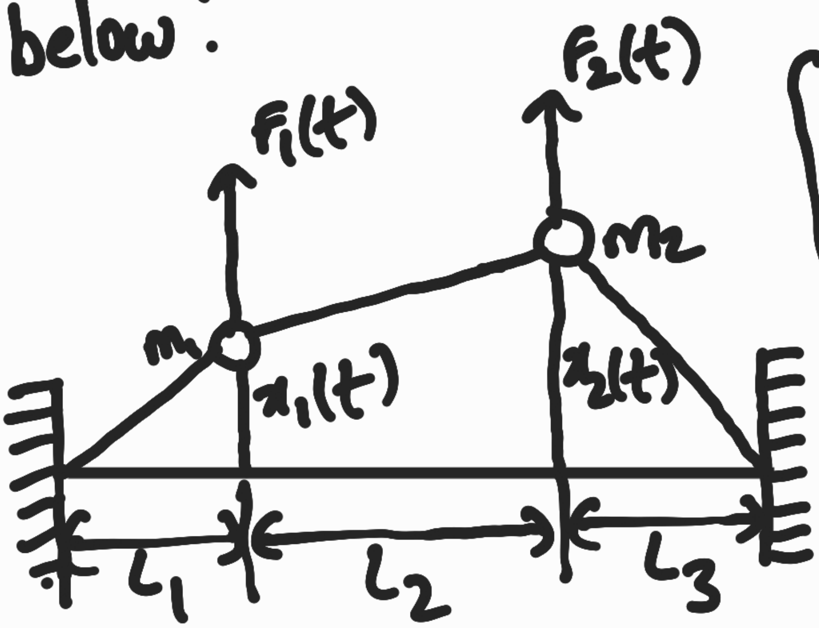
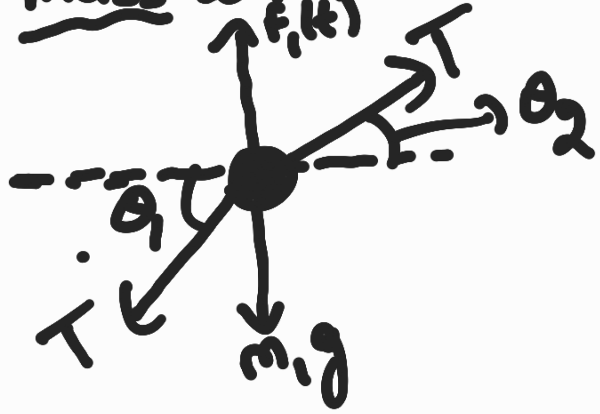
I) Consider a 2 DOF system shown in the figure below:

[Here x_1, x_2 are displacement of masses m_1 and m_2]

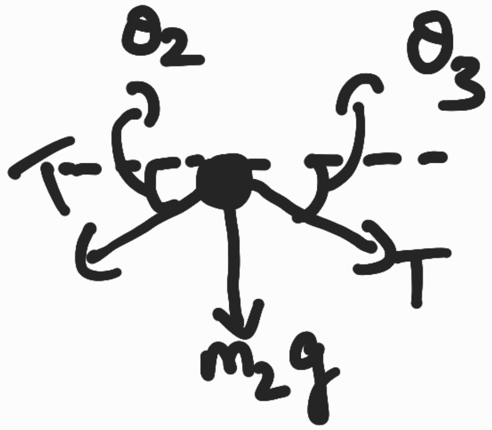


1^{st} <u>Step</u>: Write the FBD for each mass

<u>for mass m_1</u>:



<u>for mass m_2</u>:



[Assuming all strings have same Tension (T)]

Under the assumption of small displacement,

$\sin\theta \approx \tan\theta \quad \therefore \quad \sin\theta_1(t) \approx \dfrac{x_1(t)}{L_1}$

$$\sin\theta_2(t) \approx \frac{x_2(t) - x_1(t)}{L_2}$$

$$\sin\theta_3(t) \approx \frac{x_2(t)}{L_3}$$

2nd step: Write the equations of motion

For mass m_1:

$$F_1(t) - m_1 g - T\sin\theta_1(t) + T\sin\theta_2(t) = m_1 \frac{d^2 x_1(t)}{dt^2}$$

