Assignment - 2 (AE 330)

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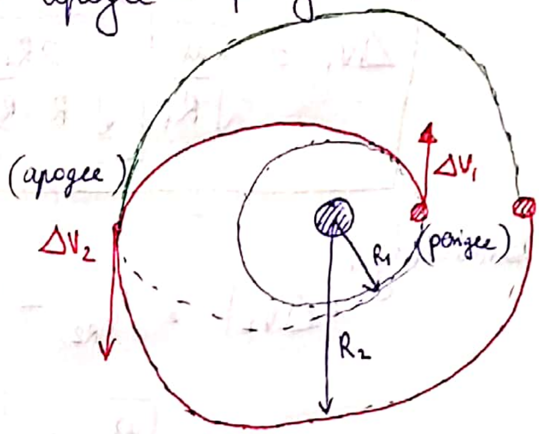
Ans 1) Before we go and solve the question, let's first derive the formulas for Hofmann transfer ellipse and their velocity increments needed at apogee & perigee.



As we can see,

Orbit 1 ⇒ Circular (R_1)

Orbit 2 ⇒ Elliptical ($2a = R_1 + R_2$)

Orbit 3 ⇒ Circular (R_2)

velocity in a circular orbit or Radius = R is equal to

$$V_R = \sqrt{\frac{\mu}{R}} \quad (\mu = GM)$$

Velocity in an elliptical orbit at a distance R & semi-major axis a, is

$$\cancel{V_e = \sqrt{\mu \frac{1}{2R} - \frac{1}{}}} \qquad V_e = \sqrt{\mu\left(\frac{2}{R} - \frac{1}{a}\right)}$$

Hence velocities at apogee & perigee comes as,

$$V_{apogee} = \sqrt{\frac{\mu}{a} \frac{1-e}{1+e}} \qquad V_{perigee} = \sqrt{\frac{\mu}{a} \frac{(1+e)}{(1-e)}}$$

Also in our Hofmmann ellipse orbit,

$$2a = R_1 + R_2$$
$$R_1 = a(1-e)$$
$$R_2 = a(1+e),$$ Hence

$$\boxed{V_{ap} = \sqrt{\frac{2\mu}{(R_1+R_2)} \frac{R_1}{R_2}}} \quad \& \quad \boxed{V_{perigee} = \sqrt{\frac{2\mu}{R_1+R_2} \frac{R_2}{R_1}}}$$

Now we can solve for expressions of ΔV_1, ΔV_2

$$\Rightarrow \quad |\Delta V_1| = V_{perigee} - V_{R1}$$

$$= \sqrt{\frac{2\mu}{(R_1+R_2)}\frac{R_2}{R_1}} - \sqrt{\frac{\mu}{R_1}}$$

$$\boxed{\Delta V_1 = \sqrt{\frac{\mu}{R_1}}\left[\sqrt{\frac{2R_2}{R_1+R_2}} - 1\right]}$$

Similarly,

$$|\Delta V_2| = V_{R2} - V_{apogee}$$

$$= \sqrt{\frac{\mu}{R_2}} - \sqrt{\frac{2\mu}{(R_1+R_2)}\frac{R_1}{R_2}}$$

$$\boxed{\Delta V_2 = \sqrt{\frac{\mu}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1+R_2}}\right)}$$

From the given data,

$R_1 = R_{earth} + h_1$; $R_2 = R_{earth} + h_2$; $\mu = G \times M_{earth}$

$= 6374 + 160$ km $\quad = 6374 + 42{,}200 \quad \mu = 3.983 \times 10^{14}$

$R_1 = 6534$ km $\quad R_2 = 48{,}574$ km

Hence,

$$\Delta V_1 = \sqrt{\frac{\mu}{R_1}}\left(\sqrt{\frac{2R_2}{R_1+R_2}} - 1\right)$$

$$= \sqrt{\frac{3.983 \times 10^{14}}{6.534 \times 10^{6}}}\left(\sqrt{\frac{2 \times 4.8574}{5.5108}} - 1\right)$$

$$\Delta V_1 = 7.8076 \times 10^{3}\,(1.3277 - 1)$$

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$$\Rightarrow \quad \Delta V_1 = 2.558 \times 10^3 \text{ m/s}$$

