AE 236: Compressible Fluid Mechanics

(Module III: Supersonic Flow Turning)

a. Oblique shock waves
b. Expansion waves

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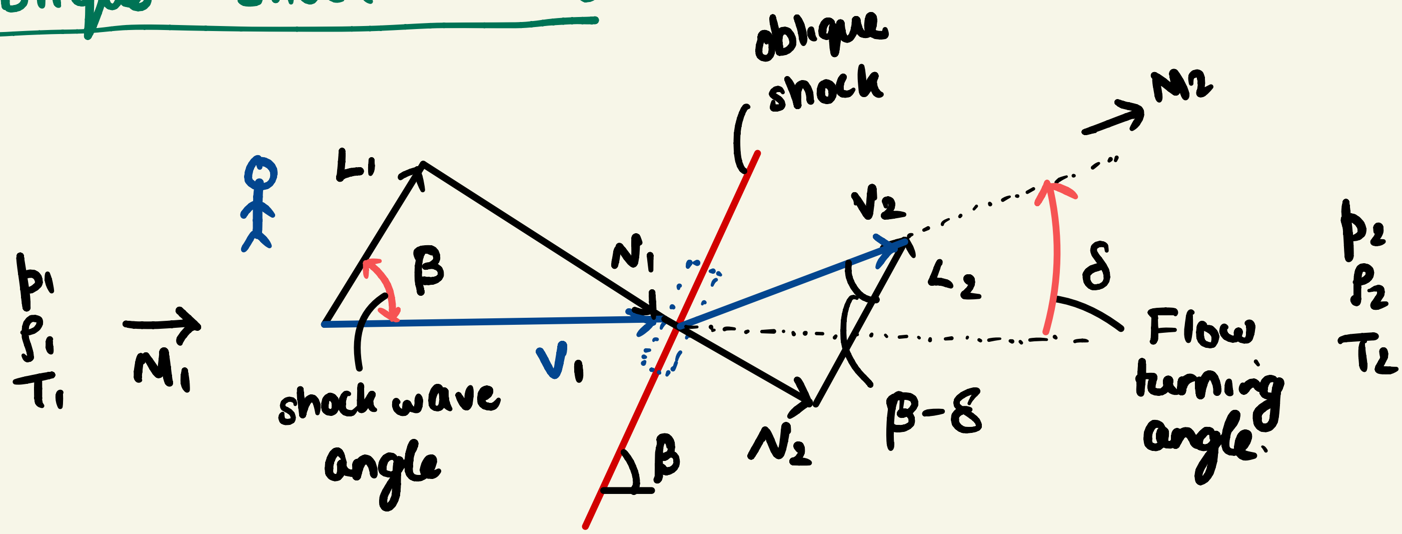
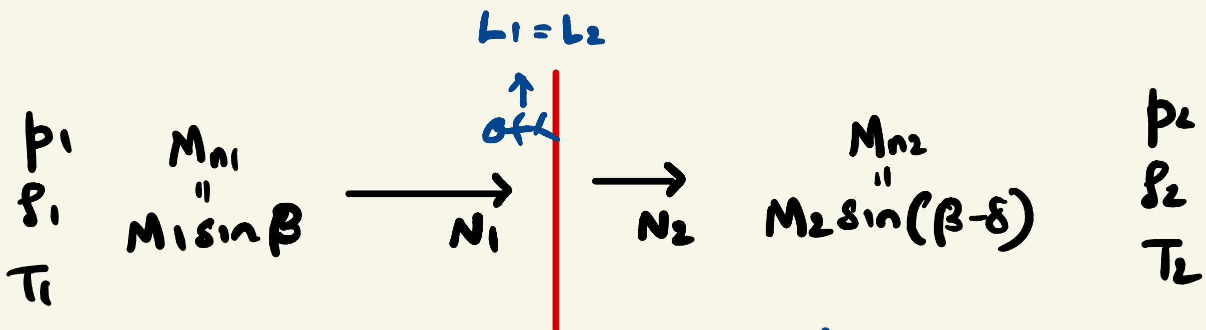
Oblique shock waves





Equivalent normal shock problem

Continuity

$$\rho_1 N_1 = \rho_2 N_2 \qquad \text{— (1)}$$

Momentum

$$p_1 - p_2 = \rho_2 N_2^2 - \rho_1 N_1^2 \qquad \text{— (2)}$$

Energy

$$\frac{2\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + V_1^2 = \frac{2\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + V_2^2$$

$$\Rightarrow \left(\frac{2\gamma}{\gamma - 1}\right)\left[\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right] = N_1^2 - N_2^2 \qquad \text{— (3)}$$

$$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1} + 1}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{p_2}{p_1}} = \frac{N_1}{N_2}$$

$$\frac{T_2}{T_1} = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{p_2}{p_1}}{\left(\frac{\gamma+1}{\gamma-1}\right) + \frac{p_1}{p_2}}$$

Rankine - Hugoniot relations for oblique shock waves

$N_1 = V_1 \sin\beta \qquad N_2 = V_2 \sin(\beta - \delta)$

$\Rightarrow$ Instead of M_1 and M_2, we can substitute $M_{n1} = M_1 \sin\beta$ and $M_{n2} = M_2 \sin(\beta - \delta)$ in the normal shock relationships

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2\beta - (\gamma-1)}{\gamma+1}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2 \sin^2\beta}{2 + (\gamma-1) M_1^2 \sin^2\beta}$$

$$\frac{T_2}{T_1} = \frac{\left[2 + (\gamma-1) M_1^2 \sin^2\beta\right]\left[2\gamma M_1^2 \sin^2\beta - (\gamma-1)\right]}{(\gamma+1)^2 M_1^2 \sin^2\beta}$$

$$M_2^2 \sin^2(\beta-\delta) = \frac{M_1^2 \sin^2\beta + 2/(\gamma-1)}{2\gamma M_1^2 \sin^2\beta/(\gamma-1) - 1}$$

From consideration of normal shock waves

$$M_1 \sin\beta \geq 1$$

For a given M_1, $\beta_{max} = 90^\circ$ Normal shock

$\beta_{min} = \sin^{-1}\frac{1}{M_1}$ Mach wave

$$\boxed{\sin^{-1}\frac{1}{M_1} \leq \beta \leq 90^\circ}$$

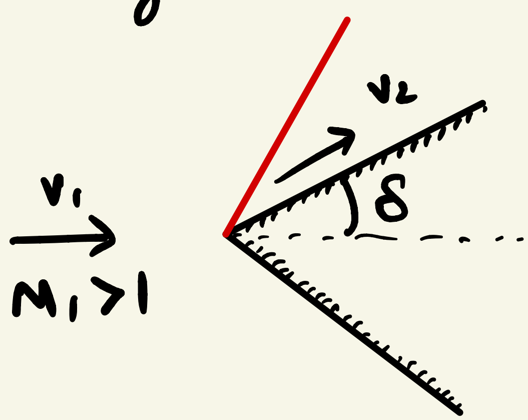
Also, from normal shock wave analysis

$$M_2 \sin(\beta - \delta) \leq 1$$

M_2 can be supersonic for oblique shock waves

Oblique shocks are one of two ways in which flow turning is achieved in supersonic flows.

For a given M_1 & δ, β is usually unknown



Flow has to remain parallel to the wall which is ensured by an oblique shock of appropriate strength/inclination

$\Rightarrow$ We need a relation connecting M_1, δ and β

We have

$$\tan\beta = \frac{N_1}{L_1}, \qquad \tan(\beta - \delta) = \frac{N_2}{L_2}$$

$$\Rightarrow \frac{\tan(\beta-\delta)}{\tan\beta} = \frac{N_2}{N_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta} = X$$

From l.h.s. and r.h.s., we get

$$\left(\frac{\tan\beta-\tan\delta}{1+\tan\beta\tan\delta}\right)\Big/\tan\beta = X$$

$$\Rightarrow \tan\delta = \frac{\tan\beta(1-X)}{1+X\tan^2\beta}$$

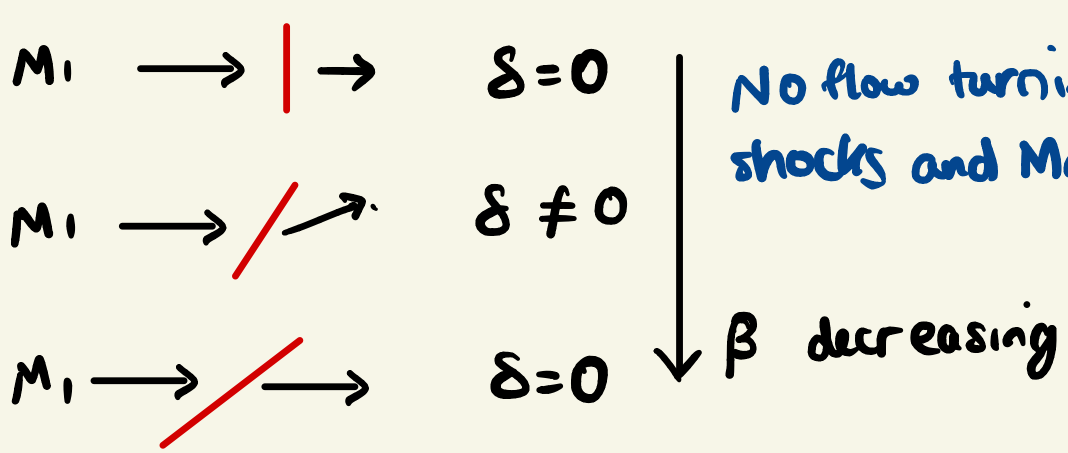
Substituting for X, we have

$$\boxed{\tan\delta = \frac{2\cot\beta(M_1^2\sin^2\beta-1)}{2+M_1^2(\gamma+\cos 2\beta)}} \quad - \circledast$$

