Lecture 29

William Rowan Hamilton
1805-1865

- Algebra
- Optics
- Mechanics

$$\widehat{\mathbf{q}} = \left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \hdashline q_4 \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{q} \\ \hdashline q_4 \end{array} \right\}$$

$$\|\widehat{\mathbf{q}}\| = \sqrt{\|\mathbf{q}\|^2 + q_4^2} = \sqrt{\mathbf{q} \cdot \mathbf{q} + q_4^2}$$

$$\widehat{\mathbf{p}} + \widehat{\mathbf{q}} = \left\{ \begin{array}{c} \mathbf{p} + \mathbf{q} \\ \hdashline p_4 + q_4 \end{array} \right\} \qquad a\widehat{\mathbf{p}} = \left\{ \begin{array}{c} a\mathbf{p} \\ \hdashline ap_4 \end{array} \right\}$$

$$\left(\widehat{\mathbf{p}} + \widehat{\mathbf{q}}\right) + \widehat{\mathbf{r}} = \widehat{\mathbf{p}} + \left(\widehat{\mathbf{q}} + \widehat{\mathbf{r}}\right) \quad \widehat{\mathbf{p}} + \widehat{\mathbf{q}} = \widehat{\mathbf{q}} + \widehat{\mathbf{p}}$$

$$\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} = \left\{ \begin{array}{c} p_4\mathbf{q} + q_4\mathbf{p} + \mathbf{p} \times \mathbf{q} \\ \hdashline p_4 q_4 - \mathbf{p} \cdot \mathbf{q} \end{array} \right\}$$

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{p}} = \left\{ \begin{array}{c} q_4\mathbf{p} + p_4\mathbf{q} + \mathbf{q} \times \mathbf{p} \\ \hdashline q_4 p_4 - \mathbf{q} \cdot \mathbf{p} \end{array} \right\}$$

$$\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} \neq \widehat{\mathbf{q}} \otimes \widehat{\mathbf{p}}$$

Example

Find the product of the quatemions

$$\widehat{\mathbf{p}} = \left\{\frac{\mathbf{p}}{p_4}\right\} = \left\{\frac{\hat{\mathrm{j}}}{1}\right\} \qquad \widehat{\mathbf{q}} = \left\{\frac{\mathbf{q}}{q_4}\right\} = \left\{\frac{0.5\hat{\mathrm{i}} + 0.5\hat{\mathrm{j}} + 0.75\hat{\mathrm{k}}}{1}\right\}$$

Details

$$\begin{aligned}
\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} &= \left\{\frac{p_4\mathbf{q} + q_4\mathbf{p} + \mathbf{p}\times\mathbf{q}}{p_4 q_4 - \mathbf{p}\cdot\mathbf{q}}\right\} \\
&= \left\{\frac{\overbrace{1\cdot\left(0.5\hat{\mathrm{i}} + 0.5\hat{\mathrm{j}} + 0.75\hat{\mathrm{k}}\right)}^{0.5\hat{\mathrm{i}}+0.5\hat{\mathrm{j}}+0.75\hat{\mathrm{k}}} + \overbrace{1\cdot\hat{\mathrm{j}}}^{\hat{\mathrm{j}}} + \overbrace{\hat{\mathrm{j}}\times\left(0.5\hat{\mathrm{i}} + 0.5\hat{\mathrm{j}} + 0.75\hat{\mathrm{k}}\right)}^{-0.5\hat{\mathrm{k}}+0.75\hat{\mathrm{i}}}}{\underbrace{1\cdot 1}_{1} - \underbrace{\hat{\mathrm{j}}\cdot\left(0.5\hat{\mathrm{i}} + 0.5\hat{\mathrm{j}} + 0.75\hat{\mathrm{k}}\right)}_{0.5}}\right\} \\
&= \left\{\frac{(0.5+0.75)\hat{\mathrm{i}} + (0.5+1.0)\hat{\mathrm{j}} + (0.75-0.5)\hat{\mathrm{k}}}{0.5}\right\}
\end{aligned}$$

