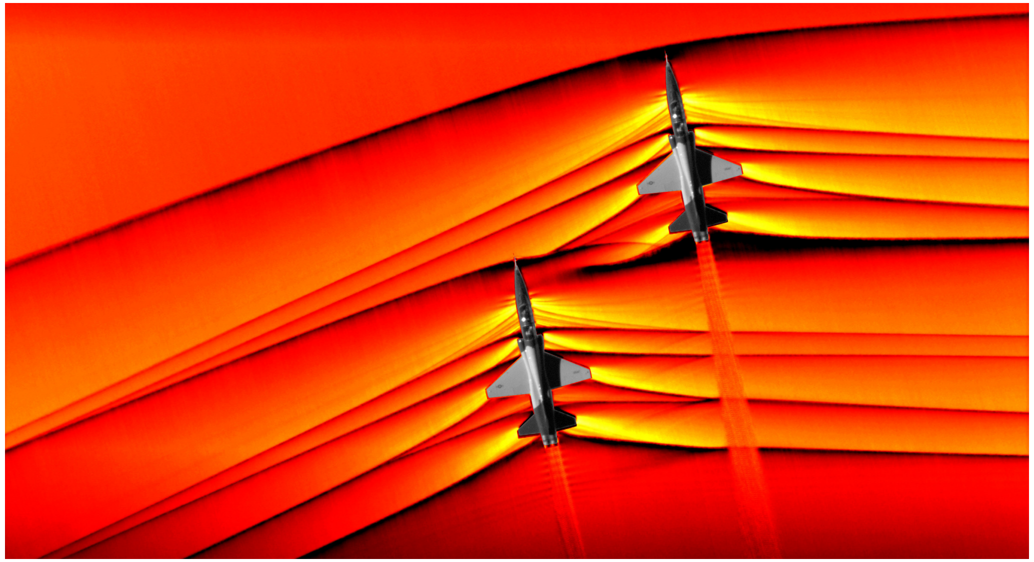
Compressible Flows
A High-Speed Tour

Aniruddha Sinha



When is a flow 'compressible'?

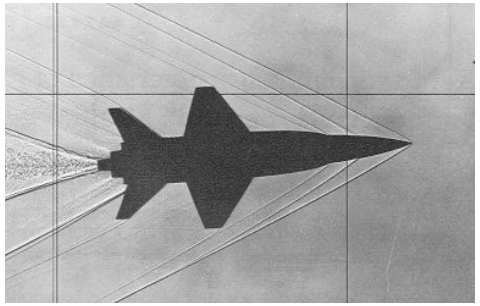
Flows in which there are significant density variations due to pressure variations (and not just due to temperature variations)

- Latter stipulation rules out liquid flows w/ convection effects, e.g.

Pressure variations are significant in high-speed flows

- Anticipated by Bernoulli's eqn. (applies to incompressible flows)
- Thus, compressibility effects become important for high-speed flows

Flow changes at one point (presence of an obstacle, say) are 'communicated' to other points by (sound) waves



- Thus, Mach no. is a natural similarity parameter

Typically, compressibility is important when freestream Mach no. > 0.3

Anderson, 2011

Thermodynamics

Because density varies in compressible flows, so does temperature

Thus thermodynamics is relevant (we use absolute temperature T)

(Specific) internal energy (energy of random molecular motion): e

(Specific) enthalpy: $h := e + p/\rho$

We consider two levels of perfection of gases:

1. (Thermally) perfect: e and h are functions of T only
   - Then, $de = C_v dT$, $dh = C_p dT$; C_v & C_p are functions of T only
2. Calorically perfect gas: specific heats (C_v and C_p) are constants
   - Then, $e = C_v T$, $h = C_p T$

Ideal gas: eqn. of state is $p = \rho R T$; R is constant for the gas

Entropy

Thermodynamic processes of particular interest:

1. Adiabatic (No heat addition or removal from system/fluid)
2. Reversible (No dissipative phenomena); e.g. inviscid, shock-free

Entropy: $ds := \delta q_{rev}/T = \delta q/T + ds_{irrev}$; $ds_{irrev} \geq 0$ (2nd law)

Isentropic process = Adiabatic + Reversible

For calorically-perfect gas: $s_2 - s_1 = C_p \ln(T_2/T_1) - R \ln(p_2/p_1)$

Integral equations relevant to aerodynamics

Specialize to inviscid flows w/o body force, heat addition, mech. work

Mass conservation: $$\frac{\partial}{\partial t}\int \rho dV + \int \rho \underline{u} \cdot d\underline{S} = 0$$

Momentum conservation (Newton's 2[nd] law) [N.B.: a vector eqn.]:

$$\frac{\partial}{\partial t}\int \rho \underline{u} d\rlap{-}V + \int (\rho \underline{u} \cdot d\underline{S})\underline{u} = -\int p d\underline{S}$$

Energy conservation (1[st] law of thermodynamics) [N.B. $V := |\underline{u}|$]:

$$\frac{\partial}{\partial t}\int \rho(e + V^2/2) d\rlap{-}V + \int \rho(e + V^2/2)\underline{u} \cdot d\underline{S} = \int \rho \dot{q} d\rlap{-}V - \int p\underline{u} \cdot d\underline{S}$$

Differential eqns of steady, inviscid, adiabatic flows

Mass cons.: $\nabla \cdot (\rho \underline{u}) = 0$

- 3D Cartesian coords.: $\partial(\rho u)/\partial x + \partial(\rho v)/\partial y + \partial(\rho w)/\partial z = 0$

Momentum cons.: $\underline{u} \cdot \nabla \underline{u} = -\nabla p/\rho$

- 2D Cartesian coords.: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$; $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$
  - Form for ***irrotational*** flows: $dp/\rho + VdV = 0$

Energy cons. (adiabatic flow of ideal gas w/o mech. work): $C_p dT = -VdV$

In *perfect* gas, C_p (& C_v) are const.; also ideal gas law is valid ($p = \rho RT$)

1st + 2nd law: ***isentropic*** => $p/\rho^{\gamma} = \text{Const.}; \quad p^{\gamma-1}/T^{\gamma} = \text{Const.}$

Total or stagnation condition

A useful result in comp. flows obtains w/ following thought experiment:

Suppose that flow at a point could be brought to rest *isentropically*

- N.B. the actual flow need not be isentropic

Resulting thermodynamic condition is called total/stagnation condition

- Denoted by ‘0’ subscript (e.g., T_0, p_0, ρ_0)

Energy eqn. in adiabatic flow w/o viscous & shaft work: $C_p dT = -V dV$

- Integrating for perfect gas (N.B. $V_0 = 0$), we have $T_0 = T + V^2/2C_p$
- N.B. for T_0 definition, (mental) adiabatic, inviscid process is sufficient

Adiabatic flow $T_0 = \text{const.}$. Isentropic flow $p_0 = \text{const.}$, $\rho_0 = \text{const.}$

Sound, Speed of Sound and Mach Number

Compressible Flows

Eqns. for Steady, Adiabatic, Inviscid 1-D Jump

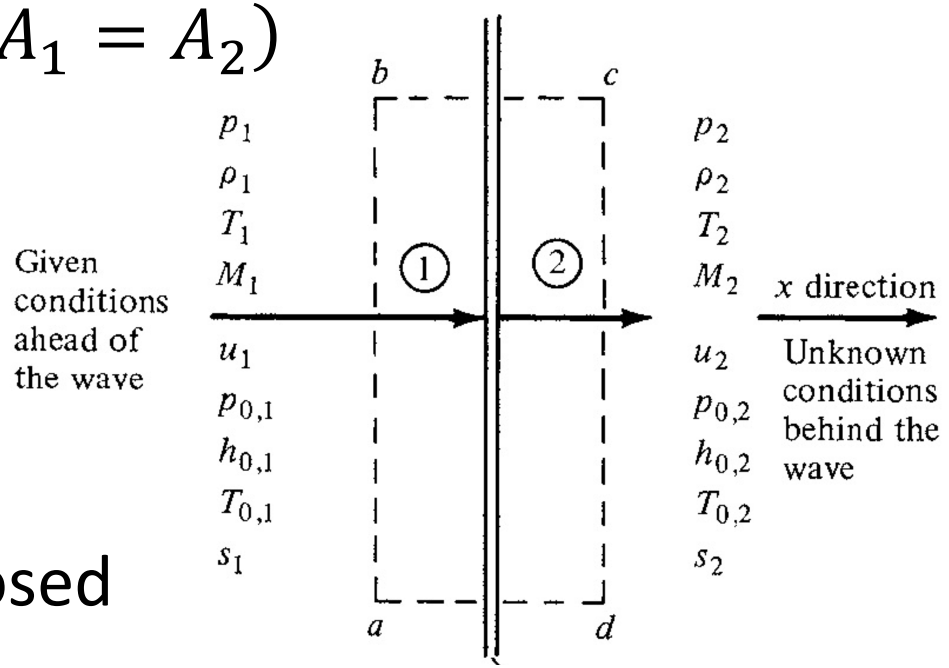
Consider 2 useful cases of discontinuity (jump) of properties in 1-D flow

- Isentropic (infinitesimal) change across jump – Sound wavefront
- Non-isentropic (finite) change across jump – Shock wavefront (later)

Cross-sectional area of CV same across jump $(A_1 = A_2)$

- Mass conservation: $\rho_1 u_1 = \rho_2 u_2$
- Mom. cons.: $\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$
- Energy cons.: $C_p T_1 + u_1^2/2 = C_p T_2 + u_2^2/2$
- Ideal gas law: $p = \rho R T$

4 unknowns (u_2, ρ_2, p_2, T_2), 4 eqns.; so well posed



Sound wave propagation (infinitesimal jump)

2nd order changes neglected due to smallness of change across jump

Mass cons.: $\rho u = (\rho + \delta\rho)(u + \delta u)$

- Linearization gives: $\delta u \approx -u\, \delta\rho/\rho$

Mom. cons.: $\rho u^2 + p = (\rho + \delta\rho)(u + \delta u)^2 + (p + \delta p)$

- Linearization, along with mass cons. gives: $u = \sqrt{\delta p/\delta\rho} \approx \sqrt{dp/d\rho}$

Flow is isentropic, since jump is infinitesimal; this means $p/\rho^\gamma = \text{Const}$

Together, these give the (almost) familiar result: $u = \sqrt{\gamma p/\rho} = \sqrt{\gamma RT}$

In steady flow, u is simply the velocity of upstream medium *relative* to infinitesimal discontinuity (called sound)

- In particular, if medium is quiescent, “speed of sound” is $a = \sqrt{\gamma RT}$

Compressible regime, $M \gtrsim 0.3$

To establish approximate demarcation, we use adiabatic energy eqn.:

