MA214 Tut-5

all problems are in order but a lot of stuff is missing so it might be a little hard to follow. use this only as a last resort.

If $P(x)$ is the interpolating polynomial of the function $f(x)$, we have:

$$P(x) = \sum_{i=0}^{n} f(x_i) L_i(x)$$

For the given equation,

$$L.H.S = \sum_{i=0}^{3} x_i^j L_i(x)$$

By comparing the above two equations, we can see that the $L.H.S$ of the given equation is the interpolating polynomial of $f(x) = x^j$.

We can also note that,

$$R.H.S = x^j = f(x)$$

So we need all the values of $j \geq 0$ such that $P(x) = f(x)$.

It is obvious that we can have such a result only when we have $j \leq 3$ because we are interpolating a polynomial function, $f(x) = x^j$, using only 4 distinct interpolation points.

Let us prove the result using the error formula

$$\text{Error term} = \frac{f^{(n+1)}(x)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Here we have $f(x) = x^j$ and $n = 3$

$$f^{(n+1)}(x) = f^{(4)}(x) = \begin{cases} 0 & j \le 3 \\ j(j-1)(j-2)(j-3)x^{j-4} & \text{otherwise} \end{cases}$$

For $j \le 3$, Error term $= 0 \implies P(x) = f(x)$

For $j > 3$, there exists some value of x such that Error term $\neq 0$.

Hence the required values of j are $0, 1, 2$ and 3.

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Problem 2

Let $x_0, x_1, \cdots, x_k$ be distinct nodes and define $g(x) := [x_0, x_1, \cdots, x_k, x]$. Prove that $g[y_0, y_1, \cdots, y_n] = f[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n]$.

Solution

First, we note that the divided difference is symmetric in the nodes.

Lemma 1. *The divided difference is a symmetric function of its arguments, that is, if $z_0, z_1, \cdots, z_k$ is a permutation of $x_0, x_1, \cdots, x_k$, then*

$$f[x_0, x_1, \cdots, x_k] = f[z_0, z_1, \cdots, z_k] \tag{1}$$

Proof. $z_0, z_1, \cdots, z_k$ is a permutation of $x_0, x_1, \cdots, x_k$, which means that the nodes $x_0, x_1, \cdots, x_k$ have only been re-labelled as $z_0, z_1, \cdots, z_k$, and hence the polynomial interpolating the function f at both these sets of nodes is the same. By definition, $f[x_0, x_1, \cdots, x_k]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $x_0, x_1, \cdots, x_k$, and $f[z_0, z_1, \cdots, z_k]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $z_0, z_1, \cdots, z_k$. Since both the interpolating polynomials are equal, so are the coefficients of x^n in them. This completes the proof. $\square$

We also note that the formula for divided difference.

Lemma 2. *The divided difference satisfies the recurrence relation*

Proof. $z_0, z_1, \cdots, z_k$ is a permutation of $x_0, x_1, \cdots, x_k$, which means that the nodes $x_0, x_1, \cdots, x_k$ have only been re-labelled as $z_0, z_1, \cdots, z_k$, and hence the polynomial interpolating the function f at both these sets of nodes is the same. By definition, $f[x_0, x_1, \cdots, x_k]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $x_0, x_1, \cdots, x_k$, and $f[z_0, z_1, \cdots, z_k]$ is the coefficient of x^n in the polynomial interpolating the function f at the nodes $z_0, z_1, \cdots, z_k$. Since both the interpolating polynomials are equal, so are the coefficients of x^n in them. This completes the proof. $\square$

We also note that the formula for divided difference.

Lemma 2. *The divided difference satisfies the recurrence relation*

$$f[x_0, x_1, \cdots, x_k] = \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0} \tag{2}$$

Proof. Refer to the lectures. $\square$

Using lemmas 1 and 2, we show that

$$
\begin{aligned}
g\left[y_i, y_{i+1}\right] &= \frac{g\left[y_{i+1}\right]-g\left[y_i\right]}{y_{i+1}-yi} && (3)\\
&= \frac{g\left(y_{i+1}\right)-g\left(y_i\right)}{y_{i+1}-yi} && (4)\\
&= \frac{f\left[x_0, x_1, \cdots, x_k, y_{i+1}\right]-f\left[x_0, x_1, \cdots, x_k, y_i\right]}{y_{i+1}-yi} && (5)\\
&= \frac{f\left[x_0, x_1, \cdots, x_k, y_{i+1}\right]-f\left[y_i, x_0, x_1, \cdots, x_k\right]}{y_{i+1}-yi} \quad \text{(Using lemma 1)} && (6)\\
&= f\left[x_0, x_1, \cdots, x_k, y_i, y_{i+1}\right] \quad \text{(Using lemma 2)} && (7)\\
& && (8)
\end{aligned}
$$

