AE240 Lecture-1

Prof. Rohit Gupta

Page No.:
Date:

Office hours - 4PM-5PM Thursday

References → H.D. Curtis - Orbital Mechanics for Engineering Students

Grading Policy
Quizzes - 30% (4 quizzes)
Midterm - 30%
End Term - 40%

Tutorials??

No attendance criteria

Vector :- Vector is an object which has both magnitude and direction

→ Length of Vector proportional to Magnitude

→ Magnitude denoted by $\|\vec{v}\|$

Unit Vector → Vector divided by magnitude & produces vector which is dimensionless and has $\|\vec{v}\| = 1$

→ Sum of vectors done by parallellogram addition

Reference Frame :- Qualitatively, reference frame is perspective from which observations are made regarding the motion of system

Rigourous :- A reference frame is a collection of atleast three non-collinear points in the 3-D Euclidian space s.t. the dist b/w any 2 points in the collection does not change w time

Doubt rotation

Inertial and Non inertial frame

→ An inertial / Newtonian reference frame is one whose points are either absolutely fixed or atmost translate relative to an absolute frame fixed at a point with the same constant velocity.

→ A non-inertial / non Newtonian reference frame is one whose points accelerate with time

Note:- In Newtons second law the acceleration must be calculated wrt. $\boxed{\text{an inertial frame of reference}}$

→ What is a co-ordinate frame.

→ A co-ordinate system provide means of measuring the observation that may be made of the motion by an observer fixed in reference frame

Ex:- Right handed co-ordinate system.

→ Given a vector A, we can write in a co-ordinate system:-

$$A = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

$$\|A\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$\hat{x}_A = \frac{A}{\|A\|} = \cos(\theta_x)\hat{i} + \cos(\theta_y)\hat{j} + \cos(\theta_z)\hat{k}$$

↓

Direction cosine

For 2 vectors A, B, AB and $\frac{A}{B}$ are undefined

→ 2 operations → Scalar Dot Product

DOT PRODUCT → The dot product of 2 vectors is a scalar defined ~~by~~ as follows

$$A \cdot B = \underbrace{\|A\|}_{\text{norm } A} \underbrace{\|B\|}_{\text{norm } B} \cos(\underbrace{\theta}_{\text{Angle}}) \quad \text{(Geometric)}$$

Commutative → $A \cdot B = B \cdot A$

For $\perp$ vectors → $\boxed{A \cdot B = 0}$ $\quad i \cdot i = 1$

$i \cdot j = 0$

$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z}$$

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→ Recap

Projection of B on a

Projection → Scalar = $B_A = B \cdot \hat{u}_A = \frac{(A \cdot B)}{||A||}$

Projection vector $\vec{B}_A = (B \cdot \hat{u}_A)\, \hat{u}_A$

$$\vec{B}_A = \frac{(A \cdot B)}{||A||^2} \vec{A}$$

If $||A|| = ||B||$ $||\vec{A}_B|| = ||B_A||$

→ Example (in slides)

CROSS PRODUCT → Cross Product of two vectors is a vector defined as follows.

$$\vec{A} \times \vec{B} = ||A||\, ||B|| \sin(\theta)\, \hat{n}_{AB}$$

Properties → $A \times B = -B \times A$ (Anti commutative)

we have $\hat{i}\, \hat{j}\, \hat{k}$ ⇒ $\hat{i} \times \hat{i} = 0$ $\hat{k} \times \hat{k} = 0$

$\hat{j} \times \hat{j} = 0$

$$\boxed{\hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}}$$

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \hat{n}_{AB} = \frac{A \times B}{||A \times B||}$$

Area of Parallelogram = $||A \times B||$ Example in slides

Page No.:
Date:

bac- cab rule →

$$\boxed{A \times (B \times C) = B(A \cdot C) - C(A \cdot B)}$$

Vector Triple Product. Prove component wise

Easy to prove simple.

