Tail probabilities

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Then for every $x > 0$,

$\mathbb{P}(X > x)$ and $\mathbb{P}(X < -x)$ are called tail probabilities.

* Note that we can use distribution function for $\mathbb{P}(X < -x) = \mathbb{F}(-x)$. I will prefer to use left tail probability for $\mathbb{P}(X < -x) = \mathbb{F}(-x)$.

* We can use right tail probability to denote $\mathbb{P}(X > x)$ for $x > 0$.

* $\mathbb{P}(|X| > x) = \mathbb{P}(X > x) + \mathbb{P}(X < -x)$, for every $x > 0$.

Theorem-3 (Moments and tail probabilities).

Let X be a random variable such that $\mathbb{E}(|X|^k) < \infty$ for some $k > 0$. Then

$$n^k \, \mathbb{P}(|X| > n) \to 0 \quad \text{as } n \to \infty.$$

that is, $\lim\limits_{n \to \infty} n^k \, \mathbb{P}(|X| > n) = 0.$

Proof. Note that

$$\mathbb{E}(|X|^k) = \int |x|^k f(x)\,dx = \int_{|x| > n} |x|^k f(x)\,dx$$

$$+ \int_{|x| \leq n} |x|^k f(x)\, dx$$

$$\geq \int_{|x| \geq n} |x|^k f(x)\, dx \geq n^k \mathbb{P}(|X| \geq n).$$

Note that

$$\mathbb{E}(|X|^k) = \int_{-\infty}^{\infty} |x|^k f(x)\, dx = \lim_{n \to \infty} \int_{-n}^{n} |x|^k f(x)\, dx \quad (*_1)$$

(Cauchy's principal value theorem)

Use the decomposition

$$\mathbb{E}(|X|^k) = \int_{|x| \geq n} |x|^k f(x)\, dx + \int_{|x| \leq n} |x|^k f(x)\, dx$$

and the fact that $0 < \int_{|x| \geq n} |x|^k f(x)\, dx < \infty$ for every $n \geq 1$. To conclude

$$\lim_{n \to \infty} n^k \mathbb{P}(|X| \geq n) \leq \lim_{n \to \infty} \int_{|x| \geq n} |x|^k f(x)\, dx$$

$$= \lim_{n \to \infty} \left(\mathbb{E}(|X|^k) - \int_{|x| \leq n} |x|^k f(x)\, dx \right)$$

$$= \mathbb{E}(|X|^k) - \lim_{n \to \infty} \int_{|x| \leq n} |x|^k f(x)\, dx$$

$$= 0.$$

A similar proof should work when X is a discrete random variable. Please try and complete it yourself. (Exercise).

Remark

Converse of the statement may not be true always.

Consider a random variable X such that

$$\mathbb{P}(X=n) = \frac{c}{n^2 \log n}, \quad n = 2, 3, \ldots$$

where $c = \left(\sum_{n=2}^{\infty} \frac{1}{n^2 \log n} \right)^{-1}$.

* Show that

$\mathbb{P}(X > n) \approx c n^{-1} (\log n)^{-1}$ that is,

$$c_l \le \frac{\mathbb{P}(X>n)}{n^{-1} (\log n)^{-1}} \le c_u \quad \text{for large enough } n \longrightarrow (*5)$$

where $0 < c_l < c_u < \infty$.

Hint: Use the idea of integral test for convergence of a series.

$\boxed{\text{IMPORTANT PROBLEM}}$

Once you show $(*5)$, it follows from the upper bound that

$$n\, \mathbb{P}(X > n) \le c_u (\log n)^{-1} \xrightarrow[n \to \infty]{} 0$$

and therefore,

$$\lim_{n \to \infty} n\, \mathbb{P}(X > n) = 0.$$

But it is straight forward to check that

$$\mathbb{E}(X) = \sum_{n=2}^{\infty} \frac{c}{\log n} = \infty.$$

Remark. Let $\lim_{n\to\infty} n^{k+\delta}\, \mathbb{P}(|X| > n) = 0$, for some $\delta > 0$, then one can show that

$$\mathbb{E}(|X|^k) < \infty. \quad \text{(Exercise)} \quad \text{(Use following Lemma)}.$$

$$\boxed{\lim_{n\to\infty} n^{k+\delta}\, \mathbb{P}(|X| > n) = 0 \text{ for some } \delta > 0.}$$

moment condition / sufficient condition for $\mathbb{E}(|X|^k)$ to exist

> We can put some restrictions on the tail probababilities of a random variable to have certain moments. Existance of some moments of a random variable puts restriction on the tail probabilities

Lemma - 1 (Tail probability and expected value) continuous

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to ([0,\infty), \mathcal{B}([0,\infty)))$ be a random variable with distribution function $\mathbb{F}$. Then

$$\mathbb{E}(X) = \int_0^\infty (1 - \mathbb{F}(x))\, dx$$

in the sense that if either side is finite, so does the other and the two are equal.