Likewise, we can show that

$$
g\left[y_i, y_{i+1}, y_{i+2}\right]=f\left[x_0, x_1, \cdots, x_k, y_i, y_{i+1}, y_{i+2}\right] \tag{9}
$$

We use induction to prove that

$$
g\left[y_0, y_1, \cdots, y_n\right]=f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n\right] \tag{10}
$$

Proof. Let us assume that $g\left[y_0, y_1, \cdots, y_n\right]=f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n\right]$. Then, using lemmas 1 and 2

$$g[y_i, y_{i+1}] = \frac{}{y_{i+1} - yi} \tag{3}$$

$$= \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - yi} \tag{4}$$

$$= \frac{f[x_0, x_1, \cdots, x_k, y_{i+1}] - f[x_0, x_1, \cdots, x_k, y_i]}{y_{i+1} - yi} \tag{5}$$

$$= \frac{f[x_0, x_1, \cdots, x_k, y_{i+1}] - f[y_i, x_0, x_1, \cdots, x_k]}{y_{i+1} - yi} \quad \text{(Using lemma 1)} \tag{6}$$

$$= f[x_0, x_1, \cdots, x_k, y_i, y_{i+1}] \quad \text{(Using lemma 2)} \tag{7}$$

$$\tag{8}$$

Likewise, we can show that

$$g[y_i, y_{i+1}, y_{i+2}] = f[x_0, x_1, \cdots, x_k, y_i, y_{i+1}, y_{i+2}] \tag{9}$$

We use induction to prove that

$$g[y_0, y_1, \cdots, y_n] = f[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n] \tag{10}$$

Proof. Let us assume that $g[y_0, y_1, \cdots, y_n] = f[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n]$. Then, using lemmas 1 and 2,

$$g[y_0, y_1, \cdots, y_n, y_{n+1}] = \frac{g[y_0, y_1, \cdots, y_{n-1}, y_{n+1}] - g[y_0, y_1, \cdots, y_{n-1}, y_n,]}{y_{n+1} - y_n} \tag{11}$$

$$\frac{f[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_{n-1}, y_{n+1}] - f[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_{n-1}, y_n]}{}$$

(8)

Likewise, we can show that

$$g\left[y_i, y_{i+1}, y_{i+2}\right] = f\left[x_0, x_1, \cdots, x_k, y_i, y_{i+1}, y_{i+2}\right] \tag{9}$$

We use induction to prove that

$$g\left[y_0, y_1, \cdots, y_n\right] = f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n\right] \tag{10}$$

Proof. Let us assume that $g\left[y_0, y_1, \cdots, y_n\right] = f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n\right]$. Then, using lemmas 1 and 2,

$$g\left[y_0, y_1, \cdots, y_n, y_{n+1}\right] = \frac{g\left[y_0, y_1, \cdots, y_{n-1}, y_{n+1}\right] - g\left[y_0, y_1, \cdots, y_{n-1}, y_n,\right]}{y_{n+1} - y_n} \tag{11}$$

$$= \frac{f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_{n-1}, y_{n+1}\right] - f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_{n-1}, y_n\right]}{y_{n+1} - y_n} \tag{12}$$

$$= f\left[x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_n, y_{n+1}\right] \tag{13}$$

This completes the proof. □

If for some i, j, $y_i = y_j$, we take y_j as a distinct node and perform $\lim y_i - y_j \to 0$ in the last step.

Problem 3

Problem 3

If $f(x) = g(x)h(x)$ then find a formula for the divided difference for f in terms of those of g and h.

Solution

$f(x) = g(x)h(x)$
Aim: To get the divided difference for f in terms of those of g and h

Let us consider the nodes $x_0, x_1, \ldots, x_n$ between $[a, b]$ $f[x_0, x_1, \ldots, x_n]$ is the coefficient of x^n in polynomial which interpolates $f(x)$ using the nodes.

