AE 308: Control Theory
AE 775: System Modeling, Dynamics and Control

Lecture 8: Block Diagram Reduction Techniques

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Introduction

- Block diagram technique is helpful to find the transfer function of the overall system.
- Objectives here are the following:
  - To study block diagrams and their underlying mathematics.
  - To obtain transfer function of system through block diagram reduction techniques.
  - To introduce signal flow graphs.
  - To obtain transfer function of the system through Mason's gain formula.

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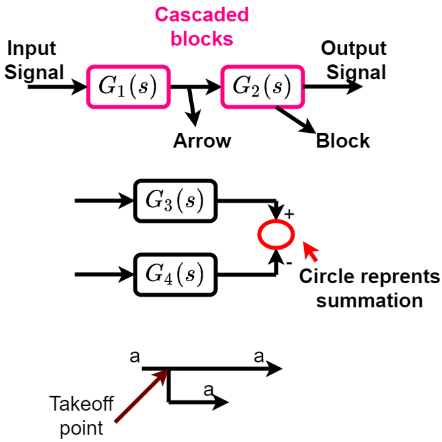
Block Diagram Representation - Introduction

Why Block Diagrams are Important?

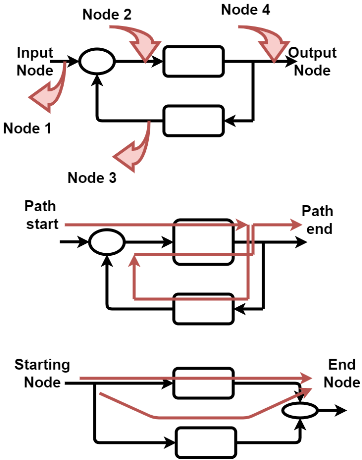
- Block diagram of a system is a pictorial representation of functions performed by each component and flow of the signals.
- In a block diagram, all system variables are linked to each other through functional blocks.
- The transfer functions of the components are present in the blocks. These blocks are connected by arrows to indicate the direction of the flow of signals.
- The signals pass only in the direction of the arrows.
- Block diagram of a control system explicitly has a unilateral property.

Block Diagram Representation - Definitions



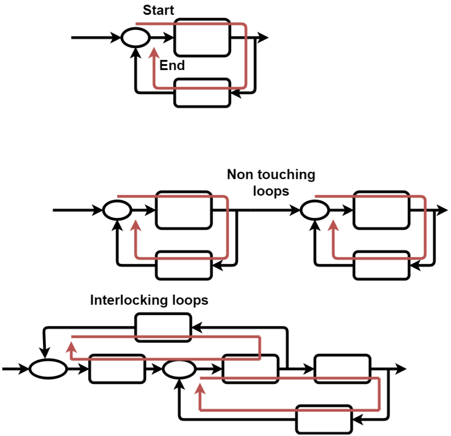
- Two blocks $G_1(s)$ and $G_2(s)$ are connected by arrows.
- The output of first block, $G_1(s)$ is the input signal to the second block $G_2(s)$.
- Blocks and arrows can create the system, however they are not enough to fully capture the complexities of feedback system.
- To capture complete system, summing junctions and take-off points are required.

Introduction - Definitions



- A **Path** is a continuous line that is created, if you place your pen on a node and trace the signal lines in the direction of the arrows.
- The **Forward path** is any path that starts at input node and ends at the output node without ever touching the same node twice.
- Two paths are considered as **Parallel**, if they both start and end at the same node while not sharing any of the same blocks.

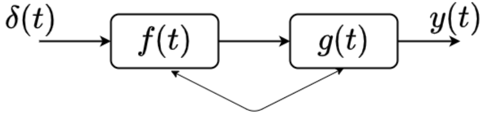
Introduction - Definitions



- A **Loop** is any path that starts and ends at the same node without ever touching any node more than once.
- Two loops can be separate from each other. This is called **Non touching loops**.
- Loops may not be fully nested, but still share some of the nodes and blocks. These are called **Overlapping/Interlocking looks**

Block Diagram Representation - Definitions

- Linear time-invariant (LTI) systems obey the properties of superposition, homogeneity and time-invariance properties.
- These properties allows to move blocks around and simplify diagrams.
- If the system is LTI, then manipulating the blocks is easy.
- As LTI systems are commutative, and obey homogeneity, superposition and time invariance, blocks can be manipulated by swapping their order without disturbing the output of system.



