AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 20: PD and PID Control Designs

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PD Controller - Introduction

Introduction

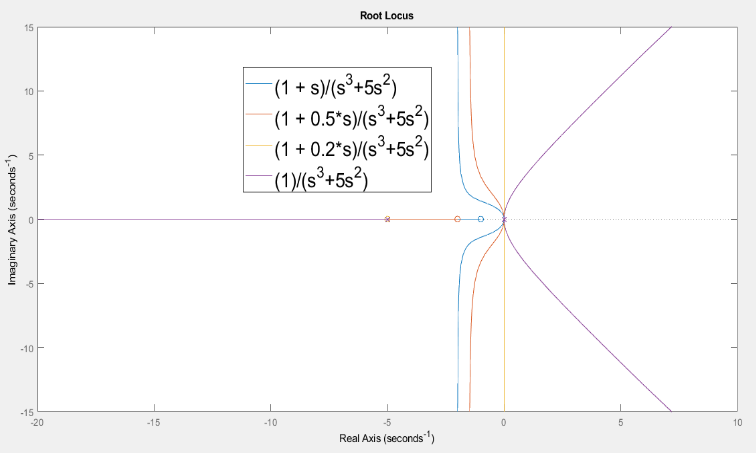
- Given below is the basic form of the PD controller

$$G_{PD} = K_p + K_d s = K_p \left(1 + T_d s\right)$$

- PD controller adds a zero at $s = -(1/T_d) = -K_p/K_d$, where K_d is the derivative gain and K_p is proportional gain
- PD controllers are used to improve the damping of dominant system behaviour, as well as speed of response in terms of rise time

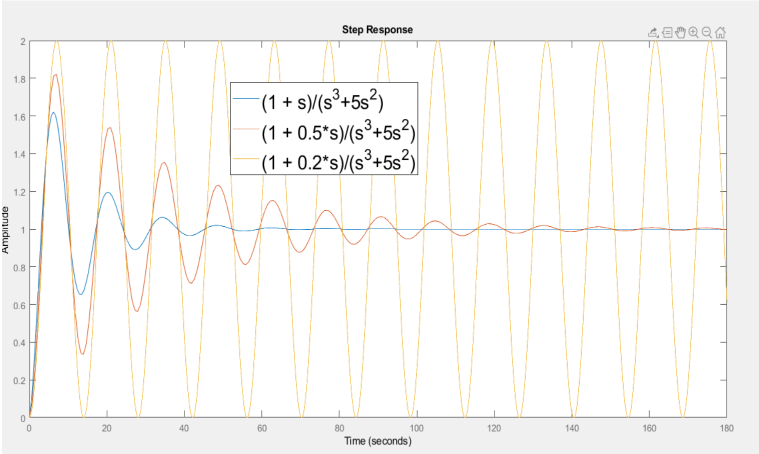
PD Controller - Introduction

Effect on Root Locus



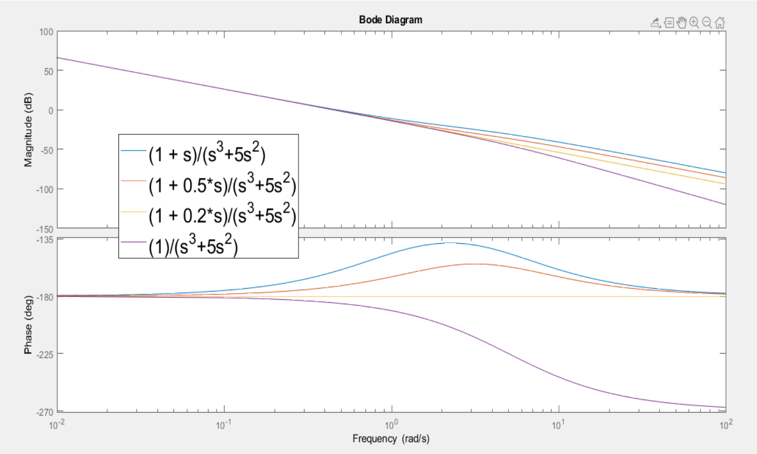
PD Controller - Introduction

Effect on Step Response



PD Controller - Introduction

Effect on Bode Diagram



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PD Controller - Control Design

Control Design using Root Locus

- Design of PD controllers is mainly concerned with determining the location of zero, based on the closed loop transient response specifications.
- In this method, we attempt to modify the root locus such that it passes through the desired dominant closed loop poles.
- The procedure makes use of angle and magnitude conditions, commonly used for drawing the root locus.
- This is done by first calculating the angle deficiency at the required dominant poles, which is used to set the zero location.
- Next, gain is determined from the magnitude condition.

