AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 2: System Modeling and Dynamics - I

Dr. Arnab Maity
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Big Picture - System Identification Problem

System identification problem

- As a practising engineer, a model of the system is not always readily available.
- The process of determining a mathematical model is called **system identification**.
- Relevant questions regarding the system identification are
  - How to model the system that we are trying to control?
  - What is relevant dynamics for the system?
  - What are mathematical equations that convert known inputs to measured outputs?
- These can be answered in two ways as the following.
- The first is referred as **black box method**. Imagine that you do not know anything about the system.
- One can subject the material in box to various inputs and measured outputs, and infer what is in the box based on the relationship between inputs and outputs.

Big Picture - System Identification Problem

- The second way is to perform through **white box method**.
- Imagine you know all the components inside the box.
- This is exactly similar to the Newton's method or determining equations of motion based on energy in the system.

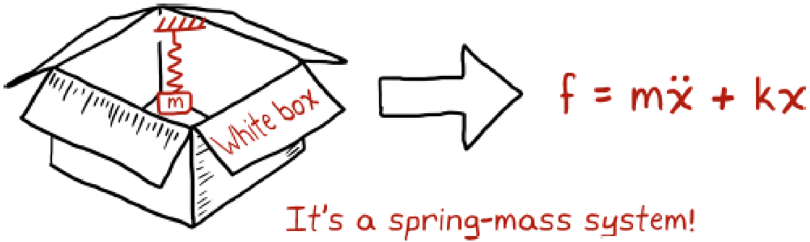


Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

System Modeling

Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- People did not know to model/ predict the dynamic behaviour of things.
- Then, this happened.

System Modeling

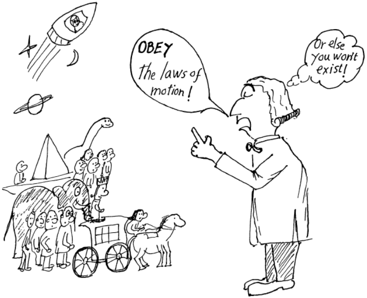


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- Science of math modeling was born in 17th century.

System Modeling

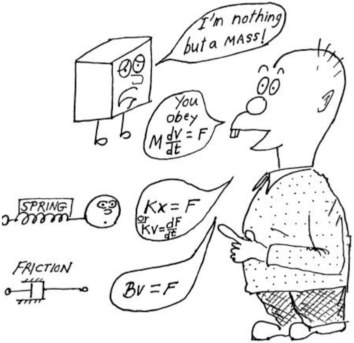


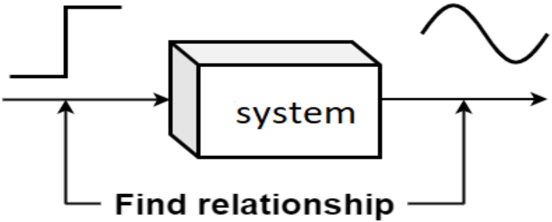
Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- The art of analyzing physical systems through math models was introduced in 18th centuries.
- M: mass, V: Velocity, F: Force, K: spring constant, B: damping coefficient

Basics of Modeling

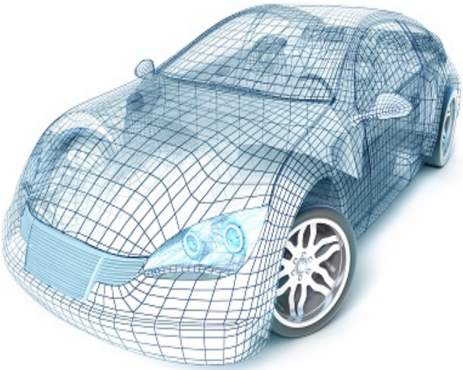
Role of Modeling

- In order to proceed with control design, we need to estimate the deficiencies that exist in the Plant/Process.
    - Examine the behavior of the plant under operating conditions
    - Requires a methodology for generating relevant responses



