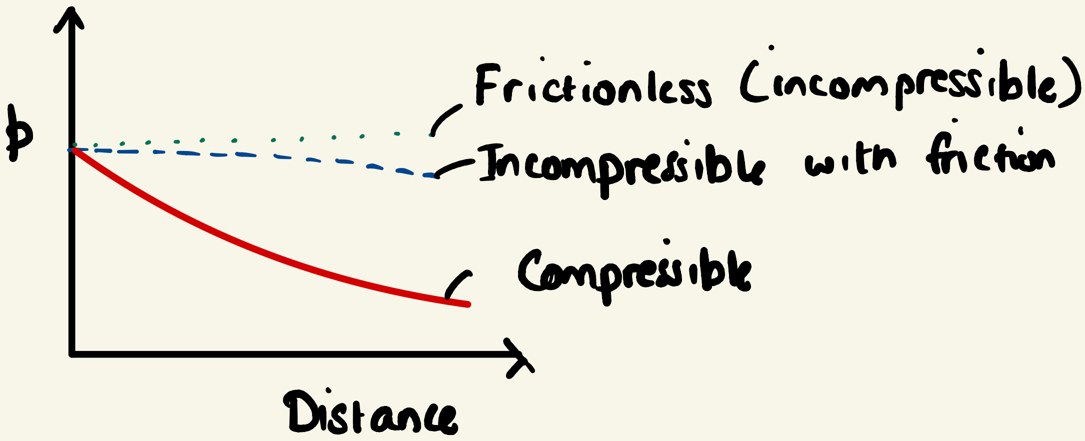
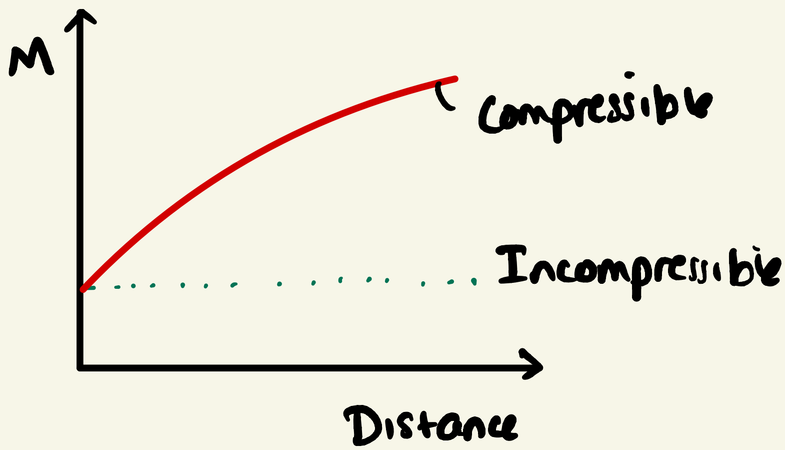
AE 236: Compressible Fluid Mechanics

(Module V : i. Adiabatic Flow with Friction)
ii Flow with heat transfer

Prof. Vineeth Nair
Dept. Aerospace Engg.
IIT Bombay

Observations





$\boxed{\text{Assumptions}}$

1. Flow is adiabatic (no heat transfer)
2. Steady
3. Constant area of cross-section

Compressible, adiabatic flow in a constant area duct with friction – FANNO FLOW

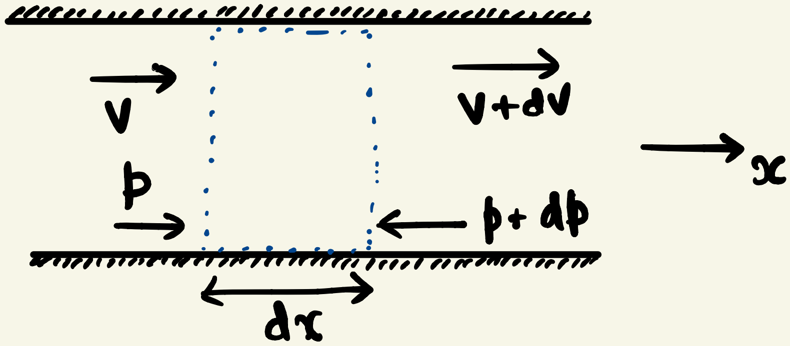
Mass flow rate (velocity out – velocity in)

= Net pressure forces – force due to wall shear stress

Let τ_w – shear stress
P – perimeter

Momentum

$$pA - (p+dp)A$$
$$- \tau_w . P . dx$$
$$= \rho V A (V + dV - V)$$



$$\Rightarrow -dp - \tau_w \frac{P}{A} dx = \rho V dV$$

For a circular duct, $P = \pi D$, $A = \frac{\pi D^2}{4}$

$$\frac{P}{A} = \frac{4}{D}$$

For non-circular ducts, define an equivalent diameter

hydraulic diameter

$$D_H = \frac{4A}{P}$$

$$-\frac{dp}{\rho V^2} - \frac{\tau_w}{\rho V^2} \frac{P}{A} dx = \frac{dV}{V} \quad — \quad (1)$$

Mass

$$\rho V A = \text{constant}$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \quad — \quad (2)$$

Energy

$$C_p T + \frac{V^2}{2} = \text{constant}$$

$$C_p dT + V dV = 0 \quad — \quad (3)$$

State

$$p = \rho R T$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \qquad — ④$$

Mach number relationship

$$M^2 = \frac{V^2}{\gamma R T}$$

$$\frac{dM}{M} = \frac{dV}{V} - \frac{1}{2}\frac{dT}{T} \qquad — ⑤$$

Unknowns dM, dV, dp, dT and $d\rho$

Eqn. ③ can be written as

$$\frac{dT}{T} + \frac{V\,dV}{C_p T} = 0 \qquad — ⑥$$

Eqn. ⑤ $\times V^2$

$$V\,dV = \frac{V^2}{2}\frac{dT}{T} + \frac{V^2}{M}\,dM \qquad V = Ma$$

$$+ M a^2 dM$$

$$V\,dV = \frac{V^2}{2}\frac{dT}{T} + M\gamma R T\,dM$$

Substitute in ⑥

$$\frac{dT}{T} + \frac{\gamma R}{C_p} M\,dM + \frac{V^2}{2C_p T}\frac{dT}{T} = 0 \qquad C_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{dT}{T} + (\gamma - 1) M\,dM + \frac{\gamma - 1}{2} M^2 \frac{dT}{T} = 0$$

$$\frac{dT}{T} = -\frac{(\gamma-1)M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M} \quad - ⑦$$

From eqns ③ & ④

$$\frac{dp}{p} = -\frac{dV}{V} + \frac{dT}{T}$$

Using eqn. ⑤

$$\frac{dp}{p} = -\frac{dM}{M} + \frac{1}{2}\frac{dT}{T}$$

Using eqn. ⑦

$$\frac{dp}{p} = -\left[1 + \frac{(\gamma-1)M^2/2}{1+\frac{\gamma-1}{2}M^2}\right]\frac{dM}{M} \quad - ⑧$$

$$-\frac{dp}{\rho V^2} - \frac{\tau_w}{\rho V^2}\frac{P}{A}dx = \frac{dV}{V} \qquad a^2 = \frac{\gamma p}{\rho} = \frac{V^2}{M^2}$$

$$\frac{dV}{V} + \frac{\tau_w}{\rho V^2}\frac{P}{A}dx + \frac{1}{\gamma M^2}\frac{dp}{p} = 0$$

Using eqn. ⑤

$$\frac{dM}{M} + \frac{1}{2}\frac{dT}{T} + \frac{1}{\gamma M^2}\frac{dp}{p} + \frac{\tau_w}{\rho V^2}\frac{P}{A}dx = 0$$

$$\boxed{\frac{dM}{M} = \frac{\gamma M^2\left[1+\frac{\gamma-1}{2}M^2\right]}{(1-M^2)}\left[\frac{\tau_w}{\rho V^2}\frac{P}{A}dx\right] \quad - (*)}$$

