Problem 1): Use the Bisection method to find the approximations $p_1, p_2, \cdots, p_5$ for the root of $f(x) = \sqrt{x} - cos(x) = 0$ on $[0, 1]$.

solution): We have $f(0) = -1; f(1) = \sqrt{1} - cos(1) = .4597 \implies f(0)f(1) < 0$

$$p_1 = \frac{0+1}{2} = .5$$

Now $f(p_1) = \sqrt{.5} - cos(.5) = -.1705 \implies f(p_1)f(1) < 0$

$$p_2 = \frac{.5+1}{2} = .75$$

Now $f(p_2) = \sqrt{.75} - cos(.75) = .1343 \implies f(p_2)f(p_1) < 0$

$$p_3 = \frac{.75+.5}{2} = .625$$

Now $f(p_3) = \sqrt{.625} - cos(.625) = -.0204 \implies f(p_3)f(p_2) < 0$

$$p_4 = \frac{.625+.75}{2} = .6875$$

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Now $f(p_3) = \sqrt{.625} - cos(.625) = -.0204 \implies f(p_3)f(p_2) < 0$

$$p_4 = \frac{.625+.75}{2} = .6875$$

Now $f(p_4) = \sqrt{.6875} - cos(.6875) = .0563 \implies f(p_4)f(p_3) < 0$

$$p_5 = \frac{.6875+.625}{2} = .65625$$

Problem 2): Use the Bisection method to find solutions accurate to within 10^{-2} for the root of $f(x) = x^3 - 7x^2 + 14x - 6 = 0$ on $[1, 3.2]$.

solution): By trail and error method we can guess $x = 3$ is a root of f, Hence we can factorize $f(x)$ as:

$$f(x) = x^3 - 7x^2 + 14x - 6 = (x-3)(x^2 - 4x + 2)$$

And the root 3 lies in the interval $[1, 3.2]$.
Let, $a_0 = 1; b_0 = 3.2$ then,

$$f(a_0) = f(1) = 1^3 - 7 * 1^2 + 14 * 1 - 6 = 2$$

$$f(b_0) = f(3.2) = (3.2)^3 - 7 * (3.2)^2 + 14 * (3.2) - 6 = -.112$$

$$\implies \qquad f(a_0) * f(b_0) < 0$$

Therefore by intermediate value theorem there exists a $c \in (a_0, b_0)$ such that $f(c) = 0$.
For accuracy within 10^{-2}, the root c should lie in the interval $[a_n, b_n]$ such that $[a_n, b_n] < 10^{-2}$. In the bisection method, the size of the range interval reduces by a factor of 2 in every iteration.

$$d_n = \frac{d_0}{2^n} = \frac{3.2 - 1}{2^n} = \frac{2.2}{2^n}$$

To get accuracy within 10^{-2}, we have the relation $d_n < 10^{-2}$.

$$\implies \qquad \frac{2.2}{2^n} < 10^{-2}$$

$$\implies \qquad 220 < 2^n$$

As $2^7 = 128$ and $2^8 = 256$ we clearly have $n > 7$ and n should be at-most 8, as n is an integer. So we need to perform our iteration at max 8th times, because at 8th iteration it has been guaranteed from our above discussion that we will reach the given accuracy level in finding the root.

$$\Longrightarrow \qquad 220 < 2^n$$

As $2^7 = 128$ and $2^8 = 256$ we clearly have $n > 7$ and n should be at-most 8, as n is an integer. So we need to perform our iteration at max 8th times, because at 8th iteration it has been guaranteed from our above discussion that we will reach the given accuracy level in finding the root.
So we will denote a_i, b_i be our interval end points such that $f(a_i)f(b_i) < 0$, $p_i = \frac{a_i+b_i}{2}$. So for $i = 1, 2, \cdots, 8$ we have the following table:

i	a_i	b_i	p_i	$f(a_i)$	$f(b_i)$	$f(p_i)$
0	1	3.2	2.1	2	-.112	1.791
1	2.1	3.2	2.65	.791	-.112	.552125
2	2.65	3.2	2.925	.552125	-.112	.08582
3	2.925	3.2	3.0625	.08582	-.112	-.05444
4	2.925	3.0625	2.99375	.08582	-.05444	.00632
5	2.99375	3.0625	3.028125	.00632	-.05444	-.02652
6	2.99375	3.028125	3.0109375	.00632	-.02652	-.01069
7	2.99375	3.0109375	3.00234375	.00632	-.01069	-.00233
8	2.99375	3.00234375	2.998046875	.00632	-.00233	.00196

