AE 240/240M/713 | *Quiz No. 2* | *Thursday, March 25, 2021*

Weightage: 10% | *Time: 0835 – 0920 Hrs.* | *Venue: Moodle*

Examination is open notes. Only calculator allowed. | *Model Solutions*

Q.1 A two-stage rocket is required to achieve a V_* of 10000 m/s. Further, I and II stage use ε of 0.1 and 0.2 respectively. However, 'I_{sp}' of both the stages is 350s. Determine π_i's of each stage, using the approximate constrained optimization method for maximizing π_* and also obtain its value. (Hint: use constraint to express π_1 in terms of π_2. $g_0 = 9.81$ m/s^2). (4)

$$V_* = 10000 m/s; \quad \varepsilon_1 = 0.1; \quad \varepsilon_2 = 0.2; \quad I_{sp1} = I_{sp2} = 350s$$

$$\ln\left[0.1+0.9\pi_1\right] = -\frac{10000}{350\times 9.81} - \ln\left[0.2+0.8\pi_2\right]$$

$$\ln\left[(0.2+0.8\pi_2)(0.1+0.9\pi_1)\right] = -2.9125 \rightarrow (0.2+0.8\pi_2)(0.1+0.9\pi_1) = 0.0543$$

$$\pi_1 = \frac{0.0604}{(0.2+0.8\pi_2)} - 0.1111; \quad \pi_* = \left\{\frac{0.0604}{(0.2+0.8\pi_2)} - 0.1111\right\}\times\pi_2$$

$$\frac{\partial\pi_*}{\partial\pi_2} = \frac{0.0604}{(0.2+0.8\pi_2)} - 0.1111 - \pi_2\times\frac{0.0483}{(0.2+0.8\pi_2)^2} = 0$$

$$0.0604(0.2+0.8\pi_2) - 0.1111(0.2+0.8\pi_2)^2 - 0.0483\pi_2 = 0$$

$$-0.0711\pi_2^2 - 0.03555\pi_2 + 0.00764 = 0$$

$$\pi_2 = 0.162; \quad \pi_1 = \cancel{0.153}\ 0.0721; \quad \pi_* = \cancel{0.025}\ 0.01168$$

Q.2 Consider a two-stage rocket with the following configuration.

$$\frac{m_{p1}}{m_{p1}+m_{s1}} = \frac{m_{p2}}{m_{p2}+m_{s2}} = 0.92; \quad m_{s1} = 4450kg; \quad I_{sp1} = 253s; \quad m_{s2} = 451kg; \quad I_{sp2} = 280s; \quad m_* = 2065kg$$

If the second stage fuel I_{sp} is changed to 350s, how much additional payload can be carried, assuming ideal burnout velocity to be constant? (Hint: all stage-wise mass values remain the same. Further, solutions, based on I_{sp} related derivative or through any other consistent methodology, in the context of question, are also admissible, as long as solution is close to the correct value.). (3)

$$\frac{\delta m_*}{\delta m_{p2}}\Big|_{dV_*=0} = -\frac{\left(\partial V_*/\partial m_{p2}\right)}{\left(\partial V_*/\partial m_*\right)} = -\frac{I_{sp1}\left(\frac{1}{m_{01}}-\frac{1}{m_{f1}}\right)+I_{sp2}\left(\frac{1}{m_{02}}\right)}{I_{sp1}\left(\frac{1}{m_{01}}-\frac{1}{m_{f1}}\right)+I_{sp2}\left(\frac{1}{m_{02}}-\frac{1}{m_{f2}}\right)}; \quad m_{p1} = 51175kg; \quad m_{p2} = 5186.5kg$$

$$m_{01} = 63327.5kg; \quad m_{02} = 7702.5kg; \quad m_{f1} = 12150.5kg; \quad m_{f2} = 2516kg$$

$$\frac{\delta m_*}{\delta m_{p2}}\Big|_{dV_*=0\, I_{sp2}=280} = -\frac{253(0.000016-0.000082)+280(0.00013)}{253(0.000016-0.000082)+280(0.00013-0.000397)} = \frac{0.0197}{0.0912} = 0.216$$

$$\frac{\delta m_*}{\delta m_{p2}}\Big|_{dV_*=0\, I_{sp2}=350} = -\frac{253(0.000016-0.000082)+350(0.00013)}{253(0.000016-0.000082)+350(0.00013-0.000397)} = \frac{0.0223}{0.0945} = 0.261$$

$$\delta m_* = (0.261-0.216)\times 5186.5 = 235.8kg$$

Q.3 Ariane rockets are capable of directly launching a small satellite in GPS constellation (i.e. circular orbit at 20,000 km altitude above Earth's surface). In a particular mission, however, the rocket underperforms by 5% on its desired injection velocity, though the injection altitude and inclinations are as desired. What kind of orbit can be expected for the satellite under these conditions? Give orbit parameters and comment on the apogee and perigee altitudes. (R_E = 6,378 km; $\mu = 3.986 \times 10^{14}$ m^3/s^2, g_0 = 9.81 m/s^2). (3)

$$V_{desired} = \sqrt{\frac{\mu}{R_E + h}} = \sqrt{\frac{3.986\times10^{14}}{26.371\times10^{6}}} = 3887.8 m/s$$

$$V_{actual} = 0.95\times V_{desired} = 0.95\times3887.8 = 3693.4 m/s$$

$$h = rV_{actual} = 26.378\times10^{6}\times3693.4 = 9.74245\times10^{10}$$

$$\varepsilon = \frac{1}{2}V_{actual}^{2} - \frac{\mu}{r} = \frac{1}{2}\times3693.4^{2} - \frac{3.986\times10^{14}}{26.378\times10^{6}} = -8.29048\times10^{6}$$

$$a = -\frac{\mu}{2\varepsilon} = \frac{3.986\times10^{14}}{2\times8.29048\times10^{6}} = 24.0396\times10^{6}$$

$$e = \sqrt{1+\frac{2\varepsilon h^{2}}{\mu^{2}}} = \sqrt{1-\frac{2\times8.29048\times10^{6}\times\left(9.74245\times10^{10}\right)^{2}}{\left(3.986\times10^{14}\right)^{2}}} = 0.097$$

Orbit is significantly elliptic with following perigee and apogee altitudes.

$$r_p = a(1-e) = 24.0396\times10^{6}\times0.903 = 21.7077\times10^{6} \rightarrow h_p = 15330 km$$

$$r_a = a(1+e) = 24.0396\times10^{6}\times1.097 = 26.3714\times10^{6} \rightarrow h_a = 19993 km$$

We see that injection point becomes the apogee, while perigee is at a significantly lower altitude.

PAPER ENDS