$$\boxed{\widehat{\mathbf{p}} \otimes \widehat{\mathbf{q}} = \left\{\frac{1.25\hat{\mathbf{i}} + 1.5\hat{\mathbf{j}} + 0.25\hat{\mathbf{k}}}{0.5}\right\}}$$

$$\widehat{\mathbf{q}}^* = \left\{\frac{-\mathbf{q}}{q_4}\right\}$$

$$\widehat{1} = \left\{\frac{\mathbf{0}}{1}\right\}$$

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{1}} = \widehat{\mathbf{1}} \otimes \widehat{\mathbf{q}} = \left\{\frac{q_4\cdot\mathbf{0} + 1\cdot\mathbf{q} + \mathbf{q}\times\mathbf{0}}{q_4\cdot 1 - \mathbf{q}\cdot\mathbf{0}}\right\} = \left\{\frac{\mathbf{q}}{q_4}\right\} = \widehat{\mathbf{q}}$$

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{q}}^* = \widehat{\mathbf{q}}^* \otimes \widehat{\mathbf{q}} = \left\{\frac{q_4(-\mathbf{q}) + q_4\mathbf{q} + \mathbf{q}\times(-\mathbf{q})}{q_4 q_4 - \mathbf{q}\cdot(-\mathbf{q})}\right\} = \left\{\frac{\mathbf{0}}{\|\widehat{\mathbf{q}}\|^2}\right\} = \|\widehat{\mathbf{q}}\|^2\,\widehat{\mathbf{1}}$$

$$\widehat{\mathbf{q}}^{-1} = \frac{\widehat{\mathbf{q}}^*}{\|\widehat{\mathbf{q}}\|^2}$$

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{q}}^{-1} = \widehat{\mathbf{q}}^{-1} \otimes \widehat{\mathbf{q}} = \widehat{\mathbf{1}}$$

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{q}}^* = \widehat{\mathbf{q}}^* \otimes \widehat{\mathbf{q}} = \widehat{\mathbf{1}} \quad \left(\text{if } \|\widehat{\mathbf{q}}\| = 1\right)$$

$$\widehat{\mathbf{q}} = \left\{ \begin{array}{c} \sin(\theta/2)\hat{\mathbf{u}} \\ \hdashline \cos(\theta/2) \end{array} \right\}$$

$$q_1 = l\sin(\theta/2) \quad q_2 = m\sin(\theta/2) \quad q_3 = n\sin(\theta/2) \quad q_4 = \cos(\theta/2)$$

$$\widehat{\mathbf{q}}^* = \left\{ \begin{array}{c} -\sin(\theta/2)\hat{\mathbf{u}} \\ \hdashline \cos(\theta/2) \end{array} \right\}$$

1. Write the quaternion as

$$\widehat{\mathbf{q}} = \left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \hdashline q_4 \end{array} \right\}$$

where $\lfloor q_1 \;\; q_2 \;\; q_3 \rfloor^T$ is the vector part, q_4 is the scalar part, and $\|\widehat{\mathbf{q}}\| = 1$.

2. Compute the direction cosine matrix of the transformation from XYZ to xyz as follows:

$$[\mathbf{Q}]_{Xx} = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_2q_1 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_3q_1 + q_2q_4) & 2(q_3q_2 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \quad (11.157)$$

$$[\mathbf{Q}]_{Xx}[\mathbf{Q}]_{Xx}{}^T = [\mathbf{Q}]_{Xx}{}^T[\mathbf{Q}]_{Xx} = [\mathbf{1}]$$

$$q_4 = \frac{1}{2}\sqrt{1 + Q_{11} + Q_{22} + Q_{33}}$$

$$q_1 = \frac{Q_{23} - Q_{32}}{4q_4} \quad q_2 = \frac{Q_{31} - Q_{13}}{4q_4} \quad q_3 = \frac{Q_{12} - Q_{21}}{4q_4}$$

$$Q_{11} + Q_{22} + Q_{33} = 3q_4^2 - (q_1^2 + q_2^2 + q_3^2)$$
$$= 4q_4^2 - 1$$

Example

(a) Write down the unit quaternion for a rotation about the x axis through an angle θ.
(b) Obtain the corresponding direction cosine matrix.

