Potential Flow Theory

Aerodynamics

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Velocity Potential

Velocity Potential

Irrotational flows can have a scalar field associated

Denote velocity vector field by $\underline{V}(\underline{r}, t)$, where $\underline{r}$ is position vector, t is time

From vector calculus theorem, a curl-free vector field (like the irrotational velocity vector field) must itself be the gradient of a scalar field; i.e.,

$$\boldsymbol{\nabla} \times \underline{V} = 0 \qquad \Longrightarrow \exists \phi \text{ s.t. } \underline{V} = \boldsymbol{\nabla} \phi$$

So, irrotational flows come equipped with a **velocity potential** $\phi(\underline{r}, t)$

Specializing to Cartesian coordinate system:

$$\boxed{\text{Cartesian coordinates:} \quad u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}}$$

Substitute this expression in vorticity expression in Cartesian coordinates to verify that vorticity indeed vanishes identically for all choices of ϕ

$$\underline{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\underline{i}} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\underline{j}} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\underline{k}}.$$

Governing equation for ϕ in incompressible flow

If a flow is **incompressible** (i.e., $\nabla \cdot \underline{V} = 0$) in addition to being irrotational, then, substituting the expression for $\underline{V}$ $(= \nabla\phi)$ in this condition, we obtain:

$$\boxed{\text{Gov. eqn. for incompressible, irrotational flows:} \quad \Delta\phi = 0}$$

I.e., velocity potential in incompressible flow satisfies (3D) Laplace eqn.

Some analytical solutions exist for the Laplace eqn.

- Depends on complexity of boundary conditions: Dirichlet/Neumann

We have the surprising result that ϕ, and hence the entire velocity field, of a flow is purely determined by geometry and kinematics, w/o explicit reference to dynamics (i.e., momentum eqn.)!

What just happened?

How is momentum conservation equation apparently rendered irrelevant?

Vector calculus operators in cylindrical coordinates

Consider an arbitrary (twice differentiable scalar field f), and a vector field

$$\underline{A} = A_r\underline{\hat{r}} + A_\theta\underline{\hat{\theta}} + A_y\underline{\hat{y}}$$

Here, r, θ and y are respectively the (cylindrical) radial, azimuthal and axial coordinates; hatted quantities are the respective unit vectors

The below vector calculus operators are in cylindrical coordinates

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\theta}} + \frac{\partial f}{\partial y}\underline{\hat{y}}$$

$$\boldsymbol{\nabla}\cdot\underline{A} = \frac{1}{r}\frac{\partial\left(rA_r\right)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_y}{\partial y}$$

$$\boldsymbol{\nabla}\times\underline{A} = \left(\frac{1}{r}\frac{\partial A_y}{\partial\theta} - \frac{\partial A_\theta}{\partial y}\right)\underline{\hat{r}} + \left(\frac{\partial A_r}{\partial y} - \frac{\partial A_y}{\partial r}\right)\underline{\hat{\theta}} + \frac{1}{r}\left\{\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right\}\underline{\hat{y}}$$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial y^2}$$

In 2D problems, cylindrical coordinates become ‘polar’ coordinates

Velocity potential in cylindrical coordinates

Cylindrical coordinate system is often used in flows with predominantly cylindrical geometry (where Cartesian coordinates complicate the solution)

Components of velocity vector in cylindrical coordinates are denoted as

$$\underline{V} = u_r \underline{\hat{r}} + u_\theta \underline{\hat{\theta}} + u_y \underline{\hat{y}}$$

For irrotational flows equipped with velocity potential $\phi(r, \theta, y)$, we have

$$\boxed{\text{Cylindrical coordinates:} \quad u_r = \frac{\partial \phi}{\partial r}, \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad u_y = \frac{\partial \phi}{\partial y}}$$

Unsurprisingly, for incompressible flows, we recover the gov. eqn. $\Delta\phi = 0$

This is because the vector calculus operators are coordinate-independent

Vector calculus operators in spherical coordinates

Consider an arbitrary (twice differentiable scalar field f), and a vector field

$$\underline{A} = A_r \underline{\hat{r}} + A_\vartheta \underline{\hat{\varphi}} + A_\vartheta \underline{\hat{\vartheta}}$$

Here, r, φ and ϑ are respectively the (spherical) radial, zenithal and azimuthal coordinates; hatted quantities are the respective unit vectors

The below vector calculus operators are in spherical coordinates

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial f}{\partial \varphi}\underline{\hat{\varphi}} + \frac{1}{r \sin\varphi}\frac{\partial f}{\partial \vartheta}\underline{\hat{\vartheta}}$$

$$\boldsymbol{\nabla} \cdot A = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r \sin\varphi}\frac{\partial (A_\varphi \sin\varphi)}{\partial \varphi} + \frac{1}{r\sin\varphi}\frac{\partial A_\vartheta}{\partial \vartheta}$$

$$\boldsymbol{\nabla} \times A = \frac{1}{r\sin\varphi}\left\{\frac{\partial (A_\vartheta \sin\varphi)}{\partial \varphi} - \frac{\partial A_\varphi}{\partial \vartheta}\right\}\underline{\hat{r}} + \frac{1}{r}\left\{\frac{1}{\sin\varphi}\frac{\partial A_r}{\partial \vartheta} - \frac{\partial (r A_\vartheta)}{\partial r}\right\}\underline{\hat{\varphi}} + \frac{1}{r}\left\{\frac{\partial\left(r A_\varphi\right)}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right\}\underline{\hat{\vartheta}}$$

$$\Delta f = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial f}{\partial \varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 f}{\partial \vartheta^2}\right]$$