So our required root is $c = p_8 = 2.998046875$

Problem 3:

solution): Notice $f(-1.5) = -10.254; f(2.5) = 232.5586 \implies f(-1.5) * f(2.5) < 0$, hence the interval $[-1.5, 2.5]$ contains at-least a root of $f(x)$. Let us denote $a_0 = -1.5; b_0 = 2.5$ and for $i = 1, 2, 3...$ we denote the iterated interval by $[a_i, b_i]$, with $f(a_i) * f(b_i) < 0$ and $p_i = \frac{a_i + b_i}{2}$. We have the following table:

i	a_i	b_i	p_i	$f(a_i)$	$f(b_i)$	$f(p_i)$
0	-1.5	2.5	.5	-10.254	232.5586	.52734
1	-1.5	.5	-.5	-10.254	.52734	-6.3281
2	-.5	.5	0	-6.3281	.52734	0

Check the second iteration $p_2 = 0$ we have $f(p_2) = 0$. Which means we have arrived at a root of $f(x)$. So our Bisection method converges to the root '0' of $f(x)$.

Problem 4): Show that the fixed point theorem does not ensure a unique fixed point of $f(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exists.

solution): Notice that $f(x)$ is continuous and differentiable over $[0, 1]$.

Problem 4): Show that the fixed point theorem does not ensure a unique fixed point of $f(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exists.

solution): Notice that $f(x)$ is continuous and differentiable over $[0, 1]$.

$$f'(x) = -3^{-x} \ln 3 < 0 \qquad \forall\, x \in [0, 1]$$

Which implies $f(x)$ is a strictly decreasing function for $x \in [0, 1]$. So the maximum value of f obtained at the point $x = 0$ and minimum value of f obtained at the point $x = 1$.

$$f(0) = 1,\ f(1) = \frac{1}{3}$$

Thus, the image of f is contained in $[\frac{1}{3}, 1] \subseteq [0, 1]$. Hence by fixed point theorem f has a unique fixed point in $[0, 1]$.

Now observe $|f'(\log_3 \ln 2.9)| = \frac{\ln 3}{\ln 2.9} > 1$, where $\log_3 \ln 2.9 \in (0, 1)$ as $e < 2.9 < e^3 \implies 1 < \ln 2.9 < 3 \implies 0 < \log_3 \ln 2.9 < 1$. Thus the second condition of fixed point theorem does not hold. So fixed point theorem does not ensure the fixed point is unique for $f(x)$ in the interval $[0, 1]$.

We now show that $f(x)$ indeed has a unique fixed point on the interval $[0, 1]$. So let's assume the opposite, i.e there exists two different fixed points x_1, x_2 of $f(x)$ on the interval $[0, 1]$, i.e:

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We now show that $f(x)$ indeed has a unique fixed point on the interval $[0,1]$. So let's assume the opposite, i.e there exists two different fixed points x_1, x_2 of $f(x)$ on the interval $[0,1]$, i.e:

$$x_1 = f(x_1) \text{ and } x_2 = f(x_2) \qquad \text{where } x_1 \neq x_2$$

Without loss of generality, we take $x_1 < x_2$.
We already have $f(x)$ is monotonically strictly decreasing on $[0,1]$, so using this;

$$f(x_1) > f(x_2)$$

But $x_1 = f(x_1)$ and $x_2 = f(x_2)$, which implies:

$$x_1 > x_2 \qquad \text{CONTRADICTION}$$

1 Q5

Method 1

$$p_n = p_{n-1} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$$

$$p_n - 21^{1/3} = p_{n-1} - 21^{1/3} - \frac{p_{n-1}^3 - 21}{3p_{n-1}^2}$$

$$p_n - 21^{1/3} = (p_{n-1} - 21^{1/3})(1 - \frac{p_{n-1}^2 + 21^{2/3} + p_{n-1}21^{1/3}}{3p_{n-1}^2})$$

$$p_n - 21^{1/3} = (p_{n-1} - 21^{1/3})^2(\frac{2p_{n-1} + 21^{1/3}}{3p_{n-1}^2})$$