If X is a discrete random variable, then

$$\mathbb{E}(X) = \sum_{n=1}^{\infty} n \, \mathbb{P}(X \geq n)$$

in the sense that if either of the sides ~~exists~~ is finite, then ~~the other does~~ so is the other and two sides are equal.

~~We prove~~ Proof-

We prove it for continuous random variable X with p.d.f. f.

Two techniques.

Analytic. [Assuming $\mathbb{E}(X) < \infty$].

$$\mathbb{E}(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty dx \, f(x) \int_0^x du.$$

$$= \int_0^\infty dx \int_0^x du \; f(x). \quad [0 < u < x < \infty].$$

$$= \int_0^\infty du \int_u^\infty dx \; f(x).$$ [We have used Fubini's theorem to interchange the limits of the integrals as the integration is finite]

$$= \int_0^\infty du \; \mathbb{P}(X \geq u).$$

$$= \int_0^\infty du \left(1 - F(u)\right).$$

Integration by parts.

Observe that

$$\int_0^n x f(x)\, dx \uparrow \lim_{n \to \infty} \int_0^n x f(x)\, dx = \mathbb{E}(X).$$

$\int x f(x) dx = x \mathbb{F}(x) - \int \mathbb{F}(x)\, dx$ using integration by parts technique. For every fixed n,

$$\int_0^n x f(x)\, dx = \left[x \mathbb{F}(x) \right]_0^n - \int_0^n \mathbb{F}(x)\, dx$$

$$= n \mathbb{F}(n) - \lim_{x \to 0} x \mathbb{F}(x) - \int_0^n \mathbb{F}(x)\, dx$$

$$= n \mathbb{F}(n) - \int_0^n \mathbb{F}(x)\, dx$$

$$\cancel{= \int_0^n \mathbb{F}(n)\, dx - \int_0^n \mathbb{F}(x)\, dx = \int_0^n (\mathbb{F}(n) - \mathbb{F}(x))\, dx}$$

$$= \cancel{n} \int_0^n (1 - \mathbb{F}(x))\, dx - n + n \mathbb{F}(n).$$

$$= \int_0^n (1 - \mathbb{F}(x))\, dx - n(1 - \mathbb{F}(n)) \longrightarrow *_6.$$

So

$$\mathbb{E}(X) = \lim_{n \to \infty} \int_0^n x f(x)\, dx$$

$$= \lim_{n \to \infty} \left[\int_0^n (1 - \mathbb{F}(x))\, dx - n(1 - \mathbb{F}(n)) \right]$$

$$= \lim_{n \to \infty} \int_0^n (1 - \mathbb{F}(x))\, dx - \lim_{n \to \infty} n(1 - \mathbb{F}(n)) \longrightarrow *_7$$

As $\mathbb{E}(|X|) = \mathbb{E}(X) < \infty$, it follows that

$$\lim_{n\to\infty} n\left(1 - F(n)\right) = \lim_{n\to\infty} n\, \mathbb{P}(X > n) = 0. \longrightarrow *8$$

By monotonicity, it follows that

$$\int_0^n \left(1 - F(x)\right) dx \uparrow \int_0^\infty \left(1 - F(x)\right) dx. \longrightarrow *9$$

that is $\displaystyle \lim_{n\to\infty} \int_0^n \left(1 - F(x)\right) dx = \int_0^\infty \left(1 - F(x)\right) dx$

Combining $*8$ and $*9$, it follows ~~that~~ from $*7$ that

$$\mathbb{E}(X) = \int_0^\infty \left(1 - F(x)\right) dx.$$

We have assumed $\mathbb{E}(X) < \infty$ and showed that

$$\mathbb{E}(X) = \int_0^\infty \left(1 - F(u)\right) du.$$

~~Let us assume $\int_0^\infty (1 - F(u))\, du < \infty$.~~

~~Then $\displaystyle\int_0^\infty (1 - F(u))\, du = \int_0^\infty du \int_u^\infty f(x)\, dx = \int_0^\infty x f(x)\, dx$~~

~~the same trick works~~ (IGNORE)