Details

(a)

$$\boxed{\widehat{\mathbf{q}} = \left\{\begin{array}{c} \sin(\theta/2) \\ 0 \\ 0 \\ \hline \cos(\theta/2) \end{array}\right\}}$$

(b)

$$[\mathbf{Q}] = \begin{bmatrix} \sin^2(\theta/2)+\cos^2(\theta/2) & 0 & 0 \\ 0 & -\sin^2(\theta/2)+\cos^2(\theta/2) & 2\sin(\theta/2)\cos(\theta/2) \\ 0 & -2\sin(\theta/2)\cos(\theta/2) & -\sin^2(\theta/2)+\cos^2(\theta/2) \end{bmatrix}$$

$$\boxed{[\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}}$$

$$\widehat{\mathbf{v}}' = \widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}} \otimes \widehat{\mathbf{q}}^*$$

$$\widehat{\mathbf{v}} = \left\{\begin{array}{c} \mathbf{v} \\ \hline 0 \end{array}\right\} \qquad \widehat{\mathbf{v}}' = \left\{\begin{array}{c} \mathbf{v}' \\ \hline 0 \end{array}\right\}$$

$$\widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}} = \left\{\begin{array}{c} \cos(\theta/2)\mathbf{v} + \sin(\theta/2)(\hat{\mathbf{u}} \times \mathbf{v}) \\ \hline -\sin(\theta/2)(\hat{\mathbf{u}} \cdot \mathbf{v}) \end{array}\right\}$$

$$\begin{aligned} \mathbf{v}' = & \overbrace{[-\sin(\theta/2)(\hat{\mathbf{u}} \cdot \mathbf{v})]}^{(\widehat{\mathbf{q}} \otimes \widehat{\mathbf{v}})_4} \overbrace{(-\sin(\theta/2)\hat{\mathbf{u}})}^{\mathbf{q}^*} + \overbrace{\cos(\theta/2)}^{q^*_4} \overbrace{[\cos(\theta/2)\mathbf{v} + \sin(\theta/2)(\hat{\mathbf{u}} \times \mathbf{v})]}^{\mathbf{q} \otimes \mathbf{v}} \\ & + \underbrace{[\cos(\theta/2)\mathbf{v} + \sin(\theta/2)(\hat{\mathbf{u}} \times \mathbf{v})]}_{\mathbf{q} \otimes \mathbf{v}} \times \underbrace{(-\sin(\theta/2)\hat{\mathbf{u}})}_{\mathbf{q}^*} \end{aligned}$$

$$= \sin^2(\theta/2)\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v}) + \left[\cos^2(\theta/2)\mathbf{v} + \cos(\theta/2)\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})\right]$$
$$+ \cos(\theta/2)\sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v}) - \sin^2(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})\times\hat{\mathbf{u}}$$

$$\mathbf{v}' = \mathbf{v}\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})\left[2\sin^2(\theta/2)\right] + (\hat{\mathbf{u}}\times\mathbf{v})[2\cos(\theta/2)\sin(\theta/2)]$$

$$\mathbf{v}' = \mathbf{v}\cos\theta + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{v})(1-\cos\theta) + (\hat{\mathbf{u}}\times\mathbf{v})\sin\theta$$