PD Control Design - Example

Example: A system is defined by the following transfer function,

$$G(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

- Design a PD controller to achieve following performance in the closed loop.
  1. 2% Settling time ≤ 4 seconds
  2. Peak time ≤ 1 second

PD Control Design - Example

Solution: Step Response of system,

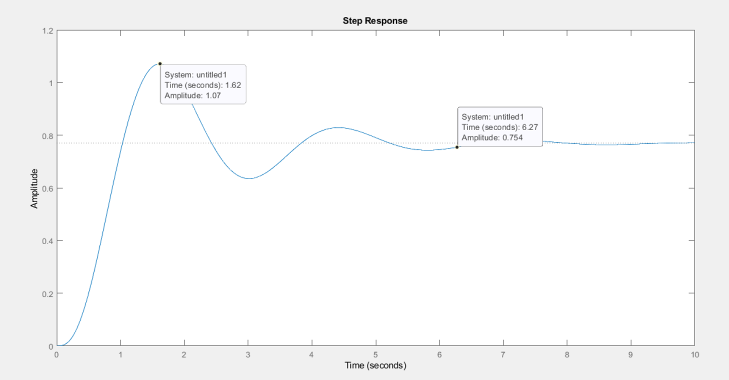


Figure: Step Response without Controller

PD Control Design - Example

- $T_s(2\%) = \frac{4}{\sigma} \leq 4 \implies \sigma \geq 1$
- $T_p = \frac{\pi}{\omega_d} \leq 1 \implies \omega_d \geq \pi$
- Therefore desired pole,

$$p_c = -\sigma \pm j\omega_d = -1 \pm j3.14$$

-

$$G(s) = \frac{20K(s+z)}{(s+1)(s+2)(s+3)}$$

PD Control Design - Example

- Substituting,

$$s = -1 + j3.14$$

- Thus,

$$G(-1 + j3.14) = \frac{20K(-1 + j3.14 + z)}{(j3.14)(j3.14 + 1)(j3.14 + 2)}$$

- Satisfying Angle Condition, $\angle G(-1 + j3.14) = -180^\circ$

$$\theta_1 = \tan^{-1}\frac{3.14}{0} = 90^\circ, \quad \theta_2 = \tan^{-1}\frac{3.14}{1} = 72.3^\circ$$
$$\theta_3 = \tan^{-1}\frac{3.14}{2} = 57.5^\circ, \quad \phi = \tan^{-1}\frac{3.14}{z - 1}$$

- $\phi = -180 + \theta_1 + \theta_2 + \theta_3 = 39.8$

PD Control Design - Example

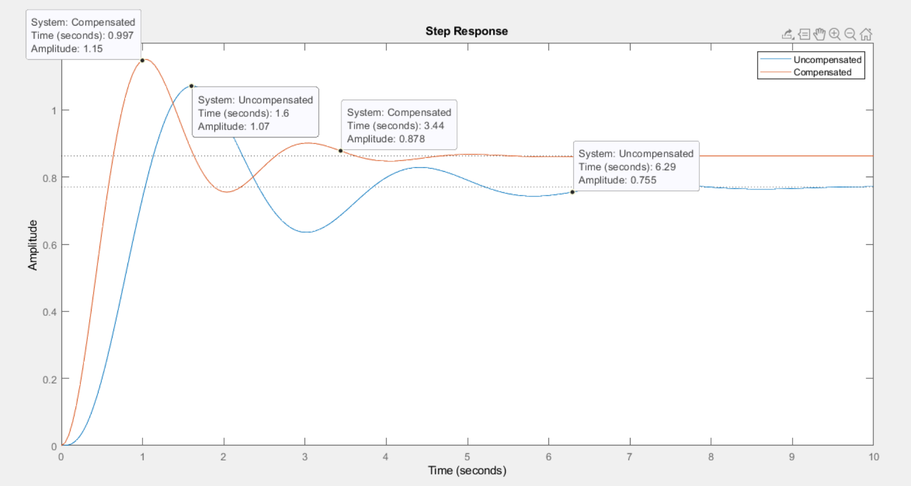
- $\tan^{-1}\frac{3.14}{z-1} = 39.8 \implies z = 4.77$
- $G_{PD} = K(s+4.77)$
- As at p_c, $|G_{PD}G| = 1$

