AE 242
Aerospace Measurements Laboratory

Time varying measurements

When we pour hot or cold liquid, how fast temperature will rise / decrease in thermometer?

Will it be instantaneous or take time to show correct temperature?

Time varying measurements

After keeping the object in the pan, how much time we should wait to get the correct measurement?

Can it be oscillatory?

Dynamic Characteristic

Time varying quantities to be measured.

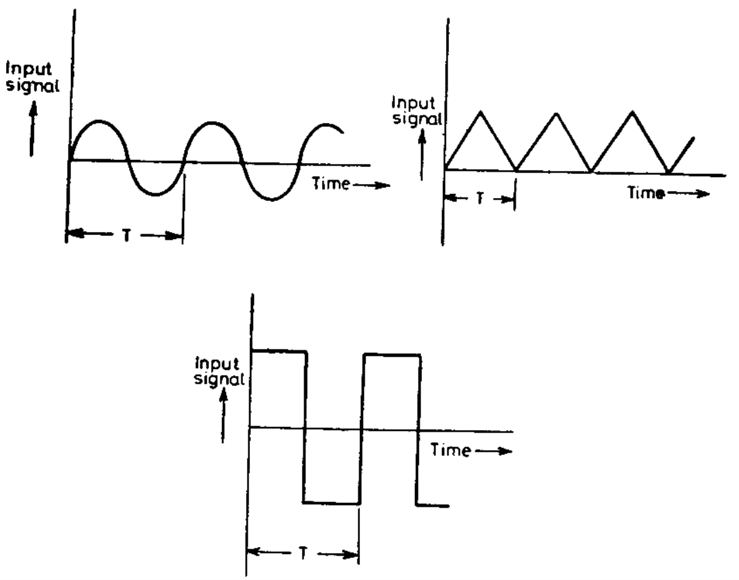
Dynamic Inputs:

Periodic input : Varying cyclically or repeating itself. Could be harmonic and non-harmonic

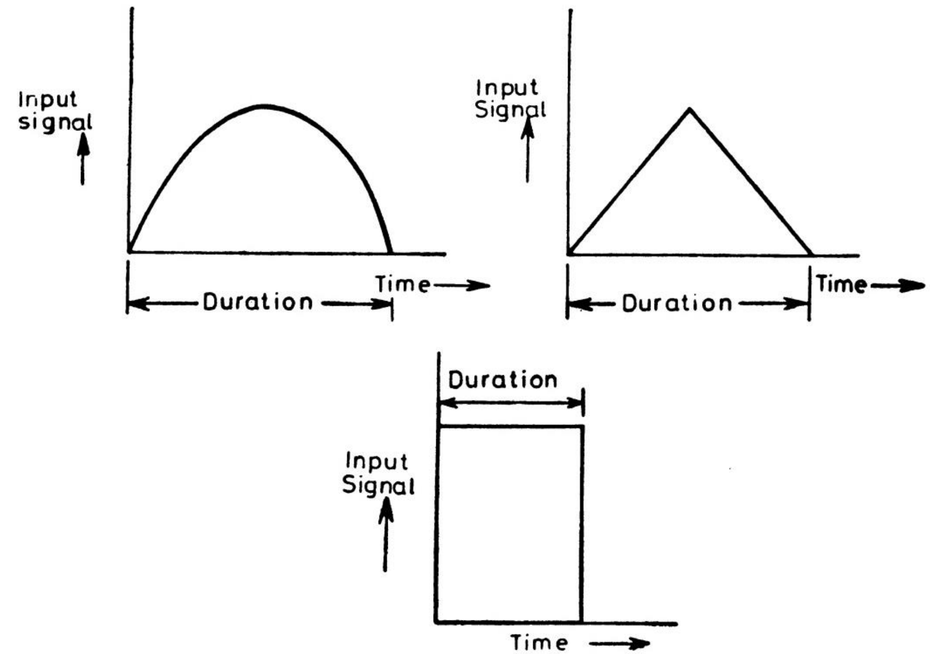
Transient input : Varying non-cyclically with time. It is of definite duration and reduces to zero after certain time.

Random input : Varying randomly with time, with no definite period and amplitude.

