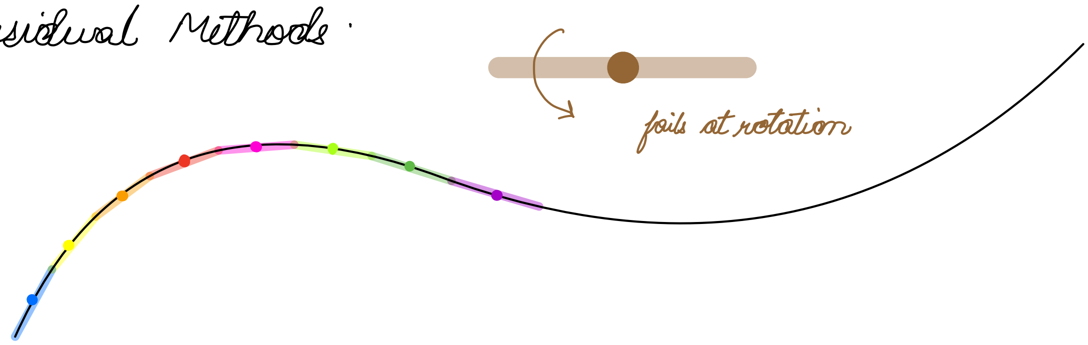
Topics	Reference	Practice
Abstraction	Sir's slides	Tutorial 1 (slide 2-9)
Usage of δ operator	Tutorial 1 (slides 11-12)	Tutorial 1 (slides 13-17)
Hamilton's Principle	Sir's notes (Notes on continuous systems)	Tutorial 1 (slides 19-29)
Euler Lagrange Equation of motion ↑ Quiz 1	Section 6.6, 6.7 SS Rao	Tutorial 1 slides (30-45)
Rayleigh Quotient ↑ Mid Sem	Section 2.5, 8.7 SS Rao Section 7.2 Meirovitch	Rayleigh Quotient - final.pdf in Announcements Practice session for Mid Sem
1 DoF ↑ Quiz 2	Unforced Chap 3 SS Rao except 3.7, 3.9, 3.10, 3.11, 3.12, 3.14, 3.15 Forced Chap 4 SS Rao except 4.8, 4.9, 4.10	Tutorial 2 Tutorial 3
2 DoF ↑ End Sem	Unforced Chap 5 SS Rao (5.1 - 5.5)	

Lumping

Weighted Residual Methods.



Distributed Coordinates

$$y = \sum q_i \phi_i(x)$$

Modal analysis
Eigenvector analysis

Ordering Schemes.

↓

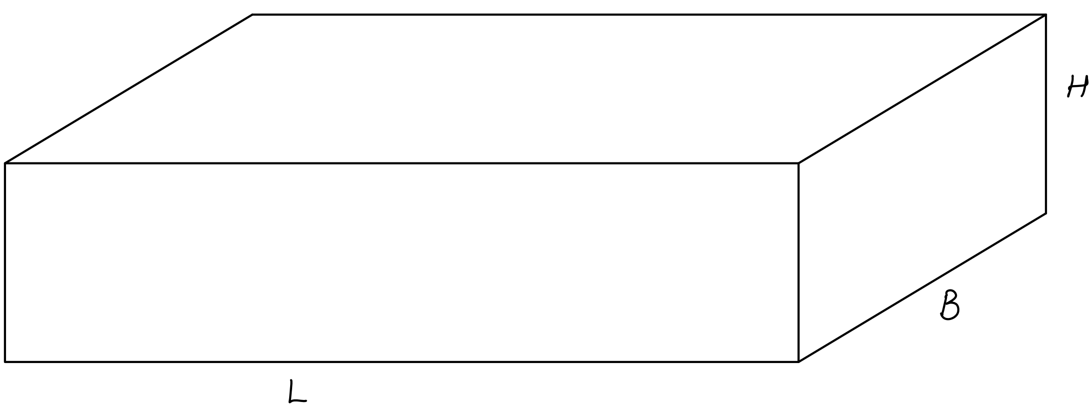
Demarkation strategies

Euler Bernoulli Beam Theory

Deformations ✓
stresses and strains ✗

Timoshenko Beam Theory

Geometric Beam Theory.



$L >> B > H$

Principle of virtual work $\longrightarrow$ Connection with FEA

Kinematics is described by dofs. [regarding kinematic boundary conditions

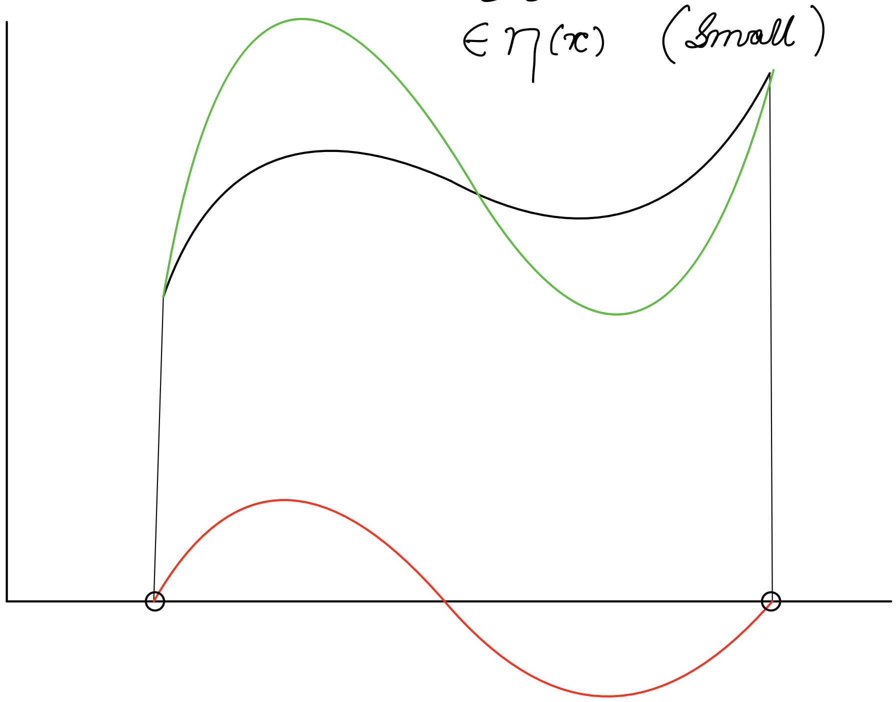
u satisfies all kinematic boundary conditions

$\tilde{u}$ also " " " " "

η is arbitrary except it has to vanish where kinematic bcs are specified

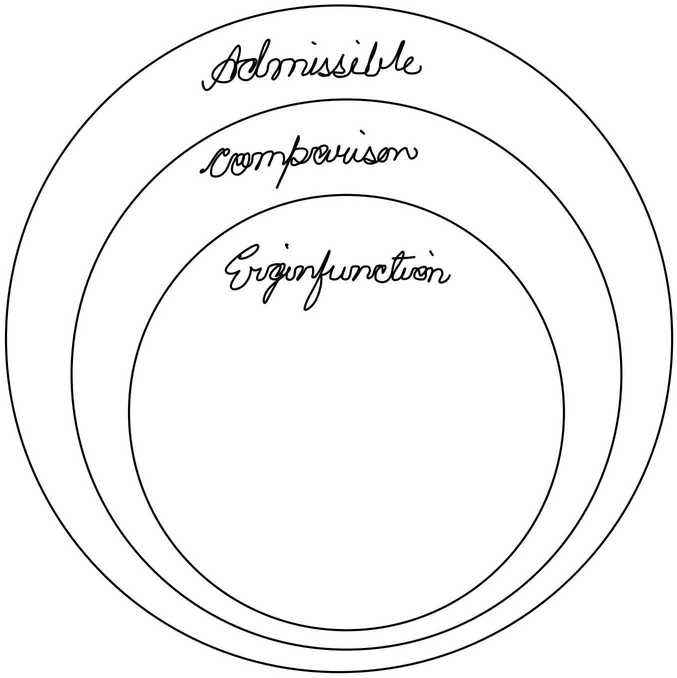
$$\tilde{u}(x) = u(x) + \underbrace{\eta(x)}_{\delta u}$$

$\in \eta(x)$ (small)



$$\delta u = \tilde{u} - u$$

where $\left.\begin{array}{l} u \equiv u(x,t) \\ \tilde{u} = \tilde{u}(x,t) \end{array}\right\}$ are admissable displacement functions



$\tilde{u}$

u

δu

$$\delta \frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial t} - \frac{\partial u}{\partial t} = \frac{\partial (\delta u)}{\partial t}$$

u is differentiable

$$\int \delta u = \int \tilde{u} - u = \int \tilde{u} - \int u = \tilde{U} - U = \delta \int u$$

u is integradable

System to particles

Constraint forces
Internal forces X Workless constraint

Statics to dynamics

$$(F - \dot{p}) \cdot \delta r = 0$$

$$\int_{t_1}^{t_2} F \,\delta r \; dt = 0 \qquad t_2 > t_1$$

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} \left(F_{ext_j} - m_j \ddot{r}_j \right) \delta r_j \; dt = 0$$

Connection with FEA ☆☆☆

Hamilton's principle
Lagrangian

Newton
PVW
Lagrangian
Hamilton

$f(x+dx)$

manifold

d on the surface
δ entirely new surface

formal way
nice mathematical sense

$$\delta u = u + \epsilon \eta$$

δ (Independent variable)
No meaning

Properties of Variational operator ☆☆☆

Variational Calculus.

The kinetic energy state funcⁿ

kinetic energy $\int_0^p v\, dp$

kinetic co energy $\int_0^v p\, dv = \frac{1}{2} m v^2$

Integradable forces

Non integradable forces

Hamilton Equation

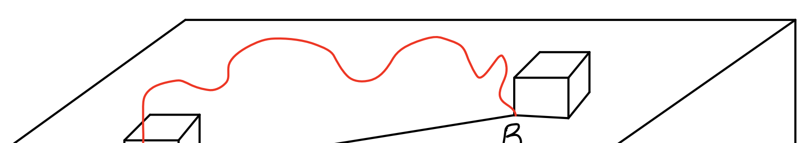
- Symbol δ
- Extended Hamilton principle
- To obtain a rule or governing eqns for the displacement field at equilibrium

$$\delta W = \delta W_c + \delta W_{NC}$$

$$\int \delta W_{NC} \neq \delta \int W_{NC} \quad \left(\begin{array}{c}\text{Not}\\ \text{always}\end{array}\right)$$

path dependent

Example friction



A conservative force does same amount of work moving an object from pt A to pt B regardless of path taken.

But the work done by a non conservative force depends on the path

Friction does more work on the block if one slide it along the indirect path across the tabletop.