Scalar Triple Product → $\vec{A} \cdot (\vec{B} \times \vec{C})$

→ Exercise in slides

To track a motion of particle need → Clock

→ Non rotating cartesian

Position

$$r(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$v(t) = \dot{r}(t)$$

Velocity $= v_x(t)\,\hat{i} + v_y(t)\,\hat{j} + v_z(t)\,\hat{k}$

Doubt is $\|\dot{s}\| = \|\dot{r}\|$

$$\cancel{\vec{v}} = \dot{\vec{r}} \qquad \boxed{\dot{S} = \|v\| = \|\dot{r}\|}$$

Note carefully $\|\dot{\vec{r}}\| \neq \dot{\|r\|}$ → Rate of change of norm

norm of rate

Both different

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→ Recap

$$\hat{u}_t = \frac{v}{\|v\|}$$

Tangential

Velocity vector

tangent

$$a = a_t \hat{u}_t + a_n \hat{u}_n$$

Tangential

↳ Normal

$$a_t = \ddot{s} = \dot{\|v\|}$$

$$a_n = \frac{\|v\|^2}{\rho}$$ ρ – Radius of Curvature

Binormal $\hat{u}_b = \hat{u}_t \times \hat{u}_n$

$$v \times a = \|v\| \hat{u}_t \times \left(\dot{\|v\|} \hat{u}_t + \frac{\|v\|^2}{\rho} \hat{u}_n \right)$$

$$\Rightarrow \frac{\|v\|^3}{\rho} \hat{u}_t \times \hat{u}_n$$

$$\Rightarrow \|v \times a\| \hat{u}_b$$

find Binormal

$$\Rightarrow \boxed{\hat{u}_b = \frac{v \times a}{\|v \times a\|}}$$

simply find Binormal

t

Right Handed

b n

→ Right handed system

$ds = \rho \, d\phi$

$\dot{s} = \rho \dot{\phi}$

~~$v = \omega x$~~

$\|v\| = \rho \dot{\phi}$

$\boxed{\vec{r} = -\hat{U}_n}$ If center = Origin

Finding center of curvature

Basically to find r_c

$$r_c = \vec{r} + \rho \hat{U}_n$$

$$\hat{U}_t = \frac{V}{\|V\|}$$

$$U_b = \frac{\vec{V} \times \vec{a}}{\|V \times \vec{a}\|}$$

→ You have both find

$\hat{U}_n = \vec{U}_b \times \vec{U}_t$

You have this

$$\rho = \frac{\|V\|^2}{a_n}$$

find $a_n = \vec{a} \cdot \hat{U}_n$

You have $\vec{r}$, ρ, $\hat{U}_n$

find co-ordinates of center of curvature

Mass → Measure of inertia of Body

force → Action of one physical body on another

Weight → force of a large mass exerted on a smaller body ⇒ weight

$$W = mg \quad \text{where} \quad \boxed{g = \frac{GM}{R^2}}$$

At diff heights =

$$g = g_0 \frac{R_E^2}{(R_e + h)^2} = \frac{g_0}{\left(1 + \frac{2h}{R_e}\right)}$$

$$= g_0 \left(1 - \frac{2h}{R_e}\right)$$

force is not a primitive concept.

Impulse of Force = $\Delta p = \int_{t_1}^{t_2} F \cdot dt$

$$\boxed{\Delta v = \frac{I_{net}}{m}}$$

Moment of force = $r \times F_{net}$

→ Product rule

$$= r \times m \frac{dv}{dt}$$

$$= \frac{d}{dt}(r \times mv) - \frac{dr}{dt} \times mv$$

=

$$= \frac{d}{dt}(r \times mv)$$

→ Angular Momentum

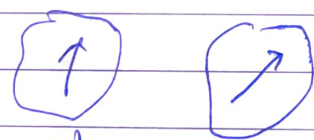
$$M) = \frac{d}{dt} H_o$$

Net angular impulse = $\Delta H = \int_{t_1}^{t_2} M_o)_{net} \, dt$

Note that both should be about O

Consider a vector fixed on stone (Rigid Body)

Eulers theorem →



A(t) A(t+dt)

★ Doubt
can IAOR be changed
NO, change vector ~~dir~~

→ Instantaneous axis of rotation.