For mass m_2:

$$F_2(t) - m_2 g - T\sin\theta_2(t) - T\sin\theta_3(t) = m_2 \frac{d^2 x_2(t)}{dt^2}$$

On arranging:

$$F_1(t) = m_1 \frac{d^2 x_1(t)}{dt^2} + m_1 g + T\sin\theta_1(t) - T\sin\theta_2(t)$$

$$F_2(t) = m_2 \frac{d^2 x_2(t)}{dt^2} + m_2 g + T\sin\theta_2(t) + T\sin\theta_3(t)$$

$m_1 g$, $m_2 g$ don't contribute to the vibration part of the problem.

$\vec{x}$ can be written as sum of equilibrium (x_e) (static) and time dependent term $(\tilde{x})$

So, the equations of motion for vibration of the masses is

$$m_1 \frac{d^2 \tilde{x}_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2}\right)\tilde{x}_1 - \frac{T}{L_2}\tilde{x}_2 = f_1$$

$$m_2 \frac{d^2 \tilde{x}_2}{dt^2} - \frac{T}{L_2}\tilde{x}_1 + \left(\frac{T}{L_2} + \frac{T}{L_3}\right)\tilde{x}_2 = f_2$$

(∴ Substitute the expressions for θ_1, θ_2, and θ_3
$(\tilde{\ })$ is removed while solving.)

$\boxed{\text{Question}}$: Obtain the free vibration response of the system shown above using the modal analysis approach. ($m_1 = m$, $m_2 = 2m$, $L_1 = L_2 = L$, $L_3 = \frac{L}{2}$)
System is subjected to the following Initial Conditions: $\vec{x}(0) = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}$, $\dot{\vec{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Sol) For $f_1, f_2 = 0$, the eqn's of motion are: $m_1 \frac{d^2 x_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2}\right)x_1 - \frac{T}{L_2}x_2 = 0$

$$m_2 \frac{d^2 x_2}{dt^2} - \frac{T}{L_2} x_1 + \left(\frac{T}{L_2} + \frac{T}{L_3}\right) x_2 = 0$$

$$\Rightarrow m\ddot{x}_1 + \frac{2T}{L} x_1 - \frac{T}{L} x_2 = 0$$

$$2m\ddot{x}_2 - \frac{T}{L} x_1 + \frac{3T}{L} x_2 = 0$$

Compare these eqn's with the Generalized vibration eqn: (Undamped and free)

$$M\ddot{x} + Kx = 0$$ [M and K are mass and Stiffness matrices]

$$\therefore M = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}$$

$$K = \begin{bmatrix} \frac{2T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & \frac{3T}{L} \end{bmatrix}.$$

<u>For a Modal analysis approach</u>,

$$\vec{x}(t) = q_1(t)\vec{u_1} + q_2(t)\vec{u_2}$$

where $q_1(t)$, $q_2(t)$ are called modal co-ordinates. $\vec{u_1}$, $\vec{u_2}$ are mode shapes corresponding to the first and second natural frequencies.

$q_1(t)$, $q_2(t)$ satisfy the decoupled equations :

$$\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0$$

$$\ddot{q}_2(t) + \omega_2^2 q_2(t) = 0$$

where $q_1(t) = C_1 \cos(\omega_1 t - \phi_1)$

$q_2(t) = C_2 \cos(\omega_2 t - \phi_2)$

Eqn's are decoupled because, $U^T M U$, $U^T K U$ are diagonalized.