Velocity potential in spherical coordinates

Spherical coordinate system is often used in flows with predominantly spherical geometry

Components of velocity vector in spherical coordinates are denoted as

$$\underline{V} = u_r \underline{\hat{r}} + u_\varphi \underline{\hat{\varphi}} + u_\vartheta \underline{\hat{\vartheta}}$$

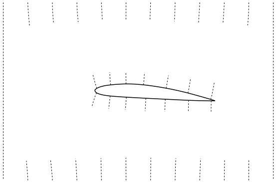
For irrotational flows equipped with velocity potential $\phi(r, \varphi, \vartheta)$, we have

$$\boxed{\text{Spherical coordinates:} \quad u_r = \frac{\partial \phi}{\partial r}, \quad u_\varphi = \frac{1}{r}\frac{\partial \phi}{\partial \varphi}, \quad u_\vartheta = \frac{1}{r \sin\varphi}\frac{\partial \phi}{\partial \vartheta}}$$

Velocity potential: Boundary conditions

Consider the uniform x-direction attached flow past a solid body

Recall: irrotationality (hence inviscidity) is necessary for existence of ϕ; hence 'no slip' can't be a condition on ϕ at any solid wall



- Because of 'no through-flow' constraint at solid wall, normal velocity (i.e., wall-normal gradient of ϕ) must vanish thereat (Neumann b.c.)

$$\text{At the body:} \quad V_n = \frac{\partial \phi}{\partial n} = 0, \quad \underline{\hat{n}} \text{ being the body normal}$$

- At infinity (i.e., far from the body), the x-component of velocity is the freestream velocity V_∞ (a Neumann b.c.) and z-component vanishes

$$\text{At infinity:} \quad \left\{ \begin{array}{l} u = \partial\phi/\partial x = V_\infty, \\ w = \partial\phi/\partial z = 0 \end{array} \right\} \quad \Longrightarrow \phi = V_\infty x + \text{constant}$$

Linearity of incompressible potential flow

In incompressible (& irrotational) flows, ϕ satisfies the Laplace equation, involving the linear Laplacian operator

Linear operator

An operator $f(\cdot)$ is linear if it satisfies superposition principle:

$$f(\cdot) \text{ is linear} \iff \left.\begin{array}{l} h_1 = f(g_1) \\ h_2 = f(g_2) \end{array}\right\} \implies f(j_1 g_1 + j_2 g_2) = j_1 h_1 + j_2 h_2$$

where $g_1, g_2 \in \text{domain}(f)$; $h_1, h_2 \in \text{range}(f)$; j_1, j_2 : arbitrary scalars

Thus, ϕ's for simple flows can be superposed to study more complex flows

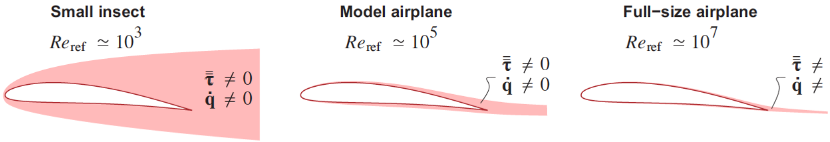
E.g., if ϕ_1 and ϕ_2 are valid velocity potentials, then so is $(j_1\phi_1 + j_2\phi_2)$

$$\Delta\phi_1 = 0 \text{ and } \Delta\phi_2 = 0 \implies \Delta(j_1\phi_1 + j_2\phi_2) = 0$$

N.B. ϕ is itself a function of position coordinates (x, y, z, say)

Potential theory applied to external aerodynamic flows

Aerodynamic flows are partly viscous (in locations of high shear), and otherwise inviscid, the two regions being *patched* together (due to Prandtl)



Incompressible potential (i.e., irrotational) flow theory is most applicable here, especially in moderate speed flows past slender immersed bodies

- Speed is low enough to preclude compressible effects (esp. shocks)
- Speed is high enough that the boundary layer is thin, and may be neglected in the first approximation
- The body isn't bluff so that flow separation is minimal

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Elementary potential flows – building blocks

Can we develop a set of 'Lego' bricks for potential flows?

Given the linearity of the potential flow problem

- We develop solutions for all fundamental 'flows'
- These can be superposed to obtain solutions for all other potential flows

There are only three fundamental flows:

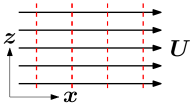
1. Uniform flow
2. Source, or sink ('line' source in 2D)
3. Irrotational vortex, CCW or CW ('line' irrotational vortex in 2D)

The problem is of course the appropriate recipe for combining the fundamental flows to model a particular chosen flow

Sign convention: Cartesian coordinates will be $x - z$ with y going into plane of figure

Cylindrical coordinates will have θ measured positive counter-clockwise!

Uniform (free) stream



$\underline{\boldsymbol{V}} = \boldsymbol{U}\hat{\underline{\boldsymbol{i}}}$

Streamlines (black solid): horizontal straight lines aligned along free stream

Equipotential lines (red dotted): vertical straight lines (orthogonal to streamlines)

ϕ is defined as $\underline{V}$ is evidently curl-free

$$\text{Relating } \underline{V} \text{ to } \phi: \qquad u = U = \frac{\partial \phi}{\partial x}, \qquad w = 0 = \frac{\partial \phi}{\partial z}$$

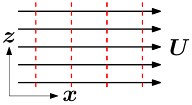
$\phi(x, z)$ is found by 2 (partial) integrations (upto arbitrary functions g & h):

$$\frac{\partial \phi}{\partial x} = U \implies \phi = \int U dx + g(z) = Ux + g(z), \quad \frac{\partial \phi}{\partial z} = 0 \implies \phi = h(x)$$

To satisfy both equations simultaneously, $\phi(x, z) = Ux + \text{constant}$

Constant is discarded for convenience as it doesn't affect velocity; $\phi = Ux$

Uniform stream (contd.)