Substitute (4) in Eqns. (7) & (8)

$$\boxed{\frac{dT}{T} = -\frac{\gamma(\gamma-1)M^4}{(1-M^2)}\left[\frac{\tau_w}{\rho V^2}\frac{P}{A}\,dx\right]} \quad -(**)$$

$$\boxed{\frac{dp}{p} = -\frac{\gamma M^2\left[1+(\gamma-1)M^2\right]}{(1-M^2)}\left[\frac{\tau_w}{\rho V^2}\frac{P}{A}\,dx\right]} \quad -(***)$$

The variation depends on $1-M^2$

$M<1 \Rightarrow dM>0,\ dT<0,\ dp<0$

$M>1 \Rightarrow dM<0,\ dT>0,\ dp>0$

There is a tendency for the flow to move towards $M=1$ – FRICTION CHOKING.

$$ds = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\frac{dp}{p}$$

$$\Rightarrow \boxed{\frac{ds}{c_p} = (\gamma-1)M^2\left[\frac{\tau_w}{\rho V^2}\frac{P}{A}\,dx\right]} \quad (****)$$

$ds>0$ always

	dM	dV	dp	dT	ds
$M<1$	+	+	–	–	+
$M>1$	–	–	+	+	+

Dimensionless wall shear stress

$$\boxed{f = \frac{\tau_w}{\frac{1}{2}\rho v^2}}$$ FANNING FRICTION FACTOR

$$\boxed{f_D = 4f}$$ DARCY FRICTION FACTOR

f = function $(Re, \epsilon/D_H, \underset{\text{weak function}}{M})$

ϵ = measure of the mean height of wall roughness.

D_H = hydraulic diameter

Eqn ⊛ becomes

$$\frac{4f\,dx}{D_H} = \frac{2(1-M^2)}{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

Integrating over a duct of length l.

$$\frac{4\bar{f}}{D_H}l = \int_{M_1}^{M_2}\frac{2(1-M^2)}{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}\frac{dM}{M}$$

$\bar{f}$ = Mean friction factor

$$\frac{4\bar{f}}{D_H}l = \frac{1}{\gamma}\left(\frac{1}{M_1^2}-\frac{1}{M_2^2}\right)+\frac{\gamma+1}{2\gamma}\ln\frac{M_1^2\left(1+\frac{\gamma-1}{2}M_2^2\right)}{M_2^2\left(1+\frac{\gamma-1}{2}M_1^2\right)}$$

Reference state * corresponding to $M=1$ at the end of the pipe (p^*, T^*, p_0^*, l^*)

l^* = length of the duct required to give $M = 1$ at the exit

$$\frac{4f}{D_H} l^* = \left(\frac{1-M^2}{\gamma M^2}\right) + \frac{\gamma+1}{2\gamma} \ln \frac{(\gamma+1)M^2}{2\left(1+\frac{\gamma-1}{2}M^2\right)} \quad — (A)$$

For pressure

$$\int_{p}^{p^*} \frac{dp}{p} = \int_{M}^{1} \left[1 + \frac{\frac{\gamma-1}{2}M^2/2}{1+\frac{\gamma-1}{2}M^2}\right] \frac{dM}{M}$$

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{(\gamma+1)/2}{1+\frac{\gamma+1}{2}M^2}\right]^{1/2} \quad — (B)$$

For temperature

$$\int_{T}^{T^*} \frac{dT}{T} = - \int_{M}^{1} \frac{(\gamma-1)M}{1+\frac{\gamma-1}{2}M^2}\, dM$$

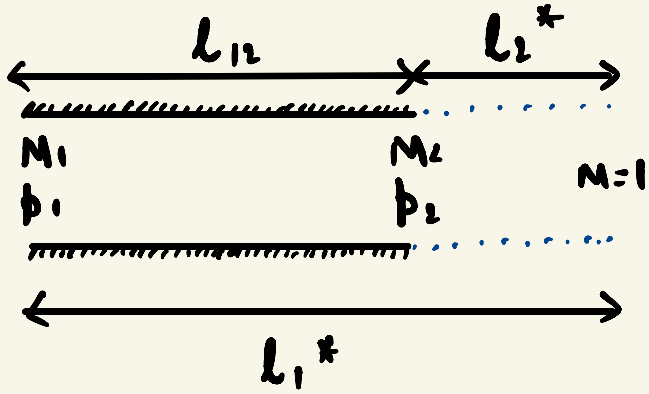
$$\frac{T}{T^*} = \frac{(\gamma+1)/2}{1+\frac{\gamma-1}{2}M^2} \quad — (C)$$

For p_0

$$\frac{p_0}{p} = \left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}} \Rightarrow \frac{p_0}{p_0^*} = \frac{p_0}{p} \cdot \frac{p^*}{p_0^*} \cdot \frac{p}{p^*}$$

$$\frac{p_0}{p_0^*} = \frac{1}{M}\left[\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad — (D)$$

A, B, C, D tabulated in friction tables for various M



Length of the pipe

$$\boxed{l_{12} = l_1^* - l_2^*}$$

Also, $p_2 = \dfrac{p_2/p^*}{p_1/p^*} \cdot p_1$

Problems

① Air flow in ϕ 5cm pipe. $M_1 = 2.5$, $M_2 = 1.5$
$\bar{f} = 0.002$, adiabatic flow. Find pipe length.

From tables at M_1 and M_2

$$\frac{4fl_1^*}{D} = 0.432 \qquad \frac{4fl_2^*}{D} = 0.136$$

$$\frac{4fl_{12}}{D} = 0.296 \Rightarrow l_{12} = 1.85\,m$$

To achieve $M=1$ at exit $l = l_1^* = 2.7m$

② Air flow: ϕ 0.3m, $Q_e = 1000 m^3/min$, $p_2 = 150 kPa$
$T_2 = 293K$, $l_{12} = 50m$, $f = 0.005$. Find M_2, p_1, T_1

$$V_e = \frac{Q_e}{A_e} = \frac{1000/60}{\pi \cdot 0.3^2/4} = 236\,m/s$$

$$a_2 = \sqrt{\gamma RT} = 343\,m/s$$

$$M_2 = \frac{V_e}{a_2} = 0.688$$

$$\frac{4\bar{f}l_2^*}{D} = 0.228 \qquad \frac{p_2}{p^*} = 1.54\ , \quad \frac{T_2}{T^*} = 1.10$$

$$\frac{4\bar{f}l_1^*}{D} = \frac{4\bar{f}l_{12}}{D} + \frac{4\bar{f}l_2^*}{D} = 3.6$$

$$\Rightarrow \quad M_1 = 0.346\ , \quad p_1/p^* = 3.14\ . \quad T_1/T^* = 1.17$$

$$p_1 = \frac{p_1/p^*}{p_2/p^*} \cdot p_2 = 306\ kPa$$

$$T_1 = \frac{T_1/T^*}{T_2/T^*} \cdot T_2 = 312\ K = 39^\circ C$$

The Fanno line

Locus of all possible states in the adiabatic flow (T_0 fixed) of a calorically perfect gas on a T-s diagram

$$\frac{ds}{C_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\frac{dp}{p}$$

$$\frac{dp}{p} = \frac{dT}{T} - \frac{dV}{V}$$

$$\frac{dT}{T} = -\frac{V\,dV}{C_p T}$$

Combining

$$\frac{ds}{C_p} = \frac{dT}{T} - \frac{\gamma-1}{\gamma}\frac{dT}{T}\left(1 + \frac{C_p T}{V^2}\right)$$

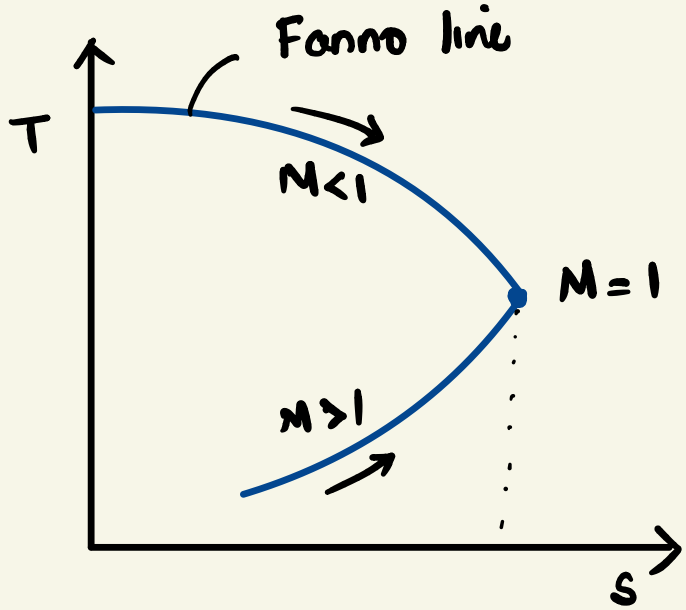
$$V^2 = 2C_p(T_0 - T)$$

$$\boxed{\frac{ds}{c_p} = \frac{1}{\gamma}\frac{dT}{T} - \frac{\gamma-1}{2\gamma}\frac{dT}{T_0 - T}} \quad \text{— (9)}$$