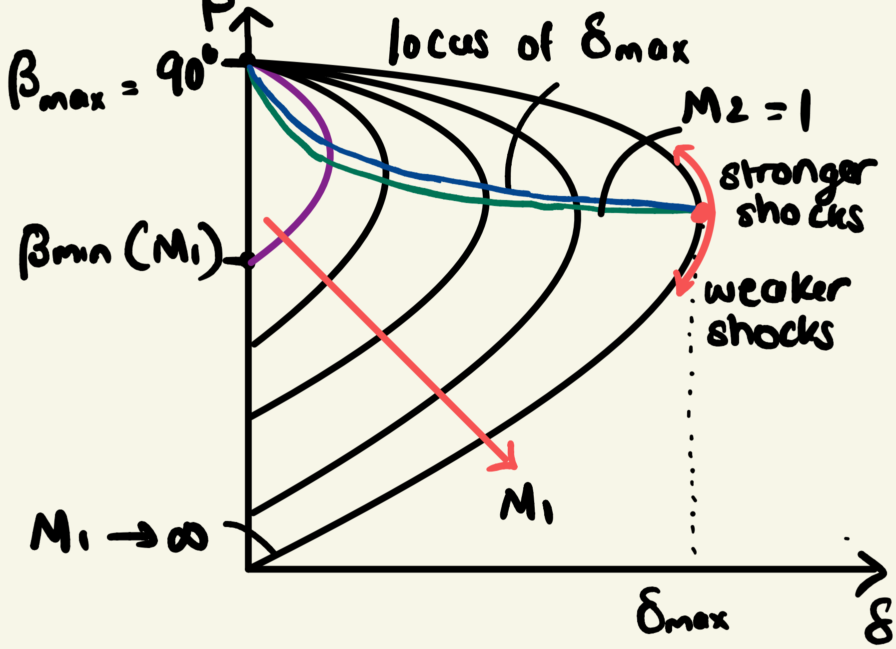
Two cases where $\delta=0$

(i) $\cot\beta = 0 \Rightarrow \beta = 90^\circ$ Normal shock

(ii) $M_1^2\sin^2\beta = 1 \Rightarrow \beta = \sin^{-1}\frac{1}{M_1}$ Mach wave



No flow turning for normal shocks and Mach waves

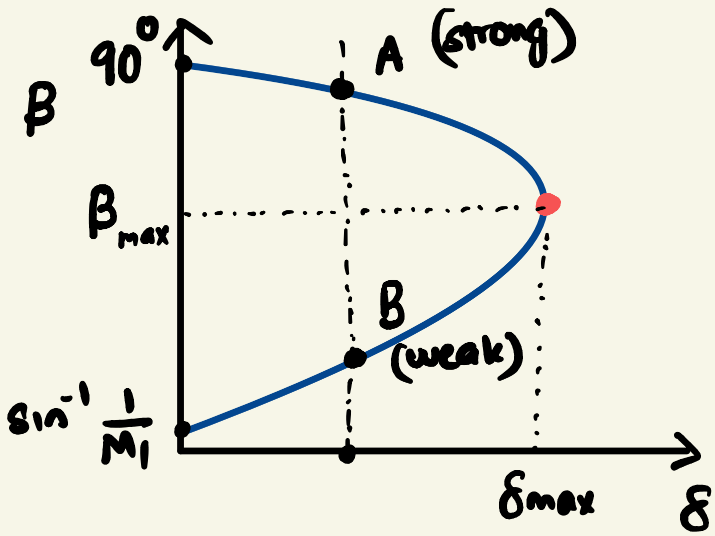


As β increases for fixed M_1, $M_{n1} = M_1 \sin\beta$ increases, p_2/p_1 increases and shocks become stronger

$\beta > \delta_{max}$ (strong shock solution) Always subsonic

$\beta < \delta_{max}$ (weak shock solution) Subsonic / supersonic

Given δ, M_1 two possible solutions for β



β_A - strong shock

β_B - weak shock

External flows — weak shock solution as the required pressure gradients cannot be maintained

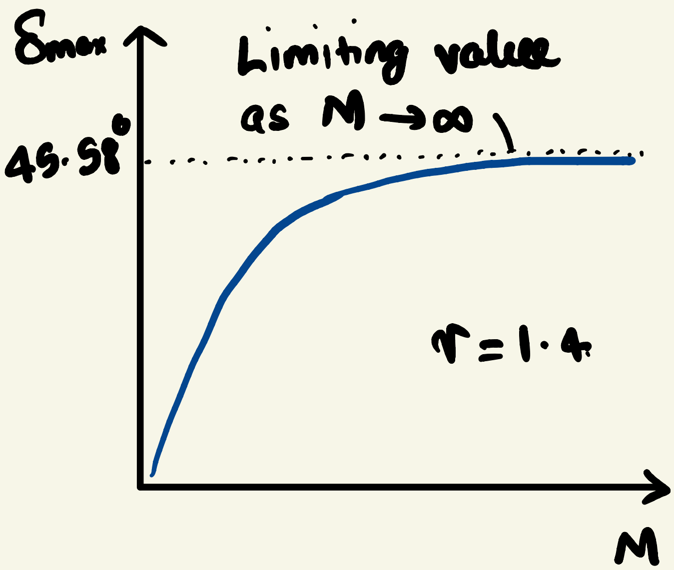
Internal flows — strong shock solution as required pressure gradients are easier to maintain

Finding δ_{max}

Set $\frac{d\delta}{d\beta} = 0$ in (*)

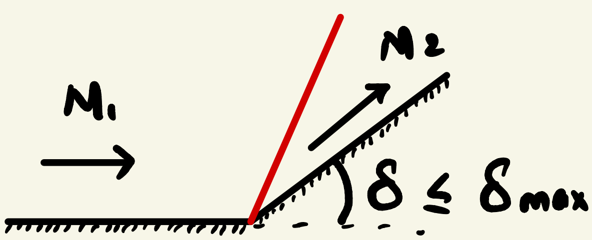
$$\Rightarrow \boxed{\sin^2\beta_{max} = \frac{\gamma+1}{4\gamma} - \frac{1}{\gamma M_1^2}\left[1 - \sqrt{(\gamma+1)\left(1 + \frac{\gamma-1}{2}M_1^2 + \frac{\gamma+1}{16}M_1^4\right)}\,\right]}$$

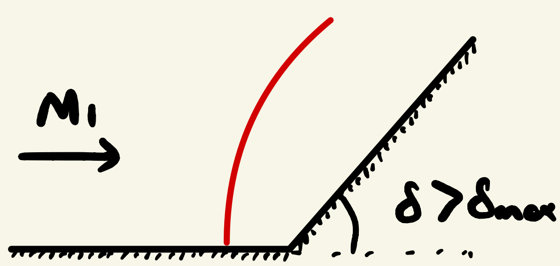
Substitute β_{max} in (*) to get δ_{max}



Tells us maximum achievable flow turning at a given M

Maximum possible turning by an oblique shock = $45.58°$

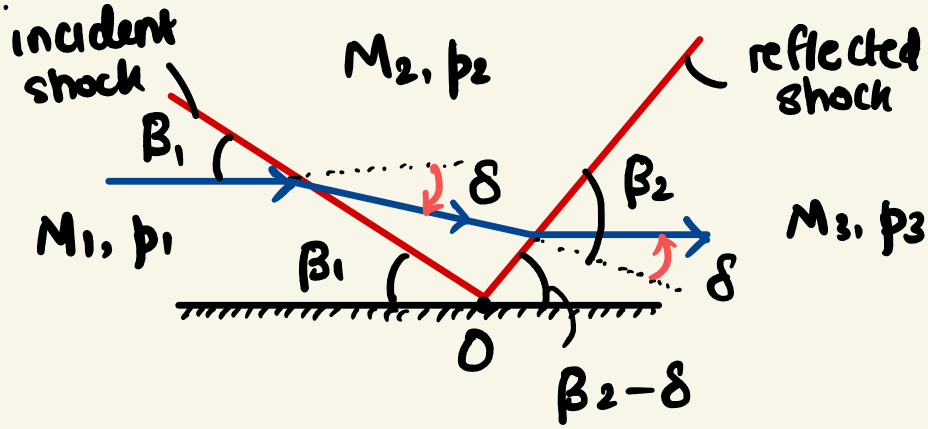




For $\delta > \delta_{max}$, we have a detached curved shock wave which consists of all possible solutions at the M_1 ($\sin^{-1}\frac{1}{M_1} \le \beta \le 90°$)

Decreasing M_1 at a given δ shifts the solution from an oblique shock to a detached curved shock.

Reflections of oblique shocks



Normal reflection

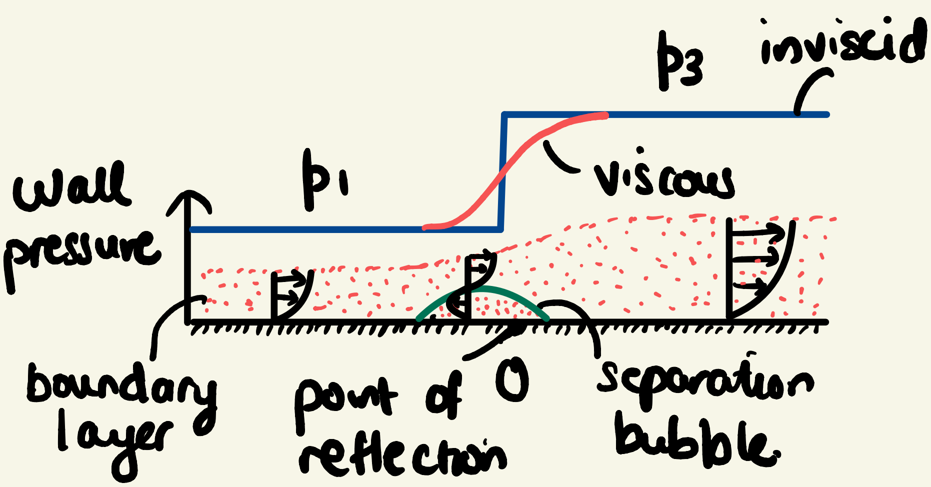
(1) For the given M_1, δ, determine β_1 M_2, p_2/p_1

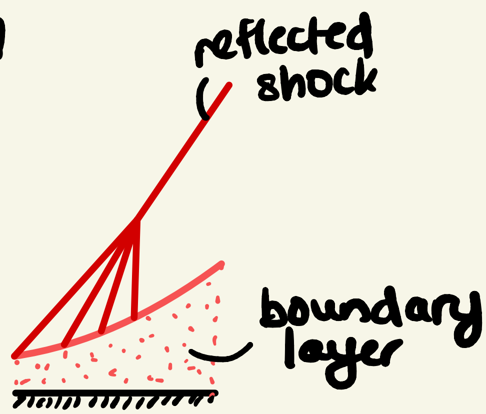
(2) For M_2 and known δ, find β_2, M_3, p_3/p_2