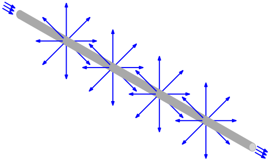
$$\underline{V} = U\hat{\underline{i}}$$

Thus, final result in Cartesian coordinates:

$$\boxed{\text{Uniform stream, } \underline{V} = U\hat{\underline{i}}: \qquad \phi = Ux}$$

What if the uniform stream is directed at an angle α, say, to x-axis?

Suppose y-axis is an infinitesimally narrow perforated pipe of length b thru which fluid issues at total rate Q, uniformly along its length and in azimuth



At any radius r, the velocity is purely radial, and related to Q as

$$u_r = \frac{Q}{2\pi rb} =: \frac{m}{r}, \qquad u_\theta = 0 \qquad u_y = 0$$

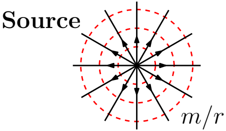
$m := Q/2\pi b$ is called source strength; positive for source, negative for sink

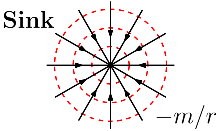
N.B.: This is automatically incompressible and irrotational

$$\nabla \cdot \underline{V} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial m}{\partial r} + \frac{1}{r}\frac{\partial(0)}{\partial \theta} = 0$$

$$\nabla \times \underline{V} = \omega_y \underline{\hat{y}} = \frac{1}{r}\left\{\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right\}\underline{\hat{y}} = \frac{1}{r}\left\{\frac{\partial(0)}{\partial r} - \frac{\partial(m/r)}{\partial \theta}\right\}\underline{\hat{y}} = \underline{0}$$

Line source/sink at origin (contd.)





Relating velocity field to velocity potential:

$$u_r = \frac{m}{r} = \frac{\partial \phi}{\partial r}, \qquad u_\theta = 0 = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$\phi(r, \theta)$ is found by integration, discarding integration constants

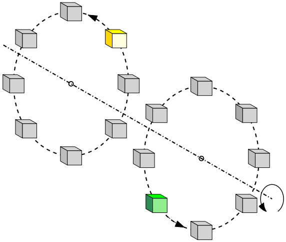
$$\boxed{\text{Line source/sink of strength } m: \qquad \phi = m \ln r}$$

Equipotential lines are circles (streamlines are radial spokes)

ϕ is singular at origin

Still, line source is a fictitious but useful model flow

Line irrotational vortex (free vortex)



(2-D) line vortex is a purely circulating steady flow: $u_\theta = f(r)$, $u_r = 0$, $u_y = 0$

Satisfies continuity for all $f(r)$ & Navier Stokes eqn. for various $f(r)$'s

But, $f(r)$ is unique for irrotationality ($\omega_y = 0$)

$$\omega_y = \frac{1}{r}\left\{\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right\} = \frac{1}{r}\left\{\frac{\partial(rf(r))}{\partial r} - \frac{\partial(0)}{\partial \theta}\right\} = 0$$

Integrating w.r.t. r, $\quad rf(r) = \text{constant}, -K, \text{say}$

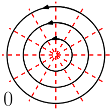
So, for irrotationality, $u_\theta = -K/r$, where K (a constant) is vortex strength

$K > 0$ signifies clockwise (CW) vortex; $K < 0$ signifies CCW vortex

Sign convention makes sense as $+y$ axis is going into figure's plane

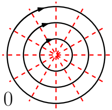
Line irrotational vortex (contd.)

Irrotational Vortex (CCW)



$u_\theta = -K/r, K < 0$

Irrotational Vortex (CW)



$u_\theta = -K/r, K > 0$

(2-D) line vortex is a purely circulating steady flow, $u_\theta = -K/r$, $u_r = 0$

$$u_r = 0 = \frac{\partial \phi}{\partial r}, \qquad u_\theta = -\frac{K}{r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$\phi(r, \theta)$ is found by integration, discarding integration constants

$$\boxed{\text{Free vortex of strength } K: \qquad \phi = -K\theta}$$

Equipotential lines are radial spokes (streamlines are circles)

ϕ has 'branch point' at $\theta = 2\pi$

Line irrotational vortex flow is 'complementary' to line source flow

Superposition of Elementary Potential Flows

Superposition of Elementary Flow Patterns

The three elementary two-dimensional flow patterns are

- Uniform stream ($\phi = Ux = Ur\cos\theta$)
- Line source/sink ($\phi = 0.5m\ln\left(x^2 + z^2\right) = m\ln r$)
- Line irrotational vortex ($\phi = -K\,\mathrm{atan}(z/x) = -K\theta$)

All three are

1. irrotational,
2. two-dimensional, and
3. incompressible

Hence, all three simultaneously satisfy 2D eqn. $\Delta\phi = 0$

These being linear PDEs, any weighted sum of solutions is also a solution

Some of these composite solutions are quite useful

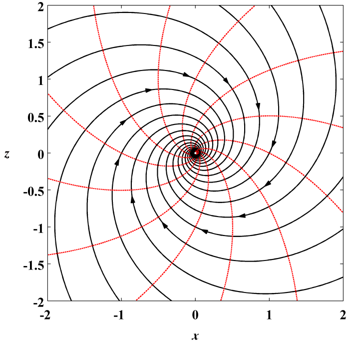
Superposition: Collocated sink & free vortex

