AE 242
Aerospace Measurements Laboratory

Second-order system

a_2 a_1, a_0 and b_0 are non zero in the generalised equation. Any instrument which can be represented using this equation is a second - order instrument.

$$\boxed{a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i}$$

Roots of this equation can be imaginary, complex, real

Above equation can be written as

$$\left(\frac{D^2}{\omega_n^2} + \frac{2\xi D}{\omega_n} + 1 \right) q_0 = K q_i$$

$$K = \frac{b_0}{a_0}$$

Static sensitivity

$$\omega_n = \sqrt{\frac{a_0}{a_2}}$$

Undamped natural frequency

$$\xi = \frac{a_1}{2\sqrt{a_0 a_2}}$$

Damping ratio

Transfer function

$$\frac{q_0}{q_i}(D) = \frac{K}{D^2/\omega_n^2 + 2\xi D/\omega_n + 1}$$

Second-order system – an example

Force measuring spring

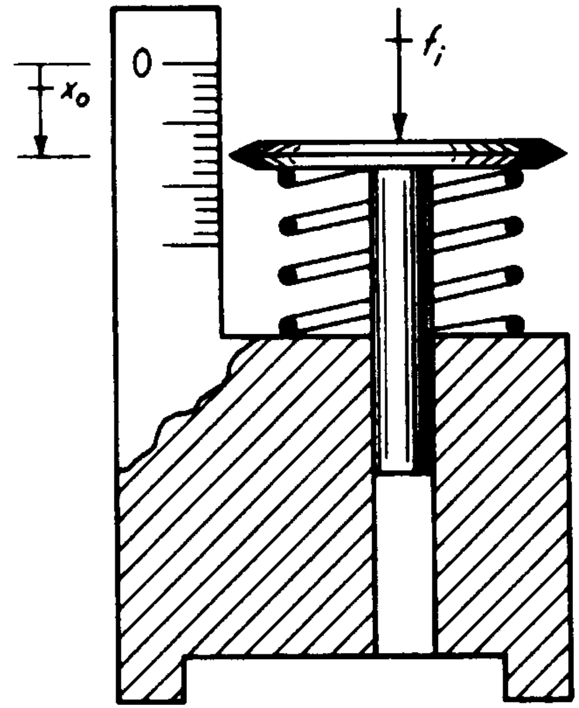
Total mass – M

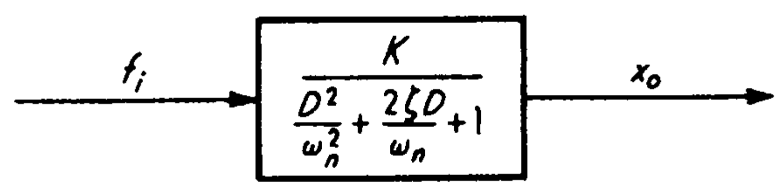
Spring constant – K_s

Damping - B

$$(\mathbf{MD}^2 + \mathbf{BD} + \mathbf{K_s})\mathbf{x}_0 = \mathbf{f}_i$$

$$\mathbf{K} = \frac{1}{\mathbf{K_s}} \qquad \omega_n = \sqrt{\frac{\mathbf{K_s}}{\mathbf{M}}} \qquad \xi = \frac{\mathbf{B}}{2\sqrt{\mathbf{K_s M}}}$$





ω_n is direct indication of speed of response

Second order systems

$$\left(\frac{D^2}{\omega_n^2}+\frac{2D\zeta}{\omega_n}+1\right)q_o = Kf_i$$

Two roots of the characteristic equation

$$-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1} \quad \text{and} \quad -\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$$

$$\text{Undamped}: \zeta = 0, \text{roots} = \pm i\omega_n$$

$$\frac{q_0/K}{q_{is}} = 1-\sin(\omega_n t+\phi)$$

Initial conditions are zero

$$q_0 = \dot{q}_0 = 0$$

$$\text{Underdamped}: 0<\zeta<1.0, \ \text{roots} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$\frac{q_0/K}{q_{is}} = 1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$$

