The gimbal-lock problem

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1 Constant speed CMGs

Based on the principle of conservation of total angular momentum, spacecrafts and satellites, as well as many other rigid-bodies in mechanical and aerospace engineering applications, use gimballed inertia wheels for the purpose of generating torque that can effect a change in the orientation of the outer rigid body to which these devices are fixed. One such mechanism is the *Control Moment Gyro* (CMG).

The figure above is a schematic of a CMG. An inertia wheel is kept spinning at an angular velocity Ω_i (here i denotes the index of the CMG.) The axis of rotation $\hat{s}_i$ of this wheel could be changed using an internal torquing mechanism around the axis $\hat{g}_i$. In a typical spacecraft, two or more such CMGs are fitted in various parts of the spacecraft body. Introducing some notation, let $\hat{s}_i$ denote a unit vector along the *spin-axis*, $\hat{g}_i$ denote a unit vector along the *gimbal-axis* and, $\hat{t}_i$ be orthogonal to both $\hat{s}_i$ and $\hat{g}_i$ with $\hat{t}_i = \hat{s}_i \times \hat{g}_i$. The subscript i denotes the ith CMG. For the purpose of this note i goes from 1 to p.

Let us now perform a short mathematical analysis. The total angular momentum of a spacecraft in gravity-free space, without any other external torques, is conserved. Let $\vec{H}_s$ denote the angular momentum of the spacecraft (the rigid body alone)and $\vec{h}_{g_i}$ denote the angular momentum of the ith gimbal. Then

$$\vec{H}_s + \vec{h}_{g_i} = \text{constant} \quad \text{or} \quad \frac{d(\vec{H}_s + \vec{h}_{g_i})}{dt} = 0$$

*Notes for SC 618

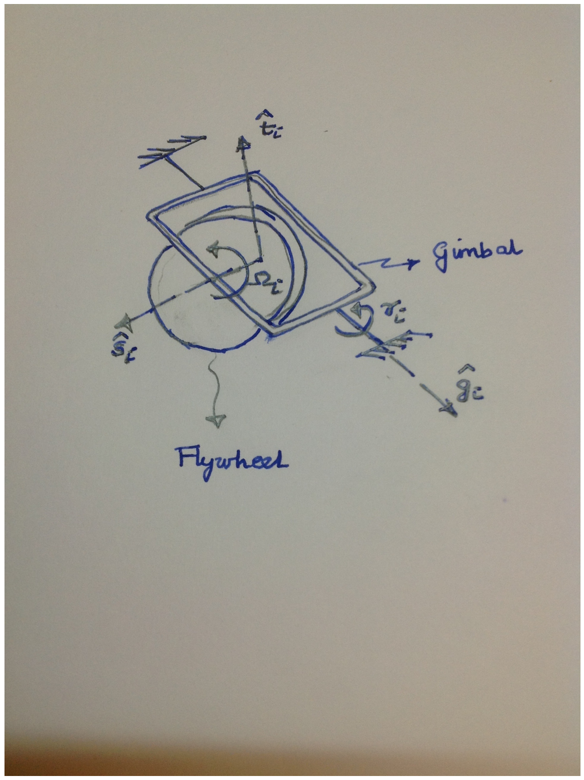

Figure 1: A schematic of a single CMG

By applying a "gimballing" torque to the CMG, we effect a change in the direction of the spin-axis $\hat{s}_i$ and as a result a change in the angular momentum of the CMG. The torque required to introduce this change creates a reaction torque, which in turn, acts on the spacecraft and changes the orientation of the spacecraft. Denoting the torque applied to the CMG as $\tau_i(\in \mathbb{R}^3)$, we have

$$\vec{\tau}_i = \frac{d\,\vec{h}_{g_i}}{dt}$$

Let us now derive an explicit expression for the angular momentum of the ith CMG. Assuming that the angular momentum of the CMG is largely contributed by the spinning wheel, and a very small amount by the rotation of the outer spacecraft, we have

$$\vec{h}_{g_i} = I_{w_i}\Omega_i\hat{s}_i \Rightarrow \frac{d\,\vec{h}_{g_i}}{dt} = I_{w_i}\Omega_i\frac{d\hat{s}_i}{dt} \tag{1}$$

where I_{w_i} denotes the inertia of the wheel about the spin axis. If the gimbal angle is denoted by γ_i and its rate of rotation by $\dot{\gamma}_i$, then

$$\frac{d\hat{s}_i}{dt} = \dot{\gamma}\hat{g}_i \times \hat{s}_i = -\dot{\gamma}_i\hat{t}_i$$

and consequently

$$\frac{d\,\vec{h}_g}{dt} = -I_{w_i}\Omega_i\dot{\gamma}_i\hat{t}_i$$

So far our analysis has been vectorial; let us now bring in a coordinate system fixed to the spacecraft. Then, in the RHS of the above equation, the unit vector $\hat{t}_i$ represented in coordinates becomes a function of γ_i, the gimbal angle. Taken together, we denote the RHS in coordinates as $C(\Omega_i, \gamma_i)\dot{\gamma}_i$ and we have, in coordinates

$$\in \mathbb{R}^3 \; — \; \tau_i = C(\Omega_i, \gamma_i)\dot{\gamma}_i$$

If there are p CMGs, then we have

$$\sum_{i=1}^{p}\tau_i = \sum_{i=1}^{p} C(\Omega_i, \gamma_i)\dot{\gamma}_i = C(\Omega_1, \dots, \Omega_p, \gamma_1, \dots, \gamma_p)\begin{bmatrix}\dot{\gamma}_1\\ \vdots \\ \dot{\gamma}_p\end{bmatrix}$$

The size of the matrix C is $3 \times p$. The matrix C denotes the coordinate representations of the unit vectors $\hat{t}_1, \dots, \hat{t}_n$. If it so happens that the configuration of all the CMGs is such that all the $\hat{t}_i$s are coplanar, then the rank of the matrix C reduces to 2 (a plane is spanned by two linearly independent vectors.) This implies that the columns of C span a 2-dimensional subspace of $\mathbb{R}^3$ and hence not every commanded torque in $\mathbb{R}^3$ is acheivable by using the gimballing action (the $\dot{\gamma}_i$s).

2 Variable speed CMGs

Since 1990s

In the past decade or more, a newer version of the conventional CMG, termed as Variable Speed CMG (VSCMG) has been introduced. In such a system, the angular velocity of the flywheel or momentum wheel can be varied as well. So, in addition to tthe control over γ, one has control over Ω_i as well. So we have

$$\frac{d\,\vec{h}_g}{dt} = I_{w_i}\dot{\Omega}_i\hat{s}_i + I_{w_i}\Omega_i\frac{d\hat{s}_i}{dt}$$

and the final torque equations now looks like

$$\sum_{i=1}^{p}\tau_i = \sum_{i=1}^{p} C(\Omega_i, \gamma_i)\dot{\gamma}_i + D(\gamma_i)\dot{\Omega}_i = [C(\mathbf{\Omega}, \gamma) \quad D(\gamma)]\begin{bmatrix}\dot{\gamma}\\ \dot{\mathbf{\Omega}}\end{bmatrix}$$

(*to be continued...*)