Let us consider the interpolating polynomial for $g(x)$ using nodes$x_0, x_1, \ldots, x_n$ be $P_g(x)$ and $h(x)$ using nodes $x_0, x_1, \ldots, x_n$ be $P_h(x)$
Using forward divided difference method for $g(x)$

$$P_g(x) = g[x_0] + g[x_0, x_1](x - x_0) + \cdots + g[x_0, x_1 \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

Using backward divided difference method for $h(x)$

$$P_h(x) = h[x_n] + h[x_{n-1}, x_n](x - x_n) + \cdots + h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_1)$$

$P(x) = P_g(x)P_h(x)$

$$\begin{aligned} &= \left(g[x_0] + g[x_0, x_1](x - x_0) + \cdots + g[x_0, x_1 \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1})\right) \\ &\qquad \left(h[x_n] + h[x_{n-1}, x_n](x - x_n) + \cdots + h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_1)\right) \end{aligned}$$

$$\begin{aligned} &= g[x_0]h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_1) + \ldots \\ &\qquad + g[x_0, x_1]h[x_0, x_1 \ldots, x_n](x - x_n) \ldots (x - x_0) + \ldots \\ &\qquad\qquad + g[x_0, x_1 \ldots, x_n]h[x_0, x_1 \ldots, x_n](x - x_n)^2 \ldots (x - x_0)^2 \end{aligned}$$

$$P_h(x) = h[x_n] + h[x_{n-1}, x_n](x - x_n) + \dots + h[x_0, x_1 \dots, x_n](x - x_n) \dots (x - x_1)$$

$$P(x) = P_g(x)P_h(x)$$

$$= \left(g[x_0] + g[x_0, x_1](x - x_0) + \dots + g[x_0, x_1 \dots, x_n](x - x_0)(x - x_1) \dots (x - x_{n-1})\right)$$
$$\left(h[x_n] + h[x_{n-1}, x_n](x - x_n) + \dots + h[x_0, x_1 \dots, x_n](x - x_n) \dots (x - x_1)\right)$$

$$= g[x_0]h[x_0, x_1 \dots, x_n](x - x_n) \dots (x - x_1) + \dots$$
$$+ g[x_0, x_1]h[x_0, x_1 \dots, x_n](x - x_n) \dots (x - x_0) + \dots$$
$$+ g[x_0, x_1 \dots, x_n]h[x_0, x_1 \dots, x_n](x - x_n)^2 \dots (x - x_0)^2$$

$$P(x) = P_n(x) + \left(g[x_0, x_1]h[x_0, x_1 \dots, x_n] + \dots + g[x_0, x_1 \dots, x_n]h[x_{n-1}, x_n]\right)$$
$$(x - x_0)(x - x_1) \dots (x - x_n) + \dots + g[x_0, x_1 \dots, x_n]h[x_0, x_1 \dots, x_n](x - x_n)^2 \dots (x - x_0)^2$$

Since the higher order terms have a factor of $(x - x_0)(x - x_1) \dots (x - x_n)$
consider $P(x) = p_n(x) + q(x)(x - x_0)(x - x_1) \dots (x - x_{n-1})$
we observe that $p_n(x)$ would also interpolate $f(x)$ and now has degree$\leq n$.
the coefficients of x^n is
$g[x_0, x_1]h[x_0, x_1 \dots, x_n] + \dots + g[x_0, x_1 \dots, x_n]h[x_{n-1}, x_n]$
Hence $f[x_0, x_1 \dots, x_n] = \sum_{r=0}^{n} g[x_0, x_1 \dots, x_r] \; h[x_r, x_{r+1} \dots, x_n]$

$$P(x) = P_n(x) + (g[x_0, x_1 \ldots, x_n]h[x_0, x_1 \ldots, x_n] + \cdots + g[x_0, x_1 \ldots, x_n]h[x_{n-1}, x_n])$$
$$(x - x_0)(x - x_1)\ldots(x - x_n) + \cdots + g[x_0, x_1 \ldots, x_n]h[x_0, x_1 \ldots, x_n](x - x_n)^2 \ldots (x - x_0)^2$$