Second order systems

$$\left(\frac{D^2}{\omega_n^2}+\frac{2D\zeta}{\omega_n}+1\right)q_o = Kf_i$$

$$\boxed{\text{Over damped}:\ \zeta > 1.0,\ \text{roots} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2-1}\right) = \frac{1}{\tau_1}, \frac{1}{\tau_2}}$$

Two roots of the characteristic equation

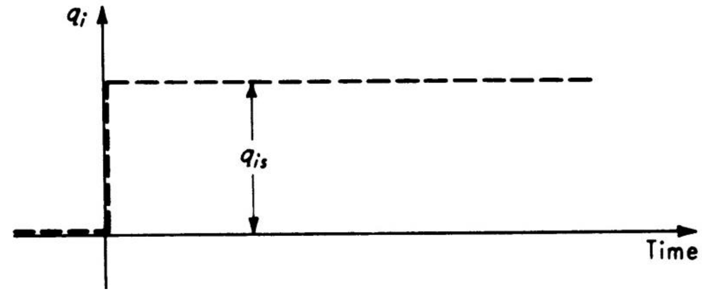
$$\boxed{\frac{q_0/K}{q_{is}} = \left(1-\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{-t/\tau_1}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{-t/\tau_2}\right)}$$

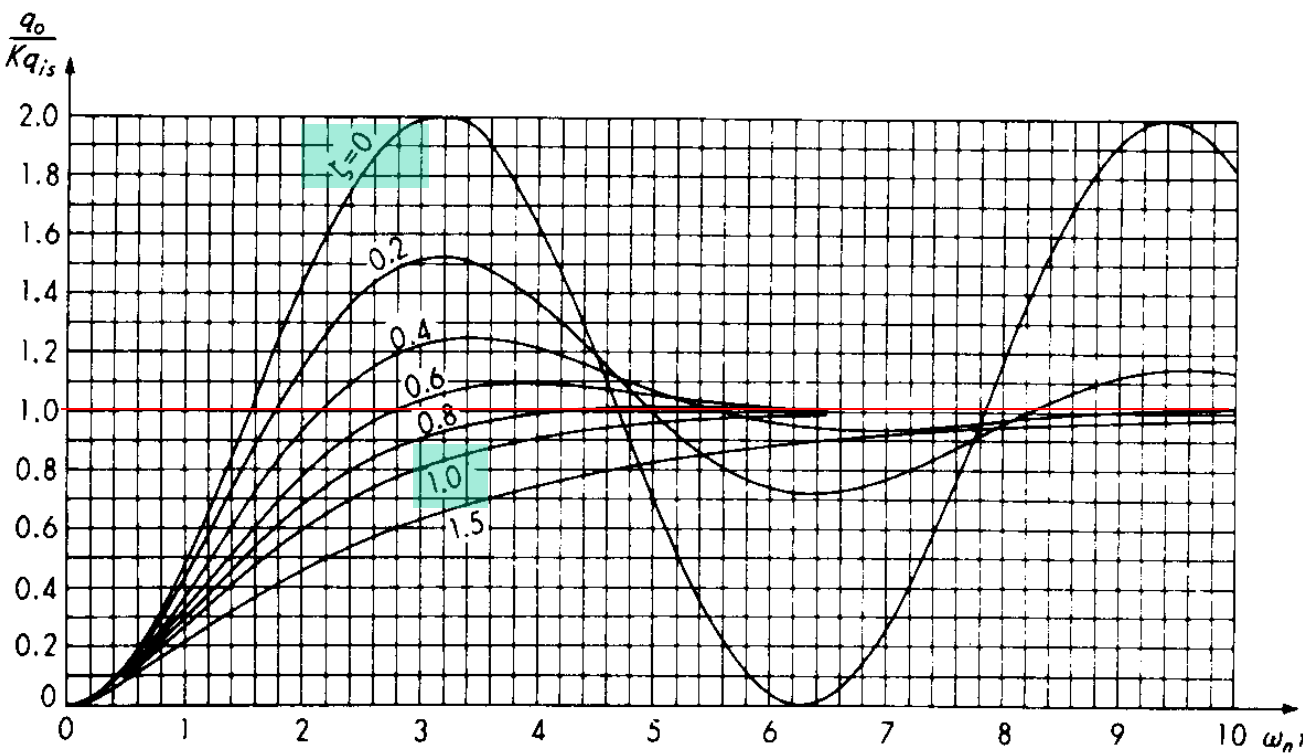
$$\text{Crtically damped}:\ \zeta = 1.0,\ \text{roots} = \omega_n$$