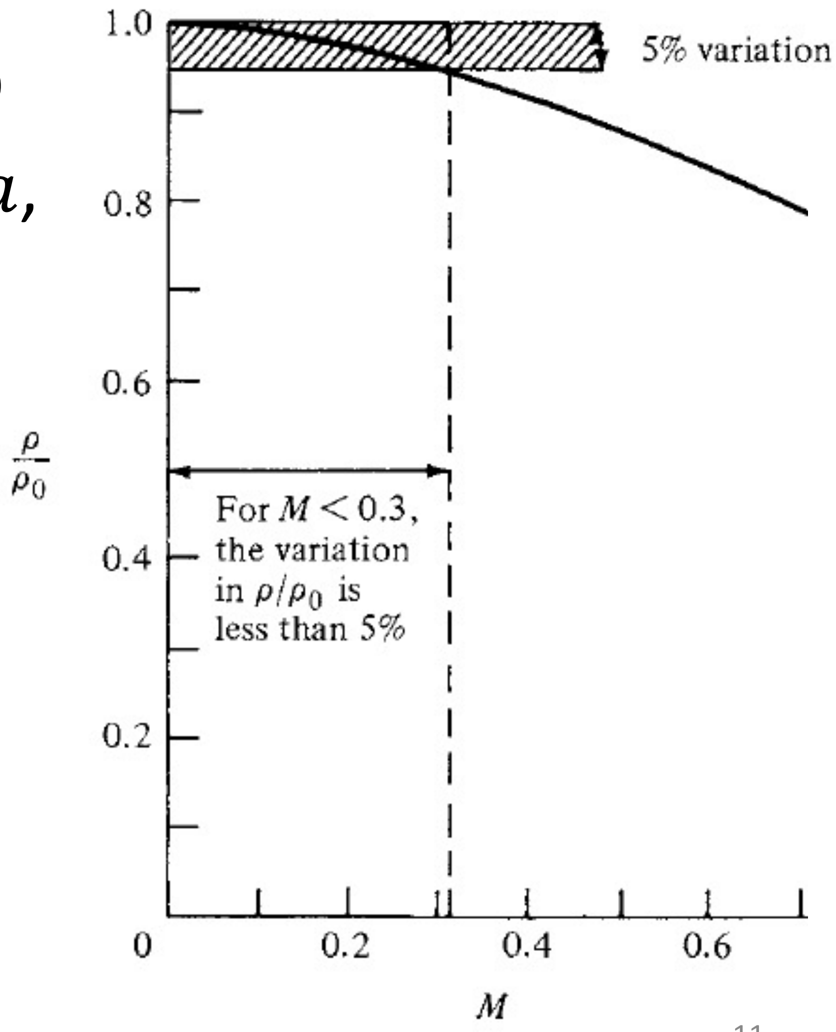
$$d\left(C_p T + V^2/2\right) = 0 \Rightarrow C_p T + V^2/2 = C_p T_0$$

Since $C_p = \gamma R/(\gamma - 1)$, $a = \sqrt{\gamma R T}$, $M := V/a$,

$$\Rightarrow T/T_0 = \left(1 + 0.5(\gamma - 1)M^2\right)^{-1}$$

But, (thought) process of reaching stagnation condition from prevailing (static) condition is isentropic; so $T/\rho^{\gamma-1} = \text{const.}$

$$\Rightarrow \frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{-\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma-1}}$$



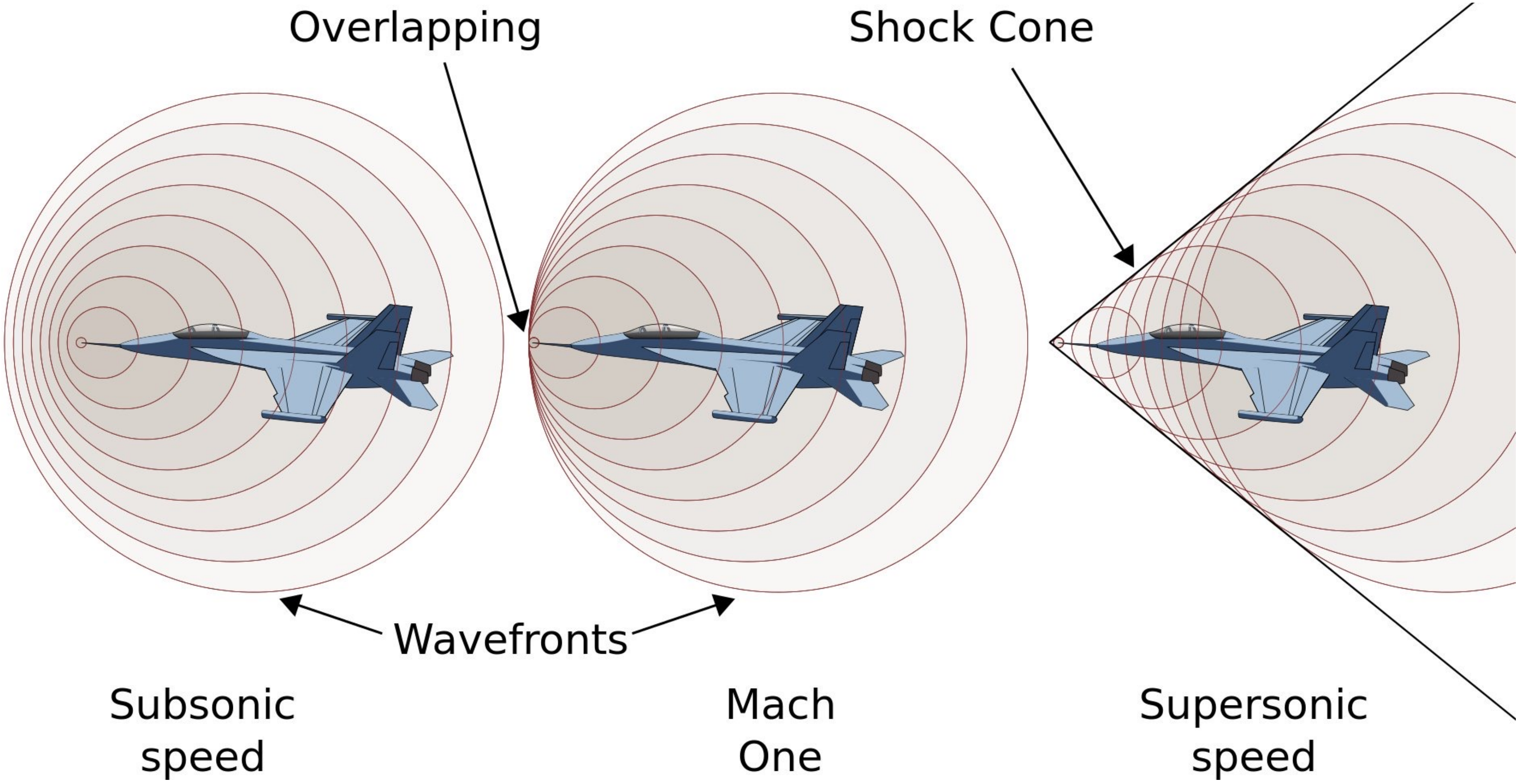
For $M \gtrsim 0.3$, $\rho/\rho_0 \lesssim 0.95$ ("non-negligible")

Anderson, 2011

Normal Shock

Compressible Flows

Information propagation at various flight speeds



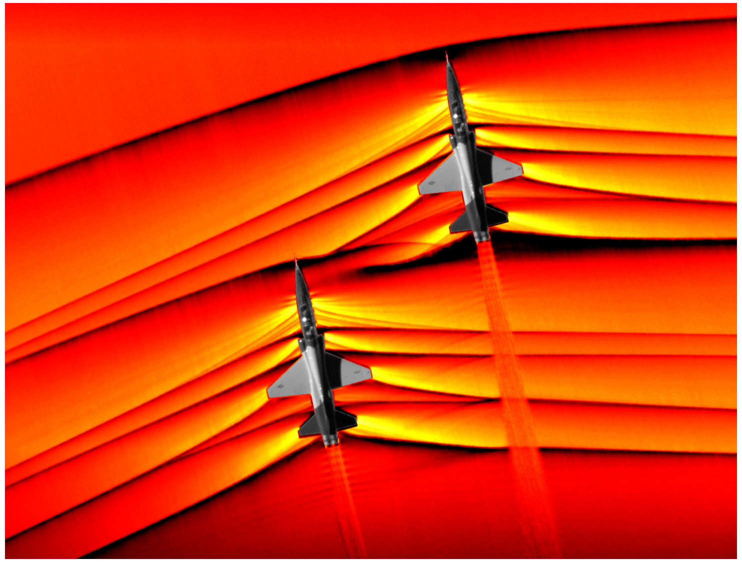
Shock wave

Changes in the flow are communicated at speed of sound

- Speed of propagation of small perturbations (sound) in the medium

$$a = \sqrt{dp/d\rho} = \sqrt{\gamma RT} = \sqrt{\gamma\, p/\rho} \quad [\because \text{sound propagates isentropically}]$$

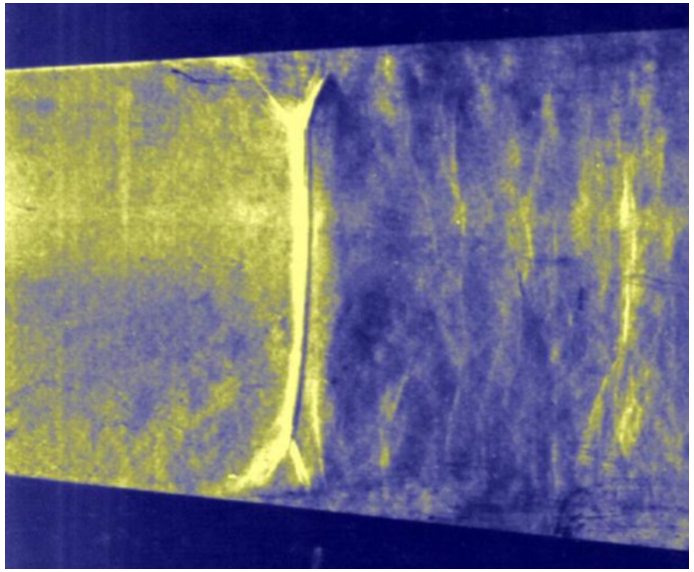
If flow/body is moving at speed $> a$ (i.e. supersonic), then a shock wave is created by nature to account for this ‘information’ change



- Extremely thin region where flow properties vary discontinuously
- Necessarily irreversible (non-isentropic) due to ‘sudden’ jump