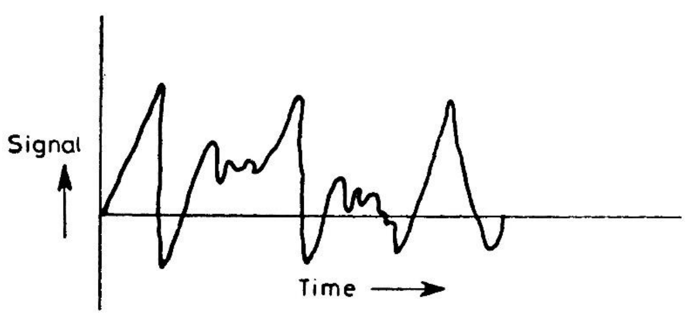
Dynamic Characteristic



Periodic

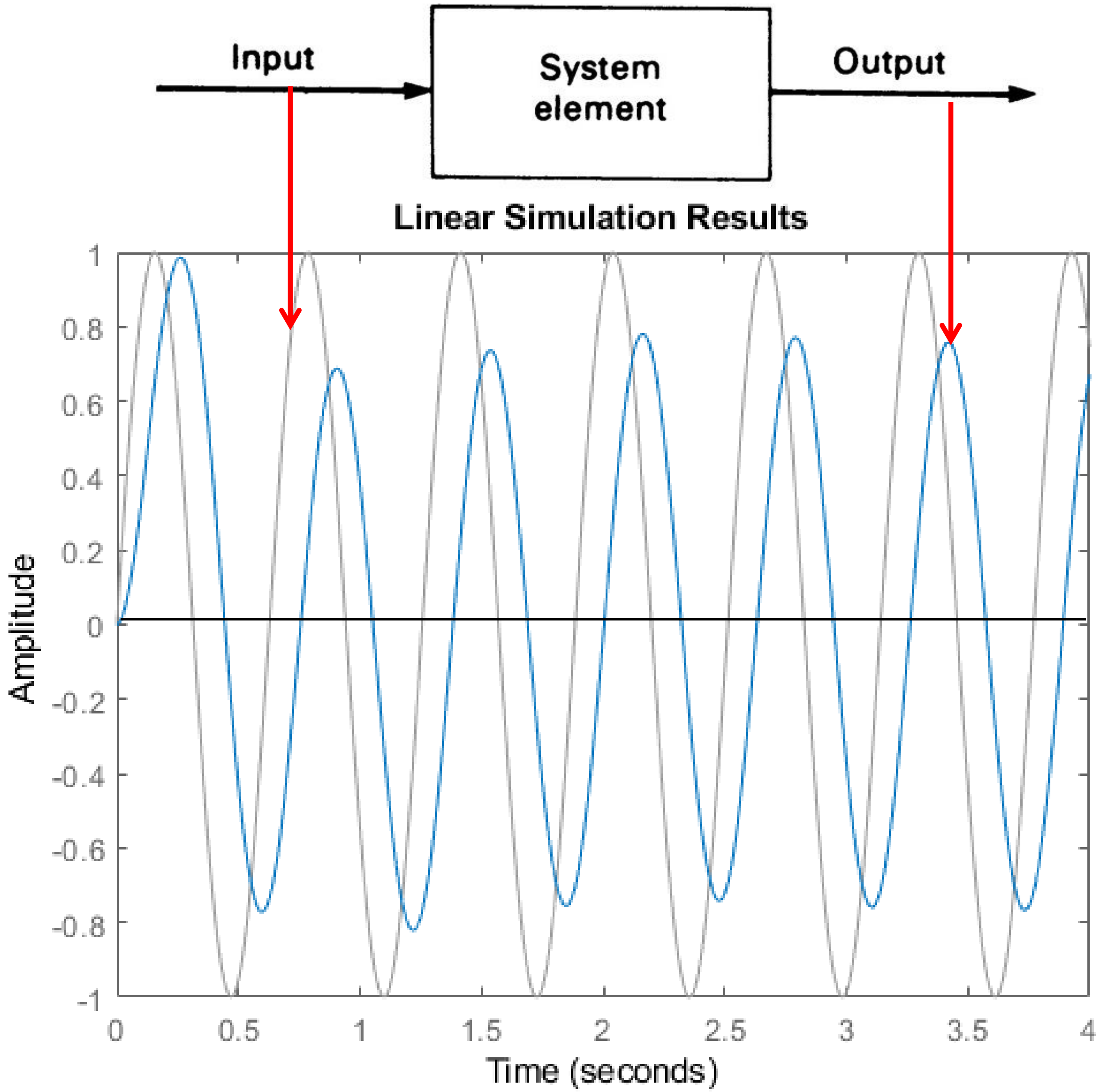


Transient

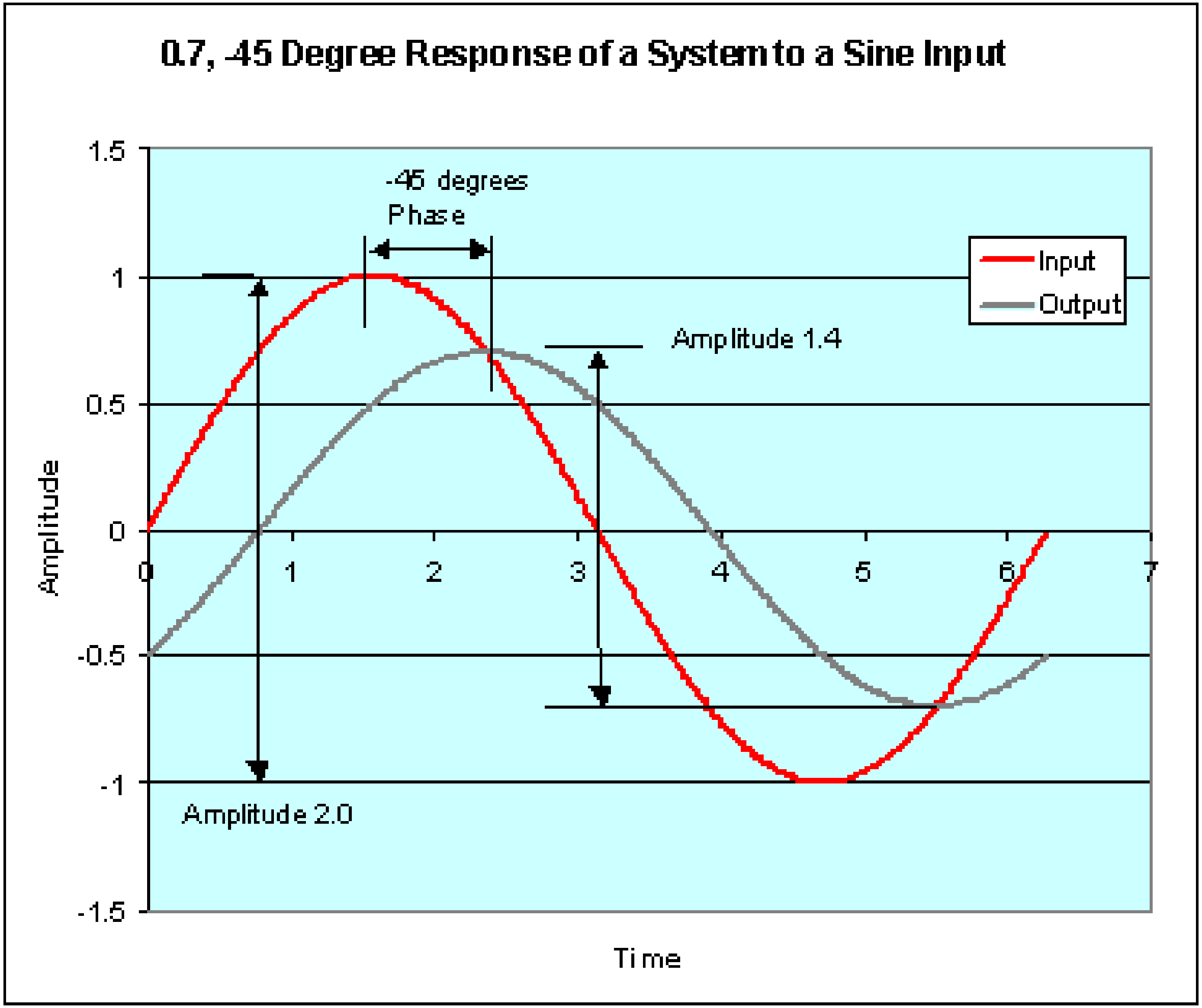


Random

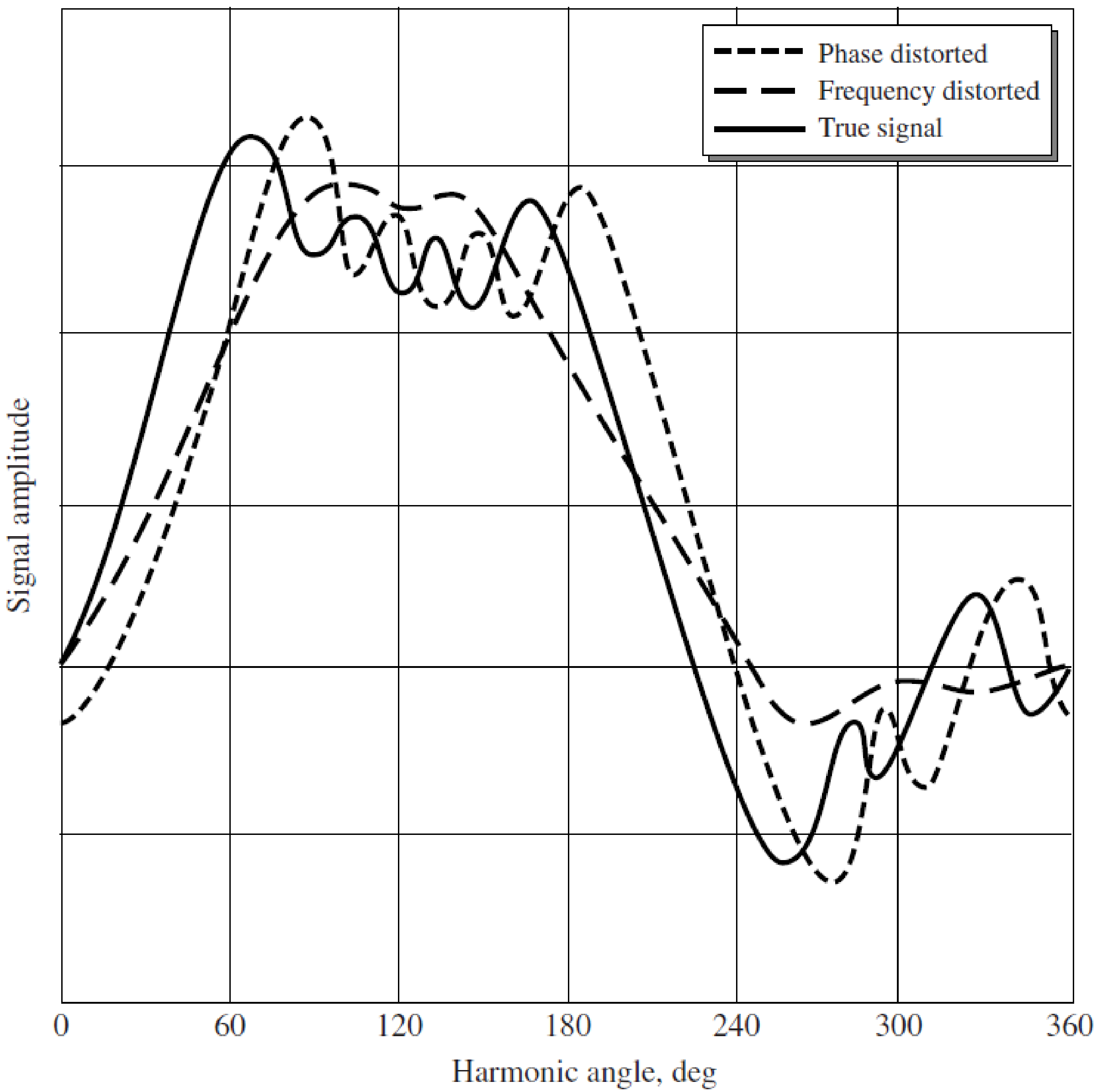
Sinusoidal input



Amplitude ratio and Phase difference



Effect of frequency response and phase shift response



Dynamic Characteristic

Dynamic characteristics of an instrument can be obtained experimentally. Theoretical model is desirable for analysis and design.

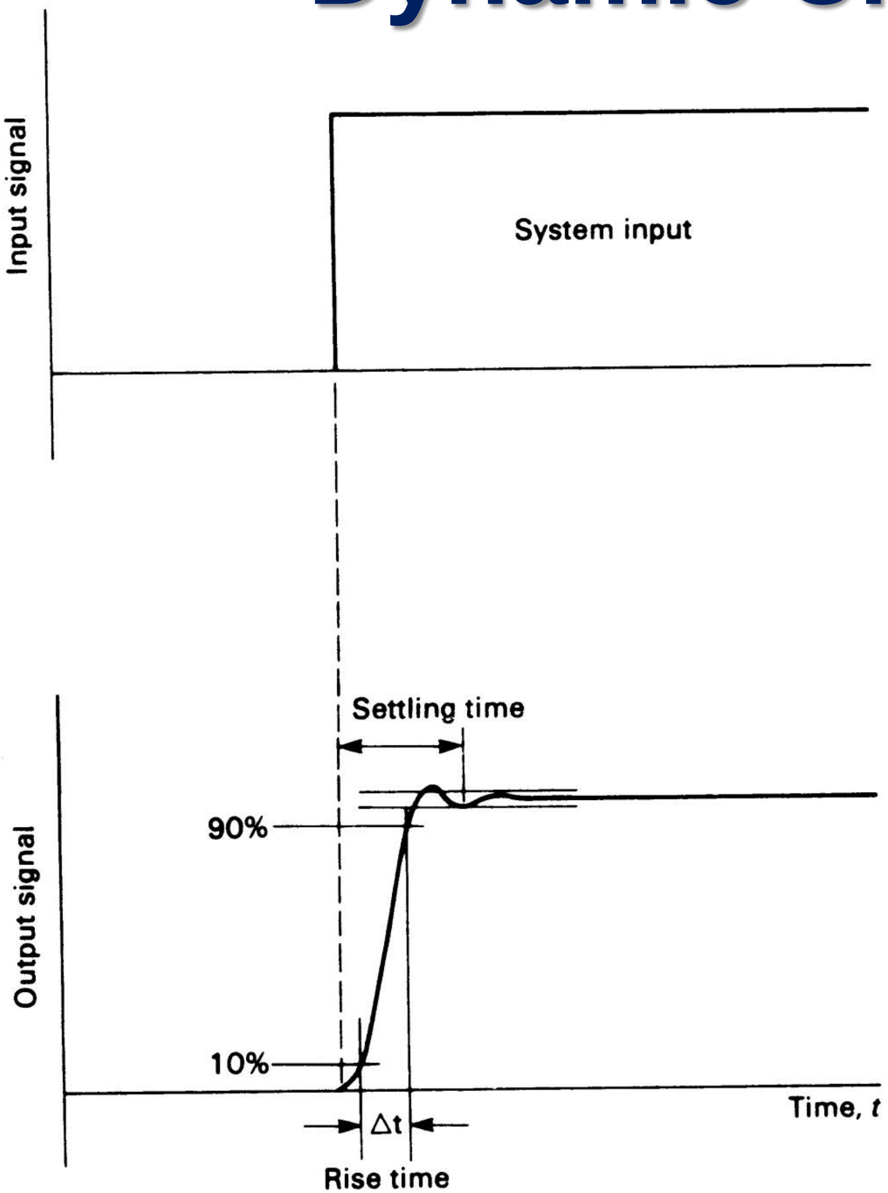
Steps in understanding dynamic behavior

a) Formulation of governing equation, relating input and output.

b) Solution of governing equations, to study various input conditions

c) To improve output response, if not satisfactory, by compensation.