Note that the above quantity is positive given $2p_{n-1} + 21^{1/3}$ is positive. Since $p_0 = 1$, subsequent $p_k - 21^{1/3} > 0$. Therefore, $p_n > 21^{1/3}$

$$\frac{p_n - 21^{1/3}}{(p_{n-1} - 21^{1/3})^2} = \frac{2}{3p_{n-1}} + \frac{21^{1/3}}{3p_{n-1}^2} < \frac{1}{21^{1/3}}$$

Now we try to prove convergence of the given series

$$|p_n - 21^{1/3}| < \frac{1}{21^{1/3}}(p_{n-1} - 21^{1/3})^2$$

$$|p_n - 21^{1/3}| < (\frac{1}{\quad})^{1+2+\dots 2^{n-1}} |p_0 - 21^{1/3}|^{2^n} = 21^{1/3}(\frac{|p_0 - 21^{1/3}|}{\quad})^{2^n}$$

$$3p_{n-1}^2$$

Note that the above quantity is positive given $2p_{n-1} + 21^{1/3}$ is positive. Since $p_0 = 1$, subsequent $p_k - 21^{1/3} > 0$. Therefore, $p_n > 21^{1/3}$

$$\frac{p_n - 21^{1/3}}{(p_{n-1} - 21^{1/3})^2} = \frac{2}{3p_{n-1}} + \frac{21^{1/3}}{3p_{n-1}^2} < \frac{1}{21^{1/3}}$$

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$$|p_n - 21^{1/3}| < \frac{1}{21^{1/3}}(p_{n-1} - 21^{1/3})^2$$

$$|p_n - 21^{1/3}| < (\frac{1}{21^{1/3}})^{1+2+\ldots 2^{n-1}} |p_0 - 21^{1/3}|^{2^n} = 21^{1/3}(\frac{|p_0 - 21^{1/3}|}{21^{1/3}})^{2^n}$$

Now we have $|p_0 - 21^{1/3}| < 2$. Thus, $|p_n - 21^{1/3}| \longrightarrow 0$ Next we find the order of convergence of the above method

$$\lim_{n\to\infty} \frac{|p_n - 21^{1/3}|}{|p_{n-1} - 21^{1/3}|^2} = \lim_{n\to\infty} (\frac{2}{3p_{n-1}} + \frac{21^{1/3}}{3p_{n-1}^2}) = \frac{1}{21^{1/3}}$$

Thus the order of convergence of the given method is 2

Method 2

$p_n = (\frac{21}{p_{n-1}})^{1/2}$ and $p_0 = 1$

Claim: $p_n = 21^{\frac{2^n - 2^{n-1} + ... + (-1)^n 2^0}{2^{n+1}}}$

We will prove this via induction. We can see that it is true for $n = 1$. Let it be true for p_{n-1}. Thus, we have

$$p_n = (\frac{21}{p_{n-1}})^{1/2} = (\frac{21}{21^{\frac{2^{n-1} + ... + (-1)^n 2^0}{2^n}}})^{1/2} = 21^{\frac{2^n - 2^{n-1} + ... + (-1)^n 2^0}{2^{n+1}}}$$

Now we can compute $S = 2^n - 2^{n-1} + ... + (-1)^n 2^0$ as the sum of a geometric series with common ratio as -2. It takes the following expression S =

$$\begin{cases} \frac{1+2^{n+1}}{3} & \text{if n is even} \\ \frac{-1+2^{n+1}}{3} & \text{if n is odd} \end{cases}$$

In both cases we get,

$$\lim_{n\to\infty} \frac{S}{2^{n+1}} = \frac{1}{3}$$

Therefore,

$$\lim_{n\to\infty} p_n \lim_{n\to\infty} 21^{\frac{S}{2^{n+1}}} = 21^{1/3}$$

Now that we have convergence, we find its order,

$$|p_n - 21^{1/3}| \quad |(\frac{21}{p_{n-1}})^{\frac{1}{2}} - 21^{1/3}| \quad 21^{1/3}\,|21^{1/6} - p_{n-1}^{1/2}| \quad 21^{1/3} \quad 1$$