$$\boxed{\Delta V_1 = 2.558 \text{ km/s}}$$

Similarly,

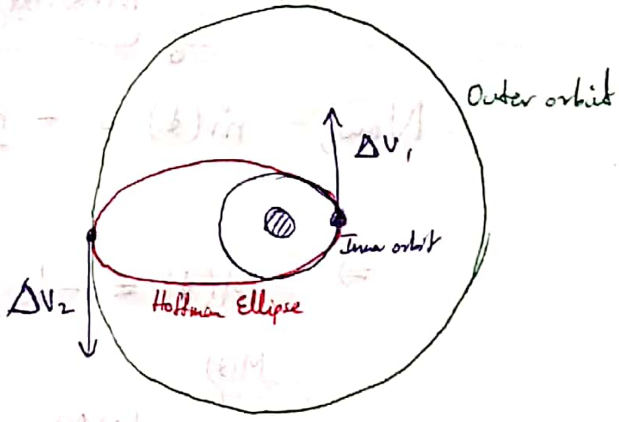
$$\Delta V_2 = \sqrt{\frac{\mu}{R_2}}\left(1 - \sqrt{\frac{2R_1}{R_1+R_2}}\right)$$

$$= \sqrt{\frac{3.983\times10^{14}}{4.8574\times10^{7}}}\left(1 - \sqrt{\frac{2\times6.374}{55.108}}\right)$$

$$= 2.863\times10^3\,(1 - 0.481)$$

$$\boxed{\Delta V_2 = 1.486 \text{ km/s}}$$

Hence, to shift from the inner circular orbit to the outer one, in minimum energy Hoffman Transfer,



$\Delta V_1 = 2.558$ km/s
$\Delta V_2 = 1.486$ km/s

Ans 2) From the data we can infer that the maximum acceleration that can be tolerated by the rocket is 50 m/s².

→ For maximum velocity achieved, the space vehicle must have $a = 50$ m/s² at all time and hence $\dot{m}(t)$ will not be constant (Burn profile) in order to satisfy these 2 constraints. Also, solving for this will give us minimum allowable burn time. $[V_{,max}$ & $t_{b,min}]$

For finding $\dot{m}(t)$ / Burn profile

Let $a_0 = 50\ m/s^2$, $\eta_p = 0.88$, $I_s = 260\,s$, $g_0 = 9.81\ m/s^2$, $M_0 = 25\,kg$

Hence we can say

$$M(t)\frac{dv}{dt} = \dot{m}(t)\, g_0 I_{sp} = M a_0$$

$\dot{m}(t) \rightarrow$ Burn profile

$M(t) \rightarrow$ Total mass at a instant t

$$\Rightarrow \quad a_0 = \frac{\dot{m}(t)\, g_0 I_{sp}}{M(t)} = \text{constant}$$

Let $\dfrac{g_0 I_{sp}}{a_0} = \text{constant } (k)$

Then, $M(t) = k\dot{m}(t)$

$$\cancel{M_0 - \int_0^t \dot{m}(t)\,dt = \dot{m}(t)}$$

Now, $\dot{m}(t) = -\dfrac{dM(t)}{dt}$

$$\Rightarrow \quad M(t) = -k\frac{dM(t)}{dt}$$

$$\int_{M_0}^{M(t)} -\frac{dM(t)}{M(t)} = \int_0^t \frac{dt}{k}$$

$$\ln\left(\frac{M_0}{M(t)}\right) = \frac{t}{k}$$

$$\boxed{M(t) = M_0 e^{-\frac{t}{k}}}$$

$$\Rightarrow \boxed{\dot{m}(t) = \frac{M_0}{k} e^{-t/k}}$$

Now, as $\eta_p = 0.88$,

$$M_{burnout} = M_o(1-\eta_p)$$

$$= M_o(0.12)$$

$$\boxed{M_b = 3 \text{ kgs}} \quad (M_o = 25 \text{ kgs})$$

Hence $t_{b,min}$,

$$t_{b,min} = k \ln\left(\frac{M_o}{M_b}\right)$$

$$t_{b,min} = \frac{g_o I_{sp}}{a_o} \ln\left(\frac{M_o}{M_b}\right) \Rightarrow \text{At const acceleration } (a_o)$$

$$t_{b,min} = \frac{9.81 \times 260}{50} \ln\left(\frac{25}{3}\right)$$

$$\boxed{t_{b,min} = 108.158 \text{ seconds}}$$

Now maximum velocity achieved,

$$V_{,max} = a_o t_{b,min}$$

$$= 50 \times 108.158$$

$$\boxed{V_{,max} = 5.408 \text{ km/s}}$$

Hence, minimum ~~maximum~~ allowable burn-time is equal to 108.158 seconds and maximum velocity is equal to 5.408 km/s