Dynamic Characteristic

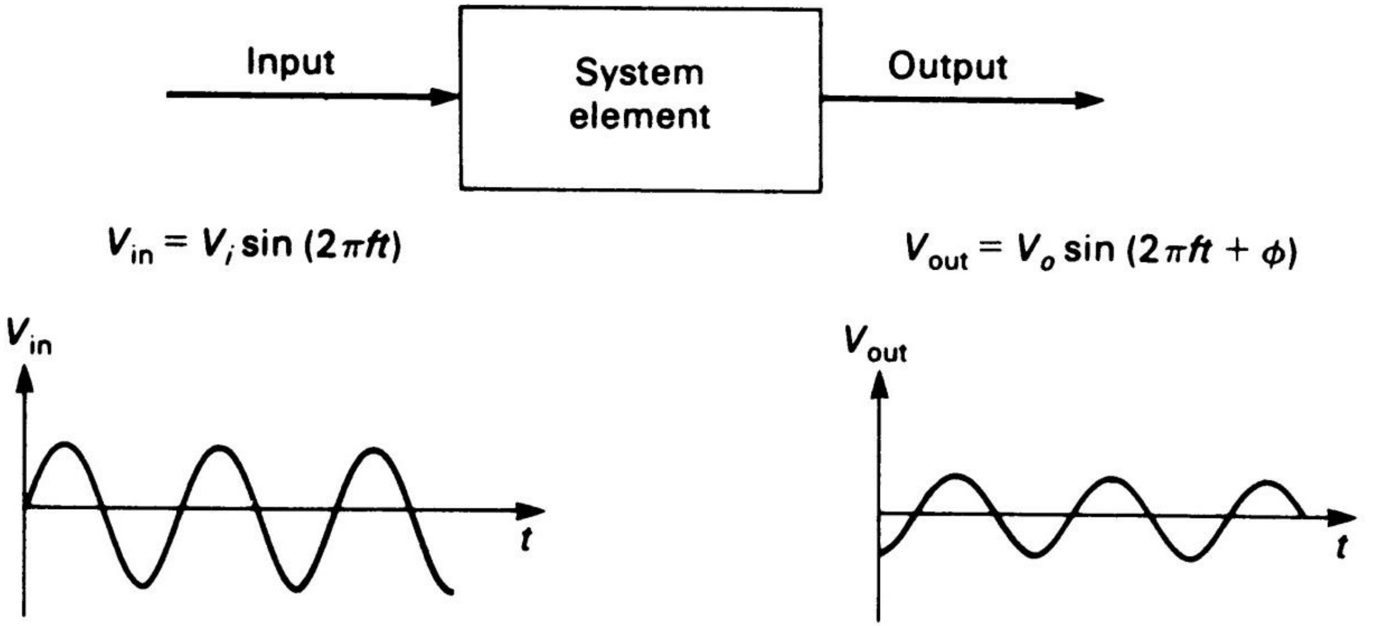


Rise Time: Time delay between the application of step input and proper output is reached. It can also be specified as time between 10% & 90% of the final output.

Overshoot: Maximum value of the output from the final output.

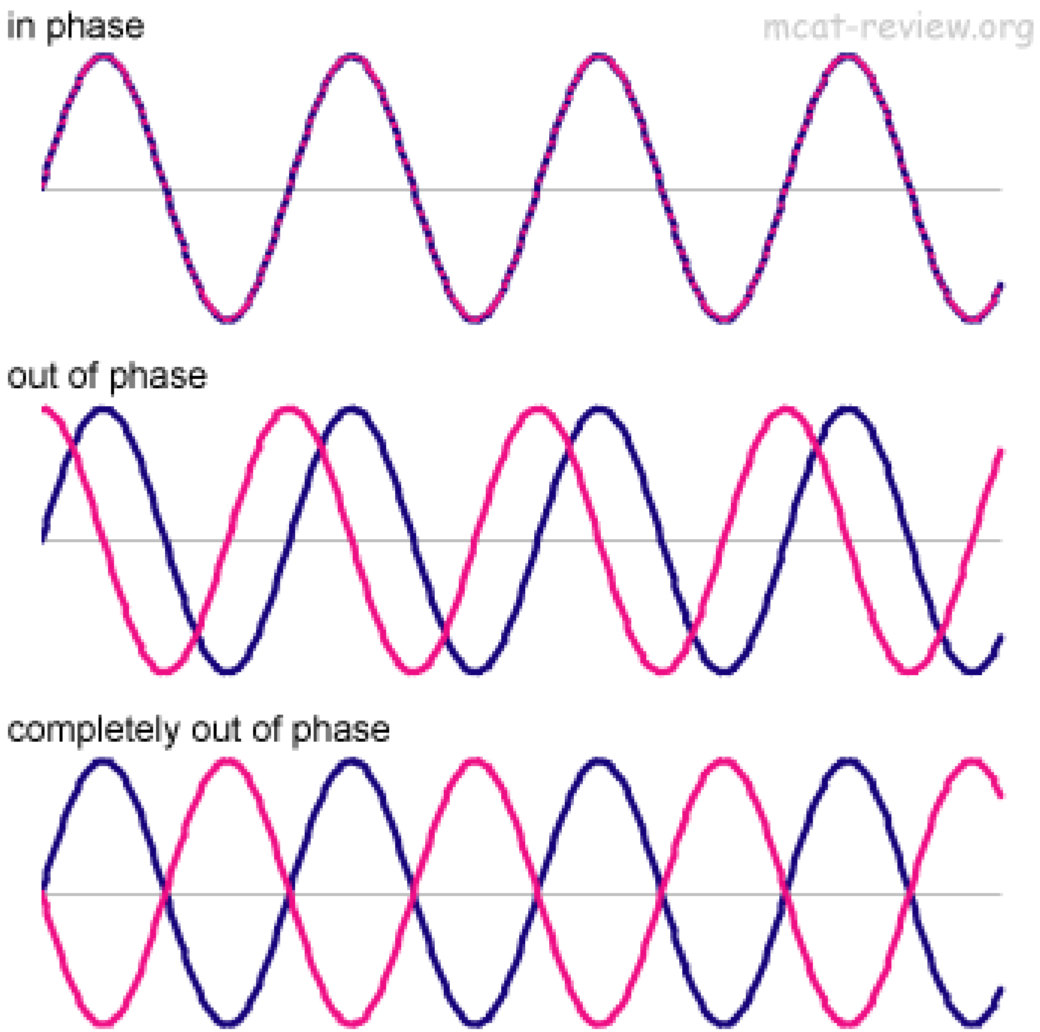
Settling time: Time required for the output to remain within some small percentage of final output.