Since the higher order terms have a factor of $(x - x_0)(x - x_1)\ldots(x - x_n)$
consider $P(x) = p_n(x) + q(x)(x - x_0)(x - x_1)\ldots(x - x_{n-1})$
we observe that $p_n(x)$ would also interpolate $f(x)$ and now has degree$\leq n$.
the coefficients of x^n is
$g[x_0, x_1]h[x_0, x_1 \ldots, x_n] + \cdots + g[x_0, x_1 \ldots, x_n]h[x_{n-1}, x_n]$
Hence $f[x_0, x_1 \ldots, x_n] = \sum_{r=0}^{n} g[x_0, x_1 \ldots, x_r] \;\; h[x_r, x_{r+1} \ldots, x_n]$

Problem 4

Construct a Hermite polynomial $H_3(x)$ for the following data for $(x, f(x), f'(x))$: (8.3,17.56492,3.116256) and (8.6,18.50515,3.151762)
If the function here is $f(x) = xlnx$ then compute $f(8.4)$ and the errors

Solution

x	$f(x)$	$f'(x)$
8.3	17.56492	3.116256
8.6	18.50515	3.151762

Given $x_0, x_1, \ldots, x_n$ points we construct a hermite polynomial $H(x)$ of degree atmost $2n+1$ where

$$H(x) = \sum_{i=0}^{1} f(x_i)H_i(x) + \sum_{i=0}^{1} f'(x_i)\hat{H}_i(x)$$

$$H_i(x) = [1 - 2(x - x_i)L_i'(x_i)](L_i(x))^2$$

$$\hat{H}_i(x) = (x - x_i)(L_i(x))^2$$

let us calculate Lagrange polynomial

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

$$L_0'(x) = \frac{1}{x_0 - x_1} \qquad L_1'(x) = \frac{1}{x_1 - x_0}$$

$$L_0(8.4) = 0.66666 \quad L_1(8.4) = 0.33333 \quad L_0'(x) = -3.3333333333 \quad L_1'(x) = 3.3333333333$$

$$H_0(x) = [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) \qquad H_1(x) = [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x)$$

$$H(x) = \sum_{i=0} f(x_i)H_i(x) + \sum_{i=0} f'(x_i)\hat{H}_i(x)$$

$$H_i(x) = [1 - 2(x - x_i)L_i'(x_i)](L_i(x))^2$$

$$\hat{H}_i(x) = (x - x_i)(L_i(x))^2$$

let us calculate Lagrange polynomial

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

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$$L_0(8.4) = 0.66666 \quad L_1(8.4) = 0.33333 \quad L_0'(x) = -3.3333333333 \quad L_1'(x) = 3.3333333333$$

$$\begin{aligned} H_0(x) &= [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) \\ &= [1 - 2(x - 8.3)(-3.3333333333)]\left(\frac{x - 8.6}{-0.3}\right)^2 \end{aligned}$$

$$\begin{aligned} H_1(x) &= [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x) \\ &= [1 - 2(x - 8.6)(3.3333333333)]\left(\frac{x - 8.3}{0.3}\right)^2 \end{aligned}$$

$$\hat{H}_0(x) = (x - x_0)L_0^2(x) \qquad \hat{H}_1(x) = (x - x_1)L_1^2(x)$$

$L_0(8.4) = 0.66666 \quad L_1(8.4) = 0.33333 \quad L_0'(x) = -3.3333333333 \quad L_1'(x) = 3.3333333333$

$$H_0(x) = [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x)$$
$$= [1 - 2(x - 8.3)(-3.3333333333)]\left(\frac{x - 8.6}{-0.3}\right)^2$$

$$H_1(x) = [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x)$$
$$= [1 - 2(x - 8.6)(3.3333333333)]\left(\frac{x - 8.3}{0.3}\right)^2$$

$$\hat{H}_0(x) = (x - x_0)L_0^2(x)$$
$$= (x - 8.3)\left(\frac{x - 8.6}{-0.3}\right)^2$$

$$\hat{H}_1(x) = (x - x_1)L_1^2(x)$$
$$= (x - 8.6)\left(\frac{x - 8.3}{0.3}\right)^2$$

$$H(x) = f(x_0)H_0(x) + f(x_1)H_1(x) + f'(x_0)\hat{H}_0(x) + f'(x_1)\hat{H}_1(x)$$

on substituting the values we get the Hermite polynomial as

$$H(x) = (-0.0020222222)x^3 + (0.11044)x^2 + (1.7008846667)x - 3.0043539553$$

$$H(8.4) = 17.8771444582 \quad f(8.4) = 17.8771463291$$

Absolute error $= |H(8.4) - f(8.4)| = 1.8709 \times 10^{-6}$ relative error$= \frac{Abs.err}{|f(8.4)|} = 1.047 \times 10^{-7}$