We now assume $\mathbb{E}(X) = \infty$. It follows from $*6$ that

$$\mathbb{E}(X) \leq \lim_{n\to\infty} \int_0^n \left(1 - F(x)\right) dx$$

that is, $\displaystyle\int_0^\infty (1-F(x))\,dx = \lim_{n\to\infty}\int_0^n (1-F(x))\,dx$

$$\geq E(X) = \infty.$$

$$\boxed{\text{If } \int_0^\infty (1-F(x))\,dx = \infty, \text{ then } E(X) = \infty.}$$

~~We try method contradiction here, that is, we try to show that~~

Note that it is enough to show that $E(X) < \infty$ implies $\displaystyle\int_0^\infty (1-F(x))\,dx < \infty$. and this we have already proved. ▨

<u>A Fact from real analysis.</u> (Integral test for a series with positive terms).

Let $f : [1,\infty) \longrightarrow [0,\infty)$ be a decreasing function such that $\lim_{x\to\infty} f(x) = 0$. Define

$$s_n = \sum_{k\geq 1}^{n} f(k) \quad \text{and} \quad t_n = \int_1^n f(x)\,dx.$$

Then $(s_n : n\geq 1)$ converges to a finite limit $\left(\sum_{k\geq 1}^\infty f(k) < \infty\right)$ if and only if $(t_n : n\geq 1)$ converges to a finite limit that is, $\displaystyle\int_1^\infty f(x)\,dx < \infty$.

THE KEY INEQUALITY

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^{n} f(x)\,dx \rightarrow (**)$$

$$\int_1^{n+1} f(x)\,dx = \sum_{k=1}^{n} \int_k^{k+1} f(x)\,dx \le \sum_{k=1}^{n} f(k) \int_k^{k+1} dx$$

$$= \sum_{k=1}^{n} f(k).$$

So we are done with the lower bound. Similarly one can derive the upper bound. The claim of the theorem follows immediately from this inequality.

Lemma - 2 (Moments lemma) [Moments and tail probabilities).

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable then

$\mathbb{E}(|X|^\alpha) < \infty$ if and only if $\sum_{n \ge 1}^{\infty} \mathbb{P}(|X| \geq n^{1/\alpha}) < \infty$.

$\boxed{\text{Proof}}$ - It follows from integral test that

$\mathbb{E}(|X|^\alpha) = \int_0^\infty \mathbb{P}(|X|^\alpha \geq x)\,dx < \infty$ if and only if

$\sum_{n \ge 1}^{\infty} \mathbb{P}(|X|^\alpha \geq n) = \sum_{n \ge 1}^{\infty} \mathbb{P}(|X| \geq n^{1/\alpha}) < \infty.$

Theorem-4 (Sufficient condition for existance of the moments in terms of tail probabilities).

Let X be a random variable with distribution function satisfying $n^{\alpha}\,\mathbb{P}(|X|>n) \longrightarrow 0$ for some $\alpha>0$. Then $\mathbb{E}(|X|^{\beta})<\infty$ for $0<\beta<\alpha$.

Proof. Note that

$$\mathbb{E}[|X|^{\beta}] = \beta\int_0^N x^{\beta-1}\,\mathbb{P}(|X|>x)\,dx + \beta\int_N^{\infty} x^{\beta-1}\,\mathbb{P}(|X|>x)\,dx \longrightarrow (**_1)$$

where N is chosen large enough so that

$$\mathbb{P}(|X|>n) \le \frac{\epsilon}{n^{\alpha}} \quad \text{for all } n\geq N.$$

and $\epsilon>0$ is an arbitrary small number. Then it follows from $(**_1)$ that

$$\mathbb{E}(|X|^{\beta}) \le N^{\beta} + \beta\epsilon\int_N^{\infty} x^{\beta-1}\,\mathbb{P}(|X|>x)\,dx$$

To deal with the integral $\int_N^{\infty} x^{\beta-1}\,\mathbb{P}(|X|>x)\,dx$ we use the previous lemma which connects the moments to the tail probability. Then we obtain.

$\int_N^{\infty} x^{\beta-1}\,\mathbb{P}(|X|>x)\,dx < \infty$ if and only if

$$\sum_{n=N}^{\infty} n^{\beta-1}\,\mathbb{P}(|X|>n) \le \sum_{n=N}^{\infty} n^{\beta-1}\frac{\epsilon}{n^{\alpha}} = \epsilon\sum_{n=N}^{\infty} n^{\beta-\alpha-1} < \infty$$

if and only if $\beta<\alpha$.

* Here we have assumed that X is a continuous