Sink of strength '$-m$' is collocated with free CW vortex of strength 'K'

Resulting velocity potential (in polar coords)

$$\phi = -m \ln r - K\theta$$

Equipotential lines are a family of logarithmic spirals (as are streamlines)

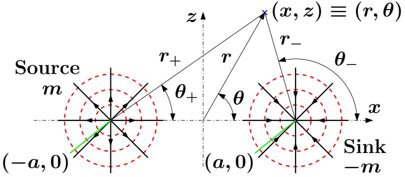


This is a fairly realistic simulation of

- A tornado (where the sink flow moves up y-axis into atmosphere)
- Or, rapidly draining bathtub vortex

At the center of real vortex (where infinite velocity is predicted), the actual flow is highly rotational and approximates solid body rotation, $u_\theta = Cr$

Superposition of source and equal sink placed apart



A source of strength $+m$ at $(x, z) = (-a, 0)$ is combined with an equal sink of strength $-m$ at $(a, 0)$

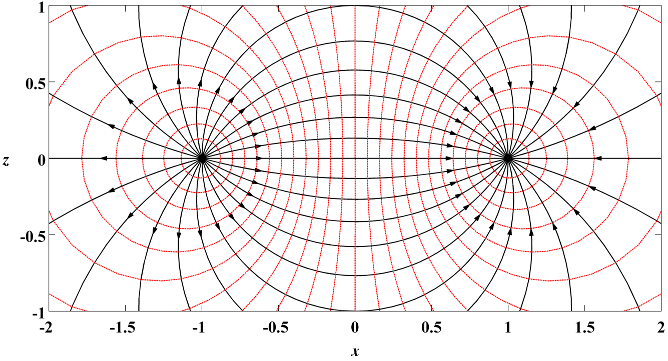
Resulting velocity potential (in Cartesian coordinates)

$$\phi = m \ln r_+ - m \ln r_- = m \ln \sqrt{(x+a)^2 + z^2} - m \ln \sqrt{(x-a)^2 + z^2}$$

Cartesian coords: $\quad \phi = \dfrac{m}{2} \ln \dfrac{(x+a)^2 + z^2}{(x-a)^2 + z^2}$

Polar coords: $\quad \phi = \dfrac{m}{2} \ln \dfrac{r^2 + a^2 + 2ar\cos\theta}{r^2 + a^2 - 2ar\cos\theta}$

Source and equal sink: flow visualization



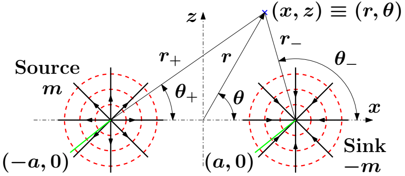
Source is at (-1,0); sink is at (1,0)

Equipotential lines (red) are displaced circles around source/sink

Streamlines (black) are circular arcs that go from source to sink

Streamlines are analogous to magnetic field lines in magnetic dipole

Almost collocated source and sink



We bring source and sink together along x-axis (i.e., $a \to 0$)

But $\lambda := 2ma$ is held constant in the limiting process

$$\phi = \lim_{a\to 0} \frac{m}{2} \ln \frac{r^2 + a^2 + 2ar\cos\theta}{r^2 + a^2 - 2ar\cos\theta} = \lim_{a\to 0} \frac{m}{2} \ln \frac{1+\beta}{1-\beta}, \quad \text{where } \beta = \frac{2ar\cos\theta}{(r^2+a^2)}$$

$$= \lim_{a\to 0} \frac{m}{2} \left[\left(\beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \cdots \right) - \left(-\beta - \frac{\beta^2}{2} - \frac{\beta^3}{3} - \cdots \right) \right]$$

$$= \lim_{a\to 0} m \left(\beta + \frac{\beta^3}{3} + \frac{\beta^5}{5} + \cdots \right) = \lambda \lim_{a\to 0} \left\{ \frac{r\cos\theta}{r^2+a^2} + \frac{4a^2}{3} \left(\frac{r\cos\theta}{r^2+a^2} \right)^3 + \cdots \right\}$$

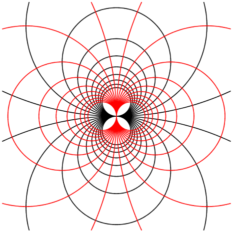
$$= \frac{\lambda\cos\theta}{r} = \frac{\lambda x}{x^2+z^2}, \qquad r > 0$$

Doublet – Almost collocated source and sink

'Doublet' results if spacing, a, between source & sink vanishes, while $\lambda := 2ma$ is held constant

Limiting procedure (see before) yields (for $r > 0$)

$$\phi = \frac{\lambda \cos\theta}{r} = \frac{\lambda x}{x^2 + z^2}$$



N.B.: Above is for doublet directed along $+x$ axis (from source to sink)

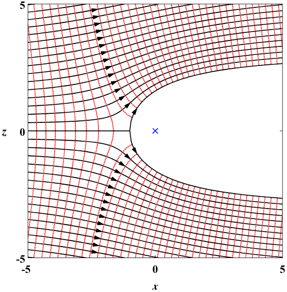
Equipotential curves are circles passing thru the origin

Doublets are useful for simulating flow over a cylinder (more later)