Problem 5

Construct a Hermite polynomial $H_3(x)$ for the following data for $(x, f(x), f'(x))$:

Absolute error $=|H(8.4)-f(8.4)| = 1.8709\times10^{-6}$ relative error$=\frac{Abs.err}{|f(8.4)|} = 1.047\times10^{-7}$

Problem 5

Construct a Hermite polynomial $H_3(x)$ for the following data for $(x, f(x), f'(x))$: (0.8,0.22363362,2.1691753) and (1,0.65809197,2.0466965).If the function here is $f(x) = sin(e^x - 2)$ then compute $f(0.9)$ and the errors

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1	0.65809197	2.0466965

$x_0 = 0.8 \quad x_1 = 1$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-1}{-0.2} = 5(1-x)$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-0.8}{0.2} = 5(x-0.8) = 5x - 4$$

$$L_0'(x) = -5 \quad L_1'(x) = 5$$

$$\begin{aligned} H_0(x) &= [1 - 2(x - x_0)L_0'(x_0)]L_0^2(x) \\ &= [1 - 2(x - \frac{4}{5})(-5)]5^2(1-x)^2 \\ &= [1 + 2(5x - 4)]25(1-x)^2 \end{aligned}$$

$$\begin{aligned} H_1(x) &= [1 - 2(x - x_1)L_1'(x_1)]L_1^2(x) \\ &= [1 - 2(x-1)5](5x-4)^2 \\ &= (11 - 10x)(5x-4)^2 \end{aligned}$$

$$= (10x - 7)25(x - 1)^2$$

and

$$\hat{H}_0(x) = (x - x_0)L_0^2(x) \qquad \hat{H}_1(x) = (x - x_1)L_1^2(x)$$

$$= (x - \frac{4}{5})5^2(1 - x)^2 \qquad = (5x - 4)^2(x - 1)$$

$$= (5x - 4)5(1 - x)^2$$

hence

$$\begin{aligned}
H_3(x) &= f(x_0)H_0(x) + f(x_1)H_1(x) + f'(x_0)\hat{H}_0(x) + f'(x_1)\hat{H}_1(x) \\
&= \Big\{(0.22363362)(10x - 7)25(x - 1)^2 + (0.65809197)(11 - 10x)(5x - 4)^2 + (2.1691753) \\
&(5x - 4)5(x - 1)^2 + (2.0466965)(x - 1)(5x - 4)^2\Big\} \\
&= \Big\{5(x - 1)^2[5(10x - 7)(0.22363362) + (5x - 4)(2.1691753)] \\
&+ (5x - 4)^2[(11 - 10x)(0.65809197) + (x - 1)(2.0466965)]\Big\} \\
&= 25[x^3(-0.1287117) + x^2(0.33527371) + x(-0.20254446) + (0.02230613)]
\end{aligned}$$

$H_3(0.9)$=0.443924795
Given $f(x) = sin(e^x - 2)$ $f(0.9) = 0.4435924388$.
Hence,
absolute error= $0.0003323562 = 3.323562 \times 10^{-4}$
Relative error=$0.00074923775 = 7.4923775 \times 10^{-4}$.

Problem 6

Use five digit rounding arithmetic and compute the table for the values of $sin(x)$ and its derivative $cos(x)$ at 0.30,0.32 and 0.35.Obtain the corresponding Hermite polynomial $H(x)$ and compute

$$H(x) = \sum_i f(x_i)H_i(x) + \sum_i f'(x_i)\hat{H}_i(x)$$

$$H_i(x) = [1 - 2(x - x_i)L_i(x_i)](L_i(x))^2$$

$$L_i^n(x) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

$$\hat{H}_i(x) = (x - x_i)(L_i(x))^2$$

given data

x	$sin(x)$	$cos(x)$
$x_0 = 0.30$	0.29552	0.95534
$x_1 = 0.32$	0.31457	0.94924
$x_2 = 0.35$	0.34290	0.93937

$$L_0(x) = \frac{(x - 0.32)(x - 0.35)}{0.02 \times 0.05} = 1000(x - 0.32)(x - 0.35)$$