Basics of Modeling

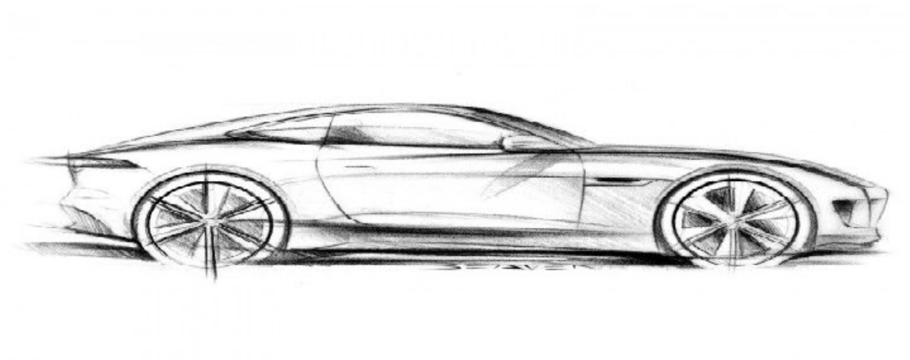
What is a Model?



- Model is a view of the system that captures the objectives to be satisfied by the system.
- Model represents an imitation of reality, in terms of those features that describe the operation of any system.

Types of Models

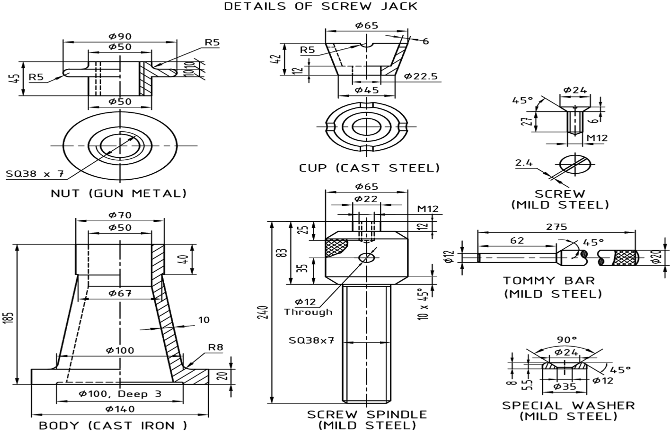
Sketch



- Most common, easy to understand
- Useful for explaining concepts

Types of Models

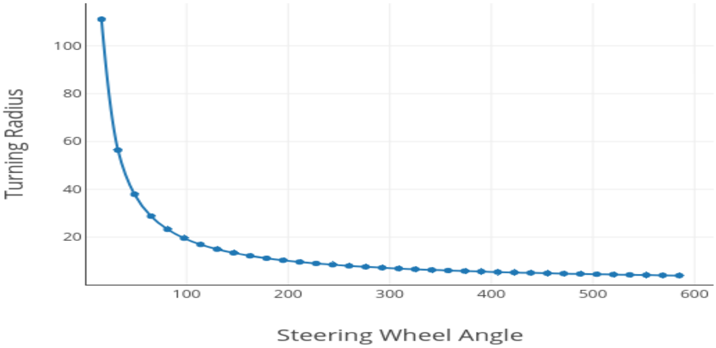
Drawing



- Drives the manufacturing process

Types of Models

Design Data



- Model is in the form of data points corresponding to system behavior.