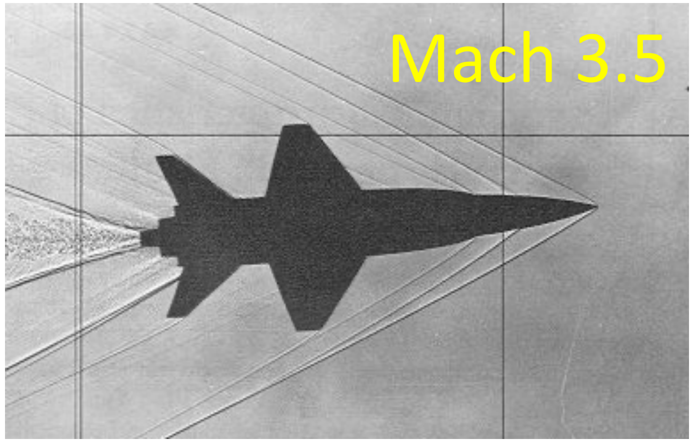
https://edition.cnn.com/videos/us/2019/03/07/nasa-shockwaves-sound-barrier-photos-newsource-orig.cnn

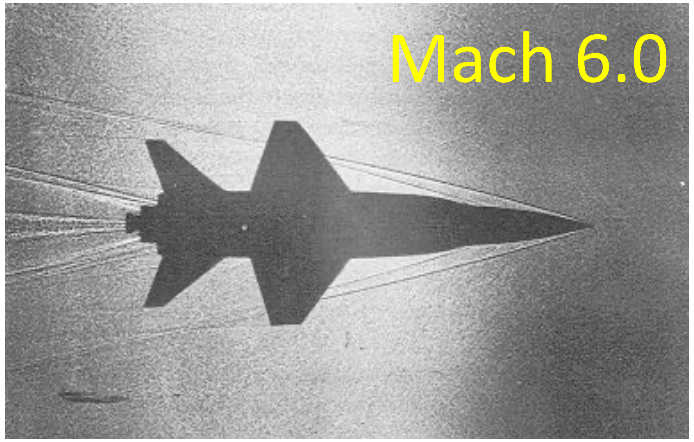
Normal and oblique shock waves



Normal shock in divergent part of Laval nozzle (G. Settles)

Essentially 1D/2D flow





Oblique shocks in free-flight models of the X-15 being fired into a wind tunnel

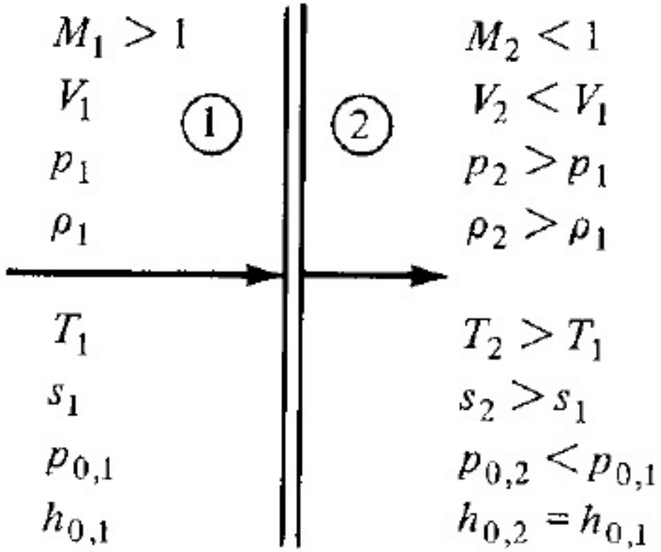
Essentially 2D/3D flow

Anderson, 2011

Normal shock: Rankine-Hugoniot 'jump' conditions

Earlier, we formulated governing eqns. for finite discontinuity (shock)

- Mass conservation: $\rho_1 u_1 = \rho_2 u_2$
- Mom. cons.: $\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$
- Energy cons.: $C_p T_1 + u_1^2/2 = C_p T_2 + u_2^2/2$
- Ideal gas law: $p = \rho R T$



4 unknowns (u_2, ρ_2, p_2, T_2), 4 eqns.; so well posed

Considerable algebra yields expressions for each of the variables downstream of the shock in terms of only the upstream Mach no. M_1

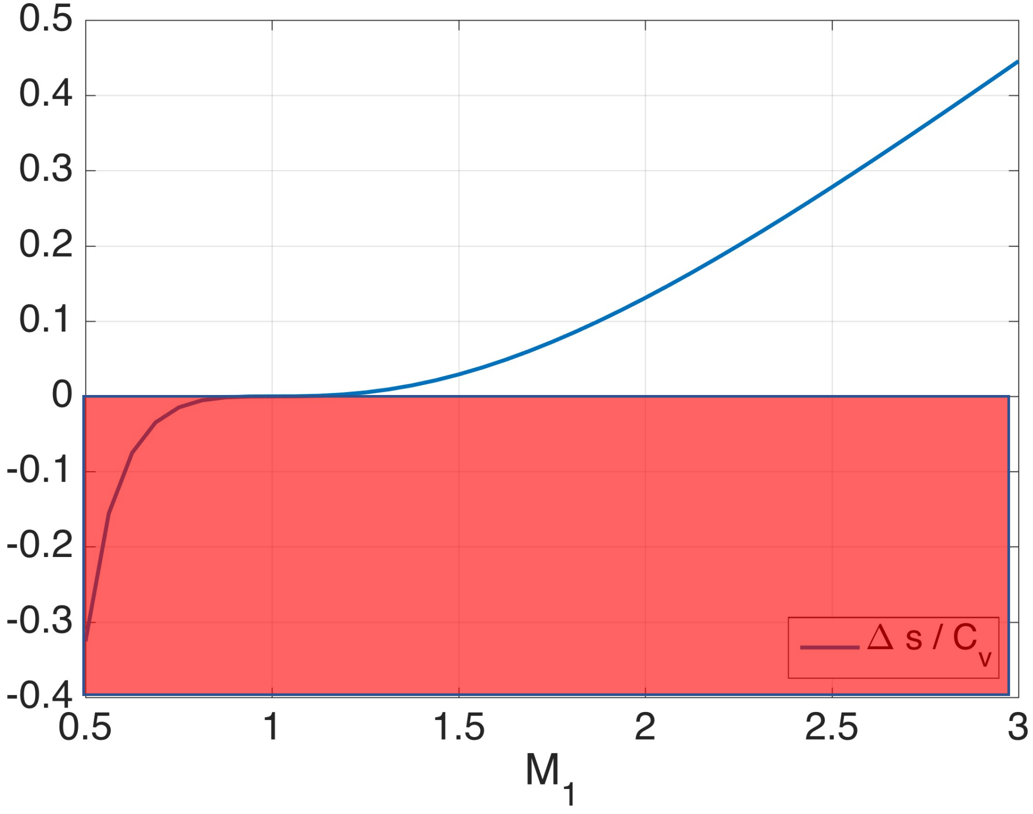
Since shocks are adiabatic, stagnation enthalpy h_0 is conserved

- For perfect gases (C_p = const.); so T_0 does not change across a shock

Normal shock: Condition for existence of shock

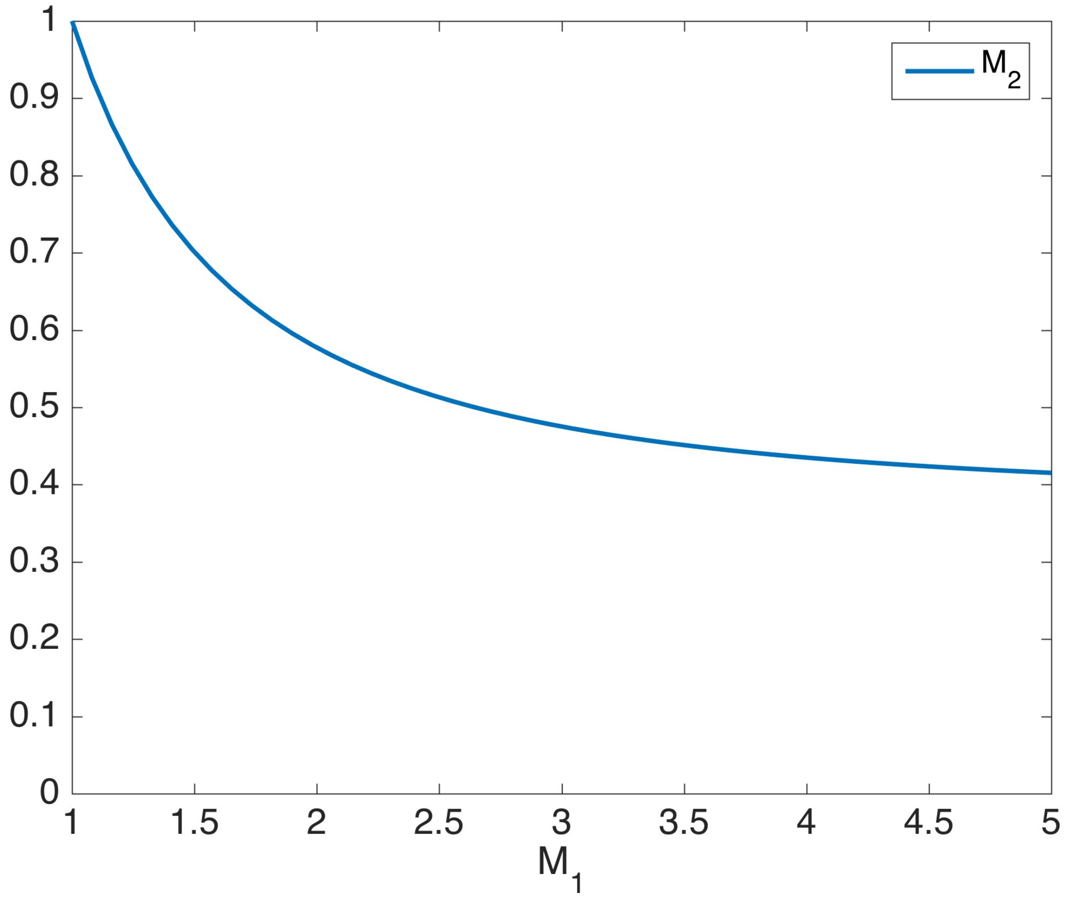
One of the outcomes of R-H conditions is the change in entropy across a shock in terms of the upstream Mach no. M_1

