AE 308: Control Theory
AE 775: System Modeling, Dynamics and Control

Lecture 9: Stability - Routh Hurwitz Criterion

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Introduction

- Stability of a system is one of the most important requirements.
- Stability of a system can be defined based on the type of the system.
- Here, the stability of Linear Time Invariant (LTI) systems will be discussed.

Definitions

- LTI system with bounded input is said to be Stable, if the system's natural response approaches to zero as $t \to \infty$.
- LTI system with bounded input is said to be Unstable, if the system's natural response grows without bound as $t \to \infty$.
- LTI system with bounded input is said to be Marginally Stable, if the system's natural response neither decays to zero nor grows but remains constant or oscillates as $t \to \infty$.

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Stability

Definitions

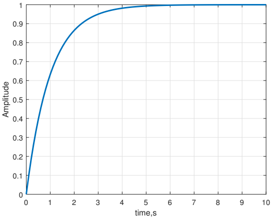


Figure: Stable system

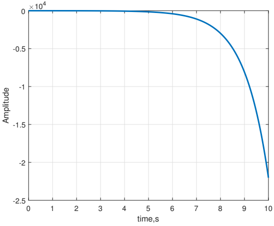


Figure: Unstable system

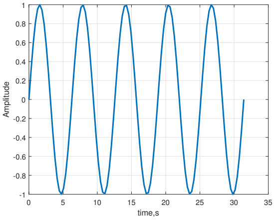


Figure: Marginally Stable

$$G(s) = \frac{1}{s+1}$$

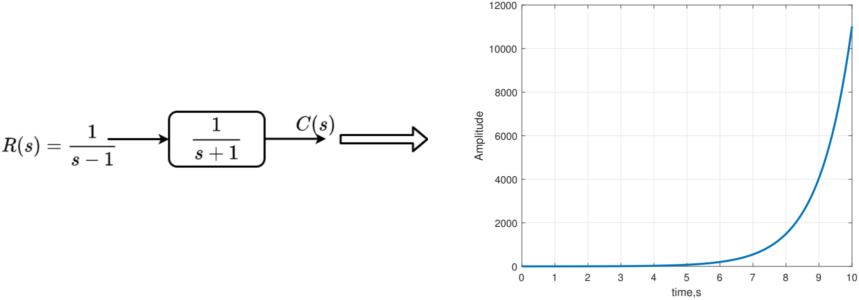
$$G(s) = \frac{1}{s-1}$$

$$G(s) = \frac{1}{s^2+1}$$

Stability - BIBO Stability

Definitions

- Consider a system



- Total response approaches to ∞, as $t \to \infty$. Response is growing unbounded due to input.
- Stability depends on input and system poles.

Stability - BIBO Stability

- As stability depends on both input and system poles, the stability can be defined based on total response

Stability Based on Total Response

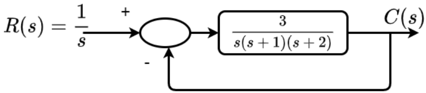
- A system is Stable, if *every* bounded input yields bounded output. This notion is Bounded Input Bounded Output (BIBO) stability.
- A system is Unstable, if *any* bounded input yields unbounded output.

System	**Pole location**
Stable	Left half of s-plane
Unstable	Right half of s-plane
Marginally stable	Imaginary axis of s- plane

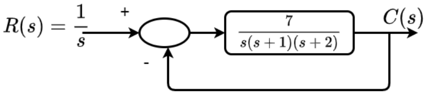
Stability - BIBO Stability

Example 1:

- Though the open loop system is stable, its closed loop system can be unstable by varying the gain. Consider the following the example:



Stable System



Unstable System

- Pole location:

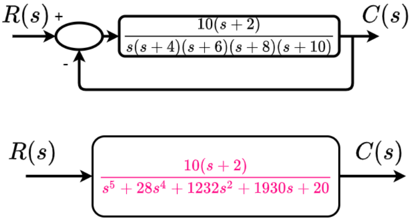
$$s_1 = -2.672$$
$$s_{2,3} = -0.1642 \pm 1.0469i$$

- Pole location:

$$s_1 = -3.087$$
$$s_{2,3} = 0.0434 \pm 1.5053i$$

Stability - BIBO Stability

Example 2:

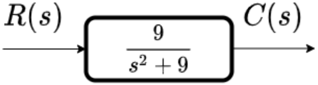


- It is not an easy task to find the closed loop poles of a system.
- Although we know the open loop poles, it is difficult to find closed loop poles of the system.