$$L_1(x) = \frac{(x - 0.30)(x - 0.35)}{-0.02 \times 0.03} = \frac{1000}{6}(x - 0.30)(x - 0.35)$$

$$(x - 0.30)(x - 0.32)$$

$$H_1(x) = [1 - 2(x - 0.32)(16.66667)](x - 0.30)^2(x - 0.35)^2\frac{10^8}{36}$$
$$= [-33.33334x + 11.66667](x - 0.30)^2(x - 0.35)^2\frac{10^8}{36}$$

$$H_2(x) = [1 - 2(x - 0.35)(80.00)](x - 0.30)^2(x - 0.32)^2 10^6$$
$$= [-160x + 57.00](x - 0.30)^2(x - 0.32)^2 10^6$$

$$\hat{H}_0(x) = (x - 0.30)(x - 0.32)^2(x - 0.35)^2 10^6$$

$$\hat{H}_1(x) = (x - 0.32)(x - 0.30)^2(x - 0.35)^2\frac{10^8}{36}$$

$$\hat{H}_2(x) = (x - 0.35)(x - 0.30)^2(x - 0.32)^2 10^6$$

Hermite polynomial for the given function $f(x) = sin(x)$ using the formula

$H(x) = 0.29552H_0(x)+0.31457H_1(x)+0.34290H_2(x)+0.95534\hat{H}_0(x)+0.94924\hat{H}_1(x)+0.93937\hat{H}_2(x)$
$H(0.34) = 0.33719$ whereas $sin(0.34) = 0.33349$

therefore Actual error$= -0.00370$
using error formula we have : $f(x) = sin(x)$ $f^{(6)}(x) = -sin(x)$
Maximum value of $f^{(6)}$ in 0.30 to 0.35 is sin(0.35) $= 0.34290$ since sin is an increasing function in 0 to $\pi/2$

$$\hat{H}_1(x) = (x-0.32)(x-0.30)^2(x-0.35)^2\frac{10^8}{36}$$

$$\hat{H}_2(x) = (x-0.35)(x-0.30)^2(x-0.32)^2 10^6$$

Hermite polynomial for the given function $f(x) = sin(x)$ using the formula

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Maximum value of $f^{(6)}$ in 0.30 to 0.35 is $\sin(0.35) = 0.34290$ since sin is an increasing function in 0 to $\pi/2$

$$error = \frac{(0.34-0.30)^2(0.34-0.32)^2(0.34-0.35)^2}{6!} \times 0.34290$$

$$\approx 3.048 \times 10^{-14} <<< \text{Actual error}$$

this means that we should use many ore decimal digits to get accurate Hermite interpolation.

Problem7

this means that we should use many ore decimal digits to get accurate Hermite interpolation.

Problem7

Compute the natural cubic spline for the following data:

x	-0.5	-0.25	0
$f(x)$	-0.0247500	0.3349375	1.1010000

$$x_0 = -0.5 \qquad x_1 = -0.25 \qquad x_2 = 0$$

$$f_0 = -0.0247500 \qquad f_1 = 0.3349375 \qquad f_2 = 1.101000$$

$S_i(x)$ is the cubic polynomial on $[x_i, x_i + 1]$ $\;i = 0, 1$
and $h = |x_{i+1} - x_i| = 0.25$

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & x_0 \leq x \leq x_1 \\ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & x_1 \leq x \leq x_2 \end{cases}$$

$$a_0 = S_0(x_0) = f_0 = -0.0247500$$
$$a_1 = S_1(x_1) = f_1 = 0.3349375$$

By the above relations we have

$$\begin{aligned} S_0''(x_0) = S_1''(x_2) &= 0 \\ 6d_0(x_0 - x_0) + 2c_0 &= 0 \\ c_0 &= 0 \end{aligned}$$

$$\begin{aligned} 6d_1(x_2 - x_1) + 2c_1 &= 0 \\ 6d_1 h + 2c_1 &= 0 \\ d_1 &= \frac{-c_1}{3h} \end{aligned}$$

$$\begin{aligned} S_0''(x_1) &= S_1''(x_1) \\ 6d_0(x_1 - x_0) + 2c_0 &= 2c_1 \\ 6d_0 h + 2c_0 &= 2c_1 \\ d_0 &= \frac{c_1}{3h} \end{aligned}$$