<u>Step 3:-</u> Solve the Eigen value Problem

$$K U = \omega^2 M U \qquad \underline{\vec{u_1}} \quad \underline{\vec{u_2}}$$

where $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ (Modal matrix)

is a matrix of eigen vectors

The non-trivial soln to this eqn exists only when $\det(K - \omega^2 M) = 0$

$$\Rightarrow \left| \begin{bmatrix} \frac{2T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & \frac{3T}{L} \end{bmatrix} - \begin{bmatrix} m\omega^2 & 0 \\ 0 & 2m\omega^2 \end{bmatrix} \right| = 0$$

$$\left|\begin{bmatrix} \frac{2T}{L} - m\omega^2 & -\frac{T}{L} \\ -\frac{T}{L} & \frac{3T}{L} - 2m\omega^2 \end{bmatrix}\right| = 0$$

$$\left(\frac{2T}{L} - m\omega^2\right)\left(\frac{3T}{L} - 2m\omega^2\right) - \frac{T^2}{L^2} = 0$$

$$\frac{6T^2}{L^2} - 7m\omega^2\frac{T}{L} + 2m^2\omega^4 - \frac{T^2}{L^2} = 0$$

$$2m^2\omega^4 - 7m\frac{T}{L}\omega^2 + \frac{5T^2}{L^2} = 0$$

This is a quadratic eq^n in $\omega^2 \Rightarrow$

$$2m^2(\omega^2)^2 - 7m\frac{T}{L}\omega^2 + \frac{5T^2}{L^2} = 0$$

$$\omega^2 = \frac{7m\frac{T}{L} \pm \sqrt{49\frac{m^2T^2}{L^2} - 40\frac{m^2T^2}{L^2}}}{4m^2}$$

$$= \frac{7\frac{mT}{L} \pm 3\frac{mT}{L}}{4m^2} = \frac{T}{mL}, \frac{5T}{2mL}$$

are the eigen values of system.

Natural frequencies are :-

$$\omega_1 = \sqrt{\frac{T}{mL}} \quad , \quad \omega_2 = \sqrt{\frac{5T}{2mL}}$$

For each ω_i,

$$K u_i = \omega_i^2 M u_i$$

Here u_i's are eigen vectors

u_{21} – 2nd particle at 1st mode

1st mode :

$$\begin{bmatrix} \frac{2T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & \frac{3T}{L} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} m\omega_1^2 & 0 \\ 0 & 2m\omega_1^2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{2T}{L} u_{11} - \frac{T}{L} u_{21} \\ -\frac{T}{L} u_{11} + \frac{3T}{L} u_{21} \end{bmatrix} = \begin{bmatrix} m\omega_1^2 u_{11} \\ 2m\omega_1^2 u_{21} \end{bmatrix}$$

$$\Rightarrow \left[\frac{2T}{L} - m\omega_1^2\right] u_{11} = \frac{T}{L} u_{21}$$

$$\left[\frac{2T}{\cancel{L}} - \cancel{m}\left(\frac{T}{\cancel{m}L}\right)\right] u_{11} = \frac{T}{L} u_{21}$$

$$u_{11} = u_{21} \quad \text{or} \quad \frac{u_{11}}{u_{21}} = 1$$

$$\Rightarrow -\frac{T}{L} u_{11} = \left(2m\left(\frac{T}{mL}\right) - \frac{3T}{L}\right) u_{21}$$

$$u_{11} = u_{21} \quad \text{or} \quad \frac{u_{11}}{u_{21}} = 1$$

From here, we can see that the ratio of eigen displacements for any given mode is constant. We can have infinitely many possibilities of u_{11}, u_{21} as long as their ratio is equal to 1.

So, for uniqueness, we chose one of the values as 1 (Means dividing by the largest displacement). This is arbitrary.

