Let's solve a few practice problems:

① (MCQ) How many non-trivial polynomials P exist such that $(x^2-ax+18)P(x)-(x^2+3x)P(x-3)=0$ $\forall a \in \mathbb{R}$?

(a) 0

(b) 1

(c) 2

(d) Infinitely many

Solution:

$$(x^2-ax+18)p(x) = x(x+3)P(x-3)$$

$$p(0)=0$$

Put $x=-3$,

$$(27+3a)\,p(-3)=0$$

Put $x=3$,

$$(27-3a)\,p(3)=0$$

If $p(j-3)=0$,

$$(j^2-aj+18)P(j) = j(j+3)P(j-3)$$

If $j \neq 0,-3$, $P(j-3)=0 \Rightarrow p(j)=0$

i.e. $P(k)=0 \Rightarrow P(k+3)=0 \quad \forall k \neq 0,-3$

We have $P(3)=0 \quad \therefore p(6)=0 \ldots$

$p(x)=0$ identically. (a)

② $P(x)$ is a real-valued polynomial of degree 2020.

Further, $P(n) = \frac{1}{n}$ for $n = 1, 2 \ldots, 2021$.

Which of the following is true? (Multiple correct)

(a) $P(2022) = \frac{1}{2022}$ (e) $P(2022) = 1$

(b) $P(2022) = \frac{2}{2022}$ (f) $P(2022) = 0$

(c) $P(2023) = \frac{1}{2023}$ (g) $P(2023) = 1$

(d) $P(2023) = \frac{2}{2023}$ (h) $P(2023) = 0$

Solution:

$P(n) = \frac{1}{n}$ for $i = 1 \ldots 2021$

$\therefore nP(n) - 1 = 0$ for $i = 1 \ldots 2021$

$\therefore$ The polynomial $xP(x) - 1$ has roots $1, 2 \ldots 2021$ and is of degree 2021.

$\therefore xP(x) - 1 = k(x-1)(x-2)\ldots(x-2021)$

Put $x = 0$

$-1 = k \times (-1)\, 2021! \Rightarrow k = 1/2021!$

$\therefore 2022\, P(2022) - 1 = \frac{1}{\cancel{(2021)!}}\ \cancel{2021 \times 2020 \times \ldots \times 1}$

$\therefore P(2022) = \frac{2}{2022} = \frac{1}{1011}$

$2023\, P(2023) - 1 = \frac{1}{\cancel{(2021)!}}\ 2022 \times \cancel{2021 \times \ldots} \times 2$

$P(2023) = 1$

(b) (g)

(3)

i	0	1	2	3	4
(x_i, y_i)	$(-2,4)$	$(-1,1)$	$(0,0)$	$(1,1)$	$(2,4)$

Of which of the following degrees does there exist a polynomial $p(x)$ s.t. $p(x_i) = y_i \;\; \forall i = 0 \ldots 4$?

(a) 1 (d) 4

(b) 2 (e) 5

(c) 3 (f) 6

Solution: (b), (e), (f)