$$\boxed{\frac{q_0/K}{q_{is}} = \left(1-(1+\omega_n t)e^{-\omega_n t}\right)}$$

Second-order system – Step response

$$\left(\frac{D^2}{\omega_n^2} + \frac{2\xi D}{\omega_n} + 1\right) q_0 = K q_{is}$$

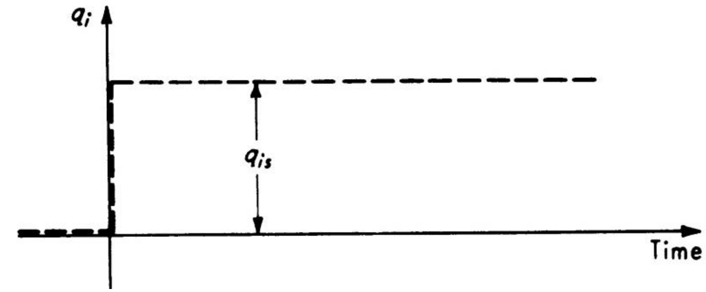




Observe the time response for different damping ratio.

Second-order system – Step response

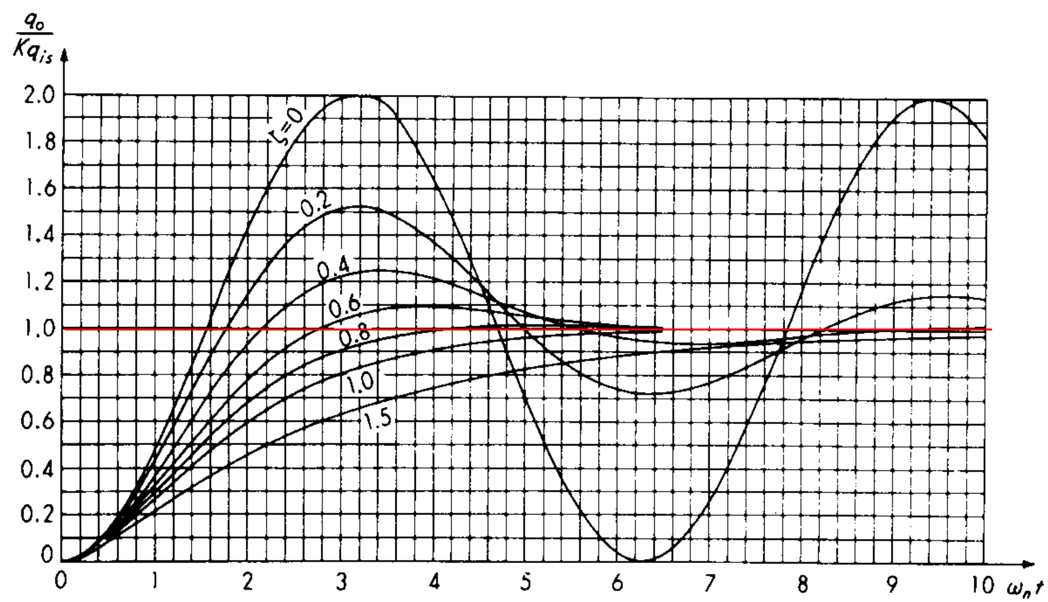
$$\left(\frac{\mathbf{D}^2}{\omega_n^2}+\frac{2\xi\mathbf{D}}{\omega_n}+1\right)\mathbf{q}_0=\mathbf{K}\mathbf{q}_{is}$$



