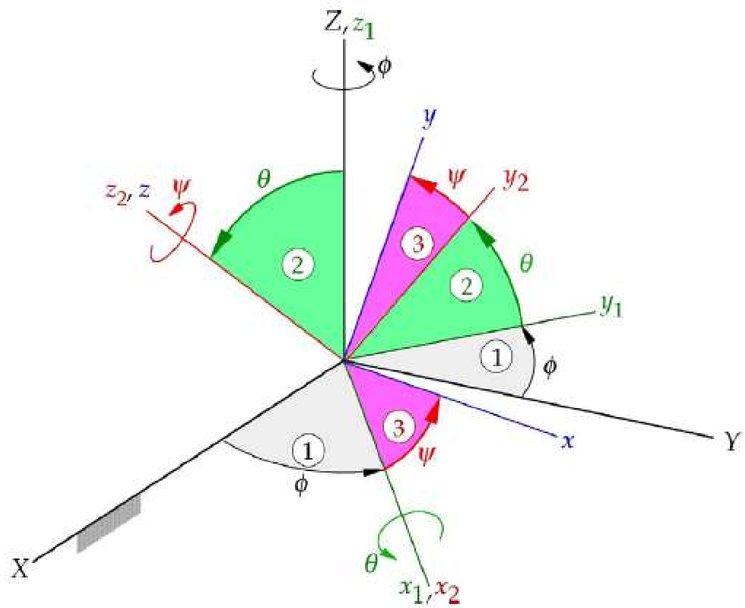
Euler Angles



$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_3(\psi)][\mathbf{R}_1(\theta)][\mathbf{R}_3(\phi)]$$

$$[\mathbf{R}_3(\psi)] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\mathbf{R}_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad [\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \sin\theta\sin\psi \\ -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\psi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{bmatrix}$$

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & -\sin\phi\cos\theta\cos\gamma - \cos\phi\sin\psi & \sin\psi\sin\theta \\ \cos\phi\cos\theta\sin\psi + \sin\phi\cos\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

- $w_p = \dot{\phi}$, $w_n = \dot{\theta}$, $w_s = \dot{\psi}$

- $w = w_x\hat{i} + w_y\hat{j} + w_z\hat{k}$

- $w = w_p\hat{K} + w_n\hat{i}_1 + w_s\hat{k}$

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = [\mathbf{R}_3(\phi)]^T \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = [\mathbf{R}_1(\theta)]^T \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = [\mathbf{R}_3(\psi)]^T \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

- $\hat{K} = \sin\theta \sin\psi \, \hat{i} + \sin\theta \cos\psi \, \hat{j} + \cos\theta \, \hat{k}$

- $\hat{i}_1 = \hat{i}_2 = \cos\psi \, \hat{i} - \sin\psi \, \hat{j}$

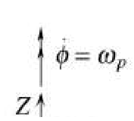
$$\boldsymbol{\omega} = (\omega_p \sin\theta \sin\psi + \omega_n \cos\psi)\hat{\mathbf{i}} + (\omega_p \sin\theta \cos\psi - \omega_n \sin\psi)\hat{\mathbf{j}} + (\omega_s + \omega_p \cos\theta)\hat{\mathbf{k}}$$

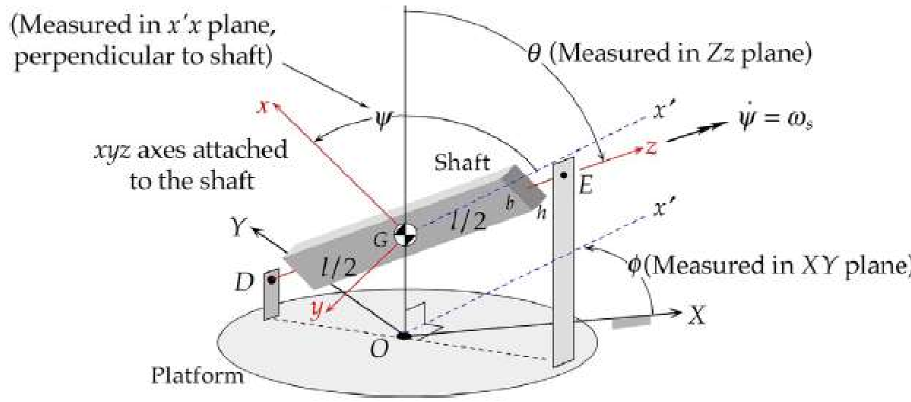
$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\sin\psi & -\sin\psi & 0 \\ \cos\theta & & 1 \end{bmatrix} \begin{Bmatrix} \omega_p \\ \omega_n \\ \omega_s \end{Bmatrix}$$