(3) Then overall pressure ratio

$$\frac{p_3}{p_1} = \frac{p_3}{p_2} \cdot \frac{p_2}{p_1}$$

(4) Find $\beta_2 - \delta$ which gives angle of reflected shock

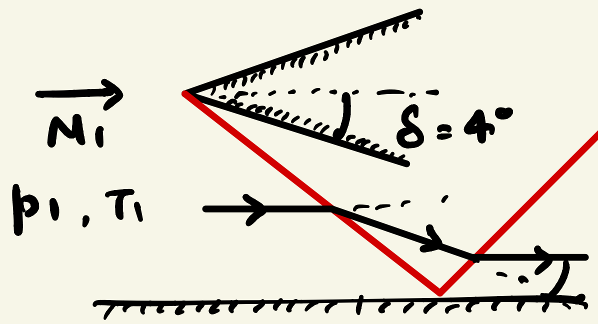




Actual form of pressure distribution depends on the type of boundary layer flow (laminar/turbulent), the thickness of the boundary layer and shock strength

Problems

① $M_1 = 2.5$, $p_1 = 60\,kPa$, $T = -20^\circ C$



$p_3 = ?$
$V_3 = ?$

For $M_1 = 2.5$, $\delta = 4^\circ$, $\beta_1 = 26.6^\circ$ (weak solution)

$$M_{n1} = M_1 \sin\beta_1 = 1.12$$

From normal shock tables at $M = 1.12$

$$M_{n2} = 0.897, \quad \frac{p_2}{p_1} = 1.336, \quad \frac{T_2}{T_1} = 1.087$$

$$M_2 = \frac{0.897}{\sin(26.6^\circ - 4^\circ)} = 2.334$$

For $M_2 = 2.334$, $\delta = 4^\circ$, $\beta_2 = 28.5^\circ$

$$M_{n2} = M_2 \sin\beta_2 = 1.113$$

From normal shock tables at $M = 1.113$

$$M_{n3} = 0.9018, \quad \frac{p_3}{p_2} = 1.297, \quad \frac{T_3}{T_2} = 1.078$$

$$M_3 = \frac{0.9018}{\sin(28.5^\circ - 4^\circ)} = 2.17$$

$$p_3 = \frac{p_3}{p_2} \cdot \frac{p_2}{p_1} \cdot p_1 = 104.0\,kPa$$

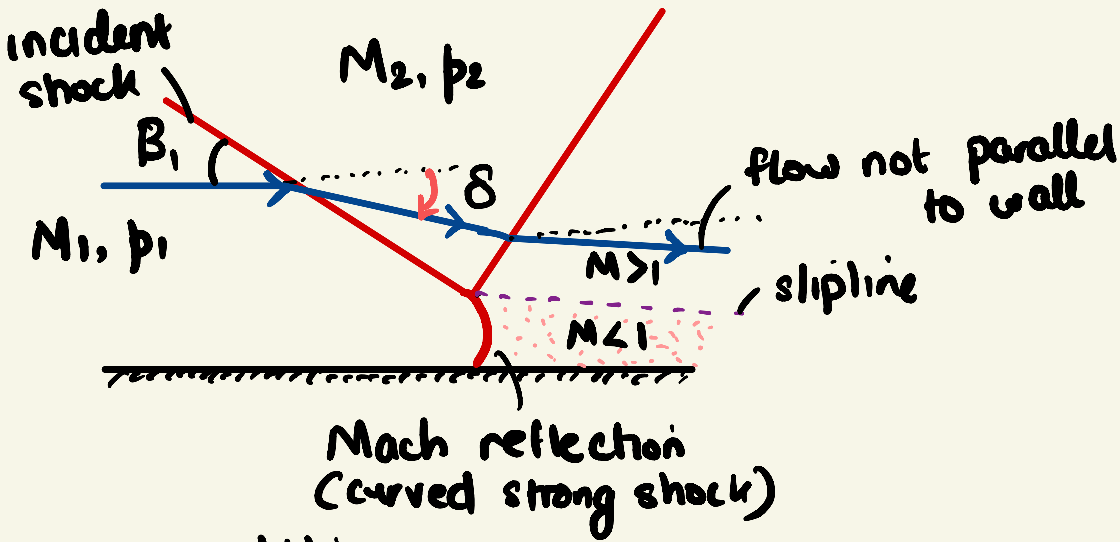
$$T_3 = \frac{T_3}{T_2} \cdot \frac{T_2}{T_1} \cdot T_1 = 296\,K$$

$$a_3 = \sqrt{\gamma R T} = 345\,m/s \Rightarrow V_3 = M_3 a_3 = 749\,m/s$$

Special cases

① What happens if $\delta > \delta_{max}(M_2)$?

We have Mach reflection with formation of sliplines



Properties of sliplines:

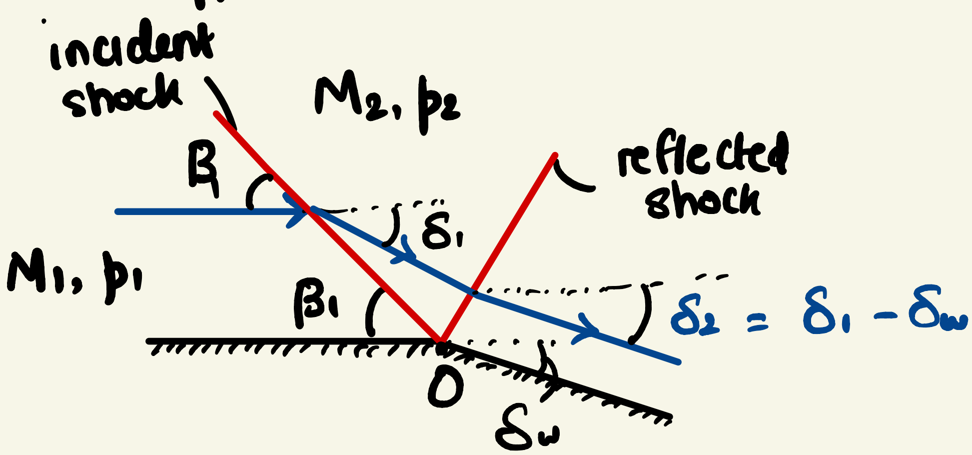
① Pressure is continuous across a slipline

② Velocity should have same direction either side of a slipline

③ ρ, T, V, s can have different values across a slipline

In the previous problem with $M_1 = 2.5$, if $\delta > 17.9°$ then we have Mach reflection

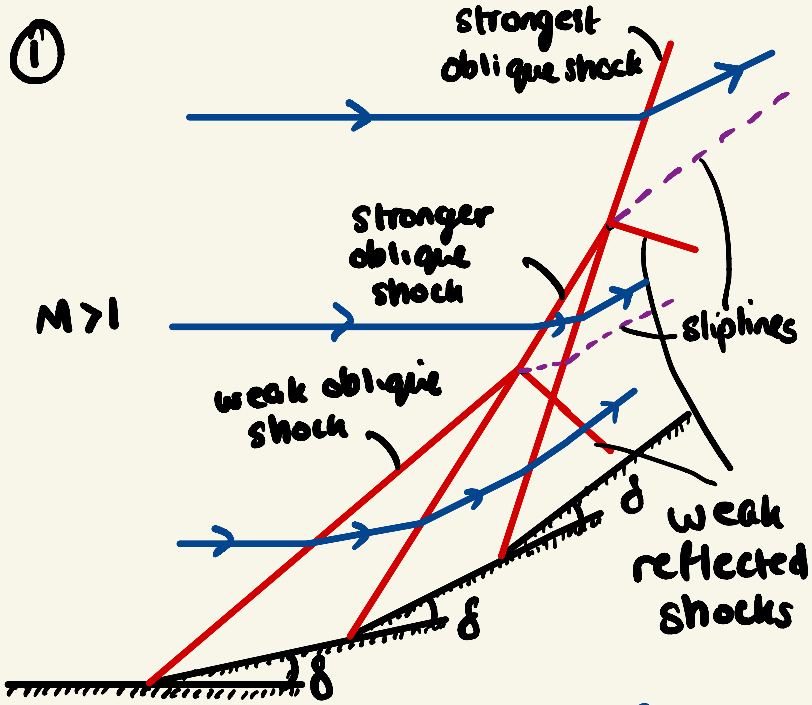
② What happens if wall is inclined ? (δ_w)



If $\delta_w = \delta_1 \Rightarrow \delta_2 = 0 \Rightarrow$ No reflected shock

Interaction of oblique shock waves

①

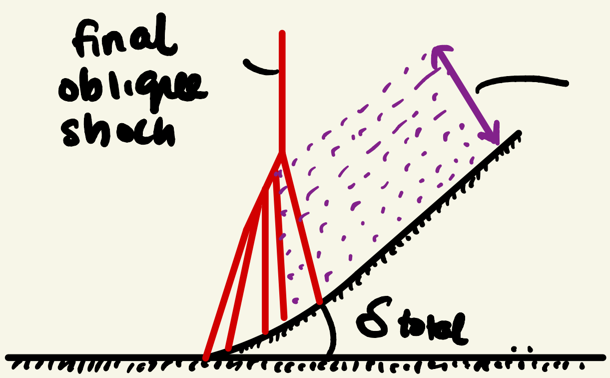


The oblique shock waves generated at each step will converge and coalesce into a single oblique shock that is stronger than any of the initial waves

Oblique shocks converge as β increases with decreasing M

Reflected shocks are formed as the weaker oblique shocks can't produce the same change as the stronger shock

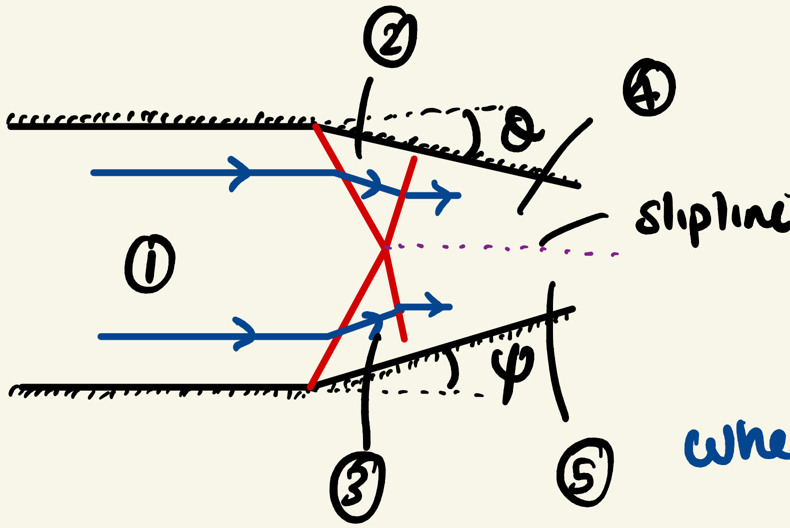
② What if we have a smooth ramp?



