Solution 1

The strain energy is given as

$$V = \frac{1}{2} \int_0^l \left[EI \left(\frac{\partial \psi}{\partial x} \right)^2 + kAG \left(\frac{\partial w}{\partial x} - \psi \right)^2 \right] dx$$

The kinetic energy of beam is given as

$$T = \frac{1}{2} \int_0^l \left[\rho A \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial \psi}{\partial t} \right)^2 \right] dx$$

The work done by the external load is 0

Application of Hamilton's principle gives

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

$$\int_{t_1}^{t_2} \Bigg\{ \int_0^l \Bigg[\underbrace{EI \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right)}_{\text{Term 1}} + \underbrace{kAG \left(\frac{\partial w}{\partial x} - \psi \right) \delta \left(\frac{\partial w}{\partial x} \right)}_{\text{Term 2}} - kAG \left(\frac{\partial w}{\partial x} - \psi \right) \delta \psi \Bigg] dx$$
$$- \int_0^l \Bigg[\underbrace{\rho A \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right)}_{\text{Term 3}} + \underbrace{\rho I \frac{\partial \psi}{\partial t} \delta \left(\frac{\partial \psi}{\partial t} \right)}_{\text{Term 4}} \Bigg] dx \Bigg\} dt = 0$$

4 Integration by parts

$$\int_{t_1}^{t_2}\int_0^{\ell} EI\frac{\partial\psi}{\partial x}\,\delta\left(\frac{\partial\psi}{\partial x}\right)dx\,dt$$

Term 1

$$=\int_{t_1}^{t_2}\left[EI\frac{\partial\psi}{\partial x}\,\delta\psi\Big|_0^{\ell}-\int_0^{\ell}\frac{\partial}{\partial x}\left(EI\frac{\partial\psi}{\partial x}\right)\delta\psi\,dx\right]dt$$

$$\int_{t_1}^{t_2}\int_0^{\ell} kAG\left(\frac{\partial w}{\partial x}-\psi\right)\delta\left(\frac{\partial w}{\partial x}\right)dx\,dt$$

Term 2

$$=\int_{t_1}^{t_2}\left[kAG\left(\frac{\partial w}{\partial x}-\psi\right)\delta w\Big|_0^{\ell}-\int_0^{\ell}kAG\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}-\psi\right)\delta w\,dx\right]dt$$

$$-\int_{t_1}^{t_2}\int_0^{\ell}\rho A\frac{\partial w}{\partial t}\,\delta\left(\frac{\partial w}{\partial t}\right)dx\,dt$$

$$=\int_{t_1}^{t_2}\int_0^{\ell}\rho A\frac{\partial^2 w}{\partial t^2}\,\delta w\,dx\,dt-\cancel{\int_0^{\ell}\left[\rho A\frac{\partial w}{\partial t}\delta w\right]_{t_1}^{t_2}dx}$$

$$-\int_{t_1}^{t_2}\int_0^{\ell}\rho I\frac{\partial\psi}{\partial t}\,\delta\left(\frac{\partial\psi}{\partial t}\right)dx\,dt$$

$$=\int_{t_1}^{t_2}\int_0^{\ell}\rho I\frac{\partial^2\psi}{\partial t^2}\,\delta\psi\,dx\,dt-\cancel{\int_0^{\ell}\left[\rho I\frac{\partial\psi}{\partial t}\delta\psi\right]_{t_1}^{t_2}dx}$$

Substituting of above equations leads to

$$\int_{t_1}^{t_2} \left\{ kAG \left(\frac{\partial w}{\partial x} - \psi \right) \delta w \Big|_0^l + EI \frac{\partial \psi}{\partial x} \delta \psi \Big|_0^l \right.$$

$$+ \int_0^l \left[-\frac{\partial}{\partial x} \left(kAG \left(\frac{\partial w}{\partial x} - \psi \right) \right) + \rho A \frac{\partial^2 w}{\partial t^2} \right] \delta w \; dx$$

$$\left. + \int_0^l \left[-\frac{\partial}{\partial x} \left(EI \frac{\partial \psi}{\partial x} \right) - kAG \left(\frac{\partial w}{\partial x} - \psi \right) + \rho I \frac{\partial^2 \psi}{\partial t^2} \right] \delta \psi \, dx \right\} dt = 0$$