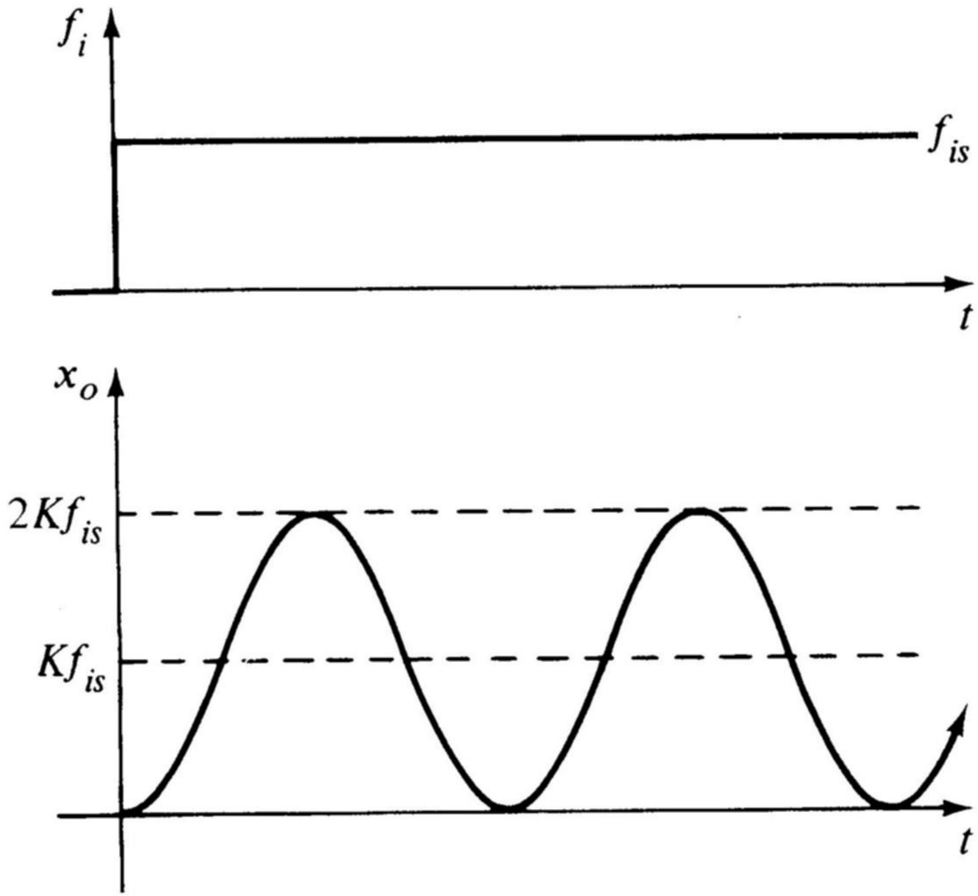
Under damping – Damping ratio less than 1. This will always give overshoot before reaching steady state

Critical damping – Damping ratio equal to 1. this will take minimum time to achieve steady state output without overshoot

Over damping – Damping ratio greater than 1.this will take more time to achieve steady state compared to critical damping output without overshoot