$$\implies K = \frac{|p_c+1| \times |p_c+2| \times |p_c+3|}{20|p_c+4.77|}$$

$$= \frac{3.14 \times 3.297 \times 3.724}{20 \times 4.906} = 0.3929$$

- $G_{PD} = 0.3929(s+4.77)$

PD Control Design - Example



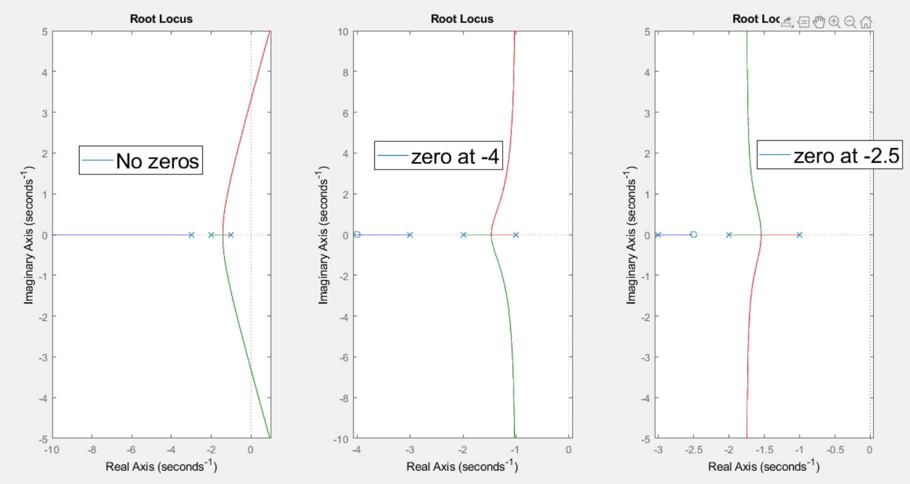
PD Control Design - Observations

Observations

- Design is meeting the setting time and peak time requirements.
- Tracking error is also reduced

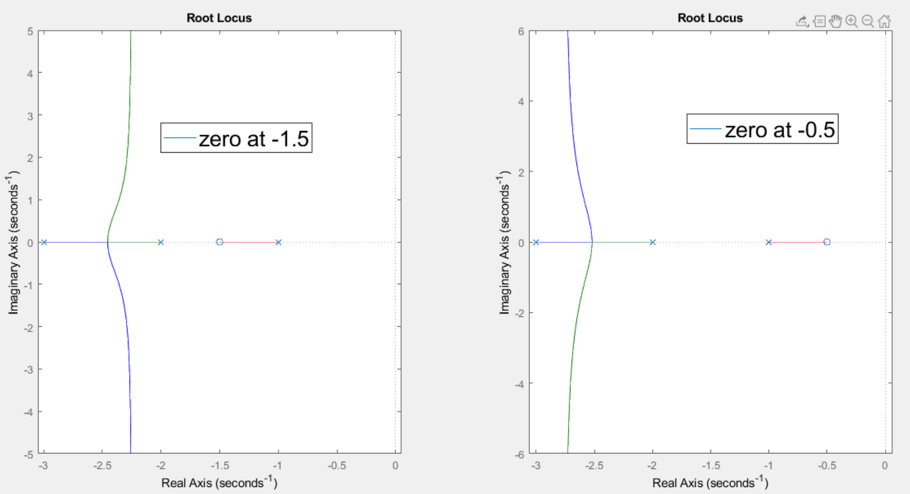
PD Control - Effects on root locus

Effect on root locus



PD Control - Effects on root locus

Effect on root locus



PD Control - Observations

Observations

- Large improvements in transient response are possible with PD controllers.
- The addition of ‘zero’ changes the root locus shape and influences both σ and ω_d, so that all attributes of the transient response are influenced.

