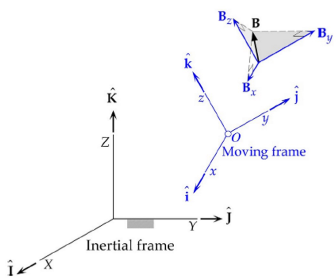
Lecture 4

- Recall that $\frac{dB}{dt} = \left.\frac{dB}{dt}\right)_{rel} + \Omega \times B$



- $$\frac{d^2B}{dt^2} = \frac{d}{dt}\left.\frac{dB}{dt}\right)_{rel} + \frac{d\Omega}{dt} \times B + \Omega \times \frac{dB}{dt}$$

$$= \frac{d}{dt}\left.\frac{dB}{dt}\right)_{rel} + \frac{d\Omega}{dt} \times B + \Omega \times \left[\left.\frac{dB}{dt}\right)_{rel} + \Omega \times B\right]$$

- $$C = \left.\frac{dB}{dt}\right)_{rel}$$

$$\frac{dC}{dt} = \left.\frac{dC}{dt}\right)_{rel} + \Omega \times C$$

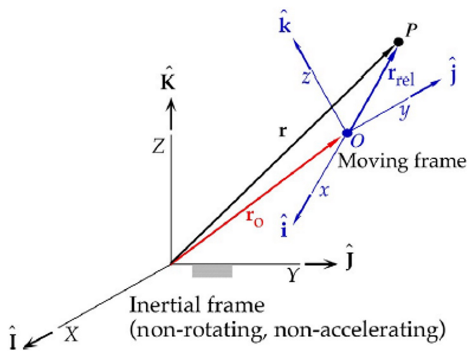
$$\frac{d}{dt}\left.\frac{dB}{dt}\right)_{rel} = \left.\frac{d^2B}{dt^2}\right)_{rel} + \Omega \times \left.\frac{dB}{dt}\right)_{rel}$$

- $$\frac{d^2B}{dt^2} = \left.\frac{d^2B}{dt^2}\right)_{rel} + \Omega \times \left.\frac{dB}{dt}\right)_{rel} + \frac{d\Omega}{dt} \times B$$

$$+ \Omega \times \left.\frac{dB}{dt}\right)_{rel} + \Omega \times (\Omega \times B)$$

$$= \left.\frac{d^2 B}{dt^2}\right)_{rel} + 2\,\Omega \times \left.\frac{dB}{dt}\right)_{rel} + \frac{d\Omega}{dt} \times B$$

$$+ \Omega \times (\Omega \times B)$$



- $r = r_0 + r_{rel}$

- r_{rel} is measured in the moving frame,
  i.e., $r_{rel} = x\hat{i} + y\hat{j} + z\hat{k}$

- $$V = \frac{dr}{dt} = V_0 + \frac{dr_{rel}}{dt}$$

$$= \underbrace{V_0}_{} + \underbrace{V_{rel}}_{\downarrow} + \Omega \times r_{rel}$$

$$\left.\frac{dr_{rel}}{dt}\right)_{rel} = \underbrace{\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}}_{},$$

- $$a = \frac{dv}{dt} = \frac{d^2 r}{dt^2} = a_0 + \frac{d^2 r_{rel}}{dt^2}$$

$$= a_O + \underbrace{a_{rel}}_{\frac{d^2 r_{rel}}{dt^2}\Big|_{rel}} + \underbrace{2\,\Omega \times \frac{dr_{rel}}{dt}\Big|_{rel}}_{\text{Coriolis acceleration (named after Gaspard-Gustave de Coriolis)}} + \frac{d\Omega}{dt} \times r_{rel} + \Omega \times (\Omega \times r_{rel})$$

– Also known as the five-term acceleration formula.

Example

At a given instant, the absolute position, velocity, and acceleration of the origin O of a moving frame are

$$\left.\begin{aligned} \mathbf{r}_O &= 100\hat{\mathbf{I}} + 200\hat{\mathbf{J}} + 300\hat{\mathbf{K}}\ (\text{m}) \\ \mathbf{v}_O &= -50\hat{\mathbf{I}} + 30\hat{\mathbf{J}} - 10\hat{\mathbf{K}}\ (\text{m/s}) \\ \mathbf{a}_O &= -15\hat{\mathbf{I}} + 40\hat{\mathbf{J}} + 25\hat{\mathbf{K}}\,(\text{m/s}^2) \end{aligned}\right\}(\text{given}) \qquad \text{(a)}$$

The angular velocity and acceleration of the moving frame are

$$\left.\begin{aligned} \boldsymbol{\Omega} &= 1.0\hat{\mathbf{I}} - 0.4\hat{\mathbf{J}} + 0.6\hat{\mathbf{K}}(\text{rad/s}) \\ \dot{\boldsymbol{\Omega}} &= -1.0\hat{\mathbf{I}} \times 0.3\hat{\mathbf{J}} - 0.4\hat{\mathbf{K}}\,(\text{rad/s}^2) \end{aligned}\right\}(\text{given}) \qquad \text{(b)}$$

The unit vectors of the moving frame are

$$\left.\begin{aligned} \hat{\mathbf{i}} &= \ \ 0.5571\hat{\mathbf{I}} + 0.7428\hat{\mathbf{J}} + 0.3714\hat{\mathbf{K}} \\ \hat{\mathbf{j}} &= -0.06331\hat{\mathbf{I}} + 0.4839\hat{\mathbf{J}} - 0.8728\hat{\mathbf{K}} \\ \hat{\mathbf{k}} &= -0.8280\hat{\mathbf{I}} + 0.4627\hat{\mathbf{J}} + 0.3166\hat{\mathbf{K}} \end{aligned}\right\}(\text{given}) \qquad \text{(c)}$$