Ans 3) Given, for a single stage rocket,

$M_o = 100000$ kg, $m_s = 9975$, $m_L = 1025$, $\Rightarrow m_p = 89000$ kg

$I_{sp} = 4000$ sec, $m_{burnout} = m_o - m_p$, $g_o = 9.81 \, m/s^2$

$$m_b = 11000$$

Hence velocity imparted to the payload (mission velocity) is equal to,
(single-stage)

$$V_{T_1} = g_0 I_{sp} \ln\left(\frac{m_0}{m_b}\right)$$

$$= 9.81 \times 400 \ln\left(\frac{100}{11}\right)$$

$$\boxed{V_{T_1} = 8.661 \text{ km/s}}$$ ⟵ Payload velocity for 1-stage rocket

Now, for the single stage, $\boxed{\varepsilon = \frac{m_s}{m_s + m_p} = 0.1008}$

& $\boxed{\lambda = \frac{m_L}{m_s + m_p} = 0.010356}$

Given, we need to create a 2-stage rocket with same m_0, structural mass (m_s) & payload mass (m_L) (Identical stages)

Hence $\lambda_1 = \lambda_2 = \lambda$ & $\varepsilon_1 = \varepsilon_2 = \varepsilon$

Now, $\frac{m_{02}}{m_{01}} = \frac{\lambda}{1+\lambda}$ —① ; $\frac{m_L}{m_{02}} = \frac{\lambda}{1+\lambda}$ —②

Hence, $\frac{\lambda}{1+\lambda} = \sqrt{\frac{m_L}{m_{01}}}$ (① × ②)

$$\lambda = \frac{\sqrt{m_L/m_{01}}}{1 - \sqrt{m_L/m_{01}}}$$ $\left(\begin{aligned} m_{01} &= m_0 = 100000 \text{ kg} \\ m_L &= m_L = 1025 \text{ kg} \end{aligned}\right)$

$$\lambda = \frac{0.101242}{1 - 0.101242}$$

$\boxed{\lambda = 0.11264}$ ⟹ $\lambda_1 = \lambda_2 = \lambda = 0.11264$

Also ⟹ $m_{02} = \sqrt{m_0 m_L}$ (from ① & ②)

$$\boxed{m_{02} \approx 10124 \text{ kg}}$$

Same structural design as before for the rocket stages will give us $\varepsilon_1 = \varepsilon_2 = \varepsilon = 0.1008$

Hence $\varepsilon_2 = \dfrac{m_{s2}}{m_{02} - m_L} = 0.1008$

$$\Rightarrow \quad m_{s2} = 0.1008\,(10124 - 1025)$$

$$\boxed{m_{s2} \approx 917 \text{ kg}}$$

$$m_{p2} = m_{02} - m_{s2} - m_L$$

$$= 10124 - 1025 - 917$$

$$\boxed{m_{p2} = 8182 \text{ kgs}}$$

As total structural mass, propellant mass is the same,

$$9975 = m_{s1} + m_{s2} \quad ; \quad 89000 = m_{p1} + m_{p2}$$

$$\Rightarrow \boxed{m_{s1} = 9058 \text{ kg}} \qquad \boxed{m_{p1} = 80818 \text{ kg}}$$

Now, $m_{b1} = m_{02} + m_{s1}$, $m_{b2} = m_L + m_{s2}$

$$\boxed{m_{b1} = 19182 \text{ kg}} \; ; \; \boxed{m_{b2} = 1942 \text{ kg}}$$

Now, the final payload velocity for a 2-stage rocket will be,

$$V_{T_2} = g_0 I_{sp_1} \ln\left(\frac{m_{01}}{m_{b1}}\right) + g_0 I_{sp_2} \ln\left(\frac{m_{02}}{m_{b2}}\right)$$

