AE 240 /AE 240M /AE 713 Practice Session's Solutions Ascent Trajectories

(a) Rectilinear Trajectories

Q.1 A rocket having a payload of 450 kg needs to achieve an ideal burnout velocity of 450 m/s. If a fuel of I_{sp} of 400s is to be used and it is known that rocket motor empty shell has a mass of 600 kg, determine the propellant and lift-off masses of the rocket, along with the effective payload fraction.

Sol. $m_{pl} = 450; \quad V_b = 450; \quad I_{sp} = 400; \quad m_s = 600;$

$V_{b|Ideal} = g_0 I_{sp} \ln \frac{m_0}{m_b};$

$450 = 9.81 \times 400 \times \ln \frac{m_0}{m_b};$

$\frac{m_0}{m_b} = 1.1215;$

$\frac{m_s+m_{pl}+m_p}{m_s+m_{pl}} = 1.1215;$

$m_p = 127.58\ kg;$

Lift-off Mass: $m_0 = m_{pl} + m_s + m_p = 1177.58\ kg$

Payload Fraction: $\frac{m_{pl}}{m_0} = \frac{450}{1177.58} = 0.3821$

Q.2 A rocket is launched vertically from the surface of the Earth and expels burnt gases at a uniform rate of 0.003 times the initial mass of rocket per second. For an effective exhaust speed of 4500 m/s, and assuming a uniform sea level gravity model (g0 = 9.805), find the speed and altitude of the rocket 15s after the lift-off. Further, determine the velocity and altitude after 80s from lift-off and the energy loss due to gravity for 80s mission.

Sol.Given parameters: $\beta = 0.003\ m_0$, $V_e = 4500$m/s, $g_0 = 9.805$

A) For 15 seconds after the lift-off:

$m(t) = m_0 - \beta t$

$m(t) = 0.955 m_0$

$V(t) = V_e \ln\left(\frac{m_0}{m(t)}\right) - g_0 t$

put values,

$V(t) = 60.1227 m/s$

$h(t) = \frac{V_e}{\beta}(m(t) ln(\frac{m(t)}{m_0}) + \beta t) - \frac{gt^2}{2}$

put all values,

$h(t) = 438.9955m$

B) For 80 seconds after the lift-off: Same approach as earlier,(here, t = 80)

$m(t) = 0.76 m_0$

$V(t) = 450.5658\, m/s$

$h(t) = 15.766\, km$

Energy loss due to gravity,

$\Delta E = -\frac{m_0 g_0 V_e}{\beta}\left[\ln \frac{m}{m_0} + \frac{\beta t}{m_0}\right]$

$\Delta E = 506.48 KJ/Kg$

Q.3 A rocket has the following lift – off parameters.

m_0 = 80 Tons, m_p = 60 Tons, I_{sp} = 300 s, g_0 = 9.81m/s2 , = 1000 kg/s. Assuming rectilinear vertical motion, determine the trajectory parameters at the end of propellant burnout, while considering the effects of gravity, altitude and atmospheric drag. $C_{D0} = 1.0$, $S_{ref} = \pi\, m^2$. Assume peak drag acceleration time to be 30s. What is the extent of correction due to drag? Also, at what time does the actual peak acceleration occur? Does it cause any error?

Sol. Given Data:

$m_0 = 80$ Ton, $m_p = 60$ Ton, $I_{sp} = 300$s, $C_D = 1.0$, $g_0 = 9.81$ m/s^2,

$\beta = 1000$ kg/s $= 1$ Ton/s, $S_{ref} = \pi\, m^2$, $t_{peak-drag} = 30$ s, $\Lambda = \frac{m_p}{m_0} = 0.75$

W.K.T for ideal condition V_b, h_b as stated below.