The absolute position, velocity, and acceleration of P are

$$\left.\begin{aligned} \mathbf{r} &= 300\hat{\mathbf{I}} - 100\hat{\mathbf{J}} + 150\hat{\mathbf{K}}(\text{m}) \\ \mathbf{v} &= 70\hat{\mathbf{I}} + 25\hat{\mathbf{J}} - 20\hat{\mathbf{K}}(\text{m/s}) \\ \mathbf{a} &= 7.5\hat{\mathbf{I}} - 8.5\hat{\mathbf{J}} + 6.0\hat{\mathbf{K}}\,(\text{m/s}^2) \end{aligned}\right\}(\text{given}) \qquad \text{(d)}$$

Find (a) the velocity $\mathbf{v}_{rel}$ and (b) the acceleration $\mathbf{a}_{rel}$ of P relative to the moving frame.

Details

(a) $V_{rel} = V - V_0 - \Omega \times r_{rel}$

$r_{rel} = r - r_0$

Solve for $\hat{I}, \hat{J}, \hat{K}$ in terms of $\hat{i}, \hat{j}, \hat{k}$

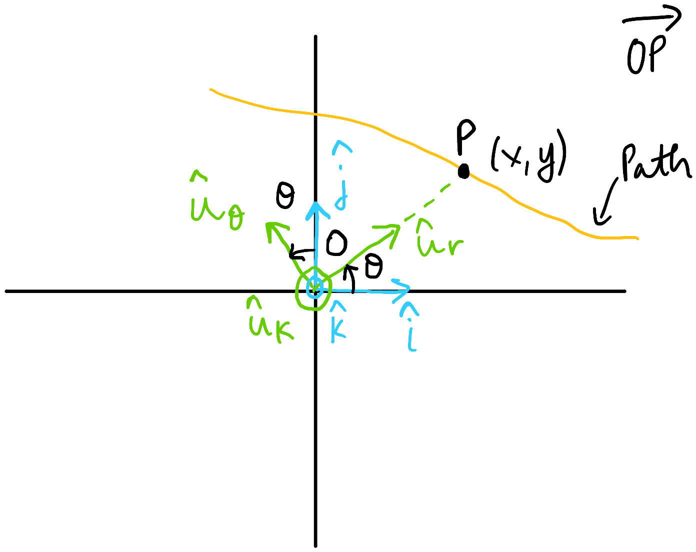
(b) $a_{rel} = a - a_0 - 2\Omega \times v_{rel} - \dot{\Omega} \times r_{rel}$
$- \Omega \times (\Omega \times r_{rel})$

Write $\hat{I}, \hat{J}, \hat{K}$ in terms of $\hat{i}, \hat{j}, \hat{k}$

Example

Find v and a in terms of r, θ and their derivatives expressed in terms of $\hat{u}_r$ and $\hat{u}_\theta$.

$\overrightarrow{OP} = r\,\hat{u}_r$



Details

$V = \cancel{V_0} + V_{rel} + \Omega \times r_{rel}$

$$= \dot{r}\hat{u}_r + \dot{\theta}\hat{u}_k \times r\hat{u}_r$$

$$= \dot{r}\hat{u}_r + r\dot{\theta}\,\hat{u}_k \times \hat{u}_r$$

$$= \dot{r}\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$$

$$a = \cancel{a_0} + a_{rel} + 2\Omega \times v_{rel} + \dot{\Omega} \times r_{rel} + \Omega \times (\Omega \times r_{rel})$$

$$= \ddot{r}\hat{u}_r + 2\dot{\theta}\hat{u}_k \times \dot{r}\hat{u}_r + \ddot{\theta}\hat{u}_k \times r\hat{u}_r + \dot{\theta}\hat{u}_k \times (\dot{\theta}\hat{u}_k \times r\hat{u}_r)$$

$$= \ddot{r}\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_k \times \hat{u}_r + r\ddot{\theta}\,\hat{u}_k \times \hat{u}_r + r\dot{\theta}^2\,\hat{u}_k \times (\hat{u}_k \times \hat{u}_r)$$

$$= \ddot{r}\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_\theta + r\ddot{\theta}\,\hat{u}_\theta + -r\dot{\theta}^2\hat{u}_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta$$

Brute Force

Brute Force

$$v = \dot{x}\,\hat{i} + \dot{y}\,\hat{j}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{u}_r \\ \hat{u}_\theta \end{bmatrix}$$

$$v = \overset{\cdot}{\overline{r\cos\theta}}\left(\cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta\right)$$
$$+\ \overset{\cdot}{\overline{r\sin\theta}}\left(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta\right)$$

$$v = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\left(\cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta\right)$$
$$+\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\left(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta\right)$$

$$v = \dot{r}\cos^2\theta\,\hat{u}_r - \cancel{\dot{r}\cos\theta\sin\theta\,\hat{u}_\theta} - \cancel{r\dot{\theta}\sin\theta\cos\theta\,\hat{u}_r}$$
$$+\ r\dot{\theta}\sin^2\theta\,\hat{u}_\theta$$
$$+\ \dot{r}\sin^2\theta\,\hat{u}_r + \cancel{\dot{r}\sin\theta\cos\theta\,\hat{u}_\theta} + \cancel{r\dot{\theta}\sin\theta\cos\theta\,\hat{u}_r}$$
$$+\ r\dot{\theta}\cos^2\theta\,\hat{u}_\theta$$

$$= \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$$