Dynamic Characteristic

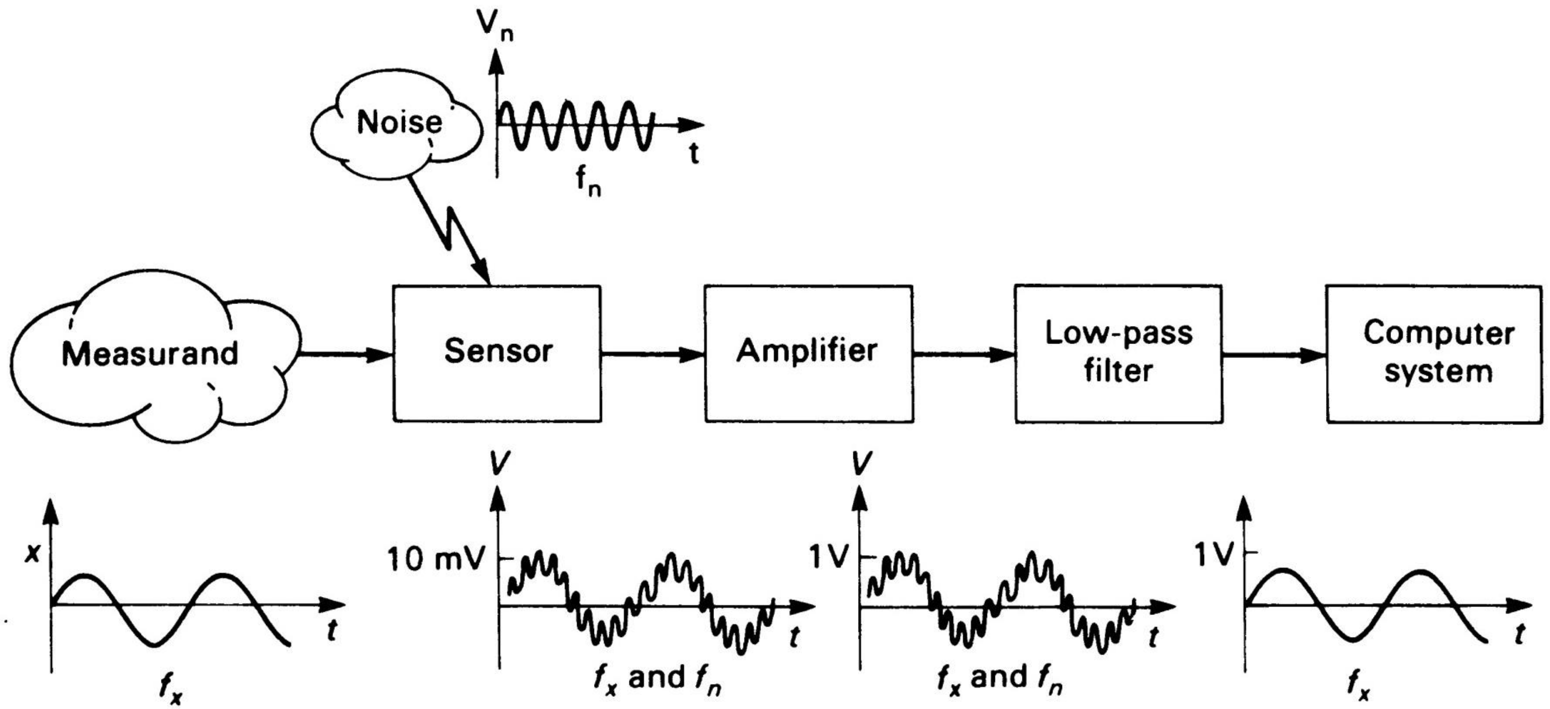


Output can differ in amplitude and phase. It is very important to study phase response of the measuring system when it is used for control, Amplitude and phase can be different for different input signal frequency.

Out of phase signal



Typical measuring system



Sensor output is a combination of sensor noise and input quantity. Noise is to be filtered such that original contents are preserved. System used for signal conditioning should have permissible noise and signal distortion parameters.

Dynamic System

Input and output can be related in simplified form:

$$\boxed{\begin{aligned} & a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-1} q_0}{dt^{n-1}} + \dots + a_1 \frac{dq_0}{dt} + a_0 q_0 = \\ & b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \dots + b_1 \frac{dq_i}{dt} + b_0 q_i \end{aligned}}$$

q_0 = Output quantity

q_i = Input quantity

a's, b's = system physical parameters assumed constant

Solution of above expression can be found out using standard mathematical techniques.

Most of the engineering systems can be simplified. No need to have such a complicated differential equation. Closed form or numerical methods can be used for solutions.

Dynamic System

Input and output can be related in simplified form:

$$a_n \frac{d^n q_0}{dt^n} + a_{n-1} \frac{d^{n-1} q_0}{dt^{n-1}} + \ldots + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_m \frac{d^m q_i}{dt^m} + b_{m-1} \frac{d^{m-1} q_i}{dt^{m-1}} + \ldots + b_1 \frac{dq_i}{dt} + b_0 q_i$$

q_0 = Output quantity

q_i = Input quantity

a's, b's = system physical parameters assumed constant

$$a_0 q_0 = b_0 q_i$$

Zeroth order system

$$a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

First order system

$$a_2 \frac{d^2 q_0}{dt^2} + a_1 \frac{dq_0}{dt} + a_0 q_0 = b_0 q_i$$

Second order system

Zero-order system

a_0 and b_0 are non zero in the generalised equation

$$\mathbf{a}_0 \mathbf{q}_0 = \mathbf{b}_0 \mathbf{q}_\mathbf{i}$$

Output, input relationship

$$\mathbf{q}_0 = \frac{\mathbf{b}_0}{\mathbf{a}_0} \mathbf{q}_\mathbf{i} = \mathbf{K} \mathbf{q}_\mathbf{i}$$

Output is related to input by a single quantity K

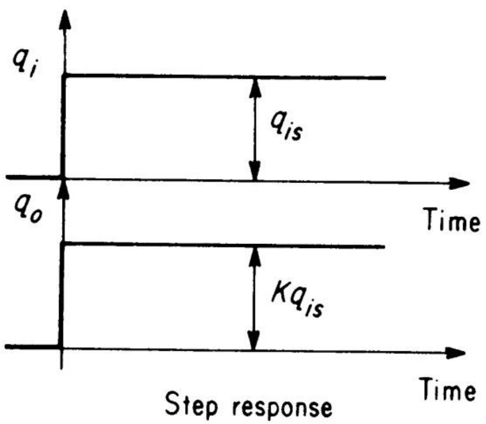
$$\mathbf{K} = \frac{\mathbf{b}_0}{\mathbf{a}_0}$$

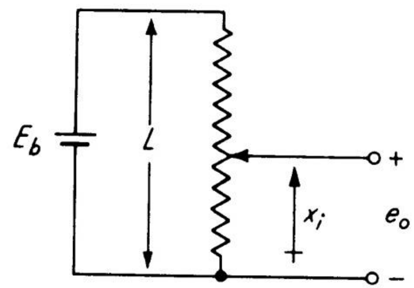
Static sensitivity, independent of input signal

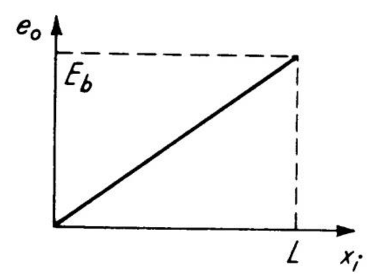
Output follows input without any distortion and lag. Zero-order instrument represents ideal or perfect dynamic performance. Zero-order system can have non-linear gain.

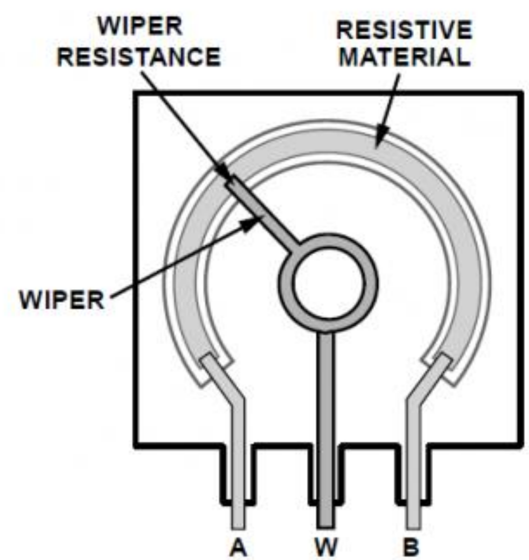
Zero-order system

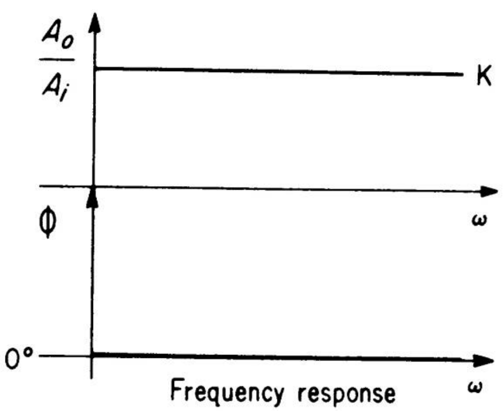
Potentiometer is a zero-order instrument for all practical purpose. Potentiometer do have a very small capacitance and inductance can become critical for very high frequency applications.

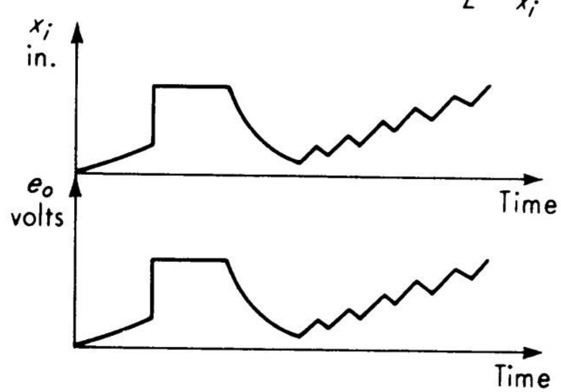












First-order system

$a_{1,}$ a_0 and b_0 are non zero in the generalised equation. Any instrument which can be represented using this equation is a first-order instrument.

$$\mathbf{a}_1 \frac{\mathbf{dq}_0}{\mathbf{dt}} + \mathbf{a}_0\mathbf{q}_0 = \mathbf{b}_0\mathbf{q}_i$$

