SC 618: Flows, derivatives and brackets

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Outline

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Linear systems - preliminaries

A linear system with an input

$$\dot{x} = Ax + Bu \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in $\mathbb{R}^n$, a vector space.

A linear autonomous system

$$\dot{x} = Ax, \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in $\mathbb{R}^n$, a vector space.

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$x(t)$ lives in $\mathbb{R}^n$, a vector space.
The right-hand side of the differential equation is termed a vector field. For the linear system, it is a linear vector field.
Linearity

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$$

Solution and flow

Solution to the set of differential equations

$$x(t) = e^{At}x_0, \quad x(0) = x_0, \quad e^{At} \triangleq I + A + A^2/2! + \ldots$$

The term $e^{At}x_0$ is termed the flow associated with the linear vector field Ax.

Nonlinear systems - preliminaries

A nonlinear system with an input

$$\dot{x} = f(x) + g(x)u \quad x(t) \in M$$

$f(\cdot), g(\cdot)$ are smooth functions, $x(t)$ lives in M, a smooth manifold.

A nonlinear autonomous system

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n$$

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The right-hand side of the differential equation is a nonlinear vector field.

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A nonlinear autonomous system

$$\dot{x} = f(x), \quad x(t) \in \mathbb{R}^n$$

$x(t)$ lives in M, a smooth manifold.
The right-hand side of the differential equation is a nonlinear vector field.
Linearity does not hold.

$$f(\alpha_1 x_1 + \alpha_2 x_2) \neq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

Solution and flow

Solution to the set of differential equations

$$x(t) = \Phi(t, x_0) \quad x(0) = x_0$$

The term $\Phi(t, x_0)$ is termed the flow associated with the nonlinear vector field $f(x)$.

Outline

Flow of a vector field

Flow of $X(x)$

The flow of the vector field $X(x)$, denoted by $\Phi(t, x_0)$, is a mapping from $(-a, a) \times U \to \mathbb{R}^n$ (where $a(> 0) \in \mathbb{R}$ and U is an open region in the state-space) and satisfies the differential equation

$$\frac{d\Phi(t, x_0)}{dt} = X(\Phi(t, x_0)) \quad \forall t \in (-a, a), x(0) = x_0 \in U.$$

over the interval $(-a, a)$ and with initial conditions starting in the region U.

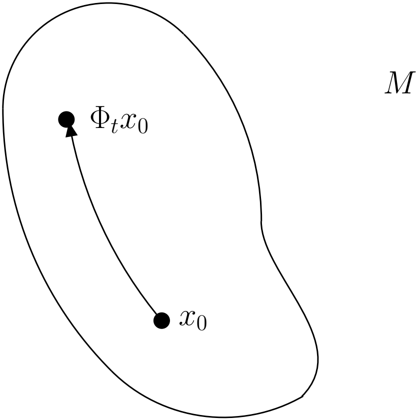

Figure: Flow of a vector field

Properties of flows

The group structure

Denote

$$\Phi_t(x_0) \stackrel{\triangle}{=} \Phi(t, x_0)$$

The set of transformations $\{\Phi_t\} : U \to \mathbb{R}^n$ satisfies the following properties.

- $\Phi_{t+s}x_0 = \Phi_t \circ \Phi_s x_0 \quad \forall t, s, t+s \in (-a, a)$ (the group binary operation.)
- $\Phi_0 x_0 = x_0$ (the group identity.)
- For a fixed $t \in (-a, a)$ we have $\Phi_t \Phi_{-t} x_0 = x_0 \Rightarrow [\Phi_t]^{-1} = \Phi_{-t}$ (existence of an inverse.)

The flow of a linear system

The group property

Remark

The three properties mentioned above impart a group structure to the set $\{\Phi_t\}$. This set is called a one-parameter (time) group of diffeomorphisms (Φ_t and its inverse are smooth mappings).