In both cases we get,

$$\lim_{n\to\infty} \frac{S}{2^{n+1}} = \frac{1}{3}$$

Therefore,

$$\lim_{n\to\infty} p_n \lim_{n\to\infty} 21^{\frac{S}{2^{n+1}}} = 21^{1/3}$$

Now that we have convergence, we find its order,

$$\frac{|p_n - 21^{1/3}|}{|p_{n-1} - 21^{1/3}|} = \frac{|(\frac{21}{p_{n-1}})^{\frac{1}{2}} - 21^{1/3}|}{|p_{n-1} - 21^{1/3}|} = \frac{21^{1/3}}{p_{n-1}^{1/2}} \frac{|21^{1/6} - p_{n-1}^{1/2}|}{|p_{n-1} - 21^{1/3}|} = \frac{21^{1/3}}{p_{n-1}^{1/2}} \frac{1}{|21^{1/6} + p_{n-1}^{1/2}|}$$

$$\lim_{n\to\infty} \frac{|p_n - 21^{1/3}|}{|p_{n-1} - 21^{1/3}|} = \frac{21^{1/3}}{21^{1/6}} \frac{1}{2 \times 21^{1/6}} = \frac{1}{2}$$

Thus the order of convergence for method 2 is 1

Since, we expect a sequence with a higher order of convergence to converge more rapidly than a sequence of lower order, the apparent speed of convergence of method 1 would be greater than that of method 2

Problem 6): Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_3.

solution): $f(x) = x^2 - 6, \qquad p_0 = 1$. Newton's method of iteration is (p_n be the nth iteration)

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{1-6}{2} = 3.5$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 3.5 - \frac{3.5^2 - 6}{2 * 3.5} = 2.6071$$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)} = 2.6071 - \frac{2.6071^2 - 6}{2 * 2.6071} = 2.4543$$

Problem 7): Let $f(x) = x^2 - 6$. Use the Secant method to find p_4, with

Problem 7): Let $f(x) = x^2 - 6$. Use the Secant method to find p_4, with $p_0 = 3$ and $p_1 = 2$.

solution): $f(x) = x^2 - 6$, $p_0 = 3$; $p_1 = 2$. Secant method of iteration is (p_n be the nth iteration)

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

solution): $f(x) = x^2 - 6, \qquad p_0 = 3;\ p_1 = 2$. Secant method of iteration is (p_n be the nth iteration)

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

$$\text{We have } f(p_0) = 3; f(p_1) = -2$$

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{-2(2-3)}{(-2-3)} = 2.4$$

$$\text{We have } f(p_1) = -2; f(p_2) = -.24$$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = 2.4 - \frac{-.24(2.4-2)}{(-.24+2)} = 2.4545$$

$$\text{We have } f(p_2) = -.24; f(p_3) = .0248$$

$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_2)}{f(p_3) - f(p_2)} = 2.4545 - \frac{.0248(2.4545-2.4)}{(.0248+.24)} = 2.4494$$

Problem 8): Let $f(x) = x^2 - 6$. Use the method of false position to find p_4, with $p_0 = 3$ and $p_1 = 2$.

solution): $f(x) = x^2 - 6$, $\quad p_0 = 3;\ p_1 = 2$. method of false positipn of iteration is (p_n be the nth iteration) exactly the Secant method and only difference here is we will check the interval where at the end points given function has opposite sign. We have:

$$f(p_0) = f(3) = 3;\ f(p_1) = f(2) = -2 \implies f(p_0) * f(p_1) < 0$$

Now we use secant method in the interval $[p_1, p_0]$. So using first iteration in false position method we get,

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 2 - \frac{-2(2-3)}{(-2-3)} = 2.4$$

$$f(p_2) = f(2.4) = -.24 \implies f(p_2) * f(p_1) > 0;\ f(p_2) * f(p_0) < 0$$

Now we use secant method in the interval $[p_2, p_0]$. So using first iteration in false position method we get,