$V_b = g_0 I_{sp} \ln \frac{m_0}{(m_0 - m_p)} - \tilde{g}\frac{m_p}{\beta}$

$h_b = \frac{m_0 g_0 I_{sp}}{\beta}\Big[(1-\Lambda)\ln(1-\Lambda) + \Lambda\Big] - \frac{1}{2}\tilde{g}\Big(\Lambda\frac{m_0}{\beta}\Big)^2 + V_0\Lambda\frac{m_0}{\beta} + h_0;\;\; \Lambda = \frac{m_p}{m_0}$

Substituting values for ideal V_b we get,

$V_b = 9.81 * 300 * \ln\left(\frac{80}{80-60}\right) - 9.81 * (\frac{60}{1})$

$V_b = 3.491 km/s$

Substituting values for ideal h_b we get,

$h_b = \frac{80*9.81*300}{1\ Ton}\left((1-0.75) * \ln(1-0.75) + 0.75\right) - \frac{1}{2} * 9.81 * \left(0.75 * \frac{60}{1}\right)^2$

$h_b = 77.325 km$

Let us find the varied gravity at the altitude of $h_b = 77.325$ km from the below eqn.

$g = \frac{\frac{g_0}{1+\frac{h_b}{R_E}} + g_0}{2}\;\; where\ R_E = 6371km$

Therefore, after substituting required values in g, we get it as

$g = 9.728 m/s2$

Let us find the value of h_b by considering the gravity effect on rocket by taking g in the eqn of h_b at t=30s. Then,

$h(t) = \frac{m_0 g_0 I_{sp}}{\beta}\left(\left(1 - \frac{\beta}{m_0}t\right)\ln\left(1-\frac{\beta}{m_0}t\right) - \left(1 - \frac{\beta}{m_0}t\right)\right) - \frac{1}{2}gt^2 - \frac{1}{2}a_D *\ t^2$

$h_b(30) = \frac{80*9.81*300}{1\ Ton}\left(\left(1-\frac{1}{80}*30\right) * \ln\left(1-\frac{1}{80}*30\right) - \left(1-\frac{1}{80}*30\right)\right) - \frac{1}{2} * 9.728 * 30^2$

$h_{b@30s} = 14.7514$ km

From the standard data of altitude vs density, we can say that

$\rho_{h_b@30s} = 0.2036\ (approx)$

As we know drag is the function of ρ,S_{ref}, v as shown below at 30 sec,

$D_{30s} = \frac{1}{2} * \rho_{30s} * v^2_{30s} *\ s_{ref} *\ C_D$

$D_{30s} = 380.6687\, kN$

From drag force we can write, $D_{30s} = m_{30s} * a_{Drag@30s}$

$m_{0@30s} = m_0 - \beta * t = 50\, Tons\;\; a_{D@30s} = D_{30}/m_{30} = 7.61334\, m/s^2$

At 30s it is given that peak drag acceleration is going to be achieved thus we can write a_D as equals to $(a_{D@30s})/2 = 3.8067 m/s^2$.

Now, By considering the effect of gravity and drag force we can find out burn out velocity burn out altitude @ the end of propellant burn out as follows. Thus,

$V_{b-drag\&g} = g_{0sp} * \ln\left(\frac{m_0}{m_0 - m_p}\right) - g * \frac{m_p}{\beta} - a_{drag} * \frac{m_p}{\beta}$

$V_{b-drag\&g} = 9.81 * 300 * \ ln\left(\frac{80}{80-60}\right) - 9.728 * \frac{60}{1} - 2.8067 * \frac{60}{1}$

$Vb - D\&g = 3.2678\, km/s$

$h_b - D\&g = \frac{m_0 g_0 I_{sp}}{\beta}\Big[(1-\Lambda)\ln(1-\Lambda) + \Lambda\Big] - \frac{1}{2}g\left(\Lambda\frac{m_0}{\beta}\right)^2 - \frac{1}{2} * a_D\left(\Lambda\frac{m_0}{\beta}\right)^2; \ \ \Lambda = \frac{m_p}{m_0}$

$h_{bD@g} = 70.620km$

(b) Constant 'q' Gravity Turn Ascent Trajectory

Q.1 A rocket with a lift-off mass of 100 tons and propellant mass of 90 tons ($I_{sp} = 250$ s) is launched directly into a gravity turn manoeuvre, with a pitch down rate of 0.3 deg./sec. It is desired that the trajectory becomes 45°to horizontal axis (neglect Earth's curvature) after some time from the launch. Find initial speed for 0.1°initial angle, which will make this manoeuvre possible. Also, determine speed, altitude and horizontal distance at the end of the manoeuvre and the time taken for it. Is the propellant sufficient to perform the above mission? If yes, determine how much is consumed. If no, how much more is needed to perform the mission? Is it necessary to take into account the effect of earth's curvature?

