1-(x^2+y^2)} = (\cos(\theta)\sin(\phi), \cos(\theta)\cos(\phi), \sin(\theta))$.

Example 3: Cylinder - $S^1 \times \mathbb{R}^1$

Cylinder

Choose cylindrical coordinates as $(\theta, z) \in ((-\pi, \pi), \mathbb{R})$ or

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Cartesian as $(x, \pm\sqrt{1-x^2}, z) = (\cos(\theta), \sin(\theta), z)$.

Tangent (velocity)vectors

- The two tuple (p, v_p) constitutes a tangent vector at $p \in M$, where $c : \mathbb{R} \supset (-a, a) \to M$ is any curve that satisfies

$$c(0) = p \qquad\qquad v_p = c'(0)$$

- The set of all tangent vectors at p is called the tangent space at p. This is a vector space and is denoted by T_pM.
- For an m-dimensional manifold M, the tangent space at each point has dimension m.

Coordinates for the tangent space

Coordinates for $M \cap U$

Let $x^1, \ldots, x^m$ be coordinates for $M \cap U$.

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Then draw curves of the form

$$\alpha_i : I \rightarrow M \cap U \quad \alpha_i(t) = (p_1, \ldots, p_i + t, \ldots, p_m) \quad i = 1, \ldots, m$$

Coordinates for T_pM

Now define

$$\frac{\partial}{\partial x^i}|_p \triangleq \frac{d\alpha_i}{dt}|_{t=0} = (0, \ldots, 1, \ldots, 0)$$

Coordinates for the tangent space

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$\frac{\partial}{\partial x^i}|_p$ defines a tangent vector at $\alpha_i(0) = p$.

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Coordinates for T_pM

Now define

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$\frac{\partial}{\partial x^i}|_p$ defines a tangent vector at $\alpha_i(0) = p$.
The set of tangent vectors

$$\{\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^m}|_p\}$$

forms a basis for the tangent space T_pM at p.

Tangent bundle

- The disjoint union of all tangent spaces to a manifold forms the tangent bundle.

$$TM = \bigcup_{p \in X} T_p M$$

- TM has a smooth manifold structure (an atlas for M induces an atlas for TM.)

Optional

- Call $\pi : TM \to M$ the canonical projection defined by

$$v_p \to p \qquad\qquad \forall v_p \in T_p X$$

- If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for X, then $\{(\pi^{-1}(U_\alpha), D\phi_\alpha)\}$is an atlas for TX.

Vector fields

- A vector field on a smooth manifold M is a map $\mathbf{X}$ that assigns to each $p \in M$ a tangent vector $\mathbf{X}(p)$ in $T_p(M)$.
- If this assignment is smooth, the vector field is called smooth or $\mathbf{C}^\infty$.

Optional

- The collection of all C^∞ vector fields on a manifold M denoted by $\mathcal{X}(M)$ is endowed with an algebraic structure as follows: Let $\mathbf{V}, \mathbf{W} \in \mathcal{X}(X)$, $a \in R$ and $f \in C^\infty(M)$
  - $\mathbf{V} + \mathbf{W}(p) = \mathbf{V}(p) + \mathbf{W}(p)$ $(\in \mathcal{X}(M))$
  - $a(\mathbf{V})(p) = a\mathbf{V}(p)$ $(\in \mathcal{X}(M))$
  - $(f\mathbf{V})(p) = f(p)\mathbf{V}(p)$ $(\in \mathcal{X}(M))$

The cotangent space

Definition

The dual space of $T_p(M)$ is called the cotangent space of X at p and denoted by $T_p^*(M)$

Differentials

For any $f \in \mathbf{C}^\infty(p)$ we define an operator $df(p) = df_p : T_p(M) \to \mathbb{R}$ called the differential of f at p by

$$df(p)(\mathbf{v}) = df_p(\mathbf{v}) = \mathbf{v}(f)$$

for every $\mathbf{v} \in T_p(M)$. Since df_p is linear, it is an element of the dual space of $T_p(M)$.