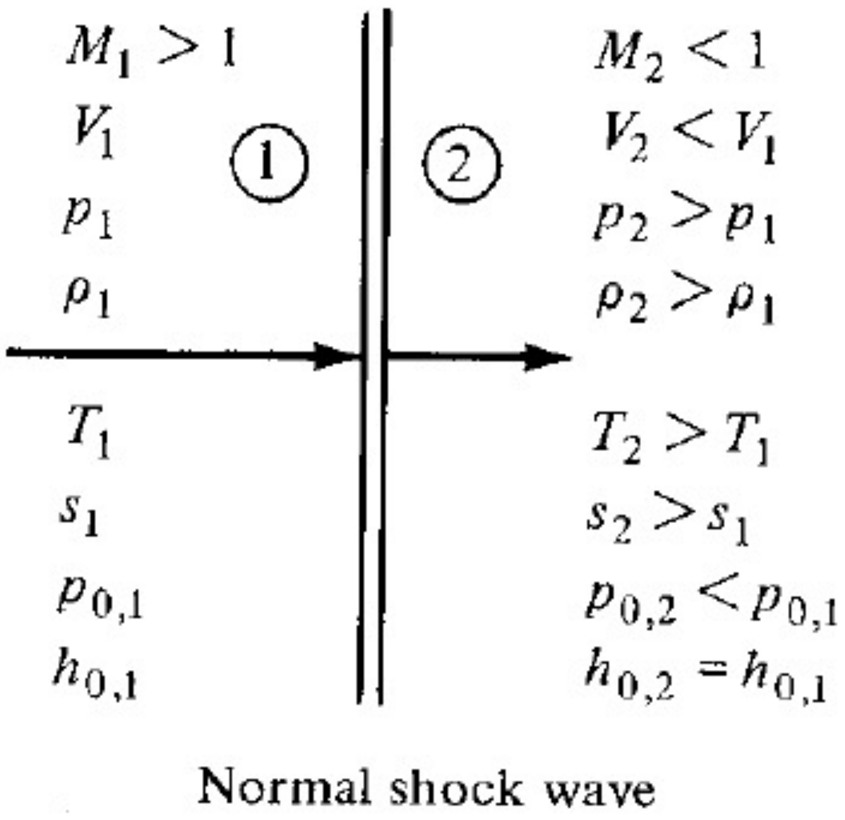
- $M_1 = 1$: essentially isentropic process, called a Mach wave
  - No different than sound wave, with infinitesimal perturbation
- Subsonic part is unphysical w/ net decrease of entropy – forbidden by 2nd law! Never show it!



So, shock (finite discontinuity) can happen only if $M_1 \geq 1$ (not subsonic)

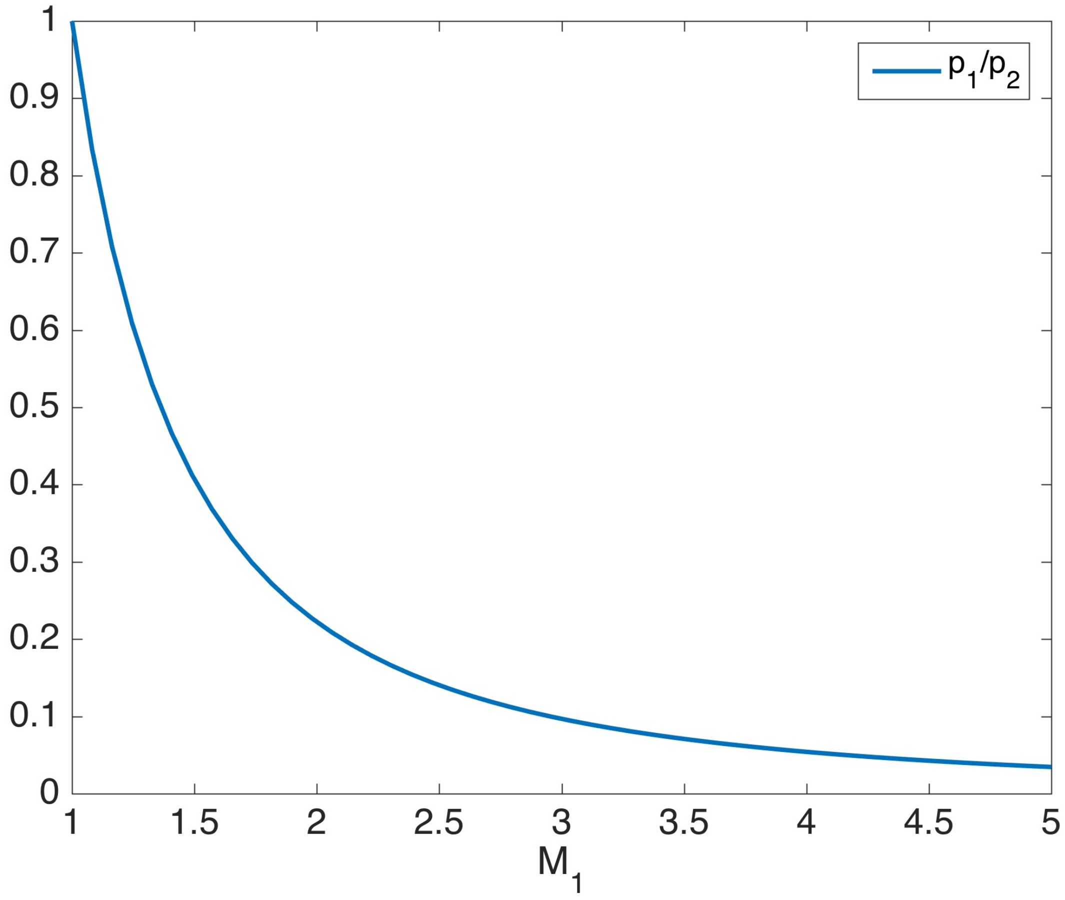
Flow variables across a normal shock: M_2

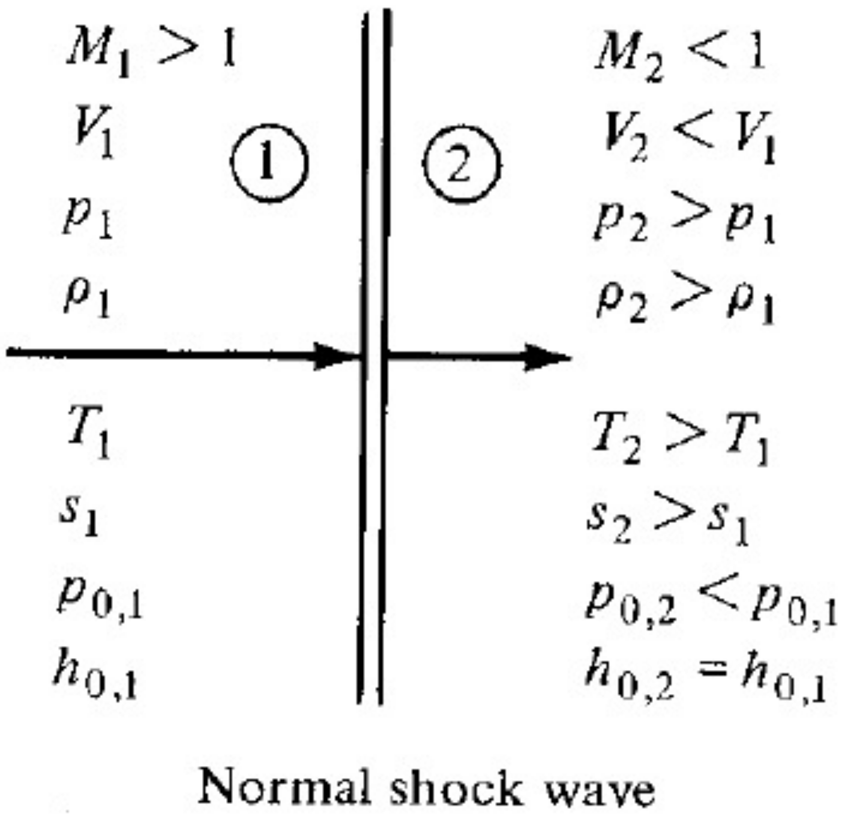




Downstream Mach no. is subsonic; more so for stronger shock (greater M_1)

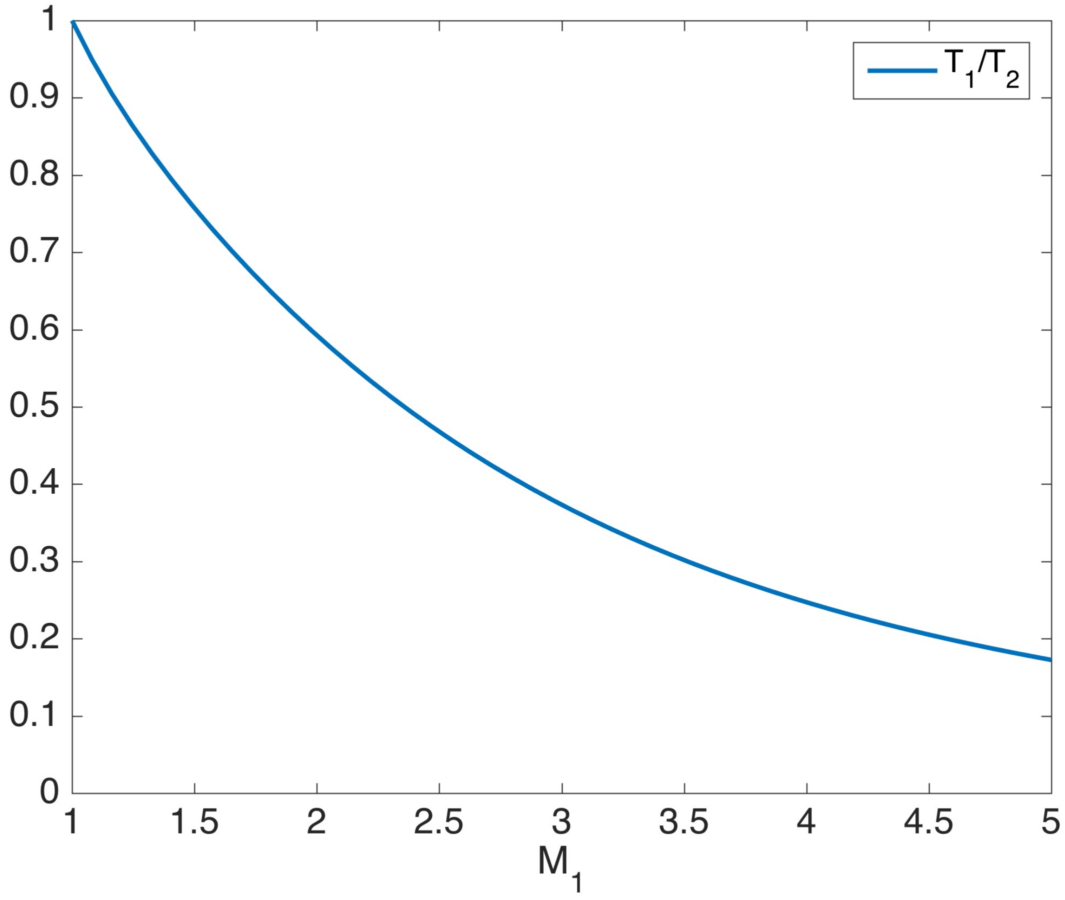
Flow variables across normal shock: p_1/p_2 (not p_2/p_1!)

