Optimal Staging Problems

Problem No. 01

A **2-stage** sounding rocket has $\varepsilon_1 = \varepsilon_2 = \mathbf{0.15}$.

Determine optimal **π's** & m_0 for a $\mathbf{m_*}$ of 10 kg, if V_* required is **4000 m/s** while burning a propellant of $\mathbf{I_{sp} = 240s}$.

Solution No. 01

The **solution** is as follows.

$$\beta = \frac{V_*}{Ng_0 I_{sp}} = \frac{4000}{2\times 9.81\times 240} = 0.8494$$

$$\pi_1 = \pi_2 = \pi = \frac{e^{-\beta} - \varepsilon}{1-\varepsilon} = \frac{0.428 - 0.15}{0.85} = 0.3267$$

$$\pi_* = \left(\frac{e^{-\beta} - \varepsilon}{1-\varepsilon}\right)^N = 0.3267^2 = 0.1067; \quad m_0 = 93.7kg$$

Problem No. 02

Consider a rocket with $\varepsilon_1 = \varepsilon_2 = 0.15$.

Determine **optimal** burnout velocity, if the mission **payload** ratio is 0.15 for an I_{sp} of **240s.**

Solution No. 02

The **solution** is as follows.

$$\pi_1 = \pi_2 = \sqrt{\pi_*} = 0.387$$

$$\begin{aligned} V_* &= -g_0 I_{sp} N \ln\left[\varepsilon + (1-\varepsilon)\pi\right] \\ &= -9.81 \times 240 \times 2 \times \ln\left[0.15 + 0.85 \times 0.387\right] \\ &= -2354.4 \times 2 \times (-1.4482) = 3466.4 m/s \end{aligned}$$

Problem No. 03

Angara 1.2, is to be **redesigned** to have a payload fraction of **0.025**.

$$\text{1-Stage: } I_{sp1} = 310s; \quad \varepsilon_1 = 0.072$$

$$\text{2-Stage: } I_{sp2} = 342.5s; \quad \varepsilon_2 = 0.089$$

If fixed stage **parameters** are as follows, determine **new** stage-wise payload **ratios.**

Solution No. 03

The **solution** is as follows.

$$\boxed{\text{Old Parameters: } \pi_1 = 0.188; \quad \pi_2 = 0.124; \quad V_* = 9633.9 m/s}$$

$$\pi_1 = \frac{-0.0776\lambda}{(\lambda + 3041.1)}; \quad \pi_2 = \frac{-0.0977\lambda}{(\lambda + 3359.9)} \rightarrow 0.025 = \frac{-0.0776\lambda}{(\lambda + 3041.1)} \times \frac{-0.0977\lambda}{(\lambda + 3359.9)}$$

$$0.025 = \frac{0.00758\lambda^2}{\left(\lambda^2 + 6401\lambda + 1.02178\times 10^7\right)} \rightarrow 0.0174\lambda^2 + 160.02\lambda + 2.5544\times 10^5 = 0$$

$$\lambda_1, \lambda_2 = -2055.9, \quad -7140.7 \rightarrow \pi_1 = 0.162; \quad \pi_2 = 0.154; \quad \pi_* = 0.029$$

$$V_{*-\text{optim}} = 4491.5 + 3846.4 = 8337.8 m/s$$

Approximate Staging Problems

Problem No. 04

Consider a 2-**stage rocket** with $\varepsilon_1 = \varepsilon_2 = 0.15$.

Determine $\boldsymbol{\pi_i}$'s if V_* is **4000 m/s** and I_{sp} is **240s.**

Solution No. 04

The **solution** is as follows.

$$\ln\left[0.15+0.85\pi_1\right]; \qquad = -\frac{4000}{240\times 9.81} - \ln\left[0.15+0.85\pi_2\right]$$

$$\ln\left[(0.15+0.85\pi_2)(0.15+0.85\pi_1)\right] = -1.699$$

$$\pi_1 = \frac{0.2152}{(0.15+0.85\pi_2)} - 0.1765; \quad \ln\pi_* = \ln(\pi_1\times\pi_2)$$

$$\frac{\partial\pi_*}{\partial\pi_2} = 0; \quad \pi_* = \left\{\frac{0.2152}{(0.15+0.85\pi_2)} - 0.1765\right\}\times\pi_2$$

$$0.1275\pi_2^2 + 0.045\pi_2 - 0.0283 = 0; \quad \pi_2 = 0.327; \quad \pi_1 = 0.326$$

$$\pi_* = 0.107; \quad \text{Exact: } \pi_1 = \pi_2 = 0.327$$

Problem No. 05

Angara 1.2, is to be redesigned to have a **burnout** velocity of 8338 m/s. If fixed stage **parameters** are as follows, determine approximate **stage-wise** payload ratios.

1-Stage: $I_{sp1} = 310s;$	$\varepsilon_1 = 0.072$
2-Stage: $I_{sp2} = 342.5s;$	$\varepsilon_2 = 0.089$

Solution No. 05

The **solution** is as follows.

$$\ln\left[0.072+0.928\pi_1\right] = -\frac{8338}{310\times 9.81} - \frac{342.5}{310}\times\ln\left[0.089+0.911\pi_2\right]$$

$$\ln\left[\left(0.089+0.911\pi_2\right)^{1.1048}\left(0.072+0.928\pi_1\right)\right] = -2.7418$$

$$\pi_1 = \frac{0.0694}{\left(0.089+0.911\pi_2\right)^{1.1048}} - 0.0776$$

$$\pi_* = \left\{\frac{0.0694}{\left(0.089+0.911\pi_2\right)^{1.1048}} - 0.0776\right\}\times\pi_2; \quad \pi_2 = 0.19; \quad \pi_1 = 0.22$$

Exact solution from Lagrange: $\pi_1 = 0.162; \quad \pi_2 = 0.154$

Option-2: π_1 from constraint: $\pi_1 = 0.173; \quad \pi_2 = 0.167$