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Linear flow

Remark

For a linear system described by

$$\dot{x} = Ax \qquad A \in \mathbb{R}^{n\times n}$$

the flow $\Phi_t x_0 = e^{At}x_0$ where $\{e^{At} : t \in (-\infty, \infty)\}$ constitutes the one-parameter group of diffeomorphisms.

Outline

A group

Definition

A group is a set $\mathcal{G}$ with a binary operation $+$ such that

- For any $x, y \in \mathcal{G}$, $x + y \in \mathcal{G}$ (Closure) and $(x + y) + z = x + (y + z)$ (Associativity)
- There exists a unique $i \in \mathcal{G}$ such that $x + i = i + x = x$ for every $x \in \mathcal{G}$ (Existence of the identity element)
- For every $x \in \mathcal{G}$ there exists a unique $y \in \mathcal{G}$ such that $x + y = i$. (Existence of inverse)

A Lie group

Definition

A smooth manifold M together with a group structure is called a Lie group G if the group operation $+$ is smooth.

$$(g,h) \to g+h \quad (\forall g,h \in G) \qquad \text{is smooth}$$

- The identity element of the Lie group is usually denoted by e.
- *Left translation* of a group

$$L_g : G \to G \qquad h \to g+h$$

- *Right translation* of a group

$$R_g : G \to G \qquad h \to h+g$$

Examples of Lie groups

- $\mathbb{R}$ or multiple copies of $\mathbb{R}$ (as $\mathbb{R}^n$) with the bianry operation being the usual component-wise addition $+$.
- The unit circle S^1 with elements denoted as $\theta(\in [0, 2\pi)$ and the binary operation being the usual addition. Similarly, multiple copies of S^1 (as $S^1 \times \ldots \times S^1$).
- The set of $n \times n$ invertible matrices with real entries with the binary operation being matrix multiplication. This group is called $GL(n, \mathbb{R})$.
- The set of $n \times n$ real-orthogonal matrices $O(n), asubsetof$GL(n,R)$.Thesetof$n $\times n$ rotation matrices $SO(n)$, a subset of $O(n, \mathbb{R})$.

Rigid body motion

Definition

Rigid body motion is characterized by two properties

- The distance between any two points remains invariant
- The orientation of the body is preserved. (A right-handed coordinate system remains right-handed)

SO(3) *and* SE(3)

- Two groups which are of particular interest to us in mechanics and control are $SO(3)$ - the special orthogonal group that represents rotations - and $SE(3)$ - the special Euclidean group that represents rigid body motions. These are Lie groups.
- Elements of $SO(3)$ are represented as 3×3 real matrices and satisfy

$$R^T R = I$$

  with $det(R) = 1$.
- An element of $SE(3)$ is of the form (p, R) where $p \in \mathbb{R}^3$ and $R \in SO(3)$.

Frames of reference or coordinate frames

- In describing rigid body motions we always fix two frames of reference. One is called the *body frame* that remains fixed to the body and the other is the *inertial frame* that remains fixed in inertial space.

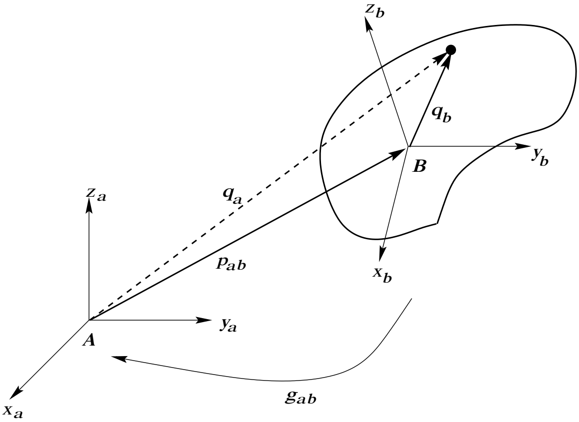


Figure: Rigid body motion

Rigid body motions and groups

- Suppose q_a and q_b are coordinates of a point q relative to frames A and B, respectively.