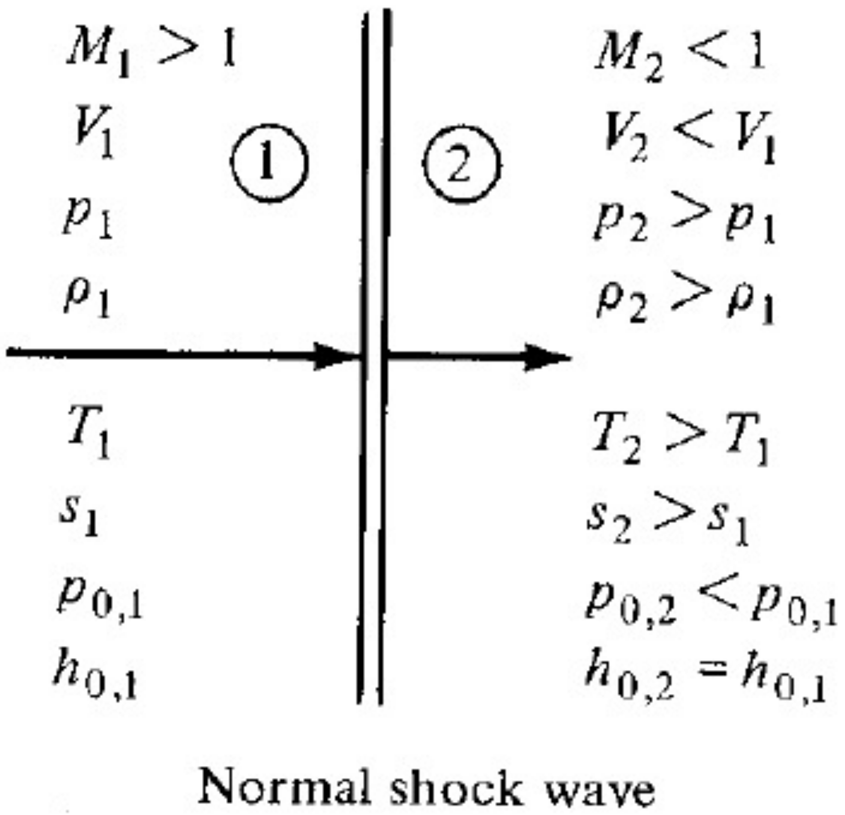




Static pressure increases thru a shock; more so for stronger shock

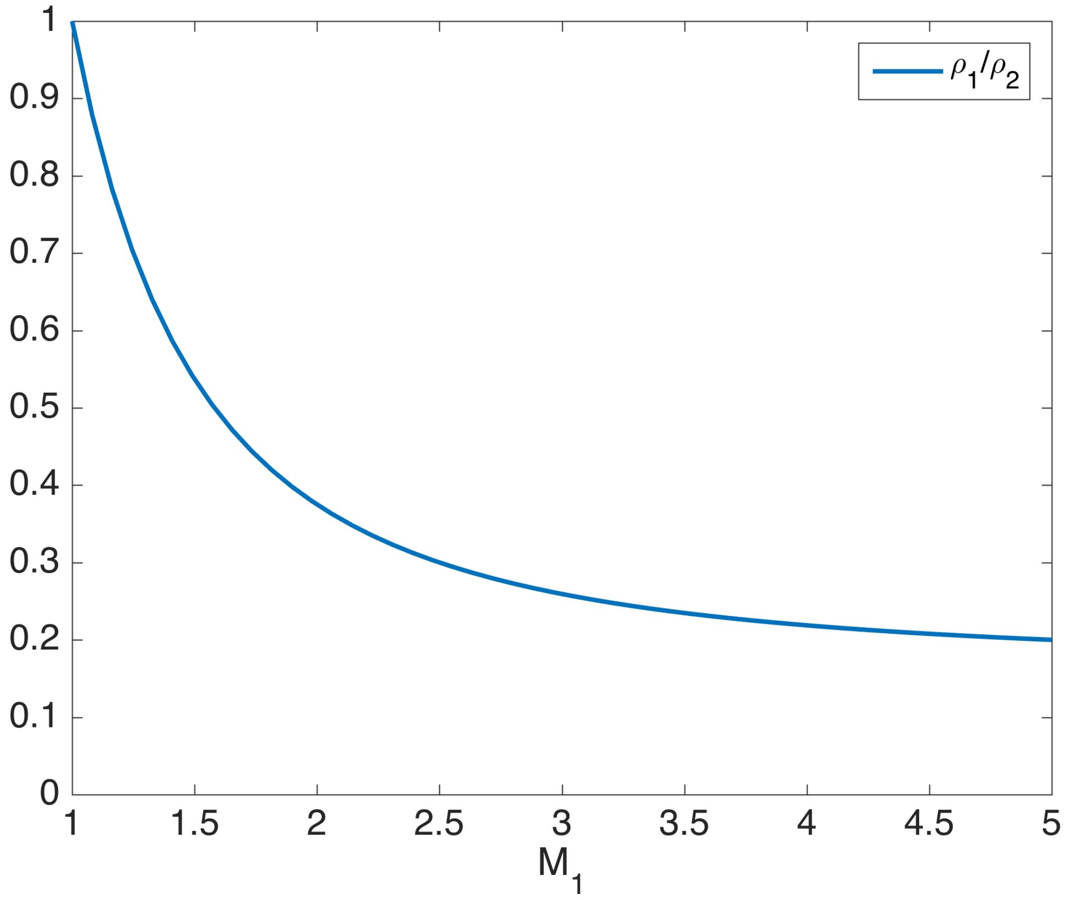
Flow variables across normal shock: T_1/T_2 (not T_2/T_1!)

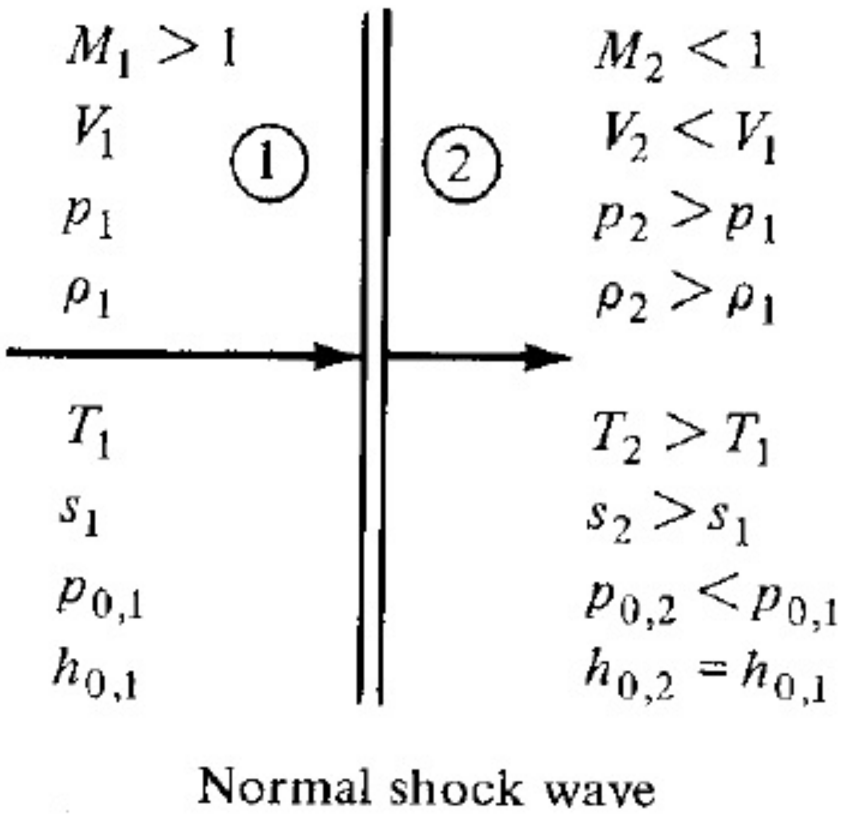




Static temperature increases thru a shock; more so for stronger shock

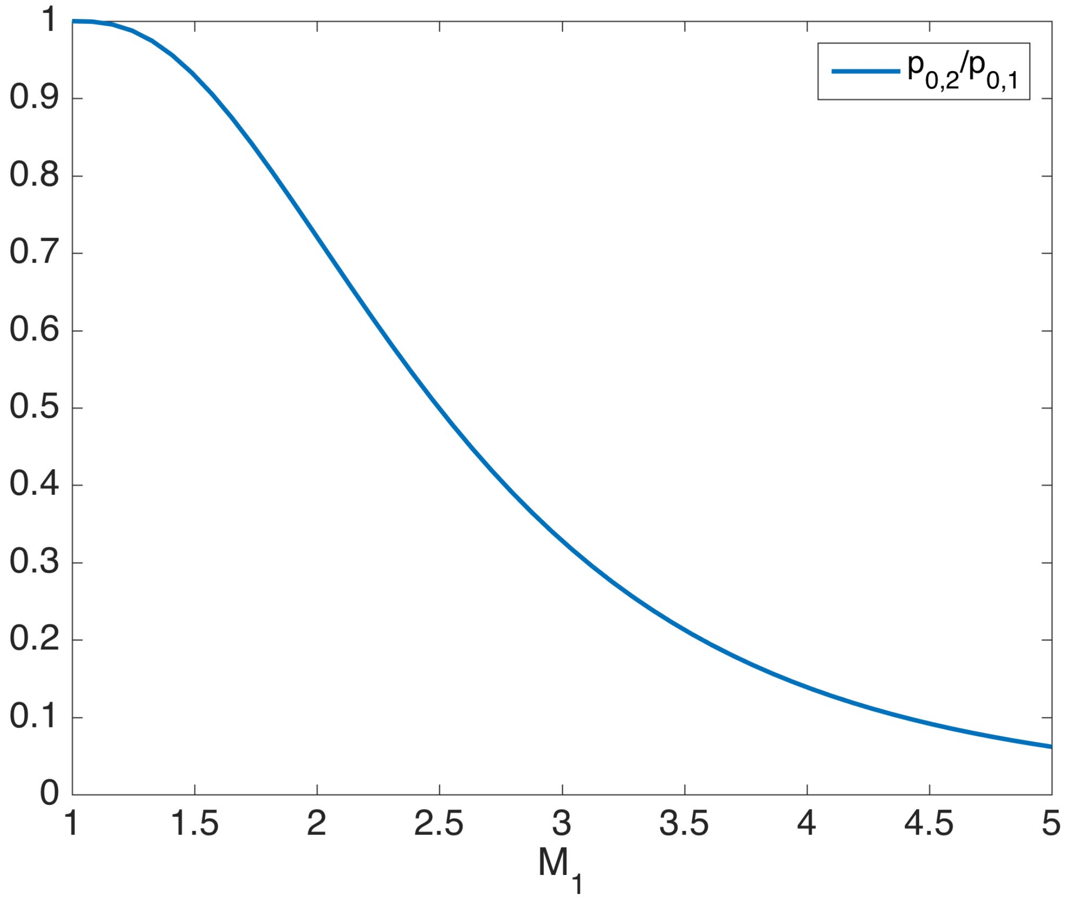
Flow variables across normal shock: ρ_1/ρ_2 (not ρ_2/ρ_1!)