The longer the path, the more work friction does

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W_{NC}\right) dt = 0$$

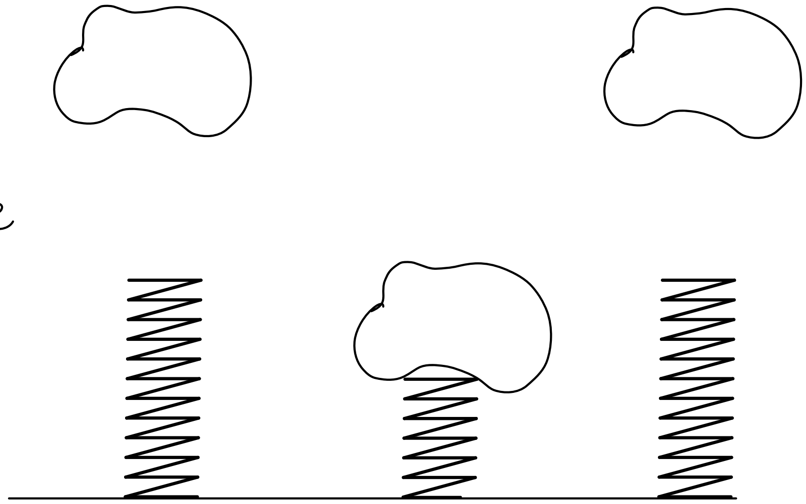
$$\int_{t_1}^{t_2} \left(\delta (T-V) + \delta W_{NC}\right) dt$$

$$\int_{t_1}^{t_2} \delta (T-V)\, dt + \cancel{\int_{t_1}^{t_2} \delta W_{NC}\, dt} = 0$$

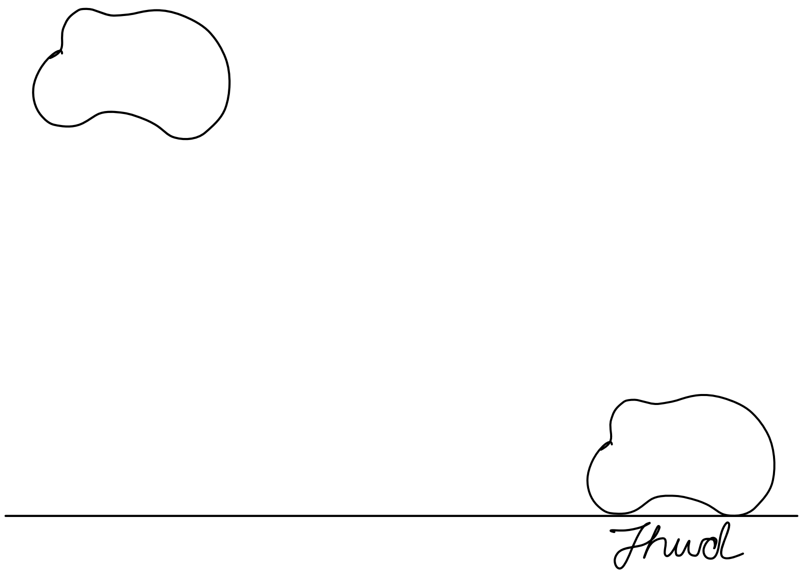
$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad \text{for } W_{NC} \equiv 0 \quad \left(\begin{array}{c}\text{For} \\ \text{conservative} \\ \text{system}\end{array}\right)$$

$$L = \text{Lagrangian} = T - V$$

Conservative system



Non Conservative system



Heat Sound deformations of ground

Lagrange's equation of motion

A more efficient way of obtaining EOM

Coordinates

DoFs

Independent

Non unique

n dof

$q_1, q_2, \ldots q_n$

$$T = T\begin{pmatrix} q_1, q_2, q_3, q_4 \ldots q_n, \\ \dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4 \ldots \dot{q}_n \end{pmatrix}$$

$$V = V\left(q_1, q_2, q_3, q_4 \ldots q_n\right)$$

Can V depend on velocity ?

Potential $\longleftrightarrow$ Integration $\longleftrightarrow$ Not Path dependent

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \delta W_{NC} \right) dt = 0$$

$$\delta T = \sum_{j=1}^{n} \frac{\partial F}{\partial q_j} \delta q_j + \sum_{j=1}^{n} \frac{\partial F}{\partial \dot{q}_j} \delta \dot{q}_j$$

$$\delta V = \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j$$

$$\delta W_{NC} = \sum_{j=1}^{n} Q_j \, \delta q_j$$

Q_j Non conservative forces.

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \, dt = \cancel{\left. \frac{\partial T}{\partial \dot{q}_j} \delta q_j \right]_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j \, dt$$

$$\delta q_j(t_1) = 0$$

$$\delta q_j(t_2) = 0$$

$$\int_{t_1}^{t_2} \left\{ \sum_{j=1}^{n} \left[\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j \right\} dt = 0$$

$$q_j = q_j(t)$$

linearly independent

$$\int_{t_1}^{t_2} \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + Q_j \right) dt = 0$$

for each Q_j $\quad j = 1, 2, 3 \dots n$

Newton's law — Balance of forces at every point

Hamilton principle
Lagrangian Equation of motion
} Weak form of equilibrium

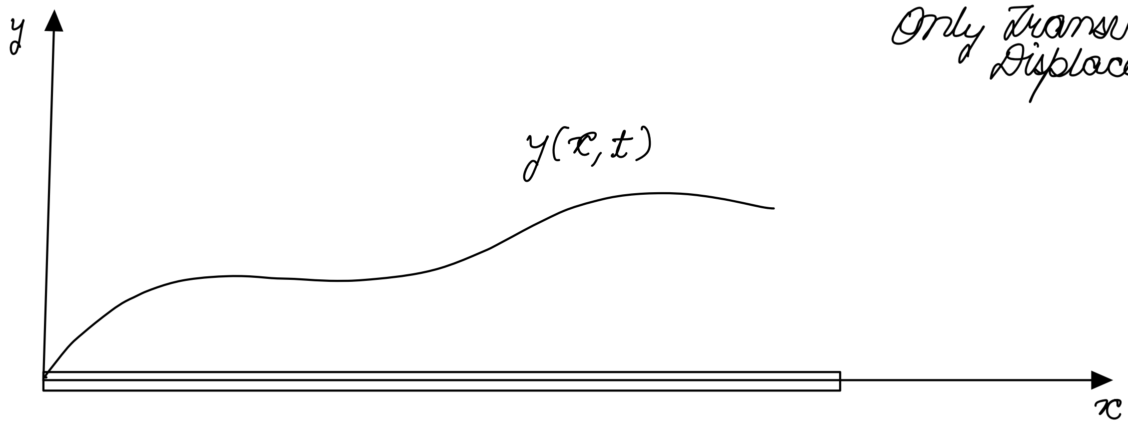
① Abstraction

② Newton's law — Balance of forces at every point

Hamilton principle
Lagrangian Equation of motion
} Weak form of equilibrium

Classical Example :—

Beams in bending



Only Transverse Displacement

Let mass per unit length $= m(x)$

$$T = \int_0^L \frac{1}{2} m\, dx \left(\frac{\partial y}{\partial t}\right)^2 \quad \therefore \int_{t_1}^{t_2} \delta T\, dt = \int_{t_1}^{t_2}\int_0^L m\, dx\, \dot{y}\, \delta\dot{y}\, dt$$

Pure bending

$$V = \int_0^L EI(x) \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \quad \therefore \delta V = \int_0^L EI\, y''\, \delta y''\, dx$$

$$T = T(\dot{y})$$

$$V = V(y'')$$

Integrate by parts

$$\delta V = \int_0^L \frac{\partial^2}{\partial x^2}\left(EI\left(\frac{\partial^2 y}{\partial x^2}\right)\right) dx\, \delta y$$

$$m\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\, \frac{\partial^2 y}{\partial x^2}\right) = 0$$

Abstraction

Bending
Axial
Torsion

Systems :- Strings
beams

Newton's laws

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

Self hanging string.

$$\frac{\partial^2 u(y,t)}{\partial t^2} = g(L-y)\frac{\partial^2 u(y,t)}{\partial y^2} - g\frac{\partial u(y,t)}{\partial y}$$

Siri's string

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right) + f(x) = \rho(x)\frac{\partial^2 y}{\partial t^2} + BCs$$

$$g(L-y) = \frac{T}{\rho}$$

Why separation of variable works ?

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2} \qquad c^2 = \text{something}$$

$$\left(\frac{\partial^2}{\partial x^2} - c^2 \frac{\partial^2}{\partial t^2}\right) y = 0$$

$$\left(\frac{\partial}{\partial x} - c \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + c \frac{\partial}{\partial t}\right) y = 0 \qquad x, t$$

Change variables

$$\xi = x - ct \qquad \frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \xi}{\partial t} = -c$$

$$\eta = x + ct \qquad \frac{\partial \eta}{\partial x} = 1 \qquad \frac{\partial \eta}{\partial t} = c$$

$y(\xi, \eta)$

$$\frac{\partial y}{\partial \xi} = \frac{\partial y}{\partial x} \overset{1}{\cancel{\frac{\partial x}{\partial \xi}}} + \frac{\partial y}{\partial t} \overset{-\frac{1}{c}}{\cancel{\frac{\partial t}{\partial \xi}}} = \frac{\partial y}{\partial x} - \frac{1}{c}\frac{\partial y}{\partial t}$$

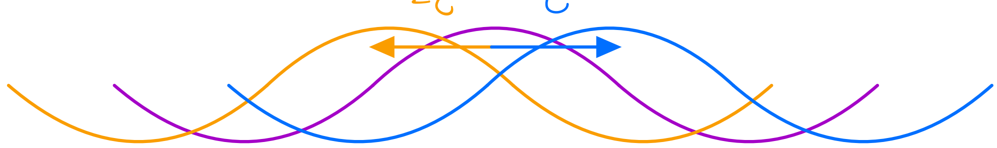
$$\frac{\partial y}{\partial \eta} = \frac{\partial y}{\partial x} \overset{1}{\cancel{\frac{\partial x}{\partial \eta}}} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial \eta} \frac{1}{c} = \frac{\partial y}{\partial x} + \frac{1}{c}\frac{\partial y}{\partial t}$$

$$c^2 \left(\frac{\partial}{\partial \xi}\right)\left(\frac{\partial}{\partial \eta}\right) y = 0$$

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0$$

$$y(\xi, \eta) = f(\xi) + g(\eta)$$

$$= f(x - ct) + g(x + ct)$$



Standing waves

fourier analysis

Course structure

Natural frequencies
Modes and shapes
Rayleigh quotient
Approximate Shapes.

Abstraction
Start from scratch
Follow procedure
Thinking

Natural frequency

Governing Eqn of string

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\rho}{T}\frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$c^2 = \frac{\rho}{T}$$

Superposition of two waves

$$\underbrace{F(x-ct)}_{\text{wave moving to the left}} + \underbrace{G(x+ct)}_{\text{wave moving to the right}} = 0$$

$C > 0$

Basis function

Linear problem

Travelling wave Standing wave

$$\frac{1}{c^2}\frac{f''(x)}{f(x)} = \frac{\ddot{g}(t)}{g(t)} = -\lambda^2$$

1. $\ddot{g} + \lambda^2 g = 0$ $\qquad g_0 e^{st}$

$$s^2 + \lambda^2 = 0$$

$$s = \pm i\lambda$$

$$g(t) = A e^{i\lambda t} + B e^{-i\lambda t}$$

$$= \tilde{A}\cos\lambda t + \tilde{B}\sin\lambda t$$

2. $f'' + \lambda^2 c^2 f$

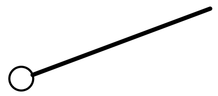
$$f(x) = P\cos c\lambda x + Q\sin c\lambda x$$

What do we know about the system?

At $t=0$, initial cond^{n}s.

At $x=0$, $x=L$ Boundary conditions

Classical BCs



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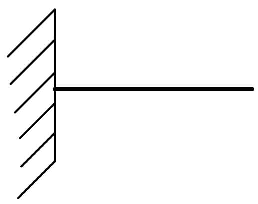
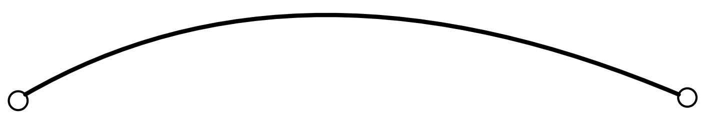


Built in
Cantilever

free

Pinned - Pinned



At $x = 0$, $x = L$ $\quad f(x) = 0$

$$f(x) = P\cos c\lambda x + Q\sin c\lambda x$$

$$f(0) = P = 0$$

$$f(L) = Q\sin c\lambda L = 0$$

Q

$$c\lambda L = n\pi$$

$$\lambda = \frac{n\pi}{cL}$$

β

$n = 1, 2, \ldots$

λ frequency.

$$g(t) = A\cos\lambda t + B\sin\lambda t$$

A B

Initial conditions

λ is not dependent on initial conditions? / No

standing wave $\xrightarrow{\lambda}$ travelling wave

$n = 1$ fundamental mode & frequency

Multiple modes

Unforced Undamped string

free vibrations problem

Rayleigh Quotient

$$T = \frac{1}{2}\int_0^L \rho(x)\left(\frac{\partial y}{\partial t}\right)^2 dx$$

$$V = \frac{1}{2}\int_0^L T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

For standing wave solution $\qquad g = g_0 e^{i\lambda t}$

$$T = \frac{1}{2}\int_0^L \rho(x)\, f^2(x)\, (\dot{g}(t))^2\, dx$$

$$T = \frac{1}{2}\int_0^L \rho(x)\, f^2(x)\, \lambda^2 (g(t))^2\, dx$$

$$V = \frac{1}{2}\int_0^L T\, (f'(x))^2\, g(t)\, dx$$

$$T_{max} = V_{max}$$

$$\frac{1}{2}\int_0^L \rho(x)\, f^2(x)\, \lambda^2 \cancel{g_0^2}\, dx = \frac{1}{2}\int_0^L T\, (f'(x))^2\, \cancel{g_0^2}\, dx$$

$$\lambda^2 = \frac{\int_0^L T(x)\, (f'(x))^2\, dx}{\int_0^L \rho(x)\, f^2(x)\, dx}$$

Obtain shape

λ_j are inherent vibration characteristics

depends on | Boundary conditions

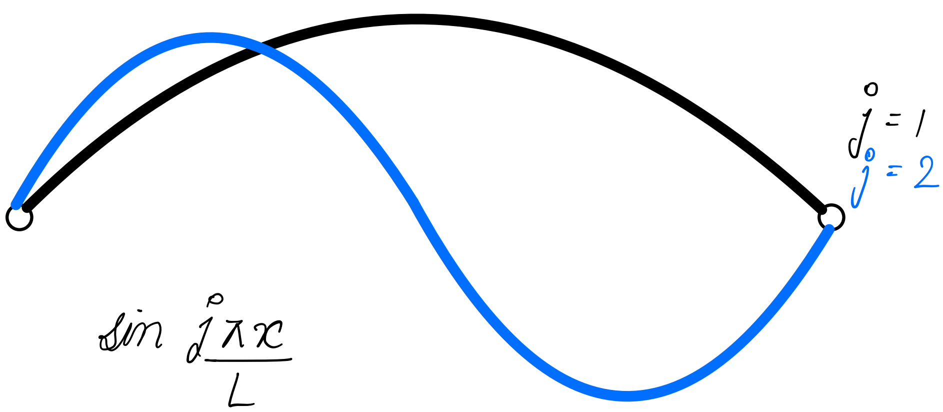
Stiffness
Mass

Extract useful info about the system.