Integrate (9) from an arbitrary value s_1 to s

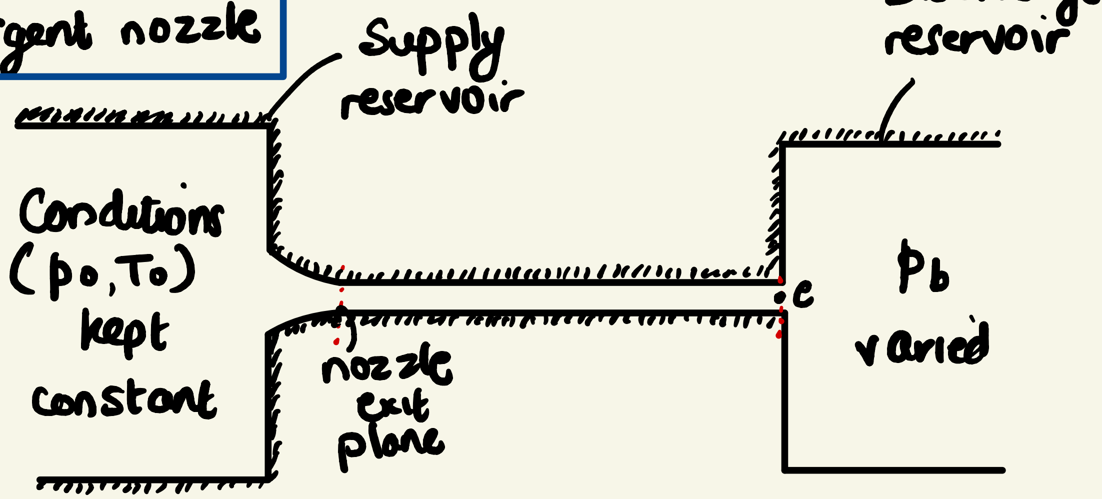
$$\boxed{\frac{s - s_1}{c_p} = \ln\left[\left(\frac{T}{T_1}\right)^{\frac{1}{\gamma}}\left(\frac{T_0 - T}{T_0 - T_1}\right)^{\frac{\gamma-1}{2\gamma}}\right]} \quad \text{— (***)}$$

FANNO LINE EQUATION



Frictional flow in a duct preceded by an isentropic nozzle

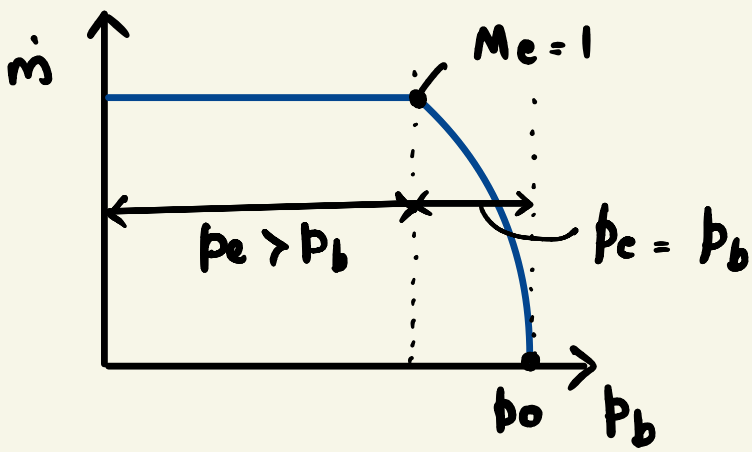
Convergent nozzle



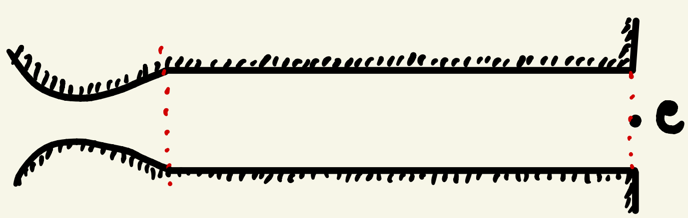
$p_b = p_0 \Rightarrow$ No flow

As $p_b \downarrow$, $\dot{m}$, $M \uparrow$ until $M_e = 1$. Once $M_e = 1$, flow becomes choked (friction choking) and further changes in p_b will not affect flow inside duct.

If p_b reduced further, correction from p_e to p_b happens through expansion waves outside the exit



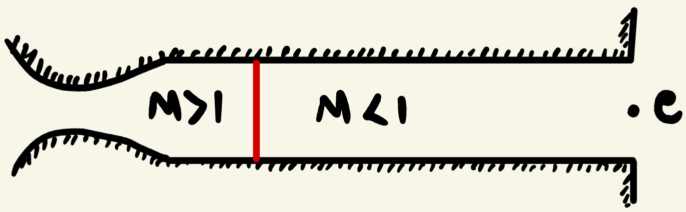
$\boxed{\text{c-d nozzle}}$



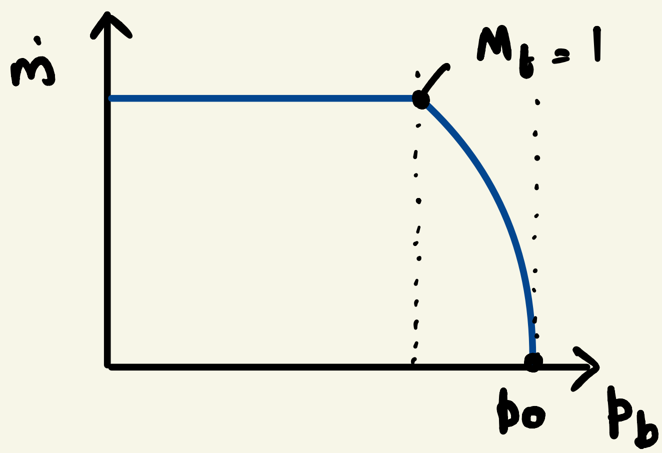
Two possibilities exist as $p_b \downarrow$

1. $M_e = 1$ at duct exit : flow choked and subsonic inside the nozzle for all p_b
2. $M_t = 1$ at a p_b much before $M_e = 1$ (more likely) : nozzle chokes before the exit.

As p_b is reduced beyond the value which first causes $M_t = 1$, a normal shock is produced in the divergent portion which moves towards the nozzle exit and enters the duct.



Once the shock enters the duct, it is located in such a manner that $p_e = p_b$. Eventually, the normal shock reaches the duct exit. Once the shock reaches the exit, flow inside the system is choked and further corrections happen through oblique shocks and expansion waves outside.



Problems

(3) Air : $p_0 = 200\,kPa$, $T_0 = 303\,K$, convergent nozzle into duct ϕ 25mm. M at nozzle exit/pipe inlet is 0.2, M at pipe end 0.8. Find pipe length, p at pipe exit. $f = 0.005$.