$$q_a = p_{ab} + R_{ab} q_b$$

  Here p_{ab} represents the position of the origin of the frame B with respect to frame A in frame A coordinates and R_{ab} is the orientation of frame B with respect to frame A .
- Appending a "1" to the coordinates of a point (to render the group operation as the usual matrix multiplication)

$$\bar{q}_a = \begin{pmatrix} q_a \\ 1 \end{pmatrix} = \begin{pmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_b \\ 1 \end{pmatrix} = \bar{g}_{ab} \bar{q}_b$$

Two results on rotations

Rotation in a plane

Claim

The rotation group $SO(2)$ can be identified with S^1 (the unit circle).

Proof:

$$S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$$

Parametrize the elements of S^1 in terms of $\theta \in [0, 2\pi]$. For each $\theta \in [0, 2\pi]$, the counter-clockwise rotation of the vectors $\{(1,0),(0,1)\}$ in $\mathbb{R}^2$ (these form a basis) by the angle θ

$$(1,0) \rightarrow (\cos\theta \quad \sin\theta) \qquad (0,1) \rightarrow (-\sin\theta \quad \cos\theta)$$

is given by the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

which is an element of $SO(2)$.

Proof (contd.)

Conversely, take an element of $SO(2)$ of the form

$$R = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

Then from the properties of an element of $SO(2)$, we have

$$a_1a_4 - a_2a_3 = 1; a_1^2 + a_3^2 = 1; a_2^2 + a_4^2 = 1; a_1a_2 + a_3a_4 = 0$$

It is possible to find a $\theta \in [0, 2\pi]$ such that that R can be represented in the form R_θ. ■

Euler's theorem

Theorem

(Euler's theorem)
Every $A \in SO(3)$ is a rotation through an angle $\theta \in S^1$ about an axis $\omega \in \mathbb{R}^3$.

Proof: Since 1 is an eigen value of A, we have $Aw = w$ where $w \in \mathbb{R}^3$ is an eigen vector. Choose two vectors e_1 and e_2 that are orthogonal to each other as well as w. So

$$< w, e_1 >= 0, \;\; < w, e_2 >= 0, \;\; < e_1, e_2 >= 0$$

The two vectors $\{e_1, e_2\}$ lie in the plane perpendicular to w and it follows that $\{w, e_1, e_2\}$ form a basis for $\mathbb{R}^3$. Since A is orthogonal, the matrix of A in this basis is of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & a_3 \\ 0 & a_2 & a_4 \end{bmatrix}.$$

Proof (contd.)

(why ?) Now

$$\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$$

is an element of $SO(2)$ and hence there exists a $\theta \in [0, 2\pi]$ such that

$$\begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

It follows that A is a rotation about w through the angle θ .

Outline

The "star" map (f_*) associated with a smooth function f

Consider a smooth map $f : X \to Y$. At each $p \in X$ we define a linear transformation as follows

$$f_{*p} : T_p(X) \to T_{f(p)}(Y)$$

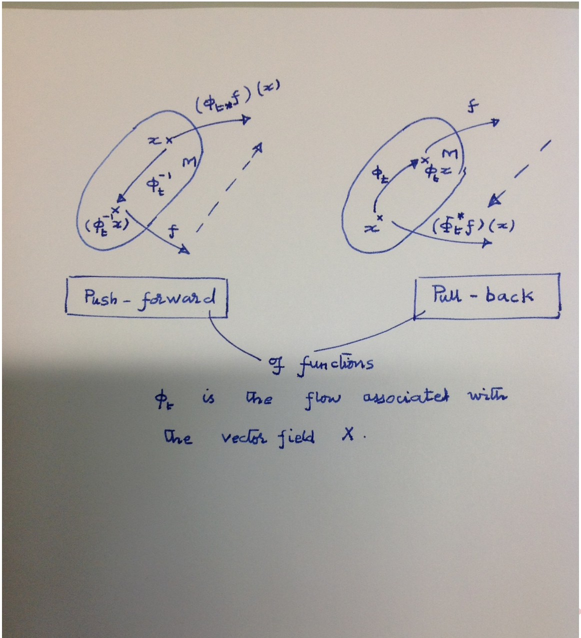
called the derivative of f at p, which is intended to serve as a "linear approximation to f near p,". Visualize this as follows.