Both the blocks can be interchanged

Block Diagram Representation - Algebra

- The output of the above system is

$$y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

- If the blocks are swapped, the output is

$$y_1(t) = \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

- We need to know show $y(t) = y_1(t)$.
- Let us define $t - \tau = u$.

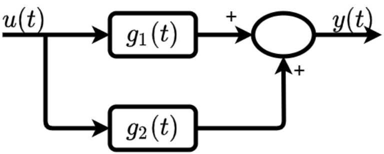
$$dt - d\tau = du$$

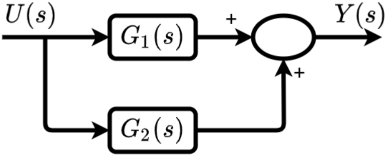
- The above substitution results in

$$\begin{aligned} y_1(t) &= -\int_{\infty}^{-\infty} g(t-u)f(u)du \\ &= \int_{-\infty}^{\infty} g(t-u)f(u)du \implies y_1(t) = y(t) \end{aligned}$$

Block Diagram Representation - Algebra

- It is convenient to create block diagrams in the s domain with transfer functions.
- In s domain, few complex time domain operations become easy.
- Let us witness this fact through an example:





Block Diagram Representation - Algebra

- The output $y(t)$ in time domain is given by

$$\begin{aligned} y(t) &= u(t) * g_1(t) + u(t) * g_2(t) \\ &= u(t) * (g_1(t) + g_2(t)) \end{aligned}$$

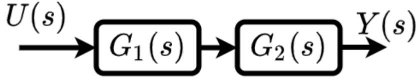
- The output $y(t)$ in s domain is given by

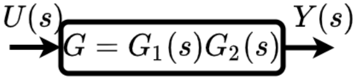
$$\begin{aligned} Y(s) &= U(s)G_1(s) + U(s)G_2(s) \\ &= U(s)\left(G_1(s) + G_2(s)\right) \end{aligned}$$

- In time domain, one needs to solve the convolution integral to obtain $y(t)$.
- In the s domain, the output can be calculated by summing two transfer functions and multiplying with the input function.

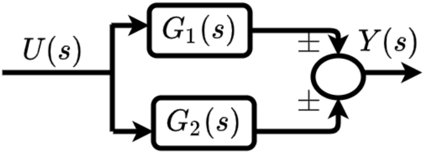
Block Diagram Representation - Systems

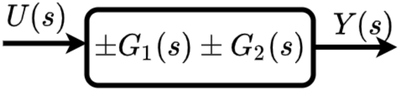
Cascaded System





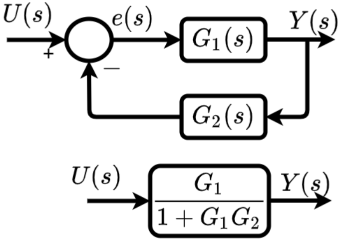
Parallel System





Block Diagram Representation - Systems

Feedback System



- From the block diagram,

$$\begin{aligned} Y(s) &= G_1(s)e(s) \\ e(s) &= U(s) - Y(s)G_2(s) \end{aligned}$$

- Substituting for $Y(s)$ in the error equation

$$\begin{aligned} e(s) &= U(s) - G_1(s)G_2(s)e(s) \\ e(s) &= \frac{U(s)}{1+G_1(s)G_2(s)} \\ Y(s) &= \frac{G_1(s)U(s)}{1+G_1(s)G_2(s)} \end{aligned}$$

