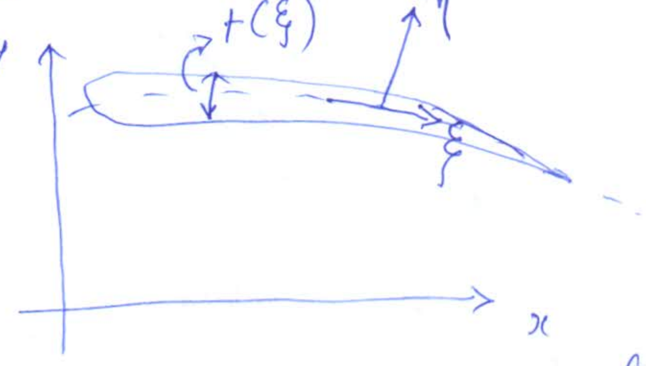
The approximation used in thin open-sections can be extended to airfoil like crosssections also.

Consider a thin open cross-sections where the thickness varies along the centre line length 'b' of the thin cross-section. Typical examples of these type of cross-sections are those of propeller and turbine blades etc.



Note that $b >> t(\xi)$

In general, following the derivation shown for narrow rectangular c/s, we can write

$$\phi = -G\theta\left[\eta^2 - \frac{t^2(\xi)}{4}\right]$$

$$\Rightarrow \quad T = \iint 2\phi \, dx\, dy$$

or equivalently

$$T = \iint 2\phi \, d\eta \, d\xi$$

$$\Rightarrow T = -2G\theta \int_{\xi=0}^{\xi=b} \int_{\eta=-\frac{t(\xi)}{2}}^{\eta=\frac{t(\xi)}{2}} \left[\eta^2 - \frac{1}{4}t^2(\xi)\right] d\eta \, d\xi$$

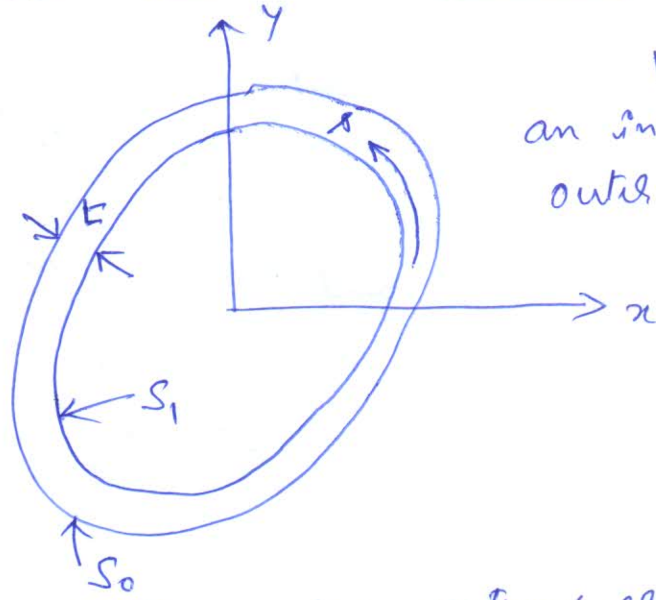
note: if centreline has significant curvature it is then necessary to introduce in the above integrand Jacobian of the fixed coordinates $x-y$ with respect to $\eta - \xi$.

$$\Rightarrow T = \frac{1}{3} G\theta \int_{\xi=0}^{b} t^3(\xi)\, d\xi$$

$$\Rightarrow \quad J = \frac{1}{3}\int_0^b t^3(\xi)\, d\xi$$

$\left(\because T = GJ\theta, \; \therefore J = \frac{1}{3}\int_0^b t^3(\xi)\, d\xi \right)$

Closed single-cell thin-walled sections



Wall enclosed by an inner contour S_1 & an outer contour S_0.

Members of closed thin-walled sections are common in aircraft structures. In the above figure, wall thickness 't' is assumed to be small compared with the total length of the complete wall contour. In general, the wall thickness is not a constant but a function of 's'.

