AE 707: Tutorial 9a. Compressible Aerodynamics

1. Consider a NACA 0012 airfoil at zero angle of attack and freestream Mach number $M_\infty = 0.79$. Assume a minimum value of $c_p = -0.43$ for the same airfoil at the same angle of attack in *incompressible* flow.

   (a) i. What is the critical Mach number for this configuration?

   ii. Hence draw a qualitative (inviscid) picture of the Mach number distribution on the airfoil.

   (b) How is the same flow field likely to look if the angle of attack is increased to 3° at the same M_∞?

   Solution:

   (a) Before actually finding the critical Mach number, let us quickly assess whether the given freestream Mach number is sub-critical or super-critical. We start by assuming that the given freestream Mach number is the critical value (i.e., $M_{cr} = M_\infty = 0.79$), and ascertain the corresponding critical pressure coefficient using the relation derived in the lectures:

$$
\begin{aligned}
c_{p,cr} &= \frac{2}{\gamma M_{cr}^2}\left[\left\{\frac{1+(\gamma-1)M_{cr}^2/2}{1+(\gamma-1)/2}\right\}^{\frac{\gamma}{\gamma-1}} - 1\right] \\
&= \frac{2}{1.4\times 0.79^2}\left[\left\{\frac{1+(1.4-1)0.79^2/2}{1+(1.4-1)/2}\right\}^{\frac{1.4}{1.4-1}} - 1\right] \\
&= -0.464.
\end{aligned}
$$

   Next, we determine the minimum pressure coefficient on the airfoil at the given freestream Mach number, using the simplest of compressibility corrections, viz. the Prandtl-Glauert compressibility correction. We find that

$$
c_{p,\min} = \frac{c^0_{p,\min}}{\sqrt{1-M_\infty^2}} = \frac{-0.43}{\sqrt{1-0.79^2}} = -0.701.
$$

   Clearly this is far lower than the critical pressure ratio predicted under the assumption that this M_∞ is the critical one. Thus, we can directly conclude that the critical Mach number must be lower than 0.79.

i. Now, we try to determine the critical Mach number numerically. For this, we have to solve the following equation

$$(c_{p,cr} =)\, \frac{2}{\gamma M_{cr}^2}\left[\left\{\frac{1+(\gamma-1)M_{cr}^2/2}{1+(\gamma-1)/2}\right\}^{\frac{\gamma}{\gamma-1}} - 1\right] = \frac{c_p^0}{\sqrt{1-M_{cr}^2}}.$$

This is of course a nonlinear equation, and its solution has to be sought iteratively.

Since we already know that $M_{cr} < 0.79$, let us initiate the iterations with $M_{cr} = 0.7$. With this, the LHS and RHS come out to be -0.78 and -0.6. Clearly, this is too low an estimate.

Next, let us guess that $M_{cr} = 0.75$. With this, the LHS and RHS come out to be -0.59 and -0.65. Now, we have gone too high!

Our next guess is a number between 0.7 and 0.75, say 0.73. This yields the LHS and RHS as -0.66 and -0.63. This guess is thus slightly too low.

Our next guess is a number between 0.73 and 0.75, say 0.74. This yields the LHS and RHS as -0.63 and -0.64. This guess is thus slightly too high.

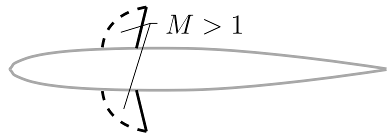
Finally, we conclude that the critical Mach number is between 0.73 and 0.74, which is sufficiently accurate for our purposes.

By the way, with a programming environment like Python, this problem can be solved with the following codelet:

```
from scipy.optimize import root_scalar
g = 1.4 #Specific heat ratio
Cp0 = -0.43 #Minimum incompressible pressure coeff.
M0 = 0.01 #‘Left’ bound for root finding
M1 = 0.79 #‘Right’ bound for root finding
# Aerodynamic relation for pressure coeff (Prandtl-Glauert rule)
CpAe = lambda M:Cp0/(1-M**2)**0.5
# Thermodynamic relation for critical pressure
CpcrTh = lambda M:2/(g*M**2)*(((1+(g-1)*M**2/2)/(1+(g-1)/2)) \
       **(g/(g-1))-1)
# Finding root of function h(M) = CpAe(M)-CpcrTh(M)’
soln = root_scalar(lambda M:(CpAe(M)-CpcrTh(M)),x0=M0,x1=M1)
# The final solution
Mcr = soln.root
```

This yields the result `Mcr = 0.737`, which is of course in the range determined by hand calculations.

ii. Since the flow is just super-critical, we expect a small pocket of supersonic flow to develop somewhere in the fore portion of the upper surface; it will be terminated by an approximately normal shock (of weak strength). Since the airfoil is symmetric and at zero angle of attack, an identical supersonic pocket will develop on the lower surface too. Below is a sketch.

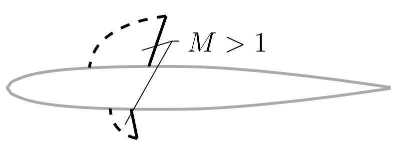


(b) At $\alpha = 3°$, the following changes will be observed in the Mach contours, vis-à-vis the $\alpha = 0°$ case:

- On the upper surface:
  - The supersonic pocket will be larger.
  - The shock will be stronger.
  - It will move further aft.
  - There will be greater acceleration of the flow.
- On the lower surface, we will observe the opposite effects:
  - The supersonic pocket will be smaller, and may vanish altogether.
  - The shock will be weaker, or may not even exist.
  - If it exists, the shock will move towards the leading edge.
  - There will be lesser acceleration of the flow.

Finally, the critical Mach number will be lowered (i.e., $M_{cr} < 0.737$).

Below is a sketch of the flow. It reiterates the ambiguity in our answer.



OR