Sol.

$m_0 = 100\ Tons; m_p = 90\ Tons; I_{sp} = 250\ s; \theta_b = 45;$

$q_0 = 0.3\ deg/s = 0.005235\ rad/s\ ; \quad \theta_0 = 0.1$

$V_0 = \frac{g_0 sin\theta_0}{q_0} = \frac{9.81\times\sin(0.1)}{0.005235} = 3.27\ m/s$

$t_b = \frac{(\theta_b - \theta_0)}{q_0} = \frac{(45-0.1)}{0.3} = 149.67\ s$

$V_b = \frac{g_0 sin\theta_b}{q_0} = \frac{9.81\times\sin(45)}{0.005235} = 1324.81\ m/s$

$h(\theta_b) = \frac{g_0}{4q_0^2}(\cos 2\theta_0 - \cos 2\theta_b)$

$h(\theta_b) = \frac{9.81}{4\times0.005235^2}(\cos 0.2 - \cos 90) = 89.45\ km$

$x(\theta_b) = \frac{g}{2q_0^2}\left[(\theta_b - \theta_0) - \frac{(\sin 2\theta_b - \sin 2\theta_0)}{2}\right]$

$x(\theta_b) = \frac{9.81}{2\times0.005235^2}\left[\left(\frac{\pi}{4} - \frac{0.1\times\pi}{180}\right) - \frac{(\sin 90 - \sin 0.2)}{2}\right] = 51.06\ km$

$\ln\frac{m_0}{m_b} = \frac{2}{q_0 I_{sp}}(\sin\theta_b \ - sin\theta_0) = \frac{2}{0.005235\times250}(\sin 45 - \sin 0.1)$

$\frac{m_0}{m_b} = e^{1.078};\ m_b = \frac{100}{e^{1.078}} = 34.03\ Tons;$

The propellant mass consumed is 65.97 Tons, hence it's sufficient for the mission.

(c) Constant Velocity Gravity Turn Trajectories

Q.1 A Launch vehicle with a mass of 5000 kg and flying at an angle of 70°with local vertical travels at the speed of 4000 m/s for a horizontal distance of 30 km. Determine the terminal angle reached at the end of the manoeuvre, along with the time taken and the amount of propellant ($I_{sp} = 300$s) burnt. Assume earth to be flat and constant sea-level gravity.

Sol.Given

$\Delta x = 30\,Km\;\; M_0 = 5000\;\; \theta_0 = 70°\;\; V_o = 4\,Km/s\;\; I_{sp} = 300s$

$\Delta x = \frac{V_0}{g_0}\Delta\theta \quad$ where $\Delta\theta = \theta(t) - \theta_0$

$\theta(t) = 71.05°$

$\Delta t = \frac{v_0}{g_0}\ln\left[\frac{tan(\frac{\theta(t)}{2})}{tan(\frac{\theta_0(t)}{2})}\right]$

$\Delta t = time\ taken = 7.926\,s$

$\frac{M}{M_0} = \left[\frac{sin(\theta(t))}{sin(\theta_0)}\right]^{\frac{-V_0}{g_0 I_{sp}}}$

M = 4956.15 Kg $= M_0 - M_p(burnt)$

$M_p = 43.85\,Kg$

(d) Constant Specific Thrust Gravity Turn Trajectories

Q.1 A rocket with a lift-off mass of 80 tons and propellant mass of 65 tons ($I_{sp} = 350$ s) is launched directly near vertically into a gravity turn trajectory, during which the specific thrust (T/m) is kept constant at 1.1g. It is desired that the trajectory becomes parallel to the local horizon (neglect Earth's curvature) at the end of burnout while consuming all the propellant. Find the speed at the 'pitch-over' angle command of 0.5°and the time taken for the manoeuvre. Also, determine the speed, altitude and horizontal distance at the end of the manoeuvre. Is it necessary to take into account the effect of earth's curvature?