Output, input relationship, three parameters $a_{1,}$ a_0 and b_0

$$\frac{\mathbf{a}_1}{\mathbf{a}_0} \frac{\mathbf{dq}_0}{\mathbf{dt}} + \mathbf{q}_0 = \frac{\mathbf{b}_0}{\mathbf{a}_0}\mathbf{q}_i$$

$$\mathbf{K} = \frac{\mathbf{b}_0}{\mathbf{a}_0}$$

Static sensitivity

$$\boxed{(\tau\mathbf{D} + 1)\mathbf{q}_0 = \mathbf{K}\mathbf{q}_i}$$

$$\tau = \frac{\mathbf{a}_1}{\mathbf{a}_0}$$

Time constant

$$\mathbf{D} = \frac{\mathbf{d}}{\mathbf{dt}}$$

Operator

First-order system

$$\frac{q_0}{q_i}(D)=\frac{K}{\tau D+1}$$

Transfer function for the first order system

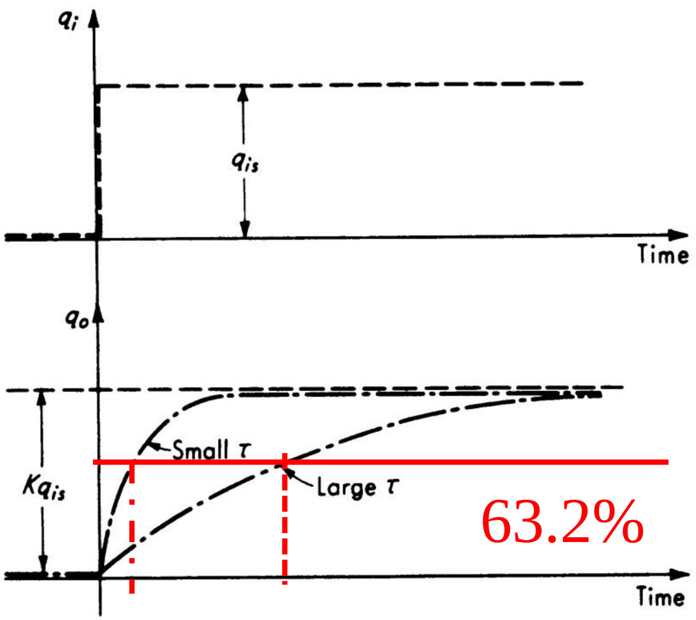
$$q_0=0; \quad at \quad t=0 \qquad\qquad q_i=q_{is}; \quad for \quad t>0$$

$$q_0=Kq_{is}(1-e^{-t/\tau})$$

Solution of the equation for a step function

$$\frac{q_0}{Kq_{is}}=(1-e^{-t/\tau})$$

In non-dimensional form



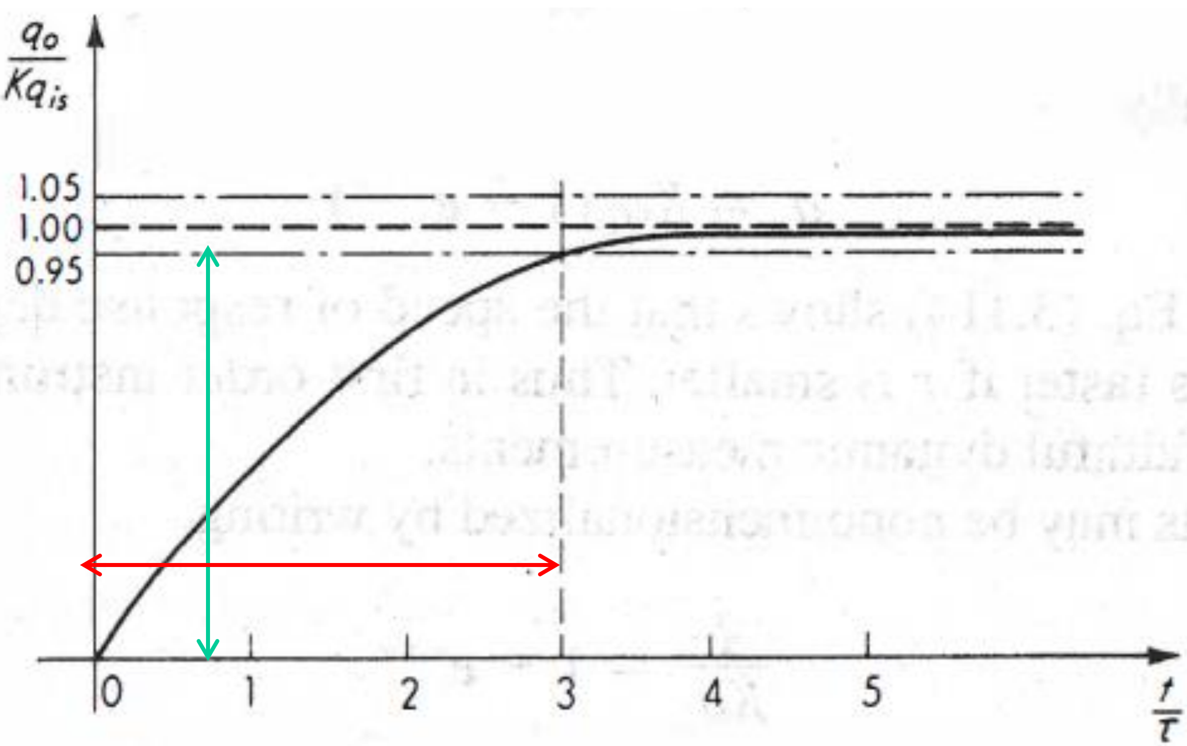
First-order system

$$e_m = q_i - \frac{q_0}{K}$$ Measurement error

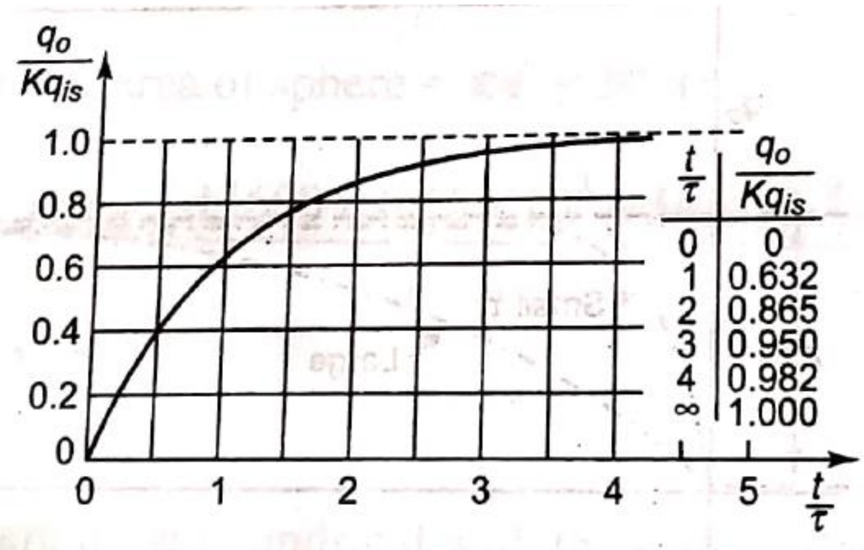
$$e_m = q_{is} - q_{is}(1 - e^{-t/\tau})$$

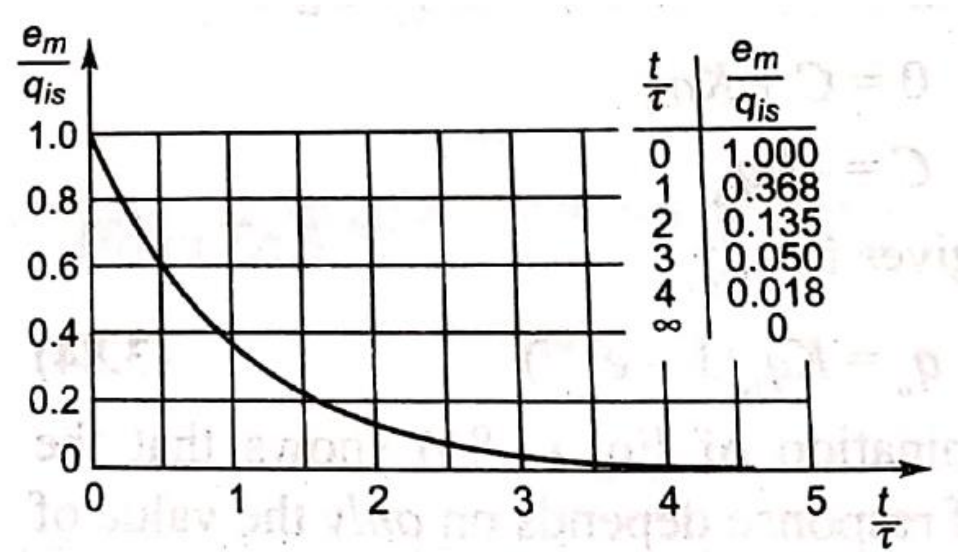
$$\frac{e_m}{q_{is}} = e^{-t/\tau}$$

In one time constant output is 63.2% of input. To achieve output within 5% of settling time, 3 time constants are required. For fast response, time constant should be less



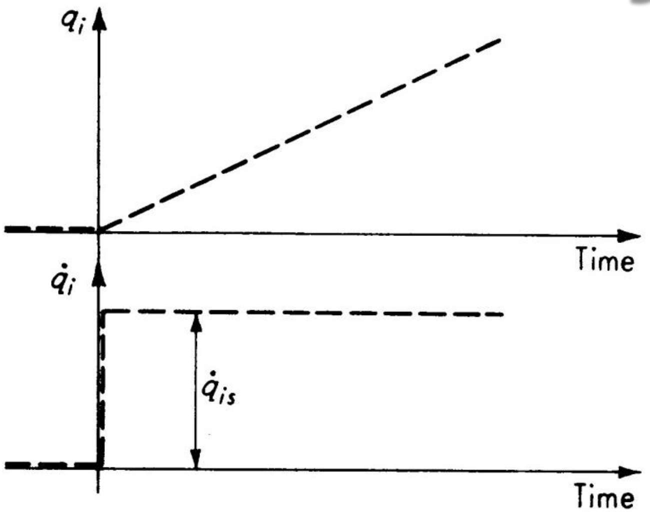
First-order system





In one time constant output is 63.2% of input. To achieve output within 5% of settling time, 3 time constants are required. Error reduces as more time is given to achieve steady state. Ideally infinite is required. For practical purpose four to five time constant is good enough. For fast response, time constant should be less.

