Strain and Kinetic Energies of Some Structural Elements

Element or member	Strain energy, π	Kinetic energy, T
1. String undergoing transverse motion	$\frac{P}{2}\int_0^l \left(\frac{\partial w}{\partial x}\right)^2 dx$ P = tension w = transverse deflection l = length	$\frac{\rho}{2}\int_0^l \left(\frac{\partial w}{\partial t}\right)^2 dx$ ρ = mass per unit length
2. Bar in tension or compression	$\frac{AE}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx$ A = cross-sectional area E = Young's modulus u = axial displacement l = length	$\frac{\rho A}{2}\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx$ ρ = density
3. Rod in torsion	$\frac{GJ}{2}\int_0^l \left(\frac{\partial \theta}{\partial x}\right)^2 dx$ GJ = torsional stiffness θ = angular deflection l = length	$\frac{\rho J}{2}\int_0^l \left(\frac{\partial \theta}{\partial t}\right)^2 dx$ ρ = density J = polar moment of inertia of cross section
4. Beam in bending	$\frac{EI}{2}\int_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$ E = Young's modulus I = moment of inertia of cross section w = transverse deflection l = length	$\frac{\rho A}{2}\int_0^l \left(\frac{\partial w}{\partial t}\right)^2 dx$ ρ = density A = area of cross section

Question 1 Determine the fundamental frequency of transverse vibration of a uniform beam fixed at both ends (Figure) using Rayleigh's method. Use the following trial functions for approximating the fundamental mode shape:

(a)

$$X(x) = C\left(1 - \cos\frac{2\pi x}{l}\right)$$

where C is a constant. This function is selected to satisfy the boundary conditions of the beam: $X(0) = 0$, $dX(0)/dx = 0$, $X(l) = 0$, and $dX(l)/dx = 0$.

(b)

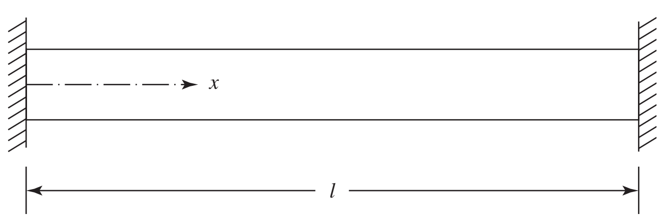
$$X(x) = C(x^2)(l - x)^2$$

with $C = w_0/24EI$.

SOLUTION The expressions for the strain and kinetic energies of a uniform beam are given by

$$\pi = \frac{1}{2}EI\int_0^l \left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]^2 dx$$

$$T = \frac{1}{2}\rho A\int_0^l \left[\frac{\partial w(x,t)}{\partial t}\right]^2 dx$$



Fixed–fixed beam.

where the transverse deflection function, $w(x,t)$, can be assumed to be harmonic:

$$w(x,t) = X(x)\cos\omega t$$

where ω is the frequency of vibration. Rayleigh's quotient for a beam bending is defined by

$$\text{maximum strain energy} = \text{maximum kinetic energy}$$

$$\pi_{\max} = \frac{1}{2}EI\int_0^l \left[\frac{d^2X(x)}{dx^2}\right]^2 dx$$

$$T_{\max} = \frac{1}{2}\rho A\omega^2\int_0^l [X(x)]^2\,dx$$

Equating $\pi_{\max}$ and $T_{\max}$, Rayleigh's quotient can be derived as

$$R(X(x)) = \omega^2 = \frac{\frac{1}{2}EI\int_0^l \left(d^2X/dx^2\right)^2 dx}{\frac{1}{2}\rho A\int_0^l [X(x)]^2\,dx}$$

(a) In this case,

$$X(x) = C\left(1-\cos\frac{2\pi x}{l}\right)$$

$$\frac{d^2X(x)}{dx^2} = C\left(\frac{2\pi}{l}\right)^2\cos\frac{2\pi x}{l}$$

$$\int_0^l\left(\frac{d^2X}{dx^2}\right)^2 dx = C^2\left(\frac{2\pi}{l}\right)^4\int_0^l \cos^2\frac{2\pi x}{l}\,dx = C^2\left(\frac{2\pi}{l}\right)^4\frac{l}{2} = \frac{8C^2\pi^4}{l^3}$$

$$\int_0^l [X(x)]^2 = C^2\int_0^l\left(1-\cos\frac{2\pi x}{l}\right)^2 dx = \frac{3C^2l}{2}$$

$$R = \omega^2 = \frac{\frac{1}{2}EI(8C^2\pi^4/l^3)}{\frac{1}{2}\rho A(3C^2l/2)} = \frac{16\pi^4}{3}\frac{EI}{\rho Al^4}$$