Second order systems – Step input



Step response of undamped second order system $\zeta = 0$

Second order systems – Step input

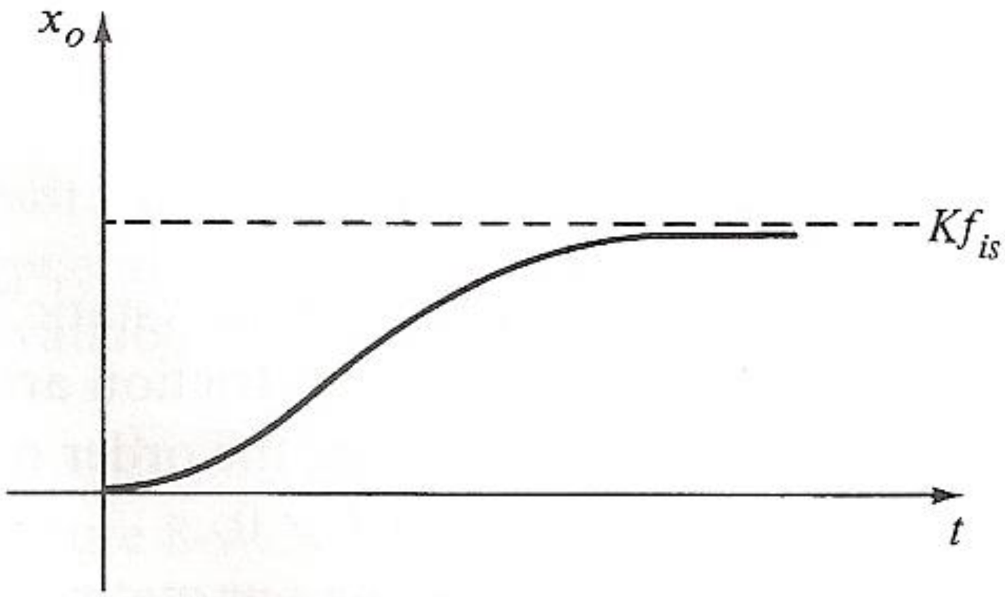


Figure 8-10 Step response of critically damped second-order system.

Step response of critically second order system ζ =1

$$q_o = Kq_{is}(1-(1+\omega_n t)e^{-\omega_n t})$$

Second order systems – ramp input

$$\left(\frac{D^2}{\omega_n^2}+\frac{2D\zeta}{\omega_n}+1\right)q_o = K\dot{q}_{is}t$$

$$-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1} \text{ and } -\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$$

Two roots of the characteristic equation

$$\text{Underdamped}: 0<\zeta<1.0,\ \text{roots} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

$$\frac{q_0}{K} = \dot{q}_{is}t - \frac{2\zeta\dot{q}_{is}}{\omega_n}\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n t\sqrt{1-\zeta^2}+\phi\right)\right]$$

Second order systems

$$\left(\frac{D^2}{\omega_n^2}+\frac{2D\zeta}{\omega_n}+1\right)q_o = K\dot{q}_{is}t$$

Two roots of the characteristic equation

$$-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1} \quad \text{and} \quad -\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$$

$$\text{Critically damped}: \zeta = 1.0, \ \text{roots} = -\omega_n$$

$$\frac{q_0}{K} = \dot{q}_{is}t - \frac{2\dot{q}_{is}}{\omega_n}\left[1-e^{-\omega_n t}\left(1+\frac{\omega_n t}{2}\right)\right]$$