Sol.

$m_0 = 80; m_p = 65; I_{sp} = 350; n_0 = 1.1; \theta_0 = 0.5; b = 90; mb = 15$

$\frac{m_0}{m_b} = e^{\left(\frac{n_0}{I_{sp}}\right)\Delta t} \rightarrow \Delta t = \ln\frac{m_0}{m_b} \times \frac{I_{sp}}{n_0} = \ln\frac{80}{15} \times \frac{350}{1.1} = 532.6\;s$

$$\Delta t = \frac{k'}{g_0}\left[\frac{\left(\tan\frac{\theta_b}{2}\right)^{n_0-1}}{n_0-1} + \frac{\left(\tan\frac{\theta_b}{2}\right)^{n_0+1}}{n_0+1} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{n_0-1}}{n_0-1} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{n_0+1}}{n_0+1}\right]$$

$$k' = \frac{\Delta t_0}{\left[\frac{\left(\tan\frac{\theta_b}{2}\right)^{n_0-1}}{n_0-1} + \frac{\left(\tan\frac{\theta_b}{2}\right)^{n_0+1}}{n_0+1} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{n_0-1}}{n_0-1} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{n_0+1}}{n_0+1}\right]}$$

$$k' = \frac{532.6\times 9.81}{\left[\frac{(\tan 45)^{0.1}}{0.1} + \frac{(\tan 45)^{2.1}}{2.1} - \frac{(\tan 0.25)^{0.1}}{0.1} - \frac{(\tan 0.25)^{2.1}}{2.1}\right]} = 1119.09$$

$$V_0 = k' \times \left[\left(\tan\frac{\theta_0}{2}\right)^{n_0-1} + \left(\tan\frac{\theta_0}{2}\right)^{n_0+1}\right]$$

$$V_0 = 1119.09 \times \left[(\tan 0.25)^{0.1} + (\tan 0.25)^{2.1}\right] = 649.91\;m/s$$

$$V_b = k' \times \left[\left(\tan\frac{\theta_b}{2}\right)^{n_0-1} + \left(\tan\frac{\theta_b}{2}\right)^{n_0+1}\right]$$

$$V_b = 1119.09 \times \left[(\tan 45)^{0.1} + (\tan 45)^{2.1}\right] = 2238.18\;m/s$$

$$h_b = \frac{k'^2}{2g}\left[\frac{\left(\tan\frac{\theta_b}{2}\right)^{2(n_0-1)}}{n_0-1} - \frac{\left(\tan\frac{\theta_b}{2}\right)^{2(n_0+2)}}{n_0+2} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{2(n_0-1)}}{n_0-1} + \frac{\left(\tan\frac{\theta_0}{2}\right)^{2(n_0+2)}}{n_0+2}\right]$$

$$h_b = \frac{1119.09^2}{2\times 9.81}\left[10 - 0.3225 - 3.372 + 7.507\times 10^{-16}\right] = 402.44\ km$$

$$x_b = \frac{2k'^2}{g_0}\left[\frac{\left(\tan\frac{\theta_b}{2}\right)^{2n_0-1}}{2n_0-1} + \frac{\left(\tan\frac{\theta_b}{2}\right)^{2n_0+1}}{2n_0+1} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{2n_0-1}}{2n_0-1} - \frac{\left(\tan\frac{\theta_0}{2}\right)^{2n_0+1}}{2n_0+1}\right]$$

$$x_b = \frac{2\times 1119.09^2}{9.81}\left[0.833 + 0.3125 - 0.0012 - 8.755\times 10^{-9}\right] = 292.23\ km$$