From friction tables, at $M = 0.2$

$$\frac{4fl_1^*}{D} = 14.533 \qquad \frac{p_1}{p^*} = 5.4555$$

From friction tables at $M = 0.8$

$$\frac{4fl_2^*}{D} = 0.07229 \qquad \frac{p_2}{p^*} = 1.2892$$

At the nozzle exit, using isentropic relations

$$\frac{p_0}{p_1} = 1.0283$$

$$\frac{4fl_{12}}{D} = \frac{4fl_1^*}{D} - \frac{4f}{D} l_2^*$$

$$l_{12} = 18.1\,m$$

$$p_2 = \frac{p_2}{p^*} \cdot \frac{p^*}{p_1} \cdot \frac{p_1}{p_0} \cdot p_0$$

$$= 45.96\,kPa$$

(b) Find p_b at which $M_e = 1$. Also find M_1.

$$\Rightarrow l_1^* = l_{12} = 18.1$$

$$\Rightarrow \frac{4fl_1^*}{D} = 14.4607$$

$$\Rightarrow M_1 = 0.2005, \quad \frac{p_1}{p^*} = 5.455$$

$$p_2 = \frac{p^*}{p_1} \cdot \left.\frac{p_1}{p_0}\right|_{M = 0.2005} \cdot p_0 = 35.66\,kPa$$

④ Flow of air from c-d nozzle into a ϕ 0.3 m pipe of length 3.5 m. At pipe inlet, M = 2, $p_1 = 101.3$ kPa. $\bar{f} = f = 0.005$

Find (a) M, p at exit of pipe if no shocks,

(b) p_b if normal shock at pipe exit,

(c) p_b if normal shock is located mid-pipe

(a) Fanno tables at $M_1 = 2$ give

$$\frac{4fl_1^*}{D} = 0.3049 \; , \quad \frac{p_1}{p^*} = 0.4083$$

$$\frac{4fl_2^*}{D} = \frac{4fl_1^*}{D} - \frac{4fl_{12}}{D} = 0.0717$$

$$\Rightarrow \quad M_2 = 1.32 \qquad \frac{p_2}{p^*} = 0.715$$

$$p_2 = \frac{p_2}{p^*} \cdot \frac{p^*}{p_1} \cdot p_1$$

$$= \frac{0.715}{0.4083} \cdot 101.3\text{ kPa} = 177.3\text{ kPa}$$

(b) M in the exit plane before the shock = 1.32

p " " " " " " " = 177.3 kPa

Using shock tables

$$\frac{p_b}{177.3} = 1.866$$

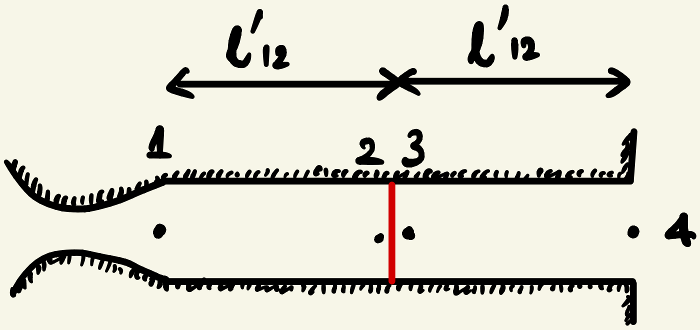
$$\Rightarrow p_b = 331\text{ kPa}$$

(c) $\frac{4fl_2^*}{D} = \frac{4fl_1^*}{2} - \frac{4fl_{12}'}{D}$ $\qquad l_{12}' = \frac{3.5}{2}$

$= 0.19$

Using Fanno tables,

$M_2 = 1.65 \qquad p_2/p^* = 0.534$



$p_2 = \frac{p_2}{p^*} \cdot \frac{p^*}{p_1} \cdot p_1 = 132.6\,kPa$

Using normal shock tables

$\frac{p_3}{p_2} = 3.01 \qquad M_3 = 0.654$

Using Fanno tables at M_3

$\frac{4fl_3^*}{D} = 0.31 \qquad p_3/p^* = 1.6$

$\frac{4fl_4^*}{D} = \frac{4fl_3^*}{D} - \frac{4fl_{12}'}{D} = 0.194$

$\Rightarrow \quad M_4 = 0.71, \quad p_4/p^* = 1.48$

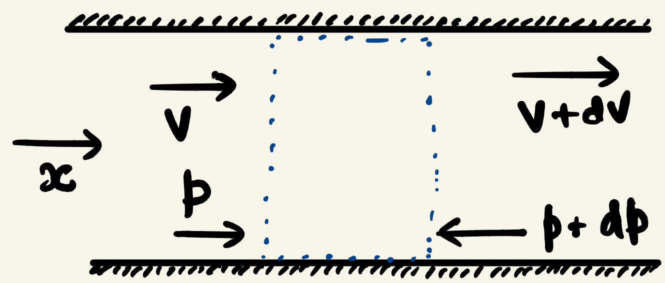
$p_4 = p_b = \frac{p_4}{p^*} \cdot \frac{p^*}{p_3} \cdot \frac{p_3}{p_2} \cdot p_2 = 369\,kPa$

Internal flows with heat addition or removal

Assumptions

1. Effects of viscosity negligible
2. Constant area
3. Steady flows
4. Gas composition does not change (γ fixed)
5. No work done.

Mass



$$\rho V = \text{constant}$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \quad — (1)$$

Momentum

$$pA - (p+dp)A = \rho V A (V + dV - V)$$

$$dp + \rho V dV = 0 \quad — (2)$$

Energy

$$\dot{m}\, dq = dQ = \dot{m}\left[C_p(T+dT) + \frac{(V+dV)^2}{2} - C_pT - \frac{V^2}{2}\right]$$

dq - heat addition per unit $\dot{m}$

$$dq = C_p dT + V dV \quad — (3)$$

Also,

$$T_0 + dT_0 = T + dT + \frac{(V+dV)^2}{2C_p}$$

$$\Rightarrow dT_0 = dT + \frac{V dV}{C_p}$$

$$dq = c_p\, dT_o \quad — \quad ④$$

Eqn. ③ / dV

$$\frac{dq}{dV} = c_p \frac{dT}{dV} + V \quad — \quad ⑤$$

$$p = \rho R T$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad — \quad ⑥$$

Using ① and ②

$$\left.\begin{aligned} -\frac{\rho V dV}{p} &= -\frac{dV}{V} + \frac{dT}{T} \\ -\frac{V dV}{RT} &= -\frac{dV}{V} + \frac{dT}{T} \end{aligned}\right\} \quad — \quad ⑦$$

$$⑦ \times \frac{T}{dV} :$$

$$\boxed{\frac{dT}{dV} = \frac{T}{V} - \frac{V}{R}} \quad — \quad (*)$$

Eqn. ⑤ becomes

$$\boxed{\frac{dq}{dV} = c_p \frac{T}{V} - \frac{V}{\gamma - 1}} \quad — \quad (**)$$

For 'small' V : $c_p \frac{T}{V} > \frac{V}{\gamma-1} \Rightarrow dq/dV > 0$

For 'large' V : $c_p \frac{T}{V} < \frac{V}{\gamma-1} \Rightarrow dq/dV < 0$

Transition from 'small' to 'large' happens when

$\frac{dq}{dV} = 0$ or $C_p \frac{T}{V} = \frac{V}{\gamma - 1}$

$$\Rightarrow \boxed{V = \sqrt{\gamma R T} = a}$$

$\Rightarrow$ $M < 1$, $dq/dV > 0$

$M > 1$, $dq/dV < 0$

$\Rightarrow$ For heat addition, $dq > 0$, flow moves towards $M = 1$ for both $M > 1$ and $M < 1$

For $dq < 0$, flow moves away from $M = 1$

M	$dq > 0$	$dq < 0$
$M < 1$	$V \uparrow$	$V \downarrow$
$M > 1$	$V \downarrow$	$V \uparrow$

$$\frac{dV}{V} = \frac{da}{a} + \frac{dM}{M} \Rightarrow \frac{dV}{V} = \frac{1}{2}\frac{dT}{T} + \frac{dM}{M} \quad — ⑧$$

From ⑥ & ⑦

$$-\frac{\rho V dV}{p} = -\frac{dV}{V} + \frac{dT}{T}.$$

$$\Rightarrow (1 - \gamma M^2)\frac{dV}{V} = \frac{dT}{T} \quad — ⑨$$

Substitute ⑨ in ⑧

$$\frac{dV}{V} = \frac{1}{2}(1 - \gamma M^2)\frac{dV}{V} + \frac{dM}{M}$$

$$\Rightarrow \left(\frac{1+\gamma M^2}{2}\right)\frac{dV}{V} = \frac{dM}{M} \quad \text{— (10)}$$

dM and dV have the same sign.