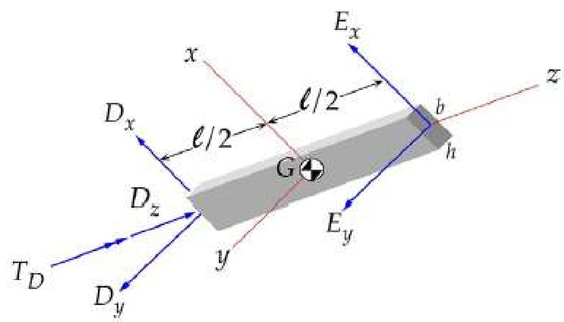
$$\begin{Bmatrix} \omega_p \\ \omega_n \\ \omega_s \end{Bmatrix} = \begin{bmatrix} \sin\psi/\sin\theta & \cos\psi/\sin\theta & 0 \\ \cos\psi & -\sin\psi & 0 \\ -\sin\psi/\tan\theta & -\cos\psi/\tan\theta & 1 \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Example

Fig. 11.24 shows a rotating platform on which is mounted a rectangular parallelepiped shaft (with dimensions b, h, and ℓ) spinning about the inclined axis DE. If the mass of the shaft is m, and the angular velocities ω_p and ω_s are constant, calculate the bearing forces at D and E as a function of ϕ and ψ. Neglect gravity, since we are interested only in the gyroscopic forces. (The small extensions shown at each end of the parallelepiped are just for clarity; the distance between the bearings at D and E is ℓ.)







Details

$$\omega_x = \omega_p \sin\theta \sin\psi, \quad \omega_y = \omega_p \sin\theta \cos\psi, \quad \omega_z = \omega_s + \omega_p \cos\theta$$

$$\dot{\omega}_x = \omega_p \omega_s \sin\theta \cos\psi, \quad \dot{\omega}_y = -\omega_p \omega_s \sin\theta \sin\psi, \quad \dot{\omega}_z = 0$$

$$F_{net} = m a_G \ , \ a_G = 0$$

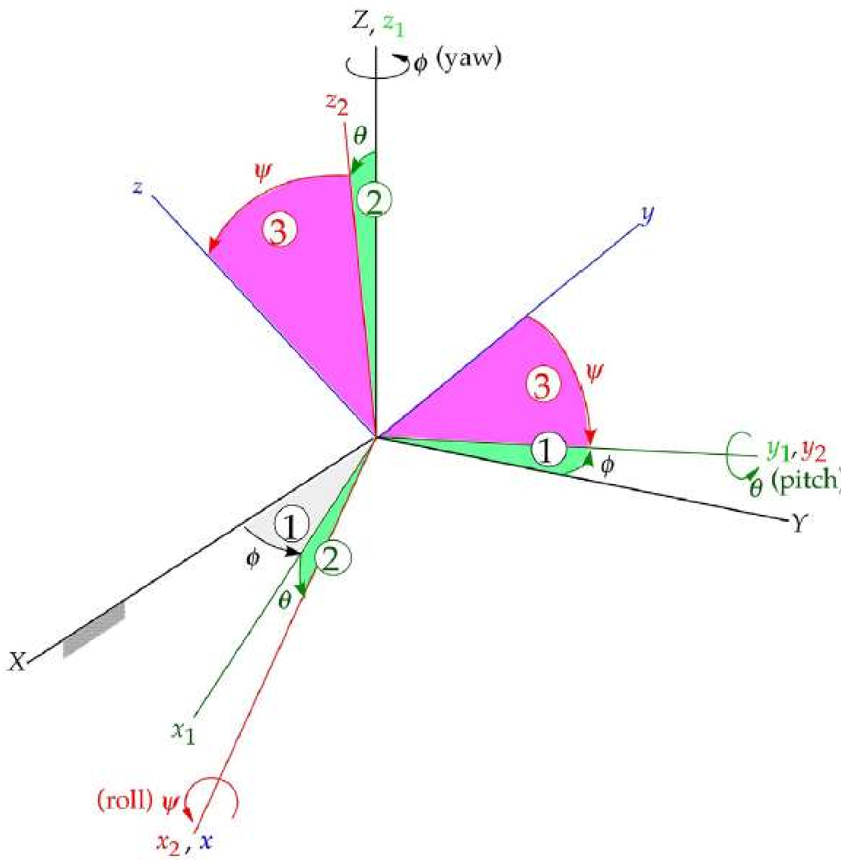
$$(D_x \hat{i} + D_y \hat{j} + D_z \hat{k}) + (E_x \hat{i} + E_y \hat{j}) = 0$$

