AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

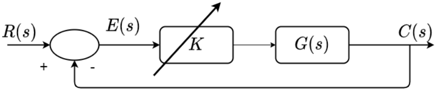
Lecture 17: Root Locus - 1

Dr. Arnab Maity
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Table of Contents

Introduction

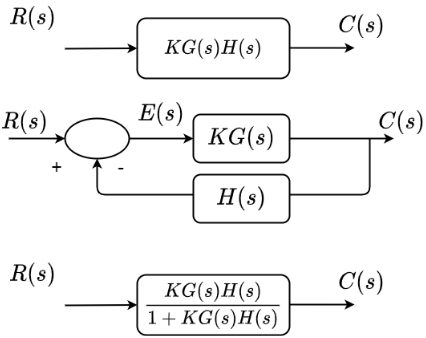
Introduction



- Root Locus, a graphical representation of closed loop poles as system parameter is varied.
- It is a powerful tool in analysing the higher order systems (greater than 2).
- Root locus is useful in describing the effect of varying gain over precentage overshoot, settling time and peak time.
- Besides transient response, root locus also gives graphical representation of system's stability.

Introduction

Introduction

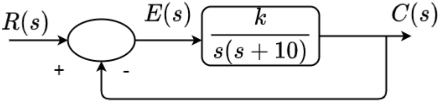


- Open loop transfer function is given by $KG(s)H(s)$.
- In general, open loop poles can be easily determined.
- To determine the poles of closed loop system, $T(s)$ the denominator has to be factorized.

Table of Contents

Root Locus

Definition



- The closed loop transfer function is given by

$$T(s) = \frac{k}{s^2 + 10s + k}$$

- Closed loop poles are given by

$$s = \frac{-10 \pm \sqrt{100 - 4k}}{2}$$

- Table is formed for various values of s by varying k.
- As the gain, k, increases in table, one pole at -10 moves towards right and another pole at 0 moves towards left.

Root Locus

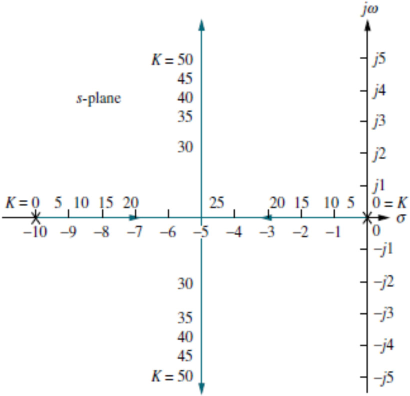
Definition

K	Pole Location 1	Pole Location 2
0	-10	0
5	-9.47	-0.53
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
50	$-5 + j5$	$-5 - j5$

- As k increases, the closed loop poles move away from open loop poles and meet at a point on real axis.
- The two poles meet at -5 for $k = 25$.
- As k increases, poles break away from real axis and move into complex plane.

Root Locus

Definition



- Figure shows *representation of path of closed loop poles as gain k is varied*.
- This graphical representation is called Root Locus.
- The root locus shows the the change in transient response as k is varied.

Root Locus -Transient Response

Observations

Gain	Pole location	Nature of response
$0 < k < 25$	Real	Overdamped
$k = 25$	Multiple	Critically damped
$k > 25$	Complex	Underdamped

- As root locus never crosses over into right half s- plane.
- Hence system is stable regardless of the value of gain, k.

Root Locus -Transient Response

Example

- Consider a second order system

$$G(s)H(s) = \frac{k}{s(s+6)}$$

- Find the range of k, for which system is overdamped, critically damped and underdamped ?

Root Locus -Transient Response

Solution

- Consider a second order system

$$G(s)H(s) = \frac{k}{s(s+6)}$$

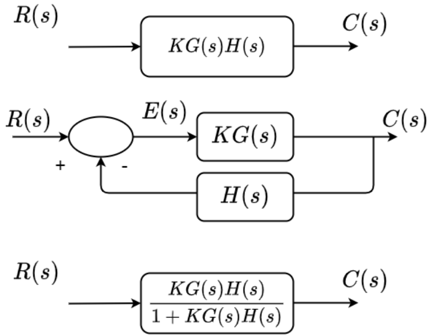
- The roots of characteristic equation are

$$s = \frac{-6 \pm \sqrt{36-4k}}{2}$$

Gain	Pole location	Nature of response
$0 < k < 9$	Real	Overdamped
$k = 9$	Multiple	Critically damped
$k > 9$	Complex	Underdamped

Table of Contents

Root Locus - Properties



- In the second order system, sketch of root locus was plotted by factorizing the characteristic equation.
- In general, factorizing the higher order polynomial without computer is a difficult task.