From eqn. (*), $dT/dV = 0$ when

$$\frac{T}{V} = \frac{V}{R} \quad \text{or} \quad V = \sqrt{RT}$$

$$\boxed{M = \frac{1}{\sqrt{\gamma}}}$$

Maximum temperature occurs when $M = \frac{1}{\sqrt{\gamma}} < 1$

$\boxed{\text{Entropy}}$

$$\frac{ds}{c_p} = \frac{dT}{T} - \left(\frac{\gamma-1}{\gamma}\right)\frac{dp}{p}$$

$$= \frac{dT}{T} - \frac{\gamma-1}{\gamma}\left(-\frac{dV}{V} + \frac{dT}{T}\right)$$

$$= \frac{1}{\gamma}\frac{dT}{T} + \frac{\gamma-1}{\gamma}\frac{dV}{V}$$

Using (*) $\quad \frac{dT}{dV} = \frac{T}{V} - \frac{V}{R}$

$$\frac{dT}{T} = \frac{dV}{V} - \frac{\gamma V^{2}\, dV}{\underbrace{\gamma R T}_{a^2}}\,\frac{1}{V} = \frac{dV}{V} - \gamma M^2 \frac{dV}{V}$$

$$\Rightarrow \frac{ds}{c_p} = \frac{1}{\gamma}\left(\frac{dV}{V} - \gamma M^2 \frac{dV}{V}\right) + \frac{\gamma-1}{\gamma}\frac{dV}{V}$$

$$= \frac{dV}{V} - M^2 \frac{dV}{V}$$

$$\boxed{\frac{ds}{C_p} = (1-M^2)\frac{dV}{V}} \quad — (11)$$

$ds>0$ for $dq>0$, $ds<0$ for $dq<0$

Heat addition moves the entropy towards a maximum at $M=1$

Problems

(5) Air flow: at some point $T = 200°C$. After a short distance T_0 increases by 1%. Find % increase in V, M. Also find ds/C_p if $M = 0.4, 0.8, 1.2, 1.6$

$$dq = C_p \, dT_0$$

$$\frac{dq}{C_p T_0} = \frac{dT_0}{T_0} = 0.01$$

Using (10)

$$\frac{dq/C_pT_0}{dV/V} = \frac{T}{T_0} - \frac{V^2}{(\gamma-1)C_pT_0}$$

$$\frac{0.01}{dV/V} = \frac{T}{T_0} - \frac{V^2}{0.4 \cdot 1007\, T_0}$$

$$\frac{T_0}{T} = 1+0.2M^2 \quad \& \quad T_0 = 473(1+0.2M^2)$$

$$\frac{dV}{V} = \frac{0.01}{(1+0.2M^2) - V^2/402.8 \cdot 473(1+0.2M^2)}$$

$$\frac{dM}{M} = \frac{(1+1.4M^2)}{2}\frac{dV}{V}$$

$$\frac{ds}{C_p} = (1-M^2)\frac{dV}{V}$$

M	dV/V (%)	dM/M (%)	ds/C_p
0.4	1.23	0.75	0.63
0.8	3.12	2.96	1.06
1.2	−2.95	−4.45	1.96
1.6	−1.17	−2.42	3.02

Equations in terms of M

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$$

$$p_1 (1 + \gamma M_1^2) = p_2 (1 + \gamma M_2^2)$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \quad \text{— (12)}$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{p_{0_2}}{p_{01}} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \frac{\left[1 + (\gamma-1)/2\, M_2^2\right]^{\gamma/\gamma-1}}{\left[1 + (\gamma-1)/2\, M_1^2\right]^{\gamma/\gamma-1}} \quad \text{— (13)}$$

$$\left.\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{V_1}{V_2} \\ \frac{\rho_2}{\rho_1} &= \frac{p_2}{p_1} \cdot \frac{T_1}{T_2} \end{aligned}\right\} \Rightarrow \frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{V_2}{V_1}$$

$$\frac{V_2}{V_1} = \frac{M_2}{M_1} \cdot \sqrt{\frac{T_2}{T_1}}$$

$$\frac{T_2}{T_1} = \frac{M_2^2 (1 + \gamma M_1^2)^2}{M_1^2 (1 + \gamma M_2^2)^2} \quad \text{— (14)}$$

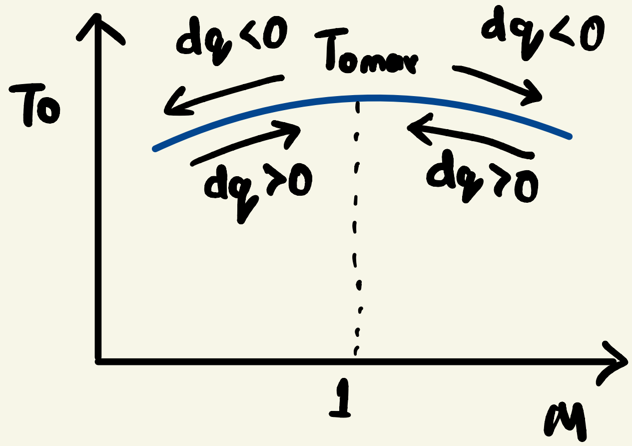
$$\Rightarrow \quad \frac{V_2}{V_1} = \frac{M_2^2(1+\gamma M_1^2)}{M_1^2(1+\gamma M_2^2)} \quad - (15)$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2(1+\gamma M_2^2)}{M_2^2(1+\gamma M_1^2)} \quad - (16)$$

$$\frac{T_{02}}{T_{01}} = \frac{T_2}{T_1}\frac{\left[1+(\gamma-1)/2\, M_2^2\right]}{\left[1+(\gamma-1)/2\, M_1^2\right]}$$

$$\frac{T_{02}}{T_{01}} = \frac{M_2^2(1+\gamma M_1^2)^2}{M_1^2(1+\gamma M_2^2)^2}\frac{\left[1+(\gamma-1)/2\, M_2^2\right]}{\left[1+(\gamma-1)/2\, M_1^2\right]} \quad - (17)$$

dq & dT_0 have the same sign



Variation of T is less straightforward.

$$\frac{T_{0max}}{T_{01}} = \frac{1}{2(1+\gamma)M_1^2}\frac{(1+\gamma M_1^2)^2}{1+(\gamma-1)M_1^2/2} = f(M_1)$$

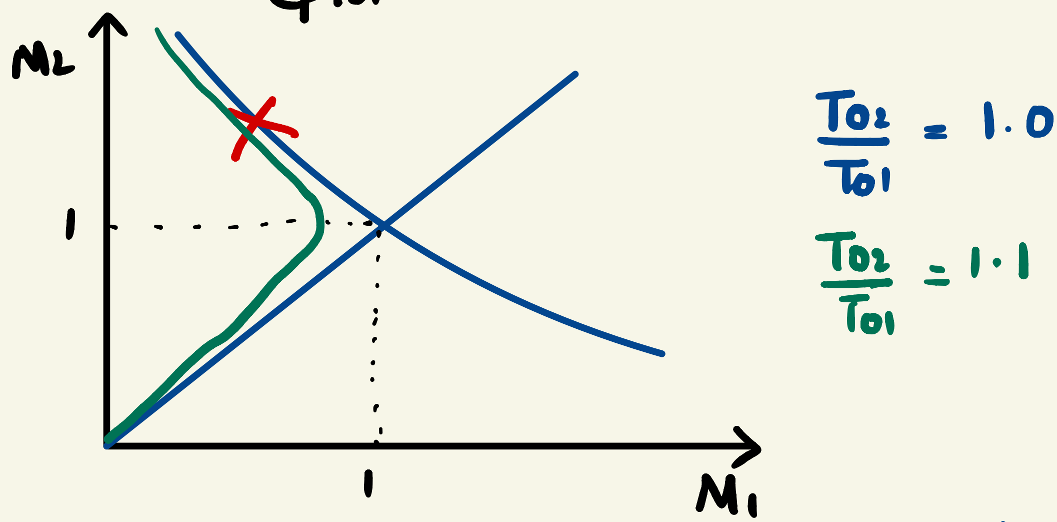
$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$\frac{s_2-s_1}{c_p} = \ln\left[\frac{M_2^2}{M_1^2}\left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^{\frac{\gamma+1}{\gamma}}\right] \quad - (18)$$

$$q = c_p (T_{02} - T_{01})$$

$$\frac{q}{c_p T_{01}} = \frac{T_{02}}{T_{01}} - 1 = \frac{M_2^2 (1+\gamma M_1^2)^2 \left[1+(\gamma-1)/2\, M_2^2\right]}{M_1^2 (1+\gamma M_2^2)^2 \left[1+(\gamma-1)/2\, M_1^2\right]} - 1 \quad -(19)$$