+ Damping

+ Forced

Discrete systems — Reduced order model



$$\sin \frac{j\pi x}{L}$$

$$\lambda_j = \frac{j\pi}{cL}$$

$$y(x,t) = f(x)\, g(t)$$

$$f(x) = \sin \frac{j\pi x}{L}$$

Pinned - Pinned

$$g(t) = e^{i\lambda_j t}\left(A_j \cos \lambda_j t + B_j \sin \lambda_j t\right)$$

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2}$$

$$y(x,t) = \sum_{j=1}^{\infty} f_j(x)\, g_j(t)$$

Linearity allows us to represent any solution as combination of individual solutions.

Mode shapes — Eigenfunctions

Natural frequency — Eigenvalues

Rayleigh Quotient

$(j-1)$ Nodes $\qquad j = 1 \qquad$ fundamental frequency.

$j > 1$ Harmonics

$j = 2$ first harmonic

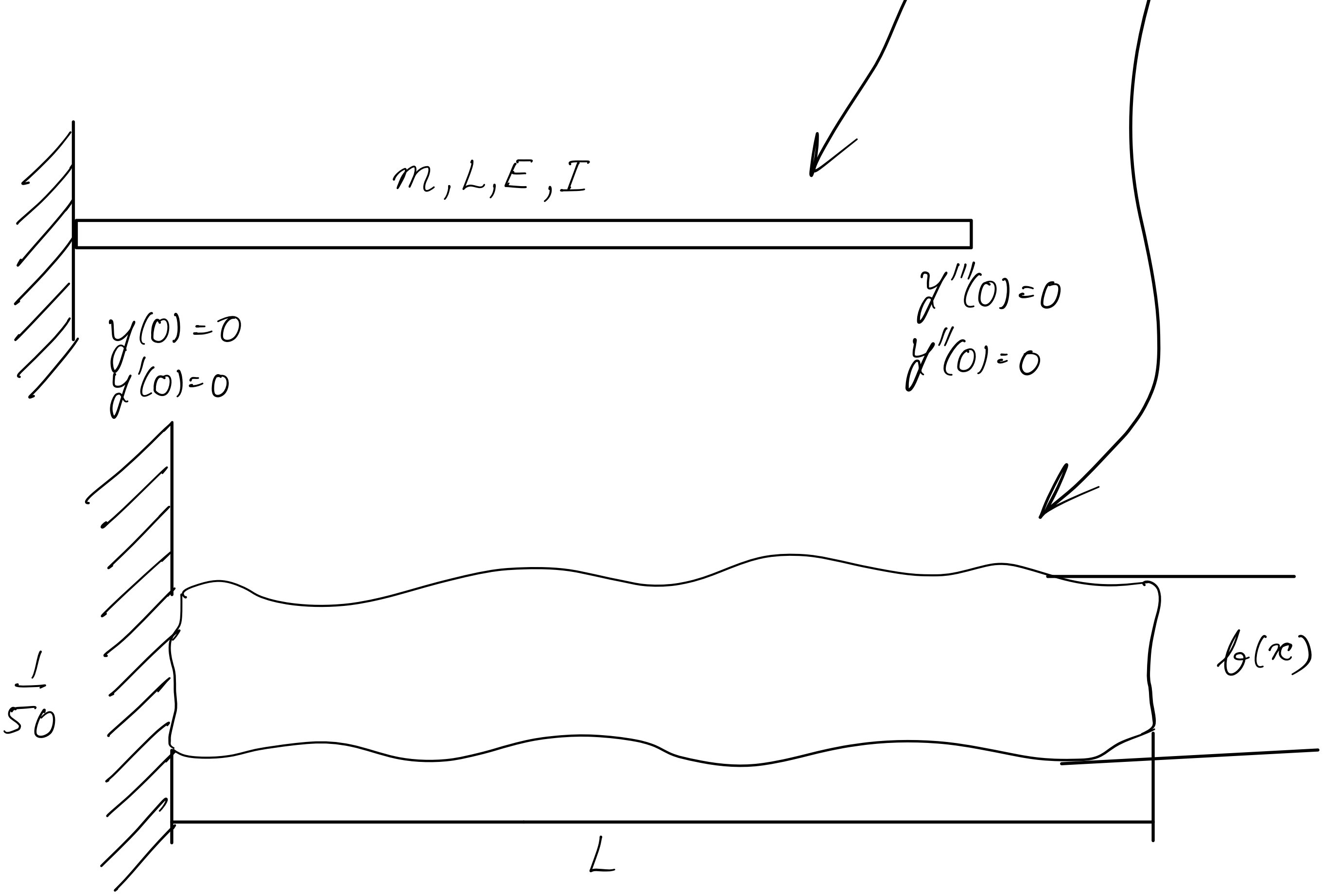
Eigenfunctions / Mode shapes are special because they are exact solution of simple problems.

Admissible functions satisfy geometric BCs

Comparison functions satisfy geometric BCs, Force or natural BCs

Eigenfunctions satisfy geometric BCs, Force or natural BCs, Governing Eqns

Ideal beam



$$\frac{b}{L} < \frac{1}{50}$$

$$y(x) = \sin^2\left(\frac{x}{L}\right)$$

$$y'(x) = \frac{2}{L}\sin\left(\frac{x}{L}\right)\cos\left(\frac{x}{L}\right)$$

$$= \frac{2}{L}\sin\left(\frac{2x}{L}\right)$$

$$y''(x) = \frac{4}{L^2}\cos\left(\frac{2x}{L}\right)$$

$$y'''(x) = -\frac{8}{L^3}\sin\left(\frac{2x}{L}\right)$$

$$y = x^2$$

$$y'(x) = 2x$$

$T \quad V$ Hamiltonian

Choose func^n $\quad \lambda_1, \lambda_2$

GBCs

NBCs — 1 NBC
2 NBC

Rayleigh Quotient of Beam.

$$V = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \frac{1}{2}k\left(w(L)\right)^2$$

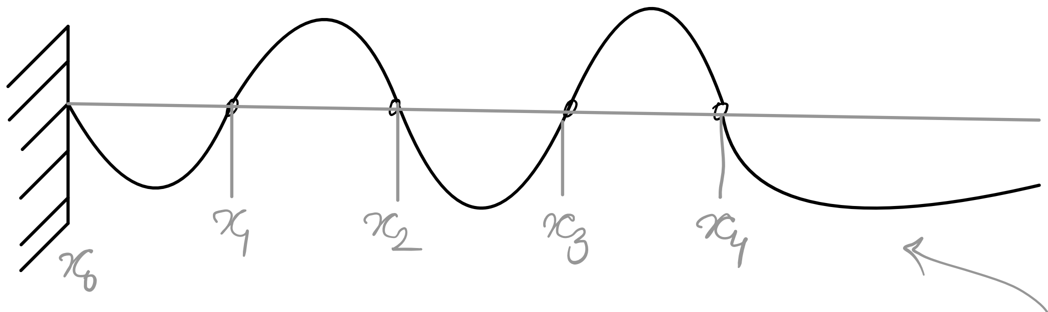
$$T = \frac{1}{2}\int_0^L m\left(\frac{\partial w}{\partial t}\right)^2 dx$$

$$w(x,t) = f(x)\cdot \boxed{\cancel{g(t)}} \rightarrow 1$$

$$V_{max} = \frac{\int_0^L \frac{1}{2}EI\left(\frac{\partial^2 f}{\partial x^2}\right)^2 dx + \frac{1}{2}k\left(f(L)\right)^2}{\frac{1}{2}\int_0^L m f^2(x)\, dx}$$

5^{th} mode
No of nodes 4

j^{th} mode
$(j-1)$ node



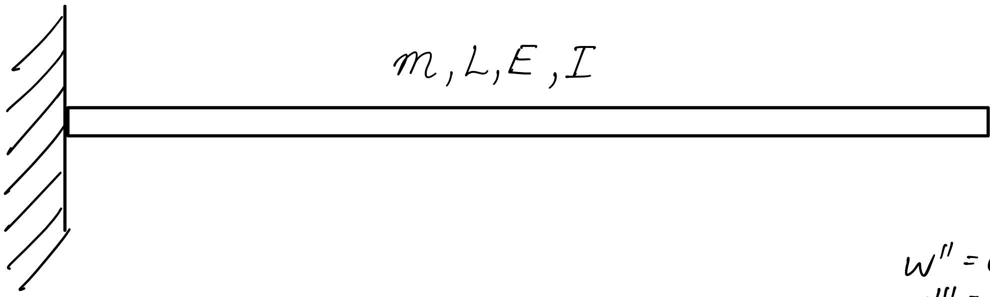
Assume $f(x)$ such that it is

$$(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_0)$$

Mid Term 8:30 - 10:30 A.M.
20th Sept

1. Abstraction
2. Hamilton's Eqn — T, V
   guess a func^n
3. Comparison func^n

Practice problems - Prepare



$W = 0$
$W' = 0$

$W'' \neq 0$
$W''' \neq 0$

$W'' = 0$
$W''' = 0$

$W \neq 0$
$W' \neq 0$

$$W = \sum_{i=1}^{N} f_i(x)\, g_i(t)$$
$$= \sum_{i=1}^{N} A_i\, F_i(x)\, e^{i\omega t}$$

$F_i(x)$ should satisfy cantilver boundary condition

$$\sin\left(\frac{\pi x}{2L}\right)$$

$\frac{\pi x}{2L}$ $\qquad \left(x - \frac{2L}{\pi}\right) x$

Chapter 8 Fundamental of Vibrations — Leonard Meirovitch

Distributed-Parameter System : Exact solutions

Chapter 7 Determination of Natural frequencies and mode shapes — S S Rao Mechanical Vibrations

$$m(x) = \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

$$U(x) = \sin\frac{\pi x}{2L}$$

$$EA(x) = \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

$$\frac{dU}{dx} = \cos\frac{\pi x}{2L}$$

$$R(U) = \omega^2 = \frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L m(x)\, U^2(x)\, dx}$$

$$= \frac{\int_0^L \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] U^2(x)\, dx}$$

$$= \frac{EA}{m}\left(\frac{\pi}{2L}\right)^2\left(\frac{(L/12\pi^2)\,(5\pi^2+6)}{(L/12\pi^2)\,(5\pi^2-6)}\right) = 3.1504\,\frac{EA}{mL^2}$$

Segue → Transition

1 and 2 dof system

Abtraction Reality → Model

Approximation

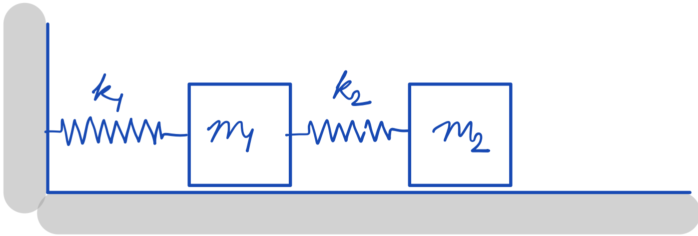
$$w(x,t) = f(x)\, g(t)$$

$$T = \int_0^L \frac{1}{2} m \left(\frac{\partial w}{\partial t}\right)^2 dx = \frac{1}{2}\left(\dot{g}(t)\right)^2 \int_0^L m \left(f(x)\right)^2 dx = \frac{1}{2} \tilde{m} \left(\dot{g}(t)\right)^2$$