Region of uniform pressure and flow direction but changing V, T, ρ, s

Final shock turns the flow by δ_{total}

③ Internal flows



Flow should be parallel to the center line as flow from ② and ③ approach it. which creates reflected shocks & sliplines

When θ = φ, sliplines vanish

What if $\delta_w > \delta_1$?

Supersonic flow

↳ turning into itself — compression (oblique shock)

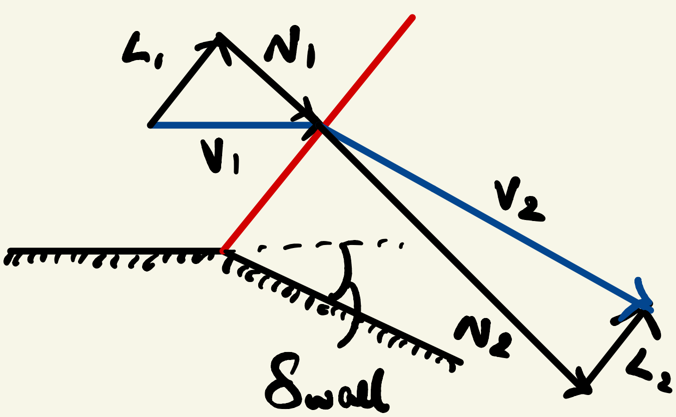
↳ turning away from itself — expansion ?

Happens when

(i) the wall is convex

(2) duct flows when pressure just at the exit is greater than outside pressure outside

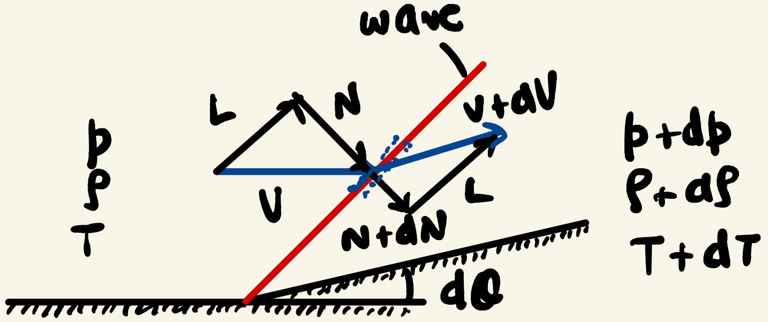
Can oblique shocks achieve this turn?



$N_2 > N_1$ which is unphysical because 2nd law of thermodynamics is violated (Expansion shocks)

Oblique shocks cannot make the flow turn this way

Solution procedure : consider an infinitesimal turn $d\theta$



$d\theta$ is infinitesimal. Also $d\rho, dp, dT, dV$ small

$\Rightarrow$ flow $\approx$ isentropic

Since flow is isentropic $d\theta$ can be +ve or -ve

↳ +ve for 'compression', -ve for 'expansion'

$\boxed{\text{Continuity}}$

$$\rho N = (\rho + d\rho)(N + dN)$$

$$\Rightarrow \rho\, dN + N\, d\rho = 0 \quad — ①$$

$\boxed{\text{Momentum}}$

$$\cancel{p} - (\cancel{p} + dp) = \rho N(\cancel{N} + dN - \cancel{N})$$

$$\Rightarrow -dp = \rho N\, dN \quad — ②$$ Euler's equation

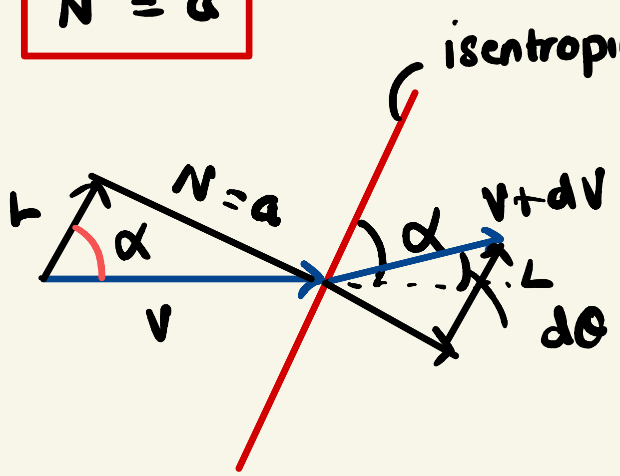
Substitute for dN from ① in ②

$$+dp = \cancel{\rho} N\left(+\frac{N\, d\rho}{\cancel{\rho}}\right)$$

$$\Rightarrow \frac{dp}{d\rho} = N^2$$

Process nearly isentropic $\left.\frac{dp}{d\rho}\right)_s = a^2 = \frac{\gamma p}{\rho}$

$$\Rightarrow \boxed{N = a}$$



α = Mach angle

$$L = V\cos\alpha = (V + dV)\cos(\alpha - d\theta)$$

$$\Rightarrow (V + dV)[\cos\alpha \underbrace{\cos d\theta}_{\approx 1} + \sin\alpha \underbrace{\sin d\theta}_{\approx d\theta}] = V\cos\alpha$$

$$\Rightarrow \cancel{V\cos\alpha} + V\sin\alpha\, d\theta + dV\cos\alpha = \cancel{V\cos\alpha}$$

$$\Rightarrow \frac{dV}{V} = -\tan\alpha \, d\theta$$

$$\boxed{\tan\alpha = \frac{1}{\sqrt{M^2-1}}}$$

$$\Rightarrow \boxed{\frac{dV}{V} = \frac{-d\theta}{\sqrt{M^2-1}}} \quad — \quad (*)$$

Energy

$$\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right) + V^2 = \left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p+dp}{\rho+d\rho}\right) + (V+dV)^2$$

$$\cancel{\left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right)} + \cancel{V^2} = \left(\frac{2\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right)\left[\cancel{1} + \frac{dp}{p} - \frac{d\rho}{\rho}\right] + \cancel{V^2} + 2V\,dV$$

$$\Rightarrow \underbrace{\left(\frac{\cancel{2}\gamma}{\gamma-1}\right)\left(\frac{p}{\rho}\right)}_{a^2}\left[\frac{dp}{p} - \frac{1}{\gamma}\frac{dp}{p}\cdot\underbrace{\frac{\cancel{\gamma p}}{\cancel{\rho}}}_{a^2}\cdot\underbrace{\frac{\cancel{d\rho}}{\cancel{dp}}}_{1/a^2}\right] = -\cancel{2}V\,dV$$

$$\Rightarrow \left(\frac{a^2}{\gamma-1}\right)\frac{dp}{p}\left[1 - \frac{1}{\gamma}\right] = -V\,dV$$

$$\Rightarrow \frac{dp}{p} = \frac{-\gamma V\,dV}{a^2} = -\frac{\gamma V^2}{a^2}\frac{dV}{V} = -\gamma M^2 \frac{dV}{V}$$

Substitute $(*)$

$$\boxed{\frac{dp}{p} = \frac{\gamma M^2}{\sqrt{M^2-1}}\, d\theta} \quad — \quad (**)$$

$$\frac{d\rho}{\rho} = \frac{d\rho}{dp}\cdot\frac{dp}{p}\cdot\frac{p}{\rho} = \frac{1}{a^2}\frac{dp}{p}\cdot\frac{a^2}{\gamma} = \frac{1}{\gamma}\frac{dp}{p}$$

$$\Rightarrow \boxed{\frac{d\rho}{\rho} = \frac{M^2}{\sqrt{M^2-1}}\, d\theta} \quad — \quad (***)$$

Entropy

$$\frac{ds}{R} = \underbrace{\frac{1}{\gamma-1}}_{C_v/R} \ln\underbrace{\left[1+\frac{dp}{p}\right]}_{p_2/p_1} - \underbrace{\frac{\gamma}{\gamma-1}}_{C_p/R} \ln\underbrace{\left[1+\frac{d\rho}{\rho}\right]}_{\rho_2/\rho_1}$$

$$\approx \frac{1}{\gamma-1}\frac{dp}{p} - \frac{\gamma}{\gamma-1}\frac{d\rho}{\rho} = 0 \quad \text{from } \text{(**) and (***)}$$

Flow process is isentropic

what is the fractional change in Mach number?

$$M^2 = \frac{V^2}{a^2} = \frac{V^2 \rho}{\gamma p}$$

$$\Rightarrow 2\frac{dM}{M} = 2\frac{dV}{V} + \frac{d\rho}{\rho} - \frac{dp}{p}$$

$$= \left[2 - M^2 + \gamma M^2\right]\frac{-d\theta}{\sqrt{M^2-1}}$$

$$\Rightarrow \boxed{\frac{dM}{M} = \left[1+\left(\frac{\gamma-1}{2}\right)M^2\right]\frac{(-d\theta)}{\sqrt{M^2-1}}} \quad - \text{(****)}$$

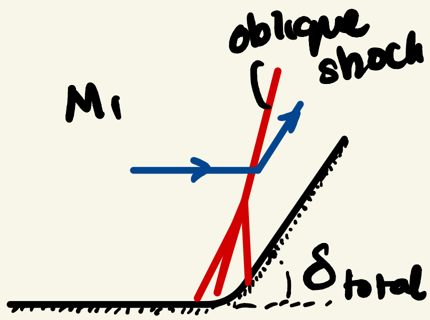
For a differential isentropic ($ds = 0$) flow turn ($d\theta$)

$$\boxed{\begin{aligned} dV &\propto -d\theta \\ dp &\propto d\theta \\ d\rho &\propto d\theta \\ dM &\propto -d\theta \end{aligned}}$$

All property changes directly proportional to $d\theta$

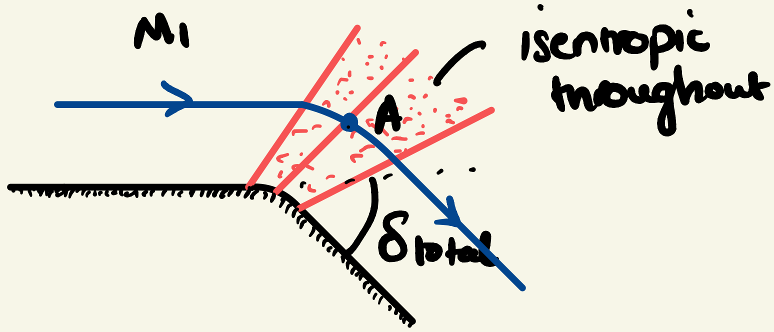
If results seem counterintuitive to intuition, that is because our intuition is SUBSONIC