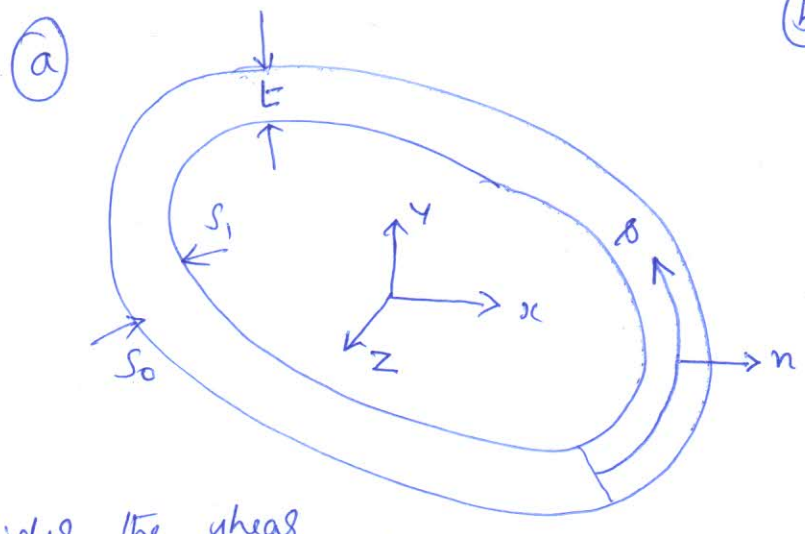
The wall thickness is enclosed by the inner contour S_1 and the outer contour S_0. Using Prandtl stress function ϕ, the stress-free boundary conditions are given by,

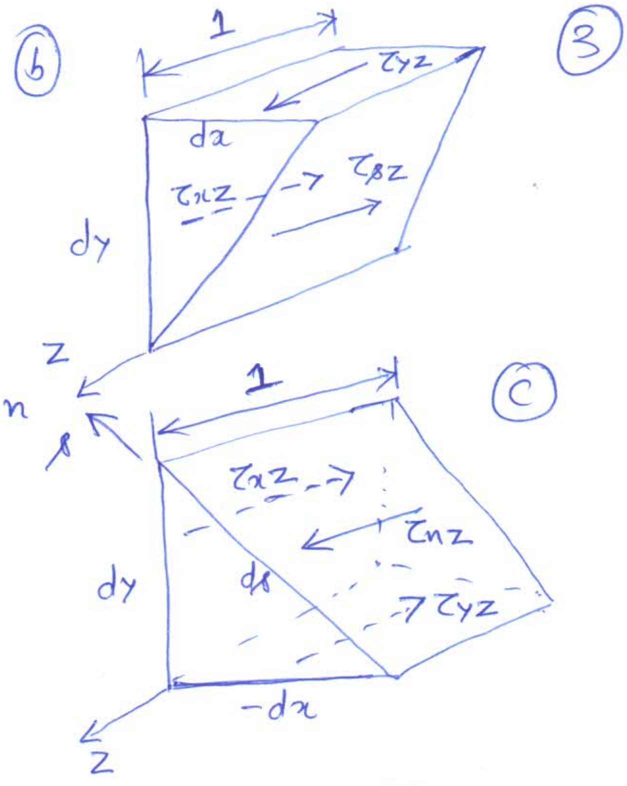
$$\frac{d\phi}{ds} = 0 \quad \text{on } S_1 \text{ and } S_0 \longrightarrow (1)$$

$$\therefore \quad \phi = C_0 \quad \text{on } S_0 \longrightarrow (2)$$

$$\phi = C_1 \quad \text{on } S_1 \longrightarrow (3)$$

where, C_0 & C_1 are two different constants and cannot be set equal to zero simultaneously as in the case of solid sections with a single boundary contour.





Consider the shear stresses in an arbitrary point on the wall section. Take an infinitesimal prismatic element of unit length in the Z-direction as shown in Fig (b). Note that the inclined surface is perpendicular to the s-direction. Balance of forces in the Z-direction gives,

$$\tau_{sz}\, dn = -\tau_{xz}\, dy + \tau_{yz}\, dx$$

$$\tau_{sz} = -\tau_{xz}\frac{dy}{dn} + \tau_{yz}\frac{dx}{dn}$$

$$= -\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial n} + \left(-\frac{\partial \phi}{\partial x}\right)\frac{\partial x}{\partial n}$$

$$\therefore \tau_{sz} = -\frac{\partial \phi}{\partial n}$$

Similarly, from (c) we have

$$\tau_{nz}\, ds = \tau_{xz}\, dy - \tau_{yz}\, dx$$

$$\tau_{nz} = \tau_{xz}\frac{\partial y}{\partial s} - \tau_{yz}\frac{\partial x}{\partial s}$$

$$= \tau_{xz}\frac{\partial y}{\partial s} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial s}$$

$$\Rightarrow \tau_{nz} = \frac{\partial \phi}{\partial s}$$

Note: negative sign is added in front of $\tau_{yz}\,dx$ to account for the fact that an increment ds is accompanied by a decrement $-dx$.