Equivalent mass

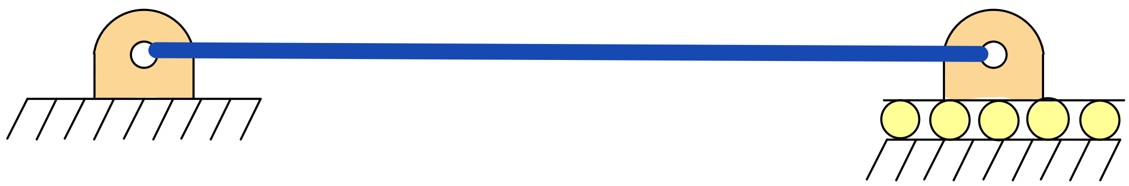
$$V = \int_0^L EI \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = \frac{1}{2}\left(g(t)\right)^2 \int_0^L EI f'' dx = \frac{1}{2} \tilde{k} \left(g(t)\right)^2$$

Equivalent spring



2 DoF Example

Linear Time Invariant Systems



$$\frac{\partial^2 y}{\partial x^2} + c^2 \frac{\partial^2 y}{\partial t^2} = 0$$

$$y(x,t) = a_1 f_1(x)\, g_1(t) + a_2 f_2(x)\, g_2(t)$$

$$T = \frac{1}{2}\int_0^L m \dot{y}^2\, dx = \frac{1}{2}\left[A_1 (\dot{q}_1(t))^2 + A_2 (\dot{q}_2(t))^2 + A_3 (\dot{q}_1(t)\dot{q}_2(t))\right]$$

$$= \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}$$

$$\text{Domain}\,[f(x)] = [0, L]$$

Similarly

$$V = \frac{1}{2}\begin{bmatrix} q_1 & q_2\end{bmatrix}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix}\begin{bmatrix} q_1 \\ q_2\end{bmatrix}$$

Continuous system $\rightarrow$ n term approximations

n dof

Check

Reality

Model

Approximation

Solution methodologies

Damping (Estimation)

Viscous	solid fluid friction
Friction	solid solid external (contact type)
Material	solid solid (internal to materials)

Most of the times, we use viscous models
deals with "small" amounts of damping
much nicer mathematically

Equivalent viscous damping

$$m\ddot{x} + c\dot{x} + kx = F$$

m, c and k are constants

Homogeneous + Particular

$m\ddot{x} + c\dot{x} + kx = 0$

Need initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$

Laplace
State space
General form

$$x(t) = A_0 e^{st}$$

$$(ms^2 + cs + k) A_0 e^{st} = 0$$

$$ms^2 + cs + k = 0$$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$c^2 > 4km$ (Purely real solutions) Door damper

$c^2 = 4km$ (Degenerate roots But real)

$c^2 < 4km$ (Complex Roots) Most examples

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

Natural frequency of undamped situation

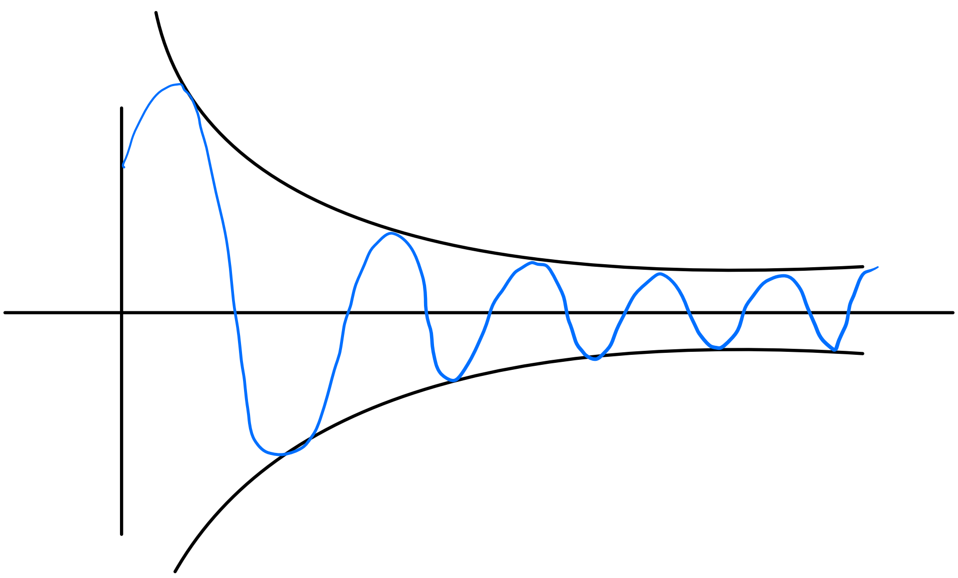
For $c^2 < 4km$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$= e^{-c/2m\, t}\left(A_1 e^{i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\, t} + A_2 e^{-i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\, t}\right)$$

Purely exponential

Purely oscillatory

m c for exponential part



frequency of damped system

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$= \sqrt{\frac{k}{m}}\left(\sqrt{1 - \left(\frac{c}{2m}\right)^2 \frac{m}{k}}\right)$$

$$= \omega_n \left(\sqrt{1 - \left(\frac{c}{2m}\right)^2 \left(\frac{m}{k}\right)}\right)$$

$$\frac{c^2}{4m^2}\,\frac{m}{k} = \frac{c^2}{4km}$$

$$\xi^2 = \frac{c^2}{4km}$$

Damping Ratio $\quad \xi = \dfrac{c}{2\sqrt{km}}$

Need to find a good reference for value of damping

$$c^2 < 4km$$

$$\frac{c^2}{4km} < 1$$

$$\boxed{\xi < 1}$$

Reference value

Underdamped System

Oscillatory & decay.

$\xi = 1$ Critical Damping

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

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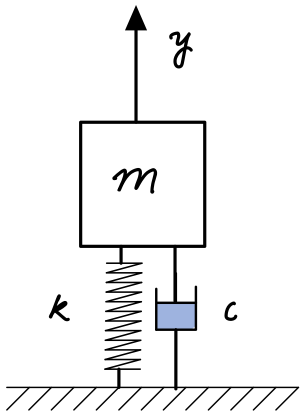
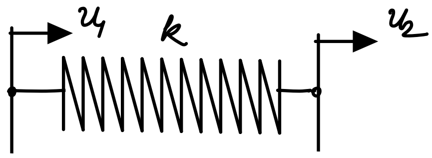
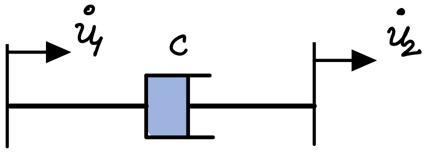
Viscous Damping



$$m\ddot{y} + c\dot{y} + ky = 0$$



$k(u_1 - u_2)$



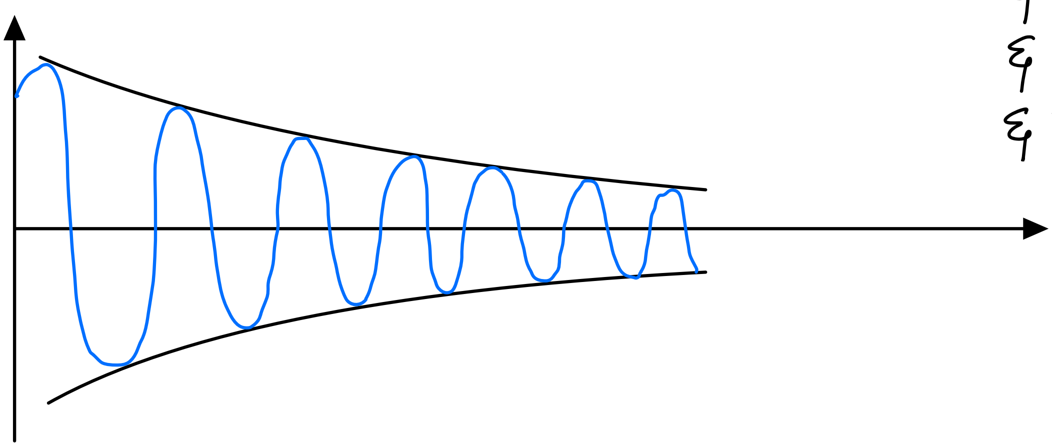
$c(u_1 - u_2)$

$\omega_n = \sqrt{\frac{k}{m}}$ Natural freq. of undamped system

$$\omega_d = \omega_n \sqrt{1-\xi^2} \qquad \xi^2 = \frac{c^2}{4km}$$

$$y(t) = e^{-\frac{c}{2m}t}\left(A_1 e^{i\sqrt{\frac{k}{m}}\sqrt{1-\xi^2}\,t} + A_2 e^{-i\sqrt{\frac{k}{m}}\sqrt{1-\xi^2}\,t}\right)$$

$$= e^{-\xi\omega_n t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$



$\xi < 1$

$\xi = 1$ (Repeated Roots)

$\xi > 1$

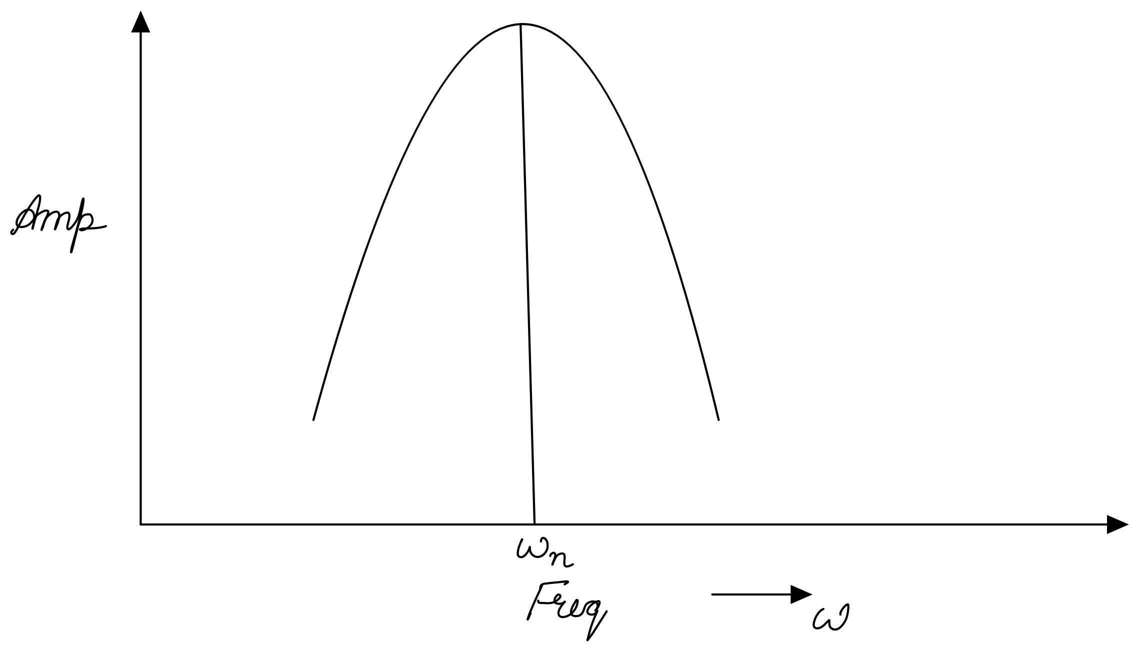
In practice, easiest option is Viscous Damping

So we estimate damping as a purely viscous damping

free vibrations — easy, but more approximate

forced vibrations — not so easy, required specific equipment and knowledge of the system, but more accurate