or

$$\omega = 22.792879\sqrt{\frac{EI}{\rho Al^4}}$$

(b) In this case,

$$X(x) = Cx^2(l-x)^2$$

$$\frac{d^2X(x)}{dx^2} = 2C(6x^2-6lx+l^2)$$

$$\int_0^l \left(\frac{d^2X}{dx^2}\right)^2 dx = 4C^2 \int_0^l (6x^2 - 6lx + l^2)^2\, dx = \tfrac{4}{5}C^2 l^5$$

$$\int_0^l (X(x))^2\, dx = C^2 \int_0^l (x^4 - 2lx^3 + x^2 l^2)^2\, dx = \tfrac{1}{630}C^2 l^9$$

$$R = \omega^2 = \frac{\frac{1}{2}EI\left(\frac{4}{5}C^2 l^5\right)}{\frac{1}{2}\rho A\left(\frac{1}{630}C^2 l^9\right)} = 504\frac{EI}{\rho A l^4}$$

or

$$\omega = 22.449944\sqrt{\frac{EI}{\rho A l^4}}$$

The exact fundamental natural frequency of the beam is given by

$$\omega_1^2 = (\beta_1 l)^2\sqrt{\frac{EI}{\rho A l^4}} = (4.730041)^2\sqrt{\frac{EI}{\rho A l^4}} = 22.373288\sqrt{\frac{EI}{\rho A l^4}}$$

and the exact fundamental natural mode is given by

$$W_1(x) = C\left[\sinh\beta_1 x - \sin\beta_1 x + \frac{\sinh\beta_1 l - \sin\beta_1 l}{\cos\beta_1 l - \cosh\beta_1 l}(\cosh\beta_1 x - \cos\beta_1 x)\right]$$

where C is a constant. It can be seen that the fundamental natural frequencies given by Rayleigh's method are very close to the exact value and larger than the exact value by only 1.875410% in the first case
0.342623% in the second case

ω_e = Exact fundamental natural frequency

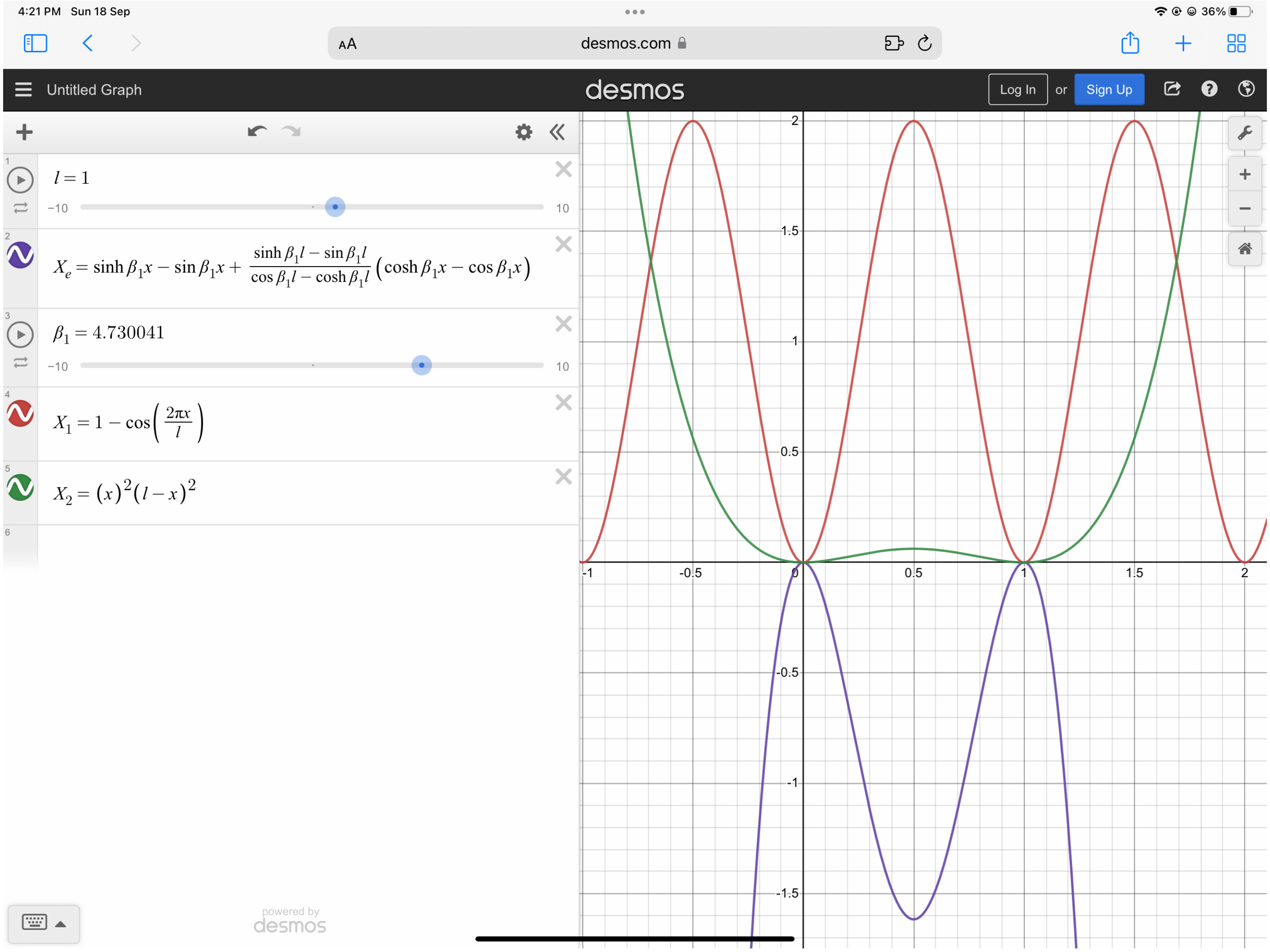
$$= 22.373288\sqrt{\frac{EI}{\rho A l^4}}$$