Block Diagram Representation - Systems

Feedback System Examples

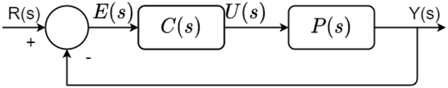


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

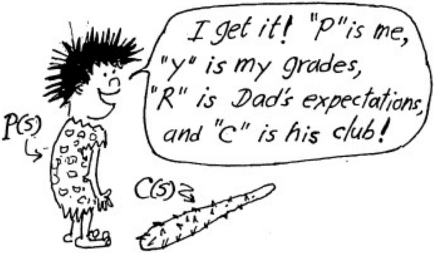
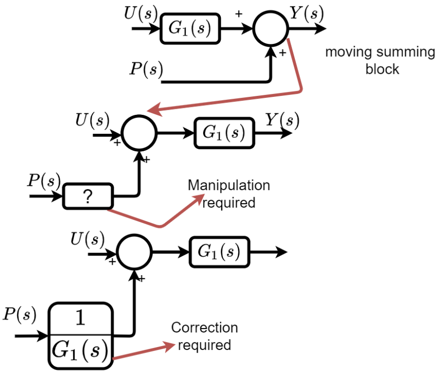


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

Block Diagram Representation - Moving a Summing Junction

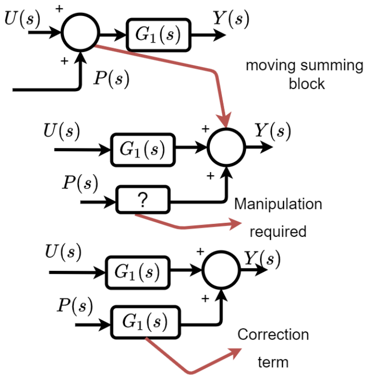
Moving a Summing Junction



- Output of the actual system is
$$Y(s) = U(s)G_1(s) + P(s)$$
- Output of the manipulated system should be same as actual system. The output of the new system is
$$Y(s) = (U(s) + P(s)X)\,G_1(s)$$
- Comparing both the equations,
$$X = \frac{1}{G_1(s)}$$

Block Diagram Representation - Moving a Summing Junction

Moving a Summing Junction



- The output of the system is

$$Y(s) = G_1(s)\,(U(s) + P(s))$$

- Output of the manipulated system is

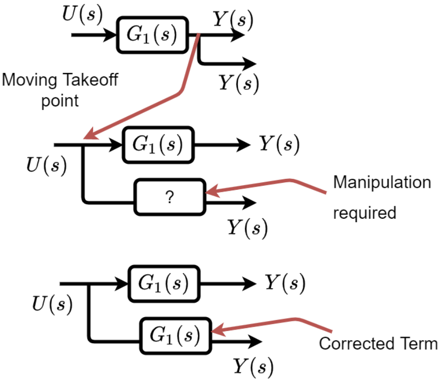
$$Y(s) = G_1(s)U(s) + P(s)X$$

- Comparing both the equations,

$$X(s) = G_1(s)$$

Block Diagram Representation - Moving a Takeoff Point

Moving a Takeoff Point



- The output of the system is

$$Y(s) = G_1(s)U(s)$$

- The output of the manipulated system is

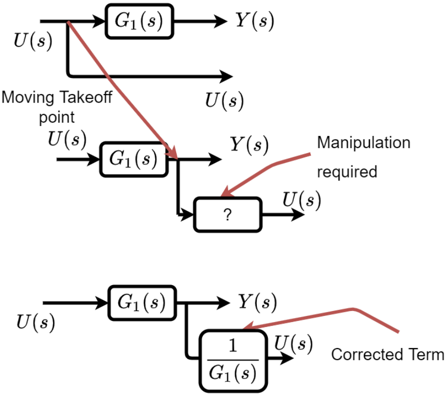
$$Y(s) = U(s)X$$

- Comparing both the equations,

$$X = G_1(s)$$

Block Diagram Representation - Moving a Takeoff Point

Moving a Takeoff Point



- The takeoff point is

$$U(s) = U(s)$$

- The takeoff point of the manipulated system is

$$U(s) = G_1(s)U(s)X$$

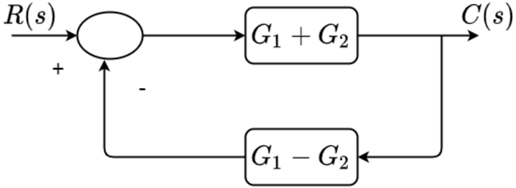
- Comparing both the equations,

$$X = \frac{1}{G_1(s)}$$

Block Diagram Representation

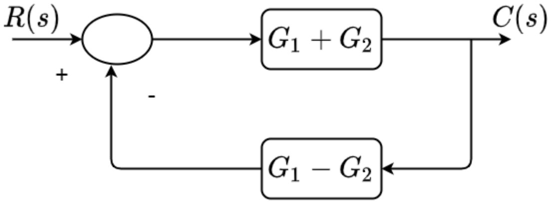
Question

- Find the overall transfer function $\frac{C(s)}{R(s)}$ of the following feedback system