Half-powered, bandwidth
Resonance peak, etc

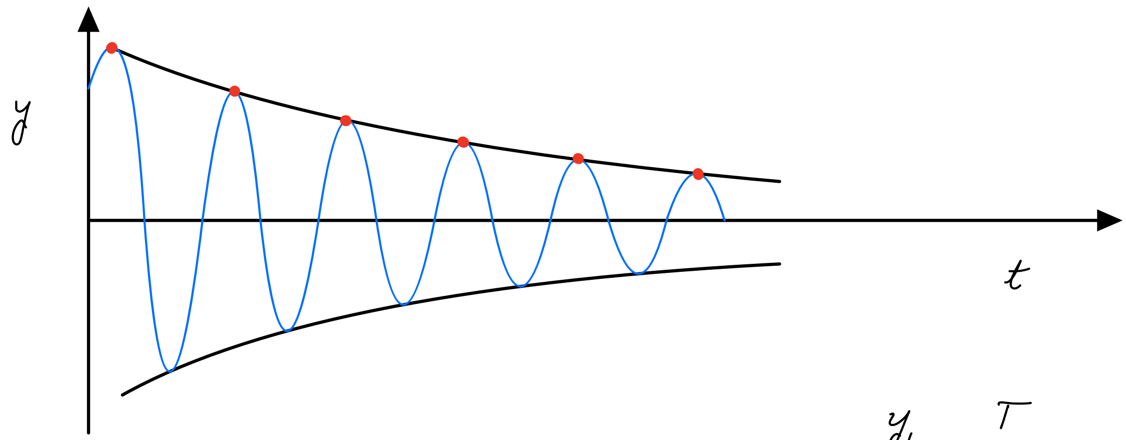


$$y(t) = e^{-\xi\omega_n t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$

$$= e^{-\xi\omega_n t}\left[A_1 \cos\omega_d t + A_2 \sin\omega_d t\right]$$

How do we obtain estimates of ξ?

$$= e^{-\xi\omega_n t} A\cos(\omega_d t + \phi)$$



$$\begin{array}{ll} y_1 & T \\ y_2 & 2T \\ y_3 & 3T \\ \vdots & \\ y_n & nT \end{array}$$

For given initial conditions,

A, ϕ is known

$$\frac{y_1}{y_2} = \frac{e^{-\xi\omega_n T}}{e^{-\xi\omega_n 2T}}$$

$$\frac{y_1}{y_n} = \frac{e^{-\xi\omega_n T}}{e^{-\xi\omega_n nT}} = e^{\xi\omega_n (n-1)T}$$

For only peaks, $\cos = 1$

T = period of oscillations of the damped system

If freq. of damped system is ω_d

$$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$\xi\omega_n (n-1)T = \frac{\xi\omega_n (n-1) \times 2\pi}{\omega_n\sqrt{1-\xi^2}}$$

Binomial Expansion

$$\cancel{(1-\xi^2)^{-1/2}}$$

$$\cancel{1+\frac{\xi^2}{2}}$$

$$\delta = \ln\left(\frac{y_j}{y_{j+n}}\right) = \frac{2\pi(n-1)\xi}{\sqrt{1-\xi^2}}$$

$$= \cancel{\frac{2\pi(n-1)\xi}{1+\frac{\xi^2}{2}}}$$

$$\delta = \frac{2\pi(n-1)\xi}{\sqrt{1-\xi^2}}$$

$\xi << 1$

$1 - \xi^2 \approx 1$

$$\delta = 2\pi(n-1)\xi$$

$$\delta^2 - \delta\xi^2 = 2\pi(n-1)\xi$$

$$\delta\xi^2 - 2\pi(n-1)\xi - \delta^2 = 0$$

Doable using simple constructs.

1 DoF

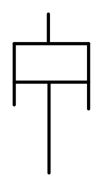
Estimate ξ

Prove that it is correct

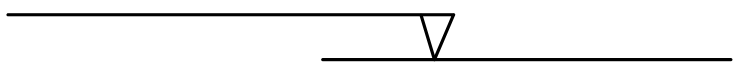
If there is deviation from expectations, discuss

$m\ddot{y} + c\dot{y} + ky = 0$

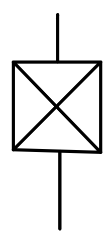
Viscous damping



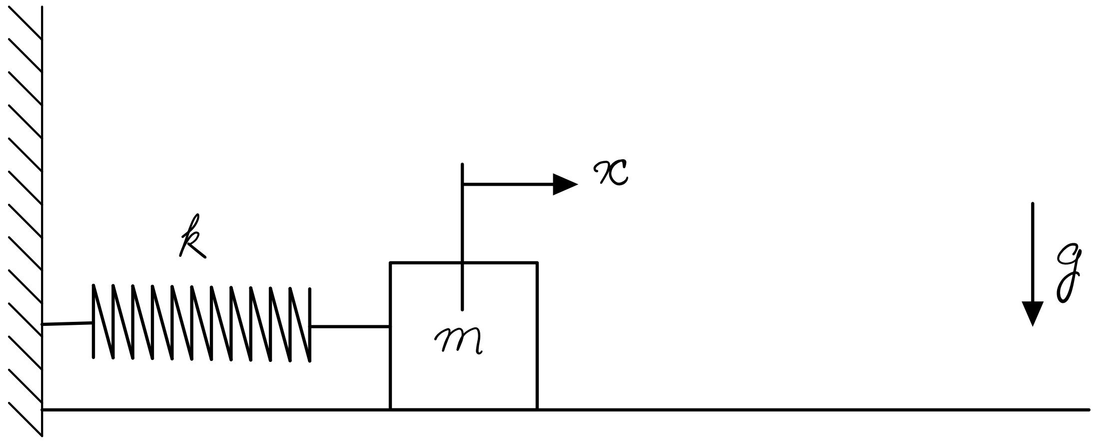
Friction damping



Material damping



Let us look at friction damping



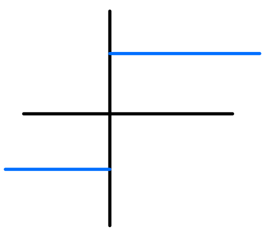
Friction between m and the surface
Force is F_d

F_d always opposes motion $x(t)$, $\dot{x}(t)$ ✓, $\ddot{x}(t)$

$$\propto -\dot{x}(t)$$

$$F_d = \mu_k N$$

Normal reaction
Coeff of friction
μ_k kinetic friction coefficient



$$-\frac{\dot{x}}{|\dot{x}|}$$ (Signum function)

not linear over the domain of interest

$$F_d = \mu_k N \left(-\frac{\dot{x}}{|\dot{x}|}\right)$$

Splitting into two domains

Split the problem into two pieces based on the sign of $\dot{x}$

Initial cond^{n}s $\quad x(0) = x_0$

$\dot{x}(0) = 0$

$\dot{x} < 0$ (moving left) $\quad$ Part 1

$$m\ddot{x} + kx = F_d$$

Once $\dot{x} > 0$ (moving right) $\quad$ Part 2

$$m\ddot{x} + kx = -F_d$$

Part 1 :-

$$\ddot{x} + \frac{k}{m}x = \frac{F_d}{m}\frac{k}{k}$$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 \frac{F_d}{k}$$

$$f_d = \frac{F_d}{k}$$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 f_d$$

Solution

$$x(t) = A\cos\omega_d t + B\sin\omega_d t + f_d$$
$$\dot{x}(t) = -\omega_d A\sin\omega_d t + B\omega_d\cos\omega_d t$$

Putting initial conditions $\quad x(t=0) = x_0$

$$A + f_d = x_0$$

$$\dot{x}(t=0) = 0$$

$$B\omega_d = 0 \qquad B = 0$$

for time $\left[0, \frac{\pi}{\omega_n}\right]$

$$x(t) = (x_0 - f_d)\cos\omega_d t + f_d$$
$$\dot{x}(t) = \omega_n (f_d - x_0)\sin\omega_d t$$

Part 2

$$x\left(t = \frac{\pi}{\omega_n}\right) = 2f_d - x_0 = -(x_0 - 2f_d)$$

$$\dot{x}\left(t = \frac{\pi}{\omega_n}\right) = 0$$

Next use

$$m\ddot{x} + kx = -F_d$$
$$x\left(\frac{\pi}{\omega_n}\right) = -(x_0 - 2f_d)$$
$$\dot{x}\left(\frac{\pi}{\omega_n}\right) = 0$$

Solution you should get

$$x(t) = (x_0 - 3f_d)\cos\omega_n t - f_d$$

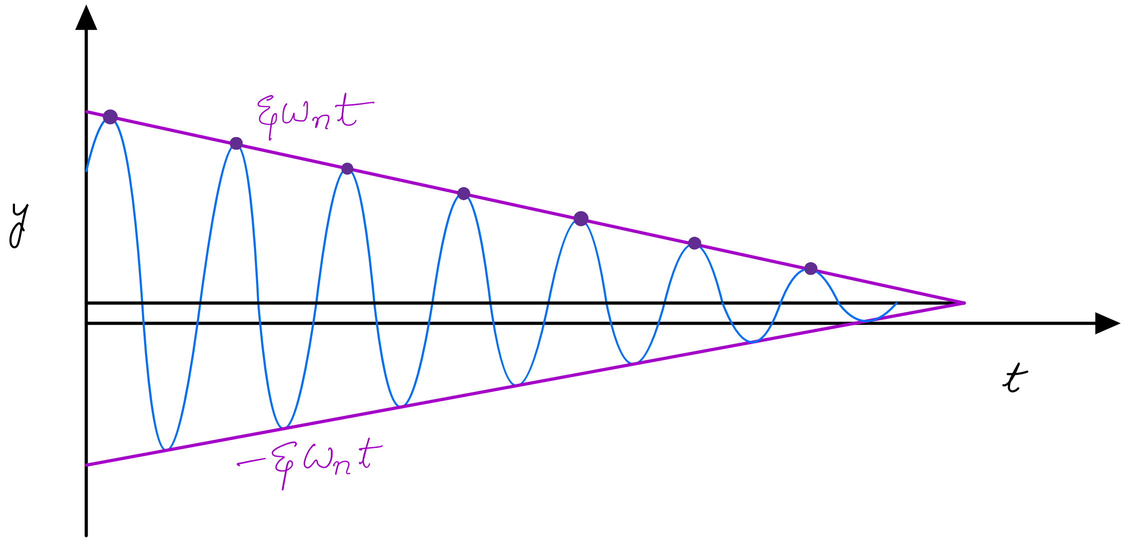
At $2\frac{\pi}{\omega_n}$, change of direction

Undamped system at $\frac{\pi}{\omega_n}$ $\frac{2\pi}{\omega_n}$

Friction damping does not affect frequency of oscillation

Amplitude reduces by $2f_d$ per half cycle

Linear envelope



Forced Systems

Nature / Characteristics of force
Duration (w.r.t. Timeperiod)
Time of response

Static
Dynamic < Periodic / Non periodic

Continuous
Discontinuous

Decay
Oscillation with or without decay
Diverging

Fourier Analysis / series
Laplace Transforms
Convolution Integral

1 DoF Project
Individual
Or less than 3 member Team

5 min video
Report
Setup (No Marks)

Combined reports acceptable

Forced vibrations

		Tools
Periodic forcing	Trigonometric (Harmonic)	Fourier based method
Non periodic forcing	Long duration / short duration	Convolution integral

Damped system subject to harmonic loading

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0 \cos(\Omega t)$$

$$\boxed{\frac{c}{m} = 2\xi\omega_n}$$

$$\xi = \frac{c}{2\sqrt{km}} = \frac{c}{2\sqrt{k}\sqrt{m}}\frac{\sqrt{m}}{\sqrt{m}} = \frac{c}{2m}\frac{\sqrt{m}}{\sqrt{k}}$$

$$2\xi\omega_n = \frac{c}{m}$$

$$c = 2\xi m\omega_n$$

$$\ddot{x} + 2\xi\omega_n + \omega_n^2 x = \frac{F_0}{k}\omega_n^2 \cos(\Omega t)$$

Homogeneous + Particular

$$e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + x_p(t)$$

$x_p(t)$? will be of form $\alpha\cos\Omega t + \beta\sin\Omega t$

$$-\Omega^2 x_p(t) + (-\alpha\Omega\sin\Omega t + \beta\Omega\cos\Omega t)\,2\xi\omega_n + \omega_n^2 x_p = \frac{F_0}{k}\omega_n^2\cos\Omega t$$

group all sine terms together, all cos terms and equals

$$x_p(t) = \underbrace{\frac{F_0}{k}}_{x_{static}} \frac{1}{G_1}\left[\frac{2\xi\Omega}{\omega_n}\sin\Omega t + \left(1-\left(\frac{\Omega}{\omega}\right)\right)^2 \cos\Omega t\right]$$

$$G_1 = \left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\xi\Omega}{\omega_n}\right)^2$$