If $u_{11} = 1$, then $u_{21} = 1$

$$\vec{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>2^{nd} Mode :-</u>

$$\begin{bmatrix} \frac{2T}{L} & -\frac{T}{L} \\ -\frac{T}{L} & \frac{3T}{L} \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} m\omega_2^2 & 0 \\ 0 & 2m\omega_2^2 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

Solving this eq^n same as above, we get the ratio of $\frac{u_{22}}{u_{12}} = -0.5$

$$u_{12} = 1, \quad u_{22} = -0.5$$

$$\vec{u_2} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

These are natural eigen vectors.

we can see that $u_1^T M u_2 = 0$

$$u_1^T K u_2 = 0$$

and $u_1^T K u_1 = \omega_1^2 u_1^T M u_1$

Now, we normalize u_1, u_2 such that

$$u_1'^T M u_1' = 1, \quad u_1'^T K u_1' = \omega_1^2$$

$$u_2'^T M u_2' = 1, \quad u_2'^T K u_2' = \omega_2^2$$

where (') indicates normalized values.

$$u_1^T M u_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ 2m \end{bmatrix}$$

$$= 3m$$

so, divide u_1 with $\sqrt{3m}$ because,

$$\frac{u_1^T}{\sqrt{3m}} M \frac{u_1}{\sqrt{3m}} = \frac{3m}{3m} = I.$$

$\therefore \vec{u}_1' = \frac{\vec{u_1}}{\sqrt{3m}}$ is written as ϕ_1

Similarly

$$u_2^T M u_2 = \begin{bmatrix} 1 & -0.5 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.5 \end{bmatrix} \begin{bmatrix} m \\ -m \end{bmatrix}$$

$$= m + \frac{m}{2} = \frac{3m}{2}$$

$$\Rightarrow \vec{u_2}' = \frac{\vec{u_2}}{\sqrt{\frac{3m}{2}}} = \phi_2$$

$$\phi_1 = \begin{bmatrix} \frac{1}{\sqrt{3m}} \\ \frac{1}{\sqrt{3m}} \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \sqrt{\frac{2}{3m}} \\ -\frac{1}{\sqrt{6m}} \end{bmatrix}$$

Step 4:- Use the initial conditions in the modal analysis approach for determining $q_1(t)$ and $q_2(t)$

There are 4 unknowns $C_1, \Phi_1, C_2,$ and Φ_2. So 4 initial conditions are required.

We know that

$$\vec{x}(t) = q_1(t)\vec{\phi_1} + q_2(t)\vec{\phi_2}$$

$\vec{x}(0) = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}$ $\dot{\vec{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are given initial conditions

Here ϕ is phase

$q_1(0) = C_1 \cos\phi_1$, $q_2(0) = C_2 \cos\phi_2$

$$\begin{bmatrix} 1.2 \\ 0 \end{bmatrix} = \frac{q_1(0)}{\sqrt{3m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{q_2(0)}{\sqrt{\frac{3m}{2}}} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$1.2 = \frac{q_1(0) + \sqrt{2}\, q_2(0)}{\sqrt{3m}}$$