(Check Slides)

$$dA = [\|A\| \sin\phi \, d\theta] \, \hat{n}$$ → vector

magnitude of dA

$$d\theta = \|\omega\| \, dt$$

$$dA = \|A\| \, \|\omega\| \sin\phi \, dt \, \hat{n}$$

$$\frac{dA}{dt} = (\cancel{\|A\|} \; \vec{A} \times \vec{\omega})$$

$$\boxed{\frac{dA}{dt} = \vec{\omega} \times \vec{A}}$$ where obviously $\vec{\omega}$ = IAOR (passing through ICOR)

ICOR where it ~~always~~ always rotates across ICOR

$\boxed{v=0}$ at ICOR

Q → find $\frac{d^2A}{dt^2} = \frac{d}{dt}\left(\frac{dA}{dt}\right)$

$$\Rightarrow \frac{d}{dt}(\vec{\omega} \times \vec{A})$$

↓ Apply ~~to~~ product rule

$$\Rightarrow \cancel{\frac{dw}{dt}}\ \frac{d^2A}{dt^2} = \frac{dw}{dt} \times A + \omega \times (\omega \times A)$$

→ Inertial Frame v/s Moving frame

Absolute v/s Relative measurements

let $\vec{v}$ be a time dependent vector and angular velocity = ~~w~~ Ω

→ ~~Inertial~~ Moving

$$B = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

Inertial

→ ~~Moving~~

$$B = B_x \hat{I} + B_y \hat{J} + B_z \hat{K}$$

Capital - Inertial (Fixed)

Moving → small case

$$\frac{dB}{dt} = \frac{dB_x}{dt}\hat{i} + \frac{dB_y}{dt}\hat{j} + \frac{dB_z}{dt}\hat{k} + B_x\frac{d\hat{i}}{dt} + B_y\frac{d\hat{j}}{dt} + B_z\frac{d\hat{k}}{dt}$$

Now $\hat{i}$ is a vector rotating with Ω and has constant magnitude.

$$\frac{di}{dt} = \Omega \times \hat{i}$$

$$\frac{dj}{dt} = \Omega \times \hat{J} \quad \text{and so on}$$

Substitute before and get

$$\text{Actual value} \left(\frac{dB}{dt}\right)_{\cancel{\text{relative}}\ \text{actual}} = \overbrace{\left.\frac{dB}{dt}\right)_{\text{relative}}}^{\frac{dB_x}{dt}\hat{i} + \frac{dB_y}{dt}\hat{j}} + \Omega \times B$$

Note - Here B is just a vector.

Translation does not change

only rotation changes vector

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→ Recap

$$\frac{dB}{dt} = \left.\frac{dB}{dt}\right|_{rel} + \Omega \times B$$

Doubt Pure translational??

→ Now we calculate →

$$\frac{d^2B}{dt^2} = \frac{d}{dt}\left(\left.\frac{dB}{dt}\right|_{rel}\right) + \frac{d\Omega}{dt} \times B + \Omega \times \frac{dB}{dt}$$

Now

write $\left.\frac{dB}{dt}\right|_{rel}$

let

$$C = \left.\frac{dB}{dt}\right|_{rel}$$

Keep interms of $\left.\frac{dB}{dt}\right|_{rel}$.

$$\frac{dC}{dt} = \left.\frac{dC}{dt}\right|_{rel} + \Omega \times C$$

⇓

(This term)

$$\frac{d}{dt}\left(\left.\frac{dB}{dt}\right|_{rel}\right) = \left.\frac{d^2B}{dt^2}\right|_{rel} + \Omega \times \left.\frac{dB}{dt}\right|_{rel}$$

$$+ \Omega \times \left(\cancel{\frac{dB}{dt} \ldots}\right)$$

Final result

$$\boxed{\frac{d^2B}{dt^2} = \left.\frac{d^2B}{dt^2}\right|_{rel} + 2\Omega \times \left.\frac{dB}{dt}\right|_{rel} + \frac{d\Omega}{dt} \times B + \Omega \times (\Omega \times B)}$$

→ See slides for figure

$$\vec{r} = \vec{r_0} + \vec{r}_{rel}$$

→ Actual Origin to moving frame

$\vec{r}_{rel}$ a vector

$$\boxed{\vec{V} = V_0 + \frac{d\vec{r}_{rel}}{dt}} \Rightarrow \vec{V} = V_0 + V_{rel} + \Omega \times r_{rel}$$

(Note → B was just a vector)

