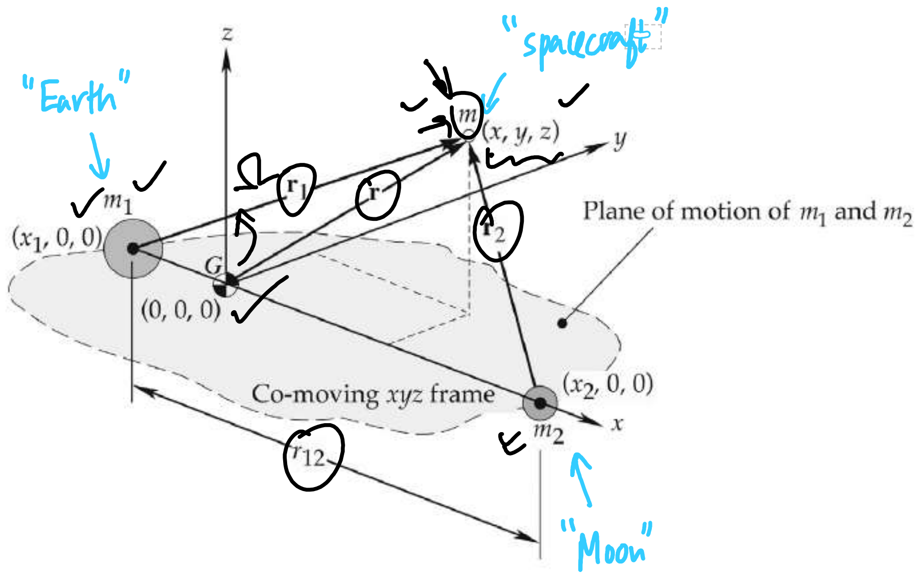
Circular Restricted Three-Body Problem



- Assume $m_1 > m_2$

- $\boldsymbol{\Omega} = \Omega \hat{k}, \ \Omega = \frac{2\pi}{T}$

- $T = \frac{2\pi}{\sqrt{\mu}} r_{12}^{3/2} \Rightarrow \Omega = \sqrt{\frac{\mu}{r_{12}^3}}$

- Recall that $\mu = GM$, where $M = m_1 + m_2$

- Locations of m_1 and m_2 on the x-axis:

$$m_1 x_1 + m_2 x_2 = 0$$

$$x_2 = x_1 + r_{12}$$

- $$x_1 = -\Pi_2 r_{12}$$
  $$x_2 = \Pi_1 r_{12}$$

  $$\Pi_1 = \frac{m_1}{m_1+m_2}$$

  $$\Pi_2 = \frac{m_2}{m_1+m_2}$$

- Since m_1 and m_2 have the same period, the larger mass (the one closest to G) has a greater orbital speed and hence, greater centripetal force.

- $m_1 >> m$, $m_2 >> m$

- The motion of m due to the gravitational fields of m_1 and m_2 has no general, closed-form solution.

- $r = x\hat{i} + y\hat{j} + z\hat{k}$

- $r_1 = (x - x_1)\hat{i} + y\hat{j} + z\hat{k}$

- $r_2 = (x - x_2)\hat{i} + y\hat{j} + z\hat{k}$

- $$\dot{r} = V_G + \Omega \times r + V_{rel}$$

$$V_{rel} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

- $$\ddot{r} = \underbrace{a_G}_{0} + \underbrace{\dot{\Omega}}_{0} \times r + \Omega \times (\Omega \times r) + 2\Omega \times V_{rel} + a_{rel}$$

$$= \Omega \times (\Omega \times r) + 2\Omega \times V_{rel} + a_{rel}$$

$$a_{rel} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

- $$\ddot{r} = \Omega\hat{k} \times \left(\Omega\hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k})\right) + 2\Omega\hat{k} \times (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$$
$$+ \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

$$= (\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\hat{i} + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\hat{j} + \ddot{z}\hat{k}$$