- Choose a parametrized curve $c(\cdot) : (-\epsilon, \epsilon) \to X$ with $c(0) = p$ and $\frac{dc}{dt}|_{t=0} = v_p$.
- Construct the curve $f \circ c$. Then define

$$f_{*p}(v_p) \triangleq T_p f \cdot v_p = \frac{d}{dt}|_{t=0}(f \circ c)(t)$$

- The rank of f at p is the rank of the Jacobian matrix at $x(p)$ and *this is independent of the choice of coordinates x.*

Push-forward and pull-back of a function



Push-forward and pull-back of a function

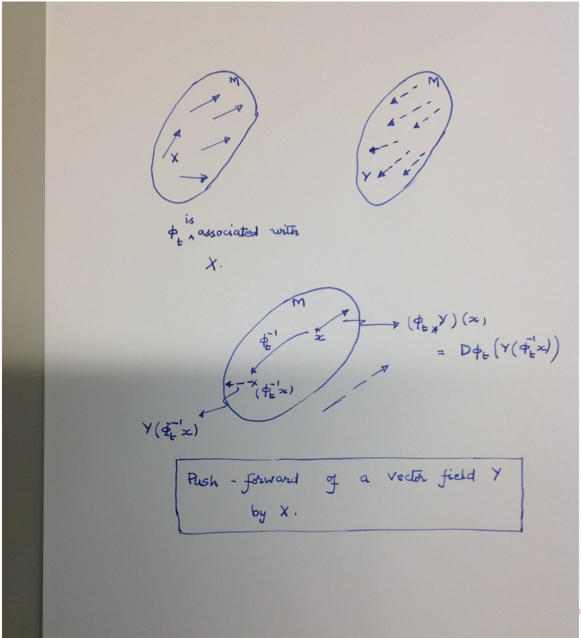
- Suppose X is a vector field on M and $f : M \to \mathbb{R}$ is a smooth function. Then the *push-forward* of the function f on M by the flow of X is the function $\Phi_{t*} f$ defined by

$$(\Phi_{t*} f)(x) \triangleq f \circ \Phi_t^{-1}(x) \quad \forall x \in M$$

- Suppose X is a vector field on M and $f : M \to \mathbb{R}$ is a smooth function. Then the *pull-back* of the function f on M by the flow of X is the function $\Phi_t^* f$ defined by

$$(\Phi_t^* f)(x) \triangleq f \circ \Phi_t(x) \quad \forall x \in M$$

Push-forward and pull-back of a vector field



Push-forward of a vector field by another vector field

Push-forward

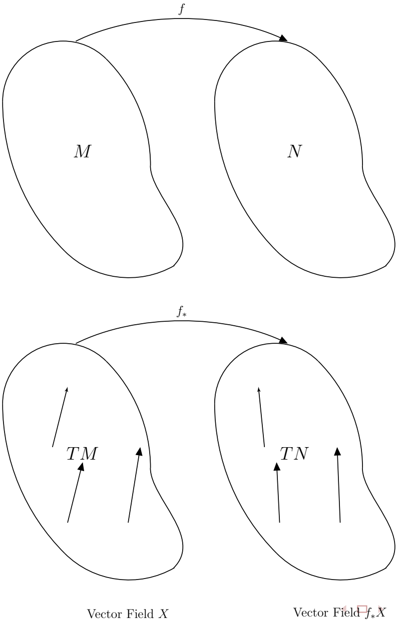
- Suppose $\Phi_t : M \to M$ is the flow associated with a vector field X, then the *push-forward of a vector field Y* on M by f is the vector field $(\Phi_{t*}Y)$ on M defined by