As $I_{sp1} = I_{sp2} = 400\,s$ (Given),

$$V_{T2} = g_0 I_{sp} \ln\left(\frac{m_{01}\, m_{02}}{m_{b1}\, m_{b2}}\right)$$

$$V_{T2} = 9.81 \times 400 \ln\left(\frac{100000}{19182} \times \frac{10124}{1942}\right)$$

$$\boxed{V_{T2} = 12.958 \text{ km/s}}$$

Hence, increment in Payload velocity, after multi-staging

$$\Delta V_T = V_{T2} - V_{T1}$$

$$= 12.958 - 8.661$$

$$\boxed{\Delta V_T = 4.297 \text{ km/s}} \Rightarrow \text{Answer}$$

Ans 4) Let's calculate $m_{oi}, m_{Li}, m_{si}, m_{pi}$ in a descending order of stage according to the given data.

4th stage,
$m_{L4} = 40$ kg
$m_{s4} = 40$ kg
$m_{p4} = 260$ kg
$m_{o4} = 340$ kg
(b-burnout) $m_{b4} = 80$ kg

$$\lambda_4 = \frac{m_{L4}}{m_{o4} - m_{L4}} \Rightarrow \lambda_4 = 0.1333$$

$$\Rightarrow \varepsilon_4 = \frac{m_{s4}}{m_{o4} - m_{L4}} \Rightarrow \varepsilon_4 = 0.1333$$

$$R_4 = \frac{m_{o4}}{m_{b4}} \Rightarrow R_4 = 4.25$$

3rd stage,
$m_{L3} = m_{o4} = 340$ kg
$m_{s3} = 250$ kg
$m_{p3} = 1700$ kg
$m_{o3} = 2290$ kg
$m_{b3} = 590$ kg

$$\lambda_3 = \frac{m_{L3}}{m_{o3} - m_{L3}} \Rightarrow \lambda_3 = 0.1743$$

$$\Rightarrow \varepsilon_3 = \frac{m_{s3}}{m_{o3} - m_{L3}} \Rightarrow \varepsilon_3 = 0.1282$$

$$R_3 = m_{o3}/m_{b3} \Rightarrow R_3 = 3.881$$

2nd stage,
$m_{L2} = m_{o3} = 2290$ kg
$m_{s2} = 550$ kg
$m_{p2} = 3500$ kg
$m_{o2} = 6340$ kg
$m_{b2} = 2840$ kg

$$\lambda_2 = \frac{m_{L2}}{m_{o2} - m_{L2}} \Rightarrow \lambda_2 = 0.5654$$

$$\Rightarrow \varepsilon_2 = \frac{m_{s2}}{m_{s2} + m_{p2}} \Rightarrow \varepsilon_2 = 0.1358$$

$$R_2 = m_{o2}/m_{b2} \Rightarrow R_2 = 2.232$$

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1^{st} stage, $m_{L1} = m_{02} = 6340\ kg$ $\quad \lambda_1 = \frac{m_{L1}}{m_{01} - m_{L1}} \Rightarrow \underline{\underline{\lambda_1 = 0.6038}}$

$m_{s1} = 1500\ kg$

$m_{p1} = 9000\ kg \Rightarrow \varepsilon_1 = \frac{m_{s1}}{m_{01} - m_{L1}} \Rightarrow \underline{\underline{\varepsilon_1 = 0.14286}}$

$m_{01} = 16840\ kg$

$m_{b1} = 7840\ kg \quad R_1 = m_{01}/m_{b1} \Rightarrow \underline{\underline{R_1 = 2.148}}$

Hence, total payload velocity imparted $\Rightarrow$

$$V_T = C_1 \ln R_1 + C_2 \ln R_2 + C_3 \ln R_3 + C_4 \ln R_4$$

Given, $C_1 = 2200 \quad C_2 = 2400, \quad C_3 = 2500 \quad C_4 = 2750$

$$V_T = (1.682 + 1.927 + 3.3902 + 3.979)\ km/s$$

$$\boxed{V_T = 10.978\ km/s}$$ $\Rightarrow$ Mission Payload Velocity that can be imparted to the Payload.