Types of Models

Schematic

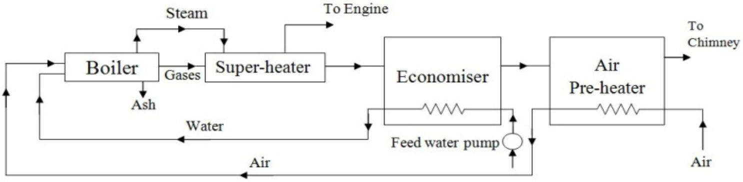


Fig- Schematic diagram of a boiler plant

- Provides overview of system process/components
- Includes data/information flow

Types of Models

Analogy

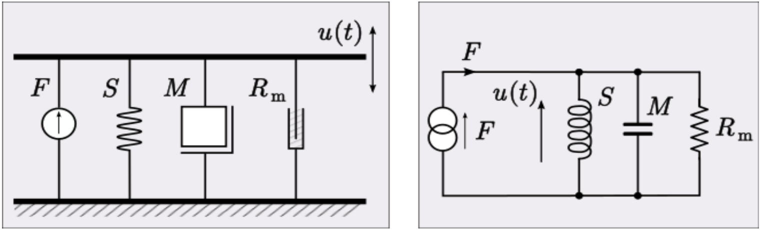
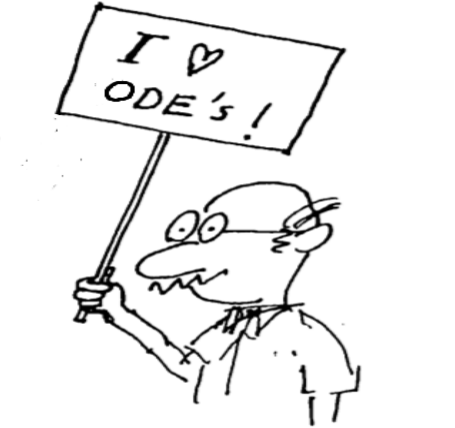


Figure: Left image: Mechanical system, Right image: Electrical system

- Brings equivalence between different disciplines
- Helps in quick assessments of performance at low costs

Types of Models

Mathematical



- Newton's second law:

$$F = Ma_{cm}$$

- Spring-mass-damper system:

$$F(t) = m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx$$

- RLC circuit:

$$\frac{dV}{dt} = \frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I}{LC}$$

- Tries to capture system features in **mathematical framework**

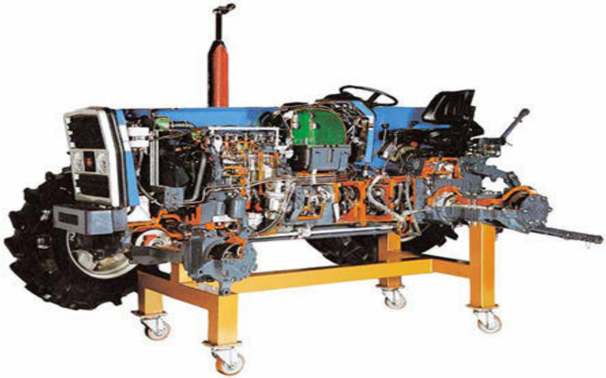
Types of Models

Mockup

- Provides full scale size feel

Types of Models

Cut Section



- Provides internal layout
- Helps in re-engineering

Types of Models

Scaled Test

- Important aid in verifying designs
- Concepts through less expensive lab level tests

Choice of Model Type

- In the context of control, models are generally mathematical or experimental.
- The choice depends on knowledge base and resources.
- **Mathematical model** is used when
    - Valid solvable theory exists
    - Necessary computational resources exist
- **Experimental model** is used when
    - Mathematical techniques are inadequate

Comparison of Model Types

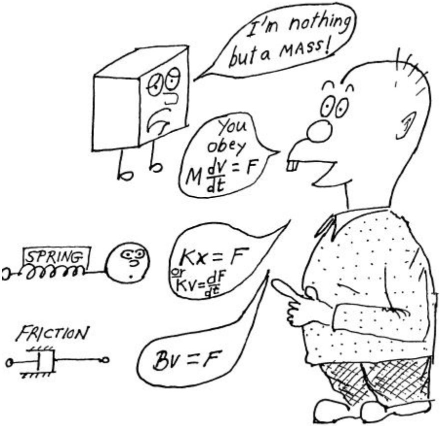
- **Mathematical Model**
  - Easy to build
  - Less expensive
  - Less accurate
- **Experimental Model**
  - More realistic
  - Difficult to synthesis
  - Expensive
- *As a first step, the models employed for control analysis and design are mathematical in nature.*

Mathematical Model

- A control system may be composed of various components
  - Mechanical
  - Thermal
  - Fluid
  - Pneumatic
  - Electrical
  - Sensors
  - Actuators
  - Computers
- Model must capture all the dynamics of these components.
- In general, such models can be created from **First Principles**.