$$\left\{ S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 \quad x_1 \le x \le x_2 \right.$$

$$a_0 = S_0(x_0) = f_0 = -0.0247500$$
$$a_1 = S_1(x_1) = f_1 = 0.3349375$$

By the above relations we have

$$S_0''(x_0) = S_1''(x_2) = 0$$
$$6d_0(x_0 - x_0) + 2c_0 = 0$$
$$c_0 = 0$$

$$6d_1(x_2 - x_1) + 2c_1 = 0$$
$$6d_1 h + 2c_1 = 0$$
$$d_1 = \frac{-c_1}{3h}$$

$$S_0''(x_1) = S_1''(x_1)$$
$$6d_0(x_1 - x_0) + 2c_0 = 2c_1$$
$$6d_0 h + 2c_0 = 2c_1$$
$$d_0 = \frac{c_1}{3h}$$

$$f_1 = S_0(x_1)$$
$$f_1 = a_0 + b_0 h + c_0 h^2 + d_0 h^3$$
$$\frac{f_1 - f_0}{h} = b_0 + c_0 h + d_0 h^2$$
$$b_0 = \frac{f_1 - f_0}{h} - \frac{c_1 h}{3}$$

$$f_2 = S_1(x_2)$$
$$f_2 = a_1 + b_1 h + c_1 h^2 + d_1 h^3$$
$$\frac{f_2 - f_1}{h} = b_1 + c_1 h - \frac{c_1 h}{3}$$
$$b_1 = \frac{f_2 - f_1}{h} - \frac{2c_1 h}{3}$$

on substituting the values we get

$$b_0 = 1.032375 \qquad b_1 = 2.2515 \qquad d_0 = 6.502 \qquad d_1 = -6.502$$

$S_0(x) = 6.502x^3 + 9.573x^2 + 5.908875x + 1.3041875$
$S_1(x) = -6.502x^3 + 3.470625x + 1.101$
Hence

$$S(x) = \begin{cases} 6.502x^3 + 9.573x^2 + 5.908875x + 1.3041875 & -0.5 \leq x \leq -0.25 \\ -6.502x^3 + 3.470625x + 1.101 & -0.25 \leq x \leq 0 \end{cases}$$

Problem 8

Compute the natural cubic spline for the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	-0.062049958	-0.28398668	0.00660095	0.24842440

Problem 8

Compute the natural cubic spline for the following data:

x	0.1	0.2	0.3	0.4
$f(x)$	-0.062049958	-0.28398668	0.00660095	0.24842440

Solution

$$x_0 = 0.1 \qquad x_1 = 0.2 \qquad x_2 = 0.3 \qquad x_3 = 0.4$$

$$y_0 = -0.62049958 \qquad y_1 = -0.28398668 \qquad y_2 = 0.00660095 \qquad y_3 = 0.24842440$$

$$S(x) = \begin{cases} S_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0 & x_0 \le x \le x_1 \\ S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1 & x_1 \le x \le x_2 \\ S_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2 & x_2 \le x \le x_3 \end{cases}$$

for $i = 1, 2$ we have

$$S_{i-1}(x_i) = S_i(x_i) \qquad S'_{i-1}(x_i) = S'_i(x_i) \qquad S''_{i-1}(x_i) = S''_i(x_i)$$

and $S(x_i) = y_i \;\; 0 \le i \le 3 \;\; S''(x_0) = S''(x_3) = 0.$

for the second derivative equations

$$S_1''(x_0) = 0 = S_2''(x_3) \quad S_0''(x_1) = S_1''(x_1) \quad S_1''(x_2) = S_2''(x_2)$$

$$6a_0(0.1) + 2b_0 = 0 = 6a_1(0.4) + 2b_2$$

$$6a_0(0.2) + 2b_0 = 6a_1(0.2) + 2b_1$$

$$6a_1(0.3) + 2b_1 = 6a_2(0.3) + 2b_2$$

$$\begin{bmatrix} 0.001 & 0.01 & 0.1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.008 & 0.04 & 0.2 & 1 & -0.008 & -0.04 & -0.2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & -0.027 & -0.09 & -0.3 & -1 \\ 0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.064 & 0.16 & 0.4 & 1 \\ 0.6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & 2 & 0 & 0 \\ 1.2 & 2 & 0 & 0 & -1.2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.8 & 2 & 0 & 0 & -1.8 & -2 & 0 & 0 \\ 0.12 & 0.4 & 1 & 0 & -0.12 & -0.4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.24 & 0.6 & 1 & 0 & -0.27 & -0.6 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \begin{bmatrix} y_0 \\ 0 \\ 0 \\ y_1 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