Now, given M_1, $\frac{q}{C_p T_{01}}$ (or $\frac{T_{02}}{T_1}$), we can find M_2



For $\frac{T_{02}}{T_{01}} = 1$ two possible solutions for $M_1 > 1$
$M_2 = 1$ (nothing happens) and $M_2 < 1$ (normal shock)

Reference conditions ($M = 1$)

$$\frac{T_0}{T_0^*} = \frac{2(\gamma+1)M^2\left[1+(\gamma-1)/2\, M^2\right]}{(1+\gamma M^2)^2} \; ; \quad \frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2}$$

$$\frac{p}{p^*} = \frac{1+\gamma}{(1+\gamma M^2)} \; ; \quad \frac{V}{V^*} = \frac{(1+\gamma)M^2}{1+\gamma M^2} \; ; \quad \frac{\rho}{\rho^*} = \frac{1+\gamma M^2}{(1+\gamma)M^2}$$

$$\frac{p_0}{p_0^*} = \frac{1+\gamma}{1+\gamma M^2}\left\{\left(\frac{2}{\gamma+1}\right)\left[1+\frac{\gamma-1}{2}M^2\right]\right\}^{\frac{\gamma}{\gamma-1}} \quad -(*)$$

$$\frac{s-s^*}{c_p} = \ln\left[M^2\left(\frac{1+\gamma}{1+\gamma M^2}\right)^{\frac{\gamma+1}{\gamma}}\right]$$

Problems

⑥ Air flow: $p_1 = 100\,kPa$, $T_1 = 10^0C$, $M_1 = 2.8$.
$dq > 0$, $M_2 = 1.3$.

(a) ~~Find~~ p_2, T_2.

From heat tables at M_1

$p_1/p^* = 0.2004$, $T_1/T^* = 0.3149$, $T_0/T_0^* = 0.6738$

From heat tables at M_2

$p_2/p^* = 0.7130$, $T_2/T^* = 0.8592$, $T_0/T_0^* = 0.958$

$$p_2 = \frac{p_2}{p^*} \cdot \frac{p^*}{p_1} \cdot p_1 = 355.8\,kPa$$

$$T_2 = \frac{T_2}{T^*} \cdot \frac{T^*}{T_1} \cdot T_1 = 772.2\,K.$$

(b) Find q_{max} if no shocks occur in the flow

For q_{max}, $M_2 = 1$

$\Rightarrow T_{02} = T_0^* = 1078.5\,K$

$$q_{max} = c_p(T_0^* - T_0)$$
$$= 1.007.(1078.5 - 726.7)$$
$$= 354.3\,kJ/kg$$

$p_2 = p^* = 499.0\,kPa$

$T_2 = T^* = 898.7\,K$

⑦ Air flow : $M_1 = 0.62$, $p_1 = 200\,kPa$, $T_1 = 350°C$

$q = -400\ kJ/kg$ transferred from air to the walls.

Find M_2, p_2, T_2.

At M_1, $\frac{T_0}{T} = 1.054 \Rightarrow T_{01} = 656.6\ K$

From heat tables @ M_1

$p_1/p^* = 1.7414$, $T_1/T^* = 0.8196$, $T_{01}/T_0^* = 0.7199$

$q = C_p(T_{02} - T_{01}) = -400\ kJ/kg$

$\Rightarrow T_{02} = 259.4\ K = -13.6°C$

$$\frac{T_{02}}{T^*} = \frac{T_{02}}{T_{01}} \cdot \frac{T_{01}}{T^*} = 50.2844$$

Looking at the tables

$M_2 = 0.2656$, $p_2/p^* = 2.184$, $T_2/T^* = 0.3365$

$$p_2 = \frac{p_2/p^*}{p_1/p^*} \cdot p_1 = 250.9\ kPa$$

$$T_2 = \frac{T_2/T^*}{T_1/T^*} \cdot T_1 = 256\ K$$

Entropy - Temperature relation

From ⊛

$$\frac{s-s^*}{C_p} = \ln\left(\frac{T}{T^*}\right) - \frac{\gamma-1}{\gamma} \ln\left(\frac{p}{p^*}\right)$$

$$M^2 = \frac{(1+\gamma)}{\gamma}\frac{p^*}{p} - \frac{1}{\gamma} \quad — ①$$

$$\left(\frac{p}{p^*}\right)^2 = \frac{1}{M^2}\left(\frac{T}{T^*}\right) \quad - (2)$$

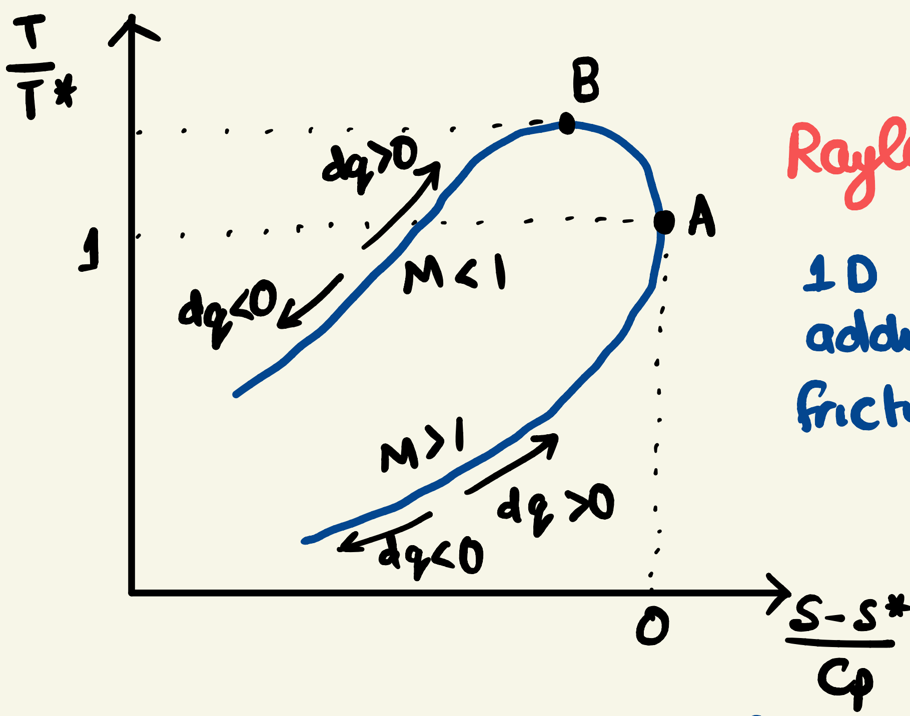
Combining (1) and (2)

$$\left(\frac{p}{p^*}\right)^2 - (1+\gamma)\frac{p}{p^*} + \gamma\frac{T}{T^*} = 0$$

or

$$\frac{p}{p^*} = \frac{(1+\gamma)}{2} \pm \frac{\sqrt{(1+\gamma)^2 - 4\gamma(T/T^*)}}{2}$$

$$\boxed{\frac{s-s^*}{c_p} = \ln\left(\frac{T}{T^*}\right) - \frac{\gamma-1}{\gamma}\ln\left\{\frac{(1+\gamma)}{2} \pm \frac{\sqrt{(1+\gamma)^2 - 4\gamma\, T/T^*}}{2}\right\}} \quad (**)$$