$\boxed{d\theta > 0}$



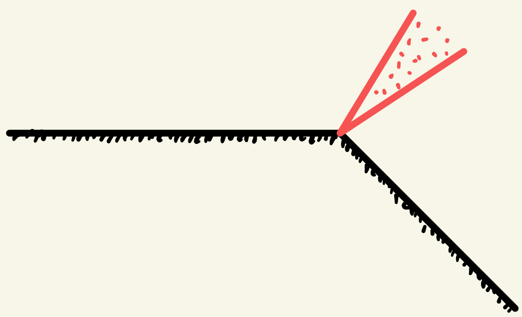
Mach waves converge

$\boxed{d\theta < 0}$

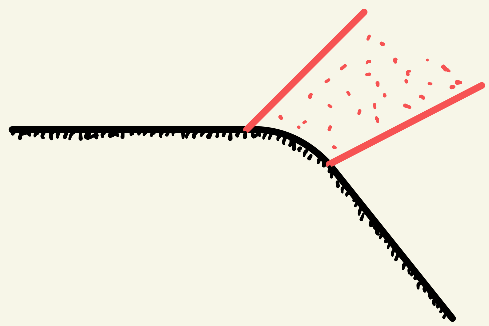


Mach waves diverge

$d\theta < 0$ produces Prandtl-Meyer flows / expansion waves



centered expansion fan



noncentered expansion fan

$$\int_{\theta_1}^{\theta_2} -d\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$

At $M = 1$, $\theta = 0$, $V = a = a^*$ Reference condition

$$\int_{0}^{\theta_2} d\theta - \int_{0}^{\theta_1} d\theta = \int_{1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M} - \int_{1}^{M_1} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$

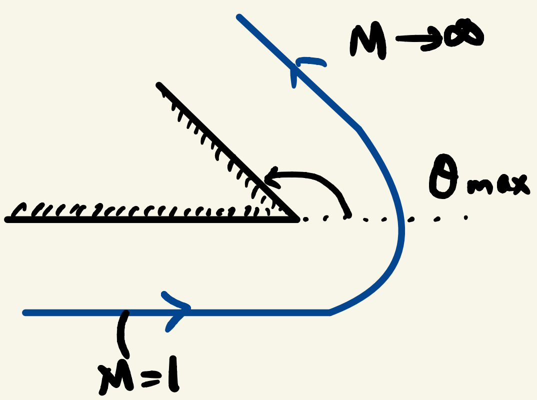
$$\Rightarrow \theta = \int_{1}^{M} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}$$ function only of M

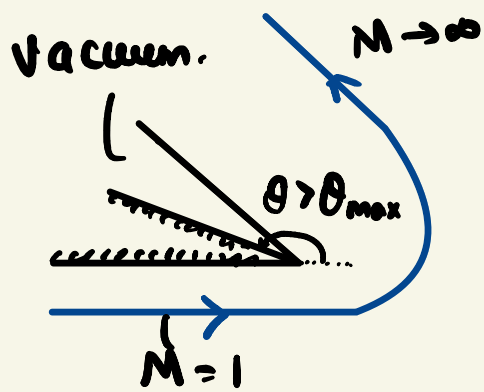
$$\boxed{\theta = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - \tan^{-1}\sqrt{M^2-1}}$$

tabulated in isentropic tables

As $M \to \infty$, $\theta_{max} = \sqrt{\frac{\gamma+1}{\gamma-1}}\,\frac{\pi}{2} - \frac{\pi}{2} = 130.5^\circ$ for $\gamma = 1.4$

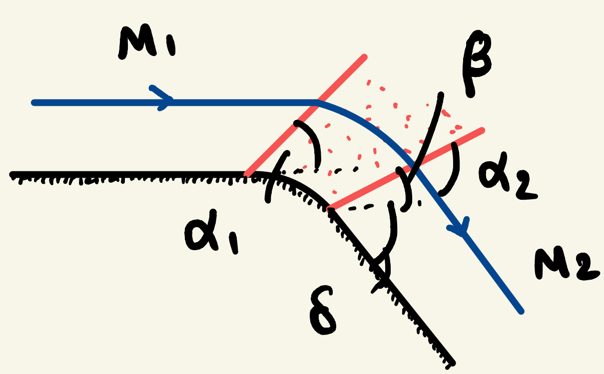
If we turn a $M=1$ flow by 130.5° we achieve $M \to \infty$ downstream.





$p, \rho, T \to 0$ downstream

Not realized physically as continuum violated, ideal gas assumption violated, liquefaction etc.



$$\alpha_1 = \sin^{-1}\frac{1}{M_1}$$

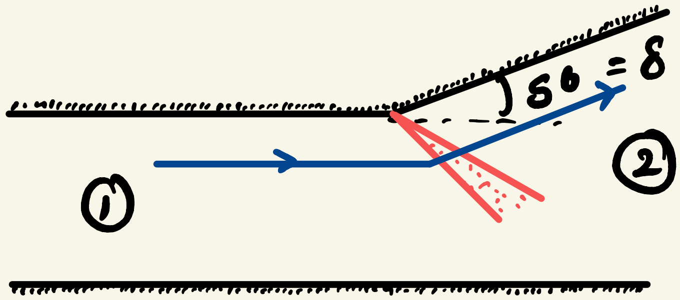
$$\alpha_2 = \sin^{-1}\frac{1}{M_2}$$

$$\beta = \alpha_2 - \delta$$

Problems

②

$M_1 = 1.8$
$T_1 = 15^\circ C$
$p_1 = 90 kPa$



$M_2 = ?$
$p_2 = ?$
$T_2 = ?$

From isentropic tables corresponding to $M_1 = 1.8$

$$\theta_1 = 20.73^\circ, \quad \frac{p_{01}}{p_1} = 5.746, \quad \frac{T_{01}}{T_1} = 1.648$$

$$\Rightarrow \theta_2 = \theta_1 + \delta$$
$$= 25.73^\circ$$

From isentropic tables corresponding to θ_2

$$M_2 = 1.98, \quad \frac{p_{02}}{p_2} = 7.585, \quad \frac{T_{02}}{T_2} = 1.784$$

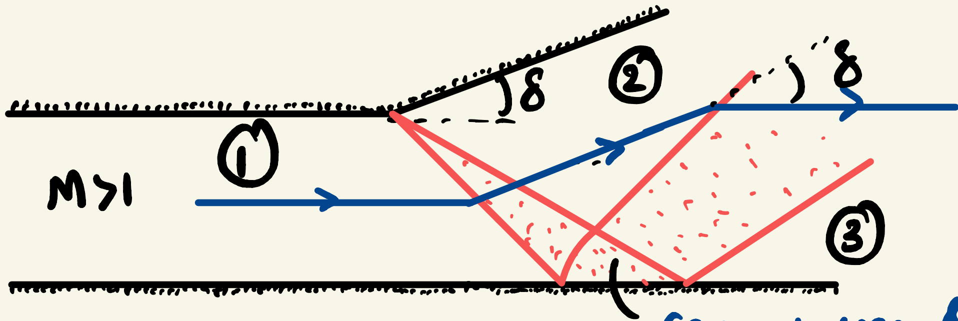
$$T_2 = \frac{T_2}{T_{02}} \cdot \frac{T_{02}}{T_{01}} \cdot \frac{T_{01}}{T_1}$$

$$= \frac{1.648}{1.784} \cdot (288) = 266 K = -7^\circ C$$

$$p_2 = \frac{p_2}{p_{02}} \cdot \frac{p_{02}}{p_{01}} \cdot \frac{p_{01}}{p_1}$$

$$= \frac{5.746}{7.585} \cdot 90 = 68.2 kPa.$$

Reflection of expansion waves



Cannot use θ equation here
(NON SIMPLE REGION)

$$\theta_2 = \theta_1 + \delta$$
$$\theta_3 = \theta_2 + \delta = \theta_1 + 2\delta$$

Problems

③ $M_1 = 2.0$, $\delta = 10°$, $p_1 = 90\,kPa$. Find M_3, p_3

From isentropic tables at $M_1 = 2.0$

$$\theta_1 = 26.38°, \quad \frac{p_{01}}{p_1} = 7.83$$

From the figure,

$$\theta_3 = \theta_1 + 2\delta = 46.38°$$

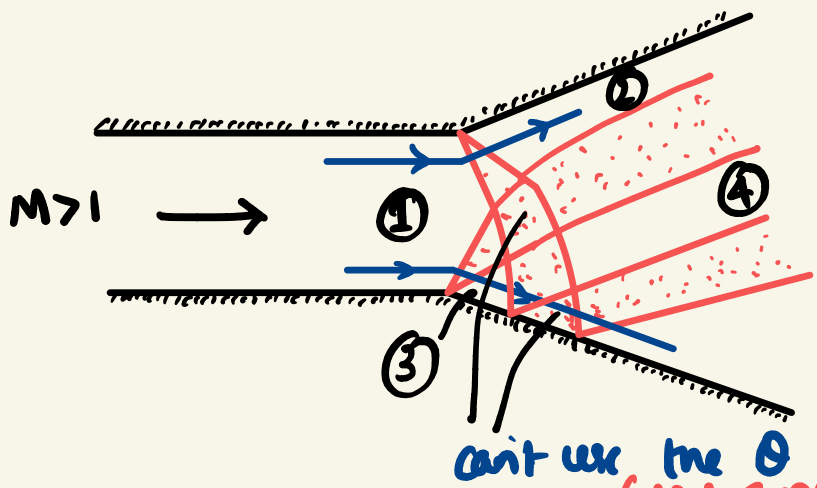
Using isentropic tables at θ_3

$$\Rightarrow M_3 = 2.83, \quad \frac{p_{03}}{p_3} = 28.41$$

$$p_3 = \frac{p_3}{p_{03}} \cdot \frac{p_{03}}{p_{01}} \cdot \frac{p_{01}}{p_1}$$

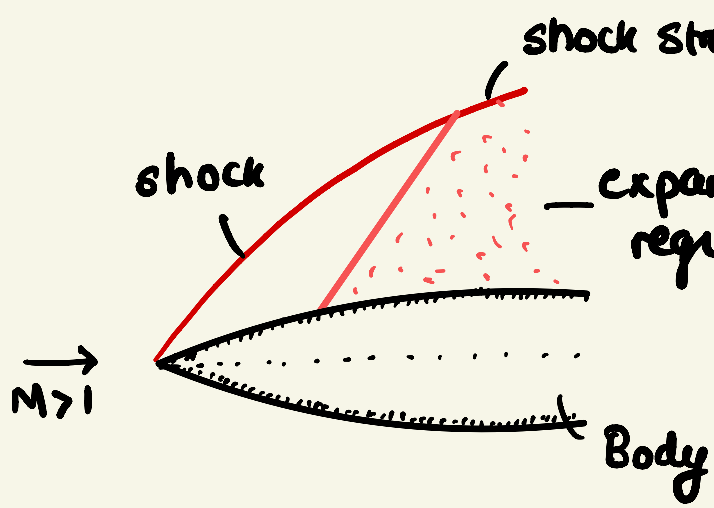
$$= \frac{7.83}{28.41} \cdot 90 = 24.81\,kPa$$