Basis and dual basis

Based on coordinates $x^1, \ldots, x^m$, the basis for the tangent space at p (T_pM) is $\{\frac{\partial}{\partial x^i}\}$ and the dual basis at p (T_p^*M) is $\{dx^i\}$

Cotangent bundle

- The disjoint union of all cotangent spaces to a manifold forms the cotangent bundle.

$$T^*M = \bigcup_{p \in X} T_p^* M$$

- T^*M has a smooth manifold structure (an atlas for X induces an atlas for T^*M.)

Optional

- Call $\pi : T^*M \to M$ the canonical projection defined by

$$v_p \to p \qquad \forall v_p \in T_p^* M$$

- If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(\pi^{-1}(U_\alpha), (D\phi)^{-1})\}$is an atlas for T^*M.

Casting a mechanical system in the manifold setting

- *Rolling wheel or coin* : Configuration variables evolve on a smooth manifold

$$\boldsymbol{q} = (x, y, \theta, \phi) \in \mathcal{M} = \mathbb{R}^1 \times \mathbb{R}^1 \times S^1 \times S^1$$

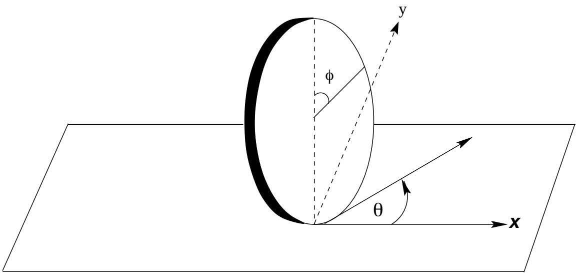

Figure: Rolling coin

- Constraints - No slip and no sliding

Velocity level constraints

- Constraints of motion are expressed as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \qquad \text{No lateral motion}$$

$$\dot{x}\cos\theta + \dot{y}\sin\theta = r\dot{\phi} \qquad \text{Pure rolling}$$

- Constraints in a matrix form

$$\begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -r \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0$$

Manifold language

- $p = (x_0, y_0, \theta_0, \phi_0)$.

$$\begin{bmatrix} \sin(\theta_0) & -\cos(\theta_0) & 0 & 0 \end{bmatrix} \in T_p^* M$$

$$\begin{bmatrix} \cos\theta_0 & \sin\theta_0 & 0 & -r \end{bmatrix} \in T_p^* M$$

$$\begin{bmatrix} v_x \\ v_y \\ v_\theta \\ v_\phi \end{bmatrix} \in T_p M$$

- This is just kinematics.
- The vector field describing the motion of the coin sits in TM.

Outline

1. Motivation

2. Mathematical preliminaries

3. Submanifolds of $\mathbb{R}^n$

4. Optional material

Two important results

The inverse function theorem

Theorem

Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ that is continuously differentiable in a region U around a point $a \in \mathbb{R}^n$, and whose derivative is non-singular at a, there exists an open set (or a neighbourhood) V containing $f(a)$ in which the function f has an inverse f^{-1} which is differentiable and further this inverse is given by

$$(Df^{-1})(y) = [Df(f^{-1}(y))]^{-1} \quad \forall y \in V$$

The implicit function theorem

Theorem

Given a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ *such that* $f(a, b) = 0$ *and* f *is continuously differentiable in a region* $U \times V$ *around* (a, b)*, and further*

$$D^i_{j+n} f(a, b) \qquad 1 \leq i, j \leq m \qquad \textit{is non-singular}.$$

Then there exists a neighbourhood $W \subset U$ *containing* a *and a neighbourhood* $P \subset V$ *containing* b*, and a differentiable function* $g(\cdot) : W \to P$ *such that*

