AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

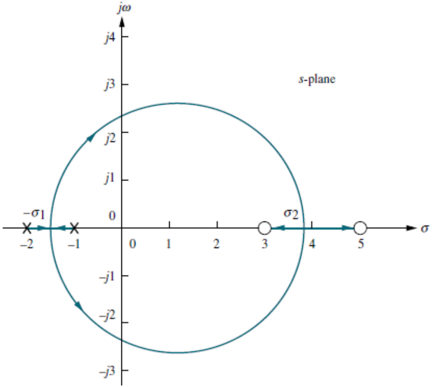
Lecture 18: Root Locus - 2

Dr. Arnab Maity
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Table of Contents

Root Locus - Sketch

Root Locus Plot



- Root loci appear to break away from real axis as system poles move from real to complex plane.
- At other times, loci appear to return to real axis as pair of complex poles becomes real.
- The point where root locus leaves the real axis σ_1 is called break away point.
- The point where root locus returns the real axis σ_2 is called break in point.

Root Locus - Sketch

Break Away Point

- As the two poles which are at -1 and -2 when $k = 0$, move towards each other as the gain increases from a value of 0.
- We conclude that gain must be maximum along the real axis at the break away point.
- Consider break in point on the real axis. When closed loop returns to real axis, the gain will continue to increase to infinity.
- The gain at the break in point is minimum along the real axis at break in point.
- At any point on root locus,

$$k = -\frac{1}{G(s)H(s)}$$

Root Locus - Sketch

Break Away Point

- For points on real axis segments of root locus, $s = \sigma$.

$$k = -\frac{1}{G(\sigma)H(\sigma)}$$

- Differentiate the above equation w.r.t σ and set the derivative to zero.
- We can find points σ, where k will be maximum and minimum.

Example

- Find the break away and in points for the following open loop transfer function,

$$G(s)H(s) = k\frac{(s-3)(s-5)}{(s+1)(s+2)}$$

Root Locus - Sketch

Break Away Point

- For any point in root locus and along real axis,

$$k = -\frac{1}{G(\sigma)H(\sigma)} = -\frac{(\sigma^2 + 3\sigma + 2)}{\sigma^2 - 8\sigma + 15}$$

- Set the derivative to zero.

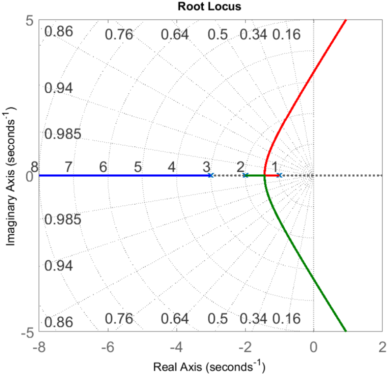
$$\begin{aligned}\frac{dk}{d\sigma} &= \left(\sigma^2 - 8\sigma + 15\right)(2\sigma + 3) - \left(\sigma^2 + 3\sigma + 2\right)(2\sigma - 8) = 0 \\ &= -11\sigma^2 + 26\sigma + 16 = 0\end{aligned}$$

- Solving for σ, we find

$$\sigma = -1.145, \quad 3.82$$

Root Locus - Sketch

$j\omega$ Crossings



- The $j\omega$ crossing is a point on root locus that separates stable operation of the system from unstable operation.
- The value of ω at the axis crossing yields the frequency of oscillation.
- The gain k at $j\omega$ crossing yields positive gain for system stability.

- To find $j\omega$ crossing, we can use R-H criterion.
- Forcing a row to zero will yield gain, k.

