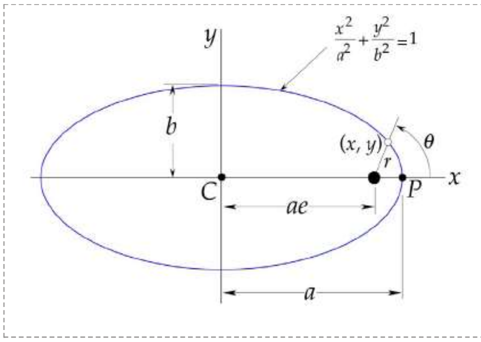
Lecture 8




- $$x = a\|e\| + \|r\|\cos\theta = a\|e\| + \left(a\frac{1-\|e\|^2}{1+\|e\|\cos\theta}\right)\cos\theta = a\frac{\|e\|+\cos\theta}{1+\|e\|\cos\theta}$$

- $$y = \|r\|\sin\theta = \left(a\frac{1-\|e\|^2}{1+\|e\|\cos\theta}\right)\sin\theta = \left(b\frac{\sqrt{1-\|e\|^2}}{1+\|e\|\cos\theta}\right)\sin\theta$$

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{\|e\|+\cos\theta}{1+\|e\|\cos\theta}\right)^2 + \left(\frac{1-\|e\|^2}{(1+\|e\|\cos\theta)^2}\right)\sin^2\theta = 1$$

- $$\varepsilon = \frac{-\mu^2}{2\|h\|^2}(1-\|e\|^2) = \underbrace{\frac{-\mu}{2a}}_{\text{does not depend on the eccentricity}} \text{ (why?)}$$

- Kepler's second law:

$$\Delta A = \frac{\|h\|}{2} \Delta t$$

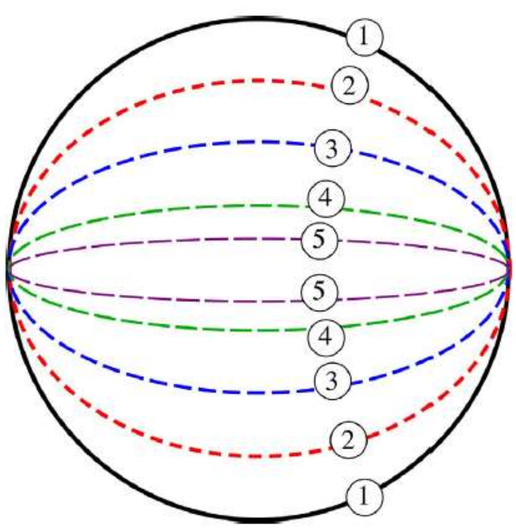
$$\pi a b = \frac{\|h\|}{2} T$$

- $$T = \frac{2\pi a b}{\|h\|} = \frac{2\pi a^2 \sqrt{1-\|e\|^2}}{\|h\|}$$

$$= \frac{2\pi}{\mu^2} \left( \frac{\|h\|}{\sqrt{1-\|e\|^2}} \right)^3$$

$$= \underbrace{\frac{2\pi}{\sqrt{\mu}}\, a^{3/2}}$$

does not depend on the eccentricity




- Kepler's third law: The period of a planet is proportional to the three-half power of its semi-major axis.

to the

- $$\frac{r_p}{r_a} = \frac{1-\|e\|}{1+\|e\|} \Rightarrow \|e\| = \frac{r_a - r_p}{r_a + r_p}$$

- $$\text{Eccentricity} = \frac{\text{Distance between the foci}}{\text{Length of the major axis}}$$

- Average distance of $m_2$ from $m_1$ in the course of one complete orbit:

$$\bar{r}_\theta = \frac{1}{n} \sum_{i=1}^{n} \|r(\theta_i)\|$$

$$= \frac{\Delta\theta}{2\pi} \sum_{i=1}^{n} \|r(\theta_i)\|$$

$$= \frac{1}{2\pi} \sum_{i=1}^{n} \|r(\theta_i)\| \Delta\theta$$

- In the limit as $n \to \infty$:

$$\bar{r}_\theta = \frac{1}{2\pi} \int_0^{2\pi} \|r(\theta)\| \, d\theta$$

$$= \frac{a(1-\|e\|^2)}{2\pi} \int_0^{2\pi} \frac{d\theta}{1+\|e\|\cos\theta}$$

$$= \frac{a(1-\|e\|^2)}{2\pi}\left(\frac{2\pi}{\sqrt{1-\|e\|^2}}\right)$$

$$= a\sqrt{1-\|e\|^2} = b$$

- Recall that $r_p = a(1-\|e\|)$ and $r_p + r_a = 2a$, and so,

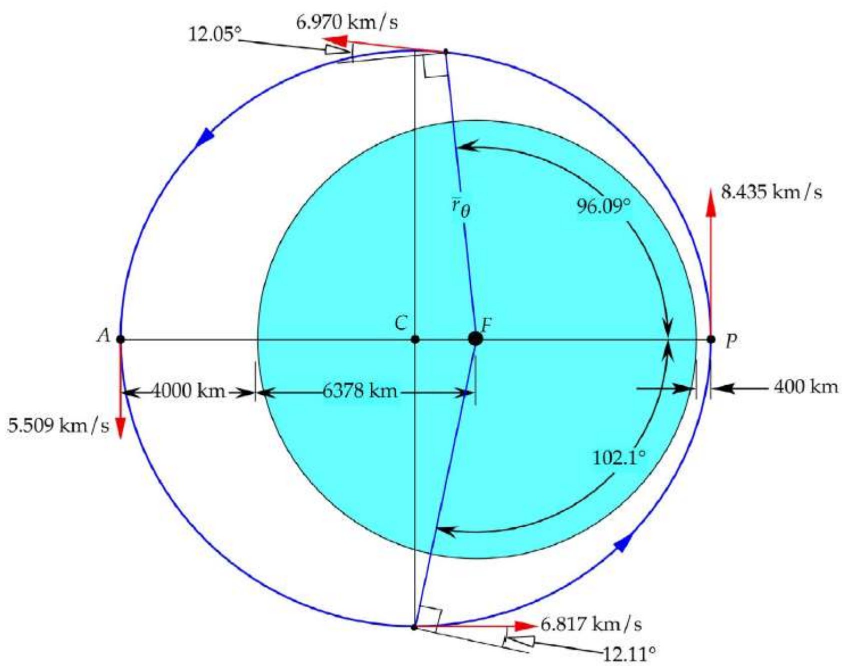
$$\bar{r}_\theta = \sqrt{r_p r_a} \quad \left(\text{Note that } \bar{r}_\theta \neq \frac{r_p + r_a}{2}\right)$$

## Example

An earth satellite is in an orbit with a perigee altitude $z_p = 400\,\text{km}$ and an apogee altitude $z_a = 4000\,\text{km}$, as shown in Fig. 2.21. Find each of the following quantities:

(a) eccentricity, $e$

(b) angular momentum, $h$




(c) perigee velocity, $v_p$

(d) apogee velocity, $v_a$

(e) semimajor axis, $a$
(f) period of the orbit, $T$
(g) true anomaly-averaged radius $\bar{r}_\theta$
(h) true anomaly when $r = \bar{r}_\theta$
(i) satellite speed when $r = \bar{r}_\theta$
(j) flight path angle $\gamma$ when $r = \bar{r}_\theta$
(k) maximum flight path angle $\gamma_{max}$ and the true anomaly at which it occurs.

Recall from Eq. (2.66) that $\mu = 398,600 \text{km}^3/\text{s}^2$ and also that $R_E$, the radius of the earth, is 6378 km.

## Details

(a) $\|e\| = \dfrac{r_a - r_p}{r_a + r_p}$ , $r_p = R_E + z_p$ , $r_a = R_E + z_a$

(b) $r_p = \dfrac{\|h\|^2}{\mu} \dfrac{1}{1 + \|e\|}$

(c) $V_p = \dfrac{\|h\|}{r_p}$

(d) $V_a = \dfrac{\|h\|}{r_a}$

(e) $a = \dfrac{r_a + r_p}{2}$

(f) $T = \dfrac{2\pi}{\sqrt{\mu}} a^{3/2}$

(g) $\bar{r}_\theta = \sqrt{r_p r_a}$

(h) $$\bar{r}_\theta = \frac{\|h\|^2}{\mu} \frac{1}{1+\|e\|\cos\theta}$$

(i) $$\frac{\|v\|^2}{2} - \frac{\mu}{\bar{r}_\theta} = -\frac{\mu}{2a}$$

(j) $$\tan\gamma = \frac{\|e\|\sin\theta}{1+\|e\|\cos\theta}$$

(k) $$\gamma = \tan^{-1}\left(\frac{\|e\|\sin\theta}{1+\|e\|\cos\theta}\right), \quad \frac{d\gamma}{d\theta} = 0$$

## Parabolic Trajectories ($\|e\|=1$)

- $$\|r\| = \frac{\|h\|^2}{\mu} \frac{1}{1+\cos\theta}$$
- $\theta \to 180^\circ$, $\|r\| \to \infty$
- $$\varepsilon = 0 \Rightarrow \frac{\|v\|^2}{2} - \frac{\mu}{\|r\|} = 0$$

$$\Rightarrow \|v\| = \sqrt{\frac{2\mu}{\|r\|}}$$

- If a body $m_2$ is launched on a parabolic trajectory, it will coast to infinity, arriving there with zero velocity relative to $m_1$.

