MA 214: Introduction to numerical analysis (2021–2022)

Re-examination Quiz 1

(1) [2 MARKS] Let $f(x) = x^3 + \sqrt{2}x^2 + \sqrt{3}x + 5$ then compute $f(e)$ using the 4-digit chopping method.

Answer: $\mathbf{40.21}$ or $\mathbf{40.22}$.

(2) [2 MARKS] Compute the solution to $x^3 - 2x^2 - 5 = 0$ in the interval $[1, 4]$ using the Newton-Raphson method with $p_0 = 2.5$.
Answer: $\mathbf{2.6906}$.

(3) [2 MARKS] The solution to $x^3 + 3x^2 - 1 = 0$ using the regula falsi method with the end-points of $[-3, -2]$ as the initial points is
Answer: $-\mathbf{2.8793}$.

(4) [2 MARKS] Let $P(x)$ be the interpolating polynomial of the smallest possible degree for the following data:

x	2	3	5
$P(x)$	2	8	25

then $P(x) = \dfrac{5x^2}{6} + \dfrac{11x}{6} - 5.$

(5) [2 MARKS] An interpolating polynomial of degree 3 interpolating the above data is

$$-\frac{x^3}{6} + \frac{5x^2}{2} - \frac{10x}{3}, \qquad \frac{11x^3}{6} - \frac{35x^2}{2} + \frac{176x}{3} - 60, \qquad x^3 - \frac{55x^2}{6} + \frac{197x}{6} - 35,$$

$$\frac{x^3}{12} + \frac{53x}{12} - \frac{15}{2} \qquad \text{or} \qquad 2 + 6(x-2) + \frac{5}{6}(x-2)(x-3) + (x-2)(x-3)(x-5).$$

Required formulae:

- Newton-Raphson iteration: $p_{n+1} = p_n - \dfrac{f(p_n)}{f'(p_n)}$.
- Regula-falsi iteration: $p_{n+1} = p_n - \dfrac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$.
- The **expected accuracy** for both the above methods is $|p_n - p_{n-1}| < 10^{-4}$.
- Lagrange interpolating polynomial: $P(x) = \sum f(x_k) L_k(x)$ where $L_k(x) = \prod_{i \neq k} \dfrac{(x - x_i)}{(x_k - x_i)}$.

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Re-examination Mid-sem

The answers should be accurate to first 7 digits after the decimal point.

(1) Consider the following fixed point iterations:

(a) $f_1(x) = x - x^3 - 4x^2 + 10$, (b) $f_2(x) = \sqrt{\left(\frac{10}{x} - 4x\right)}$,

(c) $f_3(x) = \frac{1}{2}(10 - x^3)$.

If x_1, x_2, x_3 are the respective fixed points of the above iterations then which of the x_i satisfy $x^3 + 4x^2 = 10$? [$1\frac{1}{2}$ marks]

Find the first 4 terms of the iteration f_1 starting with $x = 1.5$. [1 mark]

Find the first 4 terms of the iteration f_2 starting with $x = 1.5$. [1 mark]

Find the first 5 terms of the iteration f_3 starting with $x = 1.5$. [$2\frac{1}{2}$ marks]

(2) Consider the following data of a function f:

x	0	1	2	3
$f(x)$	8	10	6	2

Find appropriate polynomials of degrees 1, 2 and 3 to approximate the value $f(1.5)$. [1 + 1 + 1 + 3 marks]

(3) Construct the data for $x = 2, 3, 4$, $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$ and f'. [1 mark]

After constructing the correct data, find the Hermite polynomial of the smallest possible degree approximating it. [5 marks]

(4) Compute the integral $\int_{0.5}^{1} x^4 dx$ using the trapezoidal and the Simpson's $\frac{1}{3}$-rd rules. Find the absolute and the relative errors. [2 + 2 + 1 + 1 marks]

(5) For the following formula

$$\int_0^1 f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

find the degree of accuracy and find c_1, c_2, x_1, x_2. [6 marks]

Required formulae:

- Lagrange interpolating polynomial: $P(x) = \sum f(x_k) L_k(x)$ where $L_k(x) = \prod_{i \neq k} \frac{(x - x_i)}{(x_k - x_i)}$.

- Hermite polynomial: $H(x) = \sum f(x_k)H_k(x) + \sum f'(x_k)\hat{H}_k(x)$ with

  $H_k(x) = \big[1 - 2(x - x_k)L_k'(x_k)\big]L_k^2(x)$ and $\hat{H}_k(x) = (x - x_k)L_k^2(x)$.

- Trapezoidal rule: $\displaystyle\int_a^b f(x)dx \approx \frac{h}{2}\big[f(x_0) + f(x_1)\big]$.

- Simpson's $\frac{1}{3}$-rd rule: $\displaystyle\int_a^b f(x)dx \approx \frac{h}{3}\big[f(x_0) + 4f(x_1) + f(x_2)\big]$.

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Re-examination Quiz 2

- Write only the final answer in the specified place.
- The answers are expected to be correct up to first 7 digits after the decimal point.

(1) Using the composite Simpson's $\frac{1}{3}$-rd rule with $n = 4$

$\int_1^2 x \ln x dx \approx \mathbf{0.6363098}.$

(2) If the formula $\int_{-1}^{1} f(x)dx = af(-1) + bf\left(\frac{1}{2}\right) + cf(1)$ has the best possible degree of accuracy then

$c = \frac{\mathbf{-1}}{\mathbf{3}}.$

(3) If $y(t)$ is the solution to the initial value problem: $y' = y - t^2 + 1$, $0 \leqslant t \leqslant 2$, $y(0) = 0.5$ then using Euler's method with $h = 0.5$:

$y(2) \approx \mathbf{4.4375}.$

(4) If $y(t)$ is the solution to the initial value problem: $y' = 1 + (t - y)^2$, $2 \leqslant t \leqslant 3$, $y(2) = 1$ then using Euler's method with $h = 0.2$:

$y(3) \approx \mathbf{2.5388295}$ or $\mathbf{2.5388296}$.

(5) From the system of equations

$$\begin{aligned} 4x_1 - \ x_2 + \ x_3 &= 8 \\ 2x_1 + 5x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 11 \end{aligned}$$

the max$\{x_1, x_2, x_3\}-$ min$\{x_1, x_2, x_3\} = \mathbf{5.7872340}$.

Required formulae, definitions and conventions:

- Composite Simpson's $\frac{1}{3}$-rd rule:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(x_0) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + f(x_n)\right].$$

- Euler's method: $w_{i+1} = w_i + hf(t_i, w_i)$.

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Tutorial 11

(1) Apply Jacobi and Gauß-Seidel methods to solve following systems:

$$\begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \qquad \begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 6 \end{pmatrix}.$$

(2) Compute the condition numbers of the following matrices:

$$\begin{pmatrix} 3.9 & 1.6 \\ 6.8 & 2.9 \end{pmatrix}, \qquad \begin{pmatrix} 1.003 & 58.09 \\ 5.550 & 321.8 \end{pmatrix}.$$

(3) Determine the Gerschgorin disks of the following matrices:

$$\begin{pmatrix} 4 & 1 & 1 \\ 0 & 2 & 1 \\ -2 & 0 & 9 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(4) Apply the power method starting with the vector $(1,1,1)^t$ to the following matrices to compute approximations to the corresponding dominant eigenvalues:

$$\begin{pmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{pmatrix}.$$