$$0 = q_1(0) - \frac{1}{\sqrt{2}} q_2(0)$$

$$1.2\sqrt{3m} = \frac{2+1}{\sqrt{2}} q_2(0)$$

$$\Rightarrow q_2(0) = 0.4\sqrt{6m}$$

$$\Rightarrow q_1(0) = 0.4\sqrt{3m}$$

$$\dot{q}_1(t) = -C_1\omega_1 \sin(\omega_1 t - \phi_1)$$

$$\dot{q}_2(t) = -C_2\omega_2 \sin(\omega_2 t - \phi_2)$$

$$q_1(t) = C_1 \cos(\omega_1 t - \phi_1)$$

$$q_2(t) = C_2 \cos(\omega_2 t - \phi_2)$$

$$\dot{q}_1(0) = \dot{q}_2(0) \Rightarrow \sin\phi_1 = \sin\phi_2 = 0$$

$\phi_1 = 0$, $\phi_2 = 0$ can be taken

$$q_1(0) = C_1 = 0.4\sqrt{3m}$$
$$q_2(0) = C_2 = 0.4\sqrt{6m}$$

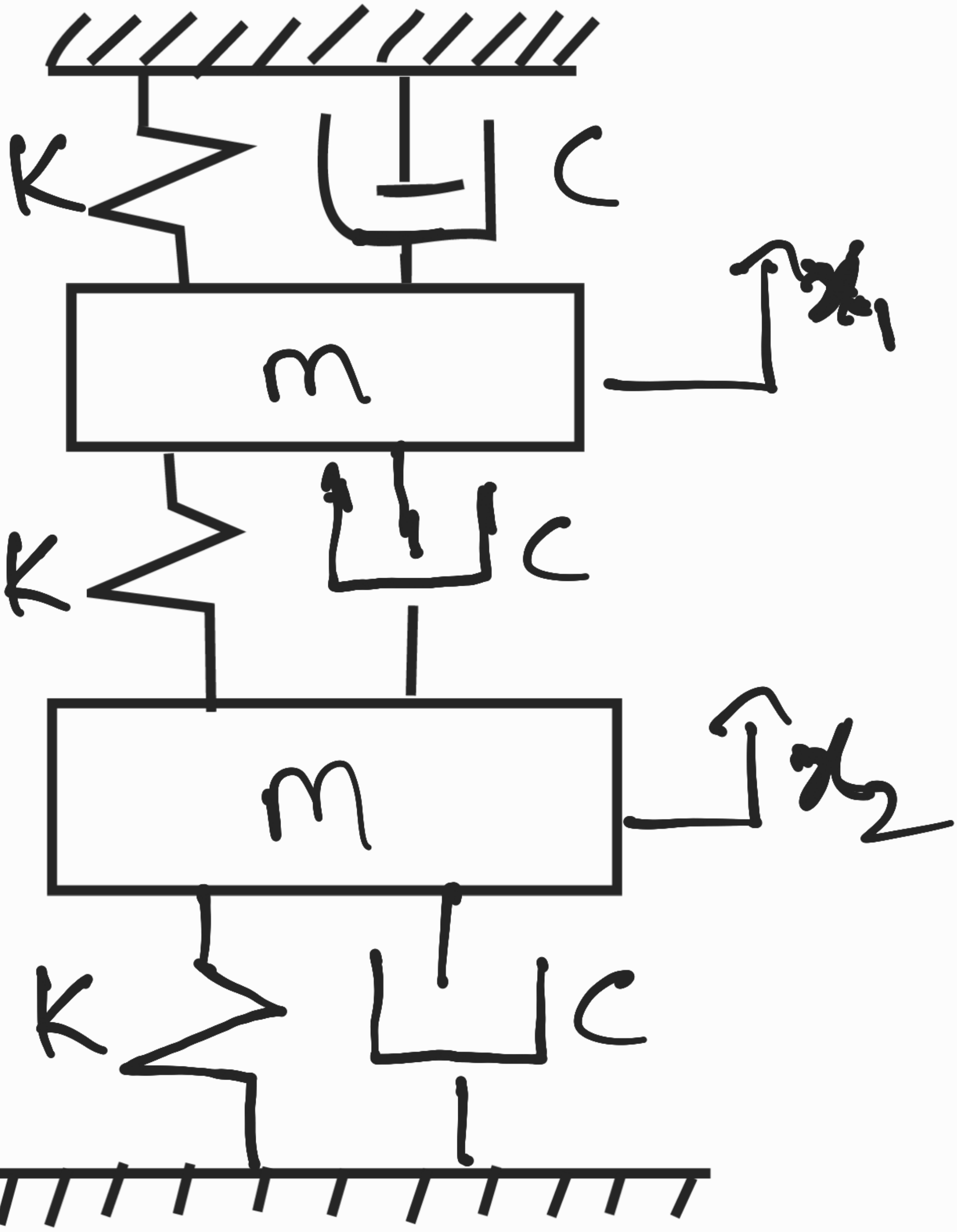
$$q_1(t) = 0.4\sqrt{3m}\cos\omega_1 t$$
$$q_2(t) = 0.4\sqrt{6m}\cos\omega_2 t$$

and
$$\vec{x}(t) = \frac{0.4\sqrt{3m}\cos\omega_1 t}{\sqrt{3m}}\begin{bmatrix}1\\1\end{bmatrix} + \frac{0.4\sqrt{6m}\cos\omega_2 t}{\sqrt{\frac{3m}{2}}}\begin{bmatrix}1\\-0.5\end{bmatrix}$$

$$\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = 0.4\cos\omega_1 t\begin{bmatrix}1\\1\end{bmatrix} + 0.8\cos\omega_2 t\begin{bmatrix}1\\-0.5\end{bmatrix}$$