Mathematical Model

First Principles



- We can employ basic laws of physics.
- All of the components/ processes involve
  - Mechanics
  - Thermodynamics
  - Fluid dynamics
  - Electrical
  - Magnetism
- This method gives idealized behavior, and ignores non-essential features (as certain assumptions are made).

Mathematical Model

First principle models are inaccurate.

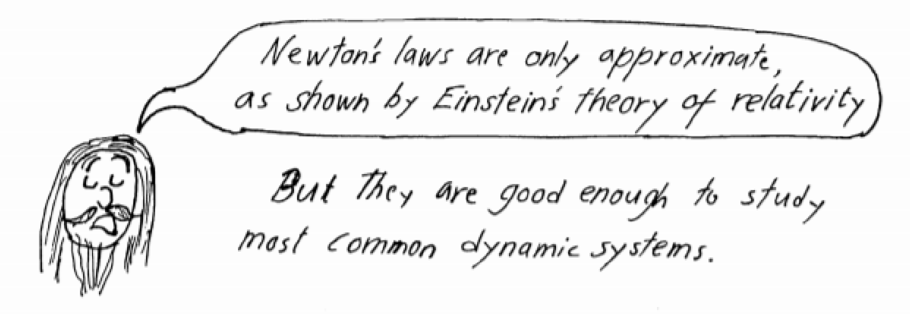


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

Mathematical Model

Mathematical models are inaccurate due to

- Neglecting less essential parameters
  - Temperature effect on resistance
  - Higher-order terms of C_l in calculating C_d
- Measurement errors
  - Mass cannot be measured accurately.
- Theories based on assumptions
  - Assumption in Bernoulli's theorem like - Fluid is incompressible, non-viscous and steady.

Mathematical Model - Mechanical System

Translational Motion

- Linear Spring
  - The force acting on the spring is directly proportional to the displacement/deformation

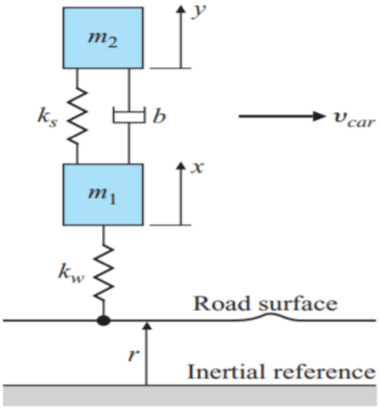
$$f(t) = ky(t)$$

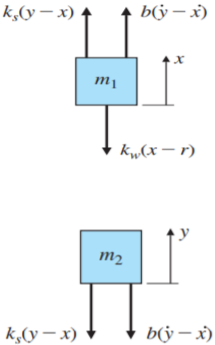
- Linear Damper
  - The force acting on the damper is directly proportional to the velocity

$$f(t) = b\frac{dy(t)}{dt}$$

Mathematical Model - Mechanical System

Two-mass system: Suspension model





Mathematical Model - Mechanical System

Two-mass system: Suspension model (cont...)