- The properties of root locus can be obtained through closed loop transfer function.

$$\boxed{T(s) = \frac{kG(s)H(s)}{1 + kG(s)H(s)}}$$

Root Locus - Properties

- The closed loop poles are obtained by solving the characteristic equation

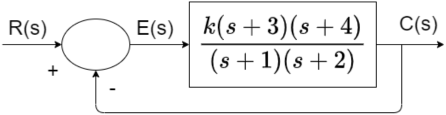
$$
\begin{aligned}
1 + kG(s)H(s) =& 0 \\
kG(s)H(s) = -1 =& 1\angle(2n+1)180^\circ, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots
\end{aligned}
$$

- Alternatively,

$$
\begin{aligned}
|kG(s)H(s)| =& 1 \\
\angle kG(s)H(s) =& (2n+1)180^\circ
\end{aligned}
$$

- If s is substituted in $G(s)H(s)$, it results in a complex number.
- If the angle of complex number is odd multiple of 180°, then that particular value of s is system pole for some particular value of k.

Root Locus - Properties



- As angle criterion is satisfied, all that remain are to satisfy magnitude criterion,

$$k = \frac{1}{|G(s)H(s)|}$$

- Consider a open loop transfer function

$$kG(s)H(s) = k\frac{(s+3)(s+4)}{(s+1)(s+2)}$$

- Closed loop transfer function is given by

$$T(s) = \frac{k(s+3)(s+4)}{(1+k)s^2 + (3+7k)s + (2+12k)}$$

Root Locus - Properties

- Consider point $s = -2 + 3j$. Let us check whether it is a part of root locus. Substitute it in open loop transfer function for angle condition

$$
\begin{aligned}
kG(s)H(s) &= k\frac{(s+3)(s+4)}{(s+1)(s+2)}\bigg|_{s=-2+3j} \\
&= k\frac{(1+3j)(2+3j)}{(-1+3j)(3j)} \\
\angle kG(s)H(s) &= \tan^{-1}\left(\frac{3}{1}\right) + \tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{3}{-1}\right) - 90^\circ \\
&= 71.57 + 56.3099 - 90 - 108.43 = -70.55^\circ
\end{aligned}
$$

- As it does not satisfy the angle condition, it does not lie on the root locus.

Root Locus - Properties

- Consider the point $s = -2 + j\frac{\sqrt{2}}{2}$. Substitute this point in open loop transfer function,

$$kG(s)H(s) = k\frac{(s+3)(s+4)}{(s+1)(s+2)}\bigg|_{s=-2+j\frac{\sqrt{2}}{2}}$$

$$= k\frac{(1+j\frac{\sqrt{2}}{2})(2+j\frac{1}{\sqrt{2}})}{\left(-1+j\frac{\sqrt{2}}{2}\right)(j\frac{\sqrt{2}}{2})}$$

$$\angle kG(s)H(s) = 35.2644 + 19.4712 - 90 - 144.7356 = -180^\circ$$

- Hence it satisfies angle condition. Through magnitude condition, k, gain is obtained

$$|kG(s)H(s)| = 1$$

$$k = \frac{1}{|G(s)H(s)|} = \frac{1.22(\frac{1}{\sqrt{2}})}{1.2247(2.12)} = 0.33$$

Table of Contents

Root Locus - Sketch

Root Locus Plot

- The follwing rules allow us to sketch the root locus with minimal information.
- The rules yield a sketch that gives insight into the behaviour of a control system.
- **Number of Branches** : The number of branches of root locus equals number of closed loop poles.
- Physically realizable systems cannot have complex coefficients. Hence resulting closed loop poles will have complex conjugate pairs.
- **Symmetry**: The root locus is symmetrical about real axis.

Root Locus - Sketch

Root Locus Plot

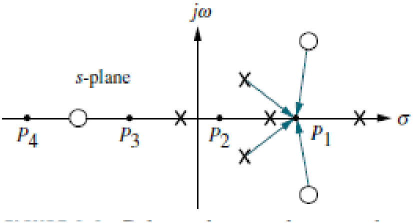


Figure: Source - "Control Systems Engineering" by Norman S Nise

- **Real axis segments** Let us determine where the real- axis segments of root locus exist through angle property.
- If an attempt is made to calculate the angular contribution of poles and zeros at each point, P_1, P_2, P_3 and P_4 along the real axis, we observe the follwing.