II) Consider a Damped Free 2 DOF System shown below

$$\begin{bmatrix} m = 10\,\text{Kg} \\ K = 10^5\,\text{N/m} \\ C = 40\,\text{Ns/m} \end{bmatrix}$$



Obtain the response of the System when the initial conditions are

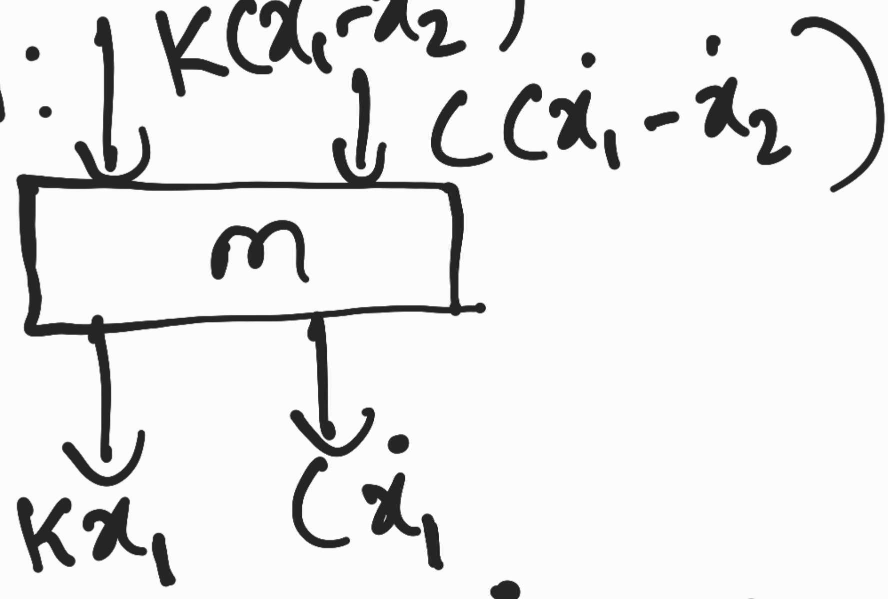
$$\vec{x}(0) = \begin{bmatrix} 10^{-3} \\ 0 \end{bmatrix} \qquad \dot{\vec{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Sol^n)

Step 1 :- Draw the FBD for each mass

From above example, we can see that gravity force can be neglected as it doesn't play a role in the vibration. It is absorbed in the static equilibrium term.

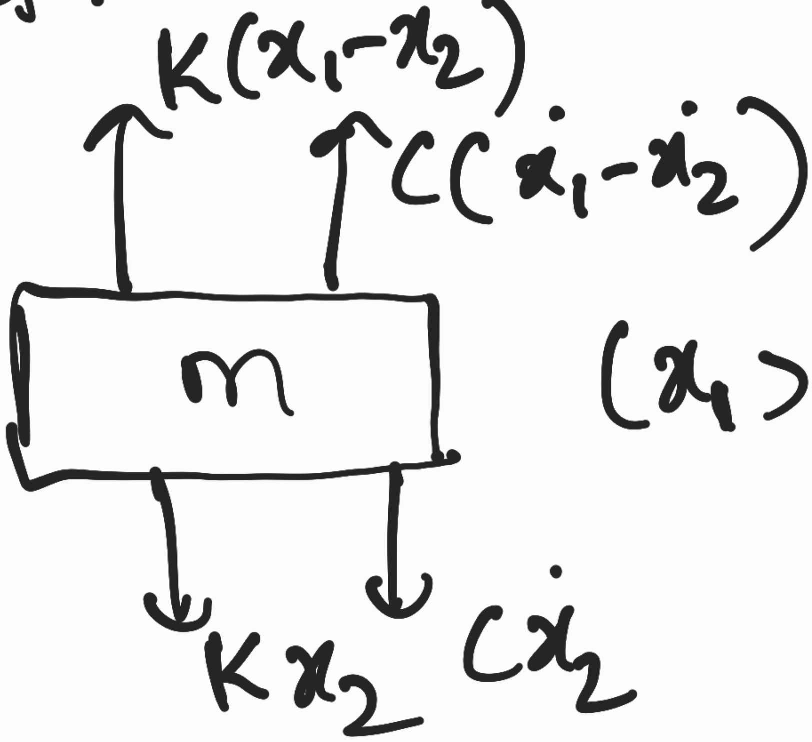
For 1st mass :