- Newton's second law:

$$m\ddot{r} = F_1 + F_2,$$

$$F_1 = -\frac{Gm_1 m}{\|r_1\|^2}\hat{u}_{r_1} = -\frac{\mu_1 m}{\|r_1\|^3} r_1$$

$$F_2 = -\frac{Gm_2 m}{\|r_2\|^2}\hat{u}_{r_2} = -\frac{\mu_2 m}{\|r_2\|^3} r_2$$

- $\ddot{r} = -\frac{\mu_1}{\|r_1\|^3} r_1 - \frac{\mu_2}{\|r_2\|^3} r_2$

- Equations of motion:

$$(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\hat{i} + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\hat{j} + \ddot{z}\hat{k} = -\frac{\mu_1}{\|r_1\|^3}\left[(x + \pi_2 r_{12})\hat{i} + \right.$$

$$- \frac{\mu_2}{\|r_2\|^3}\left[(x - \pi_1 r_{12})\right.$$

$$\ddot{x} - 2\Omega\dot{y} - \Omega^2 x = -\frac{\mu_1}{\|r_1\|^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{\|r_2\|^3}(x - \pi_1 r_{12})$$

$$\ddot{y} + 2\Omega\dot{x} - \Omega^2 y = -\frac{\mu_1}{\|r_1\|^3} y - \frac{\mu_2}{\|r_2\|^3} y$$

$$\ddot{z} = -\frac{\mu_1}{\|r_1\|^3} z - \frac{\mu_2}{\|r_2\|^3} z$$

Lagrange Points

- Equilibrium points: m has zero velocity and zero acceleration. These are also known as libration or Lagrange points.

- $\dot{x} = \dot{y} = \dot{z} = 0$ and $\ddot{x} = \ddot{y} = \ddot{z} = 0$

$$-\Omega^2 x = -\frac{\mu_1}{\|r_1\|^3}(x + \pi_2 r_{12}) - \frac{\mu_2}{\|r_2\|^3}(x - \pi_1 r_{12})$$

$$-\Omega^2 y = -\frac{\mu_1}{\|r_1\|^3} y - \frac{\mu_2}{\|r_2\|^3} y$$

$$0 = -\frac{\mu_1}{\|r_1\|^3} z - \frac{\mu_2}{\|r_2\|^3} z$$

$$0 = \left(\frac{\mu_1}{\|r_1\|^3} + \frac{\mu_2}{\|r_2\|^3}\right) z$$

$$z = 0$$

- Equilibrium points lie in the orbital plane.

- $\pi_1 = 1 - \pi_2$

- $$(1-\pi_2)(x+\pi_2 r_{12}) \frac{1}{\|r_1\|^3} + \pi_2 (x + \pi_2 r_{12} - r_{12}) \frac{1}{\|r_2\|^3} = \frac{x}{r_{12}^3}$$

$$(1-\pi_2) \frac{1}{\|r_1\|^3} + \pi_2 \frac{1}{\|r_2\|^3} = \frac{1}{r_{12}^3}$$

$$\pi_1 = \frac{\mu_1}{\mu} \quad , \quad \pi_2 = \frac{\mu_2}{\mu}$$

- $$\frac{1}{\|r_1\|^3} = \frac{1}{\|r_2\|^3} = \frac{1}{r_{12}^3}$$

$$\|r_1\| = \|r_2\| = r_{12}$$

- $$r_{12}^2 = (x + \pi_2 r_{12})^2 + y^2$$
  $$r_{12}^2 = (x + \pi_2 r_{12} - r_{12})^2 + y^2$$

- $$x = \frac{r_{12}}{2} - \pi_2 r_{12}$$
  $$y = \pm \frac{\sqrt{3}}{2} r_{12}$$

- L_4, L_5 : $x = \frac{r_{12}}{2} - \pi_2 r_{12}$, $y = \pm \frac{\sqrt{3}}{2} r_{12}$, $z = 0$

- The two primary bodies and L_4, L_5 lie at the vertices of an equilateral triangle.