Stability

Question

- Consider the following system



- Comment on the stability of the system based on natural response and BIBO definitions.

Stability

Solution

- Marginally stable: Natural response definition
- Unstable: BIBO definition
- Consider a bounded input $r(t) = \sin 3t$.

$$C(s) = \frac{27}{(s^2+9)^2}$$

- As there are repeated roots in the imaginary axis, it is not BIBO stable.
- Consider a bounded input $r(t) = \sin \omega t$, $\omega \neq 3$. Then the response is

$$C(s) = \frac{9\omega}{(s^2+9)(s^2+\omega)}.$$

- Corresponding response will have two sinusoidal signals of frequencies ω and $3\frac{rad}{s}$.

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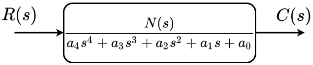
Routh Hurwitz Criterion

Methodology

- Let us discuss a method that yields system's stability without solving for poles of the system.
- Through the Routh Hurwitz (RH) criterion, we can say how many poles are present in left half, right half and imaginary axis of s-plane.
- Using this criteria, we can find number of poles, but not their coordinates.
- This method requires two steps:
  - Generate Routh's table
  - Interpret Routh's table to decide how many poles are present in left half, right half and imaginary axis of s-plane

Routh Hurwitz Criterion

Generating Routh's Table



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s			
s^0			

Table: Initial layout of RH table

- Since we are interested in the poles, let us consider the denominator.
- Begin by labelling the rows with highest powers of s from the denominator of the closed loop transfer function to s^0.
- Next start with the coefficient of highest power of s in the denominator, and list every other coefficient horizontally in the first row.

Routh Hurwitz Criterion

Generating Routh's Table (Continued...)

- In the second row, starting with the next highest power of s, fill every coefficient that was skipped in the previous row.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	b_1	b_2	0
s	c_1	0	0
s^0	d_1	0	0

Table: Completed Routh's table

$$b_1 = -\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} \qquad c_1 = -\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1}$$

$$b_2 = -\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} \qquad d_1 = -\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1}$$

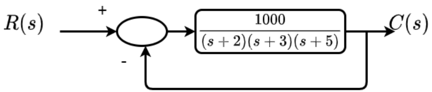
Routh Hurwitz Criterion

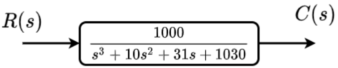
Generating Routh's Table (Continued...)

- Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column.
- The left hand column of the determinant is always the first column of previous two rows.
- The right hand column is the elements of the column above and to the right.
- The table is complete, when all the rows are completed down to s^0.

Routh Hurwitz Criterion - Example

- Create a Routh table for the following system:





- The first step is to find equivalent closed loop transfer function.
- The Routh Hurwitz criterion will be applied to the denominator of closed loop transfer function.
- Label the rows with powers of s, from s^3 to s^0.
- Start with the coefficient of the highest power and skip every other power of s.

Routh Hurwitz Criterion - Example

s^3	1	31	0
s^2	10	1030	0
s^1	b_1	b_2	b_3
s^0	c_1	c_2	0

$$b_2 = -\frac{\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0 \quad c_1 = -\frac{\begin{vmatrix} 10 & 1030 \\ b_1 & b_2 \end{vmatrix}}{b_1} = 1030$$

$$b_1 = -\frac{\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{10} = -72 \quad b_3 = -\frac{\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0 \quad c_2 = -\frac{\begin{vmatrix} 10 & 0 \\ b_1 & b_3 \end{vmatrix}}{b_1} = 0$$