Also constitute fundamental unit for simulating 3D lifting potential flows

Superposition: uniform stream & line source

A Rankine half-body shape is simulated if a uniform stream (of velocity $U\hat{\underline{i}}$) is superposed on a source of strength m (at the origin)



$$\phi = Ux + m \ln r = Ux + \frac{m}{2} \ln\left(x^2 + z^2\right),$$

$$\implies u = \frac{\partial \phi}{\partial x} = U + \frac{mx}{x^2 + z^2},$$

$$\implies w = \frac{\partial \phi}{\partial z} = \frac{mz}{x^2 + z^2}$$

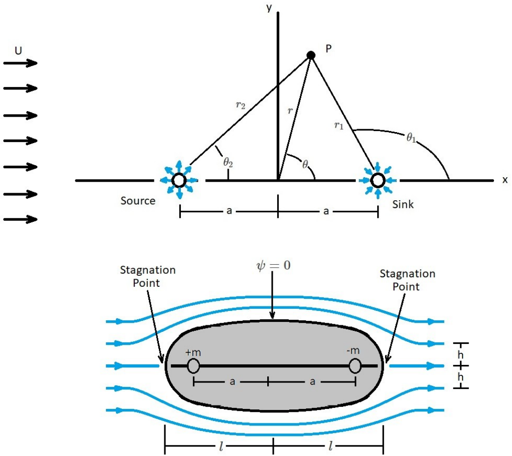
Stagnation point is at $(x = -m/U,\ z = 0)$

From stream-function analysis (not covered here), body surface is given by:

$$x = \frac{z}{\tan(\pi - Uz/m)}, \quad z \in (-\pi m/U, \pi m/U)$$

Body's half-width far downstream (i.e., at $x \to \infty$) is $\pi m/U$

Rankine oval: uniform stream + line source and sink



A closed streamline (simulating a 'full' body) forms when net volume flow rate within it goes to zero

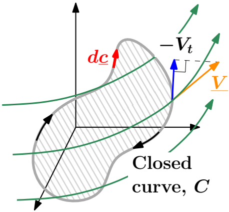
Now we are approaching realistic external aerodynamic flows

Lift from Potential Flow Theory

Circulation

Circulation, Γ: Line integral of tangential velocity around a closed curve, C

$$\Gamma := \oint_C V_t dc = \oint_C \underline{V} \cdot d\underline{c}$$



V_t: CW curve-tangential velocity at any point

$d\underline{c}$: infinitesimal (CW!) vectorial curve element

Using Stokes's theorem, we can show that

$$\Gamma = \oint_C \underline{V} \cdot d\underline{c} = \int_S (\nabla \times \underline{V}) \cdot d\underline{S} = \int_S \underline{\omega} \cdot d\underline{S}$$

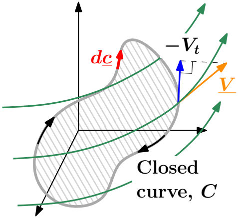
where $d\underline{S}$ is the infinitesimal vectorial area (pointing *into* the figure), and S is total area enclosed by C

Circulation is net area-integrated CW vorticity contained within curve

Circulation in irrotational flows

Let the directed element $d\underline{c} = dx_c\hat{\underline{i}} + dy_c\hat{\underline{j}} + dz_c\hat{\underline{k}}$

Potential function ϕ is defined everywhere for irrotational flows:



$$\Gamma = \oint_C \underline{V} \cdot d\underline{c} = \oint_C (\nabla\phi) \cdot d\underline{c}$$

$$= \oint_C \left(\frac{\partial\phi}{\partial x}\hat{\underline{i}} + \frac{\partial\phi}{\partial y}\hat{\underline{j}} + \frac{\partial\phi}{\partial z}\hat{\underline{k}} \right) \cdot \left(dx_c\hat{\underline{i}} + dy_c\hat{\underline{j}} + dz_c\hat{\underline{k}} \right)$$

$$= \oint_C \left(\frac{\partial\phi}{\partial x}dx_c + \frac{\partial\phi}{\partial y}dy_c + \frac{\partial\phi}{\partial z}dz_c \right) = \oint_C d\phi$$

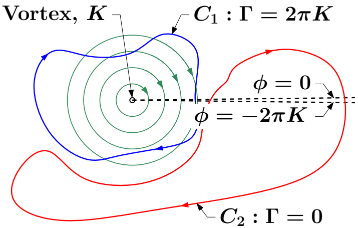
Since the integral starts and ends at the same point, we usually compute $\Gamma = 0$ in irrotational flows (e.g. sources & free streams)

Circulation in irrotational line vortex field

In irrotational flows: $\Gamma = \oint_C d\phi$

However, for a free irrotational line vortex of strength K, we have $\phi = -K\theta$:

$$\Gamma = \begin{cases} 2\pi K, & \text{for any closed CW curve enclosing the vortex center} \\ 0, & \text{otherwise} \end{cases}$$



N.B.: If curve C_1 were directed CCW, then Γ would be $-2\pi K$

In general, Γ denotes the net algebraic strength of all line vortices contained within the closed curve

Circulation around an immersed body is related to the lift on it (see later)

Uniform stream with doublet and collocated vortex

Rankine body = uniform stream + source-sink pair

Similarly, cylinder = uniform stream + doublet

Circulation is imposed by adding CW free vortex collocated with the doublet

$$\phi = Ux + \frac{\lambda \cos\theta}{r} - K\theta = Ur\cos\theta + \frac{\lambda \cos\theta}{r} - K\theta$$

Let $R := \sqrt{\lambda/U}$, which has dimension of length (recall $\lambda = 2ma = Qa/\pi b$)

Recalling, $u_\theta = -K/r$, define $\beta := K/(UR)$ as dimensionless circulation