Over damped case

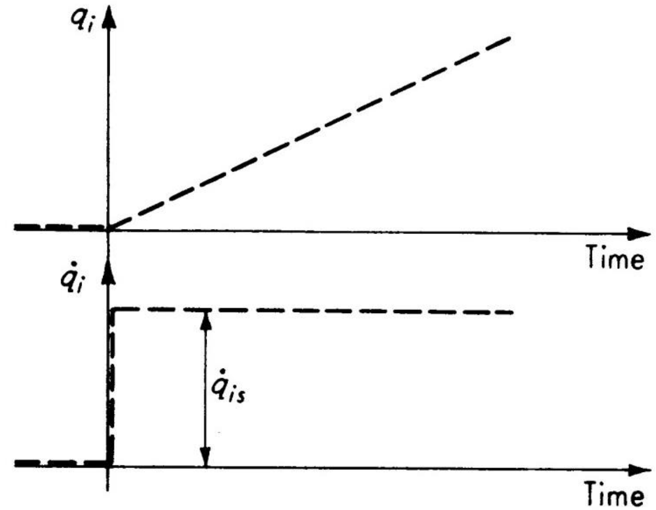
$$\frac{q_0}{K} = \dot{q}_{is}t - \frac{2\zeta\dot{q}_{is}}{\omega_n}\left(1+\frac{2\zeta^2-1-2\zeta\sqrt{\zeta^2-1}}{4\zeta\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + \frac{-2\zeta^2-1-2\zeta\sqrt{\zeta^2-1}}{4\zeta\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}\right)$$

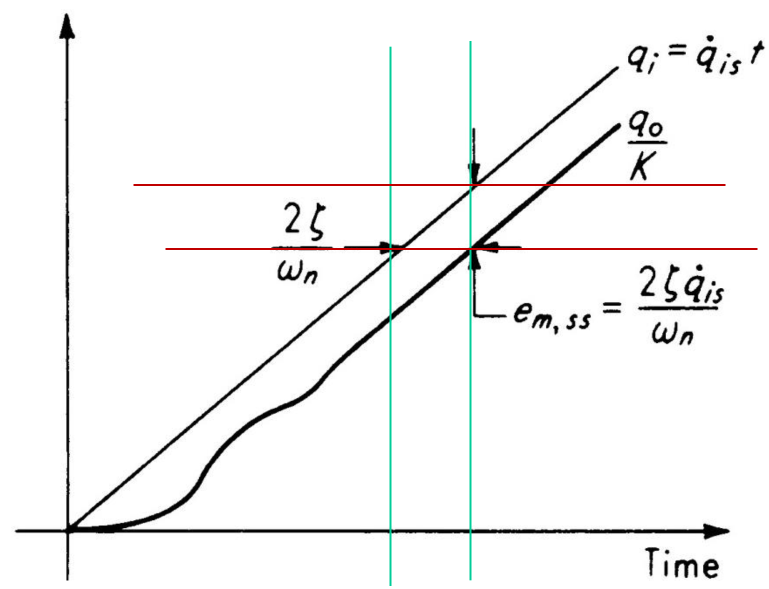
Second-order system – Ramp response

Steady state error $2\xi\dot{\mathbf{q}}_{is} / \omega_n$

Time lag $2\xi / \omega_n$

Steady state error can be reduced by reducing ξ and increasing ω_n This will increase the oscillations in the output





Second order systems

$$\left(\frac{D^2}{\omega_n^2}+\frac{2D\zeta}{\omega_n}+1\right)q_o = Kq_i\sin(\omega t)$$