Since $\tau_{nz} = \frac{\partial \phi}{\partial s} = 0$ on S_0 & S_1, and t is small, the variation τ_{nz} across the wall thickness is negligible. $\Rightarrow$ $\tau_{nz} \approx 0$ over the entire wall thickness. Only τ_{sz} is non-vanishing component.

Let ϕ be expressed in terms of s and n as,

$$\phi(s, n) = \phi_0(s) + n\,\phi_1(s) + n^2\phi_2(s) + \cdots \quad \rightarrow (4)$$

where, $$-t/2 \leq n \leq t/2$$

Considering only linear terms

$$\phi(s, n) = \phi_0(s) + n\,\phi_1(s)$$

Boundary conditions,

$$\phi(s, t/2) = \phi_0 + t/2\,\phi_1 = c_0 \quad \text{on } S_0$$

$$\phi(s, -t/2) = \phi_0 - t/2\,\phi_1 = c_1 \quad \text{on } S_1$$

$$\therefore \quad \phi_0 = \frac{1}{2}(c_0 + c_1) \quad \& \quad \phi_1 = \frac{1}{t}(c_0 - c_1)$$

$$\therefore \quad \tau_{sz} = -\frac{\partial \phi}{\partial n} = -\phi_1 = \frac{1}{t}(c_1 - c_0)$$

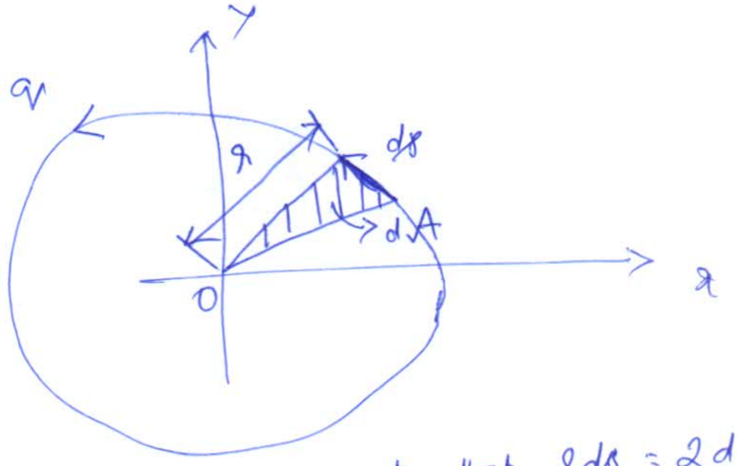
$\therefore$ Shear stress τ_{sz} on the thin-walled section is uniform over the thickness.

Define shear flow as,

$$q = \tau_{sz} t = \tau t = C_1 - C_0$$

$\Rightarrow$ regardless of wall thickness, shear flow is constant along the wall direction.

But shear flow produces torque.



$T = \oint r q \, ds$ note that $r\,ds = 2\,dA$

$\therefore T = \iint 2q\,dA = 2q\bar{A}$, where $\bar{A}$, area enclosed by the centreline of the wall thickness

