Aerospace Engineering Department, IIT Bombay
AE 308 & AE 775 - Control Theory
Tutorial 2 Solution

Q1

Linearize the nonlinear equation

$$z = x^2 + 4xy + 6y^2$$

in the region defined by $8 \leq x \leq 10, \quad 2 \leq y \leq 4.$

Solution:

Q2

Verify whether each of the following functions is linear or nonlinear.

1. $x(t-2)$
2. $x(t).x(t-2)$
3. $\frac{d}{dt}x(t)$

Solution:

Check whether each of them satisfies the homogeneity and superposition principle.

1. Linear.
2. Nonlinear.
3. Linear.

Q3

Verify whether each of the following functions is time-variant or time-invariant.

1. $x(t-2)$
2. $t.x(t)$
3. $2^{x(n)}x(n)$

Solution:

Check whether a time delay of the input equates to a time delay of the output. The final response should be equal in both cases for a system to be time-invariant.

1. Time-invariant.
2. Time-variant.
3. Time-invariant.

Q4

Perform the convolution operation between the following pair of functions:

1. $u(t)$ and $u(t)$, where $u(t)$ stands for the unit step function.

Solution:

The range of $u(\tau)$ is from 0 to ∞ while the range of $u(t-\tau)$ is from $-\infty$ to t.

For $t < 0$, there is no overlapping region between the two functions and as a result, their convolution is 0.

But for $t \geq 0$, the overlapping region of the two functions is within 0 and t. Thus, in this case

$$u(t) \star u(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)\ d\tau = \int_{0}^{t} 1.1.\ d\tau = t, \quad t \geq 0 = r(t)$$

where $r(t)$ stands for the ramp function.

Q5

Find the laplace transform of the following signals:

1. $u(t)$, where $u(t)$ stands for the unit step function.
2. t
3. e^{-at}

Problem 1.
Solution.

$$\begin{aligned} F(s) &= \int_{0^-}^{\infty} u(t)e^{-st}dt \\ &= \int_{0^-}^{\infty} 1 \cdot e^{-st}dt = -\frac{1}{s}\left[e^{-st}\right]_{0^-}^{\infty} \\ &= -\frac{1}{s}[0-1] \\ &= \frac{1}{s} \end{aligned}$$

Problem 2.
Solution

$$\begin{aligned} \int_0^{\infty} (t)e^{-st}dt &= \int_0^{\infty} -\frac{d}{ds}\left(e^{-st}\right)dt && \text{Laplace integral of } g(t) = t. \\ &= -\frac{d}{ds}\int_0^{\infty}(1)e^{-st}dt && \text{Use } \int \frac{d}{ds}F(t,s)dt = \frac{d}{ds}\int F(t,s)dt \\ &= -\frac{d}{ds}(1/s) && \text{Use } \mathcal{L}(1) = 1/s. \\ &= 1/s^2 && \text{Differentiate.} \end{aligned}$$

Problem 3. Solution

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-at}e^{-st}dt \\ \rho'(s) &- \int_0^{\infty} e^{-(s+a)t}u\iota \\ F(s) &= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} \\ F(s) &= \frac{1}{s+a} \end{aligned}$$

Q6

Evaluate the following:

1.
$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-3s+2}\right]$$

2.
$$\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s^3+s}\right]$$

3.
$$\mathcal{L}^{-1}\left[\frac{s^2+6s+8}{s^4+8s^2+16}\right]$$

Problem 1.

Solution. We factor the denominator and split the rational function into partial fractions:
$$\frac{2s-3}{(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}.$$
Multiplying both sides by $(s-1)(s-2)$ and simplifying to obtain
$$\begin{aligned}2s-3&=A(s-2)+B(s-1)\\&=(A+B)s-2A-B.\end{aligned}$$

Equating coefficients of like powers of s we obtain the system
$$\begin{cases}A+B &= 2\\ -2A-B &= \quad -3.\end{cases}$$

Solving this system by elimination we find $A=1$ and $B=1$. Now finding the inverse Laplace transform to obtain
$$\mathcal{L}^{-1}\left[\frac{2s-3}{(s-1)(s-2)}\right]=\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]+\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]=e^t+e^{2t},t\geq 0$$

Problem 2.
Solution. We factor the denominator and split the rational function into partial fractions:
$$\frac{4s^2+s+1}{s\left(s^2+1\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}.$$

Multiplying both sides by $s\left(s^2+1\right)$ and simplifying to obtain

$$\begin{aligned} 4s^2+s+1 &= A\left(s^2+1\right)+(Bs+C)s \\ &= (A+B)s^2+Cs+A. \end{aligned}$$

Equating coefficients of like powers of s we obtain $A+B=4, C=1, A=1$. Thus, $B=3$. Now finding the inverse Laplace transform to obtain

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s\left(s^2+1\right)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s}\right]+3\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] \\ &= 1+3\cos t+\sin t, t\geq 0 \end{aligned}$$

Problem 3.

Solution. We factor the denominator and split the rational function into partial fractions:

$$\frac{s^2+6s+8}{\left(s^2+4\right)^2}=\frac{B_1s+C_1}{s^2+4}+\frac{B_2s+C_2}{\left(s^2+4\right)^2}.$$

Multiplying both sides by $\left(s^2+4\right)^2$ and simplifying to obtain

$$\begin{aligned} s^2+6s+8 &= \left(B_1s+C_1\right)\left(s^2+4\right)+B_2s+C_2 \\ &= B_1s^3+C_1s^2+\left(4B_1+B_2\right)s+4C_1+C_2. \end{aligned}$$

Equating coefficients of like powers of s we obtain $B_1=0, C_1=1, B_2=6$, and $C_2=4$. Now finding the inverse Laplace transform to obtain

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s^2+6s+8}{\left(s^2+4\right)^2}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]+6\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+4\right)^2}\right]+4\mathcal{L}^{-1}\left[\frac{1}{\left(s^2+4\right)^2}\right] \\ &= \frac{1}{2}\sin 2t+6\left(\frac{t}{4}\sin 2t\right)+4\left(\frac{1}{16}[\sin 2t-2t\cos 2t]\right) \\ &= \frac{3}{2}t\sin 2t+\frac{3}{4}\sin 2t-\frac{1}{2}t\cos 2t, t\geq 0 \end{aligned}$$

Hint:-

$$\begin{aligned} &(1)\ L\{t\sin at\} = -d/dsL(\text{ sinat }) \\ &= \frac{-d}{ds}\left\{\frac{a}{s^2+a^2}\right\} \\ &= \frac{2as}{\left(s^2+a^2\right)^2} \\ &(2)\ L\{t\cos at\} = \frac{-dL}{ds}(\cos at) \\ &= \frac{-d}{ds}\left\{\frac{s}{s^2+a^2}\right\} \\ &= \frac{s^2-a^2}{\left(s^2+a^2\right)^2}. \end{aligned}$$

Please see hint carefully and arrange the formula in the given above question.