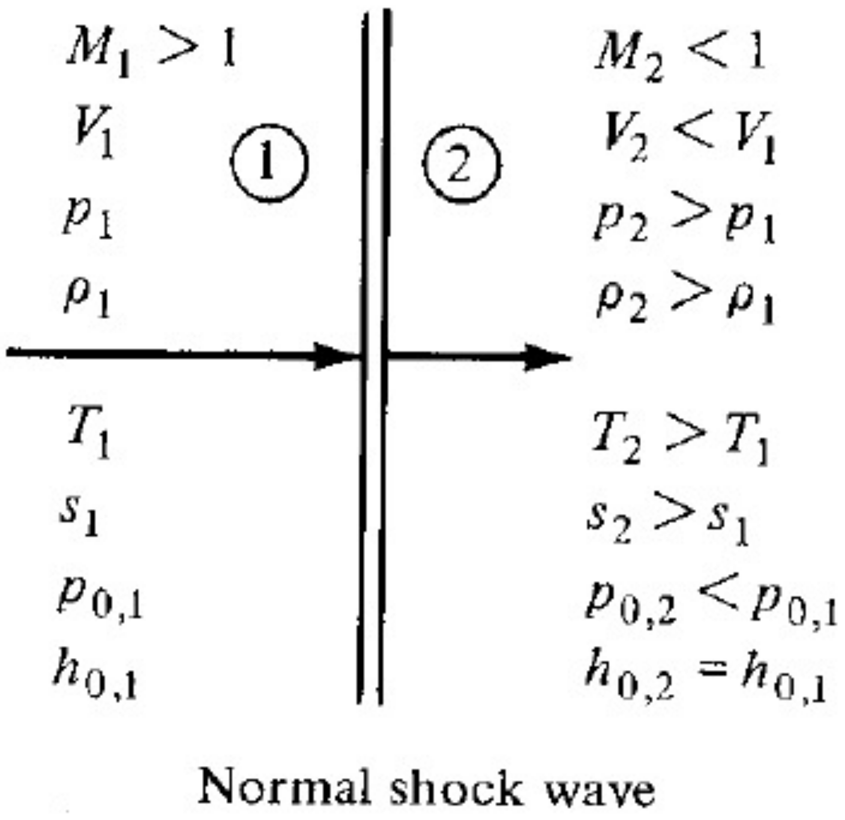




Static density increases thru a shock; more so for stronger shock

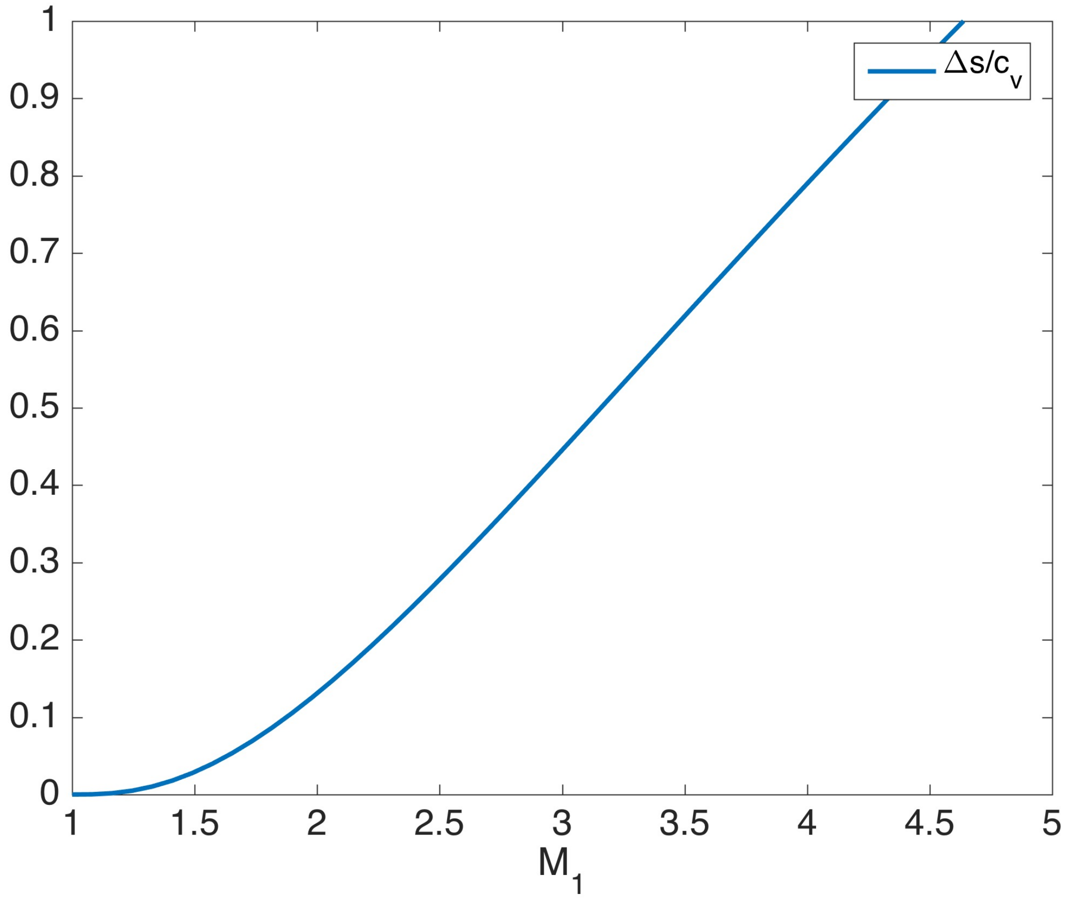
Flow variables across normal shock: $p_{0,2}/p_{0,1}$

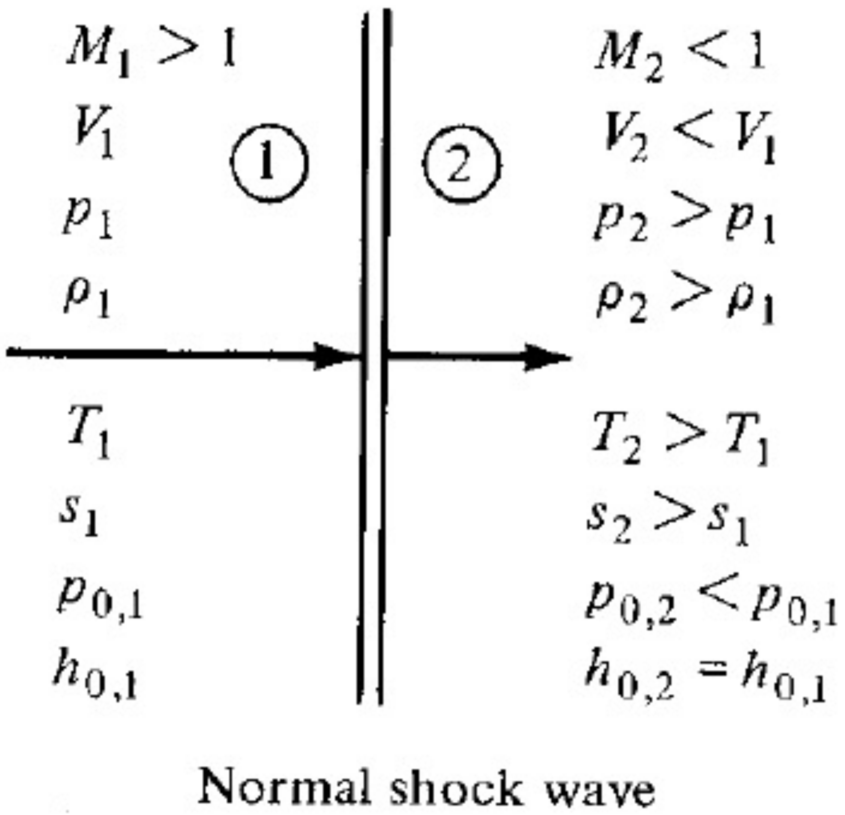




Stagnation pressure decreases thru a shock; more so for stronger shock

Flow variables across normal shock: $(s_2 - s_1)/c_v$





Entropy increases thru a shock; more so for stronger shock

Solving normal shock problems w/ tables

M	$\frac{p_2}{p_1}$	$\frac{\rho_2}{\rho_1}$	$\frac{T_2}{T_1}$	$\frac{p_{0_2}}{p_{0_1}}$	$\frac{p_{0_2}}{p_1}$	M_2
0.1000 + 01	0.1000 + 01	0.1000 + 01	0.1000 + 01	0.1000 + 01	0.1893 + 01	0.1000 + 01
0.1020 + 01	0.1047 + 01	0.1033 + 01	0.1013 + 01	0.1000 + 01	0.1938 + 01	0.9805 + 00
0.1040 + 01	0.1095 + 01	0.1067 + 01	0.1026 + 01	0.9999 + 00	0.1984 + 01	0.9620 + 00
0.1060 + 01	0.1144 + 01	0.1101 + 01	0.1039 + 01	0.9998 + 00	0.2032 + 01	0.9444 + 00
0.1080 + 01	0.1194 + 01	0.1135 + 01	0.1052 + 01	0.9994 + 01	0.2082 + 01	0.9277 + 00
0.1100 + 01	0.1245 + 01	0.1169 + 01	0.1065 + 01	0.9989 + 00	0.2133 + 01	0.9118 + 00
0.1120 + 01	0.1297 + 01	0.1203 + 01	0.1078 + 01	0.9982 + 00	0.2185 + 01	0.8966 + 00
0.1140 + 01	0.1350 + 01	0.1238 + 01	0.1090 + 01	0.9973 + 00	0.2239 + 01	0.8820 + 00
0.1160 + 01	0.1403 + 01	0.1272 + 01	0.1103 + 01	0.9961 + 00	0.2294 + 01	0.8682 + 00
0.1180 + 01	0.1458 + 01	0.1307 + 01	0.1115 + 01	0.9946 + 00	0.2350 + 01	0.8549 + 00
0.1200 + 01	0.1513 + 01	0.1342 + 01	0.1128 + 01	0.9928 + 00	0.2408 + 01	0.8422 + 00
0.1220 + 01	0.1570 + 01	0.1376 + 01	0.1141 + 01	0.9907 + 00	0.2466 + 01	0.8300 + 00

Given upstream Mach no., read off ratio of variables across shock.

Given ratio of variables, read off upstream/downstream Mach no.