Similarly acceleration → $a = \frac{dv}{dt} = a_0 + \frac{d^2 r_{rel}}{dt^2}$

Derivation

$$a = \frac{dv}{dt} = \frac{d^2 r}{dt^2} = a_0 + \frac{d^2 r_{rel}}{dt^2}$$

vector relative

↓

substitute

Final result

$$a = a_0 + \underbrace{a_{rel}}_{\left.\frac{d^2 r_{rel}}{dt^2}\right|_{rel}} + 2\Omega \times \left.\frac{dr_{rel}}{dt}\right|_{rel} + \frac{d\Omega}{dt} \times r_{rel} + \Omega \times (\Omega \times r_{rel})$$

Question → Given r_0, v_0, a_0
Given Ω $\dot{\Omega}$ also

→ Anything with subscript measured in relative

→ Do everything in terms of one consistent notation

Note → All the above are vectorial equations holding

⇓

Be consistent, use only one frame (on both sides).

⇓

If measurements asked in relative frame, use

Curvilinear → Polar co-ordinates are ~~fixed~~

We have origin → fixed

$V = V_0 + V_{rel} + \Omega \times r_{rel}$

$\boxed{V_0 = 0}$ for this case...

$\cancel{\Omega = \Omega \hat{k}}$

$\Omega = \dot{\theta}\hat{k}$

In terms of gen $V = V_{rel} + \dot{\theta}\hat{k} \times (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$\boxed{V = V_{rel} + -\dot{\theta}\sin\theta\,\hat{i} + \dot{\theta}\cos\theta\,\hat{j}}$$

$V_{rel} = \dot{r}\,\hat{u}_r$

In terms of $\hat{u}_r$

$V = V_{rel} + \Omega \times r_{rel}$

$V = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{k} \times \hat{u}_r$

$$\boxed{V = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta}$$

For acceleration $a = \cancel{a_0}^{0} + a_{rel} + 2\Omega \times v_{rel}$
$+ \dot{\Omega} \times r_{rel}$
$+ \Omega \times (\Omega \times r_{rel})$

$\boxed{\dot{\Omega} = \ddot{\theta}\,\hat{u}_k}$

$a_{rel} = \ddot{r}\,\hat{u}_r \;\bar{-}\; 2\Omega\dot{r}\,\hat{u}_r$

Derivation →

→ Use
bit of cross product
and stuff.

$$\boxed{\text{Final Result} \rightarrow (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{u}_\theta}$$

Brute Force (can check)

$v = \dot{x}\hat{i} + \dot{y}\hat{j} \cancel{+ \dot{z}\hat{k}}$ Substitute $x = r\cos\theta$

$y = r\sin\theta$

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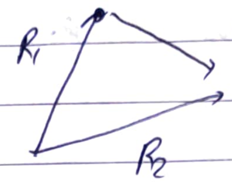
$\to$ Attraction b/w two masses

1 - Dim of $R_2 - R_1$



R_1 - $x_1\, y_1\, z_1$

R_2 - $x_2\, y_2\, z_2$

COM $\to$ $R_G = \dfrac{m_1 R_1 + m_2 R_2}{m_1 + m_2}$ $\Rightarrow$ $v_G = \dfrac{m_1 \dot{R_1} + m_2 \dot{R_2}}{m_1 + m_2}$

$\Rightarrow$ $a_G = \dfrac{m_1 \ddot{R_1} + m_2 \ddot{R_2}}{m_1 + m_2}$

$r = R_2 - R_1 = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

Newtons Law

$F_{12} = \dfrac{G m_1 m_2}{\|r\|^2}\, \hat{u}_r$

Force on 1

$F_{12} = m_1 \ddot{R_1}$ $\qquad$ $F_{21} = m_2 \ddot{R_2}$

$\boxed{a_G = 0}$ $\to$ COM does not accelerate.

$\downarrow$

$R_G \Rightarrow$ inertial frame

$\ddot{R_1} = \dfrac{G m_2\, r}{\|r\|^3}$ $\qquad$ $\ddot{R_2} = \dfrac{G m_1\, r}{\|r^3\|}$

this $r \Rightarrow$ vector $R_2 - R_1$

Now $\ddot{R}_1 = \ddot{x}_1 \hat{I} + \ddot{y}_1 \hat{J} + \ddot{z}_1 \hat{k}$

$\Downarrow$ compare both

$$\ddot{x}_1 = \frac{Gm_2}{||r||^3}(x_2 - x_1) \quad \text{same for } y, z \text{ etc}$$

$\downarrow$ obtain solution to DE's

Conservative Force -

$\downarrow$

Force can be obtained as grad of some potential fn

$\nearrow$ V=scalar

$$F = -\nabla V$$

$\Downarrow$

Where $\boxed{V = -\frac{Gm_1m_2}{||r||}}$

Doubt

Potential Fn

V ⇒ function of $x_1, x_2, y_1, y_2, z_1, z_2$

For force on 1 → Changeable variables $x_2\ y_2\ z_2$

$\downarrow$

$$F_{12} = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial z_2}$$

→ In an inertial frame → COM always in straight line.