Root Locus - Sketch

$j\omega$ Crossings- Example

- Find the frequency and gain, k for which root locus crosses imaginary axis for the following system.

$$T(s) = \frac{k(s+3)}{s^4 + 7s^3 + 14s^2 + (8+k)s + 3k}$$

- Form the Routh table

s^4	1	14	$3k$
s^3	7	$(8+k)$	
s^2	$\frac{90-k}{7}$	$3k$	
s	$\frac{-k^2-65k+720}{90-k}$		
s^0	$21k$		

Root Locus - Sketch

$j\omega$ Crossings - Example

- The row corresponding to s is made zero to find k

$$-k^2 - 65k + 720 = 0$$

- From the above equation k is evaluated as

$$k = 9.65$$

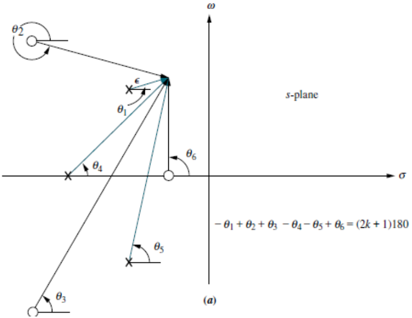
- Forming the polynomial corresponding to s^2 with $k = 9.65$, we have

$$\frac{90-k}{7}s^2 + 3k = 0 = 80.35s^2 + 202.7 = 0$$

- s is found to be $s = \pm 1.59$. Thus root locus crosses the imaginary axis ± 1.59 with the gain of 9.65.

Root Locus - Sketch

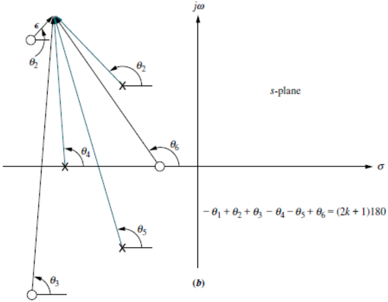
Angle of Arrival and Departure



- Let us calculate the root locus departure angle from the complex poles and arrival angle to the complex zeros.
- If we assume a point on root locus ϵ close to a complex pole, the sum of angles drawn from all finite poles and zeros to this point is odd multiple of 180°.
- Thus only unknown angle in the sum is the angle drawn from the pole that is ϵ close.

Root Locus - Sketch

Angle of Arrival and Departure



- From the previous figure,

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6$$
$$=(2k+1)180$$

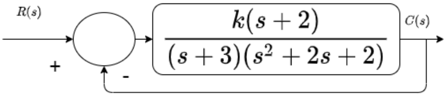
- In similar lines, we assume ϵ close to complex zero. Then,

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6$$
$$=(2k+1)180^{\circ}$$

Root Locus - Sketch

Angle of Arrival and Departure - Example

- Find the angle of departure from complex pole for the following system



- Complex pole is $-1 \pm j$. Calculate the sum of angles drawn to a point ϵ, close to the complex pole $-1+j$.
- The sum of angles at a point near to complex pole $-1+j$ is

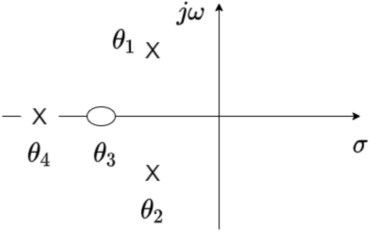
$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = 180$$

- Angles corresponds to

$$\theta_1 \rightarrow \text{Angle of departure} \quad \theta_2 \rightarrow \text{Angle to } \epsilon \text{ point from } -1-j$$
$$\theta_3 \rightarrow \text{Angle from zero at } -2 \quad \theta_4 \rightarrow \text{Angle from pole at } -3.$$

Root Locus - Sketch

Angle of Arrival and Departure - Example



- From figure, angle $\theta_2 = 90^\circ$
- Angle θ_3 is given by

$$\theta_3 = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ$$

- Angle θ_4 is given by

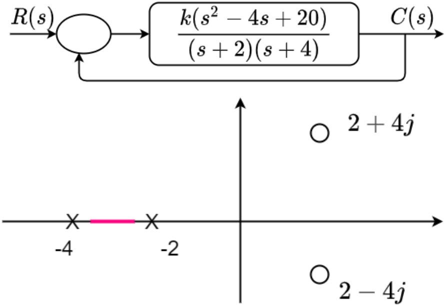
$$\theta_4 = \tan^{-1}\left(\frac{1}{2}\right) = 26.5651^\circ$$

