AE 308: Control Theory
AE 775: System Modelling, Dynamics & Control

Lecture 14: Bode Plot

Dr. Arnab Maity
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Bode Plot - Introduction

Bode Plot

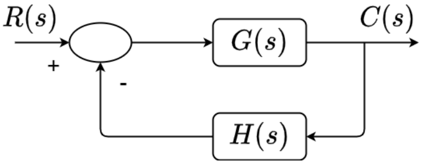
- In Bode plots or Bode diagrams, the log-magnitude and phase frequency response curves are as functions of $\log \omega$.

Features

1. Because the magnitude of $G(j\omega)$ in the Bode plot is expressed in dB, product and division factors in $G(j\omega)$ become additions and subtractions, respectively. The phase relations are also added and subtracted from each other algebraically.
2. The magnitude plot of the Bode plot of $G(j\omega)$ can be approximated by straight-line segments, which allow the simple sketching of the plot without detailed computation.

Bode Plot - Introduction

Closed loop control system



- Closed loop transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

- Thus characteristic equation of the closed loop system is

$$CE = 1 + G(s)H(s) = 0$$

- $G(s)H(s)$ is called as Loop Transfer Function.

Asymptotic Bode Plot - Introduction

Asymptotic Bode Plots

- Sketching Bode plots can be simplified, because they can be approximated as a sequence of straight lines.
- Straight-line approximation of the Bode plot is relatively easy to construct.
- The data necessary for the other frequency-domain plots, such as the polar plot and the magnitude-versus-phase plot, can be easily generated from the Bode plot.

Asymptotic Bode Plot - Introduction

Asymptotic Bode Plots (cont...)

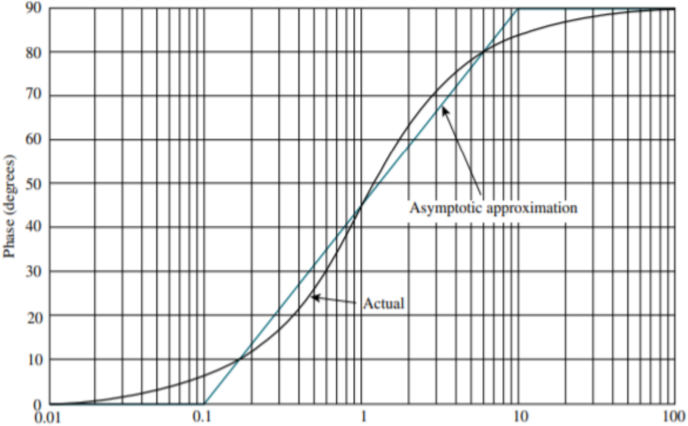


Figure: *Source - "Control Systems Engineering", by N. S. Nise*

Bode Plot - Example

Example 1

- Loop transfer function is

$$G(s)H(s) = \frac{(s+20)}{(s+2)(s+400)}$$

- Bring in standard form

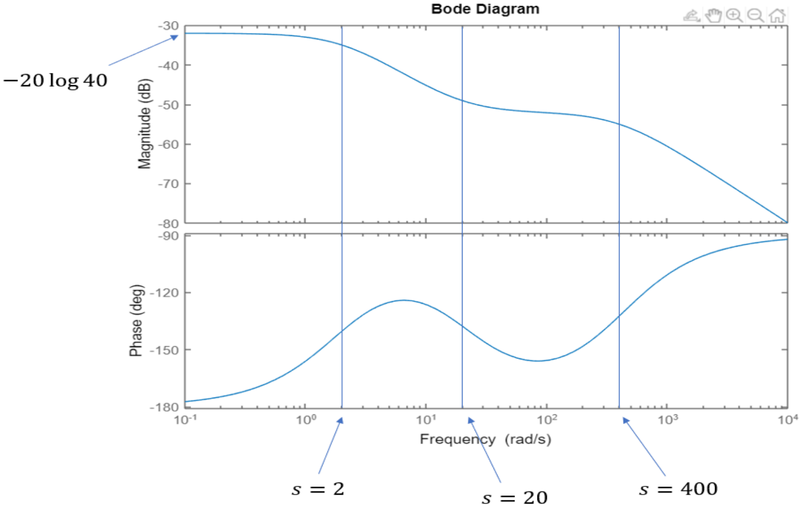
$$G(s)H(s) = \frac{\left(\frac{s}{20}+1\right)}{40\left(\frac{s}{2}+1\right)\left(\frac{s}{400}+1\right)}$$

- Here,

$$k = \frac{1}{40}$$

Bode Plot - Example

Example 1 (cont...)