Damped system

$x_h(t) \rightarrow 0$ as $t \rightarrow \infty$

Only x_p remains

long time record

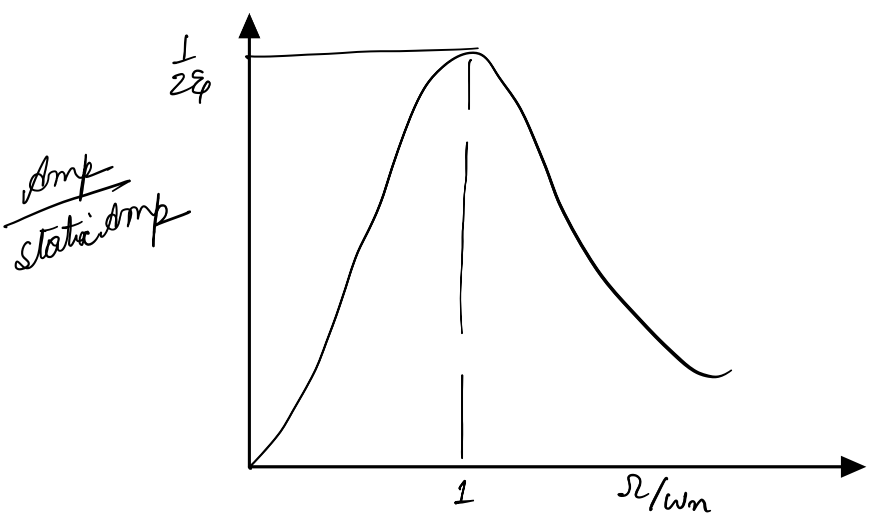
Response depends only on forcing, not initial cond^{n}s

$$x_p(t) = \frac{F_0/k}{\sqrt{G_1}} \sin(\Omega t - \phi)$$

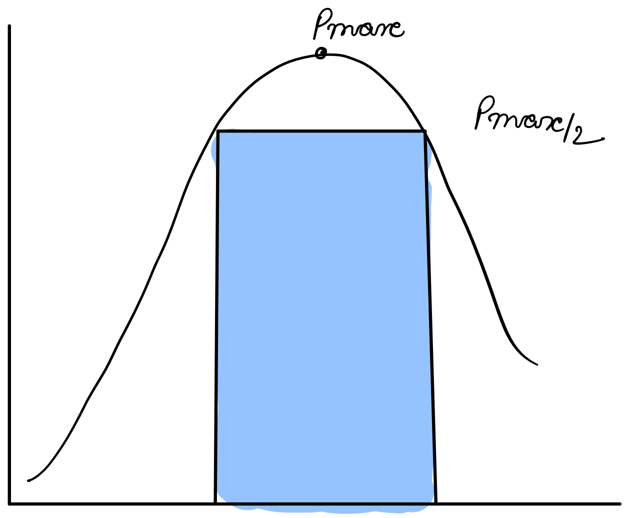
What happens when $\Omega \rightarrow \omega_n$ (Resonance)

$G_1 = 4\xi^2$ Amp $= \frac{F_0}{k}\left(\frac{1}{2\xi}\right)$

Amplitude $\rightarrow \infty$ for undamped system



Half power method



Work done by the forces in the system (frequency Ω)

Inertial	$m\ddot{x}$
Stiffness	$c\dot{x}$
Damping	kx
Applied force	$F_0 \cos \Omega t$

$x_p(t) = A\cos(\Omega t - \phi)$

Multiple terms in the forcing freq. ?

$F(t) = F_1 \cos \Omega_1 t + F_2 \cos \Omega_2 t + \cdots$

linear time invariant
superposition

Fourier series expansion

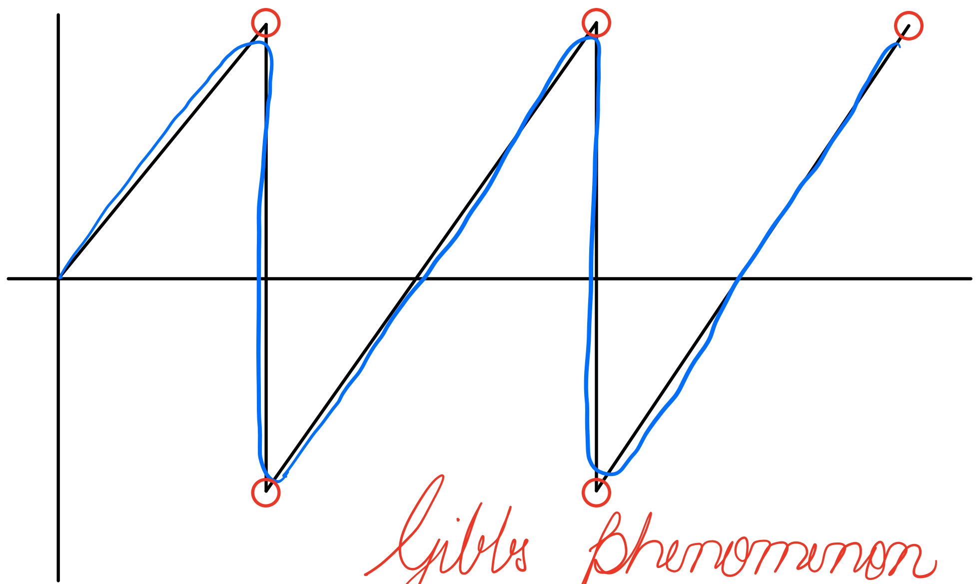
Tutorial
How to quick recap ?

Periodic but not harmonic

$F(t) = \sum$ Fourier series expansion

Tutorials

There always be an error



C_0 but not C_1
Gibbs phenomenon.

Gibbs phenomenon

Laplace in tutorial

Non periodic

Before that

Methods to tackle

Periodic loads
- One term harmonic
- Multiple terms (Fourier)
- General smooth
- General non smooth (Fourier series fail, Gibbs phenomenon) — Tutorial problem

Superposition + linear combination

Stage 1 Fourier series expansion

Stage 2 Individual solution

Non periodic loads
- Smooth — Duhamel Integral
- General — Superposition

For tutorials

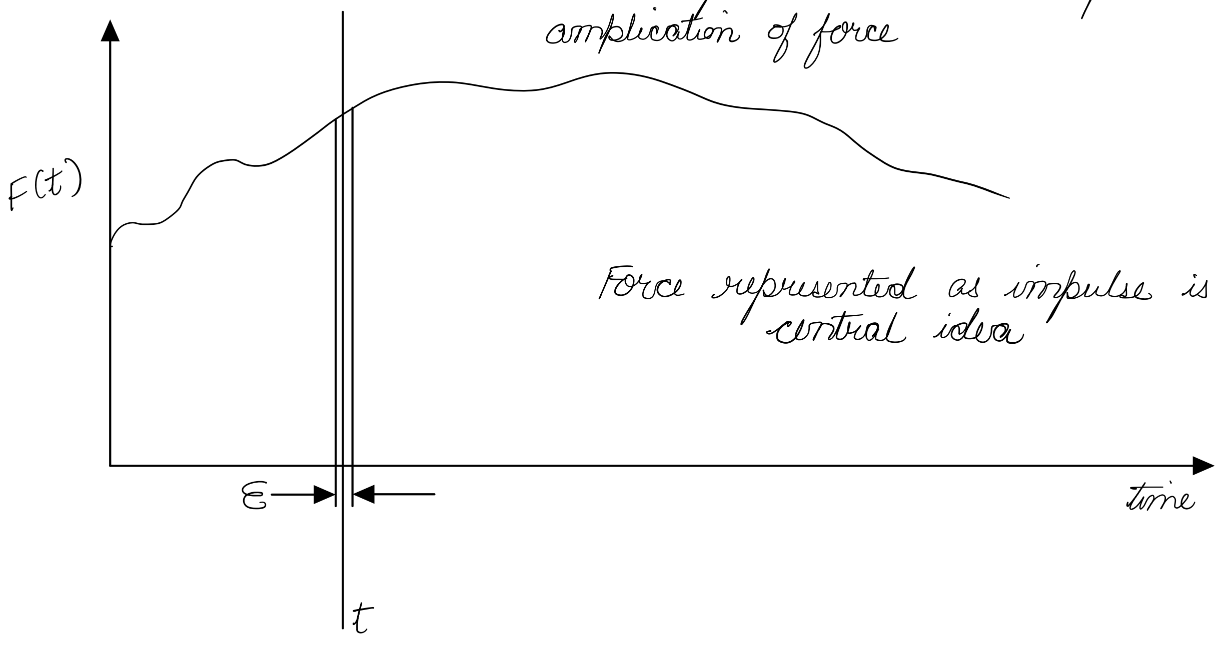
Tutorial packages

General non periodic forcing

Convolution Integral

Unit Impulse

Unit of Impulse F × time
change is (linear) momentum
time segment is "small" compared to duration of application of force



Force represented as impulse is the central idea

Determine response of the system to a unit impulse at some arbitrary time.

Scale to match impulse due to force (linearity)

Sum over all impulses that make up the force (superposition)

Define unit impulse based on the Dirac-Delta function.

$$\delta(t-\tau) = 0 \qquad t \neq \tau$$

$$\int_0^\infty \delta(t-\tau)\, dt = 1 \qquad t = \tau$$

Casuality of time.

Define Impulse of $F(t)$ at a time 't'

$$\hat{F} \triangleq \int_t^{t+\Delta t} F(\tau)\, d\tau$$

Based on the def. of Dirac Delta

$$\int_{-\infty}^{\infty} \delta(t-\tau)\, dt = 1$$

We want

$$\int_t^{t+\Delta t} F(\tau)\, d\tau = 1 \qquad \text{as} \quad \Delta t \to 0$$

$$\int_t^{t+\Delta t} \left(m\ddot{x} + c\dot{x} + kx\right) d\tau = \int_t^{t+\Delta t} F(\tau)\, d\tau$$

$$\int_t^{t+\Delta t} (m\ddot{x})\, d\tau = m\,\dot{x}(\tau+\Delta\tau) - m\,\cancel{\dot{x}(\tau)}^{\,0}$$

$$\int_t^{t+\Delta t} c\dot{x} \quad d\tau = c\,x(\tau+\Delta\tau) - c\,x(\tau) = 0$$

$$\int_t^{t+\Delta t} kx \quad d\tau = k \qquad = 0$$

$$\lim_{\Delta t \to 0} \quad m\dot{x}(t+\Delta t) = \int_t^{t+\Delta t} F(t)\, dt = 1$$

$$\dot{x}(t^+) = \frac{1}{m}$$

Response to a Unit Impulse

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$x(0) = 0$$

$$\dot{x}(0) = 1/m$$

Let impulse applied at $t = 0$ — will change later to unit impulse applied at time τ

$$x(t) = e^{-\xi\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

$$A = 0 \qquad B = \frac{1}{m\omega_d}$$

$$x(t) = \frac{e^{-\xi \omega_n t} \sin \omega_d t}{m \omega_d}$$

Impulse applied at $t = \tau$, sys at rest prior to impulse

Replace t with $t - \tau$

Unit Impulse Response funcⁿ $g(t-\tau)$

$$x(t-\tau) = \frac{1}{m\omega_d} e^{-\xi \omega_n (t-\tau)} \sin \omega_d (t-\tau)$$