Now, $\tau_{sz} = \tau = \frac{q}{t}$ & $\gamma = \frac{\tau}{G} = \frac{q}{Gt}$

$\therefore$ strain energy density is given by,

$$W = \frac{1}{2}\tau\gamma = \frac{q^2}{2Gt^2}$$

$\therefore$ Total strain energy stored in the bar (per unit length)

$$U = \oint W t \, ds$$
$$= \oint \frac{q^2}{2Gt}\, ds$$

The work done by the torque T through the twist angle θ is given by,

$$W_e = \frac{1}{2} T\theta = \frac{1}{2} 2q\bar{A}\theta = q\bar{A}\theta$$

But, $W_e = U$

$$\therefore \quad q\bar{A}\theta = \frac{q^2}{G} \oint \frac{ds}{t}$$

$$\therefore \quad \theta = \frac{q}{2\bar{A}G} \oint \frac{ds}{t}$$

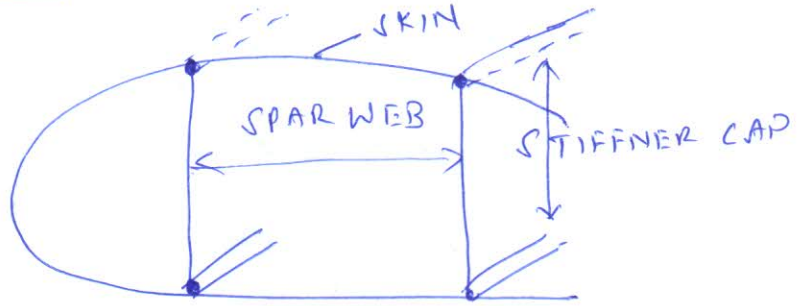
Also, $T = GJ\theta$

$$\therefore \quad \frac{T}{GJ} = \frac{q}{2\bar{A}G} \oint \frac{ds}{t}$$

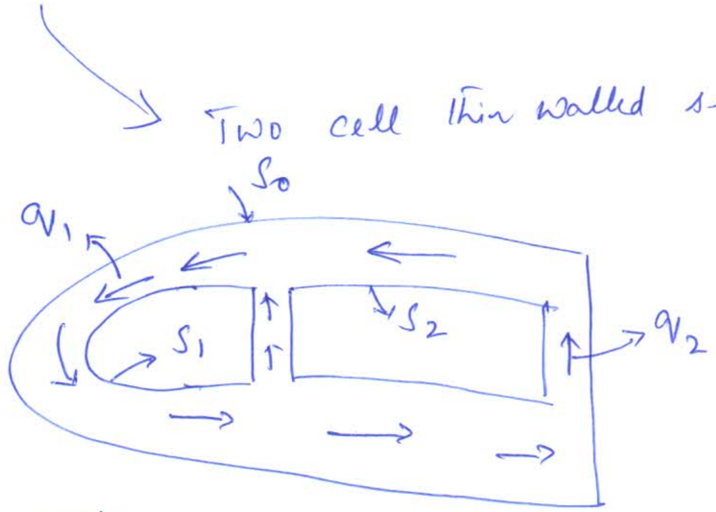
$$\therefore \quad J = \frac{2\bar{A}T}{q \oint \frac{ds}{t}}$$

$$\Rightarrow \quad J = \frac{4\bar{A}^2}{\oint \frac{ds}{t}}$$

MULTICELL THIN-WALLED SECTIONS



Two cell thin walled section



We can write,

$$\phi(S_0) = C_0 \quad ; \quad \phi(S_1) = C_1 \quad \& \quad \phi(S_2) = C_2$$

For each cell shear flow is considered positive if it forms a counterclockwise torque about the cell and its value is equal to the value of ϕ on the inside contour minus that on the outside contour, i.e,

$$q_1 = C_1 - C_0$$

$$q_2 = C_2 - C_1$$

$$q_{12} = C_1 - C_2 \quad \text{or} \quad q_{12} = q_1 - q_2$$

Total torque is equal to torque ~~the~~ contribution from two-cell

$$T = 2\bar{A}_1 q_1 + 2\bar{A}_2 q_2$$

$$\theta_1 = \frac{1}{2\bar{A}_1 G} \oint_{cell\,1} q \frac{ds}{t} \; ; \; \theta_2 = \frac{1}{2\bar{A}_2 G} \oint_{cell\,2} q \frac{ds}{t}$$

$$\theta_1 = \theta_2 = \theta$$

Note:

$$T = G \bar{J} \theta$$

$$\therefore \bar{J} = \frac{T}{G\theta} = \frac{4\bar{A}_1 \sum \bar{A}_i q_i}{\oint_{cell\,1} q\,(ds/t)} \quad \text{for n-cell}$$

$$\because T = \sum_{i=1}^{n} 2\bar{A}_i q_i$$

for n-cell