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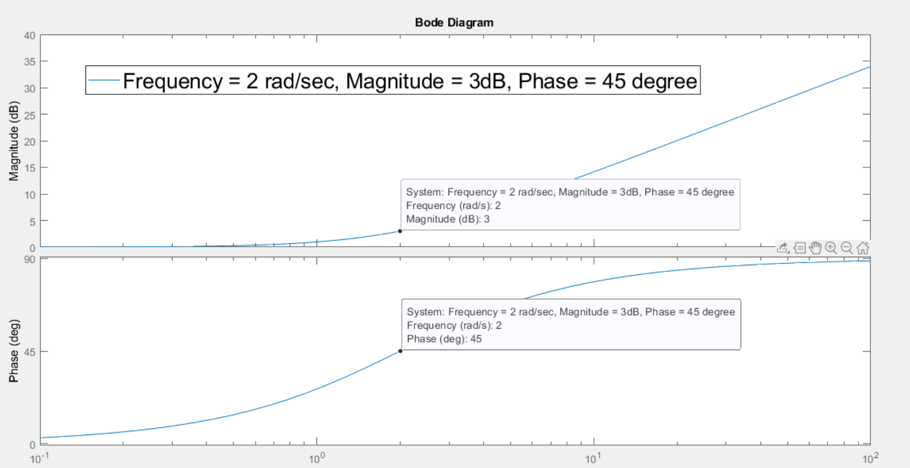
PD Controller - Control Design

Control Design using Bode Plots

- Design of PD – controller in frequency domain is primarily governed by the requirements on PM.
- In addition, a condition is put that DC gain remains unchanged. Therefore, in case there are requirements also on error constants, these are satisfied first, before designing PD
- The general form of PD in this case is $K_p(1 + T_d s)$,where corner frequency $1/T_d$ is chosen such that the positive phase to be added, occurs close to the GCO.

PD Controller - Control Design

Bode Plot of $(1 + 0.5s)$



PD Control Design - Observations

Observations

- We see that at frequencies $> 1/T_d$, increase in phase is accompanied by an increase in gain as well.
- This has the effect of pushing the GCO of the compensated system to a higher value.
- Thus, the design of PD controller has to take this fact into account and add the required additional phase at the new GCO.
- This also results in a kind of iteration as the additional phase is actually calculated at the original GCO.

PD Control Design - Example

Example: A system is defined by the following transfer function,

$$G(s) = \frac{K_x}{s(s^2 + 4.2s + 14.4)}$$

- Design a PD controller to achieve following performance in the closed loop.
  1. $K_v \geq 3$
  2. $GM > 6dB$, $PM > 30°$

PD Control Design - Example

Solution:

- First step is to achieve the specified K_v which can be done by making $K_x \geq 43.2$ **(How??)**

PD Control Design - Example

Solution:

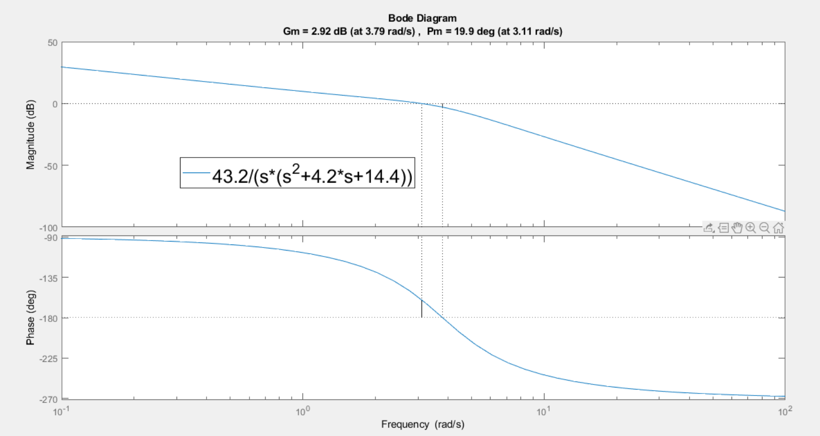
- First step is to achieve the specified K_v which can be done by making $K_x \geq 43.2$ **(How??)**