→ General n-body problem → No closed form solution, chaotic in nature.

→ change of vector r

$$\ddot{r} = \ddot{R}_2 - \ddot{R}_1$$

$$\ddot{r} = \frac{-G(m_1+m_2)\,\hat{u}_r}{\|r\|^2}$$

Gravitational Parameter

$$= \frac{-\mu\, r}{\|r\|^3} \rightarrow \text{vector}$$

$$\boxed{\ddot{r} = \frac{-\mu r}{\|r\|^3}}$$

→ Moving frame on mass m_1

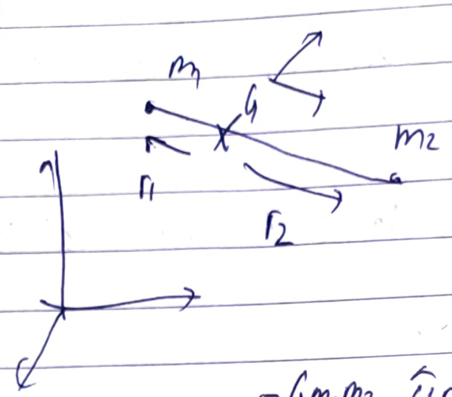
(How does motion of m_2 evolve wrt m_1)

As seen by m_1

Doubt by $\Omega = \dot{\Omega} = 0$
What do we mean

So acc. to m_1 $\ddot{r} = \frac{-\mu r}{\|r\|^3}$

In frame fixed to m_1



$$\frac{-Gm_1m_2}{||r||^2}\hat{U}_r = m_2\ddot{r}_2$$

com

$$\boxed{m_1r_1 + m_2r_2 = 0}$$

* Derivation Imp

$$r_1 = \frac{-m_2}{m_1} r_2$$

$r_1 r_2$ ↓ vectors

Also $\hat{U}_r = \frac{r_2}{||r_2||}$

$$\boxed{r = \left(\frac{m_1+m_2}{m_1}\right) r_2}$$

↓ substitute Back

If G → origin then from G ↓ → vectors r_2

$$\Rightarrow \boxed{\frac{-Gm_1^3 m_2}{(m_1+m_2)^2||r_2||^3} r_2 = m_2\ddot{r}_2}$$

In com frame

similarly $\boxed{\ddot{r}_1 = \frac{-\mu'' r_1}{||r_1||^3}}$ ⇐ $\boxed{\ddot{r}_2 = \frac{-\mu'}{||r_2||^3} r_2}$ where $\mu' = \left(\frac{m_1}{m_1+m_2}\right)^3 \mu$

→ ~~doubt~~ to do f.p. tutorial

→ Some part ~~good~~ is not in slides

* In G frame what do we define axes as

* Office Hours → ICOR → How does axes rotate ??

[Lec-5] Do we assume No initial Acc ??

[Lec-3] Does absolute angular vel ~~per~~ ⇒ Translate + Rotate

→ For a vector, does, ~~absolute~~ pure translational not change it ??

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AE240 Lecture-6

+ Angular momentum of Body m_2 relative to m_1

$$\cancel{H_2} \; H_{2/1} = \vec{r} \times m_2 \dot{\vec{r}}$$

$$h = \frac{H_{2/1}}{m_2} = r \times \dot{r}$$

So basically we have

$$\{ h = r \times \dot{r}$$

specific relative angular momentum.

$$\frac{dh}{dt} = \cancel{\dot{r} \times \dot{r}}^{\,0} + r \times \ddot{r}^{\,0}$$

$$= 0$$

conservation of angular mom.

$\frac{dh}{dt}$ = does not change

Doubt
m_1 frame is non-inertial frame, is it fine??

