ANALYTICAL AND GEOMETRIC MECHANICS

TUTORIAL-1

Systems & Control Engineering

§ 1. Exercises

1. **Question.** (***Submersions***) Comment whether the following maps are submersions or not.

(1-a) $\mathbb{R}^2 \setminus \{(0,0)\} \ni (x, y) \longmapsto f(x, y) := x^2 + y^3 \in \mathbb{R}$

(1-b) $\mathbb{GL}(2) \ni A \longmapsto f(A) := \det A \in \mathbb{R}$ where $\mathbb{GL}(2) = \{M \in \mathbb{R}^{2\times 2} \mid \det M \neq 0\}$.

2. **Question.** (***Immersions***) Comment whether the following maps are immersions or not.

(2-a) $t \in \mathbb{R} \longmapsto f(t) := (\sin(t), \cos(t)) \in \mathbb{R}^2$.

(2-b) $t \in \mathbb{R} \setminus \{0\} \longmapsto f(t) := (t^2, t^4) \in \mathbb{R}^2$

(2-c) $t \in [0, 2\pi[\longmapsto f(t) := (\sin(t), \cos(t)) \in \mathbb{R}^2$.

3. **Question.** (***Embedding***) Comment which of the maps given in Question 2, are embedding..

4. **Question.** (***Submanifolds***) Comment which of the following sets are embedded submanifolds, also give their dimension.

(4-a) $V \subset \mathbb{R}^n$, V is a subspace with $\dim V = m$.

(4-b) $\mathsf{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$

(4-c) $\mathsf{SO}(n) := \{A \in \mathbb{R}^{n\times n} \mid A^\top A = AA^\top = I, \det A = 1\}$

(4-d) $\mathsf{T}^2 = \mathsf{S}^1 \times \mathsf{S}^1$

5. **Question.** (***Product Manifolds***) Let $M_1 \subset \mathbb{R}^{n_1}$ be an embedded submanifold of dimension m_1, and $M_2 \subset \mathbb{R}^{n_2}$ be an embedded submanifold of dimension m_2. Show that $M_1 \times M_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a submanifold with dimension $m_1 + m_2$.

6. **Question.** (***Lie Groups***) Identify which of the following sets along with the respective group operator qualify as lie group

(6-a) $G := \mathbb{R}^n \quad g(x, y) := x + y$

(6-b) $G := \mathbb{R} \quad g(x, y) := xy$.

(6-c) $G := G_1 \times G_2 \quad g((x_1, y_1), (x_2, y_2)) = (g_1(x_1, x_2), g_2(y_1, y_2))$ where G_1, G_2 are lie groups with group operators g_1 and g_2.

7. **Question.** (***Tangent Spaces***) Compute the associated tangent space to a point x on each of the following manifolds

(7-a) $\mathsf{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\|^2 = 1\}$

(7-b) $V \subset \mathbb{R}^n$, V is a subspace with dimension $V = m$

(7-c) $U \subset \mathbb{R}^n$, U is an open set.

Date: February 2, 2022.

§ 1. Exercises

1. **Question.** (***Submersions***) Comment whether the following maps are submersions or not.

(1-a) $\mathbb{R}^2 \setminus \{(0,0)\} \ni (x,y) \longmapsto f(x,y) := x^2 + y^3 \in \mathbb{R}$

(1-b) $\mathbb{GL}(2) \ni A \longmapsto f(A) := \det A \in \mathbb{R}$ where $\mathbb{GL}(2) = \{M \in \mathbb{R}^{2\times 2} \mid \det M \neq 0\}$.

$f: U \to V$ is a submersion if

for all $x \in U$ $Df(x)$ is an onto map. (Surjective)

$f: U \to V$ is an immersion if

for all $x \in U$ $Df(x)$ is an injective map

1) $R^2 \to R$: $(x, y) \longrightarrow x^2 + y^3$

$Df(x,y) = [2x, \ 3y^2]$ ✓

rank $Df(x,y) = 1$ for all $(x,y) \neq 0$

2) $GL(R,2) \longmapsto R$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} =$ det (A)

$\det(A) = ad - bc$

$Df = [d \ \ -c \ \ -b \ \ a]$

unless $a, b, c, d = 0$

$\text{rank}(Df) = 1 \Rightarrow$ it's a submersion

3) $f: R \to R^2$ and it be a submersion ✗

$f(x,y) = x = [1, 0]$

$f(x,y) = (x, 0)$

$f: \mathbb{R}^2 \to \mathbb{R}^2 \quad f(x,y) = (x,0)$

[Diffeomorphism $\subseteq$ Submersions.]