$$K_v = \lim_{s \to 0} sG(s)$$

$$\implies \frac{K_x}{14.4} \geq 3, \quad K_x \geq 43.2$$

PD Control Design - Example

Bode plot of $\frac{43.2}{s(s^2+4.2s+14.4)}$



- Both GM and PM are below the desired values.

PD Control Design - Example

- PM is to be increased by 10° at 3.11 rad/sec.
- Approximate solution for PD controller can be the one which add 15° phase.

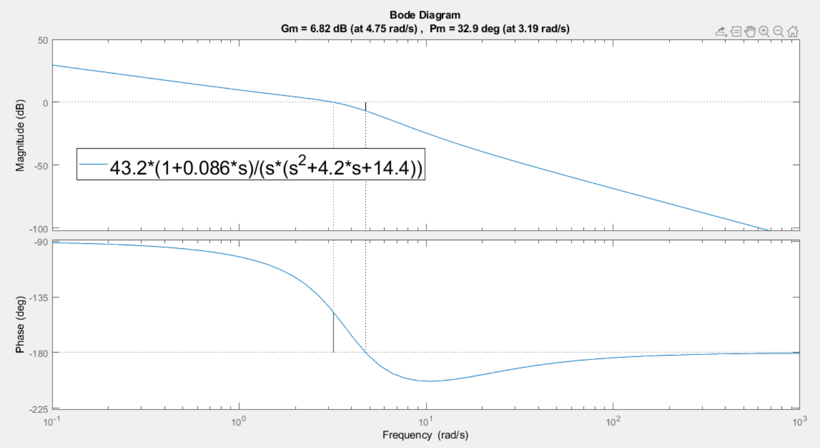
$$\angle(1 + T_d s)|_{\omega=3.11} = 15^\circ$$

$$\implies 3.11 T_d = \tan 15^\circ, \quad T_d = 0.086$$

$$G_{PD} = 1 + 0.086 s$$

PD Control Design - Example

Bode plot of $\frac{43.2(1+0.086s)}{s(s^2+4.2s+14.4)}$



- All requirements are met.

PD Controller - Drawbacks

Drawbacks

- PD controllers are improper transfer functions and hence, reduce relative degree $(n - m)$, and may result in unexpected changes.
- Further, we may also wish to preserve $(n - m)$ in order to ensure a desired slope of high frequency asymptote in bode plot.
- Therefore, we need an alternative to PD control to ensure $(n - m)$, which is the lead compensator.

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Lead Compensator - Structure

Structure

- Lead compensator structure is as shown below.

$$G_{Lead}(s) = K_c \frac{\alpha(Ts+1)}{\alpha Ts+1} = K_c \frac{(s+1/T)}{s+1/\alpha T}, \quad \alpha < 1$$

- Here, K_c is compensator gain, T is the compensator time constant and (α) is a parameter that decides the amount of lead added by the compensator.
- We see that above form will preserve relative degree.

Lead Compensator - Features

Features

- Lead compensator adds a zero at $s = -1/T$ and a pole at $s = -1/(\alpha T)$, to the plant, so that (n – m) is constant.
- Further, as a bonus, we also get additional design degree of freedom, to better achieve the specifications.
- When $\alpha \to 0$, pole lies at $-\infty$, resulting in PD controller.
- Also, if $\alpha \to 0$ and $T \to \infty$, the zero moves towards the origin, leading to a pure D control.
- DC gain of lead compensator is $K_c\alpha$, and is usually kept 1.0, which fixes K_c once α is determined.