Interaction of expansion waves



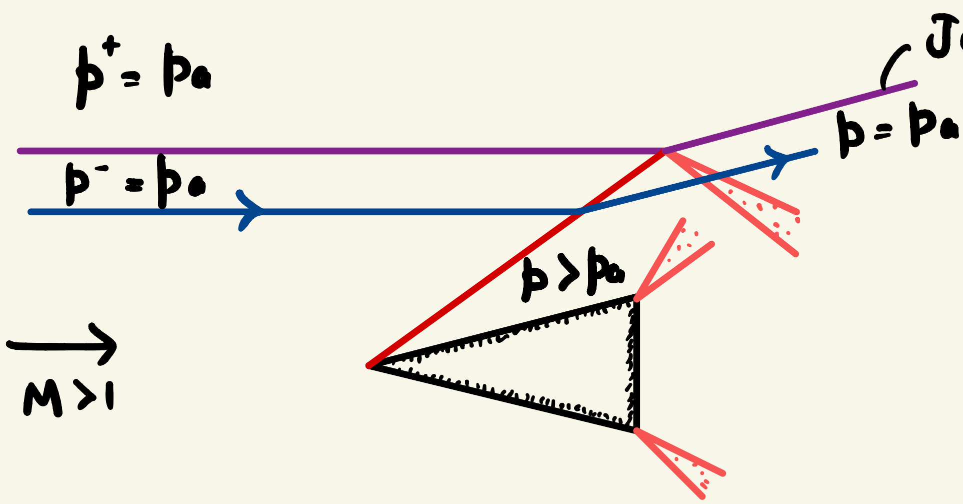
Region ④ has uniform properties with no slipstreams as the entire flow is isentropic

Interaction of expansion waves with shock



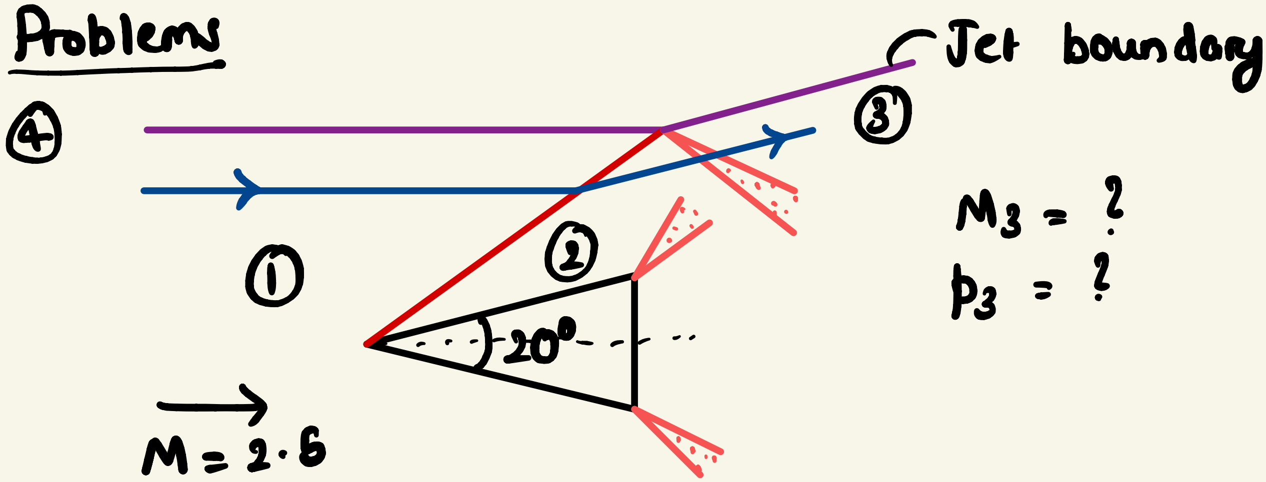
Expansion fans weaken the shock on interaction

Interaction of an oblique shock with a jet boundary



Oblique shock interacting with a jet boundary produces an expansion wave and vice versa.

Problems



$M_3 = ?$
$p_3 = ?$

$\longrightarrow$
$M = 2.5$
$p_1 = 100\,kPa$

At $M_1 = 2.5$, $\delta = 10°$, using oblique shock tables

$$M_2 = 2.086, \quad \frac{p_2}{p_1} = 1.866$$

Isentropic tables at $M_2 = 2.086$ give

$$\frac{p_{02}}{p_2} = 8.945$$

We need $p_3 = p_1$

$$\frac{p_{03}}{p_3} = \underbrace{\frac{p_{03}}{p_{02}}}_{1} \cdot \frac{p_{02}}{p_2} \cdot \frac{p_2}{p_3 \} p_1}$$

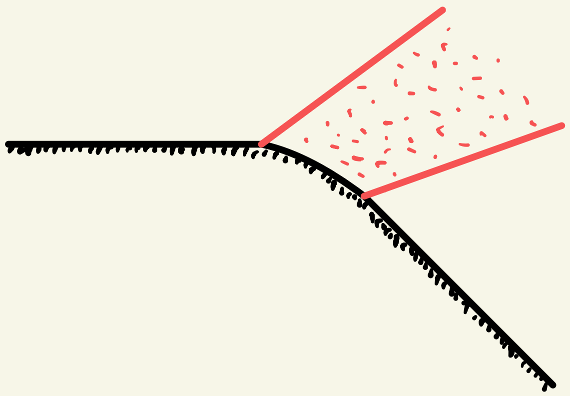
$$= 1 \times 8.945 \cdot 1.866$$

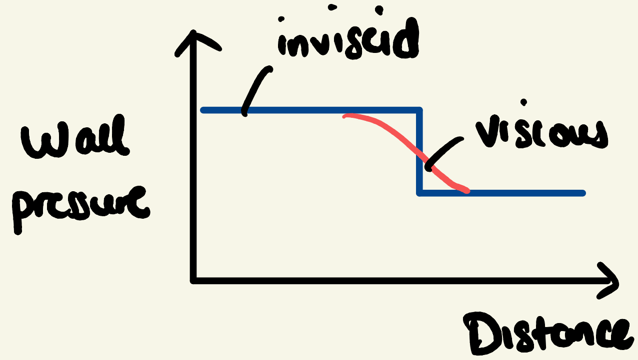
$$= 16.70$$

Using isentropic tables corresponding to p_{03}/p_3

$M_3 = 2.485$
$p_3 = p_1 = 100\,kPa$

Boundary layer effects

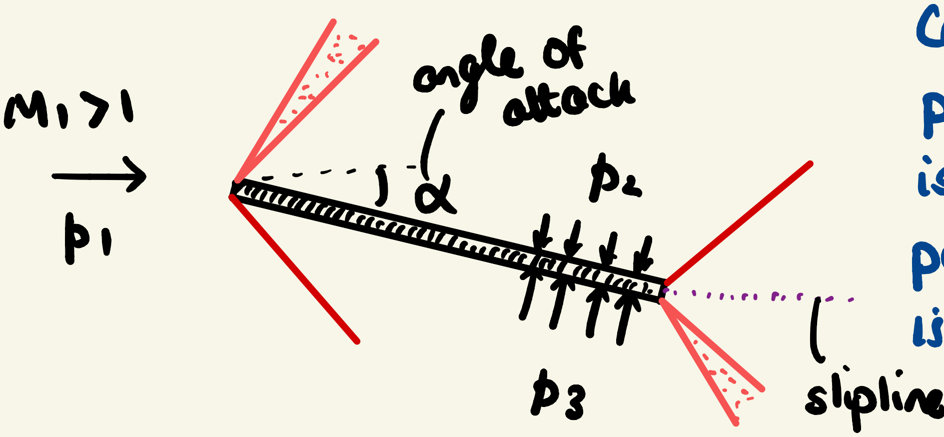




Pressure gradient depends on the type of boundary layer (laminar / turbulent) and its thickness

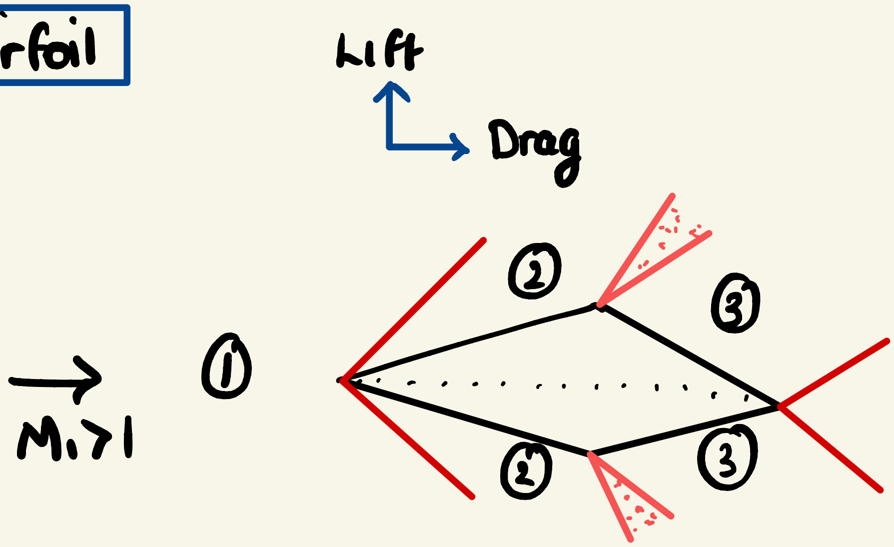
Shock-expansion theory

Flat plate



Component of force perpendicular to flow is LIFT and component parallel to the flow is DRAG (WAVE DRAG)

Airfoil



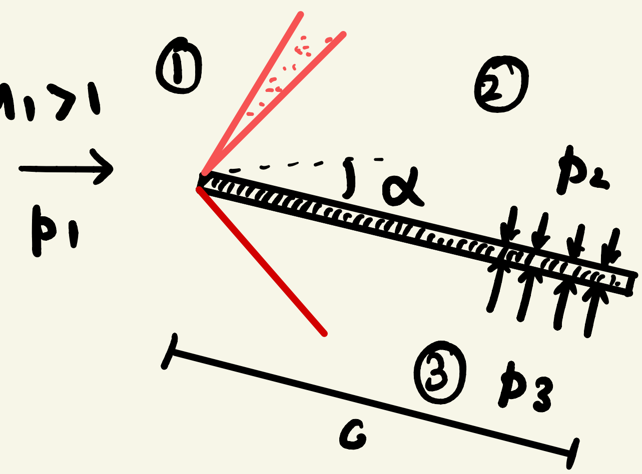
Diamond airfoil

Problems

(5) Wing modelled as flat plate of width 0.25 m : $\alpha = 3^\circ$, $M_1 = 2.5$, $p_1 = 60$ kPa. Estimate lift and drag / span.