2. **Question.** (***Immersions***) Comment whether the following maps are immersions or not.

(2-a) $t \in \mathbb{R} \longmapsto f(t) := (\sin(t), \cos(t)) \in \mathbb{R}^2 \to (t, t^3)$

(2-b) $t \in \mathbb{R} \setminus \{0\} \longmapsto f(t) := (t^2, t^4) \in \mathbb{R}^2 \leftarrow \{t \in \mathbb{R} \longmapsto f(t) = (t^2, t^4)\}$ is not an immersion]

(2-c) $t \in [0, 2\pi[\longmapsto f(t) := (\sin(t), \cos(t)) \in \mathbb{R}^2. \leftarrow$

$[0, 2\pi) \longmapsto f(t) = (\sin t, \cos t)$

Q 2(a) $\quad f(t) := [\sin(t), \cos(t)]$

$]a, b[= (a, b)$

$Df(t) = \begin{bmatrix} \cos t \\ -\sin(t) \end{bmatrix}$, rank $(Df) = 1$ for all t

and hence it is an immersion.

$Df(t): \mathbb{R} \to \mathbb{R}^2$

$f(x+t) = f(x) + Df(x)t + o(t)$, $\quad \mathbb{R}^n$

$\frac{o(t)}{\|t\|} \to 0 \quad t \to 0$

$t \in \mathbb{R}^n$.

[$f: U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$]

$Df(t): \mathbb{R}^n \to \mathbb{R}^m$.

$v \longmapsto Df(x).v$

Surjective $\Rightarrow$ Submersion

Injective $\Rightarrow$ Immersion

Surjective & Injective (Bijective) $\Rightarrow$ Diffeomorphism.

3. **Question.** (***Embedding***) Comment which of the maps given in Question 2, are embedding..

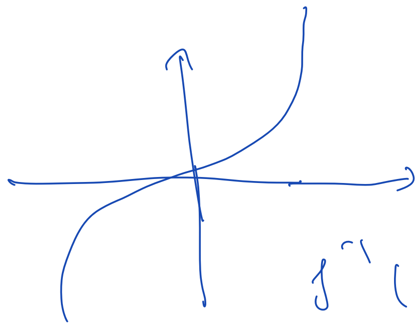
$(0, 2\pi) \ni t \longmapsto (\sin t, \cos t)$

$f: U \to f(U)$ needs to be an homeomorphism.

f, f^{-1} are well defined and are continuous on U & $f(U)$.

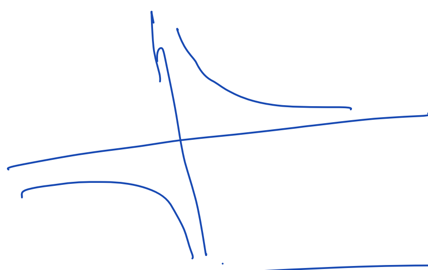
$f: t \longmapsto (t, t^3)$ immersion ✓

embedding ?



$f^{-1}(t, t^3) = t$

$f: [t \longmapsto 1/t]$



Manifolds: a set M is called a manifold iff $M \subset \mathbb{R}^n$

a) for each $a \in M$, open set $U \ni a$, & submersion $\varphi: U \longrightarrow \mathbb{R}^k$ such that $M \cap U = f^{-1}(c)$ where $c \in \mathbb{R}^k$

a) manifolds, also give their dimension.

(4-a) $V \subset \mathbb{R}^n$, V is a subspace with $\dim V = m$.