Block Diagram Representation

Solution



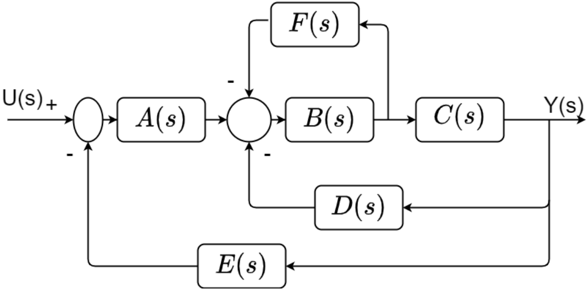
- The transfer function is given as

$$\frac{C(s)}{R(s)} = \frac{G_1 + G_2}{1 + (G_1 + G_2)(G_1 - G_2)}$$

Block Diagram Representation - Example

Question

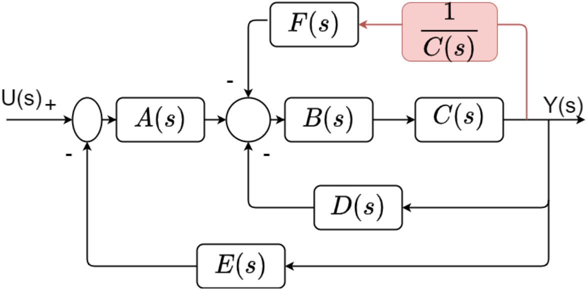
- Find the overall transfer function of the following system



Block Diagram Representation - Example

Solution

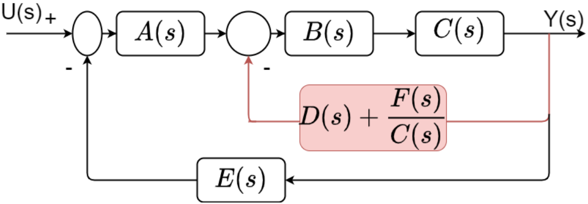
- **Step 1:** Move the takeoff point between B and C block ahead of block $C(s)$.



Block Diagram Representation - Example

Solution

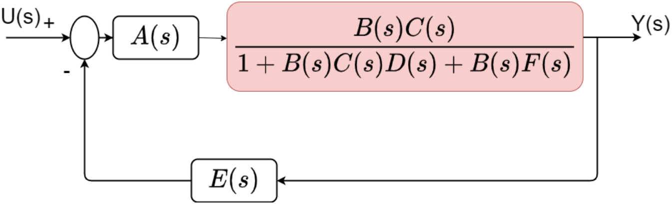
- **Step 2:** F is cascaded with $\frac{1}{C(s)}$, which in turn is parallel to $D(s)$



Block Diagram Representation - Example

Solution

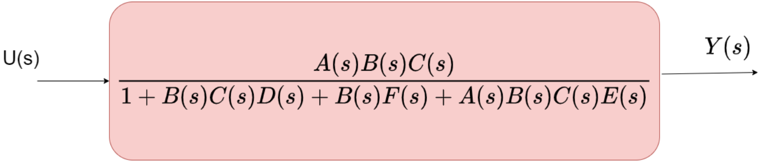
- **Step 3:** The negative feedback including $B(s)$, $C(s)$ and $D(s) + \frac{F(s)}{C(s)}$ is reduced.