→ If $r \times \dot{r}$ = constant → remains zero at all points

→ Given $h = r \times \dot{r}$ and $\hat{u}_h = \frac{h}{\|h\|}$

$\hat{u}_h$ → remains in the same direction

→ Path of m_2 around m_1 ⇒ Lies in a plane

→ Resolve velocity into 2-components

↓

$$v_r = \|v_r\| \hat{u}_r$$
$$v_\theta = \|v_{\theta\perp}\| \hat{u}_\perp$$

Doubt → Non inertial Frame

Substiture $v = v_r \hat{u}_r + v_\perp \hat{u}_\perp$

in ~~$\dot{r} \times r$~~ $r \times \dot{r}$

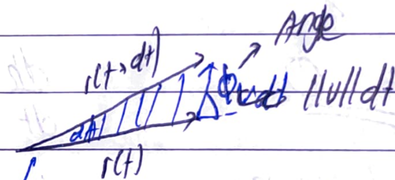
$r \hat{u}_r$

$$\Rightarrow \boxed{h = \|r\| \, \|v_\perp\| \, \hat{h}}$$

and

$$\|h\| = \|r\| \, \|v_\perp\|$$

In differential time dt



Area of Δ

$$\Rightarrow \frac{1}{2} AB \sin(c)$$

$$dA = \frac{1}{2} \|v\| \, dt \, \|r\| \sin\phi$$

$$= \frac{1}{2} \|r\| \underbrace{\|v\| \sin\phi}_{\|v_\perp\|} \, dt$$

$$dA = \frac{1}{2} \|r\| \, \|v_\perp\| \, dt$$

$$\boxed{\frac{dA}{dt} = \frac{1}{2}\|h\| = \text{constant}}$$

→ Keplers second law. (Law of equal areas)

→ Recall $\quad r \cdot r = \|r\|^2$

$$\dot{\overline{r \cdot r}} = 2\|r\| \, \dot{\overline{\|r\|}}$$

$$\cancel{2} r \cdot \dot{r} = \cancel{2} \|r\| \, \dot{\overline{\|r\|}}$$

$$\boxed{r \cdot v = \|r\| \, \|\dot{r}\|} \rightarrow \text{Imp result}$$

$$\cancel{\|r\|} \, \|v\| \sin\phi = \cancel{\|r\|} \, \|\dot{r}\|$$

⇒ We have $\quad \ddot{r} = -\frac{\mu}{\|r\|^3} r$

$$\boxed{\ddot{r} \times h = -\frac{\mu}{\|r\|^3} r \times h}$$

⇓

$$\cancel{\frac{d}{dt}(\dot{r} \times h)} = \frac{d}{dt}(\dot{r} \times h) = \ddot{r} \times h + \cancel{\dot{r} \times \dot{h}}$$

$$\dot{h} = 0$$

⇓

$$\frac{d}{dt}(\dot{r} \times h) = -\frac{\mu}{\|r\|^3} r \times h \qquad h = (r \times \dot{r})$$

To simplify $\quad \frac{1}{\|r\|^3} r \times (r \times \dot{r}) = \frac{1}{\cancel{\|r\|^3}} \cancel{\|r\|}$

→ Derivation

$$\frac{d}{dt}(\dot r \times h) = \frac{d}{dt}\left(\frac{\mu r}{\|r\|}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\dot r \times h - \frac{\mu r}{\|r\|}\right) = 0$$

$$\boxed{\dot r \times h - \frac{\mu r}{\|r\|} = C}$$ → Laplace's vector

⇓

Take dot product with h

$$\cancelto{0}{(\dot r \times h)\cdot h} - \cancelto{0}{\frac{\mu (r\cdot h)}{\|r\|}} = C h$$

⇓

$$C \cdot h = 0$$

C = ~~in the~~ C is in the plane ⊥ to h

Page No.:
Date:

$$\Rightarrow \frac{r}{\|r\|} + \underbrace{\frac{c}{u}}_{\text{eccentricity vector}} = \frac{\dot{r} \times h}{u}$$

eccentricity vector. ⇒ Denote by e

↓

Dot with r

$$\frac{r \cdot r}{\|r\|} + r \cdot e = \frac{r \cdot (\dot{r} \times h)}{u}$$

$A \cdot (B \times C)$
⇓
$(A \times B) \cdot C$

$$\Rightarrow \|r\| + r \cdot e = \frac{\|h\|^2}{u}$$

↓ Rearrang

Doubt
what is true eccentricity and how do we do??