Asymptotic Bode Plot - Analysis

Features

- All bode plots show specific pattern for two limiting frequency points - 0 and ∞.
- This is typically in the form of low and high frequency asymptotes, which provide DC gain $G(0)$ and relative degree (i.e. $n - m$) of $G(s)$.
- We also find that changes in the asymptote angles occur around frequencies that correspond to poles and zeros.

Asymptotic Bode Plot - Analysis

Features (cont...)

- Poles and zeros are seen as points where slope of magnitude plot changes and termed as **corner frequencies**.
- $G(0)$ and k are seen as intercept, which is $20 \log k$, and slope, which is $-20L$ dB/decade for $\omega = 0$ where $G(s) = k/s^L$.
- $(n - m)$ is seen as slope of curve for $\omega \to \infty$, which is $-20(n - m)$ dB/decade.

Asymptotic Bode Plot - Example

Question

- Draw a bode plot for transfer function with unity feedback

$$G(s) = 100\frac{(s+1)}{(s+10)(s+100)}$$

Step 1: Rewrite the transfer function in proper form

$$G(s)H(s) = \frac{100}{10 \times 100}\frac{(s+1)}{\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$$

Asymptotic Bode Plot - Example

Step 2: Separate the transfer function into its constituent parts

- A constant of 0.1
- A pole at $s = -10$
- A pole at $s = -100$
- A zero at $s = -1$

Asymptotic Bode Plot - Example

Step 3: Draw the Bode diagram for each part

- The constant is the cyan line (a quantity of 0.1 is equal to -20 dB). The phase is constant at 0 deg.
- Pole at 10 rad/sec is the green line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. Phase is 0 deg up to $1/10$ times of the break frequency, i.e. 1 rad/sec, then drops linearly down to -90 deg at 10 times of the break frequency, i.e. 100 rad/sec.
- Pole at 100 rad/sec is the blue line. It is 0 dB up to the break frequency, then drops off with a slope of -20 dB/dec. Phase is 0 deg up to $1/10$ times of the break frequency, i.e. 10 rad/sec, then drops linearly down to -90 deg at 10 times of the break frequency, i.e. 1000 rad/sec.
- Zero at 1 rad/sec is the red line. It is 0 dB up to the break frequency, then rises at 20 dB/dec. Phase is 0 deg up to $1/10$ times of the break frequency, i.e. 0.1 rad/sec, then rises linearly to 90 deg at 10 times of the break frequency, i.e. 10 rad/sec.