ω_1 = frequency using function $C\left(1 - \cos\frac{2\pi x}{l}\right)$

$$= 22.792879\sqrt{\frac{EI}{\rho A l^4}}$$

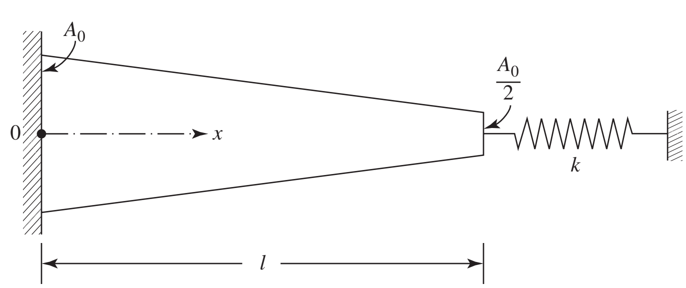
ω_2 = frequency using function $C\,(x\,(L - x))$

$$= 22.449944\sqrt{\frac{EI}{\rho A l^4}}$$



Question 2 Find the fundamental natural frequency of longitudinal vibration of the tapered bar fixed at $x = 0$ and connected to a linear spring of stiffness k at $x = l$ shown in (Figure) using Rayleigh's method. Assume the variation of the cross-sectional area of the bar to be $A(x) = A_0\,(1 - x/2l)$ and use the trail function $X(x) = C\sin(\pi x/2l)$ for the mode shape.

SOLUTION The expressions for the strain and kinetic energies of a uniform bar, including the strain energy due to the deformation of the spring at $x = l$, can be



expressed as

$$\pi = \frac{1}{2}E\int_0^l A(x)\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}ku^2(l,t)$$

$$T = \frac{1}{2}\rho\int_0^l A(x)\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx$$

where the longitudinal deflection function, $u(x, t)$, is assumed to be harmonic:

$$u(x,t) = X(x)\cos\omega t$$

$$\pi_{\max} = \frac{1}{2}E\int_0^l A_0\left(1-\frac{x}{2l}\right)\left[\frac{dX(x)}{dx}\right]^2 dx + \frac{1}{2}kX^2(l)$$

$$T^*_{\max} = \frac{1}{2}\rho\int_0^l A_0\left(1-\frac{x}{2l}\right)X^2(x)\,dx$$

Using

$$X(x) = C\sin\frac{\pi x}{2l}$$

$$\frac{dX}{dx}(x) = \frac{C\pi}{2l}\cos\frac{\pi x}{2l}$$

we can obtain

$$\int_0^l A_0\left(1-\frac{x}{2l}\right)\left(\frac{C\pi}{2l}\right)^2\cos^2\frac{\pi x}{2l}\,dx + \frac{1}{2}k(C)^2 = \frac{A_0C^2\pi^2}{8l}\left(\frac{3}{4}+\frac{1}{\pi^2}\right)+\frac{k}{2}C^2$$

$$\int_0^l A_0\left(1-\frac{x}{2l}\right)C^2\sin^2\frac{\pi x}{2l}\,dx = A_0C^2\frac{l}{2}\left(\frac{3}{4}-\frac{1}{\pi^2}\right)$$

Rayleigh's quotient is given by

$$R = \omega^2 = \frac{\pi_{\max}}{T^*_{\max}} = \frac{(EA_0C^2\pi^2/16l)\left(\frac{3}{4}+1/\pi^2\right)+kC^2/2}{(\rho A_0C^2l/4)\left(\frac{3}{4}-1/\pi^2\right)}$$

$$= \frac{1}{\rho A_0 l^2}(3.238212EA_0 + 3.063189kl)$$

Thus, the natural frequency of vibration is given by

$$\omega = \left[\frac{1}{\rho A_0 l^2}(3.238212EA_0 + 3.063189kl)\right]^{1/2}$$