From the 1^{st} stage $\Rightarrow t_{b1} = 50\ sec$, $\dot{m}_1 =$ const

Hence, $\dot{m}_1 = \frac{m_{p1}}{t_{b1}} \Rightarrow \boxed{\dot{m} = 180\ kg/s}$

Hence, acceleration of the rocket at take off, $(M = m_{01})$

$$a_{TO} = \frac{\dot{m}_1 C_1}{m_{01}} - g_0$$

$$= \left(\frac{180 \times 2200}{16840} - 9.81\right) m/s^2$$

$$\boxed{a_{TO} = 13.705\ m/s^2}$$

Hence, acceleration at take off has a value of

$$\underline{\underline{a_{TO} = 13.705\ m/s^2}}$$

Ans 5) For a rocket in a drag-free environment, velocity as f(t) is, (SINGLE STAGE)

$$V(t) = g_0 I_{sp} \ln\left(\frac{m_0}{m(t)}\right) - \tilde{g}_p t$$

[Assuming 1-D motion upwards along the radial direction]

Burnout Height / Altitude

$g_0 \rightarrow$ Earth's standard gravity : 9.81 m/s²

$\tilde{g}_p \rightarrow$ or simply $\tilde{g}$ is the average value of deceleration faced by the rocket because of planet's or terrestrial body's gravitation.

$$m(t) = m_0 - \dot{m}t \; ; \quad \dot{m} = \frac{m_p}{t_b} \Rightarrow \left[\frac{m_0}{m_0 - \frac{m_p t}{t_b}}\right] \Rightarrow \left(\frac{\frac{t_b}{n}}{\frac{t_b}{n} - t}\right) \Rightarrow \left(\frac{k}{k-t}\right)$$

where $k = \underline{\underline{t_b/n}}$

Now, burn out ~~velocity~~ altitude can be found as,

$$h_b = \int_0^{t_b} V(t).dt$$

$$h_b = g_0 I_{sp} \int_0^{t_b} \ln\left(\frac{k}{k-t}\right).dt - \tilde{g}\int_0^{t_b} t.dt$$

$$h_b = g_0 I_{sp}\left[\int_0^{t_b} \ln k.dt - \int_0^{t_b} \ln(k-t).dt\right] - \frac{\tilde{g}t_b^2}{2}$$

$$h_b = g_0 I_{sp}(\ln k)t_b - \frac{\tilde{g}t_b^2}{2} + g_0 I_{sp}\Big[(k-t)\ln(k-t) - (k-t)\Big]\Big|_0^{t_b}$$

Let $h_b = g_0 I_{sp} Z - \frac{\tilde{g}t_b^2}{2}$

$\Downarrow$

$$Z = t_b \ln k + \left[(k-t_b)\ln(k-t_b) - \cancel{(k-t_b)}\right] - \left[k\ln k - \cancel{k}\right]$$

$$Z = t_b + (k-t_b)\ln(k-t_b) - (k-t_b)\ln k$$

$$\boxed{Z = t_b + (k-t_b)\ln\left[\frac{k-t_b}{k}\right]}$$

In terms of $\eta = \frac{m_p}{m_0}$ & t_b,

As $k = \frac{t_b}{\eta}$

$$Z = t_b + \left(\frac{t_b}{\eta} - t_b\right) \ln(1-\eta)$$

$$\boxed{Z = t_b\left[1 + \left(\frac{1-\eta}{\eta}\right)\ln(1-\eta)\right]}$$

Putting value of Z in h_b expression,

$$\boxed{h_b = g_0 I_{sp} t_b \left[1 + \left(\frac{1-\eta}{\eta}\right)\ln(1-\eta)\right] - \frac{\tilde{g}\, t_b^2}{2}}$$