$$\left.M_G\right)_{net} = \frac{\ell}{2} \hat{k} \times (E_x \hat{i} + E_y \hat{j}) + \left(-\frac{\ell}{2} \hat{k}\right) \times (D_x \hat{i} + D_y \hat{j} + D_z \hat{k}) + T_D \hat{k}$$

$$= D_y \ell \hat{i} - D_x \ell \hat{j} + T_D \hat{k}$$

$$\left.M_x\right)_{net} = A\dot{\omega}_x + (C-B)\omega_y \omega_z$$
$$\left.M_y\right)_{net} = B\dot{\omega}_y + (A-C)\omega_z \omega_x$$
$$\left.M_z\right)_{net} = C\dot{\omega}_z + (B-A)\omega_x \omega_y$$

Yaw, Pitch and Roll Angles



$$[\mathbf{Q}]_{Xx} = [\mathbf{R}_1(\psi)][\mathbf{R}_2(\theta)][\mathbf{R}_3(\phi)]$$

$$[\mathbf{R}_1(\psi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \quad [\mathbf{R}_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[\mathbf{R}_3(\phi)] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\psi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\theta\cos\psi \end{bmatrix}$$

$$[\mathbf{Q}]_{xX} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\sin\psi - \cos\phi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{bmatrix}$$

$$\boldsymbol{\omega} = \omega_{\text{yaw}}\hat{\mathbf{K}} + \omega_{\text{pitch}}\hat{\mathbf{j}}_2 + \omega_{\text{roll}}\hat{\mathbf{i}}$$

$$\omega_{\text{yaw}} = \dot{\phi} \qquad \omega_{\text{pitch}} = \dot{\theta} \qquad \omega_{\text{roll}} = \dot{\psi}$$

$$\begin{Bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{\mathbf{i}}_2 \\ \hat{\mathbf{j}}_2 \\ \hat{\mathbf{k}}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{Bmatrix}$$

$$\hat{\mathbf{K}} = -\sin\theta\hat{\mathbf{i}} + \cos\theta\sin\psi\hat{\mathbf{j}} + \cos\theta\cos\psi\hat{\mathbf{k}}$$

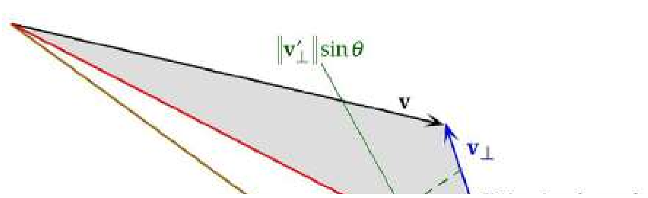
$$\hat{\mathbf{j}}_2 = \cos\psi\hat{\mathbf{j}} - \sin\psi\hat{\mathbf{k}}$$

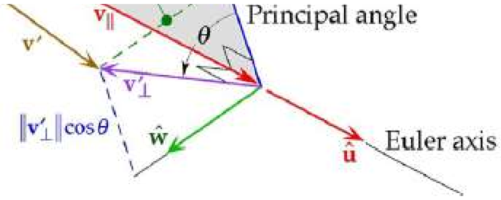
$$\begin{aligned} \boldsymbol{\omega} = &\left(-\omega_{\text{yaw}}\sin\theta + \omega_{\text{roll}}\right)\hat{\mathbf{i}} + \left(\omega_{\text{yaw}}\cos\theta\sin\psi + \omega_{\text{pitch}}\cos\psi\right)\hat{\mathbf{j}} \\ &+ \left(\omega_{\text{yaw}}\cos\theta\cos\psi - \omega_{\text{pitch}}\sin\psi\right)\hat{\mathbf{k}} \end{aligned}$$

$$\begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} -\sin\theta_{\text{pitch}} & 0 & 1 \\ \cos\theta_{\text{pitch}}\sin\psi_{\text{roll}} & \cos\psi_{\text{roll}} & 0 \\ \cos\theta_{\text{pitch}}\cos\psi_{\text{roll}} & -\sin\psi_{\text{roll}} & 0 \end{bmatrix} \begin{Bmatrix} \omega_{\text{yaw}} \\ \omega_{\text{pitch}} \\ \omega_{\text{roll}} \end{Bmatrix}$$

$$\begin{Bmatrix} \omega_{\text{yaw}} \\ \omega_{\text{pitch}} \\ \omega_{\text{roll}} \end{Bmatrix} = \begin{bmatrix} 0 & \sin\psi_{\text{roll}}/\cos\theta_{\text{pitch}} & \cos\psi_{\text{roll}}/\cos\theta_{\text{pitch}} \\ 0 & \cos\psi_{\text{roll}} & -\sin\psi_{\text{roll}} \\ 1 & \sin\psi_{\text{roll}}\tan\theta_{\text{pitch}} & \cos\psi_{\text{roll}}\tan\theta_{\text{pitch}} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Quaternions