Rayleigh line

1D flow with heat addition and negligible friction

Reaching M=1 chokes the flow – thermal choking

$T_B = T_{max} = T|_{dT/dM=0}$ gives $M_B = \frac{1}{\sqrt{\gamma}}$

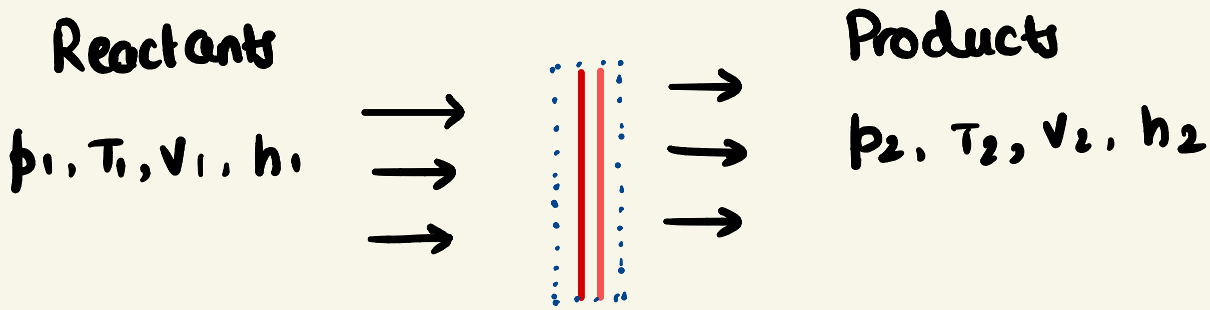
$$\boxed{\frac{T_{max}}{T^*} = \frac{(1+\gamma)^2 M_B^2}{(1+\gamma M_B^2)^2} = \frac{(1+\gamma)^2}{4\gamma}}$$

	q	T_0	M	p	V	s
$M<1$	+	+	+	−	+	+
$M>1$	+	+	−	+	−	+
$M<1$	−	−	−	+	−	−
$M>1$	−	−	+	−	+	−

Combustion waves

A composite wave consisting of a shock wave sustained by the release of chemical energy in a combustion zone immediately behind the shock.

Heat addition results from the release of chemical energy.



$\boxed{\text{Mass}}$

$$\rho_1 V_1 = \rho_2 V_2 = \dot{m} \qquad \dot{m} = \text{mass flow / unit area}$$

$\boxed{\text{Momentum}}$

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$$

$$\Rightarrow p_1 + \frac{\dot{m}^2}{\rho_1} = p_2 + \frac{\dot{m}^2}{\rho_2}$$

$$\dot{m}^2 = \frac{p_2 - p_1}{(1/\rho_1) - (1/\rho_2)} \quad — (1)$$

From (1), we have

$$\boxed{\begin{aligned} v_1 &= \frac{1}{\rho_1}\sqrt{\frac{p_2 - p_1}{(1/\rho_1) - (1/\rho_2)}} \\ v_2 &= \frac{1}{\rho_2}\sqrt{\frac{p_2 - p_1}{(1/\rho_1) - (1/\rho_2)}} \end{aligned}} \quad — (2)$$

Energy

$$c_p T_1 + \frac{v_1^2}{2} + q' = c_p T_2 + \frac{v_2^2}{2} \quad — (3)$$

From (2)

$$v_2^2 - v_1^2 = \frac{p_2 - p_1}{(1/\rho_1) - (1/\rho_2)}\left[\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2}\right]$$

$$v_2^2 - v_1^2 = (p_1 - p_2)\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] \quad — (4)$$

From (3)

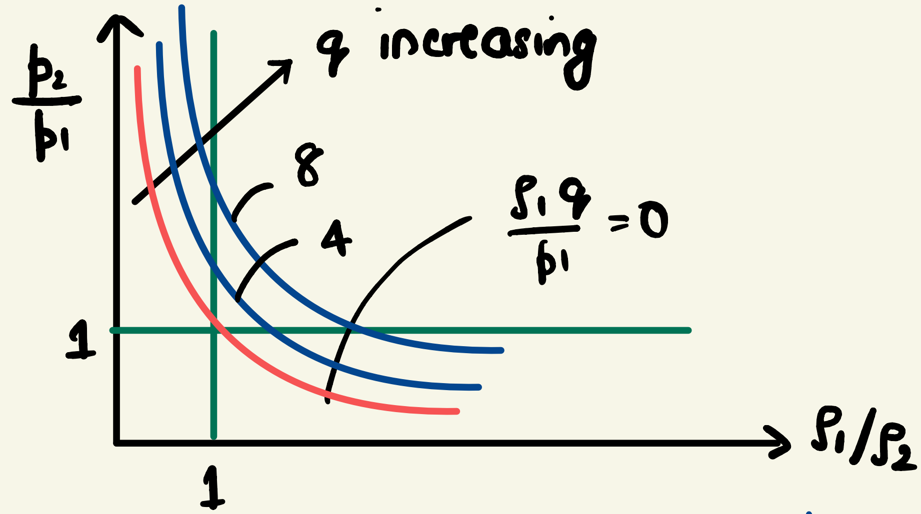
$$\begin{aligned} v_2^2 - v_1^2 &= 2\frac{\gamma R}{\gamma - 1}(T_1 - T_2) + 2q' \\ &= \frac{2\gamma}{\gamma - 1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right] + q \quad — (5) \end{aligned}$$

$$(p_1 - p_2)\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\right] = \frac{2\gamma}{\gamma - 1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right] + q \quad — (6)$$

(6) $\times\ \rho_1/\rho_2$

$$\boxed{\left[1 - \frac{p_2}{p_1}\right]\left[\frac{\rho_1}{\rho_2} + 1\right] = \frac{2\gamma}{\gamma - 1}\left[1 - \left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)\right] + \frac{\rho_1 q}{p_1}} \quad — (7)$$

$q = 0$ gives normal shock relation



$p_2/p_1 = 1$ and $\rho_2/\rho_1 = 1$ represents the origin

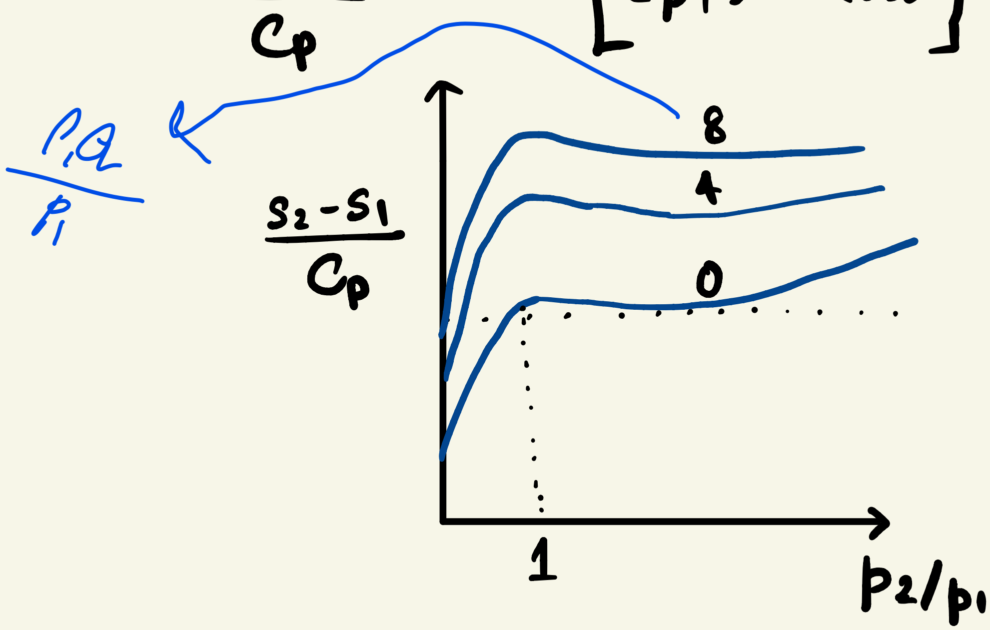
Q. What are the possible final states for various q?