If $\hat{F}$ is the magnitude due to force at time $t-\tau$

Scale $x(t-\tau)$ using $\hat{F}$

$x(t-\tau)$ unit impulse response at time τ

$g(t-\tau)$

Duhamel's integral approach for arbitrary excitations

Response to Impulse

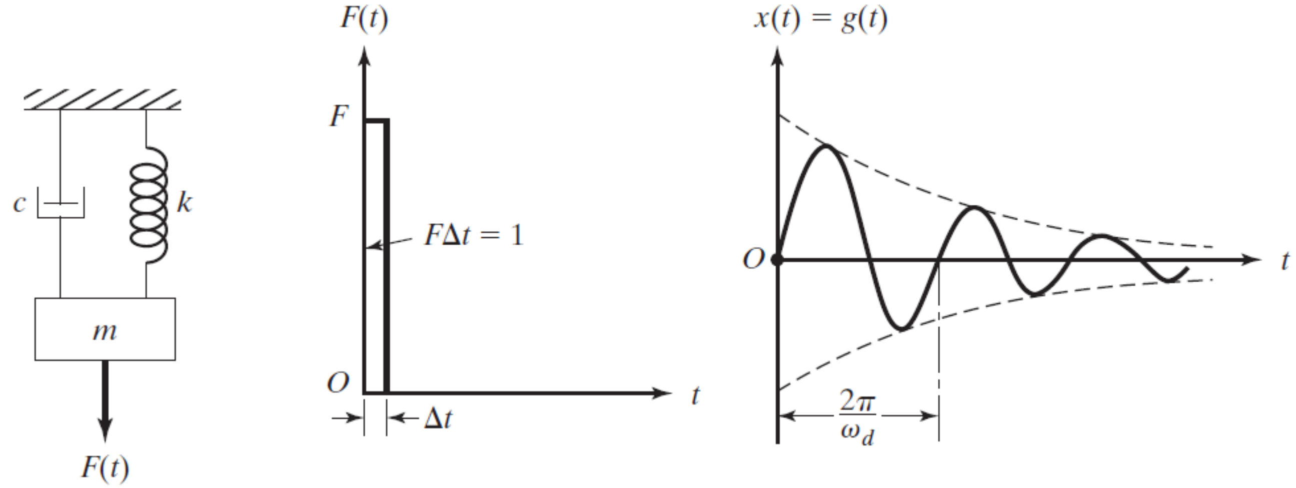
If the mass is at rest before the unit impulse is applied ($x = \dot{x} = 0$ for $t < 0$ or at $t = 0^-$), we obtain, from the impulse-momentum relation,

$$\text{Impulse} = \hat{f} = 1 = m\dot{x}(t = 0) - m\dot{x}(t = 0^-) = m\dot{x}_0$$

Thus the initial conditions are given by

$$x(t = 0) = x_0 = 0$$
$$\dot{x}(t = 0) = \dot{x}_0 = \frac{1}{m}$$

$\hat{f} = 1$



Response to Impulse

Consider a viscously damped spring-mass system subjected to a unit impulse at $t = 0$, as shown in Figs. 4.6(a) and (b). For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

is given by Eq. (2.72a) as

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}}$$

Most important formula

where

$$\boxed{\zeta = \frac{c}{2m\omega_n}}$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

Hence,

$$x(t) = \boxed{g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t}$$

Response to Impulse

Previous slide equation gives the response of a single-degree-of-freedom system to a unit impulse, which is also known as the impulse response function, denoted by $g(t)$.

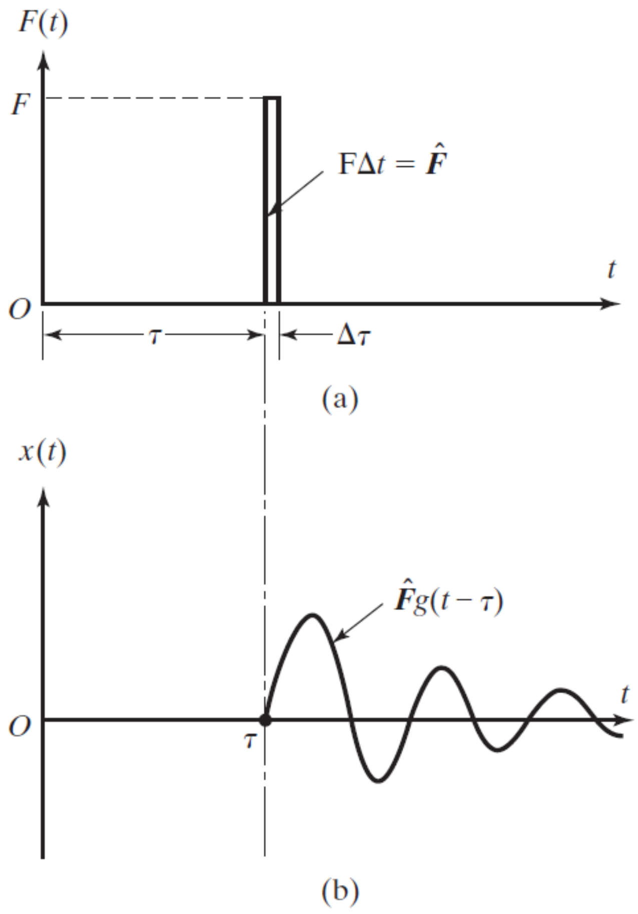
If the magnitude of the impulse is F instead of unity, the initial velocity x_0 is F/m and the response of the system becomes

$$x(t) = \frac{\mathrm{F}e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t = \hat{F}g(t)$$

If the impulse F is applied at an arbitrary time $t = \tau$, as shown in Fig. 4.7(a), it will change the velocity at $t = \tau$ by an amount F/m. Assuming that $x = 0$ until the impulse is applied, the displacement x at any subsequent time t, caused by a change in the velocity at time τ, with t replaced by the time elapsed after the application of the impulse-that is, $t - \tau$. Thus we obtain

$$x(t) = \hat{F}g(t - \tau)$$

This is shown in Figure



Response to an impulse

$\longrightarrow$ the principle of impulse and momentum states that impulse equals change in momentum:

$$Impulse = F\Delta t = m(v_2 - v_1)$$

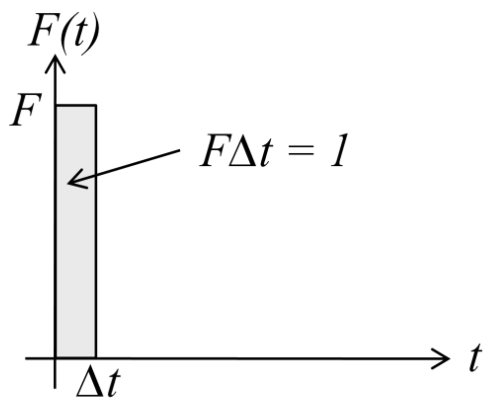
or:

$$\underline{F} = \int_t^{t+\Delta t} F dt = m\dot{x}_2 - m\dot{x}_1$$

A unit impulse is defined as:

$$\underline{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F dt = 1$$

Now consider the response of an undamped system to a unit impulse. Recall that the free vibration response is given by:



$$x(t) = x_0 cos\omega_n t + \frac{\dot{x}_0}{\omega_n} sin\omega_n t$$

If the mass starts from rest, we can get the velocity just after impulse as:

$$\dot{x}_0 = \frac{\underline{f}}{m} = \frac{1}{m}$$

and the response becomes:

$$x(t) = \frac{1}{m\omega_n} sin\omega_n t$$

for a non-unit impulse, the response is:

$$x(t) = \frac{\underline{F}}{m\omega_n} sin\omega_n t$$

For an underdamped system, recall that the free response was gven by:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} sin\omega_d t\right]$$

For a unit impulse, the response for zero initial conditions is:

$$x(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} sin\omega_d t = g(t)$$

where $g(t)$ is known as the *impulse response function*. For a non-unit impulse, the response is:

$$x(t) = \frac{\underline{F}e^{-\zeta\omega_n t}}{m\omega_d} sin\omega_d t = \underline{F}g(t)$$

If the impulse occurs at a delayed time $t = \tau$, then

$$x(t) = \frac{\underline{F}e^{-\zeta\omega_n (t-\tau)}}{m\omega_d} sin\omega_d (t-\tau) = \underline{F}g(t-\tau)$$

If two impulses occur at two different times, then their responses will superimpose.

Example: For a system having $m = 1$ kg; $c = 0.5$ kg/s; $k = 4$ N/m; F = 2 Ns obtain the response when two impulses are applied 5 seconds apart.

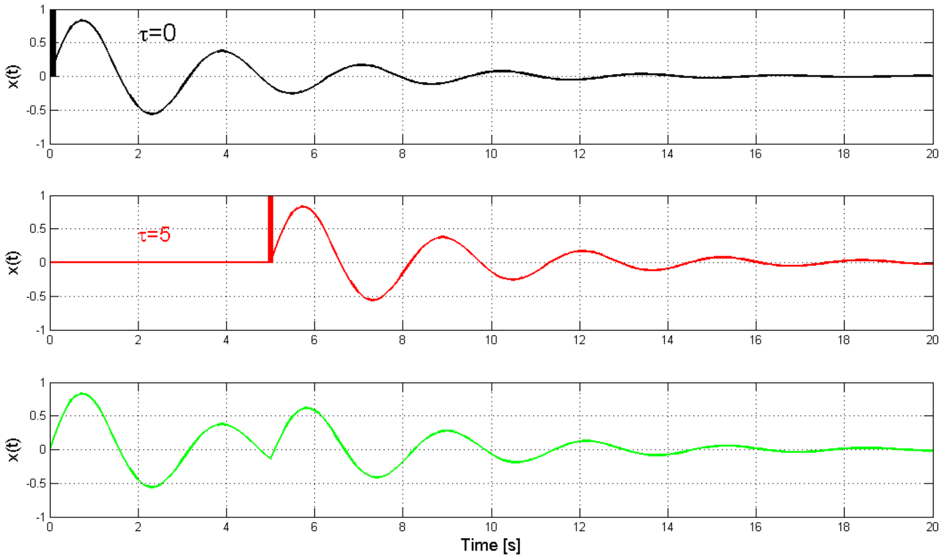
Solution: here we have $\omega_n = 2\frac{\text{rad}}{\text{s}}, \zeta = 0.125, \omega_d = 1.984\frac{\text{rad}}{\text{s}}$ so the solutions become:

$$x_1(t) = \frac{2e^{-0.25t}}{1.984} \sin 1.984t \qquad t > 0$$

$$x_2(t) = \frac{2e^{-0.25(t-\tau)}}{1.984} \sin[1.984(t-\tau)] \qquad t > 5$$

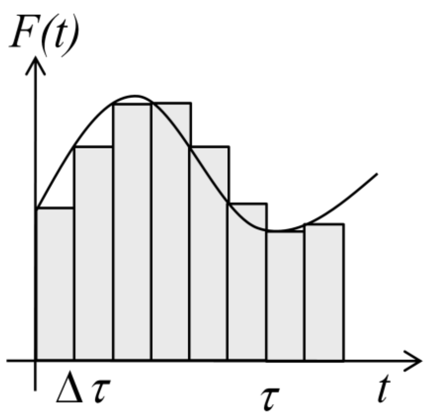
And the total response is:

$$x(t) = \begin{cases} \dfrac{2e^{-0.25t}}{1.984}\sin 1.984t & 0 < t < 5 \\ \dfrac{2e^{-0.25t}}{1.984}\sin 1.984t + \dfrac{2e^{-0.25(t-\tau)}}{1.984}\sin[1.984(t-\tau)] & 5 < t < 20 \end{cases}$$



Response to an Arbitrary Input

The input force is viewed as a series of impulses. The response at time t due to an impulse at time τ is:



$$x(t) = F(\tau)\Delta\tau g(t-\tau)$$

The total response at time t is the sum of all responses:

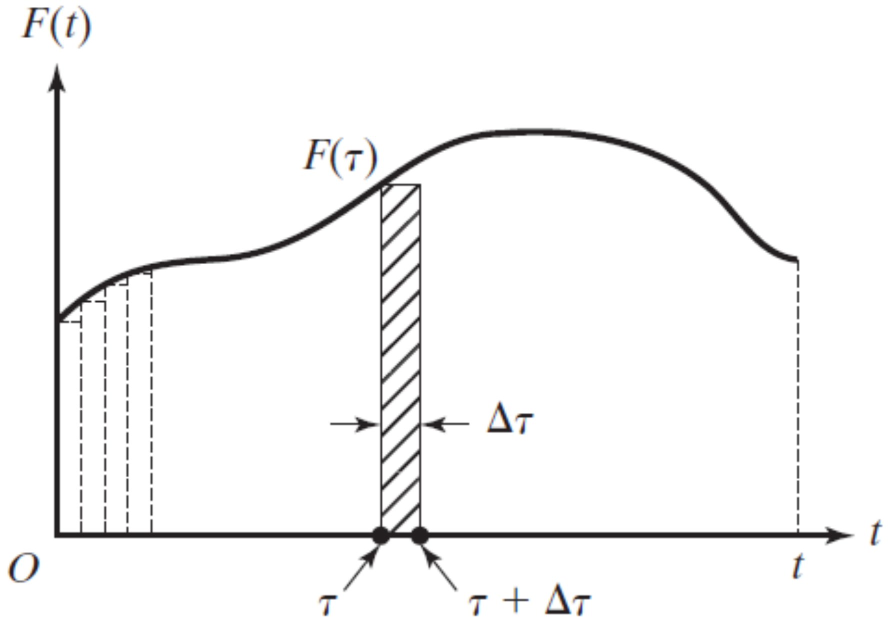
Response to General Forcing Condition

We consider the response of the system under an arbitrary external force $F(t)$. This force may be assumed to be made up of a series of impulses of varying magnitude. Assuming that at time τ, the force $F(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $F(\tau)\Delta\tau$. At any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at t due to this impulse alone with $\hat{\mathrm{F}} = F(\tau)\Delta\tau$:

$$\Delta x(t) = F(\tau)\Delta\tau g(t - \tau)$$

The total response at time t can be found by summing all the responses due to the elementary impulses acting at all times τ :

$$x(t) \simeq \sum F(\tau)g(t - \tau)\Delta\tau$$



$$x(t) = \sum F(\tau)g(t-\tau)\Delta\tau$$

Hence

$$x(t) = \int_0^t F(\tau)g(t-\tau)d\tau$$

Duhamel Integral formula.

