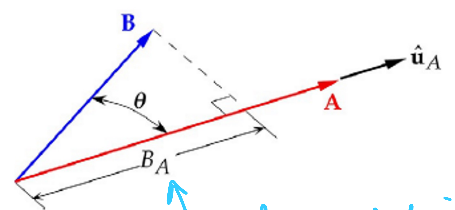
Lecture 2

- Recall that $A \cdot B = \|A\| \|B\| \cos\theta$
$$= A_x B_x + A_y B_y + A_z B_z$$

If $A = B$, then $\|A\| = \sqrt{A \cdot A}$

- The dot product can be used to project one vector onto the line of action of another:



scalar projection of B onto A

Now, $B_A = \|B\| \cos\theta$ and
$$B \cdot \hat{u}_A = \|B\| \cos\theta$$

So, $B_A = B \cdot \hat{u}_A = B \cdot \dfrac{A}{\|A\|} = \dfrac{A \cdot B}{\|A\|}$

Similarly, $A_B = \dfrac{A \cdot B}{\|B\|}$

- Note carefully, that $\vec{B}_A = B_A \hat{u}_A = \left(\dfrac{A \cdot B}{\|A\|^2}\right) A$

and similarly, $\vec{A}_B = \left(\dfrac{A \cdot B}{\|B\|^2}\right) B$

- Observe that if $\|A\| = \|B\|$, then $A_B = B_A$

Example

Let $\mathbf{A} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 18\hat{\mathbf{k}}$ and $\mathbf{B} = 42\hat{\mathbf{i}} - 69\hat{\mathbf{j}} + 98\hat{\mathbf{k}}$. Calculate
(a) the angle between $\mathbf{A}$ and $\mathbf{B}$;
(b) the projection of $\mathbf{B}$ in the direction of $\mathbf{A}$;
(c) the projection of $\mathbf{A}$ in the direction of $\mathbf{B}$.

Details

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(a) $\theta = \cos^{-1}\left(\frac{A \cdot B}{\|A\|\|B\|}\right)$

$A \cdot B = 1 \cdot 42 + 6 \cdot (-69) + 18 \cdot (98) = 1392$

$\|A\| = \sqrt{1^2 + 6^2 + 18^2} = 19$

$\|B\| = \sqrt{42^2 + (-69)^2 + 98^2} = 127$

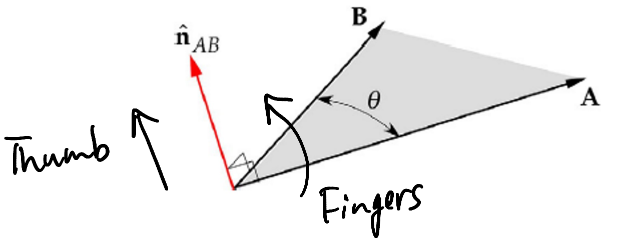
$\theta = 54.77^{0}$

(b) $B_A = \frac{A \cdot B}{\|A\|} = \frac{1392}{19} = 73.26$

(c) $A_B = \frac{A \cdot B}{\|B\|} = \frac{1392}{127} = 10.96$

- The cross product of two vectors is a vector defined as follows:

$$\boxed{A \times B = \|A\| \|B\| \sin\theta \, \hat{n}_{AB}}$$



$$A \times B = -(B \times A) \quad \text{(anti-commutative)}$$

- $\hat{i}, \hat{j}, \hat{k}$

  $\hat{i} \times \hat{i} = 0, \quad \hat{j} \times \hat{j} = 0, \quad \hat{k} \times \hat{k} = 0$

  $\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$

- Given two vectors A and B,

$$A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$
$$B = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\boxed{A \times B = (A_y B_z - A_z B_y)\hat{i} - (A_x B_z - A_z B_x)\hat{j} + (A_x B_y - A_y B_x)\hat{k}}$$

or, $$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- Note that $\hat{n}_{AB} = \dfrac{A \times B}{\|A \times B\|}$.

- Note that the area of the parallelogram spanned by A and B $= \|A \times B\|$.

Note that $A \times B = 0 \iff$ one vector

- Note that $A \times B = 0 \iff$ one vector is a scalar multiple of the other.

Example

Let $A = -3\hat{i} + 7\hat{j} + 9\hat{k}$ and $B = 6\hat{i} - 5\hat{j} + 8\hat{k}$.
Find a unit vector, which lies in the plane of A and B and is perpendicular to A.

Details

Compute $D = A \times B \Rightarrow A \perp D,\ B \perp D$.
Compute $C = D \times A \Rightarrow C \perp D,\ C \perp A$.