From eqn ② if $p_2 > p_1$, $\rho_2 > \rho_1$ and
if $p_2 < p_1$, $\rho_2 < \rho_1$

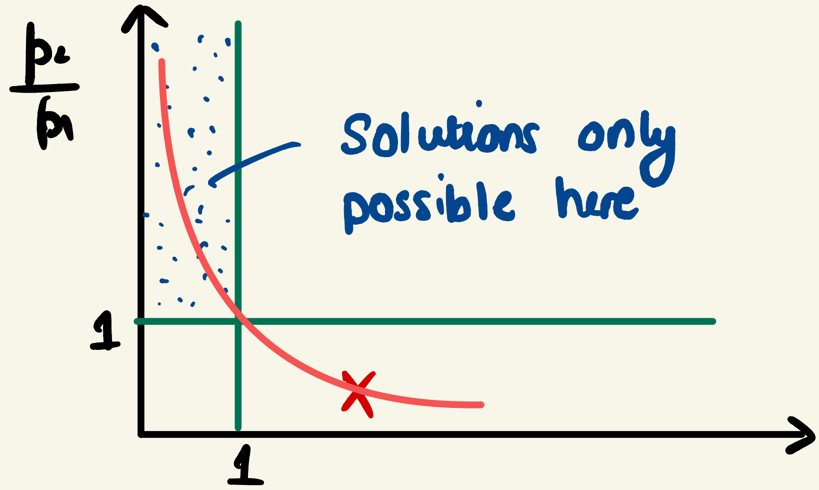
Solution can only lie in the second and fourth quadrants

From entropy considerations,

$$\frac{s_2 - s_1}{c_p} = \ln\left[\left(\frac{p_2}{p_1}\right)^{1/\gamma}\left(\frac{\rho_1}{\rho_2}\right)\right]$$



These curves have a minimum point in the range $1 < \rho_1/\rho_2 < 0$



For $q=0$,
$\Delta s > 0$ when $p_2/p_1 > 1$
and
$\Delta s < 0$ when $p_2/p_1 < 1$.

Only upper branch is permitted for $q=0$

For $q>0$, $\Delta s > 0$ even when $p_2/p_1 < 1$ over a range i.e. both upper and lower branches are permitted

Cases for which $q>0$ and $p_2/p_1 > 1$ are called detonation waves and cases with $q>0$, $p_2/p_1 < 1$ are called deflagration waves (or flames)

For detonation waves, it was experimentally verified observed that conditions behind the wave correspond to those at minimum local entropy point
↳ Chapman - Jouguet detonation waves

$$\frac{\overbrace{s_2 - s_1}^{s}}{c_p} = \frac{1}{\gamma} \ln\left(\frac{p_2}{p_1}\right) + \ln \frac{\rho_1}{\rho_2} \quad \text{— (7)}$$

Differentiate (7) with ρ_1/ρ_2

$$\frac{d(s/C_p)}{d(\rho_1/\rho_2)} = \frac{1}{\gamma(p_2/p_1)}\frac{d(p_2/p_1)}{d(\rho_1/\rho_2)} + \frac{1}{(\rho_1/\rho_2)} = 0$$

$$\Rightarrow \quad \frac{d(p_2/p_1)}{d(\rho_1/\rho_2)} = -\frac{\gamma(p_2/p_1)}{(\rho_1/\rho_2)} \qquad \text{— (8)}$$

Differentiate (*) w.r.t. ρ_1/ρ_2 for fixed $\rho_1 q/p_1$

$$1+\frac{\gamma+1}{\gamma-1}(p_2/p_1) = \left[1-\frac{\gamma+1}{\gamma-1}\left(\frac{\rho_1}{\rho_2}\right)\right]\frac{d(p_2/p_1)}{d(\rho_1/\rho_2)} \qquad \text{— (9)}$$

Substitute (8) in (2)

$$1+\frac{\gamma+1}{\gamma-1}(p_2/p_1) = -\frac{\gamma(p_2/p_1)}{\rho_1/\rho_2} + \frac{\gamma(\gamma+1)}{\gamma-1}(p_2/p_1)$$

$$\boxed{\frac{\rho_1}{\rho_2} = \frac{\gamma(p_2/p_1)}{(\gamma+1)(p_2/p_1)-1}} \qquad \text{— (**)}$$

$$M_1^2 = \frac{v_1^2}{a_1^2} = \frac{v_1^2\rho_1}{\gamma p_1}$$

From (2), we get

$$\boxed{M_1^2 = \frac{1}{\gamma}\frac{(p_2/p_1)-1}{1-\rho_1/\rho_2} = \frac{1}{\gamma}\left[(\gamma+1)\left(\frac{p_2}{p_1}\right)-1\right]} \qquad \text{— (***)}$$

and

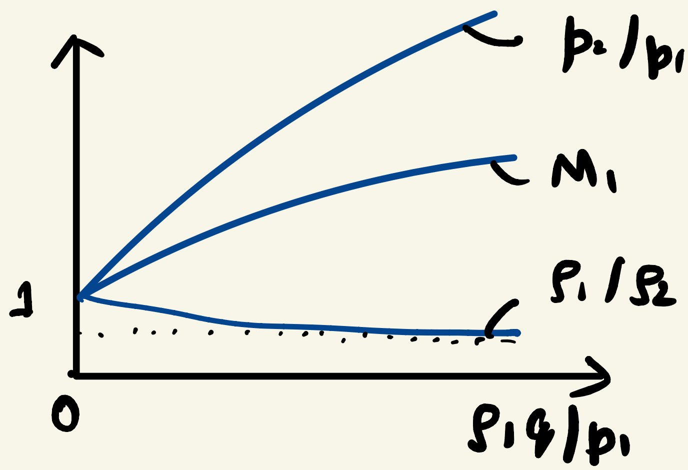
$$\boxed{v_2^2 = \left(\frac{p_1\rho_1}{\rho_2^2}\right)\frac{p_2/p_1-1}{1-\rho_1/\rho_2}} \qquad \text{— (****)}$$

$$M_2^2 = \frac{v_2^2}{\gamma p_2/\rho_2}$$

$$\boxed{M_2^2 = \frac{\rho_1/\rho_2}{\gamma(p_2/p_1)}\,\frac{p_2/p_1 - 1}{1-\rho_1/\rho_2}} \quad \text{— (***)}$$

Substituting for ρ_1/ρ_2 from (**)

$M_2^2 = 1$, $M_2 = 1$ at minimum entropy point
Mach number behind a C-J wave is always 1.



For high values of p_2/p_1, we obtain

$$\boxed{\frac{\rho_1}{\rho_2} = \frac{\gamma}{\gamma+1}}$$

Eqn. (*) becomes

$$-\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}+1\right) = -\left(\frac{2\gamma}{\gamma+1}\right)\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right) + \frac{\rho_1 q}{p_1}$$

Substituting for ρ_1/ρ_2, we get

$$\boxed{\frac{p_2}{p_1} = (\gamma-1)\,\rho_1 q/p_1}$$

Also,

$$\boxed{v_1^2 = \frac{p_2/\rho_1}{1-\rho_1/\rho_2} = (\gamma+1)\frac{p_2}{p_1}}$$

Problems

⑧ Long insulated duct : $p_1 = 120\,kPa$, $T_1 = 20°C$. detonation wave propagates due to ignition at one end. Find velocity of wave and velocity of gas behind the wave. $q = 2\,MJ/kg$, $\gamma = 1.4$, $\rho_1/\rho_2 = 0.617$

$$\rho_1 = \frac{p_1}{RT_1} = 1.427\,kg/m^3, \quad \frac{\rho_1 q}{p_1} = 23.78$$

Eqn ④ gives

$$\left[1 - \frac{p_2}{p_1}\right](1 + 1.427) = 7\left[1 - \left(\frac{p_2}{p_1}\right)1.427\right] + 23.78$$

$$\Rightarrow p_2/p_1 = 7.73 \quad \text{or} \quad p_2 = 927.6\,kPa$$

$$M_1^2 = \frac{1}{\gamma}\,\frac{p_2/p_1 - 1}{1 - \rho_1/\rho_2} \Rightarrow M_1 = 3.53$$

$$V_1 = 3.53 \times \sqrt{1.4 \times 287 \times 293} = 1211.2\,m/s$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} \Rightarrow T_2 = 1397.4\,K.$$

We know $M_2 = 1$

$$\Rightarrow V_2 = 1 \times \sqrt{1.4 \times 287 \times 1397.4}$$

$$= 749.3\,m/s$$