$$v_4' = \overbrace{[-\sin(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v})]}^{(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}})_4}\overbrace{[\cos(\theta/2)]}^{q^*_4} - \overbrace{[\cos(\theta/2)\mathbf{v} + \sin(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})]}^{\mathbf{q}\otimes\mathbf{v}}\cdot\overbrace{[-\sin(\theta/2)\hat{\mathbf{u}}]}^{\mathbf{q}^*}$$
$$= -\sin(\theta/2)\cos(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v}) + \cos(\theta/2)\sin(\theta/2)(\hat{\mathbf{u}}\cdot\mathbf{v}) + \sin^2(\theta/2)(\hat{\mathbf{u}}\times\mathbf{v})\cdot\hat{\mathbf{u}}$$
$$= 0$$

$$\widehat{\mathbf{q}}\otimes\left(\widehat{\mathbf{q}}^*\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}\right)\otimes\widehat{\mathbf{q}}^* = \left(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^*\right)\otimes\widehat{\mathbf{v}}\otimes\left(\widehat{\mathbf{q}}\otimes\widehat{\mathbf{q}}^*\right) = \widehat{\mathbf{1}}\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{1}} = \left(\widehat{\mathbf{1}}\otimes\widehat{\mathbf{v}}\right)\otimes\widehat{\mathbf{1}} = \widehat{\mathbf{v}}\otimes\widehat{\mathbf{1}} = \widehat{\mathbf{v}}$$

Example

Consider the vector $\mathbf{v} = v\hat{\mathbf{j}}$. Using the quaternion and corresponding direction cosine matrix in Example 11.22, carry out the following operations and interpret the results geometrically:
(i) $\widehat{\mathbf{v}}' = \widehat{\mathbf{q}}\otimes\widehat{\mathbf{v}}\otimes\widehat{\mathbf{q}}^*$
(ii) $\{\mathbf{v}'\} = [\mathbf{Q}]\{\mathbf{v}\}$
where

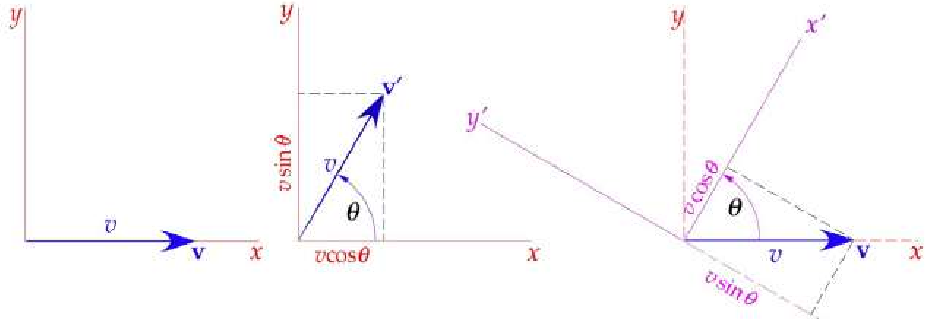
$$\widehat{\mathbf{v}} = \left\{\begin{array}{c} v\hat{\mathbf{j}} \\ \hline 0 \end{array}\right\} \qquad \widehat{\mathbf{q}} = \left\{\begin{array}{c} \sin(\theta/2)\hat{\mathbf{i}} \\ \hline \cos(\theta/2) \end{array}\right\} \qquad [\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Details

(i) $\hat{v}' = \left\{\begin{array}{c} v' \\ \hline 0 \end{array}\right\}$

$$v' = v\cos\theta\,\hat{j} + \hat{i}(\hat{i}\cdot\hat{j})(1-\cos\theta) + (\hat{i}\times v\hat{j})\sin\theta$$
$$= v\cos\theta\,\hat{j} + v\sin\theta\,\hat{k}$$