Asymptotic Bode Plot - Example

Step 4: Add the results from step 3

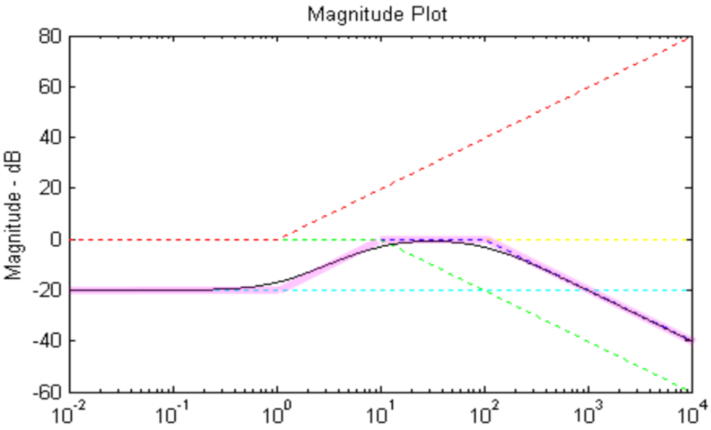


Figure: *Source - "https://lpsa.swarthmore.edu/Bode/BodeExamples.html"*

Asymptotic Bode Plot - Example

Step 4: Add the results from step 3

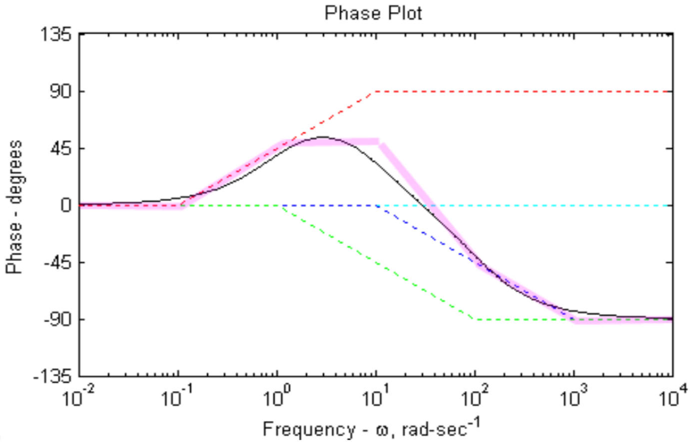


Figure: *Source - "https://lpsa.swarthmore.edu/Bode/BodeExamples.html"*

Bode Plot - Drawbacks

Drawbacks

- Bode plot consists of two graphics, which need to be interpreted together.
- However, in some cases, there is a need to see complete frequency response in a single graphic and Nyquist plot addresses this need.

Concept of Crossover

Gain crossover (GCO) point

- The gain-crossover point on the frequency-domain plot of $G(j\omega)$ is the point at which $|G(j\omega)| = 1$ or $|G(j\omega)|_{dB} = 0$.
- The frequency at the gain-crossover point is called the gain-crossover frequency ω_{GCO}.

Phase crossover (PCO) point

- The phase-crossover point on the frequency-domain plot of $G(j\omega)$ is the point at which $\angle G(j\omega) = -180^0$.
- The frequency at the phase-crossover point is called the phase crossover frequency ω_{PCO}.

Bode Plot - Crossover Points

Gain and Phase crossover points

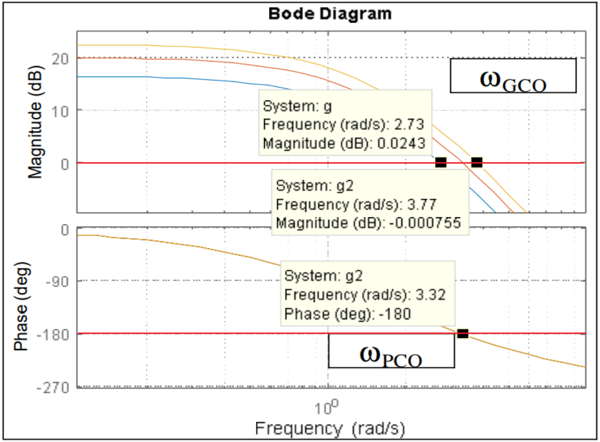


Figure: *Source - "System Modeling Dynamics and Control, Lecture Notes", by Prof. Ashok Joshi*

Crossover - Features

Features

- ω_{GCO} and ω_{PCO} contain stability information as much as that these are related to the corresponding location of closed loop poles in s-plane.
- Therefore, magnitude and phase at PCO and GCO are used as quantitative measures of relative stability of the closed loop system, also called **margins**.

Stability Margins - Introduction

Gain Margin

- Gain Margin (GM) is the amount of gain that can be added to plant, before unity feedback closed loop system becomes unstable.
- GM is the reciprocal of the magnitude at ω_{PCO} and is given by

$$\frac{1}{|G(j\omega_{PCO})|}$$

- Gain margin is considered positive, if $|G(j\omega_{PCO})| < 1$.