Governing Equation of motion

$$\boxed{\begin{aligned} -\frac{\partial}{\partial x} \left[kAG \left(\frac{\partial w}{\partial x} - \psi \right) \right] + \rho A \frac{\partial^2 w}{\partial t^2} &= 0 \\ -\frac{\partial}{\partial x} \left(EI \frac{\partial \psi}{\partial x} \right) - kAG \left(\frac{\partial w}{\partial x} - \psi \right) + \rho I \frac{\partial^2 \psi}{\partial t^2} &= 0 \end{aligned}}$$

Boundary conditions

$$kAG \left(\frac{\partial w}{\partial x} - \psi \right) \delta w \Big|_0^l = 0$$

$$EI \frac{\partial \psi}{\partial x} \delta \psi \Big|_0^l = 0$$

Simply supported Beam

$$w(0,t) = 0$$
$$w(l,t) = 0$$
(Essential Boundary conditions)

$$kAG \left(\frac{\partial w}{\partial x} - \psi \right) \cancel{\delta w \Big|_0^l} \overset{0}{} = 0$$

$$EI \frac{\partial \psi}{\partial x} \delta \psi \Big|_0^l = 0$$

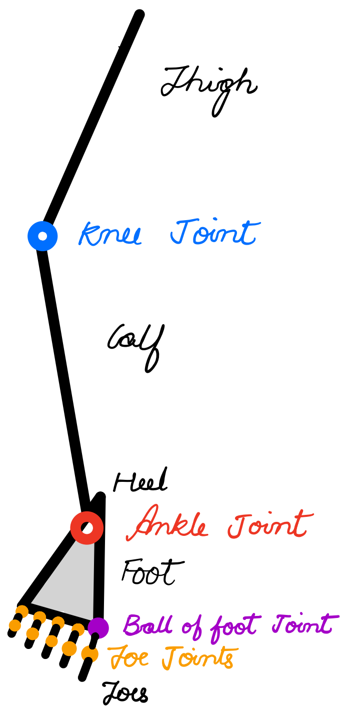
$$EI \left. \frac{\partial \psi}{\partial x} \right|_{x=l} = 0$$

$$EI \left. \frac{\partial \psi}{\partial x} \right|_{x=0} = 0$$

(Natural Boundary condition)

Solution 2

There can be multiple solutions



Leg is divided into
- Thigh Part
- Calf + Shin
- Foot + Heels

Foot should be discretized much finer having toes, toes joints and ball of the foot

Thighs, Calf + Shin, Foot, Toes can be considered as :-

① Rigid

② Continuous system such as
- beam having high flexural stiffness
- bar having high axial stiffness
- shaft having high torsional stiffness

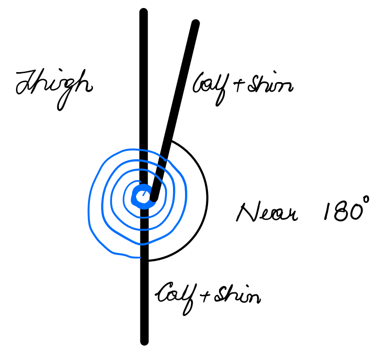
Since all of the above are mostly parts supported by continuous bone.

Knee Joint should be restricted as shown in figure.

Some mechanism had to be shown.

Pin Joint + Torsional spring

↓ Mean position (standing)



Ankle Joint

Same as knee Joint but with additional Torsional movement (This is must)

Ball of the foot joint same as knee joint with appropriate angle restrictions

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Toes joints same as ball of the foot as well as knee joints with appropriate angle restriction.