Anderson, 2011 (Appendix B/C)

Oblique Shocks

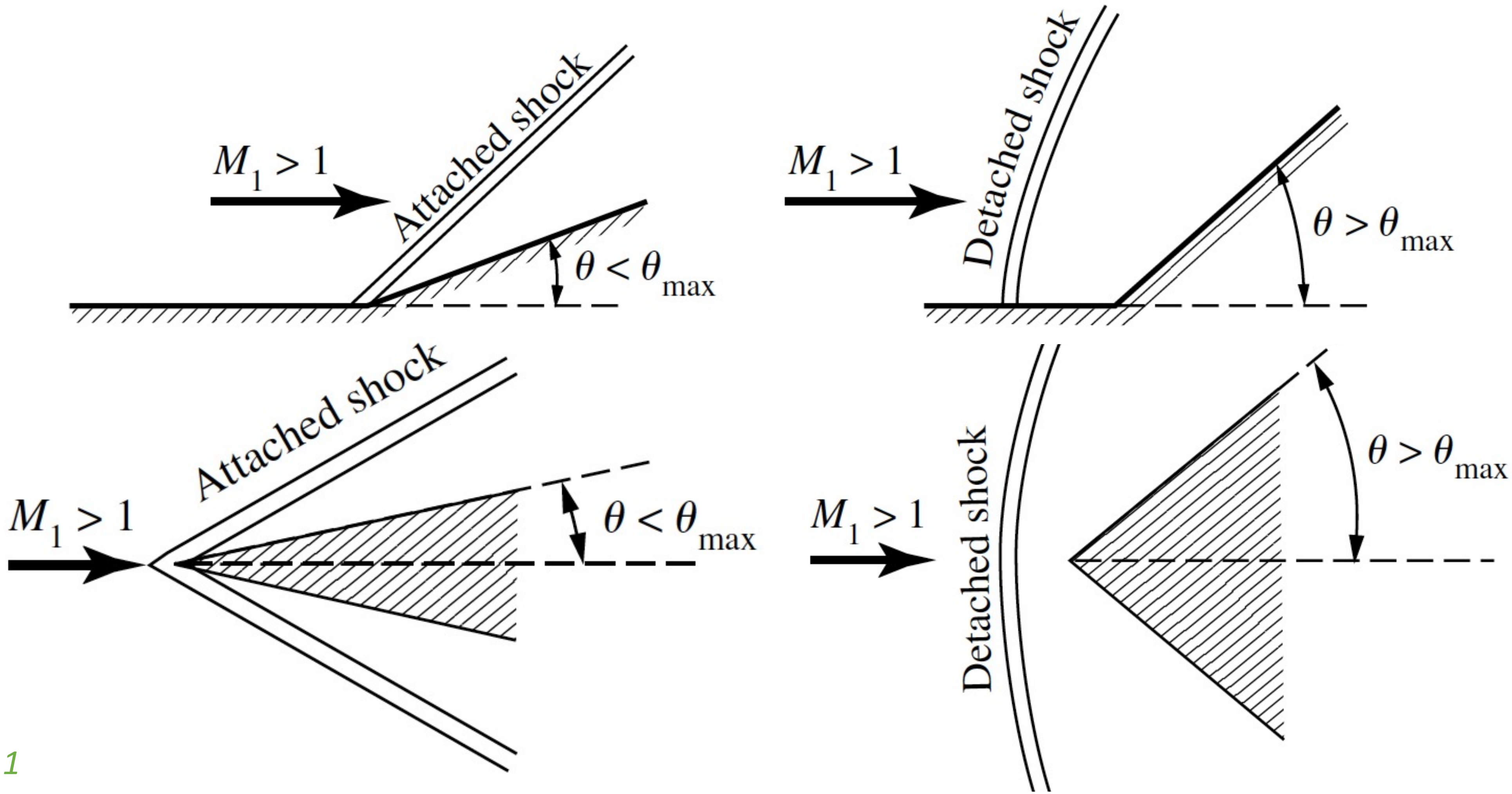
Compressible Flows

Oblique shocks: Attached vs. detached shocks

Oblique shocks are caused by supersonic flow being turned in on itself

Wedge

Cone



Anderson, 2011

Oblique shock waves

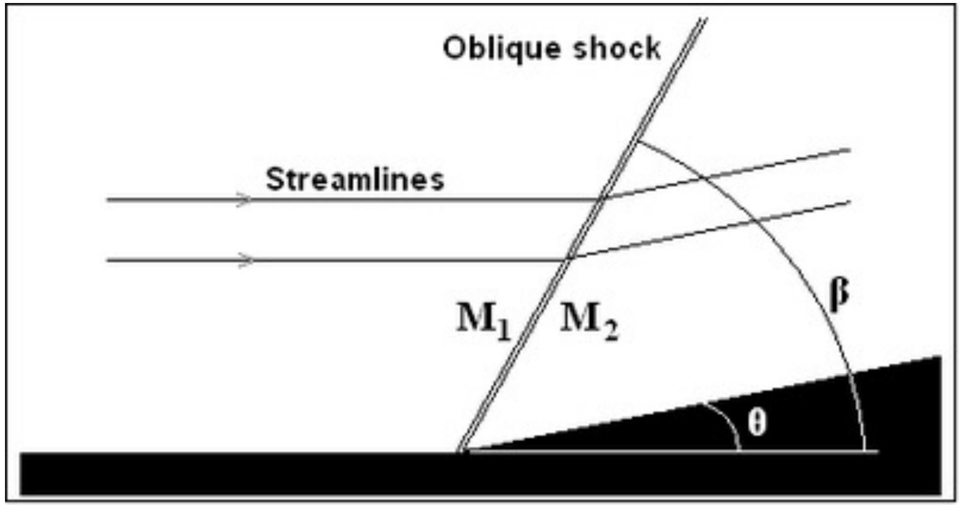
Tangential component of velocity (not of Mach no.!) remains same across shock

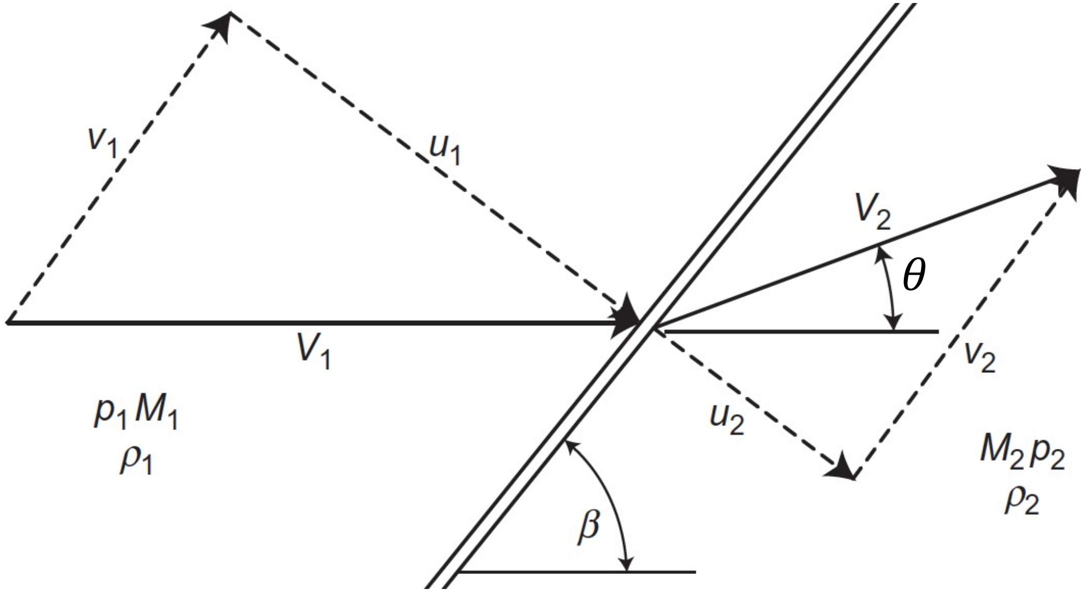
Normal component of Mach no. changes, as if it were across a normal shock

Flow is ‘refracted’ by angle θ given by

$$\theta = \tan^{-1}\left[2\cot\beta\left(\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma + \cos 2\beta) + 2}\right)\right]$$

Most often, wedge angle (θ) is known along with M_1, and β needs to be found





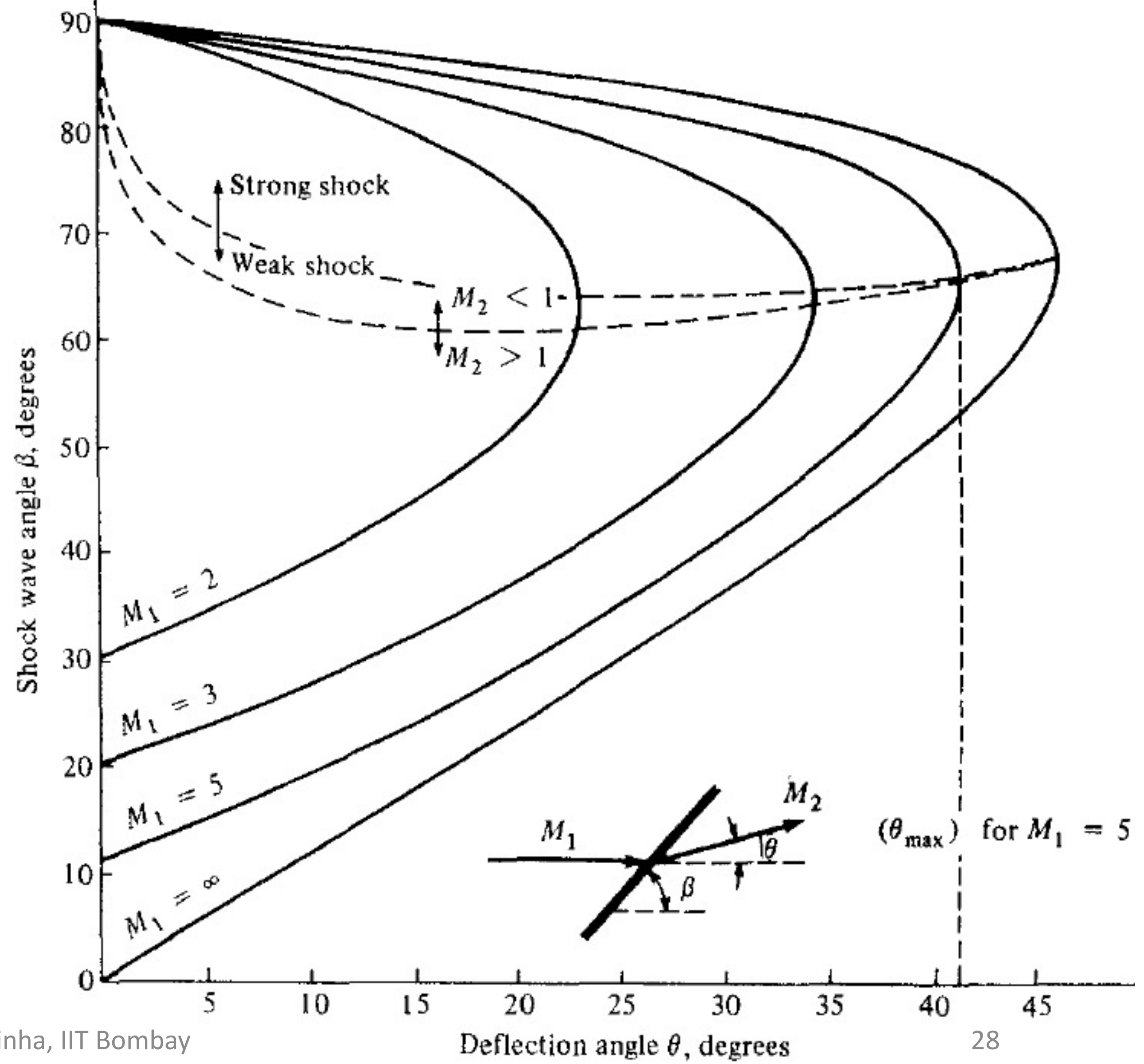
Houghton et al., 2013

Oblique shock wave: $\theta - \beta - M$ curves

For every M_1, there is a $\theta_{\max}$; detached bow shock if $\theta > \theta_{\max}$

