AE 240/338/713 Quiz No. 2 Tuesday, March 21, 2017 Weightage: 10%

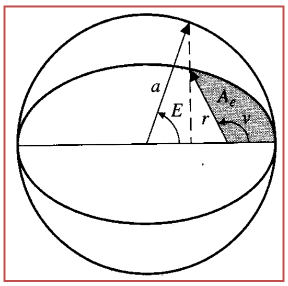
Time: 1900 – 2025 Hrs. Venue: LC 001/LT001/LT002

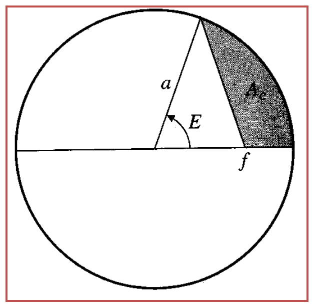
Examination is closed notes. Only calculator is allowed.

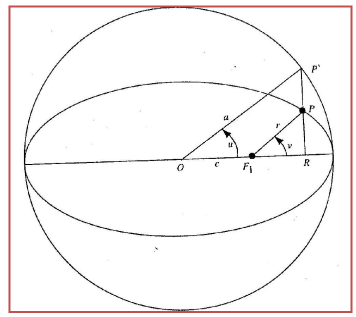
Model Solutions

Q.1 (a) Derive the expressions (a) for time of flight between any two points on an elliptic geometry in terms of eccentric anomaly 'E', semi-major axis 'a' and gravitational parameter 'μ', and (b) for true anomaly 'θ' in terms of eccentricity 'e' and eccentric anomaly 'E', using the auxiliary circle approach and concepts of mean motion, and mean / eccentric anomaly.(Hint: Use orbital time period expression given below.) (2)

$$T^2 = \frac{4\pi^2}{\mu} a^3$$







$$A_c = \frac{1}{2} a^2 (E - e\sin E); \quad A_e = \frac{b}{a} A_c = \frac{ab}{2}(E - e\sin E); \quad n = \frac{2\pi}{T} = \frac{2\pi}{\left(2\pi\sqrt{a^3}\Big/\sqrt{\mu}\right)};$$

$$A_e = \frac{\pi ab \cdot \Delta t}{\left(2\pi\sqrt{a^3}\Big/\sqrt{\mu}\right)} = \frac{n \cdot \Delta t \cdot ab}{2}; \quad n \cdot \Delta t = (E - e\sin E) = M$$

$$u = E; \quad \cos E = \frac{OR}{OP'} = \frac{OF_1 + FR}{a}; \quad \cos E = \frac{c + r\cos\theta(\nu)}{a}$$

$$\cos E = \frac{ae + r\cos\theta}{a}; \quad r = \frac{p}{1 + e\cos\theta} = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

$$\cos E = \frac{ae(1 + e\cos\theta) + a\left(1 - e^2\right)\cos\theta}{a(1 + e\cos\theta)} = \frac{e + \cos\theta}{1 + e\cos\theta} \rightarrow \tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \cdot \tan\frac{\theta}{2}$$

(b) Next, using the above TOF relations, find time to travel from $\theta_A = 100^o$ to $\theta_B = 140^o$ on a geocentric ellipse which has semi-major axis of 12,000 km and eccentricity of 0.45. (1)

$$e = 0.45; \quad \theta_A = 100^o; \quad \theta_B = 140^o; \quad \tan\frac{E_A}{2} = \sqrt{\frac{0.55}{1.45}} \cdot \tan 50^o = 0.734$$

$$E_A = 72.55^o = 1.266 rad; \quad \tan\frac{E_B}{2} = \sqrt{\frac{0.55}{1.45}} \cdot \tan 70^o = 1.692; \quad E_B = 118.8^o = 2.074 rad$$

$$a = 12\times10^6 m; \quad n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{3.986\times10^{14}}{\left(12\times10^6\right)^3}} = 0.00048 rad/s$$

$$(E_B - e\sin E_B) - (E_A - e\sin E_A) = 1.68 - 0.30 = 1.38 = n \cdot \Delta t \rightarrow \Delta t = 2865.2 s$$

Q.2 (a) Derive the expression for time of flight, for a parabolic trajectory, in terms of parameters ‘μ’, ‘h’ and ‘θ’. (Hint: Use the following relations as applicable). (1)

$$r=\frac{p}{1+\cos\theta};\quad p=a\left(1-e^2\right);\quad h=\sqrt{\mu p}=rv$$

$$TOF=t_B-t_A=\frac{1}{h}\int_{\theta_A}^{\theta_B} r^2 d\theta;\quad r=\frac{p}{1+\cos\theta}=\frac{\left(h^2/2\mu\right)}{\cos^2\left(\theta/2\right)};$$

$$t_B-t_A=\frac{h^3}{4\mu^2}\int_{\theta_A}^{\theta_B}\frac{d\theta}{\cos^4\left(\theta/2\right)}=\frac{h^3}{2\mu^2}\int_{\theta_A/2}^{\theta_B/2}\sec^4\left(\theta/2\right)d\left(\theta/2\right);$$

$$\sec^4\left(\theta/2\right)=\sec^2\left(\theta/2\right)\left[1+\tan^2\left(\theta/2\right)\right]$$

$$t_B-t_A=TOF_{Parabola}=\frac{h^3}{2\mu^2}\left(\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right)_{\theta_A}^{\theta_B}$$