- $v = v_{\parallel} + v_{\perp}$

- $v_{\parallel} = (v \cdot \hat{u})\,\hat{u}$

- $v_{\perp} = v - (v \cdot \hat{u})\,\hat{u}$

- $\|v_{\perp}'\| = \|v_{\perp}\|$

- $v_{\parallel}' = (v \cdot \hat{u})\,\hat{u}$

- $\hat{w} = \hat{u} \times \dfrac{v_{\perp}}{\|v_{\perp}\|}$

- $v_{\perp}' = \|v_{\perp}'\| \cos\theta \dfrac{v_{\perp}}{\|v_{\perp}\|} + \|v_{\perp}'\| \sin\theta\, \hat{u} \times \dfrac{v_{\perp}}{\|v_{\perp}\|}$

- $v_{\perp}' = \cos\theta\, v_{\perp} + \sin\theta\, \hat{u} \times v_{\perp}$

- $\hat{u} \times v_{\perp} = \hat{u} \times (v - v_{\parallel}) = \hat{u} \times v$

- $v_{\perp}' = \cos\theta\,[v - (v \cdot \hat{u})\,\hat{u}] + \sin\theta\, \hat{u} \times v$

- $v' = v_{\parallel}' + v_{\perp}'$

- $v' = (v \cdot \hat{u})\,\hat{u} + \cos\theta\,[v - (v \cdot \hat{u})\,\hat{u}] + \sin\theta\, \hat{u} \times v$

$$= \cos\theta\, v + (1-\cos\theta)(\hat{u}\cdot v)\,\hat{u} + \sin\theta\, \hat{u}\times v$$ (Rodrigues' rotation formula)

$$\begin{aligned}
\hat{\mathbf{i}} &= \cos\theta\hat{\mathbf{I}} + (1-\cos\theta)\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{I}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{I}}\\
\hat{\mathbf{j}} &= \cos\theta\hat{\mathbf{J}} + (1-\cos\theta)\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{J}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{J}}\\
\hat{\mathbf{k}} &= \cos\theta\hat{\mathbf{K}} + (1-\cos\theta)\left(\hat{\mathbf{u}}\cdot\hat{\mathbf{K}}\right)\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{u}}\times\hat{\mathbf{K}}
\end{aligned}$$

$$\hat{\mathbf{u}} = l\hat{\mathbf{I}} + m\hat{\mathbf{J}} + n\hat{\mathbf{K}} \qquad l^2 + m^2 + n^2 = 1$$

$$\begin{aligned}
\hat{\mathbf{i}} &= \left[l^2(1-\cos\theta) + \cos\theta\right]\hat{\mathbf{I}} + [lm(1-\cos\theta) + n\sin\theta]\hat{\mathbf{J}} + [ln(1-\cos\theta) - m\sin\theta]\hat{\mathbf{K}}\\
\hat{\mathbf{j}} &= [lm(1-\cos\theta) - \sin\theta]\hat{\mathbf{I}} + \left[m^2(1-\cos\theta) + \cos\theta\right]\hat{\mathbf{J}} + [mn(1-\cos\theta) + l\sin\theta]\hat{\mathbf{K}}\\
\hat{\mathbf{k}} &= [ln(1-\cos\theta) + m\sin\theta]\hat{\mathbf{I}} + [mn(1-\cos\theta) - l\sin\theta]\hat{\mathbf{J}} + \left[n^2(1-\cos\theta) + \cos\theta\right]\hat{\mathbf{K}}
\end{aligned}$$

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} l^2(1-\cos\theta)+\cos\theta & lm(1-\cos\theta)+n\sin\theta & ln(1-\cos\theta)-m\sin\theta \\ lm(1-\cos\theta)-n\sin\theta & m^2(1-\cos\theta)+\cos\theta & mn(1-\cos\theta)+l\sin\theta \\ ln(1-\cos\theta)+m\sin\theta & mn(1-\cos\theta)-l\sin\theta & n^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$