First-order system - Ramp response



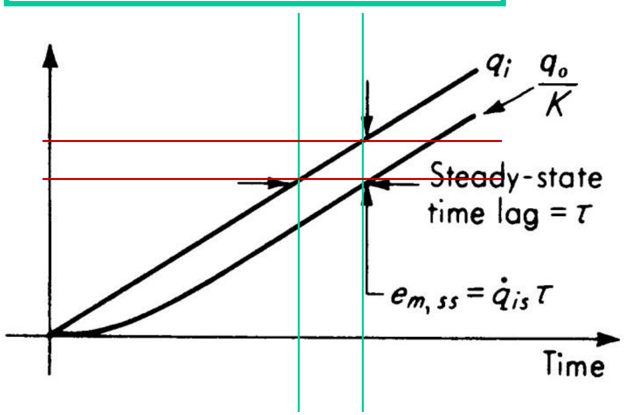
$$\mathbf{q_i} = \begin{cases} \mathbf{q_0} = 0 & \mathbf{t} \le 0 \\ \dot{\mathbf{q}}_{\mathbf{is}}\mathbf{t} & \mathbf{t} \ge 0 \end{cases}$$

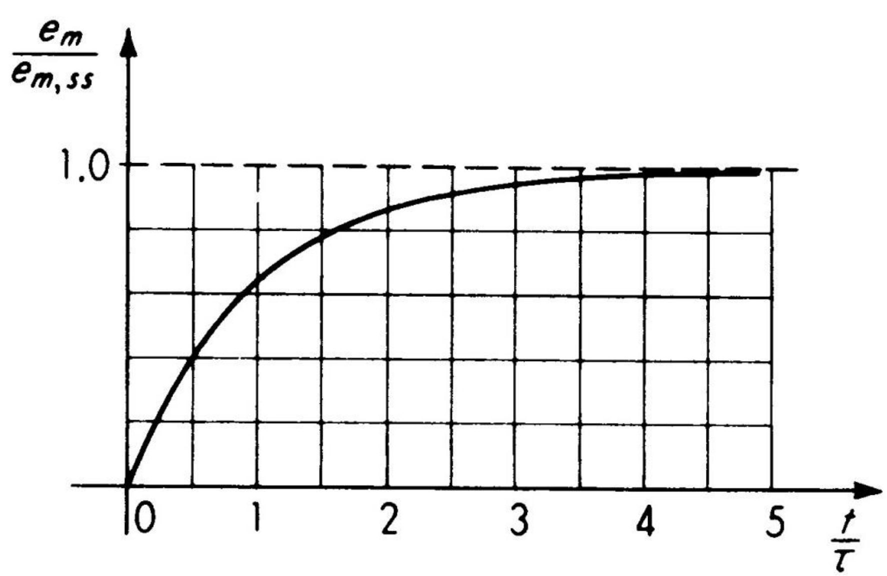
$$\mathbf{e_m} = \underbrace{-\dot{\mathbf{q}}_{\mathbf{is}}\boldsymbol{\tau}\mathbf{e}^{-\mathbf{t}/\tau}}_{\text{Transient error } e_{m,t}} + \underbrace{\dot{\mathbf{q}}_{\mathbf{is}}\boldsymbol{\tau}}_{\text{Steady state error } e_{m.ss}}$$

First term gradually disappears – transient error. Second term persist for ever – steady state error.

$$\mathbf{q_0} = \mathbf{K}\dot{\mathbf{q}}_{\mathbf{is}}(\boldsymbol{\tau}\mathbf{e}^{-\mathbf{t}/\tau} + \mathbf{t} - \boldsymbol{\tau})$$

$$\boxed{\frac{q_0}{K} = \dot{q}_{is}(\tau e^{-t/\tau} + t - \tau)}$$

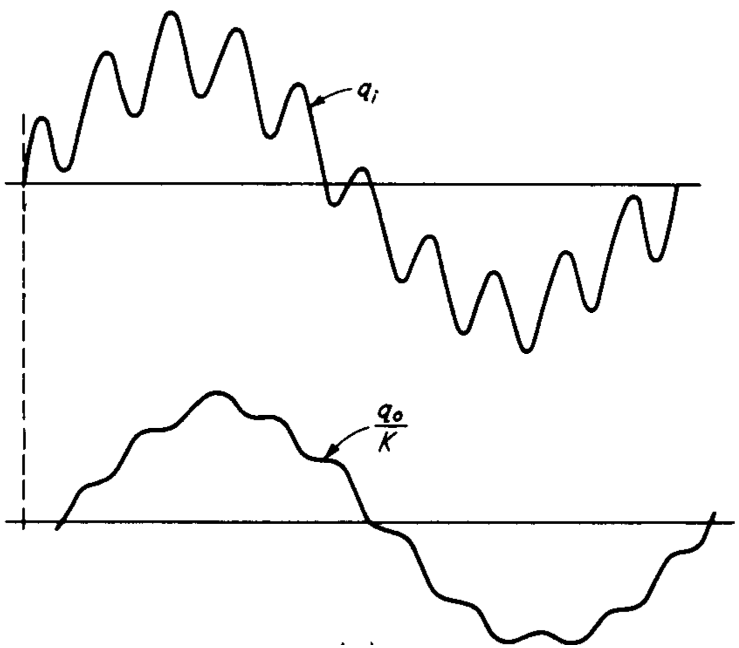




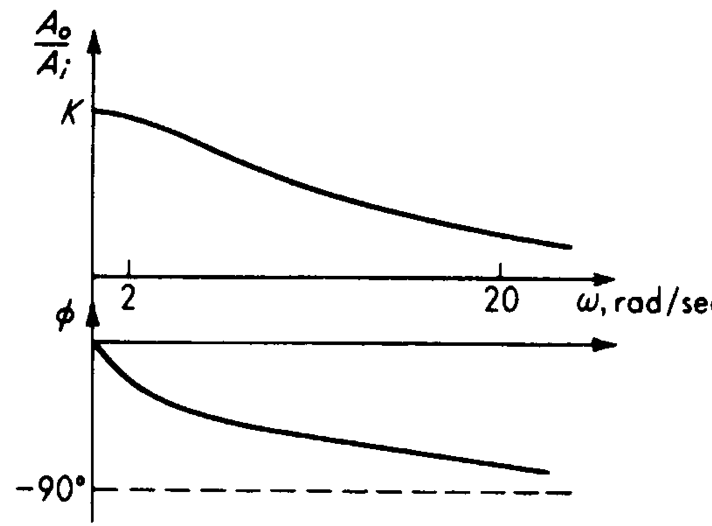
First-order system – sinusoidal inputs

For sinusoidal input, input and output can be given as

$$\frac{\mathbf{q}_0}{\mathbf{q}_i}(\mathbf{i}\omega) = \frac{\mathbf{K}}{\mathbf{i}\omega\tau + 1} = \frac{\mathbf{K}}{\sqrt{\omega^2\tau^2 + 1}} \angle[\tan^{-1}(-\omega\tau)]$$

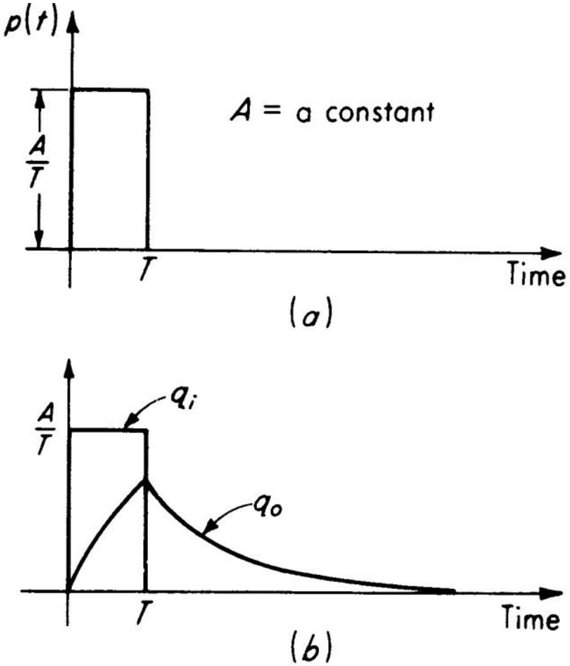


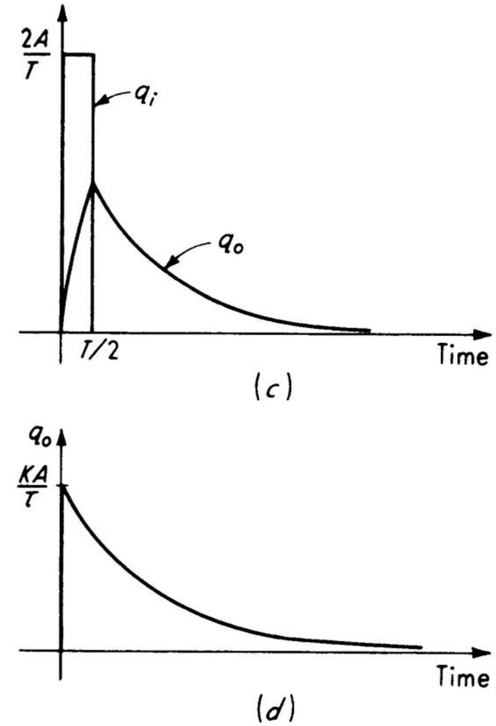
Inadequate frequency response



Output gain and phase depends on frequency.

First-order system – Impulse response





Same response as step input till the pulse is high. It never reaches the required output.