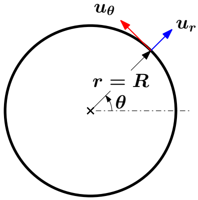
Dimensionless velocity potential: $\boxed{\dfrac{\phi}{UR} = \left(\dfrac{r}{R} + \dfrac{R}{r}\right)\cos\theta - \beta\theta}$

Uniform stream, doublet and collocated vortex: Velocity

$$\phi = UR\left[\left(\frac{r}{R} + \frac{R}{r}\right)\cos\theta - \beta\theta\right]$$

$$\implies u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta,$$

$$\text{Also, } u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left\{\left(1 + \frac{R^2}{r^2}\right)\sin\theta + \beta\frac{R}{r}\right\}$$



N.B.: $r = R$ defines a closed curve (a circle) on which normal velocity is u_r

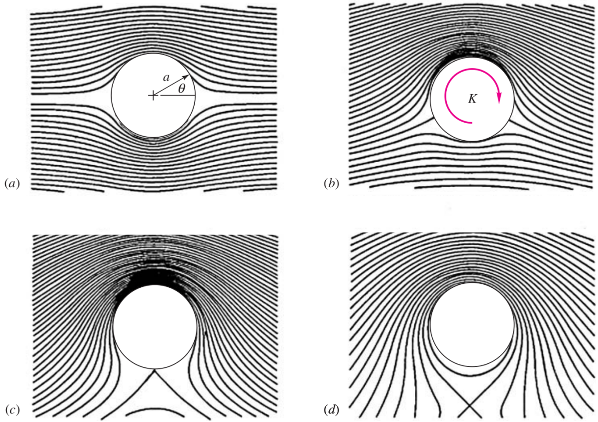
But $u_r(r = R, \theta) = 0$; so there is no normal velocity on this closed curve

Thus, this combination of uniform flow, doublet and vortex simulates flow over a circular cylinder

In particular, on the 'cylinder' surface, the flow is purely tangential, with velocity $u_\theta = -U(\beta + 2\sin\theta)$

N.B.: At any point in the flow, u_r is independent of the vortex strength introduced, whereas u_θ is affected by it

Simulating cylinders with various circulation



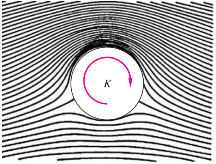
$\beta (= K/(UR))$ is (a) 0, (b) 1, (c) 2 & (d) 3

Recall: on cylinder surface (i.e., at $r = R$), $u_r = 0$, $u_\theta = -U(\beta + 2\sin\theta)$

The stagnation points on the surface (if $\beta < 2$) are at $\theta_s = \text{asin}(-\beta/2)$

White, Fluid Mechanics, McGraw-Hill, 2009

Potential flow modeling viscous effect of rotating cylinder!



This flow resembles the one that *may be* obtained if a cylinder rotates in a stream of fluid

Correctly models quickening of flow (closer streamlines) where surface velocity adds to freestream, and slowing of flow at opposite circumferential point

But effect of rotating cylinder is transmitted to surrounding fluid by viscous effect, which is not modelled by potential flow theory!

Apparently, appropriate amount of vorticity can be superposed so as to model viscous effects in potential flow

Pressure on the cylinder surface from Bernoulli's principle

Velocity is decoupled from pressure, and has been found from potential flow theory independently (w/o apparent reference to momentum conservation)

Actually, momentum conservation is used implicitly to conclude that an initially irrotational flow remains irrotational in the absence of viscous effects

Since the flow is steady, incompressible, inviscid and irrotational everywhere (except origin, which is 'within' cylinder body), Bernoulli's principle applies

Indeed, pressure can be found from the solved velocity field using Bernoulli's principle (which is a restatement of the momentum conservation eqn.)

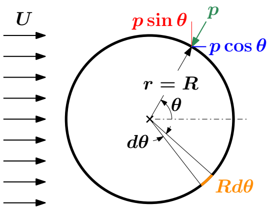
$$p + \rho V^2/2 = \text{constant} = p_\infty + \rho U^2/2 \quad (\text{since } V_\infty = U)$$

Recall: on the 'cylinder' surface, $u_r = 0$ and $u_\theta = -U(\beta + 2\sin\theta)$

By Bernoulli's principle, the pressure on the cylinder surface is

$$p = p_\infty + \frac{\rho}{2}\left\{U^2 - \left(u_r^2 + u_\theta^2\right)\right\} = p_\infty + \frac{\rho U^2}{2}\left\{1 - (\beta + 2\sin\theta)^2\right\}$$

$$p(r=\beta,\theta) = p_\infty + \frac{\rho U^2}{2}\{1-(\beta+2\sin\theta)^2\}$$



Integrated effect of constant pressure (e.g. p_∞) vanishes by Gauss divergence theorem:

$$\iint p_\infty d\underline{S} = \iiint \nabla p_\infty d\mathcal{V} = 0$$

Drag force (force along stream dirn.) per unit length of the cylinder is

$$D' = -\int_{-\pi}^{+\pi} (p-p_\infty)\, R\cos\theta d\theta = -\int_{-\pi}^{+\pi} \frac{\rho U^2}{2}\{1-(\beta+2\sin\theta)^2\} R\cos\theta d\theta$$

$$= -\frac{\rho U^2 R}{2}\int_{-\pi}^{+\pi}\left(1-\beta^2-4\sin^2\theta-4\beta\sin\theta\right)\cos\theta d\theta$$

$$= 0$$

D'Alembert's paradox

Drag force on the cylinder is zero!