,

Routh Hurwitz Criterion

Interpretation of Routh Table

- We have generated Routh table. Now let us see what can be commented on stability based on the table.
- The number of sign changes in the first column is equal to number of poles in right half of the s-plane.
- In the above example, there is a sign change from 10 $(+)$ to b_1 $(-)$ and another sign change from b_1 $(-)$ to c_1 $(+)$.
- Hence there are two right half poles for the above system.
- System is Unstable, as it has two poles in right half of s-plane.
- The poles of closed loop system are

$$s_1 = -13.4136$$
$$s_{2,3} = 1.7068 \pm 8.5950i$$

Routh Hurwitz Criterion - Special Cases

Zero Only in the First Column

- If the first element is zero, division by zero would be required.
- A polynomial which has reciprocal roots of original polynomial has its roots distributed in the same left half, right half and imaginary axis of s-plane.
- Using this fact, the above problem can be handled.
- Let us show that the polynomial we are looking for, the one with reciprocal roots, is simply the original polynomial with its coefficients written in reverse order.

Routh Hurwitz Criterion - Special Cases

Zero Only in the First Column

- Consider the equation,

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

- If s is replaced by $\frac{1}{d}$, then d will have roots which are reciprocal of s.
- By substituting this, we get

$$\begin{aligned} \frac{1}{d^n} + a_{n-1}\frac{1}{d^{n-1}} + \cdots + a_1\frac{1}{d} + a_0 = 0 \\ \frac{1}{d^n}\left[1 + a_{n-1}d + \cdots + a_1 d^{n-1} + a_0 d^n\right] = 0 \end{aligned}$$

- Thus, the polynomial with reciprocal roots is a polynomial with coefficients written in reverse order.

Routh Hurwitz Criterion - Special Cases

Example

- Comment on the stability of the following system

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

- Form a Routh's table

s^5	1	3	5
s^4	2	6	3
s^3	0	$\frac{7}{2}$	0

- As there is zero in the first column, we have to proceed with the alternative procedure.
- The new polynomial, which has roots equal to reciprocal of s, is

$$P(d) = 3d^5 + 5d^4 + 6d^3 + 3d^2 + 2d + 1$$

Routh Hurwitz Criterion - Special Cases

Example (Continued...)

- Form a new Routh's table for the new polynomial as

d^5	3	6	2
d^4	5	3	1
d^3	4.2	1.4	0
d^2	1.33	1	0
d	-1.75	0	0
d^0	1	0	0

- There are no zeros in the first column.
- As there are two sign changes in first column, the system has two right half poles and hence the system is unstable.
- The poles of the system are

$$\begin{aligned} s_1 &= -1.6681 \\ s_{2,3} &= -0.5088 \pm 0.7020i \\ s_{4,5} &= 0.3429 \pm 1.5083i \end{aligned}$$

Routh Hurwitz Criterion - Special Cases

Entire Row is Zero

- While forming a Routh's table, we find that entire row consists of zero.
- Let us look at an example that demonstrates how to construct and interpret the Routh table when entire row has zeros.

Example

- Determine the stability of the following system

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Routh Hurwitz Criterion - Special Cases

Example (Continued...)

- Form a Routh table for the above system as

s^5	1	6	8
s^4	$\cancel{7}$ 1	$\cancel{42}$ 6	$\cancel{56}$ 8
s^3	0	0	0
s^2			
s			
s^0			

- Consider a row immediately above to the rows which has complete zeros.
- Form an auxiliary polynomial using the entries as coefficients

$$P(s) = s^4 + 6s^2 + 8$$

- Differentiate the above equation w.r.t. s,

$$\frac{dP(s)}{ds} = 4s^3 + 12s$$

Routh Hurwitz Criterion - Special Cases

Example (Continued...)

s^5	1	6	8
s^4	1	6	8
s^3	$\cancel{4}$ 1	$\cancel{12}$ 3	0
s^2	3	8	0
s	$\frac{1}{3}$	0	0
s^0	8	0	0

- Coefficients of the auxiliary equation is used to fill the rows which has zeros.
- As there are no sign changes in first column, the system has no poles in right half of s-plane.
- Hence the system is stable.
- The pole locations are

$$\begin{aligned} s_1 &= -7 \\ s_{2,3} &= \pm 2i \\ s_{4,5} &= \pm 1.4142i \end{aligned}$$

Routh Hurwitz Criterion - Relative Stability

Relative Stability

- Through RH criterion, absolute stability of system can be derived.
- Relative stability gives the degree of stability or how close to instability.
- As shown earlier in one of the examples, by varying gains, the system becomes unstable. This change in gain gives some degree of stability.
- Let us see how to obtain relative stability through RH criterion.