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_0)}{f(p_2) - f(p_0)} = 2.4 - \frac{-2.4(2.4-3)}{(-.24-3)} = \frac{22}{9}$$

$$f(p_3) = f(2.444) = -\frac{2}{81} = -.0247 \implies f(p_3) * f(p_2) > 0;\ f(p_3) * f(p_0) < 0$$

Now we use secant method in the interval $[p_3, p_0]$. So using first iteration in false position method we get,

$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_0)}{f(p_3) - f(p_0)} = \frac{22}{9} - \frac{-\frac{2}{81}(\frac{22}{9} - 3)}{(-\frac{2}{81} - 3)} = \frac{120}{49} = 2.449$$

Problem 9): An object falling vertically through air is faces various resistance as well as gravitational force.Assume that an object with mass a is dropped from a height d_0 and that the height of the object after t seconds is

$$d_t = d_0 - \frac{ag}{k}t + \frac{a^2 g}{k^2}\left(1 - e^{-\frac{kt}{a}}\right)$$

Where $g = 9.80665\ m/s^2$ and $k = .24\ kg/m$ represents the coefficient of air resistance.If an object of mass $.25\ kg$ is dropped from a height of $300\ meters$ then find, to within $.01$ error, the time it takes this object to hit the ground.

solution): We need to determine t such that $d_t = 0$.

$$\text{Consider } f(t) = d_0 - \frac{ag}{k}t + \frac{a^2 g}{k^2}\left(1 - e^{-\frac{kt}{a}}\right)$$

We need to find t' such that $f(t') = 0$. The first derivative of $f(t)$ gives us $f'(t) = \frac{ag}{k}\left(e^{-\frac{kt}{a}} - 1\right) \implies$ for $t > 0$, $f'(t) < 0 \implies f(t)$ is strictly decreasing $\forall\, t \in [0, \infty)$ so if it has a solution it will be the only solution. Now $f(0) = d_0 > 0$ and $f\left(\frac{d_0 + 3\frac{a^2 g}{k^2}}{\frac{ag}{k}}\right) =$

solution): We need to determine t such that $d_t = 0$.

$$\text{Consider } f(t) = d_0 - \frac{ag}{k}t + \frac{a^2g}{k^2}\left(1 - e^{-\frac{kt}{a}}\right)$$

We need to find t' such that $f(t') = 0$. The first derivative of $f(t)$ gives us $f'(t) = \frac{ag}{k}\left(e^{-\frac{kt}{a}} - 1\right) \implies$ for $t > 0$, $f'(t) < 0 \implies f(t)$ is strictly decreasing $\forall\, t \in [0, \infty)$ so if it has a solution it will be the only solution. Now $f(0) = d_0 > 0$ and $f\left(\frac{d_0 + 3\frac{a^2g}{k^2}}{\frac{ag}{k}}\right) = \frac{a^2g}{k^2}\left(e^{-\frac{kt}{a}} - 2\right) \leq -\frac{a^2g}{k^2} < 0$

$$\text{As } e^{-\frac{kt}{a}} \leq 1 \;\forall\, t \geq 0$$

So by Intermediate value theorem $f(t)$ attains a root in the interval $\left[0, \frac{d_0 + 3\frac{a^2g}{k^2}}{\frac{ag}{k}}\right]$, and the root is unique(as $f(t)$ is strictly decreasing). First we will solve $f(t) = 0$ ignoring the exponential term. It gives:

$$d_0 - \frac{ag}{k}t + \frac{a^2g}{k^2} = 0 \implies 300 + \frac{.25^2 * 9.80665}{.24^2} = \frac{.25 * 9.80665}{.24}t \implies t = 30.4095$$

Consider the interval $[30.40, 30.41]$. We have, $f(30.40) = .096$; $f(30.41) = -.0052 \implies f(30.40)f(30.41) < 0$. So our required root will be in the interval $[30.40, 30.41]$.

Using Bisection method we can choose

$$p_1 = \frac{30.40 + 30.41}{2} = 30.405$$

Now our root(t') either lies in the interval $[30.40, 30.405]$, or in the interval $[30.405, 30.41]$. In either case as length of both the two intervals is $.005$ and $p_1 = 30.405$, we can conclude

$$|t' - p_1| \leq .005 < .01$$

So $p_1 = 30.405$ is our required root, within $.01$ error.