Lead Compensator - Features

- Phase is given by,

$$\phi = tan^{-1}\omega T - tan^{-1}\alpha\omega T$$

- Maximum phase occurs at,

$$\frac{d\phi}{d\omega} = 0 \rightarrow \omega_m = \frac{1}{\sqrt{\alpha}T}, \text{ as } \frac{d^2\phi}{d\omega^2}|_{\omega=\omega_m} < 0$$

- Substituting $\omega = \omega_m = \frac{1}{\sqrt{\alpha}T}$ in phase equation, we obtain

$$\tan\phi_m = \frac{1-\alpha}{2\sqrt{\alpha}} \rightarrow \sin\phi_m = \frac{1-\alpha}{1+\alpha}$$

Lead Compensator - Example

Example : Find maximum phase and the frequency at which it occurs of the following system,

$$G(s) = \frac{0.01(1+s)}{0.01s+1}$$

Lead Compensator - Example

Solution:

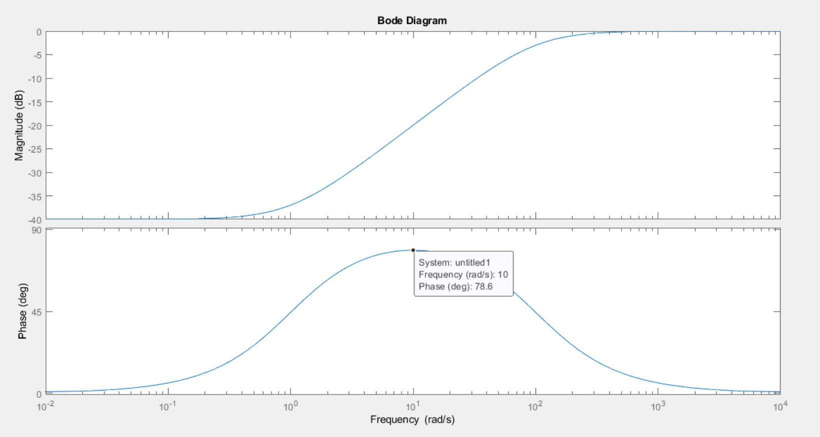

Figure: $\omega_m = 10 rad/sec$, $\phi_m = 78.6°$

Limitations of P, PI, PD

Limitations

- With the design of P, PI and PD controllers, we are in a position to ensure a wide range of tracking and transient responses for any given plant.
- However, the above assurance is usually under the condition that either tracking or transient response features drive the design of control element.
- In reality, we are likely to encounter a combination of steady-state and transient response specifications, so that employing any one of these would not be adequate.

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PID controller - Introduction

Introduction

- PID controller aims to achieve both tracking and transient response simultaneously and, hence, is a better option in comparison to either PI or PD.
- This is so because it includes all three actions which help in achieving a wide range of performance.
- Further, it manages the overall design effort well, while increasing the overall design degrees of freedom.

PID controller - Zeigler - Nichols

Zeigler - Nichols PID Design

- Zeigler - Nichols is a methodology for designing PID controllers, based on the specific assumptions about the unit step response of the plant.
- The controller transfer function is rewritten in terms of the overall gain K_p and time constants T_i and T_d , as shown below.

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right)$$

- There are two methods for arriving at the controller.

Ziegler Nichols - First Method

First Method

- This method applies if the response to a step input exhibits an S-shaped curve

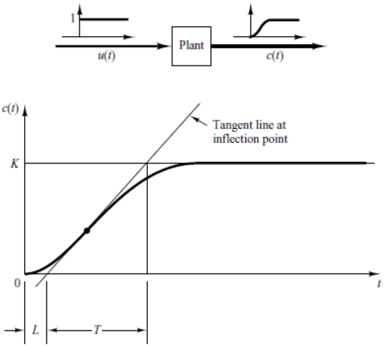

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Ziegler Nichols - First Method

- The S-shaped curve is characterized by two constants, delay time $L$and time constant T.
- The delay time and time constant are determined by drawing a tangent line at the inflection point of the S-shaped curve and determining the intersections of the tangent line with the time axis and line $c(t) = K$
- Transfer function may then be approximated by a first-order system with a transport lag as follows:

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts+1}$$

Ziegler Nichols - First Method

- Ziegler and Nichols suggested to set the values of K_p,T_i,T_d according to the following table,

Type of Controller	K_p	T_i	T_d
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

Ziegler Nichols - First Method

- PID controller tuned by the first method of Ziegler–Nichols rules gives:

$$
\begin{aligned}
G_c(s) &= K_p\left(1+\frac{1}{T_i s}+T_d s\right)\\
&= 1.2\frac{T}{L}\left(1+\frac{1}{2Ls}+0.5Ls\right)\\
&= 0.6T\frac{(s+\frac{1}{L})^2}{s}
\end{aligned}
$$

- Thus, the PID controller has a pole at the origin and double zeros at $s=-1/L$.