For region ① at $M_1 = 2.5$

$$\frac{p_{01}}{p_1} = 17.09, \ \theta_1 = 39.13^\circ$$



$$\Rightarrow \theta_2 = \theta_1 + \alpha = 42.13^\circ$$

From isentropic tables,

$$M_2 = 2.63, \ \frac{p_{02}}{p_2} = 20.93$$

$$\Rightarrow p_2 = \frac{p_2}{p_{02}} \cdot \overbrace{\frac{p_{02}}{p_{01}}}^{1} \cdot \frac{p_{01}}{p_1} \cdot p_1 = \frac{17.09}{20.93} \cdot 60$$

$$= 49.02 \text{ kPa}$$

For region ③, $\delta = \alpha = 3^\circ$, $M_1 = 2.5$

$$\beta = 26^\circ, \ M_{n1} = 2.5 \sin 26^\circ = 1.096$$

From shock tables at M_{n1}

$$\frac{p_3}{p_1} = 1.28 \Rightarrow p_3 = 74 \text{ kPa}$$

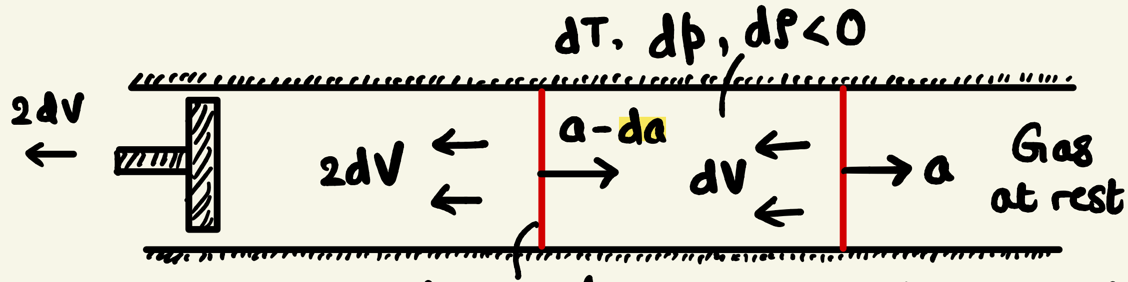
Lift per unit span $= (p_3 - p_2) \cdot c \cdot 1 \cdot \cos\alpha$

$= 6.23$ kN/m

Drag per unit span $= (p_3 - p_2) \cdot c \cdot 1 \cdot \sin\alpha$

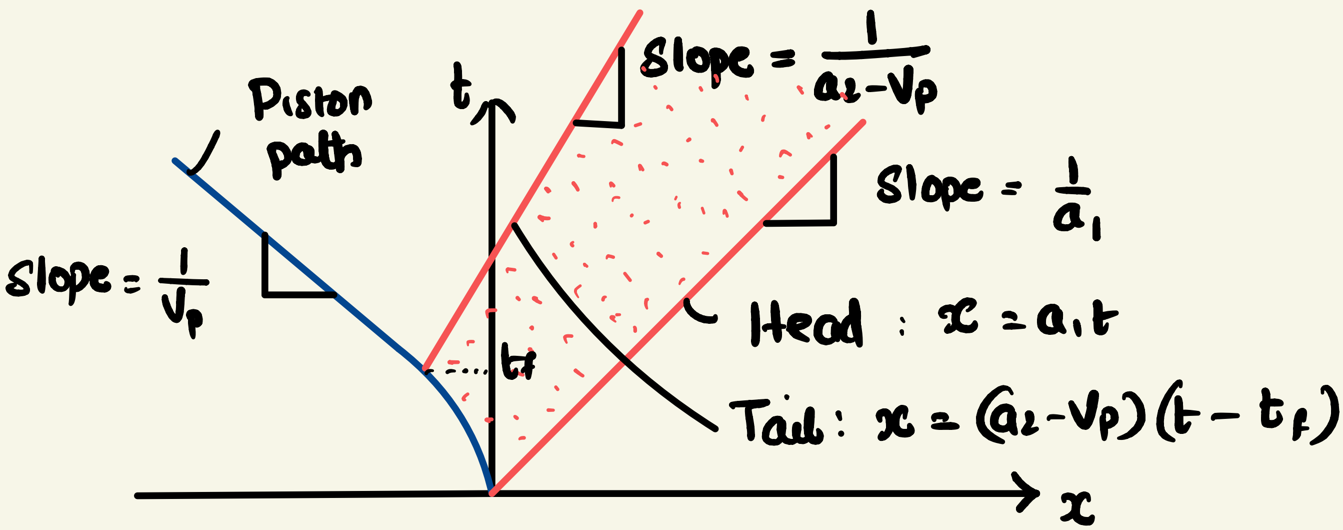
$= 0.33$ kN/m

Unsteady expansion waves

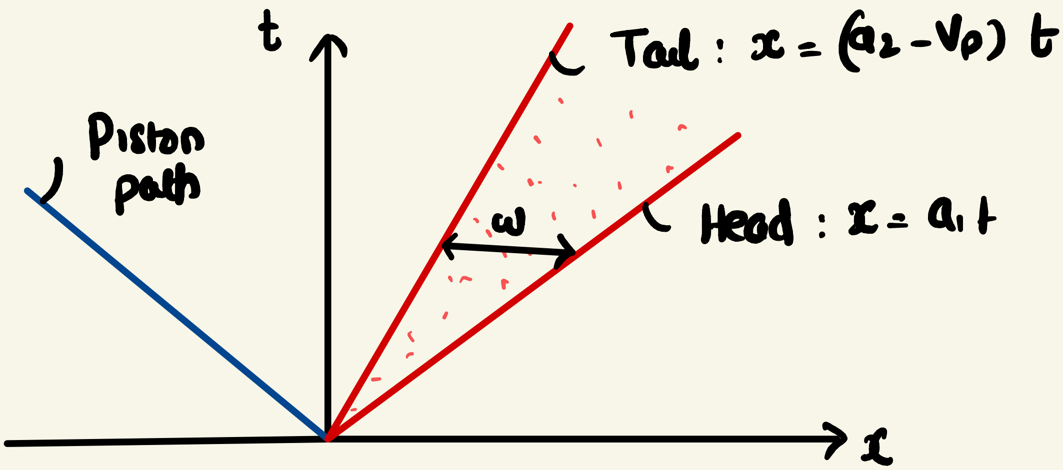


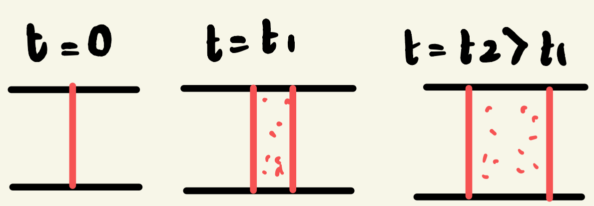
Wave velocity w.r.t. duct : $a - da - dV$
" " " flow : $a - da$

If piston velocity accelerated from 0 to V_p



If acceleration were immediate from 0 to V_p

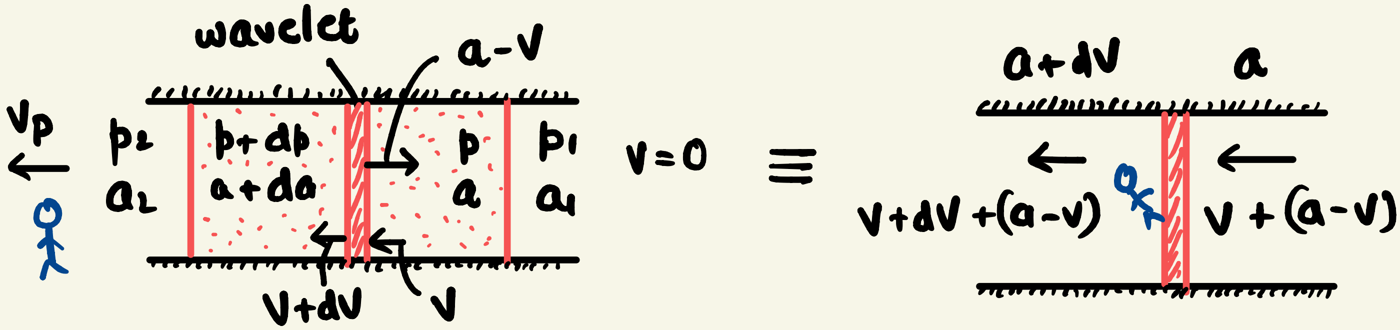




Width of the expansion wave keeps increasing with time

$$w = [a_1 - (a_2 - V_p)]\, t$$

Consider the flow relative to a wavelet of the expansion wave



$\boxed{\text{Momentum}}$

$$p-(p+dp) = \rho a\left[(a+dV)-a\right]$$

$$-dp = \rho a\, dV$$

Since flow is isentropic $\dfrac{dp}{d\rho} = a^2$

$$\Rightarrow -\frac{d\rho}{\rho} = \frac{dV}{a}$$

Integrating across the wave

$$\int_0^{V_p} dV = -\int_{\rho_1}^{\rho_2} a\,\frac{d\rho}{\rho}$$

$$\boxed{\frac{a}{a_1} = \left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}}}$$

$$V_p = -\int_{\rho_1}^{\rho} a_1\left(\frac{\rho}{\rho_1}\right)^{\frac{\gamma-1}{2}}\frac{d\rho}{\rho}$$

$$= -\left[\left(\frac{2}{\gamma-1}\right)\frac{a_1}{\rho_1^{\frac{\gamma-1}{2}}}\left(\rho_2^{\frac{\gamma-1}{2}} - \rho_1^{\frac{\gamma-1}{2}}\right)\right]$$