$$(\Phi_{t*}Y)(x) = T_{(\Phi_t^{-1}x)}[Y(\Phi_t^{-1}x)] \quad \forall x \in M$$

- In coordinates

$$(\Phi_{t*}Y)(x) = (D\Phi_t)(Y(\Phi_t^{-1}x))$$

Push forward of vector fields under a diffeomorphism f



Push-forward of vector fields under a diffeomorphism f

Push-forward

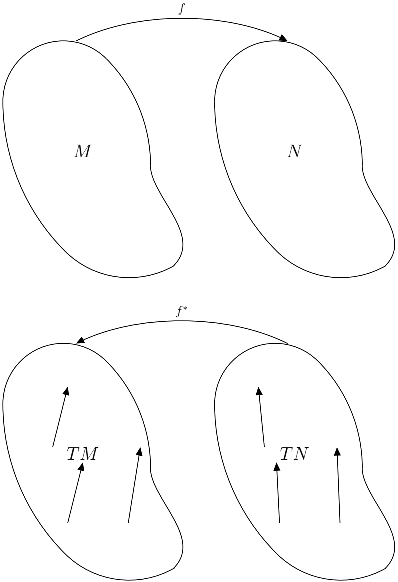
- Suppose $f : M \to N$ is a diffeomorphism, then the *push-forward* of a vector field X on M by f is the vector field $f_* X$ on N defined by

$$(f_* X)(f(x)) = T_x f(X(x)) \quad \forall x \in M$$

- In coordinates

$$y = f(x) \qquad (f_* X)(y) = Df(x).X(x) = \frac{dy}{dx} \cdot X(x)$$

Pull back of a vector field under a diffeomorphism f



Vector Field f^*Y Vector Field Y

Pull-back of vector fields under a diffeomorphism f

The pull-back

- Suppose $f : M \to N$ is a diffeomorphism, then the *pull-back* of a vector field Y on N by f is the vector field f^*Y on M defined by

$$f^*Y = (f^{-1})_*Y = Tf^{-1} \circ Y \circ f$$

- In coordinates

$$y = f(x) \qquad (f_*X)(y) = Df(x).X(x) = \frac{dy}{dx} \cdot X(x)$$

Outline

Operations on vector fields

The Gradient

Consider a smooth function $g(\cdot) : U \to \mathbb{R}$. The gradient of such a function, denoted by ∇g, is defined as

$$\nabla g(x) = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}$$

alternate notation: $\mathrm{grad}(g)$.

The Lie derivative of a function

The Lie derivative

The Lie derivative of a function f along X is

$$(\mathcal{L}_X f)(x) = \frac{d}{dt}|_{t=0}(\Phi_t^* f)(x) = \frac{d}{dt}|_{t=0} f \circ \Phi_t(x)$$

In coordinates we have the familiar

$$(\mathcal{L}_X f)(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} X(x)$$

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Alternate notation

$$(Xf)(x) = \frac{d}{dt}|_{t=0} f \circ \Phi_t(x) = \lim_{t \to 0} \frac{f(\Phi_t(x)) - f(x)}{t}$$

High school physics

The cross product

- Vector space $\mathbb{R}^3$ and the cross-product operation $\times$.
  - $(\alpha_1 a_1 + \alpha_2 a_2) \times b = \alpha_1(a_1 \times b_1) + \alpha_2(a_2 \times b_2)$ - linearity. (*holds in the second argument as well.*)
  - $a \times b = -b \times a$ - skew-commutative.
  - $a \times (b \times c) + c \times (a \times b) + b \times (c \times a) = 0$ - the Jacobi-Lie identity.