$$\boxed{\|r\| = \frac{\|h\|^2}{u} \frac{1}{1 + \|e\| \cos(\theta)}}$$

– u, h = constants

Keplers first law → Planets follow elliptical orbit

Note $\|e\|$ = constant
e → vector = constants vector

Changes = θ, and $\|r\|$

✓ Done

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→ Orbital Equation describes conic sections

→ Circular ($\|e\| = 0$)
→ Elliptical ($\|e\| < 1$)
→ Parabolic ($\|e\| = 1$)
→ Hyperbolic ($\|e\| > 1$)

Velocity normal component = $\boxed{v_\perp = \|r\|\dot{\theta}}$

$$\|h\| = \|r\|\, v_\perp = \|r\|^2 \dot{\theta}$$

$$\Downarrow$$

Substitute $\|r\|$

$$\boxed{v_\perp = \frac{\|h\|}{\|r\|} = \frac{\mu}{\|h\|}(1 + \|e\| \cos\theta)}$$

$$\dot{\theta} = \frac{\|h\|}{\|r\|^2}$$

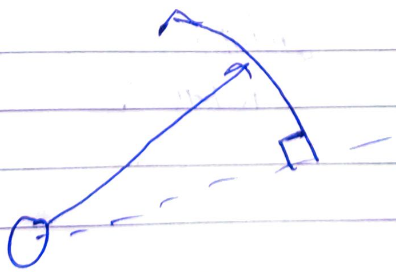
$$v_r = \dot{\|r\|} = \frac{d}{dt}\left(\frac{\|h\|^2}{\mu} \frac{1}{1 + \|e\| \cos\theta}\right)$$

(θ → only)

Simplify

$$\boxed{v_r = \frac{\mu}{\|h\|} \|e\| \sin\theta}$$

→ Figure



$$\|r\| = \frac{\|h\|}{\mu} \frac{1}{(1 + \|e\|\cos\theta))}$$

~~$\|r\|$ max when $\cos\theta = -1$~~
$\theta = 180°$

At $0° = \cos\theta = 1$
↓ Denom → min

$\|r\|$ = min

At $\theta = 0°$ $\Rightarrow$ Perihelion

$$r_p = \frac{\|h\|}{\mu} \frac{1}{1+\|e\|}$$

$$\tan(\gamma) = \frac{V_r}{V_\perp}$$ ($\gamma \rightarrow$ Flight path angle)

~~$\tan(\gamma) =$~~

$$\boxed{\tan(\gamma) = \frac{\|e\| \sin(\theta)}{1+\|e\| \cos(\theta)}}$$

At periapses $\rightarrow$ $\theta = 0°$ $V_r = 0$

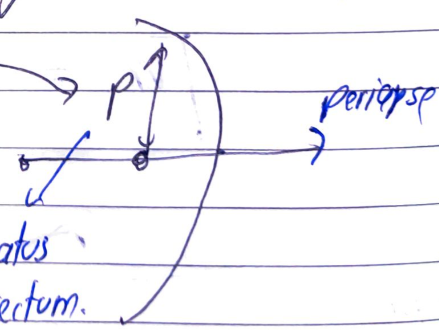
$$\boxed{\gamma = 0}$$

$V_r > 0$ $(0 < \theta < 180°)$
$V_r < 0$ $(180 < \theta < 360°)$

$\rightarrow$ Periapse is the symmetric line (Apse line)

$\Downarrow$

$$\boxed{p = \frac{\|h\|^2}{\mu}} \rightarrow p$$



We had $\ddot{r} = \dfrac{-\mu}{\|r\|^3}\, r$

Finding $\ddot{r}\cdot\dot{r} = \dfrac{-\mu}{\|r\|^3}\, r\cdot\dot{r}$

$$\Downarrow$$

$$\ddot{r}\cdot\dot{r} = \frac{1}{2}\,\overset{\cdot}{\widehat{\dot{r}\cdot\dot{r}}} = \frac{1}{2}\,\overset{\cdot}{\widehat{\|v\|^2}} =$$