For each $\theta \leq \theta_{\max}$, there are 2 solutions of β

- Smaller β solution -> higher M_2 (>1, usually), thus called *weak*
  - Usually favoured by nature
- Larger β solution -> lower M_2 (<1, usually), thus called *strong*
  - Occurs if backpressure is somehow increased



Anderson, 2011

Solving oblique shock problems

Typically, θ and M_1 are given, and $M_2, p_2/p_1, T_2/T_1$, etc. are asked for

1) First check from $\theta - \beta - M$ curves that $\theta \le \theta_{\text{max}}$ for this M_1
   - If not, then a detached shock exists; the solution is complicated
2) For $\theta \le \theta_{\text{max}}$, find weak-solution of β from the $\theta - \beta - M$ curves
3) Find normal component of upstream Mach no., $M_{n1} = M_1 \sin\beta$
4) For this, find normal component of downstream Mach no. M_{n2} from the *normal* shock tables
   - Also find desired ratios $p_2/p_1, T_2/T_1, p_{0,2}/p_{0,1}$, etc.
5) Find downstream Mach no. as $M_2 = M_{n2}/\sin(\beta - \theta)$

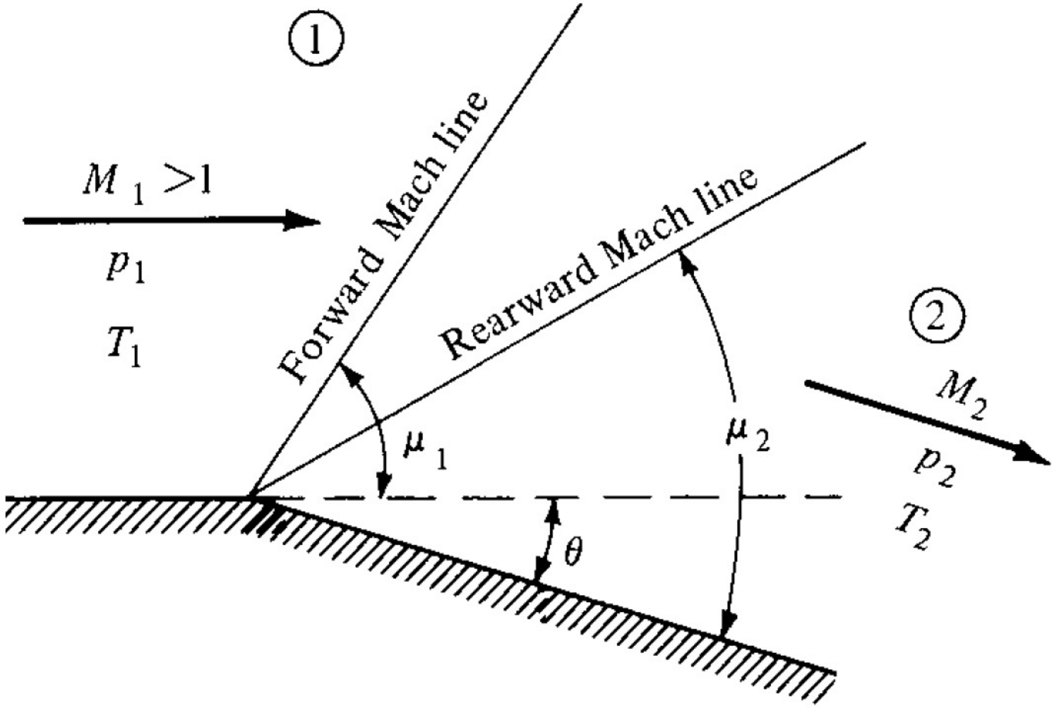
Expansion Fans

Compressible Flows

Prandtl-Meyer expansion fan

Expansion fans are caused by supersonic flow being turned away from itself

- This results in continuous (hence reversible & isentropic) change of flow properties
- Mach line is an infinitesimally weak shock wave, with sonic flow normal to it both upstream and downstream



Anderson, 2011

Prandtl-Meyer expansion fan

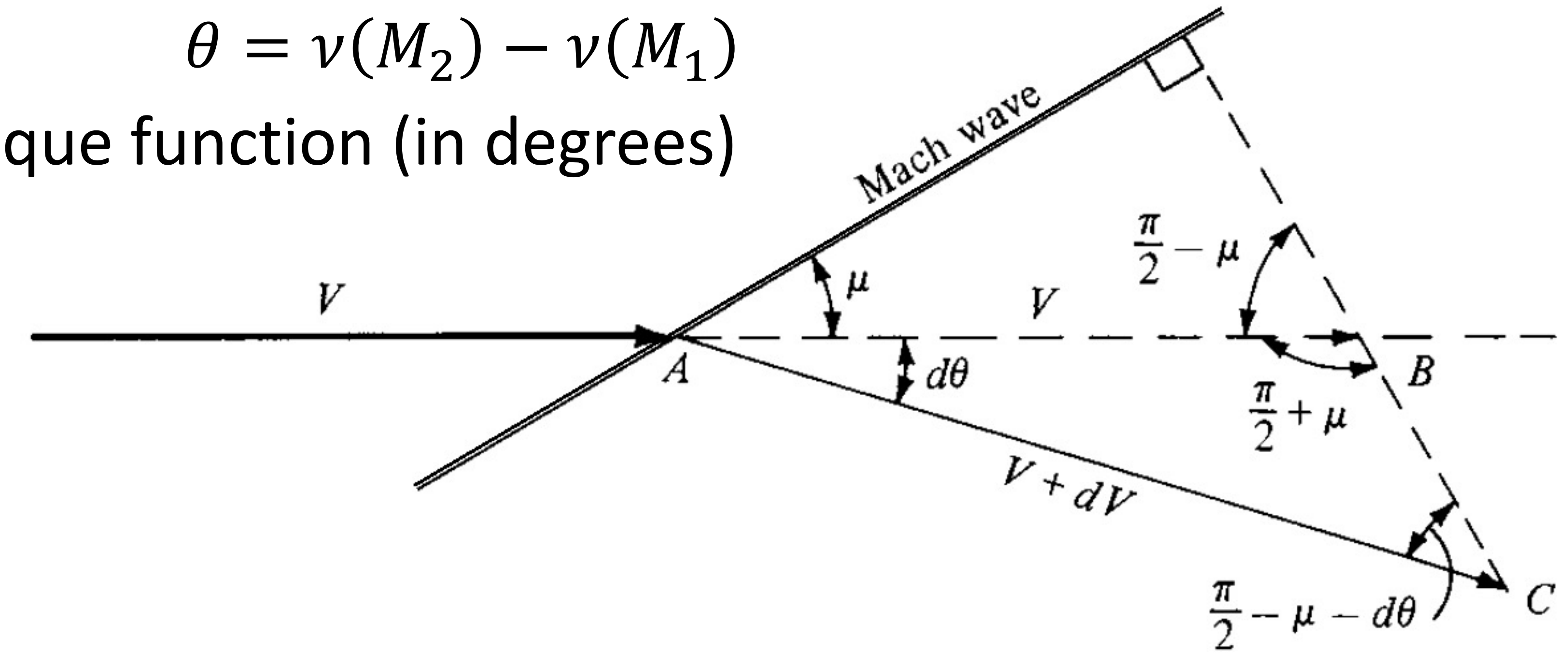
Infinitesimal change in velocity (dV) due to an infinitesimal deflection ($d\theta$) across a wave of vanishing strength (Mach wave) is

$$d\theta = \sqrt{M^2 - 1}\,dV/V$$

Expressing dV in terms of dM and integrating, Prandtl & Meyer found

$$\theta = \nu(M_2) - \nu(M_1)$$

where $\nu(M)$ is a unique function (in degrees)



Anderson, 2011

Solving expansion fan problems

Typically, θ and M_1 are given, and M_2, p_2/p_1, T_2/T_1, etc. are asked for

1) For given M_1 obtain $\nu(M_1)$ from Prandtl-Meyer tables (as below*)
2) Find $\nu(M_2)$ from $\theta = \nu(M_2) - \nu(M_1)$
3) Obtain M_2 for above $\nu(M_2)$ from Prandtl-Meyer tables again
4) Expansion fans being isentropic, $p_{0,2} = p_{0,1}$, $T_{0,2} = T_{0,1}$; thus find

$$\frac{T_2}{T_1} = \frac{T_{0,1}/T_1}{T_{0,2}/T_2} = \frac{1 + 0.5(\gamma - 1)M_1^2}{1 + 0.5(\gamma - 1)M_2^2}$$

$$\frac{p_2}{p_1} = \frac{p_{0,1}/p_1}{p_{0,2}/p_2} = \left(\frac{1 + 0.5(\gamma - 1)M_1^2}{1 + 0.5(\gamma - 1)M_2^2}\right)^{\frac{\gamma}{\gamma-1}}$$

M	ν	μ
0.1000 + 01	0.0000	0.9000 + 02
0.1020 + 01	0.1257 + 00	0.7864 + 02
0.1040 + 01	0.3510 + 00	0.7406 + 02
0.1060 + 01	0.6367 + 00	0.7063 + 02
0.1080 + 01	0.9680 + 00	0.6781 + 02

Anderson, 2011 (Appendix C/D)

End of Topic

Compressible Flows – A High-Speed Tour