Block Diagram Representation - Example

Solution

- **Step 4:** Final negative feedback block can be reduced



- The overall transfer function is

$$\frac{Y(s)}{U(s)} = \frac{A(s)B(s)C(s)}{1+B(s)C(s)D(s)+B(s)F(s)+A(s)B(s)C(s)E(s)}$$

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Signal Flow Graph Based Modeling- Introduction

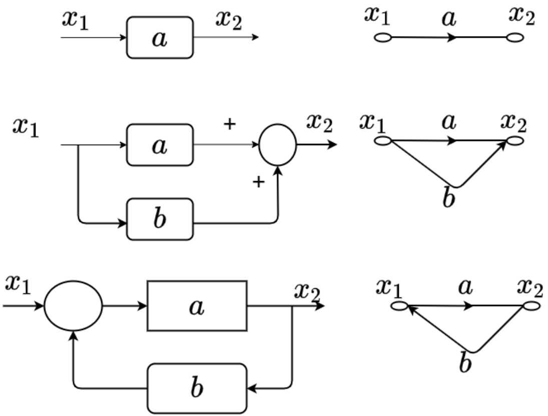
Introduction

- Let us discuss a technique for reducing the signal flow graph to single transfer function that relates the output of the system to its input.
- One of the disadvantages of the block diagram reduction techniques requires successive application of fundamental relationships in order to arrive at the overall transfer function.
- Reducing a signal flow graph to single transfer function can be performed by Mason's gain formula.
- Mason's gain formula has several components that must be evaluated.

Signal Flow Graph Based Modeling - Equivalence

Block Diagram and Signal Flow Graph Equivalence

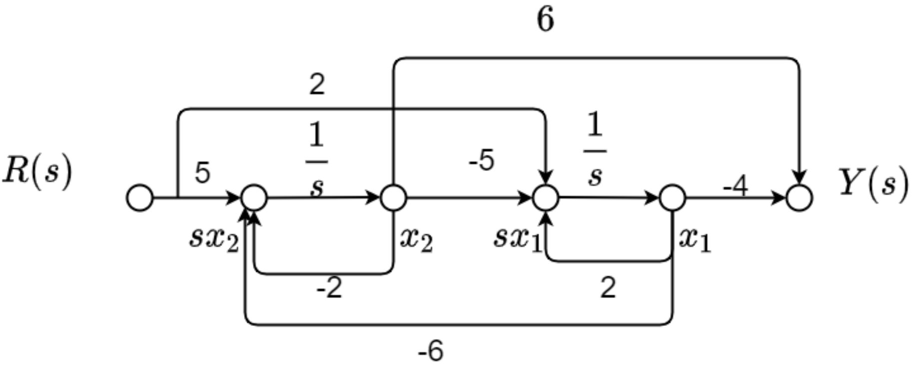
- Elements of signal flow graph have their equivalent representation in the block diagram.



Signal Flow Graph Based Modeling - Example

Question

- Represent the following signal flow graph using state space equations



Signal Flow Graph Based Modeling - Example

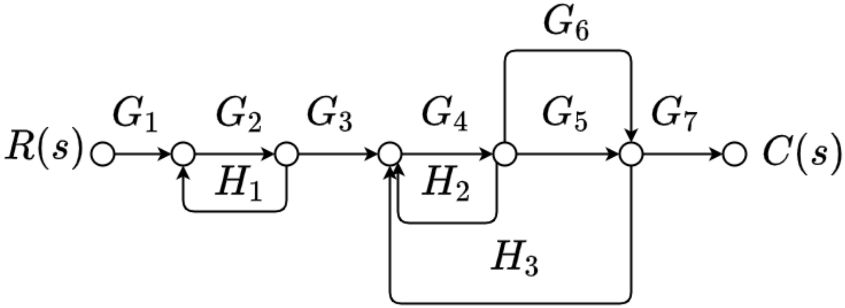
Solution

- State equation for the signal flow graph

$$\dot{x}_1 = 2x_1 - 5x_2 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 5r$$
$$y = -4x_1 + 6x_2$$

Signal Flow Graph Based Modeling - Definitions

Signal Flow Graph Definitions



- **Loop Gain:** The product of branch gains found by traversing a path that starts at a node and ends at the same node.