Also ⇒ $\boxed{r\cdot\dot{r} = \|r\|\,\overset{\cdot}{\widehat{\|r\|}}}$ $\boxed{\overset{\cdot}{\widehat{\|r\|^{-1}}} = \dfrac{1}{\|r\|^2}\,\overset{\cdot}{\widehat{\|r\|}}}$

Now $\dfrac{1}{2}\,\overset{\cdot}{\widehat{\|v\|^2}} = \ddot{r}\cdot\dot{r} = \dfrac{-\mu}{\|r\|^3}\,\|r\|\,\overset{\cdot}{\widehat{\|r\|}}$

$$= \frac{-\mu}{\|r\|^2}\,\overset{\cdot}{\widehat{\|r\|}}$$

$$= \overset{\cdot}{\widehat{-\mu\,\|r\|^{-1}}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\|v\|^2 + \frac{\mu}{\|r\|}\right) = 0$$

✩ Doubt
↓
But this is non inertial frame.

$$\Downarrow$$

$$\boxed{\underbrace{\frac{1}{2}\|v\|^2}_{\text{Relative K.E per unit mass}} + \underbrace{\frac{\mu}{\|r\|}}_{\text{Relative PE per unit mass}} = \underbrace{\mathcal{E}}_{\text{Total ME per unit mass.}}}$$

Kinda like conservation of energy.

At periapses. $\theta = 0°$

$$\mathcal{E} = \mathcal{E}_p = \frac{\|v_p\|^2}{2} - \frac{\mu}{r_p}$$

$$= \frac{\|v_\perp\|^2}{2}$$

$$= \frac{\|h\|^2}{2 r_p^2} - \frac{\mu}{r_p}$$

$$\mathcal{E} = \frac{-\mu^2}{2\|h\|^2}\left(1 - \|e\|^2\right)$$

Energy of space craft of mass m

$$\boxed{E = m\mathcal{E}}$$

Circular Orbits ($\|e\| = 0$)

$$\boxed{\|r\| = \frac{\|h\|^2}{\mu}}$$

$$\|h\| = \|r\|\|v_\perp\| = \|r\|\,\|v\| \qquad \boxed{v_r = 0}$$

Substitute

$$V_{circular} = \sqrt{\frac{\mu}{\|r\|}}$$

Time period $= \frac{2\pi r}{v} = \frac{2\pi \|r\|^{3/2}}{\sqrt{\mu}}$

$\mathcal{E} = -\frac{\mu^2}{2\|h\|^2}$

Norm.

$= -\frac{\mu}{2}\frac{\mu}{\|h\|^2} = -\frac{\mu}{2r}$

→ As radius decreas → Larger orbit, more energy. (Less negative)

$\mu = G(M_{earth} + M_{sat})$

$\boxed{\mu \approx G\, M_{earth}}$

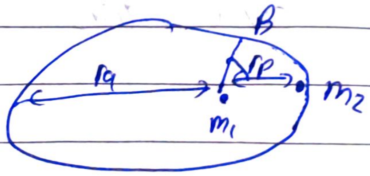
→ Geostationary orbit → Same ω as earth.

$r_{GEO} = 42164\ km \quad \omega_e = 72.9218 \times 10^{-6}$

Elliptical Orbits → $0 < \|e\| < 1$

$\boxed{r_a = \frac{\|h\|^2}{\mu}\frac{1}{1-\|e\|}}$ → Ap   One mass at Focus.

$2a = r_p + r_a = \frac{\|h\|^2}{\mu}\frac{1}{1-\|e\|^2}$



$\Rightarrow \|r\| = \frac{a(1-\|e\|^2)}{1+\|e\|\cos\beta}$

$CF = a - FP$

Focal distance $= ae$ / $a\|e\|$

At $\theta = 0^\circ$ $\quad r_p = a(1-\|e\|)$

~~at~~ $\boxed{a\|e\| = r_B \cos(\cancel{180}\, 180^\circ - B)}$ —— 5

~~Hell~~ Substitute here

$$\boxed{r_B = \frac{a(1-\|e\|^2)}{1+\|e\|\cos(B)}}$$

We get $\boxed{\|e\| = -\cos(B)}$ → for that pt. find B in terms of $\|e\|$

also you finally ~~get~~ resubstitute this in 5

You get $\boxed{r_B = a}$

$$\boxed{b = a\sqrt{1-\|e\|^2}}$$

Congratulations, you proved conic sections