on solving we get

$$a_0 = -8.99579832 \qquad a_1 = -0.94630872 \qquad a_2 = 9.94210526$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & -0.027 & -0.09 & -0.3 & -1 \\
0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.064 & 0.16 & 0.4 & 1 \\
0.6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.4 & 2 & 0 & 0 \\
1.2 & 2 & 0 & 0 & -1.2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.8 & 2 & 0 & 0 & -1.8 & -2 & 0 & 0 \\
0.12 & 0.4 & 1 & 0 & -0.12 & -0.4 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.24 & 0.6 & 1 & 0 & -0.27 & -0.6 & -1 & 0
\end{bmatrix}
\begin{bmatrix} c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

on solving we get

$$
\begin{array}{lll}
a_0 = -8.99579832 & a_1 = -0.94630872 & a_2 = 9.94210526 \\
b_0 = 2.69874477 & b_1 = -2.13095238 & b_2 = -11.93051380 \\
c_0 = 3.18521341 & c_1 = 4.15115115 & c_2 = 7.09102091 \\
d_0 = -0.95701357 & d_1 = -1.02140673 & d_2 = -1.31539611
\end{array}
$$

Problem 9

Compute the cubic spline for the data in the above problem and $f'(0.1) = 3.58502082$ and $f'(0.4) = 2.16529366$.

Solution

$$S_0(x) = a_0x^3 + b_0x^2 + c_0x + d_0$$
$$S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$$
$$S_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$$

So we get Let $x_0 = 0.1, x_1 = 0.2$ and so on The continuity equations equations are:

$$a_0(0.1)^3 + b_0(0.1)^2 + c_0(0.1) + d_0 = -0.62049958$$
$$a_0(0.2)^3 + b_0(0.2)^2 + c_0(0.2) + d_0 = -0.28398668$$
$$a_1(0.2)^3 + b_1(0.2)^2 + c_1(0.2) + d_1 = -0.28398668$$
$$a_1(0.3)^3 + b_1(0.3)^2 + c_1(0.3) + d_1 = 0.00660095$$
$$a_2(0.3)^3 + b_2(0.3)^2 + c_2(0.3) + d_2 = 0.00660095$$
$$a_3(0.4)^3 + b_3(0.4)^2 + c_{(}0.4) + d_3 = 0.24842440$$

The first derivative equations are

$$3a_0(0.1)^2 + 2b_0(0.1) + c_0 = 3.58502082$$
$$3a_0(0.2)^2 + 2b_0(0.2) + c_0 = 3a_1(0.2)^2 + 2b_1(0.2) + c_1$$
$$3a_1(0.3)^2 + 2b_1(0.3) + c_1 = 3a_2(0.3)^2 + 2b_2(0.3) + c_2$$

$$6a_0(0.2) + 2b_0 = 6a_1(0.2) + 2b_1$$

$$6a_1(0.3) + 2b_1 = 6a_2(0.3) + 2b_2$$

As a matrix :

$$\begin{bmatrix} 0.001 & 0.01 & 0.1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.008 & 0.04 & 0.2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.027 & 0.09 & 0.3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.064 & 0.16 & 0.4 & 1 \\ 0.03 & 0.2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.12 & 0.4 & 1 & 0 & -0.12 & -0.4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.27 & 0.6 & 1 & 0 & -0.27 & -0.6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.64 & 0.8 & 1 & 0 \\ 1.2 & 2 & 0 & 0 & -1.2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.8 & 2 & 0 & -1.8 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} -0.6204995 \\ -0.28398668 \\ -0.28398668 \\ 0.00660095 \\ 0.00660095 \\ 0.24842440 \\ 3.5850208 \\ 0 \\ 0 \\ 2.16529366 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above equations we get the coefficients

$$[-5.4278927], [-0.02776892], [3.75341136], [-0.99013505], [14.33700811], [-11.88670941],$$

$$[6.12519946], [-1.14825426], [1.13523922], [-5.59455431], [5.914384], [-1.29485582$$