Comment: the cross-product of two linearly independent vectors in $\mathbb{R}^3$ yields a vector in a new *direction.*

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Comment: the cross-product of two linearly independent vectors in $\mathbb{R}^3$ yields a vector in a new *direction.*

An alternate notation

$$a \times b \leftrightarrow \hat{a}b \qquad \hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

The Lie derivative of a vector field

The pull back of a vector field

The Lie derivative of Y along X is

$$\mathcal{L}_X Y \triangleq \frac{d}{dt}\Big|_{t=0} \Phi_t^* Y$$

where Φ is the flow of X.

The Lie derivative of a vector field

The pull back of a vector field

The Lie derivative of Y along X is

$$\mathcal{L}_X Y \triangleq \frac{d}{dt}|_{t=0} \Phi_t^* Y$$

where Φ is the flow of X.
Explicitly

$$(\mathcal{L}_X Y)(x) = \frac{d}{dt}|_{t=0} (D\Phi_t(x))^{-1} \cdot Y(\Phi_t(x))$$

The Lie bracket

In coordinates we have the familiar expression

$$\frac{d}{dt}|_{t=0} (D\Phi_t(x))^{-1} \cdot Y(\Phi_t(x)) = \frac{\partial Y}{\partial x} X(x) - \frac{\partial X}{\partial x} Y(x) = [X, Y](x)$$

Operation on vector fields

The Jacobi-Lie bracket

The Jacobi-Lie bracket of two vector fields is an operation between two vector fields that yields another vector field. For two vector fields X and Y, both defined from U to $\mathbb{R}^n$, it is defined as

$$[X, Y] = (\mathcal{L}_X Y) = (DY) \cdot X - (DX) \cdot Y$$

and satisfies the following properties (for any three vector fields X, Y, Z)

- $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$ - linearity in the first argument (also hold for the second argument)
- $[X, Y] = -[Y, X]$ - skew-commutative.
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [X, Z]] = 0$ - the Jacobi-Lie identity

More properties

The Jacobi-Lie bracket

Let X generate the flow $\{\Phi_t\}$ and Y generate the flow $\{\Psi_t\}$. Then $[X, Y] = 0$ if and only if $\Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t$ for all $s, t \in \mathbb{R}$.

Outline

The Lie algebra - $\mathfrak{so}(3)$

3×3 skew-symmetric matrices

Recall

$$\omega \times x \leftrightarrow \hat{\omega} x \qquad \hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The eigen values of $\hat{\omega}$ are $0, \pm \|\omega\| \, i$ (Hint: The trace of a matrix is the sum of its eigen values.)

Claim

Exponential of a skew-symmetric matrix is a rotation matrix

To show $e^{\hat{\omega}} \in SO(3)$

$$(e^{\hat{\omega}})(e^{\hat{\omega}})^T = (e^{(\hat{\omega}-\hat{\omega})}) = I \Rightarrow det(e^{\hat{\omega}}) = \pm 1$$

Now from $\omega = 0$, $e^{\hat{\omega}} = I$ and $\det(I) = 1$. The determinant is a continuous function of the elements of the matrix

$$\Rightarrow det(e^{\hat{\omega}}) = 1$$

Properties of the Lie algebra - $\mathfrak{so}(3)$

It is a vector space of dimension 3.
The tangent space of the identity of $SO(3)$ i. e. $T_e\, SO(3) = \mathfrak{so}(3)$.
The bracket operation $[\cdot, \cdot] : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$ satisfies

- $[\alpha\hat{x} + \beta\hat{y}, \hat{z}] = \alpha[\hat{x}, \hat{z}] + \beta[\hat{y}, \hat{z}]$ - linearity in the first argument (also hold for the second argument)
- $[\hat{x}, \hat{z}] = -[\hat{z}, \hat{x}]$ - skew-commutative.
- $[\hat{x}, [\hat{y}, \hat{z}]] + [\hat{z}, [\hat{x}, \hat{y}]] + [\hat{y}, [\hat{z}, \hat{x}]] = 0$ - the Jacobi-Lie identity

Notice that the cross product relates as

$$[\hat{x}, \hat{z}] = \widehat{x \times z}$$