(4-b) $\mathsf{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\}$

(4-c) $\mathsf{SO}(n) := \{A \in \mathbb{R}^{n\times n} \mid A^\top A = AA^\top = I, \det A = 1\}$

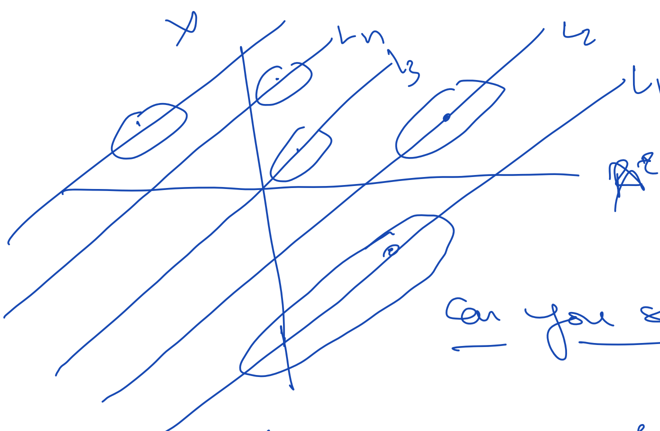
(4-d) $\mathsf{T}^2 = \mathsf{S}^1 \times \mathsf{S}^1$

5. **Question.** (***Product Manifolds***) Let $M_1 \subset \mathbb{R}^{n_1}$ be an embe

b) $S^n = \{x \in R^{n+1} \mid \underline{\|x\|^2 = 1}\}$. $\quad U = R^n$, $M \cap U = M$

$\varphi: R^{n+1} \to R \qquad \varphi(x) = \|x\|^2$

$S^n = \{ f^{-1}\{1\}$ $\Rightarrow$ this is a manifold.

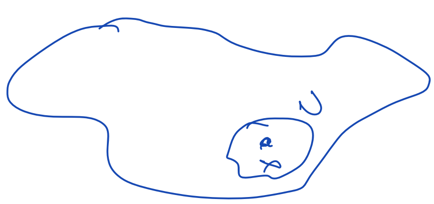


$[M = \bigcup L_i]$

Can you show that M is a Manifold.

L_1 is a manifold.

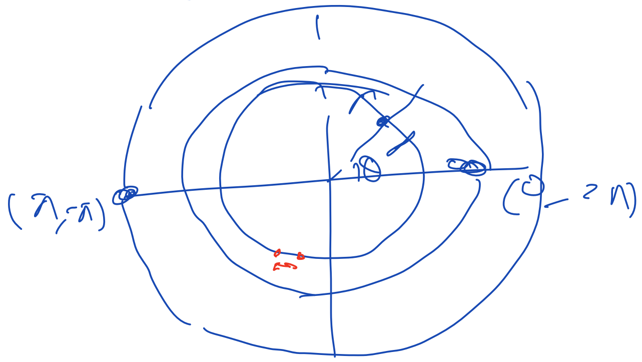
b) Immersion | Embedding condition:- for any $x \in M$, $\exists\, U \ni x$, an $V \subseteq R^m$ & an Immersion φ such that $\varphi(V) = M \cap U$



$M \cap U = \varphi(V)$

⇒ Generally φ: must be an embedding.

$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

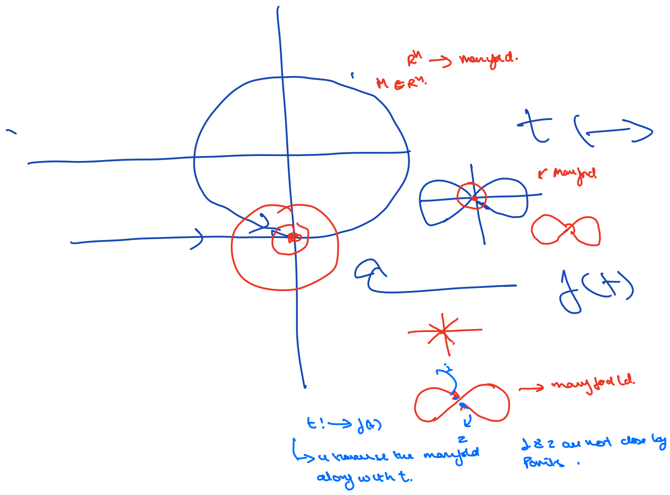


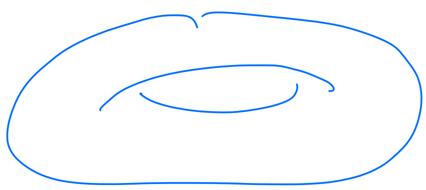
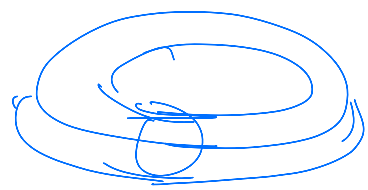
$(x,y) = (\cos\theta, \sin\theta)$