For an underdamped system:

$$\boxed{x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} sin\omega_d(t-\tau)\, d\tau}$$

Note: this does not consider initial conditions. This type of formula is called the *convolution integral* or the *Duhamel integral*. For base excitation, the resulting response is

$$z(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)e^{-\zeta\omega_n(t-\tau)}\sin(\omega_d(t-\tau))d\tau$$

$$\boxed{m\ddot{x} + c\dot{x} + kx} = \boxed{F(t)}$$

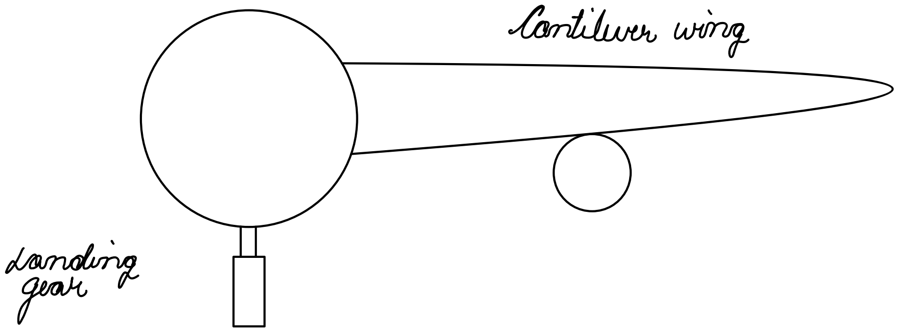
$x(t) =$ Complementary func^n $+$ Particular solution

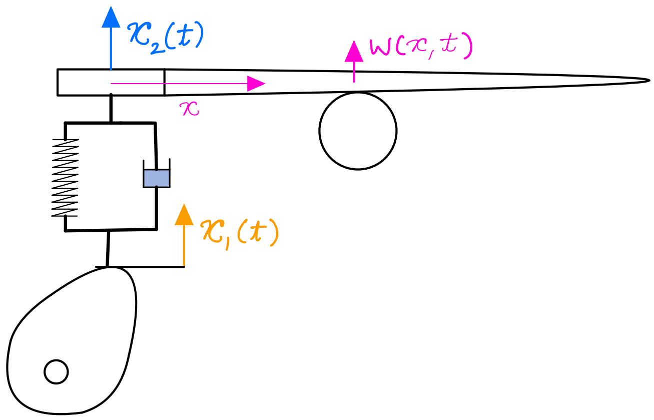
Limits of Duhamel Integral

$F(t)$ is integrable

Final exam tougher than previous Mid Sem quizzes

Example of Final term Question

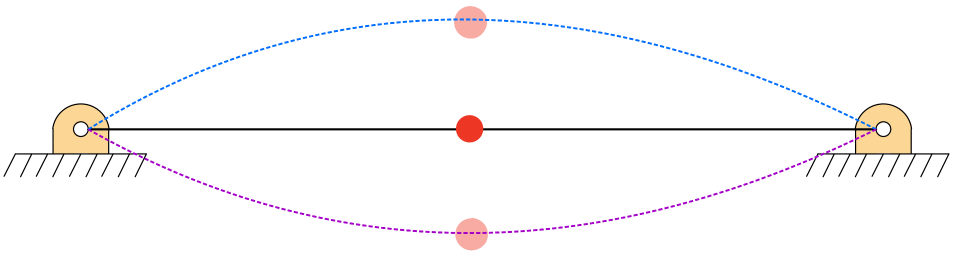




2 DOF

1DOF → 2DOF Big jump conceptually

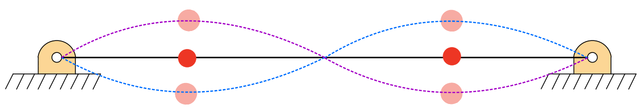
2DOF → NDOF Bookkeeping

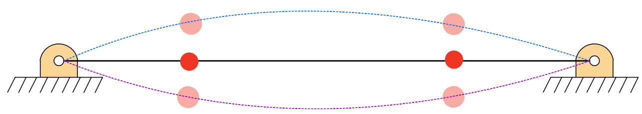


1 DoF

↓

2 DoF



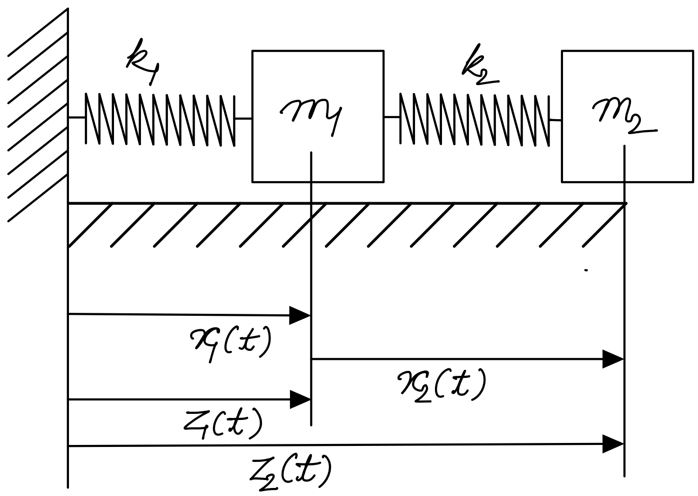


Synchronous shapes or arrangements

Mode shapes

Coordinate Transformation

Example :-



$(x_1(t), x_2(t))$
$(z_1(t), z_2(t))$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 = \frac{1}{2} m_1 \dot{z}_1^2 + \frac{1}{2} m_2 \dot{z}_2^2$$

$$V = \frac{1}{2} k_1 x_1 + \frac{1}{2} k_2 x_2 = \frac{1}{2} k_1 z_1^2 + \frac{1}{2} k_2 (z_2 - z_1)^2$$

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$\begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Coordinate
Transformation
Matrix

Idea :- There exist q_1 q_2 such that

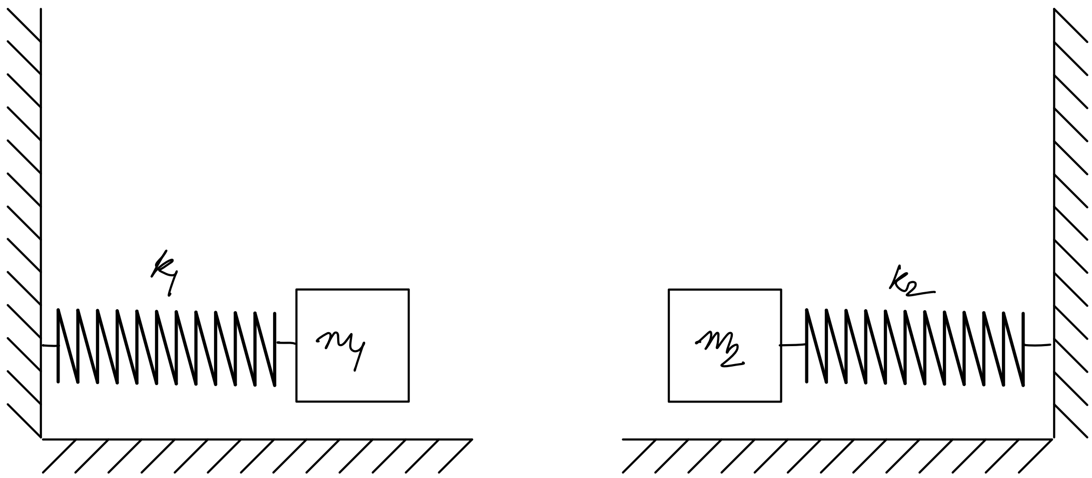
$$\begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Principle coordinates

$$[M]\{\ddot{X}\} + [K]\{X\} = \{0\}$$

Subjected to $\begin{bmatrix} X(0) = X_0 \\ \dot{X}(0) = \dot{X}_0 \end{bmatrix}$

$$\{X(t)\} = X_0 \, e^{st}, \quad s \rightarrow s_1, s_2$$



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 \qquad \omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k x_2^2 \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

Response

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\omega_{n_1} t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\omega_{n_2} t}$$

Mode shapes / Natural freq.
Damping

Free vibration on 2 DOF

Syllabus end

Special solutions which are synchronous.

$$\ddot{f}(t)\; u^T M u + f(t)\; u^T K u = 0$$

$$\lambda = \frac{u^T K u}{u^T M u}$$

Equation of $f(t)$

$$\ddot{f}(t) + \lambda f(t) = 0$$

For vibratory solution $\lambda > 0$

$$\lambda = \omega^2$$

$$\ddot{f} = -\omega^2 f$$

$$M \ddot{f} u + k f u = 0$$

$$f(-\omega^2 M u + k u) = 0$$

For non trivial solution of $f(t)$

$$\omega^2 M u = k u$$

$$(M^{-1} K)\, u = \omega^2 u$$

M is +ve definate
K is +ve definate

u is eigenvectors of $M^{-1}K$

ω^2 are eigenvalues of $M^{-1}K$

2DoF $\quad u_1 \quad u_2$

$\omega_1 \quad \omega_2$

NDoF system $\quad u_1 \quad u_2 \quad \cdots \quad u_N$

$\omega_1 \quad \omega_2 \quad \cdots \quad \omega_N$

$$x(t) = A\, u_1 e^{i(\omega_1 t - \phi_1)} + B\, u_2 e^{i(\omega_2 t - \phi_2)}$$

A, B are real.

u_1, u_2 are called the mode shapes

ω_1, ω_2 are called the natural frequency

We can prove

$$u_1^T M u_2 = 0 \qquad u_1^T K u_2 = 0$$

$$u_2^T M u_1 = 0 \qquad u_2^T K u_1 = 0$$

$$M\ddot{x} + Kx = 0$$

$$I\ddot{x} + M^{-1}Kx = 0$$

$$x = [u_1 \quad u_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = u_1 q_1(t) + u_2 q_2(t)$$

$$x = [U]\{q(t)\}$$

$$[U]\{\ddot{q}\} + [M^{-1}K][U]\{q\} = 0$$

$$\underbrace{[U]^T[U]}_{I}\{\ddot{q}\} + [U]^T[M^{-1}K][U]\{q\} = 0$$

$$\{\ddot{q}\} + \left[\ \ddots \ \Lambda \ \ddots\ \right]\{q\} = 0$$

1/2 hour

Practice Session

2 DoF System Undamped Unforced.

$$u_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$$

$$u_2 = \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix}$$

T V

Equation of Motion

Responses

Damping in NDoF systems

$$M\ddot{X} + KX = 0$$

$M^{-1}K$ ✓

$$M\ddot{X} + C\dot{X} + KX = 0$$

Import Ques : Does the C matrix allow diagolisation ?

If it allows, choose C to enable diagonisation

Proportional Damping

Rayleigh Damping $C = aM + bK$ choose $a, b \in R$ to fit damping

$$M\ddot{x} + (aM + bK)\dot{x} + Kx = 0$$

$$\zeta_j = \frac{1}{2}\left(\frac{a}{\omega_j} + b\,\omega_j\right)$$

$$(KM^{-1}C) = (KM^{-1}C)^T$$

then C is proportional

Exams point of view

Generalised Rayleigh damping

$$C = M^{1/2}\left[\sum_{i=0}^{n-1} a_{ij}\left[M^{-1/2}KM^{1/2}\right]^{i/j}\right]M^{1/2}$$

Caughy Damping