Ziegler Nichols - Second Method

Second Method

- In the second method, we first set $T_i = \infty$ and $T_d = 0$. Using the proportional control action only, increase K_p from 0 to a critical value K_{cr} at which the output first exhibits sustained oscillations.
- If the output does not exhibit sustained oscillations for whatever value K_p may take, then this method does not apply.
- Thus, the critical gain K_{cr} and the corresponding period P_{cr} are experimentally determined.

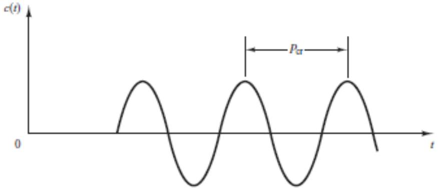

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Ziegler Nichols - Second Method

- Ziegler and Nichols suggested that we set the values of the parameters K_p , T_i , and T_d according to the table

Type of Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

Ziegler Nichols - Second Method

- PID controller tuned by the second method of Ziegler–Nichols rules gives:

$$\begin{aligned} G_c(s) &= K_p \left(1 + \frac{1}{T_i s} + T_d s\right) \\ &= 0.6K_{cr}\left(1 + \frac{1}{0.5P_{cr}s} + 0.125P_{cr}s\right) \\ &= 0.075K_{cr}P_{cr}\frac{(s+\frac{4}{P_{cr}})^2}{s} \end{aligned}$$

- Thus, the PID controller has a pole at the origin and double zeros at $s = -4/P_{cr}$.

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PID Controller Design - Bode

PID control design using Bode

- Zeigler-Nichols method of tuning broadly aims to arrive at a stable closed loop system with acceptable transient response.
- However, a more focused design can be done using frequency domain methods, which take into account the design specifications.
- In this method, following generic form of the PID controller is assumed

$$G_{PID}(s) = \frac{K(as+1)(bs+1)}{s}$$

PID Controller Design - Example

Example: Consider a system as given below.

$$G(s) = \frac{1}{s^2 + 1}$$

- Design a PID controller so that K_v is 4, PM is at least 50° and GM is more than $10dB$.

PID Controller Design - Example

Solution:

- PID controller is assumed as,

$$G_{PID}(s) = \frac{K(as+1)(bs+1)}{s}$$

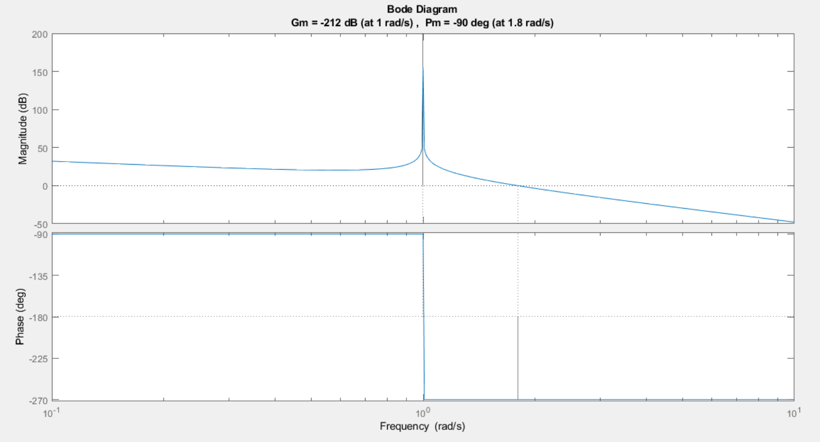
- First step is to achieve the specified K_v which can be done by making $K = 4$

$$K_v = \lim_{s \to 0} sG(s)G_{PID}(s)$$

$$\implies \frac{K}{1} = 4, \quad K = 4$$

PID Controller Design - Example

Bode plot of $\frac{4}{s(s^2+1)}$



- We see that we need to add a large positive phase at the GCO.