$\theta \longmapsto \varphi_1(\theta) = (\cos\theta, \sin\theta)$

$\theta \in (0, 2\pi)$, $\varphi_2 : (-\pi, \pi)$, $\theta = (\cos\theta, \sin\theta)$

$t \longmapsto f(t) \in \mathbb{R}^2$



5. **Question.** (*Product Manifolds*) Let $M_1 \subset \mathbb{R}^{n_1}$ be an embedded submanifold of dimension m_1, and $M_2 \subset \mathbb{R}^{n_2}$ be an embedded submanifold of dimension m_2. Show that $M_1 \times M_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a submanifold with dimension $m_1 + m_2$.

$\Rightarrow$ $x \in M_1$, $y \in M_2$, $(x,y) \in M_1 \times M_2$

Since M_1 & M_2 are submanifold $\exists$ U_1, U_2, V_1, V_2

& φ_1, & φ_2 $[U_1 \ni x \quad U_2 \ni y]$

s.t $\left.\begin{array}{l} \varphi_1(V_1) = M_1 \cap U_1 \\ \varphi_2(V_2) = M_2 \cap U_2 \end{array}\right\}$

$\varphi : V_1 \times V_2 \longrightarrow U_1 \times U_2$

$\varphi(x,y) := [\underbrace{\varphi_1(x)}_{\text{homeomorphism on } U_1}, \varphi_2(y)]$ homeomorphis U_2

components homeomorphisms $\Rightarrow$ φ is an homeomorphism

Claim

1) $\varphi(V_1 \times V_2) = (M_1 \times M_2) \cap (U_1 \times U_2)$ $\Rightarrow$ φ is an embedding.

2) φ is an immersion.

3) φ is an embedding.

$\varphi(V_1 \times V_2) = \varphi_1(V_1) \times \varphi_2(V_2)$

$= (M_1 \cap U_1) \times (M_2 \cap U_2)$

$= (M_1 \times M_2) \cap (U_1 \times U_2)$

2) φ is an immersion

$D\varphi = \begin{bmatrix} D\varphi_1 & 0 \\ 0 & D\varphi_2 \end{bmatrix}$ Since $D\varphi_1$ is injective
$D\varphi_2$ is inj

$\Rightarrow$ $D\varphi$ is injective $\Rightarrow$ φ is immersion.

$\Rightarrow$ $M_1 \times M_2$ is a submanifold.

$T_2 = S_1 \times S_1$ $\Rightarrow$ it is a manifold.

Note: $S_1 \times S_1 \neq S^2$

Trivial result: Every open set of $\mathbb{R}^n$ is a regular manifold

6. **Question.** (*Lie Groups*) Identify which of the following sets along with the respective group operator qualify as lie group

(6-a) $G := \mathbb{R}^n \quad g(x, y) := x + y$

(6-b) $G := \mathbb{R} \quad g(x, y) := xy.$ $\mathbb{R} \setminus \{0\}$

(6-c) $G := G_1 \times G_2 \quad g((x_1, y_1), (x_2, y_2)) = (g_1(x_1, x_2), g_2(y_1, y_2))$ where G_1, G_2 are lie groups with group operators g_1 and g_2.

a) $(G, g(x,y))$ set with a binary operation attached to it satisfying:

1) for all $x, y \in G$, $g(x,y) \in G$.

2) Identity element: e.

$g(x, e) = g(e, x) = x.$

3) Inverse element: for all $x \in G$ $\exists$ $y \in G$.

s.t. $g(x,y) = g(y,x) = e.$

4) Associativity

$g(x, g(y,z)) = g(g(x,y), z))$

Often we denote.