Signal Flow Graph Based Modeling - Definitions

Signal Flow Graph Definitions

- There are four loops in the above signal flow graph and their gain are

$$L_1 = G_2 H_1$$
$$L_2 = G_4 H_2$$
$$L_3 = G_4 G_5 H_3$$
$$L_4 = G_4 G_6 H_3$$

- **Forward Path Gain:** The product of gains found by traversing a path from input node to a output node. Forward path gains in the above signal flow graph are

$$F_1 = G_1 G_2 G_3 G_4 G_5 G_7$$
$$F_2 = G_1 G_2 G_3 G_4 G_6 G_7$$

Signal Flow Graph Based Modeling - Definitions

Signal Flow Graph Definitions

- **Non Touching Loop:** Loops that do not have nodes in common. Non touching loops in the above signal flow graph are $(L_1L_2,\ L_1L_3,\ L_1L_4)$.
- **Non Touching Loop Gain:** The product of non touching loop gains is taken two, three, four or more at a time. In the above signal graph, all three of a non touching loop gains taken two at a time are

$$P_1 = L_1L_2$$
$$P_2 = L_1L_3$$
$$P_3 = L_1L_4$$

Signal Flow Graph Based Modeling - Mason's Gain Formula

Mason's Gain Formula:

- The transfer function, $\frac{C(s)}{R(s)}$ of a system represented by a signal flow graph, is

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$

where,

k: Number of forward paths

T_k: The k_{th} forward path gain

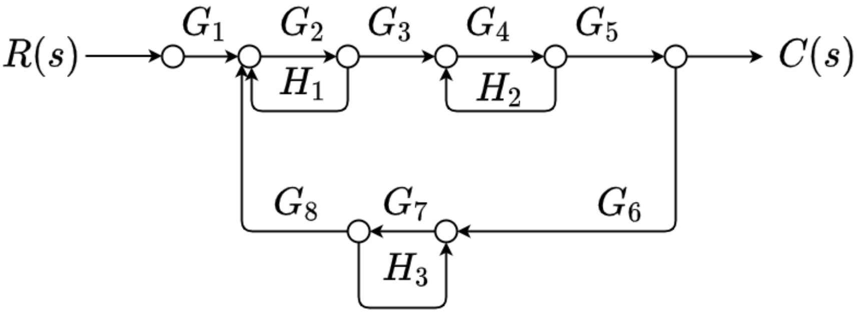
Δ= 1-$\sum$loop gains+$\sum$ non touching loops taken two at a time-$\sum$ non touching loops taken three at a time+ ...

Δ_k = Value of Δ for the part of graph not touching the k^{th} forward path.

Signal Flow Graph Based Modeling - Example

Question

- Find the transfer function, $\frac{C(s)}{R(s)}$ for the signal flow graph shown in the figure.



Signal Flow Graph Based Modeling - Example

Solution

- **Step 1:** Identify the forward path gain. In this case, there is only one forward path, i.e.

$$T_1 = G_1 G_2 G_3 G_4 G_5$$

- **Step 2:** Identify the loops and calculate its gain. In this case, there are four loops, i.e.

$$L_1 = G_2 H_1,\ L_2 = G_4 H_2,\ L_3 = G_2 G_3 G_4 G_5 G_6 G_7 G_8,\ L_4 = G_7 H_3$$

- **Step 3:** Identify nontouching loops taken two at a time. In our case, L_1, L_2, L_4 are non touching. The gains of non touching loops are

$$P_1 = L_1 L_2,\ P_2 = L_1 L_4,\ P_3 = L_2 L_4$$

Signal Flow Graph Based Modeling - Example

Solution

- **Step 4:** Identify non touching loops taken three at a time are L_1, L_2, L_4. The product of gain taken three at a time is

$$M = L_1 L_2 L_4$$

- **Step 5:** Identify Δ and Δ_k. Δ is given by

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_2 + L_1 L_4 + L_2 L_4) - (L_1 L_2 L_4)$$

- **Step 6:** Δ_k is obtained by eliminating the loop gains that touch the forward path T_k from Δ

$$\Delta_1 = 1 - L_4$$

Signal Flow Graph Based Modeling - Example

Solution

- **Step 7:** Substituting all these terms in Mason's gain formula, we obtain the transfer function as

$$\frac{C(s)}{R(s)} = \frac{T_1 \Delta_1}{\Delta}$$

References I

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