Stability Margins - Bode Plot

Gain Margin

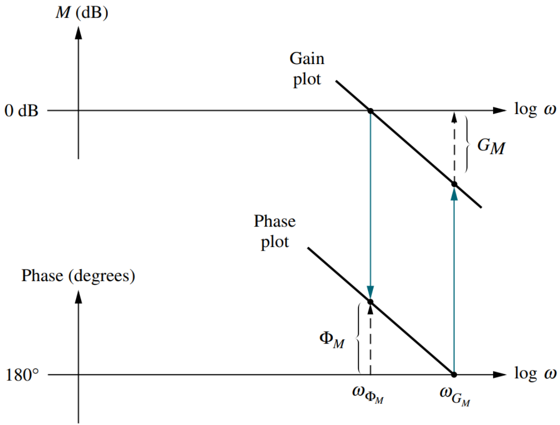


Figure: *Source - "Control System Engineering", by Norman S. Nise*

Stability Margins - Introduction

Phase Margin

- Similarly, phase margin (PM) is the amount of phase lag (or negative angle) that can be added to the plant before the closed loop system becomes unstable.
- As negative angle is measured clockwise from positive real axis, PM is defined as $180^0 + \angle G(j\omega_{GCO})$.
- PM is treated as positive if negative angle is $< 180^0$.

Stability Margins - Bode Plot

Phase Margin

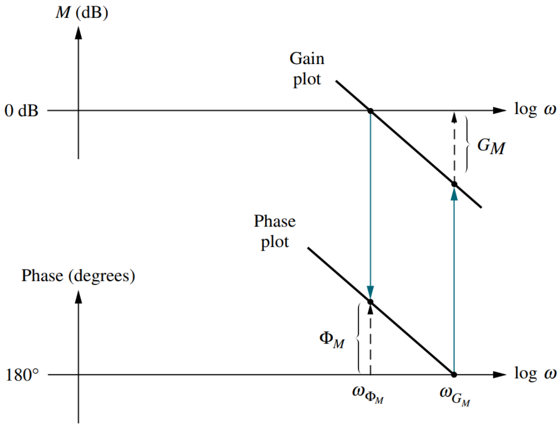


Figure: *Source - "Control System Engineering", by Norman S. Nise*

Stability Margins - Example

Example 1

- Determine **GCO**, **PCO**, **GM** and **PM** for following plant and predict the stability of closed loop system. Also, correlate with closed loop pole locations

$$G(s) = \frac{10}{s(s+1)(s+5)}$$

Stability Margins - Example

Example 1

- Determine **GCO**, **PCO**, **GM** and **PM** for following plant and predict the stability of closed loop system. Also, correlate with closed loop pole locations

$$G(s) = \frac{10}{s(s+1)(s+5)}$$

Solution

- Let find GCO

$$|G(j\omega)| = \left| \frac{10}{j\omega(j\omega+1)(j\omega+5)} \right| = \frac{10}{\omega\sqrt{\omega^2+1}\sqrt{\omega^2+25}}$$

$$\frac{10}{\omega_{GCO}\sqrt{\omega_{GCO}^2+1}\sqrt{\omega_{GCO}^2+25}} = 1$$

Stability Margins - Example

Example 1 (cont...)

- Simplifying further

$$\omega_{GCO}^6 + 26\omega_{GCO}^4 + 25\omega_{GCO}^2 - 100 = 0$$

$$\omega_{GCO}^2 = 1.506 \rightarrow \boldsymbol{\omega_{GCO} = 1.227}$$

- Let find PCO

$$\angle G(j\omega) = -90^0 - \tan^{-1}(\omega) - \tan^{-1}(0.2\omega)$$

$$-90^0 - \tan^{-1}(\omega_{PCO}) - \tan^{-1}(0.2\omega_{PCO}) = -180^0$$

$$\boldsymbol{\omega_{PCO} = 2.25}$$

Stability Margins - Example

Example 1 (cont...)