- Parabolic paths are therefore called escape trajectories.

- Escape velocity:

$$V_{esc} = \sqrt{\frac{2\mu}{\|r\|}}$$

$$= \sqrt{2}\, V_{circular}$$

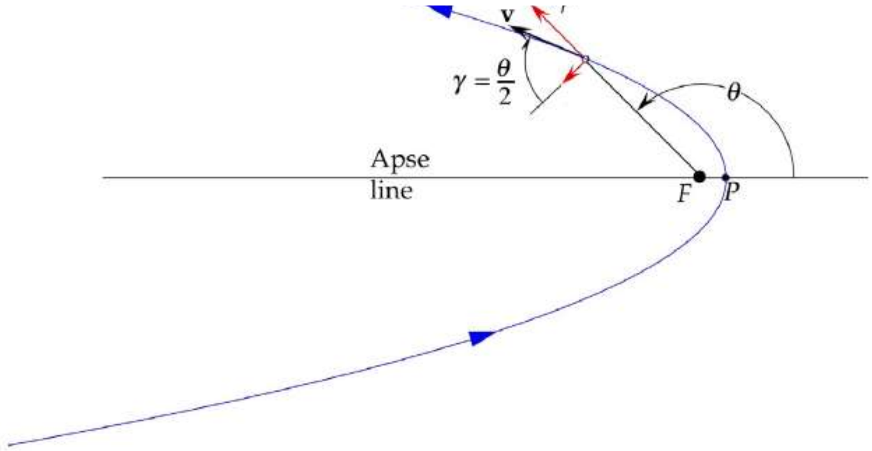
- Escape from a circular orbit requires a velocity boost of 41.4%.

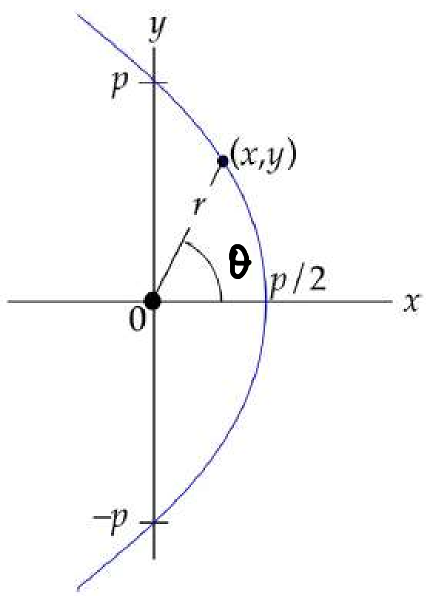
- Recall that $\tan \gamma = \dfrac{\sin\theta}{1+\cos\theta}$

$$= \frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2}$$

$$= \tan \theta/2$$

v..



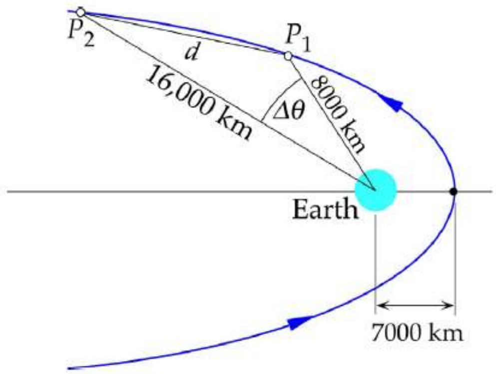

- $x = \|r\| \cos\theta$ , $y = \|r\| \sin\theta$, $\|r\| = \dfrac{p}{1+\cos\theta}$

$$x = \frac{p\cos\theta}{1+\cos\theta} \quad , \quad y = \frac{p\sin\theta}{1+\cos\theta}$$

$$\frac{x}{p/2} + \left(\frac{y}{p}\right)^2 = 1 \Rightarrow x = \frac{p}{2} - \frac{y^2}{2p}$$

<u>Example</u>

The perigee radius of a satellite in a parabolic geocentric trajectory of Fig. 2.24 is 7000 km. Find the distance $d$ between points $P_1$ and $P_2$ on the orbit, which are 8000km and 16, 000km, respectively, from the center of the earth.




## Details

$$r_p = \frac{\|h\|^2}{2\mu}$$

$$8000 = \frac{\|h\|^2}{\mu} \frac{1}{1+\cos\theta_1}$$

$$16000 = \frac{\|h\|^2}{\mu} \frac{1}{1+\cos\theta_2}$$

$$\Delta\theta = \theta_2 - \theta_1$$

Law of cosines: $d^2 = 8000^2 + 16000^2 - 2\cdot 8000\cdot 16000\cdot \cos\Delta\theta$