Exact impulse is infinite amplitude, derivative of step function

Second-order system

a_2 a_1, a_0 and b_0 are non zero in the generalised equation. Any instrument which can be represented using this equation is a second - order instrument.

$$\boxed{\mathbf{a}_2 \frac{\mathbf{d}^2\mathbf{q}_0}{\mathbf{dt}^2} + \mathbf{a}_1 \frac{\mathbf{dq}_0}{\mathbf{dt}} + \mathbf{a}_0\mathbf{q}_0 = \mathbf{b}_0\mathbf{q}_\mathbf{i}}$$

Above equation can be written as

$$\left(\frac{\mathbf{D}^2}{\omega_\mathbf{n}^2} + \frac{2\xi\mathbf{D}}{\omega_\mathbf{n}} + 1\right)\mathbf{q}_0 = \mathbf{K}\mathbf{q}_\mathbf{i}$$

$$\mathbf{K} = \frac{\mathbf{b}_0}{\mathbf{a}_0}$$

Static sensitivity

$$\omega_\mathbf{n} = \sqrt{\frac{\mathbf{a}_0}{\mathbf{a}_2}}$$

Undamped natural frequency

$$\xi = \frac{\mathbf{a}_1}{2\sqrt{\mathbf{a}_0\mathbf{a}_2}}$$

Damping ratio

Transfer function

$$\frac{\mathbf{q}_0}{\mathbf{q}_\mathbf{i}}(\mathbf{D}) = \frac{\mathbf{K}}{\mathbf{D}^2/\omega_\mathbf{n}^2 + 2\xi\mathbf{D}/\omega_\mathbf{n} + 1}$$

Second-order system – an example

Force measuring spring

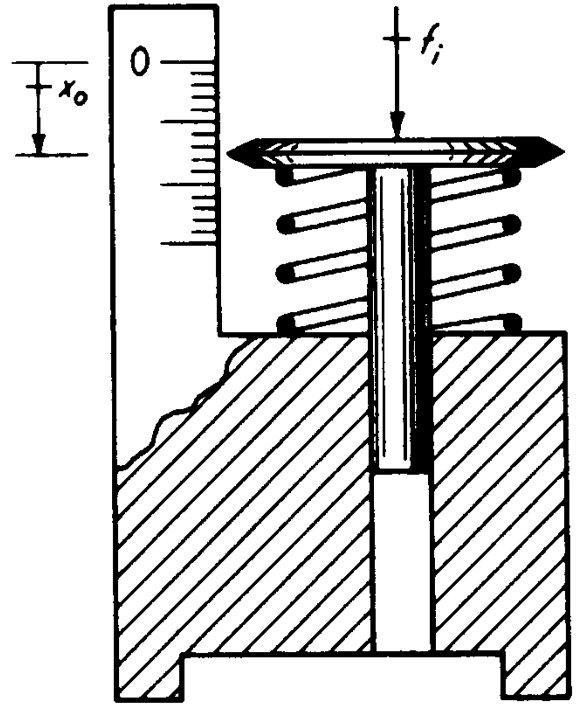
Total mass – M

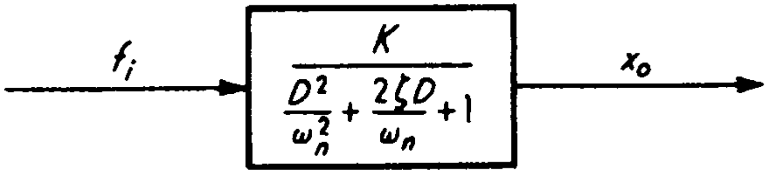
Spring constant – K_s

Damping - B

$$(\mathbf{MD}^2 + \mathbf{BD} + \mathbf{K}_s)\mathbf{x}_0 = \mathbf{f}_i$$

$$\mathbf{K} = \frac{1}{\mathbf{K}_s} \qquad \omega_n = \sqrt{\frac{\mathbf{K}_s}{\mathbf{M}}} \qquad \xi = \frac{\mathbf{B}}{2\sqrt{\mathbf{K}_s\mathbf{M}}}$$

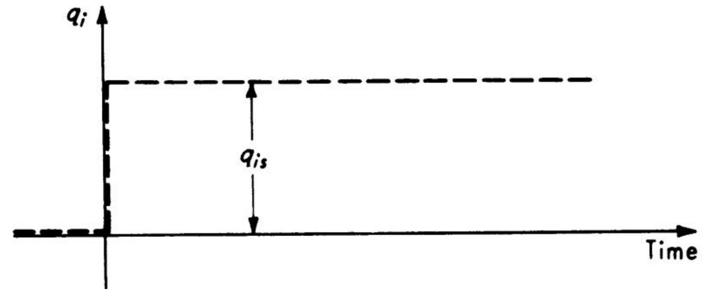


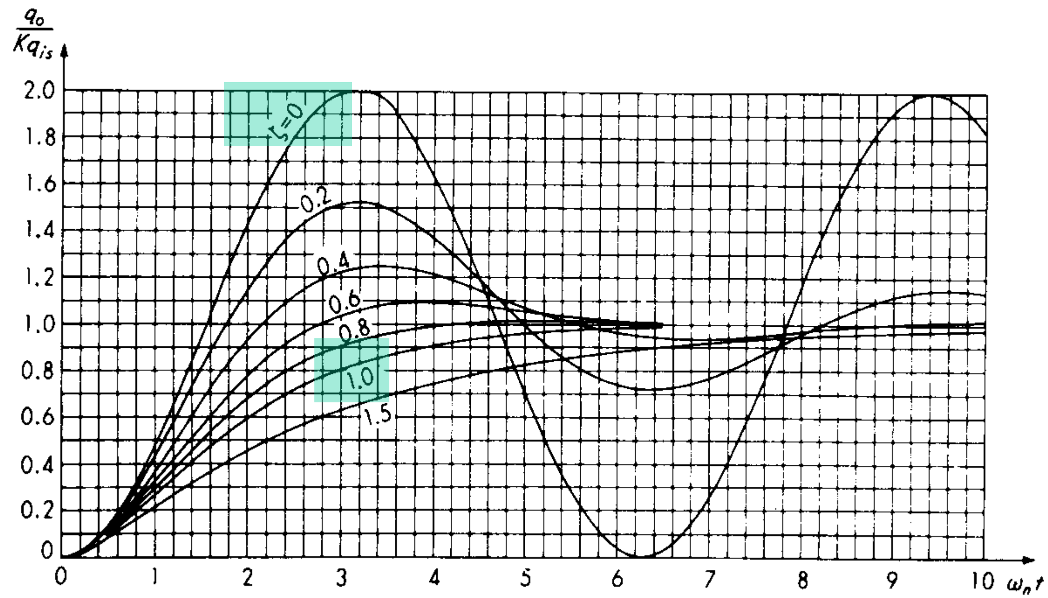


ω_n is direct indication of speed of response

Second-order system – Step response

$$\left(\frac{\mathbf{D}^2}{\omega_n^2}+\frac{2\xi\mathbf{D}}{\omega_n}+1\right)\mathbf{q}_0=\mathbf{K}\mathbf{q}_{is}$$





Second-order system – Step response

$$\frac{q_0}{Kq_{is}} = -\frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + -\frac{\xi - \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t} 1$$

Over damped case

$$\frac{q_0}{Kq_{is}} = -(1 + \omega_n t)e^{-\xi\omega_n t} + 1$$

Critically damped case

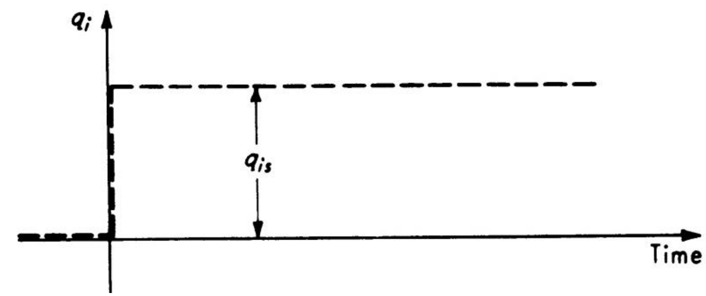
$$\frac{q_0}{Kq_{is}} = -\frac{e^{-\xi\omega_n t}}{\sqrt{1 - \xi^2}}\left(sin\sqrt{1 - \xi^2}\omega_n t + \phi\right) + 1$$

$$\phi = \sin^{-1}\sqrt{1 - \xi^2}$$

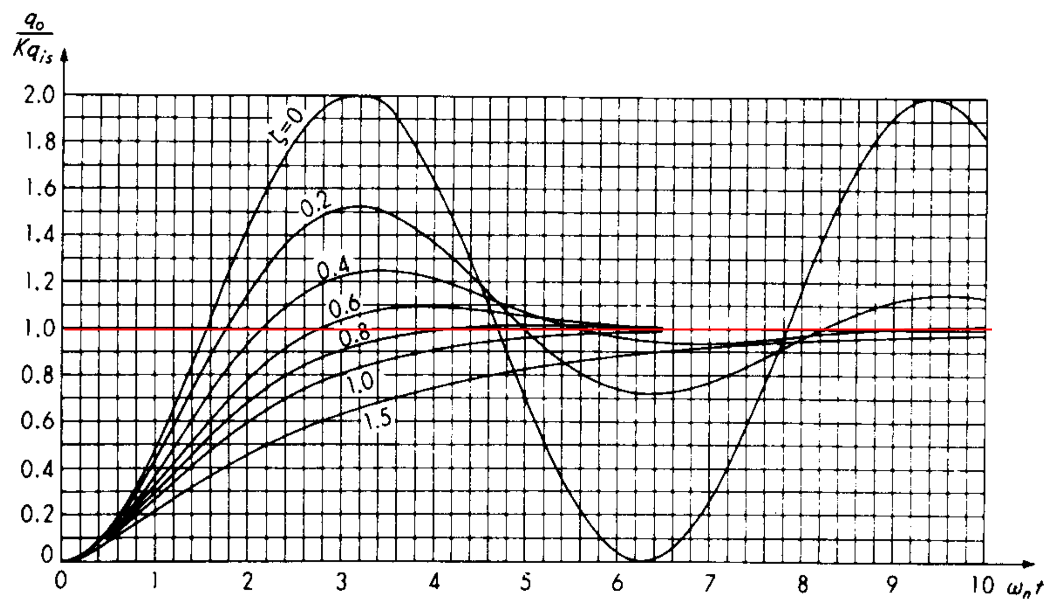
underdamped damped case

Second-order system – Step response

$$\left(\frac{\mathbf{D}^2}{\omega_n^2}+\frac{2\xi\mathbf{D}}{\omega_n}+1\right)\mathbf{q}_0=\mathbf{K}\mathbf{q}_{is}$$