$$m\ddot{x}_1 = -Kx_1 - c\dot{x}_1 - Kx_1 + Kx_2 - c\dot{x}_1 + c\dot{x}_2$$

$$m\ddot{x}_1 + 2Kx_1 - Kx_2 + 2c\dot{x}_1 - c\dot{x}_2 = 0$$

For 2^{nd} mass :



$(x_1 > x_2)$

$$m\ddot{x}_2 = Kx_1 - 2Kx_2 + C\dot{x}_1 - 2C\dot{x}_2$$

$$m\ddot{x}_2 - Kx_1 + 2Kx_2 - C\dot{x}_1 + 2C\dot{x}_2 = 0$$

Matrix Formulation :-

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2C & -C \\ -C & 2C \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$+\begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$C = \begin{bmatrix} 2C & -C \\ -C & 2C \end{bmatrix}$$

$$K = \begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}$$

<u>Step 2</u>:- Check whether C can be expressed in terms of M, K as $C = \alpha M + \beta K$

Here, we can see that it is possible because all are 2×2 square matrices.

Step 3 :- Now solve for Eigen values and Eigen vectors needed to decouple the undamped eq^n.

$$\det(K - \omega^2 M) = 0$$

$$\left| \begin{bmatrix} 2K - \omega^2 m & -K \\ -K & 2K - \omega^2 m \end{bmatrix} \right| = 0$$

$$4K^2 - 4Km\omega^2 + m^2\omega^4 - K^2 = 0$$

$$m^2\omega^4 - 4Km\omega^2 + 3K^2 = 0$$

$$\omega^2 = \frac{4mK \pm \sqrt{16K^2m^2 - 12m^2K^2}}{2m^2}$$

$$= \frac{4mK \pm 2mK}{2m^2} = \frac{K}{m} \text{ or } \frac{3K}{m}$$

If $\omega^2 = \frac{K}{m}$, the eigen vector $\vec{u_1}$ will satisfy the eigen value eq^n.

$$K\vec{u_1} = \omega_1^2 M \vec{u_1}$$

$$\begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} \omega_1^2 m & 0 \\ 0 & \omega_1^2 m \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}$$

$$2Ku_{11} - Ku_{21} = \omega_1^2 m\, u_{11}$$

$$Ku_{21} = \left(2K - m\omega_1^2\right) u_{11}$$

$$\frac{u_{21}}{u_{11}} = \frac{2K - m\omega_1^2}{K}$$

Let $u_{11} = 1$, $u_{21} = \dfrac{2K - m\omega_1^2}{K}$

$$\therefore \vec{u_1} = \begin{bmatrix} 1 \\ \dfrac{2K - m\omega_1^2}{K} \end{bmatrix}$$

Similarly,

$$K\vec{u_2} = \omega_2^2 M \vec{u_2}$$

$$\begin{bmatrix} 2K & -K \\ -K & 2K \end{bmatrix}\begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} \omega_2^2 m & 0 \\ 0 & \omega_2^2 m \end{bmatrix}\begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

$$\vec{u_2} = \begin{bmatrix} 1 \\ \dfrac{2K - m\omega_2^2}{K} \end{bmatrix} \quad \left(\because u_{12} = 1 \right) \quad \frac{u_{22}}{u_{12}} = \frac{2K - m\omega_2^2}{K}$$

