MA 214: Introduction to Numerical Analysis

Department of Mathematics, Indian Institute of Technology Bombay

Mid-Semester Examination

Marks: 25

Date: 24-02-2019

Time: 2 hours

Instructors: S. Baskar and S. Sivaji Ganesh

Instructions:

(1) Write your Name, Roll Number, and Tutorial Batch clearly on your answer book as well as every supplement you may use. A penalty of -1 mark will be awarded for failing to do so.

(2) Number the pages of your answer book and make a question-page index on the front page. A penalty of -1 mark will be awarded for failing to do so.

(3) The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.

(4) Only scientific calculators are allowed. Any kind of programing device is not allowed.

(5) Formulas used need not be proved but needs to be stated clearly.

(6) The question paper contains 7 questions. Answer all the questions.

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(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(\sin x)$. Use Taylor's theorem to find a real number $\epsilon > 10^{-2}$ such that the inequality

$$|f(x) - 1| < 10^{-2}$$

holds for all $x \in (-\epsilon, \epsilon)$. [3 Marks]

(2) Define the notion of little oh (small oh) in the context of sequences of real numbers. Prove or disprove the following using the definition of little oh:

$$\frac{1}{\ln n} = o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

[3 Marks]

(3) For each $x \in (0, \infty)$, does the process of evaluating the function

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

stable or unstable? Justify your answer. [4 Marks]

(4) Let A be a diagonally dominant matrix of size 2×2. Show that Naive Gaussian elimination method to solve the system of linear equations $Ax = b$ is applicable. [3 Marks]

-P.T.O.-

(5) Use Cholesky factorization to solve the system of equations

$$\begin{cases} x_1 + 3x_3 = 1, \\ 2x_1 - x_2 = 3, \\ x_1 + 2x_3 = -1. \end{cases}$$

[4 Marks]

(6) The eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -2 & 3 & 3 \\ 5 & -4 & 3 \\ -5 & 5 & -2 \end{pmatrix}.$$

are $\lambda_1 = -7$, $\lambda_2 = -2$, $\lambda_3 = 1$, and
$v_1 = (0, -1, 1)^T$, $v_2 = (1, 1, -1)^T$, $v_3 = (1, 1, 0)^T$.

Answer the following questions:

(a) To which eigenvalue (and the corresponding eigenvector) does the power method converge (theoretically) if we take the initial guess $x^{(0)} = (1, 1, 1)^T$? Justify your answer without performing the iterations. [2 Marks]

(b) Give the general form of the iterative sequences of power method. Perform two iterations of the power method for the above matrix with $x^{(0)} = (-1, -1, 1)^T$. [2 Marks]

(7) Let A be an $n \times n$ matrix and P be an $n \times n$ invertible matrix such that such that $P^{-1}AP = D$, where D is the diagonal matrix given by

$$D = \text{diag}(d_1, d_2, \cdots, d_n),$$

where $d_j \in \mathbb{R}$, $j = 1, 2, \cdots, n$. For a given $n \times n$ matrix B show that the eigenvalues of $A + B$ lie in the union of the disks

$$\{\lambda \in \mathbb{C} : |\lambda - d_i| \leq \kappa_\infty(P) \, \|B\|_\infty\}, \quad i = 1, 2, \cdots, n.$$

[4 Marks]

[**Note:** The set of all eigenvalues of the matrix $A + B$ is the same as the set of all eigenvalues of the matrix $P^{-1}(A + B)P$.]

— End of Question Paper —