- Let find GM

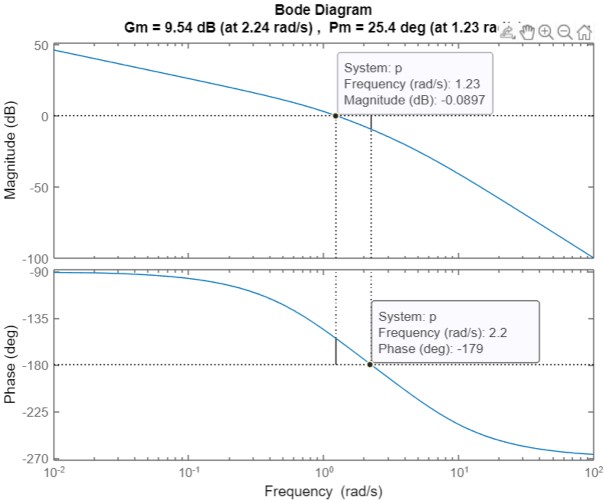
$$
\begin{aligned}
GM &= \left|\frac{1}{G(j\omega_{PCO})}\right| \\
&= \left|\frac{1}{G(2.25j)}\right| = 3.03 = \mathbf{9.63\ dB}
\end{aligned}
$$

- Let find PM

$$
\begin{aligned}
PM &= 180^0 + \angle G(j\omega_{GCO}) \\
&= 180^0 + \angle G(1.227j) = \mathbf{25.4^0}
\end{aligned}
$$

Stability Margins - Example

Example 1 - Verification



Infinite Stability Margins - Introduction

Infinite GM

- There are situations where the phase plot either does not cross 180^0, or phase cross over occurs at $\omega = \infty$.
- In such cases, the GM becomes undefined and is commonly interpreted to be infinite.
- Implication of such a situation is that no amount of increase in gain will destabilize the closed loop system.

Infinite PM

- There are situations where the magnitude plot either does not cross 0 dB, or gain cross over occurs at $\omega = \infty$.