This is a special case of D'Alembert's paradox that states

D'Alembert's paradox

According to inviscid theory, the drag on any body immersed in a uniform stream is identically zero

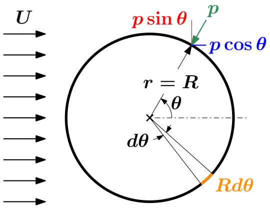
The reasons are of course the neglect of both the contributors to incompressible 2D drag:

1. Viscous drag, and
2. Form drag due to flow separation behind non-slender bodies that actually causes large pressure differences between windward and leeward sides

Conclusion: Don't look to potential flow theory to estimate profile drag

Lift on the cylinder

$$p(r=\beta,\theta) = p_\infty + \frac{\rho U^2}{2}\{1-(\beta+2\sin\theta)^2\}$$



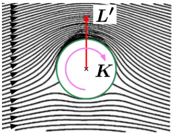
Lift force (force normal to stream dirn.) per unit length of the cylinder is

$$L' = -\int_{-\pi}^{+\pi} (p - p_\infty)\, R\sin\theta d\theta$$

$$= -\frac{\rho U^2 R}{2}\int_{-\pi}^{+\pi}\left(1-\beta^2-4\sin^2\theta-4\beta\sin\theta\right)\sin\theta d\theta$$

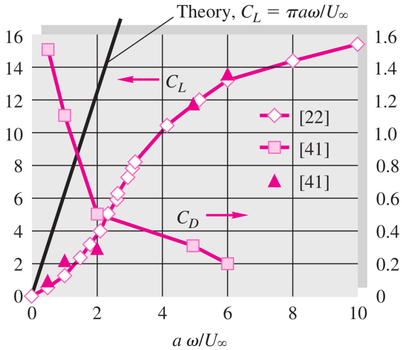
$$= 2\pi\rho U^2 R\beta = 2\pi\rho U^2 R(K/UR) = (2\pi K)\rho U$$

$$= \rho U\Gamma$$



Positive sign reflects the fact that lift is upward for CW circulation when flow is from left to right

Comparison of rotating cylinder theory with data



Drag and lift of a rotating cylinder of large aspect ratio at $Re_D = 3800$, after Tokumaru and Dimotakis [22] and Sengupta et al. [41]

Unaccounted flow separation behind the cylinder causes large errors in even the *lift* predictions from inviscid theory

The theory is still useful as a stepping stone for simulating lift on slender body flows, e.g. in flow over airfoils and wings

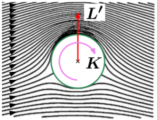
White, Fluid Mechanics, McGraw-Hill, 2009

Kutta-Joukowsky theorem – Airfoil theory

Magnitude of sectional lift on circ. cyl. $|L'| = \rho U |\Gamma|$

In left-to-right flow, lift is upward for CW circulation

This is a special case of Kutta-Joukowsky theorem



Kutta-Joukowsky theorem

According to inviscid theory, the lift per unit span of a cylinder of any shape immersed in a uniform incompressible stream of speed U and density ρ equals $\rho U \Gamma$, where Γ is the total circulation around the body

The direction of the lift is 90° from the stream direction, rotating opposite to the circulation

Kutta-Joukowsky theorem is the basis of inviscid airfoil theory since airfoil shapes can be represented by a suitable combination of vortices and sources immersed in an appropriate uniform stream

A Digression into Kinematics

Helmholtz decomposition

In general, a vector field (exemplified by the velocity vector field $\underline{V}$) can be linearly decomposed into

- A part with non-zero divergence but curl-free $\underline{V}_\sigma$ (i.e., $\nabla \times \underline{V}_\sigma = \underline{0}$),
- A part with non-zero curl but divergence-free $\underline{V}_\omega$ (i.e., $\nabla \cdot \underline{V}_\omega = 0$),
- The remainder that is both curl-free and divergence-free $\underline{V}_b$ (i.e., $\nabla \cdot \underline{V}_b = 0$ and $\nabla \times \underline{V}_b = \underline{0}$) [the 'b' refers to boundary influence]

$$\underline{V} = \underbrace{\underline{V}_\sigma}_{\text{Irrotational part}} + \underbrace{\underline{V}_\omega}_{\text{`Incompressible' part}} + \underbrace{\underline{V}_b}_{\text{`Incompressible' \textit{and} irrotational part}}$$

This is Helmholtz decomposition; it applies to all *well-behaved* vector fields

- Sufficiently smooth, and
- Rapidly-decaying far enough away (from region of interest)

Also called fundamental theorem of vector calculus

Helmholtz decomposition recipe

The curl-free $\underline{V}_\sigma$ must be the gradient of a *scalar* potential (ϕ, of course)

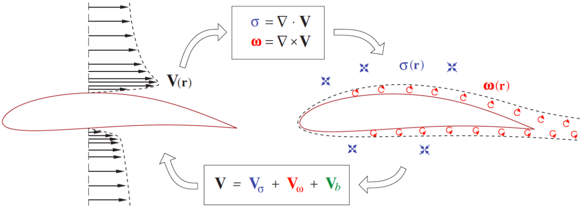
Similarly, the divergence-free $\underline{V}_\omega$ must be the curl of a *vector* potential $\underline{B}$

We have defined dilatation as $\sigma := \nabla \cdot \underline{V}$, and vorticity as $\underline{\omega} := \nabla \times \underline{V}$