Routh Hurwitz Criterion - Relative Stability

- We usually require information about relative stability of the system.
- A useful approach for examining relative stability is to shift s axis and proceed with RH criterion. Substitute

$$s = \hat{s} - \sigma_1 \,.$$

- Use this substitution in characteristic equation, write the polynomials in terms of $\hat{s}$.
- The number of sign changes results in poles right to the vertical line $s = -\sigma_1$.

Routh Hurwitz Criterion - Relative Stability

Example

- Determine the range of K such that the following characteristic equation has poles more negative than -1

$$s^3 + 3(K+1)s^2 + (7K+5)s + 4K + 7 = 0$$

- As we are interested in roots more negative to -1, substitute $s = \hat{s} - 1$

$$(\hat{s}-1)^3 + 3(K+1)(\hat{s}-1)^2 + (7K+5)(\hat{s}-1) + 4K + 7 = 0$$

- Simplifying the equation,

$$\hat{s}^3 + 3K\hat{s}^2 + (K+2)\hat{s} + 4 = 0$$

- Form a Routh table corresponding to the above polynomial

Routh Hurwitz Criterion - Relative Stability

Example (Continued...)

$\hat{s}^3$	1	$(K+2)$
$\hat{s}^2$	$3K$	4
$\hat{s}$	$\frac{3K(K+2)-4}{3K}$	0
$\hat{s}^0$	4	0

Table: Routh Table

- The first column should not have any sign changes.
- From $\hat{s}^2$

$$K > 0$$

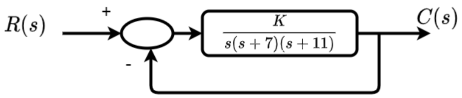
- From $\hat{s}$

$$3K(K+2) - 4 > 0$$
$$3K^2 + 6K - 4 > 0$$
$$(K - 0.5275)(K + 2.5275) > 0$$

- Hence, the range of K is $K > 0.5275$.

Routh Hurwitz Criterion - Stability Design

Example - Stability Design



- Find the range of gain K, for which system becomes stable, unstable and marginally stable.

- First find the closed loop transfer function

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

- Next form the Routh table

Routh Hurwitz Criterion - Stability Design

Example - Stability Design (Continued ...)

s^3	1	77
s^2	18	K
s	$\frac{1386-K}{18}$	0
s^0	K	0

- K should be positive for first column to have no sign changes.
- The third row first element should also be positive to have no sign change. Hence

$$1386 - K > 0 \implies K < 1386$$

- For the system to be Stable, K should be

$$0 < K < 1386$$

- If $K > 1386$, there are two sign changes in the first column.

Routh Hurwitz Criterion - Stability Design

Example - Stability Design (Continued ...)

- Hence, if $K > 1386$, the system is Unstable.
- System to be marginally stable, an entire row in Routh's table should be zero.
- The row corresponding to s will have zeros, if

$$1386 - K = 0 \implies K = 1386$$

- Consider the auxilary equation,

$$P(s) = 18s^2 + 1386$$
$$\frac{dP(s)}{ds} = 36s$$

Routh Hurwitz Criterion - Stability Design

Example - Stability Design (Continued ...)

s^3	1	77
s^2	18	K
s	36	0
s^0	1386	0

- As there are no sign changes in first column, the system is marginally stable.
- The pole locations are

$$s_1 = -18$$
$$s_{2,3} = \pm 8.775i$$

Routh Hurwitz Criterion

Question

- Consider the general third-order polynomial equation

$$P(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$$

- Find the condition such that all roots are negative through RH criterion.
- Assume all the coefficients are positive.

Routh Hurwitz Criterion

Solution

s^3	a_0	a_2
s^2	a_1	a_3
s^1	$-\frac{(a_0a_3-a_1a_2)}{a_1}$	0
s^0	a_3	0

- To have all negative roots, the column corresponding to s row should be positive.
- Hence,

$$a_0a_3 - a_1a_2 < 0$$

$$\boxed{a_1a_2 > a_0a_3}$$

References I

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