Solution 3

Recall that admissible functions satisfy the geometric boundary conditions but not the natural boundary conditions and equation of motion. Comparison functions satisfy both geometric and natural boundary conditions but not the equation of motion.

Comparison functions for Simply supported beam

$W(x) = f(x)$ such that $f(0) = 0$, $f(l) = l$

Comparison function for Cantilever beam

$W(x) = g(x)$ such That

$$g(0) = 0$$

$$\frac{dg}{dx}(0) = 0$$

$$\frac{d^2 g}{dx^2}(l) = 0$$

$$\frac{d^3 g}{dx^3}(l) = 0$$

Solution 4

ⓐ

$$T = \frac{1}{2}\int_0^L \rho A(x)\left(\frac{\partial w}{\partial t}\right)^2 dx$$

$$V = \frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

$$\int_{t_1}^{t_2} \delta T\,dt = \int_{t_1}^{t_2} \delta\left[\frac{1}{2}\int_0^L \rho A(x)\left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}M\left(\frac{\partial w(L,t)}{\partial t}\right)^2\right]dt$$

$$= \int_{t_1}^{t_2}\left[\int_0^L \rho A(x)\,\frac{\partial w}{\partial t}\,\delta\left(\frac{\partial w}{\partial t}\right)dx + M\,\frac{\partial w(L,t)}{\partial t}\,\delta\left(\frac{\partial w(L,t)}{\partial t}\right)\right]dt$$

$$= \int_{t_1}^{t_2}\left(\int_0^L \rho A(x)\,\frac{\partial w}{\partial t}\left[\frac{\partial \delta w}{\partial t}\right]dx + M\,\frac{\partial w(L,t)}{\partial t}\left[\frac{\partial \delta w(L,t)}{\partial t}\right]\right)dt$$

$$= \boxed{\underbrace{\int_0^L\int_{t_1}^{t_2} \rho A(x)\,\frac{\partial w}{\partial t}\left(\frac{\partial \delta w}{\partial t}\right)dt\,dx}_{\text{Expression 1}}} + \boxed{\underbrace{\int_{t_1}^{t_2} M\,\frac{\partial w(L,t)}{\partial t}\left(\frac{\partial \delta w(L,t)}{\partial t}\right)dt}_{\text{Expression 2}}}$$

Using integration by parts to expression 1

$$\int_0^L \int_{t_1}^{t_2} \rho A(x) \frac{\partial w}{\partial t} \left(\frac{\partial \delta w}{\partial t} \right) dt\, dx = -\int_0^L \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\rho A(x) \frac{\partial w}{\partial t} \right) \delta w \, dt\, dx$$

$$+ \int_0^L \cancel{\left[\rho A(x) \frac{\partial w}{\partial t} \delta u \right]_{t_1}^{t_2}} dx$$

Using integration by parts to expression 2

$$\int_{t_2}^{t_1} M \frac{\partial w}{\partial t}(L,t) \left(\frac{\partial \delta w(L,t)}{\partial t} \right) dt = -\int_{t_1}^{t_2} M \frac{\partial^2 w(L,t)}{\partial t^2} \delta w(L,t)\, dt$$

$$+ \cancel{\left[M \frac{\partial w(L,t)}{\partial t} \delta w(L,t) \right]_{t_1}^{t_2}}$$

$$-\int_{t_1}^{t_2} \delta V \, dt = -\int_{t_1}^{t_2} \delta \int_0^L \frac{EI(x)}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx\, dt$$

$$= -\int_{t_1}^{t_2} \int_0^L EI(x) \left(\frac{\partial^2 w}{\partial x^2} \right) \delta \left(\frac{\partial^2 w}{\partial x^2} \right) dx\, dt$$

$$= -\int_{t_1}^{t_2} \int_0^L EI(x) \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 \delta w}{\partial x^2} \right) dx\, dt$$