- L_1, L_2, L_3 are found by setting $y = 0$ as well as $z = 0$

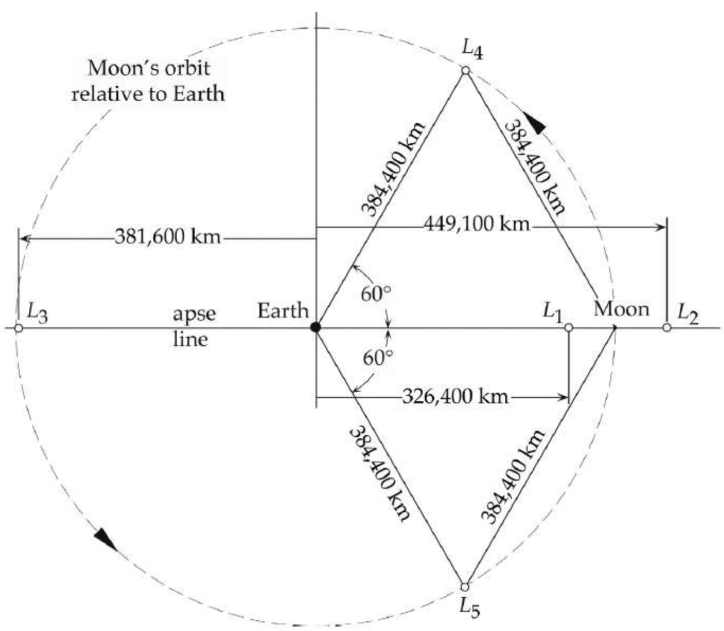
- $$r_1 = (x + \pi_2 r_{12})\hat{i} \qquad |x + \pi_2 r_{12}|$$
  $$r_2 = (x - \pi_1 r_{12})\hat{i} = (x + \pi_2 r_{12} - r_{12})\hat{i}$$
  $$|x + \pi_2 r_{12} - r_{12}|$$

- $$(1 - \pi_2)\frac{x + \pi_2 r_{12}}{[illegible]} + \pi_2 \frac{(x + \pi_2 r_{12} - r_{12})}{[illegible]} - \frac{1}{r^3} x = 0$$

$|x+\pi_2 r_{12}|$ $\quad$ $|x+\pi_2 r_{12}-r_{12}|$ $\quad$ r_{12}

– $\xi = \dfrac{x}{r_{12}}$

– $f(\pi_2,\xi) = (1-\pi_2)\dfrac{\xi+\pi_2}{|\xi+\pi_2|^3} + \pi_2\dfrac{(\xi+\pi_2-1)}{|\xi+\pi_2-1|^3} - \xi$



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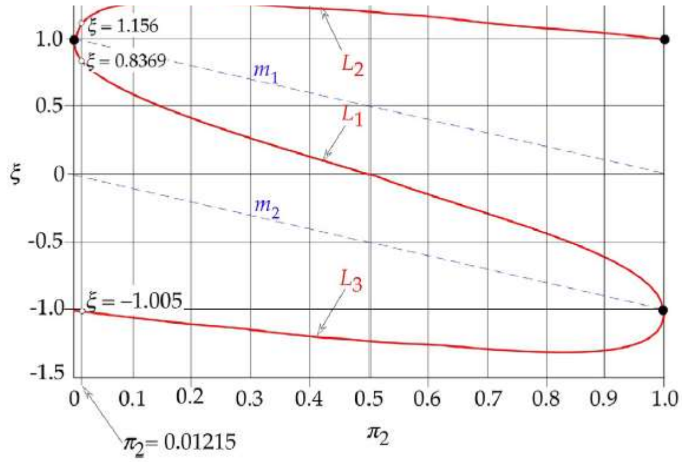

- If an equilibrium point is stable, then a small mass occupying that point will tend to return to that point if it is nudged out of position.

- The perturbation results in a small oscillation (orbit) about the equilibrium point.

- Thus, objects can be placed in small orbits (called halo orbits) around stable equilibrium points without requiring much in the way of station keeping.

- On the other hand, if a body located at an unstable equilibrium point is only slightly perturbed, then it will oscillate in a divergent fashion, drifting eventually completely away from that point.

- L_1, L_2, L_3 unstable

stable

L_4 L_5 stable

$$\frac{m_1}{m_2} + \frac{m_2}{m_1} \geqslant 25 \qquad \frac{m_1}{m_2} \text{ exceeds } 24.96$$

- Earth-moon system 81.3