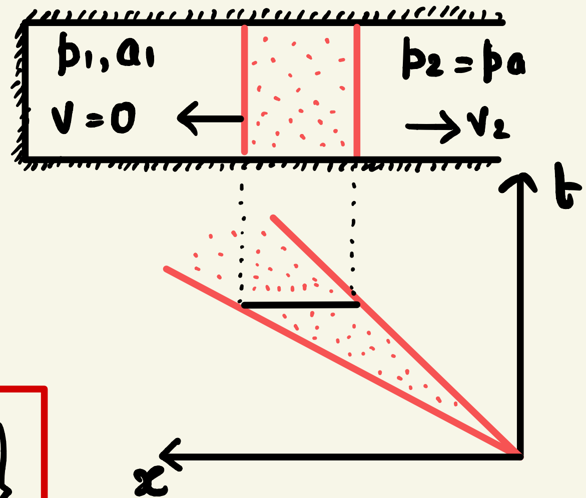
$$= -\left(\frac{2}{\gamma-1}\right)a_1\left[\left(\frac{a_2}{a_1}\right)-1\right]$$

$$V_p = \frac{2a_1}{\gamma-1} - \frac{2a_2}{\gamma-1}$$

$$\boxed{\frac{a_2}{a_1} = 1 - \left(\frac{\gamma-1}{2}\right)\frac{V_p}{a_1}} \qquad \boxed{\frac{p_2}{p_1} = \left(\frac{a_2}{a_1}\right)^{\frac{2\gamma}{\gamma-1}}} \qquad \boxed{\frac{\rho_2}{\rho_1} = \left(\frac{a_2}{a_1}\right)^{\frac{2}{\gamma-1}}}$$

Practical scenarios

① Tube sealed at both ends, contains gas at $p_1 > p_a$, one end suddenly opened



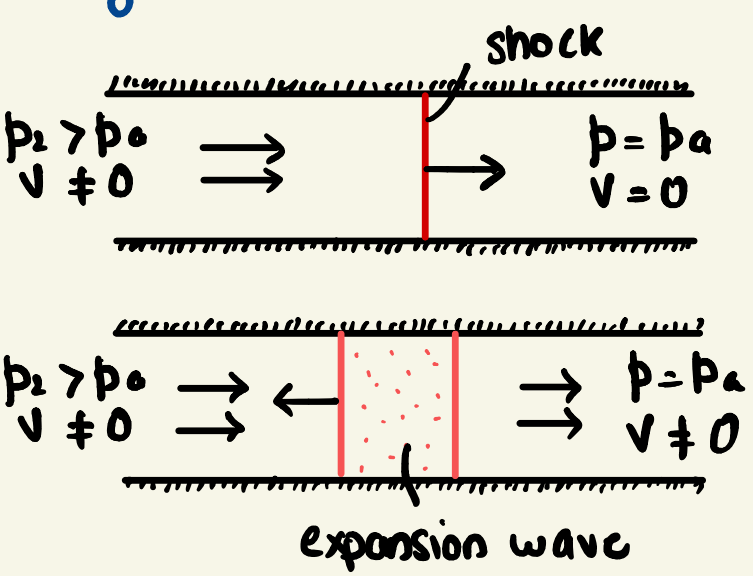
$$\frac{p_a}{p_1} = 1 - \left(\frac{\gamma-1}{2}\right)\frac{V_2}{a_4}$$

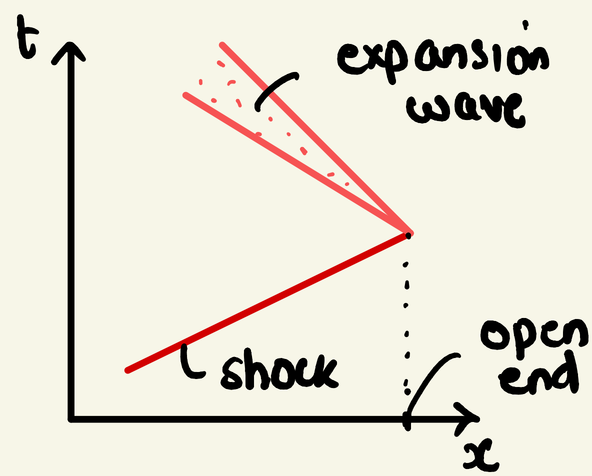
$$\boxed{V_2 = \left(\frac{2a_1}{\gamma-1}\right)\left\{1 - \left(\frac{p_a}{p_1}\right)^{\frac{\gamma-1}{2\gamma}}\right\}}$$

Maximum velocity that can be generated by an unsteady expansion wave propagating into a gas at rest

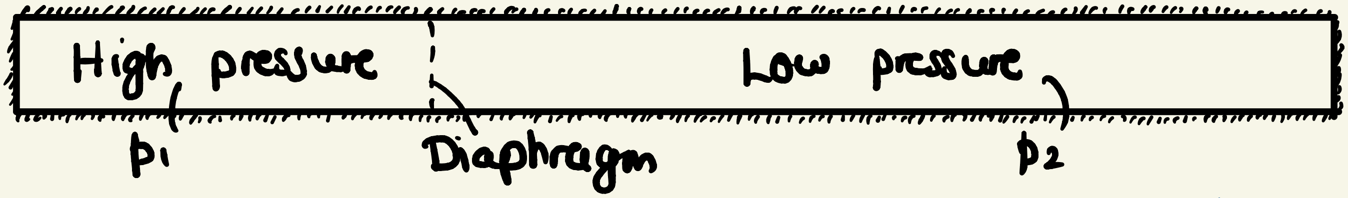
$$\boxed{V_{2,max} = \frac{2a_4}{\gamma-1}}$$

② Moving shock reaching the end of an open duct generates an expansion wave

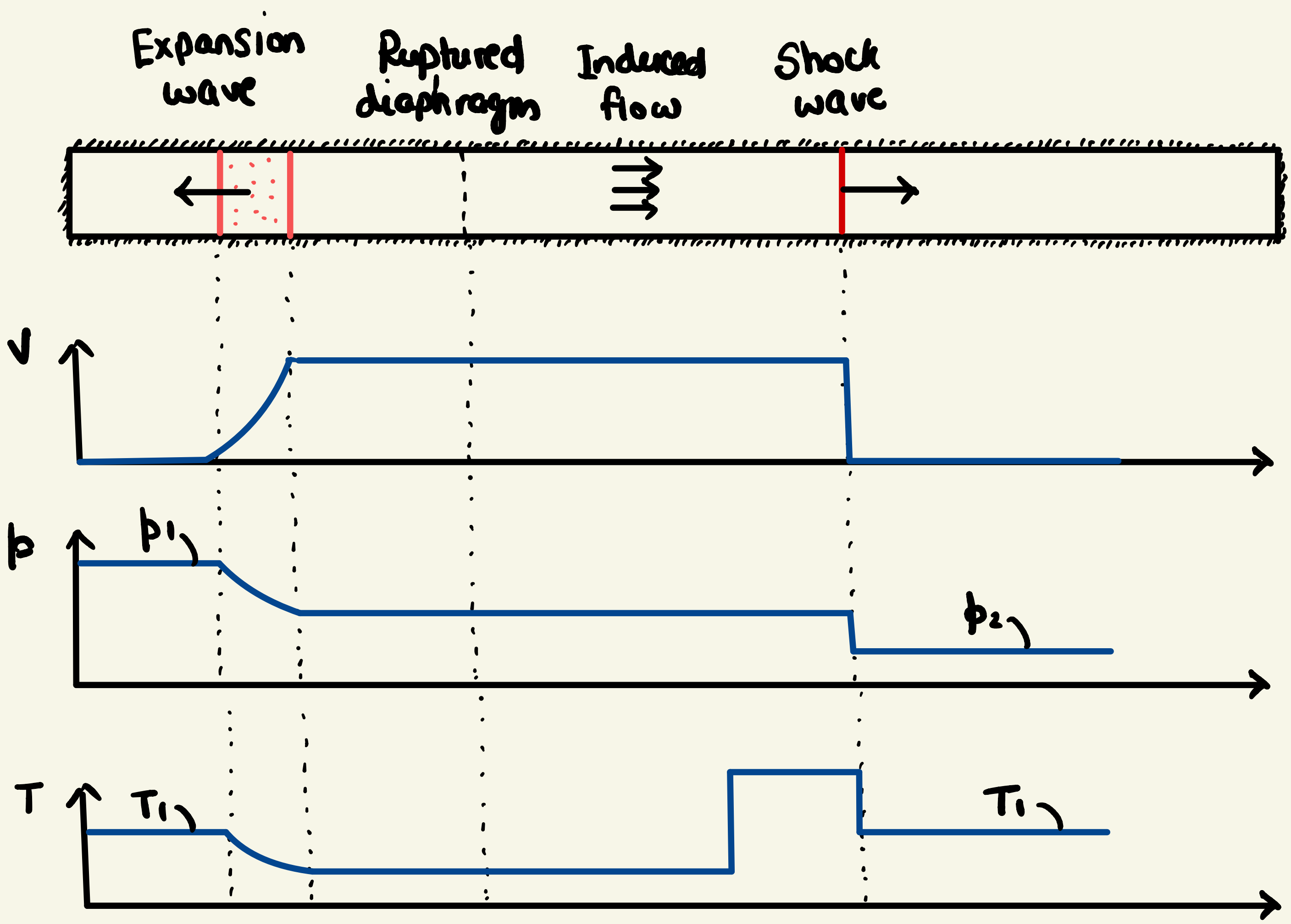




③ Shock tube - A long tube of constant area divided into two sections: high pressure gas on one side and low pressure gas on the other



when diaphragm breaks, a shock propogates into low-pressure section and an expansion wave propogates into high-pressure section generating a region of uniform velocity between them



Induced velocity is determined by noting that the velocity and pressure behind the shock wave must be equal to the velocity and pressure behind the expansion wave

Problems

(6) Pipe 4m long, $p = 200 kPa$, $T = 30°C$ ruptures at end. Find velocity of air if $p_a = 103 kPa$. Also find speed of tail of expansion wave and time taken for the head of the wave to reach pipe end.

$$V_p = \left(\frac{2a_1}{\gamma - 1}\right)\left[1 - \left(\frac{p_a}{p_1}\right)^{\frac{\gamma-1}{2\gamma}}\right]$$

$$a_1 = \sqrt{\gamma R T} = \sqrt{1.4 \cdot 287 \cdot 303} = 348.9 \text{ m/s}$$

$$\Rightarrow V_p = 157.8 \text{ m/s}$$

$$\frac{a_2}{a_1} = 1 - \left(\frac{\gamma - 1}{2}\right)\frac{V_2}{a_1} = 0.91$$

$$\Rightarrow a_2 = 317.5 \text{ m/s}$$

Speed of the tail $= a_2 - V_2 = 159.7$ m/s

Time taken for head to reach pipe end

$$t = \frac{4}{a_1} = 11.5 \text{ ms}$$