(b) A spacecraft is put on a parabolic trajectory from a circular orbit at 400 km altitude above earth’s surface and travels to a distance of 386,000 km from centre of earth. Determine the time taken to reach this point and the residual velocity at this location. Also, determine the velocity impulse required to start the manoeuvre. (2)

$$r_{circle}=6378+400=6778km;\quad v_c=\sqrt{\frac{3.986\times10^{14}}{6.778\times10^6}}=7668.6m/s$$

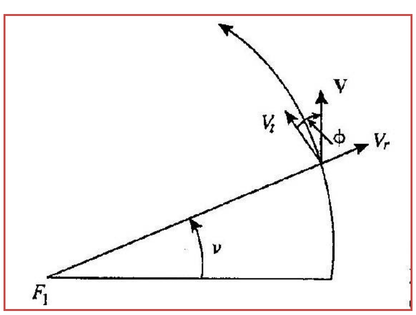
$$v_{parabola}=\sqrt{\frac{2\times3.986\times10^{14}}{6.778\times10^6}}=10845.1m/s;\quad \Delta v=3176.5m/s$$

$$r_B=3.86\times10^8m;\quad h=10845.1\times6.778\times10^6=7.351\times10^{10};\quad v_B=\frac{7.351\times10^{10}}{3.86\times10^8}=190.4m/s$$

$$\theta_B=\cos^{-1}\left(\frac{\left(7.351\times10^{10}\right)^2}{3.986\times10^{14}\times3.86\times10^8}-1\right)=164.75^o=2.876rad$$

$$TOF_{Parabola}=\frac{\left(7.351\times10^{10}\right)^3}{2\times\left(3.986\times10^{14}\right)^2}\left(\tan82.37+\frac{1}{3}\tan^3 82.37\right)=1250.07\times146.39=182997.9s=50.8h$$

Q.3 (a) Derive the relation between elevation angle ‘φ’ and true anomaly ‘θ’ for a space object moving on a conic section trajectory. (Hint: Use the conic relations given in Q.2 above). (1)



$$\tan\phi=\frac{V_r}{V_t}=\frac{\dot r}{r\dot\theta};\quad r=\frac{p}{1+e\cos\theta};\quad \dot r=\frac{pe\dot\theta\sin\theta}{\left(1+e\cos\theta\right)^2};$$

$$\tan\phi=\frac{\dfrac{e\,\cancel{p}\cancel{\dot\theta}\sin\theta}{(1+e\cos\theta)^{\cancel{2}}}}{\dfrac{\cancel{p\dot\theta}}{\cancel{(1+e\cos\theta)}}};\quad \tan\phi=\frac{e\sin\theta}{\left(1+e\cos\theta\right)}$$

(b) Next, an incoming object is sighted at an altitude of 36,000 km above earth, with a velocity of 4500 m/s and -50° elevation angle. What kind of trajectory is described by the object and what is the closest point of approach with respect to earth's surface? (1)

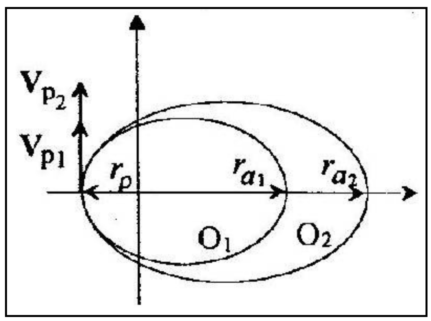
$$r = 42.378\times10^{6}\,m;\quad h = 42.378\times10^{6}\times4500\times\cos\left(-50^{o}\right) = 1.2258\times10^{11}$$

$$\varepsilon = \frac{4500^{2}}{2} - \frac{3.986\times10^{14}}{42.378\times10^{6}} = 719176.2 > 0 \rightarrow \text{ Hyperbolic}$$

$$a = -\frac{3.986\times10^{14}}{2\times719176.2} = -2.771\times10^{8};\quad e = \sqrt{1+\frac{2\times7.192\times10^{5}\times\left(1.2258\times10^{11}\right)^{2}}{\left(3.986\times10^{14}\right)^{2}}} = 1.0658$$

$$r_{p} = 2.771\times10^{8}\times0.0658 = 18.246\times10^{7};\quad \text{Perigee Altitude} = 11868.5 \text{ km}$$

Q.4 (a) Derive the expression for 'ΔV', to be given to a spacecraft in an elliptic orbit O_1 at its perigee, in order to raise the apogee corresponding to the orbit O_2, as per the figure given below. (1)



$$v_{p1} = \sqrt{2\left(\frac{\mu}{r_{p}} - \frac{\mu}{\left(r_{p}+r_{a1}\right)}\right)};\quad v_{p2} = \sqrt{2\left(\frac{\mu}{r_{p}} - \frac{\mu}{\left(r_{p}+r_{a2}\right)}\right)};\quad \Delta v_{raising} = v_{p2} - v_{p1}$$

(b) Determine the velocity impulse needed at perigee of an elliptic orbit with perigee of 100 km altitude and apogee of 150 km altitude above earth, in order to raise the apogee to 250 km altitude. (1)

$$r_{p} = 6.478\times10^{6};\quad r_{a1} = 6.528\times10^{6};\quad r_{a2} = 6.628\times10^{6}$$

$$v_{p1} = \sqrt{2\times3.986\times10^{8}\times\left(0.1544-0.0768\right)} = 7860.8\,m/s$$

$$v_{p2} = \sqrt{2\times3.986\times10^{8}\times\left(0.1544-0.0763\right)} = 7890.5;\quad \Delta v_{raising} = 29.7\,m/s$$

Data: R_E = 6,378 km; $\mu = 3.986\times10^{14}$ m^3/s^2

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