- Solving for θ_1 results in

$$\begin{aligned}-\theta_1 - \theta_2 + \theta_3 - \theta_4 &= 180\\ -\theta_1 - 90 + 45 - 26.56 &= 180\\ \theta_1 &= -251.56^\circ\end{aligned}$$

Root Locus - Sketch

- Sketch the root locus for the following system



- Number of Branches : As there are two poles, there are two root locus branches.
- Real Axis Segments : Root locus along real axis exist to the odd number of poles and zeros.

- It can be seen from the pole zero plot, root locus exist between poles -2 and -4.

Root Locus - Sketch

- Asymptotes : There are no asymptotes as the two branches of root locus reaches two open loop zeros as $k \to \infty$
- Break away point : To find the break away point, set

$$k = -\frac{1}{G(\sigma)H(\sigma)}$$
$$\frac{dk}{d\sigma} = 0 \implies 10s^2 - 24s - 152 = 0$$

- Solving the above equation results in $s = -2.88$. Hence the root locus breaks at $s = -2.88$.
- $j\omega$ crossing: Consider the characteristic equation

$$\begin{aligned} 1 + G(s)H(s) =& s^2 + 6s + 8 + ks^2 - 4ks + 20k = 0 \\ =& (1+k)s^2 + (6-4k)s + 8 + 20k = 0 \end{aligned}$$

Root Locus - Sketch

- Form a Routh table

s^2	$(1+k)$	$8+20k$
s	$(6-4k)$	
s^0	$(8+20k)$	

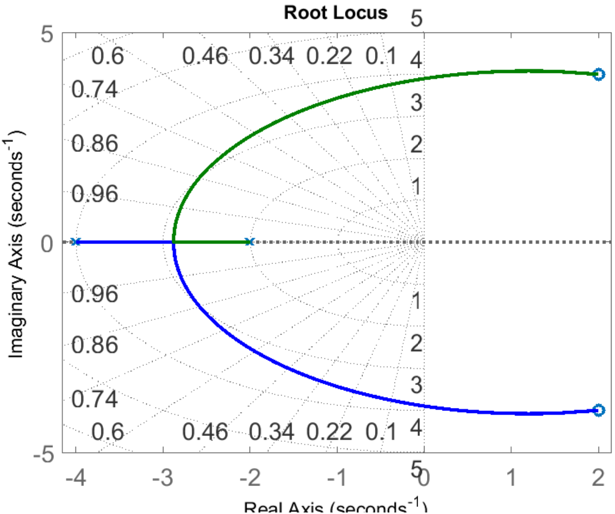
- First column should have no sign changes,

$$6-4k>0 \implies k>1.5$$
$$8+20k>0 \implies k>-0.4$$

- We consider only positive gain. Hence system is unstable for $k>1.5$.
- Consider the polynomial corresponding to s^2,

$$(1+k)s^2+(8+20k)\Big|_{k=1.5}=0 \implies s=\pm j3.9$$

Root Locus - Sketch



Root Locus - Sketch

Example

- Find the $j\omega$ crossing, gain k and break in point for the following system

$$G(s) = \frac{k(s-2)(s-4)}{s^2+6s+25}$$

Root Locus - Sketch

Solution

- Find the $j\omega$ crossing, gain k and break away point for the following system

$$G(s) = \frac{k(s-2)(s-4)}{s^2+6s+25}$$

- The characteristic equation is

$$1 + G(s)H(s) = (1+k)s^2 + (6-6k)s + 25 + 8k = 0$$

- Form the Routh table

s^2	$(1+k)$	$25+8k$
s	$(6-6k)$	
s^0	$(25+8k)$	

Root Locus - Sketch

Solution

- The row corresponding to s should be zero for $j\omega$ crossing. Hence

$$6 - 6k = 0 \implies k = 1$$

- Consider the row corresponding to s^2 to find frequency at which it is crossing $j\omega$ axis.