$g(x,y)$ by $x \cdot y.$

$S^1 \to$ group, $SO(3)$ is a group.

is set of inverse matrices under multiplication is a group

$I : (x) \longrightarrow y$ s.t $g(x, y) = e$

A group is a lie group if I, $g(x \cdot y)$ are smooth.

$\longrightarrow$ lie group is a manifold in its own right and also a sub manifold in appropriate euclidean space.

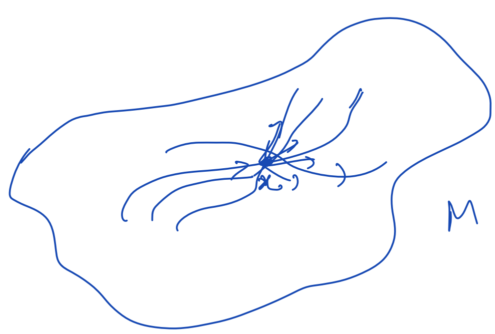
groups with group operators g_1 and g_2.

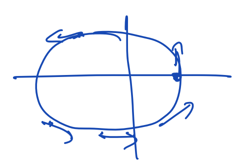
7. **Question.** (***Tangent Spaces***) Compute the associated tangent space to a point x on each of the following manifolds

(7-a) $S^2 = \{x \in \mathbb{R}^3 \mid \|x\|^2 = 1\}$

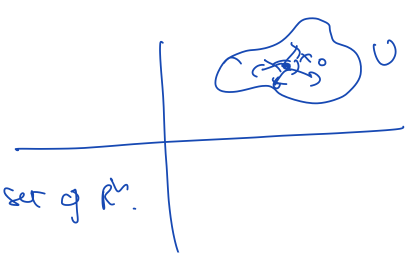
(7-b) $V \subset \mathbb{R}^n$, V is a subspace with dimension $V = m$

(7-c) $U \subset \mathbb{R}^n$, U is an open set.



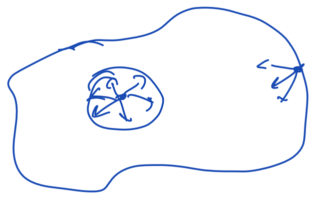


$T_x M = \{$ all velocity vectors that a curve lying on M and passing through x_0 can possess $\}$.



$T_x U = \{R^n\}$

U is an open set of R^n.



$S^2 = \{ \|x\|^2 = 1 \}$.

$x(t)$ on S^2

$\|x(t)\|^2 = 1$

$$\frac{d}{dt} \|x(t)\|^2 = 0$$

$$x^T(t) \cdot \frac{d(x)}{dt} = 0$$

$$x^T(t) \cdot v = 0$$

$$T_x S^2 = \{ v \in R^3 \mid x^T v = 0 \}$$

For an manifold defined as

$$M = \{ x \in R^n \mid f(x) = c \}$$

$$T_x M = [\mathrm{Ker}\,(Df(x))]$$

$$M = \{ \varphi(x) \mid x \in R^m \}$$ immersion only.

$$T_x M = \mathrm{Im}\,(D\varphi)$$

eg.: $S^1 = \{(\sin\theta, \cos\theta) \mid \theta \in [0, 2\pi[\}$.

$$T_\theta S^1 = [(\cos\theta, -\sin\theta)].$$

Tangent space $\Rightarrow$ $(x, T_x M)$ $\rightarrow$ Bond vector notation

$$TM :\rightarrow \bigcup_{x \in M} [x, T_x M]$$ $\rightarrow$ shows that this is a manifold.

Proof that TM is a manifold..

$$SO(N) = \{ A^T A = A A^T = I \}$$

$$\Rightarrow \frac{d}{dt} (A^T(t) \; A(t)) = 0$$

$$A^T(t) \frac{d A(t)}{dt} + \frac{d A^T(t)}{dt} . A(t) = 0$$

$$A^T . v + v^T A = 0$$

$$\underline{T_A SO(N) = \{ v \in R^{n \times n} \mid A^T v + v^T A = 0 \}}$$

Dimension of a Manifold.

$$= \dim(M) = \dim(T_x M)$$

For a connected Manifold $\dim(T_{x_1} M) = \dim(T_{x_2} M)$

x_1, x