Helmholtz gave the recipe for partitioning ($\underline{r}$ denotes position vector):

$$\phi(\underline{r}) = -\iiint \frac{\sigma(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} d^3\underline{r}', \quad \underline{V}_\sigma(\underline{r}) = \nabla\phi = \frac{1}{4\pi}\iiint \sigma(\underline{r}')\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} d^3\underline{r}',$$

$$\underline{B}(\underline{r}) = \iiint \frac{\underline{\omega}(\underline{r}')}{4\pi|\underline{r}-\underline{r}'|} d^3\underline{r}', \quad \underline{V}_\omega(\underline{r}) = \nabla\times\underline{B} = \frac{1}{4\pi}\iiint \underline{\omega}(\underline{r}')\times\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3} d^3\underline{r}'$$



Drela, Flight Vehicle Aerodynamics, MIT Press, 2014

Interpretation of divergent part of velocity field

$$\underline{V}_\sigma(\underline{r}) = \iiint \sigma(\underline{r}') \left\{ \frac{1}{4\pi|\underline{r}-\underline{r}'|^2} \left(\frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|} \right) \right\} d^3\underline{r}' =: \iiint \sigma(\underline{r}')\underline{K}_\sigma(\underline{r}|\underline{r}')d^3\underline{r}'$$

Kernel $\underline{K}_\sigma(\underline{r}|\underline{r}')$ gives velocity vector at $\underline{r}$ due to unit volume source at $\underline{r}'$

The velocity is directed along the separation vector $(\underline{r}-\underline{r}')$ ($=:\underline{s}$, say)

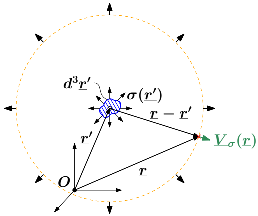
Magnitude scales as inverse square of separation: $|\underline{K}_\sigma(\underline{r}|\underline{r}')| = 1/4\pi|\underline{r}-\underline{r}'|^2$

Dilatation is rate of increase of volume of fluid per unit volume

At point $\underline{r}'$, dilatation $\sigma(\underline{r}')$ over volume $d^3\underline{r}'$ is emitting flow isotropically (radially outward) at volume rate $\sigma(\underline{r}')d^3\underline{r}'$ – volume source

Flow speed at a point separated by s from it is this volume rate divided by area of sphere $4\pi s^2$

Vectorially add effect from all volume sources



Interpretation of rotational part of velocity field

$$\underline{V}_\omega(\underline{r}) = \frac{1}{4\pi} \iiint \underline{\omega}(\underline{r}') \times \frac{\underline{r} - \underline{r}'}{|\underline{r} - \underline{r}'|^3} d^3\underline{r}'$$

Same kernel appears, but with slightly different effect due to cross product

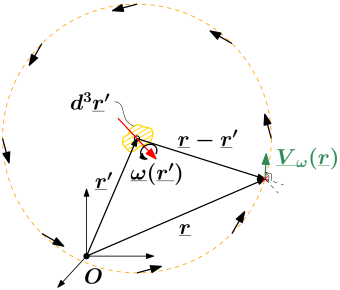
The velocity is directed orthogonal to both the separation vector $(\underline{r} - \underline{r}')$ and the vorticity axis

Vorticity gives volumetric vortical flow rate per unit volume orthogonal to its axis

Flow speed is not same everywhere on the sphere centered at $\underline{r}'$ and passing thru $\underline{r}$

It depends on how far $\underline{r}$ is from the vortical axis

Vectorially add effect from all volume sources



Implication of 'boundary influence' part of velocity field

In external aerodynamics, $\underline{V}_b$ is the (constant) freestream velocity

- Volume sources are present close to aircraft, modelling flow compressibility, but their effect decay with square of distance from aircraft so that $\underline{V}_\sigma(\underline{r} \to \infty) \to 0$
- Vorticity sources are in boundary layer or wake, but again $\underline{V}_\omega(\underline{r} \to \infty) \to 0$

In more general cases, $\underline{V}_b$ is not constant, but can be found

Since $\underline{V}_b$ is curl-free, there is scalar potential ϕ_b such that $\underline{V}_b = \boldsymbol{\nabla}\phi_b$

But, since $\underline{V}_b$ is divergence-free, we have

$$\boldsymbol{\nabla} \cdot \underline{V}_b = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi_b = \Delta\phi_b = 0$$

I.e., ϕ_b satisfies Laplace equation, whose solution is determined completely by appropriate boundary conditions

Helmholtz decomposition and Potential flow theory

Helmholtz decomposition gives $\underline{V}$ from σ and $\underline{\omega}$ distribution in flow domain

$$\underline{V}(\underline{r}) = \frac{1}{4\pi}\iiint \sigma(\underline{r}')\frac{\underline{r}-\underline{r}'}{\left|\underline{r}-\underline{r}'\right|^3}d^3\underline{r}' + \frac{1}{4\pi}\iiint \underline{\omega}(\underline{r}')\times\frac{\underline{r}-\underline{r}'}{\left|\underline{r}-\underline{r}'\right|^3}d^3\underline{r}' + \underline{V}_b$$

In potential flows, dilatation and vorticity are concentrated at a few locations (inside or on surface of body)

- Rest of flow domain (i.e., 'outer' flow) is excluded from the integrals
- This simplifies the computations significantly

In fact, all that we have developed in potential flow theory could also have been derived directly from Helmholtz' decomposition theorem

- In 2D, dilatation distribution becomes line source, and vorticity distribution becomes line irrotational vortex
- These are 'singularities', in the sense that σ and $\underline{\omega}$ are infinite on the line, but have finite integrated effect (m and K, respectively)
- Stream function is identified as the 2D variant of vector potential $\underline{B}$

End of Topic