For maximum altitude, h_{max}

$$\boxed{h_{max} = h_b + \frac{V_b^2}{2\tilde{g}_{h_b}}}$$

*** NOTE:- $\tilde{g}_{h_b}$ = Average gravity at altitude h_b

Now, for our numerical problem, $\eta = 0.8$, $t_b = 25$ sec, Location = Moon

$I_{sp} = 180$ s, $g_{moon} = g_0/6$

ASSUMPTION:- As burn time is only 25 seconds, we assume that throughout the burn time, gravity acting on the body remains constant as there will not be significant altitude rise in that duration. Hence,

$$h_b = g_0 I_{sp} t_b \left[1 + \left(\frac{1-\eta}{\eta}\right)\ln(1-\eta)\right] - \frac{g_0 t_b^2}{6 \times 2}$$

$$h_b = 9.81 \times 180 \times 25 \left[1 + \frac{0.2}{0.8}\ln 0.2\right] - \frac{9.81}{12} \times 25 \times 25$$

$$h_b = 4.4145 \times 10^4 (0.5976) - 0.511 \times 10^3$$

$\cancel{\boxed{h_b = 2{,}127 \text{ km}}}$ $\boxed{h_b = 25.87 \text{ km}}$

Burnout Velocity :-

$$V_b = g_0 I_{sp} \ln\left(\frac{m_0}{m_0 - m_p}\right) - \tilde{g} t_b$$

$$\boxed{V_b = g_0 I_{sp} \ln\left(\frac{1}{1-n}\right) - \tilde{g} t_b}$$

Putting our values

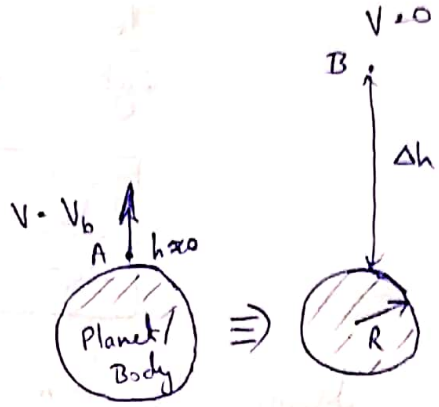
$$V_b = 9.81 \times 180 \ln\left(\frac{1}{0.2}\right) - \frac{9.81}{6} \times 25$$

$$V_b = 2.842 \times 10^3 - 0.041 \times 10^3$$

$$\boxed{V_b = 2.801 \text{ km/s}}$$

Total height / Altitude :-

Total Altitude can be found as sum of burnout height & Δh defined in the picture alongside



By energy conservation,

$$(PE + KE)_A = (PE + KE)_B$$

$$-\frac{\mu m}{R} + \frac{1}{2} m V_b^2 = -\frac{\mu m}{R + \Delta h}$$

$$\frac{1}{2} V_b^2 = \frac{\mu \Delta h R}{R^2 (R + \Delta h)}$$

$$(R + \Delta h) V_b^2 = 2 g R \Delta h \qquad (\text{As } \mu/R^2 = g)$$

$$\Delta h = \frac{V_b^2 R}{2gR - V_b^2}$$

$$\text{or} \quad \Delta h = \frac{V_b^2}{2g - \frac{V_b^2}{R}} \implies \boxed{\Delta h = \frac{V_b^2}{2g\left(1 - \frac{V_b^2}{2gR}\right)}}$$

Hence,

$$\boxed{h_{,max} = h_b + \frac{V_b^2}{2g\left(1 - \frac{V_b^2}{2gR}\right)}}$$

Cases Dif $\frac{V_b^2}{2gR} \ll 1$,

$$\boxed{h_{,max} = h_b + \frac{V_b^2}{2g}} \quad \left(\text{for small } \frac{V_b^2}{2gR}\right)$$

Case 2) if $\frac{V_b^2}{2gR} \geq 1$, $h_{,max} \to \infty$

For our case, we need to first find the value for $V_b^2/2gR$

$R_{(moon)} = 1731.1 \times 10^3$ m

$$\frac{V_b^2}{2gR} = \frac{(2.801)^2 \times \cancel{10^6}}{2 \times \frac{9.81}{6} \times 1.731 \times \cancel{10^6}} = \frac{7.846}{5.66}$$

$$\boxed{\frac{V_b^2}{2gR} = 1.386 > 1}$$

Hence, the $h_{,max}$ value is $+\infty$ as technically the burnout velocity is greater than the escape velocity of the moon.

$\boxed{V_b = 2.801 \text{ km/s}}$

$V_{escape\,(moon)} = 2.38$ km/s (from google)

Hence, ~~$h_b = 2.127$ km~~ ~~$h_{,max} = +\infty$~~

ANSWERS $\Rightarrow$ $h_b = 25.87$ km, $h_{,max} = +\infty$