$$f(x, g(x)) = 0 \quad \forall x \in W$$

Derivations: an alternate viewpoint to tangent vectors

- A tangent vector $\mathbf{v}$ to a point p on a surface will assign to every smooth real-valued function f on the surface a "directional derivative" $\mathbf{v}(f) = \bigtriangledown f(p) \cdot \mathbf{v}$.
- Draw a curve $\alpha(.) : M = \mathbf{R} \rightarrow \mathbf{R}$ on the plane. Let $\alpha \in \mathbf{C}^{\infty}$. Consider $\frac{d\alpha(t)}{dt}|_{t=p}$. What is this expression ? It takes a curve $\alpha(\in \mathbf{C}^{\infty})$ to the reals at a point p.
- A tangent vector (derivation) to a differentiable manifold M at a point p is a real-valued function $\mathbf{v}$: $C^{\infty} \rightarrow R$ that satisfies
  - (**Linearity**) $\mathbf{v}(af + bg) =$ a $\mathbf{v}$(f) + b $\mathbf{v}$(g)
  - (**Leibnitz Product Rule**) $\mathbf{v}(fg) = f(p)\mathbf{v}(g) + \mathbf{v}(f)g(p)$ for all $f, g \in \mathbf{C}^{\infty}(X)$ and all $a, b \in R$
- A vector field $\mathbf{X}$ on a differentiable manifold M is a linear operator that maps smooth functions to smooth functions. Mathematically

$$\mathcal{X} : \mathbf{C}^{\infty}(X) \rightarrow \mathbf{C}^{\infty}(X)$$

Maps between manifolds and "the linear approximation"

Consider a smooth map $f : M \to N$. At each $p \in M$ we define a linear transformation

$$f_{*p} : T_p(M) \to T_{f(p)}(N)$$

called the derivative of f at p, which is intended to serve as a "linear approximation to f near p," defined as

- For each $\mathbf{v} \in T_p(M)$ we define $f_{*p}(\mathbf{v})$ to be the operator on $\mathbf{C}^\infty(f(p))$ defined by $(f_{*p}(\mathbf{v}))(g) = \mathbf{v}(g \circ f)$ for all $g \in \mathbf{C}^\infty(f(p))$.

Lemma

(Chain Rule) Let $f : M \to N$ and $g : N \to P$ be smooth maps between differentiable manifolds. Then $g \circ f : M \to P$ is smooth and for every $p \in X$

$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$$

An alternate viewpoint of maps between manifolds

The derivative of f at p could be constructed as follows. Once again $f(\cdot) : M \to N$. (The notation f_{*p} is now changed to $T_p f$.)

- Choose a parametrized curve $c(\cdot) : (-\epsilon, \epsilon) \to X$ with $c(0) = p$ and $\dot{c}(0) = v_p$.
- Construct the curve $f \circ c$. Then define

$$T_p f \cdot v_p = \frac{d}{dt}|_{t=0}(f \circ c)(t)$$

- With coordinates, we have

$$T_{x(p)}(y \circ f \circ x^{-1}) = \frac{\partial}{\partial x^j}(y \circ f \circ x^{-1})|_{x(p)} \qquad \text{The Jacobian at } x(p)$$

- The rank of f at p is the rank of the Jacobian matrix at $x(p)$ and *this is independent of the choice of the charts*

Critical points and regular points

Consider a smooth map $f : M \to N$

- A point $p \in M$ is called a critical point of f if $T_p f$ is not onto.
- A point $p \in M$ is called a regular point of f if $T_p f$ is onto.
- A point $y \in N$ is called a *critical value* if $f^{-1}(y)$ contains a critical point. Otherwise, y is called a regular value of f.

Submersion theorem

Theorem

If $f : X^n \to Y^k$ is a smooth map, and $y \in f(X) \subset Y$ is a regular value of f, then $f^{-1}(y)$ is a regular submanifold of X of dimension $n - k$.

- Example: $f(x, y) = x^2 + y^2 - 1$. Take $f^{-1}(0)$.
- Show that $O(n) = \{A \in M_n(\mathbb{R}) | A^T A = I\}$ is a submanifold of $M_n(\mathbb{R})$. What is its dimension ?
  Hint: Consider a map $f : \mathbb{R}^{n\times n} \to S^{n\times n}$ (symmetric matrices). Let $f(A) = A^T A$ and examine the value I.