Now using $m = 10\,kg$, $K = 10^5\,N/m$, we get $\omega_1 = 100$ rad/sec, $\omega_2 = 173.2$ rad/sec

$$\vec{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{u_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now normalize the eigen vectors such that $\phi_1^T M \phi_1 = 1$, $\phi_2^T M \phi_2 = 1$

$$\phi_1^T K \phi_1 = \omega_1^2, \quad \phi_2^T K \phi_2 = \omega_2^2$$

$$u_1^T M u_1 = 2m = 20$$

$$u_2^T M u_2 = 2m = 20$$

$$\therefore \vec{\phi_1} = \frac{\vec{u_1}}{\sqrt{2m}} = \frac{\vec{u_1}}{\sqrt{20}}, \quad \vec{\phi_2} = \frac{\vec{u_2}}{\sqrt{2m}} = \frac{\vec{u_2}}{\sqrt{20}}$$

$$\phi = \frac{1}{\sqrt{20}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

<u>Step 4:-</u> Now solve for Damping vector ξ_i.

$$C = \alpha M + \beta K$$

where U is modal matrix

$$U = [\vec{\phi}_1 \; \vec{\phi}_2]$$

$$U^T C U = \alpha U^T M U + \beta U^T K U$$

$$= \alpha I + \beta \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{20} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c & 3c \\ c & -3c \end{bmatrix} \quad (\because c = 40)$$

$$= \frac{1}{20} \begin{bmatrix} 2c & 0 \\ 0 & 6c \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 40 & 0 \\ 0 & 3(40) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} \beta 10^4 & 0 \\ 0 & \beta (173.2)^2 \end{bmatrix}$$

$$4 = \alpha + 10^4 \beta$$

$$12 = \alpha + (173.2)^2 \beta$$

solving for α and β we get

$$\beta = \frac{8}{(173.2^2 - 10^4)} \simeq 4 \times 10^{-4}$$

$$\alpha = 3.5 \times 10^{-4}$$

$$\alpha + \beta \omega_1^2 = 2 \zeta_1 \omega_1$$

$$\zeta_1 = \frac{\alpha + \beta \omega_1^2}{2 \omega_1} = \frac{\cancel{4}\,2}{\cancel{2} \times 100}$$

$$= 0.02 \text{ Nm/s}$$

$$\zeta_2 = \frac{\alpha + \beta\omega_2^2}{2\omega_2} = \frac{12}{2(173.2)}$$

$$= 0.0346 \text{ Nm/s}$$

Step5: Apply the initial conditions.

$$\begin{bmatrix} 10^{-3} \\ 0 \end{bmatrix} = \frac{q_1(0)}{\sqrt{20}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{q_2(0)}{\sqrt{20}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\sqrt{20}\, 10^{-3} = q_1(0) + q_2(0)$$

$$0 = q_1(0) - q_2(0)$$

$$2q_2(0) = 10^{-3}\sqrt{20}, \quad q_2(0) = 5 \times 10^{-4} \times \sqrt{20}$$

$$q_1(0) = 5 \times 10^{-4} \times \sqrt{20}$$

Step6: Write the damped sol^n

$$q_1(t) = q_1(0)\, e^{-\zeta_1\omega_1 t}\left(\cos\omega_{1d}t + \frac{\zeta_1}{\sqrt{1-\zeta_1^2}}\sin\omega_{1d}t\right)$$

$$q_2(t) = q_2(0)\, e^{-\zeta_2\omega_2 t}\left(\cos\omega_{2d}t + \frac{\zeta_2}{\sqrt{1-\zeta_2^2}}\sin\omega_{2d}t\right)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Finally
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\sqrt{20}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

Also** solve the examples:
Meirovitch (2001) - Ex: 5.2, Ex: 7.5, 7.6
(Although it won't be asked in Exam, you can look at Example 7.10 for forced viscous damping system)