$$(1+k)s^2 + 25 + 8k\Big|_{k=1} = 0 \implies s^2 = -\frac{33}{2} = \pm j4.06$$

- To find the break in point, consider

$$k = -\frac{1}{G(s)H(s)}$$

$$k = -\frac{s^2 + 6s + 25}{s^2 - 6s + 8}$$

Root Locus - Sketch

Solution

- To find the break in point along real axis, $s = \sigma$ and set

$$\frac{dk}{d\sigma} = 0$$

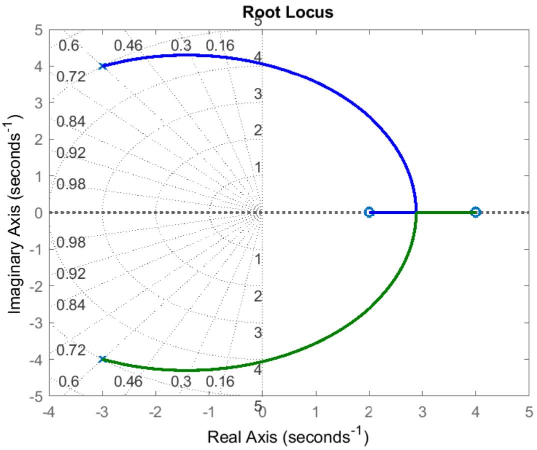
- The resulting equation in s is

$$12s^2 + 34s - 198 = 0$$

- Solving the above equation for s results in break in point. $s = 2.89$

Root Locus - Sketch

Solution



Root Locus - Pole Sensitivity

- Root locus is a plot of closed loop poles as system parameter (gain) is varied.
- We would like to know the extent to which changes in parameter value affects the performance of the system.
- Root sensitivity: It is defined as fractional change in system's closed loop pole to fractional change in system parameter.

$$S_{s:k} = \frac{\frac{\delta s}{s}}{\frac{\delta k}{k}} = \frac{k}{s}\frac{\delta s}{\delta k}$$

- The actual change in poles can be approximated as

$$\Delta s = s S_{s:k} \frac{\Delta k}{k}$$

- s is actual pole and k is actual gain.

Root Locus - Pole Sensitivity

Example

- Find the root sensitivity of the following system at $s = -9.47$ $(k = 5)$. Also calculate change in pole location for 10% change in k.

$$T(s) = s^2 + 10s + k$$

- Differentiate the characteristic equation w.r.t k.

$$2s\frac{\delta s}{\delta k} + 10\frac{\delta s}{\delta k} + 1 = 0$$
$$\frac{\delta s}{\delta k} = -\frac{1}{2s + 10}$$

- Sensitivity is given by

$$S_{s:k} = \left.\frac{k}{s}\frac{\delta s}{\delta k}\right|_{s=-9.47,k=5} = -0.059$$

Root Locus - Pole Sensitivity

Example

- The actual change in poles is given by

$$
\begin{aligned}
\Delta s =& s S_{s:k} \frac{\Delta k}{k} \\
=& -9.47(-0.059)(0.1) = 0.055
\end{aligned}
$$

- Hence closed loop poles will move 0.055 units right for 10% change in gain.

References I

- Katsuhiko Ogata: "*Modern Control Engineering*", Pearson Education, Inc., Upper Saddle River, New Jersey, Fifth Edition, 2010.
- Farid Golnaraghi and Benjamin C. Kuo: "*Automatic Control Systems*", John Wiley & Sons, Inc., New Jersey, Ninth Edition, 2010.
- Norman S. Nise: "*Control Systems Engineering*", John Wiley & Sons, Inc., New Jersey, Sixth Edition, 2011.