Observations

- At each point the angular contribution of pair of complex poles or zeros is zero.

Root Locus - Sketch

Root Locus Plot

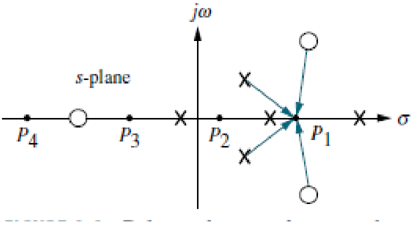


Figure: Source - "Control Systems Engineering" by Norman S Nise

- Let the complex zeros be at $-3 \pm i$. Complex poles at $-0.5 \pm 0.5i$. Let us calculate the angle contribution at P_1 due to complex poles and zeros. Assume $P_1 = 2$.

- Consider the angle of $G(s)$

$$G(s) = \left. \frac{(s+3+i)\,(s+3-i)}{(s+0.5+i)\,(s+0.5-i)} \right|_{s=2}$$

Root Locus - Sketch

Root Locus Plot

- Angle evaluated at $P_1 = 2$ is given by

$$\begin{aligned}\angle G(s) =& \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{-1}{5}\right) - \tan^{-1}\left(\frac{1}{2.5}\right) - \tan^{-1}\left(\frac{-1}{2.5}\right)\\ =&0\end{aligned}$$

- Hence the angle contributed by complex poles and zeros are zero.
- Consider the angle contribution of poles to the right of P_1. Let the pole towards right of P_1 be 5.
- Hence contribution of angle at point P_1 due to pole at 5 is given by

$$G(s) = \left.\frac{1}{(s-5)}\right|_{s=2}$$

$$\angle G(s) = -\tan^{-1}\left(\frac{0}{-3}\right) = -180^\circ$$

Root Locus - Sketch

Root Locus Plot

- Consider a pole to the left of P_1. Let the pole to the left of P_1 be 1. item The angle contribution of pole 1 to the point P_1 is

$$G(s) = \left.\frac{1}{s-1}\right|_{s=2}$$
$$\angle G(s) = -\tan^{-1}\left(\frac{0}{1}\right) = 0^\circ$$

- Similarly consider a zero along the real axis left to the point P_1. Consider a zero along axis to be located at $s = -3$.
- The angle contributed by zero to the point P_1 is

$$G(s) = \left.(s+3)\right|_{s=2}$$
$$\angle G(s) = \tan^{-1}\left(\frac{0}{5}\right) = 0^\circ$$

Root Locus - Sketch

Root Locus Plot

Observations

- The contribution of the open loop poles poles and open loop zeros to the left of respective point is zero.
- The conclusion is that the only contribution to the angle at any of the points comes from the open loop, real axis poles and zeros that exist to the right of respective point.

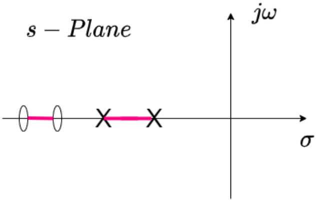
- If we calculate the angle at each point using only open loop, real axis poles and zeros to the right of each point, we observe the following.

Observations

- The angles on the real axis alternate between 0° and 180°.
- The angle is 180° for regions of real axis that exist to left of odd number of poles and/ or zeros.

Root Locus - Sketch

Root Locus Plot



- **Real -axis Segments** On the real axis, for $k > 0$ the root locus exists to the left of an odd number of real axis, finite open loop poles and/or finite open loop zeros.

- **Starting and Ending Points** Let us examine closed loop transfer function for small and large gains, k. Consider closed loop transfer function as

$$G(s) = k\frac{N_G}{D_G}, \quad H(s) = \frac{N_H}{D_H}$$

$$T(s) = \frac{kN_GN_H}{kN_GN_H + D_GD_H}$$

Root Locus - Sketch

Root Locus Plot

- For smaller values of gain, k, the charcteristic equation becomes

$$D_G D_H = 0$$

- For larger values of gain, $k \to \infty$, the $T(s)$ results in

$$T(s) = \frac{N_G N_H}{N_G N_H + \frac{D_G D_H}{k}}$$

- Characteristic equation becomes,

$$N_G N_H = 0$$

Rules

The root locus begins at poles of $G(s)H(s)$ and ends at zeros of $G(s)H(s)$.

Root Locus - Sketch

Root Locus Plot

- **Behaviour at infinity** Let us examine the root locus if number of poles are more than number of zeros.