Under damping – Damping ratio less than 1. This will always give overshoot before reaching steady state



Critical damping – Damping ratio equal to 1. this will take minimum time to achieve steady state output without overshoot

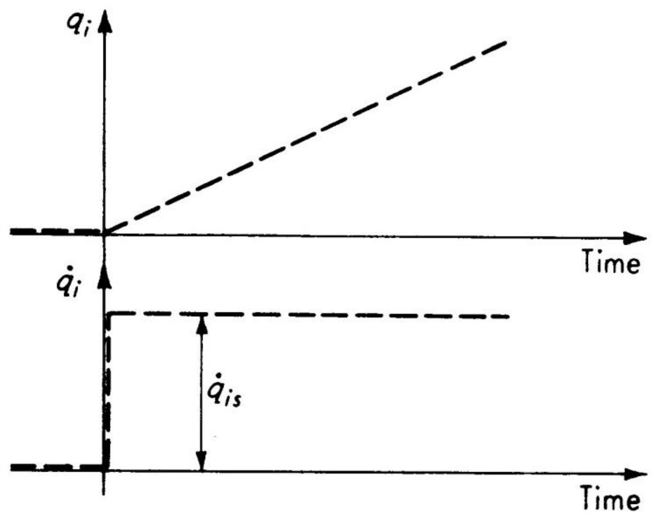
Over damping – Damping ratio greater than 1.this will take more time to achieve steady state compared to critical damping output without overshoot

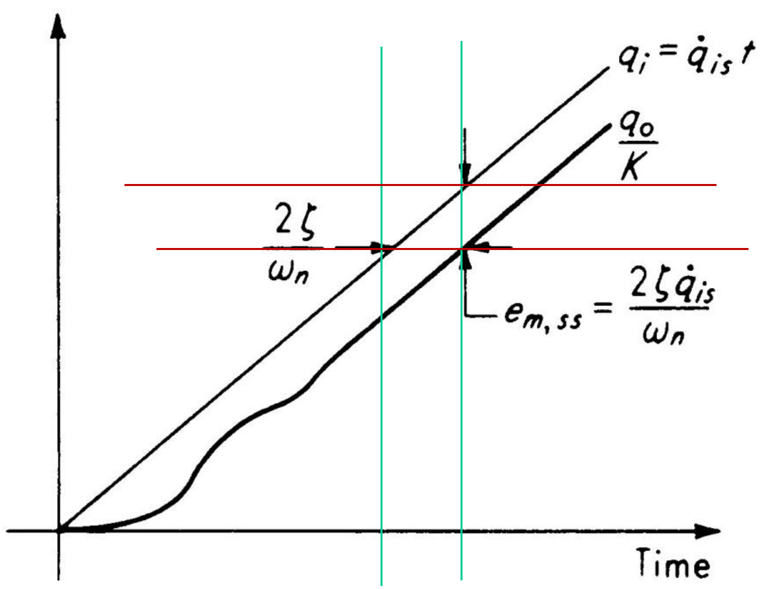
Second-order system – Ramp response

Steady state error $2\xi \dot{\mathbf{q}}_{is} / \omega_n$

Time lag $2\xi / \omega_n$

Steady state error can be reduced by reducing ξ and increasing ω_n This will increase the oscillations in the output





Second-order system – Ramp response

$$\frac{q_0}{K} = \dot{q}_{is}t - \frac{2\xi\dot{q}_{is}}{\omega_n}\left(1 + \frac{2\xi^2 - 1 - 2\xi\sqrt{\xi^2-1}}{4\xi\sqrt{\xi^2-1}}e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + \frac{-2\xi^2 + 1 - 2\xi\sqrt{\xi^2-1}}{4\xi\sqrt{\xi^2-1}}e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t}\right)$$

Over damped case

$$\frac{q_0}{K} = \dot{q}_{is}t - \frac{2\dot{q}_{is}}{\omega_n}\left[1 - e^{-\xi\omega_n t}\left(1 + \frac{\omega_n t}{2}\right)\right]$$

Critically damped case

$$\frac{q_0}{K} = \dot{q}_{is}t - \frac{2\xi\dot{q}_{is}}{\omega_n}\left[1 - \frac{e^{-\xi\omega_n t}}{2\xi\sqrt{1-\xi^2}}\left(sin\sqrt{1-\xi^2}\,\omega_n t + \phi\right)\right]$$

$$\phi = \tan^{-1}\frac{2\xi\sqrt{1-\xi^2}}{2\xi^2 - 1}$$

underdamped damped case

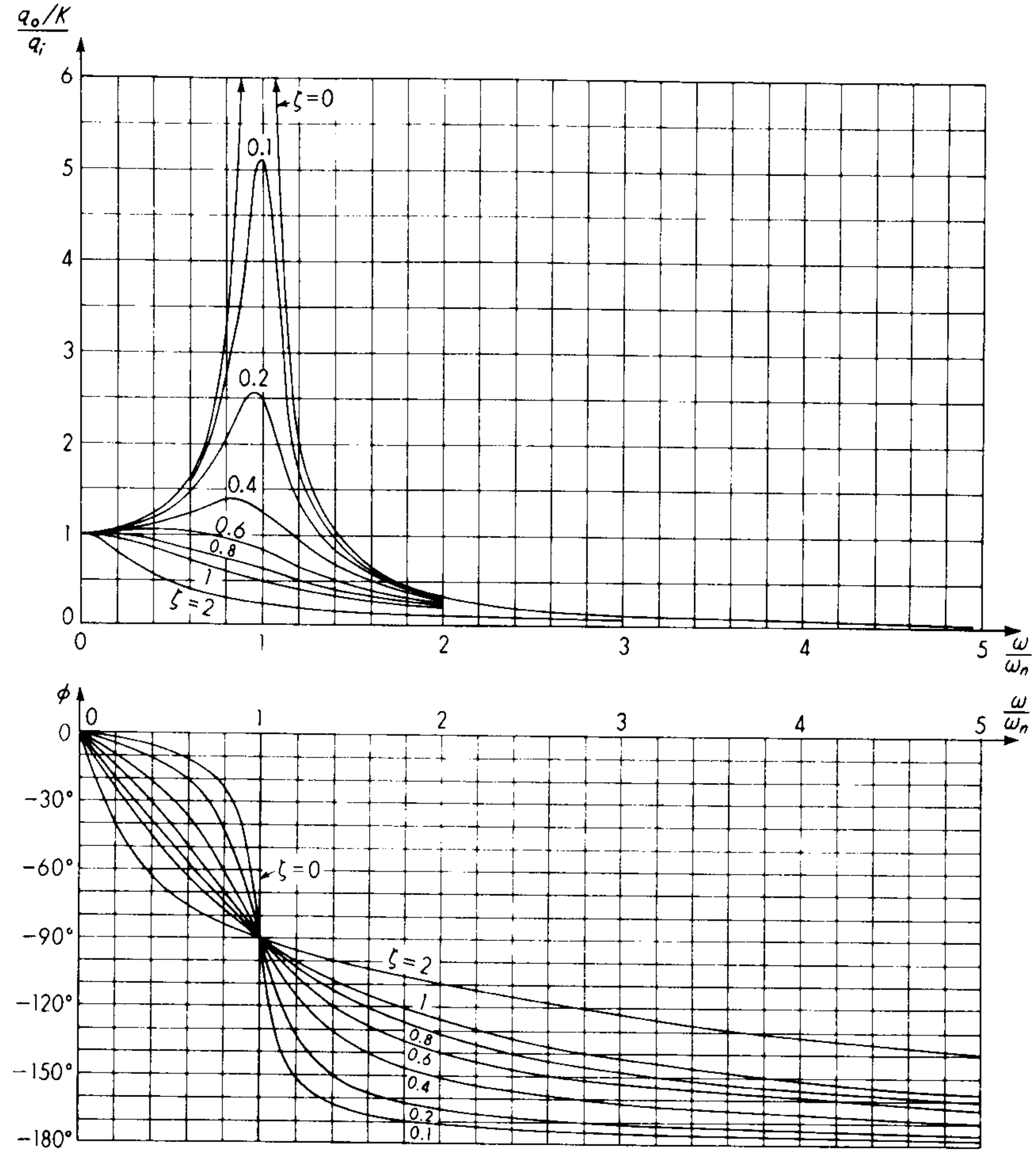
Second-order system

Flat response for damping of 0.6- 0.7

Linear variation of phase for above damping

Damping in the above range is a good choice for second order system

Frequency up to 0.4 of natural frequency



Second-order system – Frequency response

$$\frac{q_0}{q_i}(i\omega) = \frac{K}{(i\omega/\omega_n)^2 + 2\xi(i\omega/\omega_n) + 1}$$

$$\frac{q_{0/K}}{q_i}(i\omega) = \frac{1}{\sqrt{[1-(\omega/\omega_n)^2]^2 + 4\xi^2(\omega/\omega_n)^2}} \angle\phi$$

$$\phi = \tan^{-1}\frac{2\xi}{\omega/\omega_n - \omega_n/\omega}$$