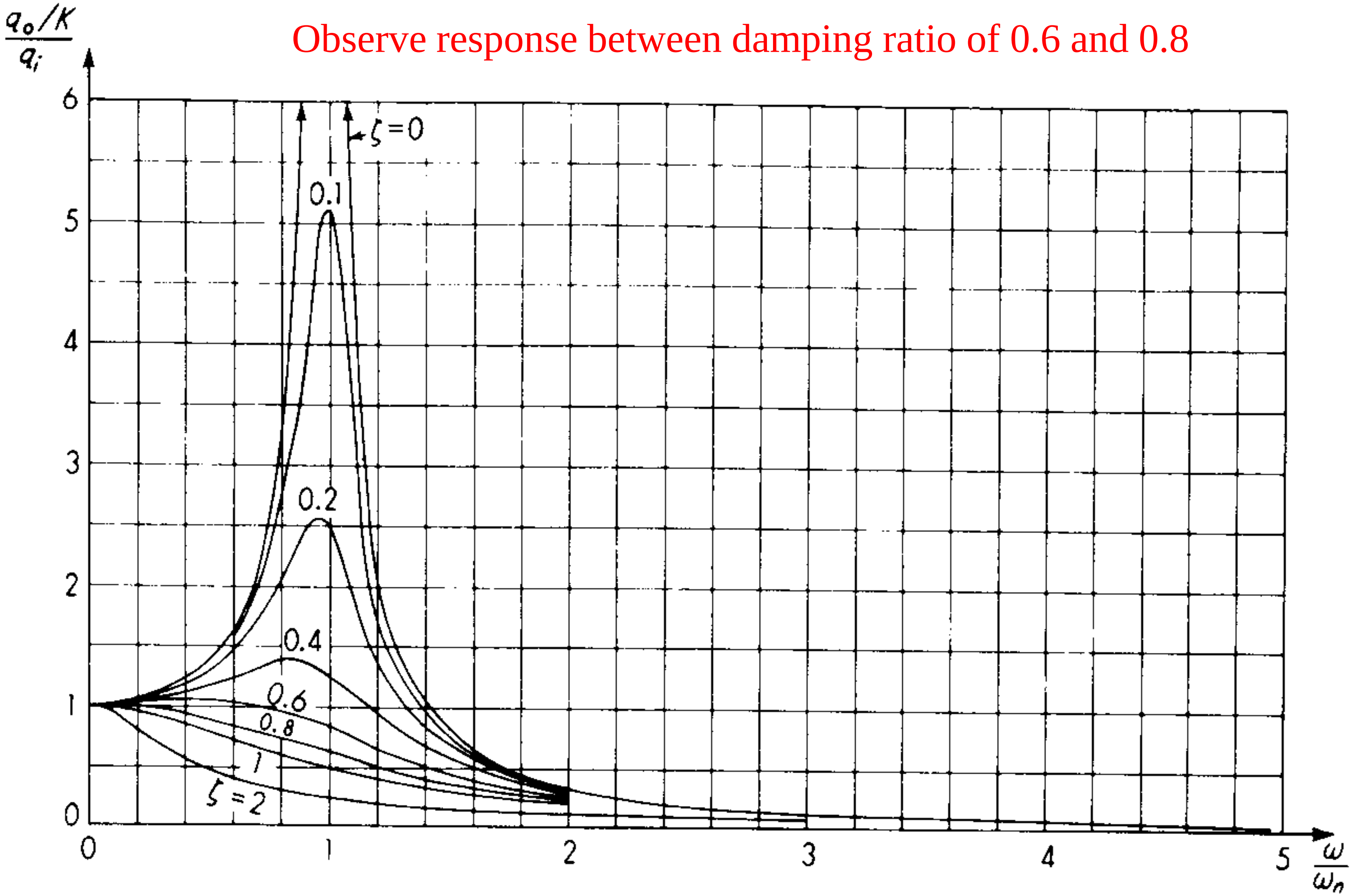
$$\boxed{\frac{q_0}{q_i}(i\omega)=\frac{K}{(i\omega/\omega_n)^2+2\zeta i\omega/\omega_n+1}}$$

$$\boxed{\begin{aligned}\frac{q_0/K}{q_i}(i\omega)&=\frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2+4\zeta^2(\omega/\omega_n)^2}}\angle\phi\\ \phi&=\tan^{-1}\frac{2\zeta}{\omega/\omega_n-\omega_n/\omega}\end{aligned}}$$

Output is dependent on input frequency. It will peak close to natural frequency upto some damping ratio. Phase difference is also dependent on the input frequency.

Second-order system – frequency response

Observe response between damping ratio of 0.6 and 0.8



Second-order system – frequency response

Amplitude ratio strongly related to small and large damping ratio.

Flat response for damping of 0.6- 0.7

Linear variation of phase for above damping

Damping in the above range is a good choice for second order system

Frequency up to 0.4 of natural frequency