Rules

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of asymptotes is given by real axis intercept, σ_a and θ_a

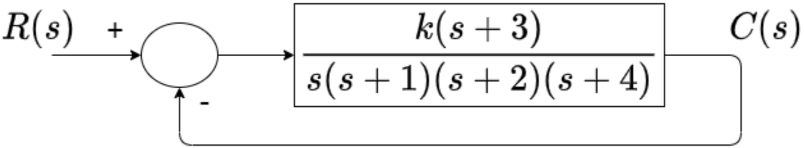
$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\text{Number of finite poles} - \text{Number of finite zeros}}$$

$$\theta_a = \frac{(2m+1)\pi}{\text{Number of finite poles} - \text{Number of finite zeros}}, \quad m = 0, 1, 2$$

Root Locus - Sketch

Example

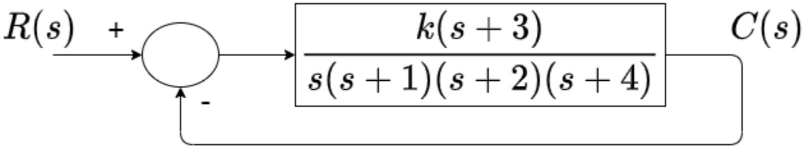
- Sketch the root locus for the follwing system



Root Locus - Sketch

Solution

- Sketch the root locus for the follwing system



- Let us start by calculating the asymptotes. Real intercept is given by

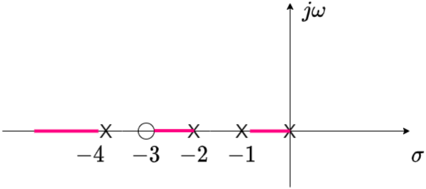
$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\theta_a = \frac{(2m+1)\pi}{\text{Number of finite poles} - \text{Number of finite zeros}}$$

$$= \frac{\pi}{3}, \pi, \frac{5\pi}{3} \quad m = 0, 1, 2$$

Root Locus - Sketch

Root Locus Plot

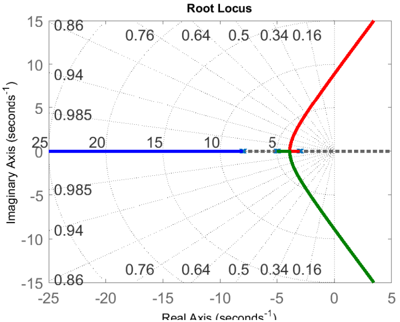


- Consider Real-axis segments rule, then part of root locus in real axis is as shown in the figure.
- As there are more number of poles than zero, asymptotes will tell us how to reach as $k \to \infty$

- The complete root locus is as shown in the figure.
- Utilizing all the rules, root locus of the above system is obtained.

Root Locus - Sketch

Root Locus Plot



- The real axis segments lie to the left of an odd number of open loop poles and zeros.
- The locus starts at open loop poles and ends at open loop zeros.
- The complete details of the sketch will discussed in the upcoming classes.

Root Locus - Sketch

Example

- Find the root locus in real axis segment for the following system

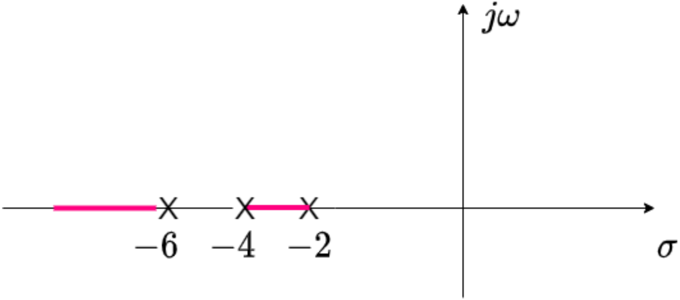
$$G(s) = \frac{k}{(s+2)(s+4)(s+6)}$$

Root Locus - Sketch

Solution

- Find the root locus in real axis segment for the following system

$$G(s) = \frac{k}{(s+2)(s+4)(s+6)}$$



References I

- Katsuhiko Ogata: "*Modern Control Engineering*", Pearson Education, Inc., Upper Saddle River, New Jersey, Fifth Edition, 2010.
- Farid Golnaraghi and Benjamin C. Kuo: "*Automatic Control Systems*", John Wiley & Sons, Inc., New Jersey, Ninth Edition, 2010.
- Norman S. Nise: "*Control Systems Engineering*", John Wiley & Sons, Inc., New Jersey, Sixth Edition, 2011.