Infinite Stability Margins - GM

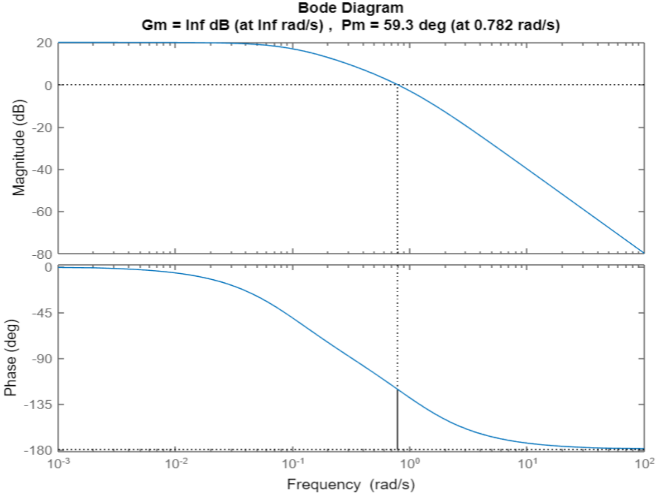
Infinite GM

- Consider a system

$$G(s) = \frac{1}{(s+0.1)(s+1)}$$

Infinite Stability Margins - GM

Infinite GM



Infinite Stability Margins - PM

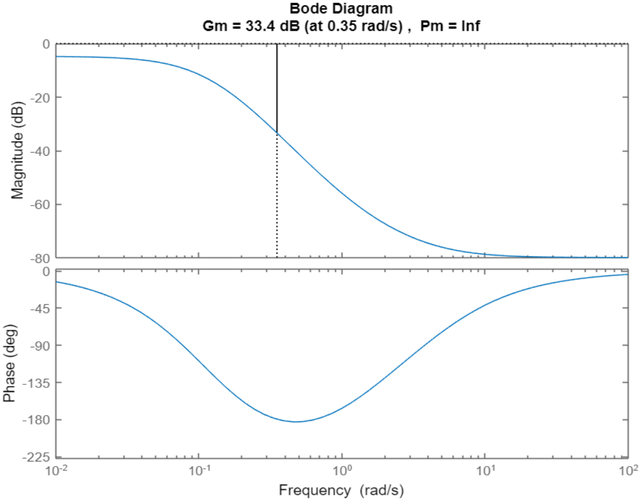
Infinite PM

- Consider a system

$$G(s) = \frac{0.0001(s+1)(s+2)(s+5)}{(s+0.12)^3}$$

Infinite Stability Margins - PM

Infinite PM



Infinite Stability Margins - GM and PM

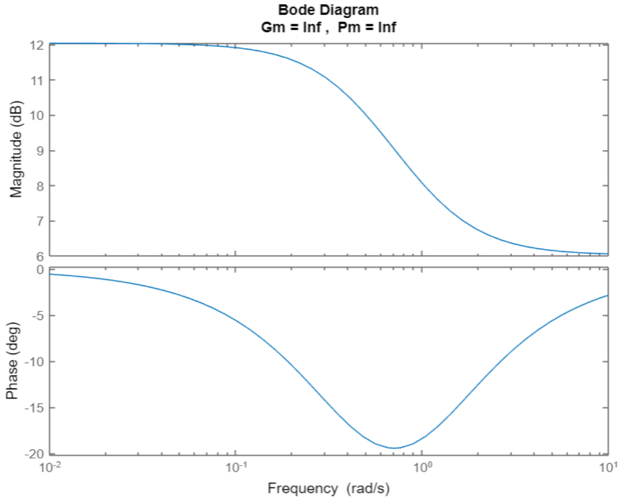
Infinite GM and PM

- Consider a system with unity feedback

$$G(s) = \frac{2(s+1)}{(s+0.5)}$$

Infinite Stability Margins - GM and PM

Infinite GM and PM



Non Minimum Phase System - GM and PM

Non minimum phase - GM and PM

- Phase characteristics of non-minimum phase systems are significantly different and hence it is expected that both GM and PM would get affected.
- Let, consider a system

$$G(s) = \frac{2(s+1)e^{-s}}{(s+0.5)}$$

- We can employ 1^{st} order Pade's approximation to get the rational transfer function as

$$G(s) = \frac{2(s+1)(1-0.5s)}{(s+0.5)(1+0.5s)}$$

Non Minimum Phase System - GM and PM

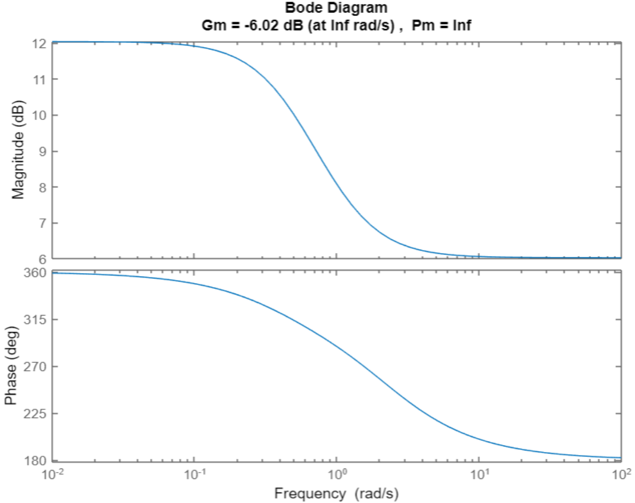
Non minimum phase - GM and PM (cont...)

- After solving for the GCO, PCO, GM and PM, we get

$$\begin{aligned}\omega_{PCO} &= \infty \\ \omega_{GCO}^2 &= -0.083 \rightarrow \omega_{GCO} = \text{No} \\ GM &= -6.02\text{dB}\end{aligned}$$

Non Minimum Phase System - GM and PM

Non minimum phase - GM and PM (cont...)



Summary

- Bode plot consists of two graphics, which need to be interpreted together.
- Magnitude plot changes at poles and zeros and these frequencies are called as corner frequency.
- Infinite gain and phase margins provide greater design freedom.
- Non-minimum phase behaviour has a significant impact on the stability margins.

References I

- Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini: "*Feedback Control of Dynamic Systems*", Pearson Education, Inc., Upper Saddle River, New Jersey, Seventh Edition, 2015.
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- Ashok Joshi: "*System Modeling Dynamics and Control*", Lecture Notes, IIT Bombay, Mumbai, 2019.