Using Integration by parts once wrt x

$$= \int_{t_1}^{t_2} \left[\int_0^L \frac{\partial}{\partial x}\left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) dx - \left[EI(x) \frac{\partial^2 w}{\partial x^2} \delta\left(\frac{\partial w}{\partial x}\right) \right]_0^L \right] dt$$

Using integration by parts again wrt x

$$= -\int_{t_1}^{t_2} \left[\int_0^L \left(\frac{\partial^2}{\partial x^2}\left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) dx \, \delta w + \left[EI(x) \frac{\partial^2 w}{\partial x^2} \delta\left(\frac{\partial w}{\partial x}\right) \right]_0^L \right. \right.$$

$$\left. - \left[\frac{\partial}{\partial x}\left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) \delta w \right]_0^L \right] dt = 0$$

Using essential Boundary conditions

$w(0,t) = 0 \qquad \delta w \big|_{x=0} = 0$

$\frac{dw}{dx}(0,t) = 0 \qquad \delta\left(\frac{dw}{dx}\right)\Big|_{x=0} = 0$

$$= -\int_{t_1}^{t_2} \left[\int_0^L \left(\frac{\partial^2}{\partial x^2}\left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) dx \, \delta w + \left[EI(x) \frac{\partial^2 w}{\partial x^2} \delta\left(\frac{\partial w}{\partial x}\right) \right]_{x=L} \right. \right.$$

$$\left. - \left[\frac{\partial}{\partial x}\left(EI(x) \frac{\partial^2 w}{\partial x^2} \right) \delta w \right]_{x=L} \right] dt = 0$$

Hamilton Principle

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0$$

$$-\int_0^L\int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\rho A(x)\frac{\partial w}{\partial t}\right)\delta w\, dt\, dx - \int_{t_1}^{t_2} M\frac{\partial^2 w(L,t)}{\partial t^2}\delta w(L,t)\, dt$$

$$-\int_{t_1}^{t_2}\left[\int_0^L\left(\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)dx\,\delta w + \left[EI(x)\frac{\partial^2 w}{\partial x^2}\delta\left(\frac{\partial w}{\partial x}\right)\right]_{x=L}\right.\right.$$

$$\left.-\left[\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)\delta w\right]_{x=L}\right]dt = 0$$

Governing Equation

$$\rho A(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) = 0$$

Boundary Conditions (Natural)

$$\left[EI(x)\frac{\partial^2 w}{\partial x^2}\right]_{x=L} = 0$$

$$\left[M\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)\right]_{x=L} = 0$$

(b)

$$T = \frac{1}{2}\int_0^L \rho A(x)\left(\frac{\partial w}{\partial t}\right)^2 dx$$

$$V = \frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$$

Assuming $w(x,t)$ to be harmonic

$$w(x,t) = X(x)\cos\omega t$$

where ω is the frequency of vibration

Rayleigh's Quotient for a beam bending is defined as

Maximum strain energy = Maximum kinetic energy

$$V_{max} = \frac{1}{2}\int_0^L EI(x)\left(\frac{d^2X}{dx^2}\right)^2 dx$$

$$T_{max} = \frac{1}{2}\int_0^L \rho A(x)\,\omega^2 X^2\,dx$$

Equating $V_{max} = T_{max}$

$$R(X) = \omega^2 = \frac{\frac{1}{2}\int_0^L EI(x)\left(\frac{d^2X}{dx^2}\right)^2 dx}{\frac{1}{2}\int_0^L \rho A(x)\,X^2\,dx}$$

$X(x)$

$$EI(x) = E_0 I_0\left(1-\left(\frac{x}{12L}\right)^3\right)$$

$$A(x) = A_0\left(1-\frac{x}{3L}\right)$$

Exact Solution

$\downarrow$

Comparison funcn

$\downarrow$

Admissible funcn ✓✓

$$X(0) = 0$$

$$\frac{dX}{dx}(0) = X'(0) = 0$$

Calculate first frequency

(c) Modify $X(x)$ such that it cuts the $x \in (0, L)$ once!

Calculate second frequency