So, $A, B, C \perp D \Rightarrow A, B, C$ are coplanar

Hence, $\hat{u}_c = \dfrac{C}{\|C\|}$.

bac - cab Rule

$$\underbrace{A \times (B \times C)}_{\text{vector triple product}} = B(A\cdot C) - C(A\cdot B)$$

$$= (A\cdot C)B - (A\cdot B)C$$

$$= (A_xC_x + A_yC_y + A_zC_z)B - (A_xB_x + A_yB_y + A_zB_z)C$$

Proof

$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$B = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$C = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

x - component

$(A_xC_x + A_yC_y + A_zC_z)B_x$

$(A_xC_x + A_yC_y + A_zB_z)C_x$

$-(A_xB_x + A_yB_y + A_zB_z)$

$= (A\cdot C)B_x$

$C = C_x i + C_y j + \ldots$

Let us find the x-component of the RHS

$$= (A \cdot C) B_x - (A \cdot B) C_x$$

$$(A \cdot C) B_x - (A \cdot B) C_x$$

Let us now compute the x-component of the LHS using

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix}$$

x-component

$$A_y (B_x C_y - B_y C_x) - A_z (B_z C_x - B_x C_z)$$

$$= A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z \pm A_x B_x C_x$$

$$= A_y C_y B_x + A_z C_z B_x - A_y B_y C_x - A_z B_z C_x$$

$$= (A \cdot C) B_x - (A \cdot B) C_x$$

- Another useful identity

$$\underbrace{A \cdot (B \times C)}_{\text{scalar triple product}} = (A \times B) \cdot C$$

$\downarrow$

Geometric meaning?

Example

If $F = E\times\{D\times[A\times(B\times C)]\}$, then use the bac-cab rule to reduce this expression to one involving only dot products.

Details

$$F = E\times\{D\times[B(A\cdot C) - C(A\cdot B)]\}$$

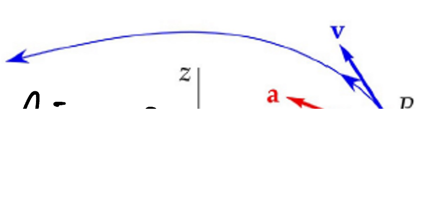
$$= (A\cdot C)[E\times(D\times B)] - (A\cdot B)[E\times(D\times C)]$$

$$= (A\cdot C)[D(E\cdot B) - B(E\cdot D)] - (A\cdot B)[D(E\cdot C) - C(E\cdot D)]$$

$$= D[(A\cdot C)(E\cdot B) - (A\cdot B)(E\cdot C)] - (A\cdot C)(E\cdot D)B + (A\cdot B)(E\cdot D)C$$

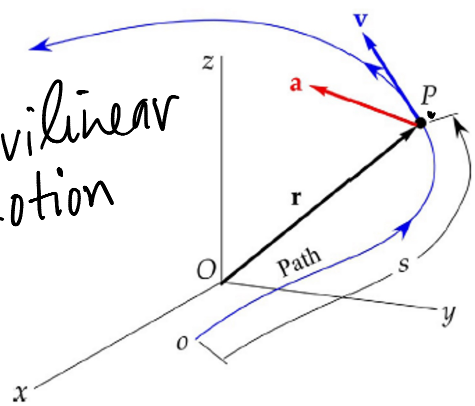
- To track the motion of a particle P, we need a clock and a non-rotating Cartesian coordinate system.

$\hat{u}_t$



$$V = \|V\|\,\hat{u}_t$$

curvilinear motion



$$V = \|V\| \, u_t$$

$$ds = \|V\| \, dt$$

$$\dot{s} = \|v\| = \|\dot{r}\|$$

$$r(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$\begin{aligned} V(t) &= \dot{r}(t) \\ &= \dot{x}(t)\,\hat{i} + \dot{y}(t)\,\hat{j} + \dot{z}(t)\,\hat{k} \\ &= V_x(t)\,\hat{i} + V_y(t)\,\hat{j} + V_z(t)\,\hat{k} \end{aligned}$$

$$a_x(t) = \frac{d}{dt} V_x(t), \ldots$$

$$\begin{aligned} a(t) &= \dot{v}(t) \\ &= a_x(t)\,\hat{i} + a_y(t)\,\hat{j} + a_z(t)\,\hat{k} \end{aligned}$$

– Note carefully, that $\|\dot{r}\| \neq \dot{\widehat{\|r\|}}$

<u>Example:</u> $r(t) = \hat{i} + t\hat{j} + t^2\hat{k}$ (m)

$$\dot{r}(t) = \hat{j} + 2t\,\hat{k}$$

$$\|\dot{r}(1)\| = \sqrt{1^2 + 2^2} = \sqrt{5} \ (\text{m/sec})$$

$$\dot{\widehat{\|r(t)\|}} = \frac{d}{dt}\sqrt{1^2 + t^2 + t^4}$$

$$\|r'(t)\| = \frac{d}{dt}\sqrt{1+t^2+t^4}$$

$$= \frac{1}{2\sqrt{1+t^2+t^4}} \cdot (2t+4t^3)$$

At time $t=1$ sec, the above expression is equal to $\sqrt{3}$ (m/sec).