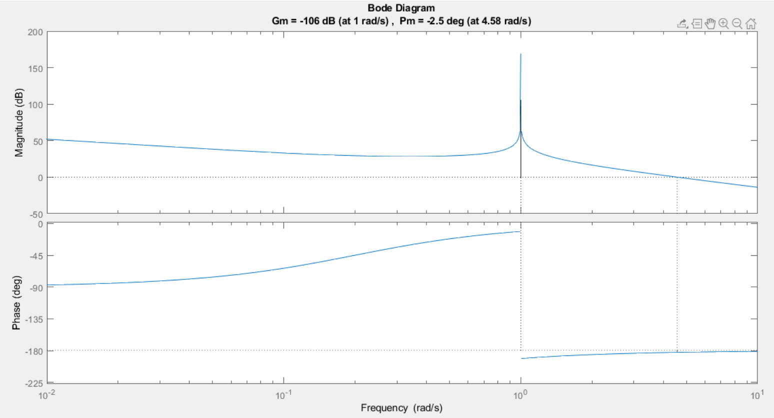
PID Controller Design - Example

- Large positive phase at the GCO can be added by first choosing value of 'a' to be large (say 5) (i.e. zero at -0.2)
- This acts as the PI controller, as shown below.

$$G''(s) = \frac{5s+1}{s} \times \frac{4}{s^2+1}$$

PID Controller Design - Example

Bode plot of $G''(s)$



- We see that, while GCO increases from 1.8 to 4.58, PCO remains unchanged.

PID Controller Design - Example

- We now choose ‘b’ so that PM requirement is met. This also will change PCO by adding more positive phase in low frequency regime.
- Approximate solution for PD controller can be the one which add 50° phase.

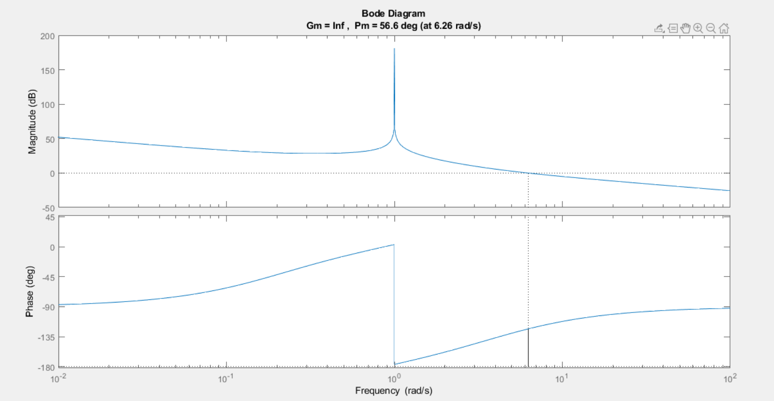
$$\angle(1+bs)|_{\omega=4.58} = 50^\circ$$

$$\implies 4.58b = \tan 50^\circ, \quad b = 0.26$$

$$G'''(s) = (1+0.26s)\frac{5s+1}{s} \times \frac{4}{s^2+1}$$

PID Controller Design - Example

Bode plot of $G'''(s)$



PID Controller Design - Observations

Observations

- Desired conditions are met
- We see that, as GCO increases to 6.14, PM is more than required.
- Further, as there is no PCO, GM becomes infinite.

References

- Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini: "*Feedback Control of Dynamic Systems*", Pearson Education, Inc., Upper Saddle River, New Jersey, Seventh Edition, 2015.
- Katsuhiko Ogata: "*Modern Control Engineering*", Pearson Education, Inc., Upper Saddle River, New Jersey, Fifth Edition, 2010.
- Farid Golnaraghi and Benjamin C. Kuo: "*Automatic Control Systems*", John Wiley & Sons, Inc., New Jersey, Ninth Edition, 2010.
- Ashok Joshi: "*System Modeling Dynamics and Control*", Lecture Notes, IIT Bombay, Mumbai, 2019.