(ii) $v' = v\cos\theta\,\hat{j}' - v\sin\theta\,\hat{k}'$



Before rotation

$v_x = v$

$v_y = v$

After vector rotation

$v'_x = v\cos\theta$

$v'_y = v\sin\theta$

After frame rotation

$v_{x'} = v\cos\theta$

$v_{y'} = -v\sin\theta$

$$\dot{\hat{\mathbf{q}}} = \left\{ \begin{array}{c} \dfrac{d}{dt}\left[\hat{\mathbf{u}}\sin(\theta/2)\right] \\ \hline \dfrac{d}{dt}\cos(\theta/2) \end{array} \right\} = \left\{ \begin{array}{c} \dot{\hat{\mathbf{u}}}\sin(\theta/2) + \hat{\mathbf{u}}(\dot{\theta}/2)\cos(\theta/2) \\ \hline -(\dot{\theta}/2)\sin(\theta/2) \end{array} \right\}$$

$$\dot{\hat{\mathbf{u}}} = \frac{1}{2}[\hat{\mathbf{u}} \times \boldsymbol{\omega} - \cot(\theta/2)\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \boldsymbol{\omega})]$$

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\left\{ \begin{array}{c} \sin(\theta/2)\hat{\mathbf{u}} \times \boldsymbol{\omega} + \cos(\theta/2)\boldsymbol{\omega} \\ \hline -\dot{\theta}\sin(\theta/2) \end{array} \right\}$$

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\left\{ \begin{array}{c} \mathbf{q} \times \boldsymbol{\omega} + q_4\boldsymbol{\omega} \\ \hline -\boldsymbol{\omega} \cdot \mathbf{q} \end{array} \right\}$$

$$\dot{\widehat{\mathbf{q}}} = \frac{1}{2}\widehat{\mathbf{q}} \otimes \widehat{\boldsymbol{\omega}}$$

$$\widehat{\boldsymbol{\omega}} = \left\{ \begin{array}{c} \boldsymbol{\omega} \\ \hline 0 \end{array} \right\}$$

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\left\{ \begin{array}{c} (q_2\omega_3 - q_3\omega_2) + q_4\omega_1 \\ (q_3\omega_1 - q_1\omega_3) + q_4\omega_2 \\ (q_1\omega_2 - q_2\omega_1) + q_4\omega_3 \\ \hline -\omega_1 q_1 - \omega_2 q_2 - \omega_3 q_3 \end{array} \right\} = \frac{1}{2}\left[\begin{array}{ccc|c} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ \hline -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{array} \right] \left\{ \begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \hline q_4 \end{array} \right\}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\widehat{\mathbf{q}}\right\} = \frac{1}{2}[\boldsymbol{\Omega}]\left\{\widehat{\mathbf{q}}\right\}$$

$$[\boldsymbol{\Omega}] = \left[\begin{array}{ccc|c} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ \hline -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{array}\right]$$

$$\left\{\widehat{\mathbf{q}}\right\} = \exp\left(\frac{[\boldsymbol{\Omega}]}{2}t\right)\left\{\widehat{\mathbf{q}}_0\right\}$$

$$\exp\left(\frac{[\boldsymbol{\Omega}]}{2}t\right) = \begin{bmatrix} \cos\frac{\omega t}{2} & \frac{\omega_z}{\omega}\sin\frac{\omega t}{2} & -\frac{\omega_y}{\omega}\sin\frac{\omega t}{2} & \frac{\omega_x}{\omega}\sin\frac{\omega t}{2} \\ -\frac{\omega_z}{\omega}\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} & \frac{\omega_x}{\omega}\sin\frac{\omega t}{2} & \frac{\omega_y}{\omega}\sin\frac{\omega t}{2} \\ \frac{\omega_y}{\omega}\sin\frac{\omega t}{2} & -\frac{\omega_x}{\omega}\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} & \frac{\omega_z}{\omega}\sin\frac{\omega t}{2} \\ -\frac{\omega_x}{\omega}\sin\frac{\omega t}{2} & -\frac{\omega_y}{\omega}\sin\frac{\omega t}{2} & -\frac{\omega_z}{\omega}\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{bmatrix}$$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$