- Force balance provides

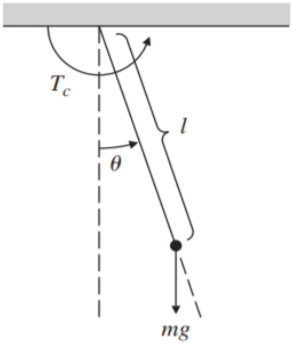
$$\begin{aligned} b(\dot{y}-\dot{x}) + k_s(y-x) - k_w(x-r) &= m_1\ddot{x} \\ -k_s(y-x) - b(\dot{y}-\dot{x}) &= m_2\ddot{y} \end{aligned}$$

- Some rearrangement gives

$$\begin{aligned} \ddot{x} + \frac{b}{m_1}(\dot{x}-\dot{y}) + \frac{k_s}{m_1}(x-y) + \frac{k_w}{m_1}x &= \frac{k_w}{m_1}r \\ \ddot{y} + \frac{b}{m_2}(\dot{y}-\dot{x}) + \frac{k_s}{m_2}(y-x) &= 0 \end{aligned}$$

Mathematical Model - Mechanical System

Rotational motion - Pendulum



- The moment of inertia about pivot point is
$$I = ml^2$$
- The equation of motion can be obtained using torque balance
$$T_c - mgl\sin\theta = I\ddot{\theta}$$
- On rearranging
$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$$

Mathematical Model - Electrical System

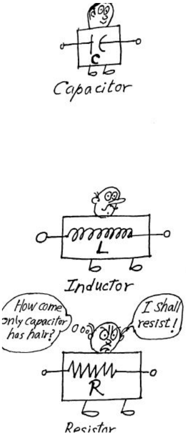


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- Elements of electrical circuits:

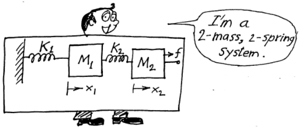
$$i = c\frac{dv}{dt}$$

$$v = L\frac{di}{dt}$$

$$v = Ri$$

i: current (Amperes), v: voltage (volts), R: resistor (ohms), c: capacitor (Columbs), L: inductance (Henries)

Mathematical Model - Electrical System



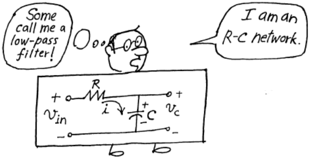


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- Input: f (force) & Output: x_2

$$m_1\ddot{x}_1 = k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = f + k_2(x_1 - x_2)$$

- Input: v_{in} & Output: v_c

$$i = c\frac{dv}{dt}$$
$$v_c = v_{in} + ri$$
$$v_c = v_{in} + rc\frac{dv}{dt}$$

Mathematical Model - Electrical System

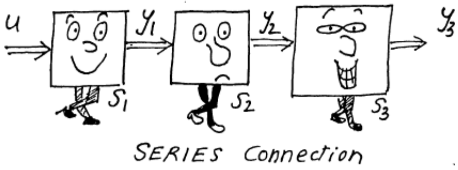
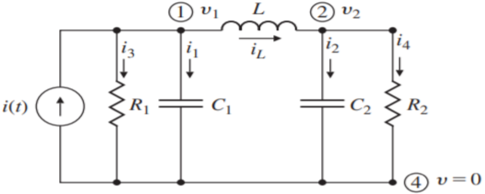


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- System is represented by block or box which has inputs and outputs.
- Systems can be connected to each other in series to form a new system.

Mathematical Model - Electrical System

Electrical Circuit

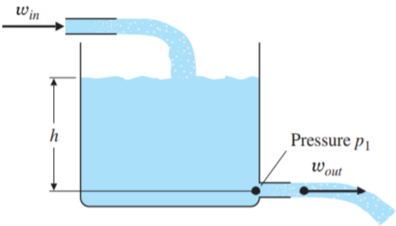


$i_3 = \dfrac{v_1}{R_1}$	$i_1 = C_1\dfrac{dv_1}{dt}$	$i_2 = C_2\dfrac{dv_2}{dt}$	$i_4 = \dfrac{v_2}{R_2}$	$v_1 - v_2 = L\dfrac{di_L}{dt}$

$i(t) = \dfrac{v_1}{R_1} + C_1\dfrac{dv_1}{dt} + i_L$	$i_L = C_2\dfrac{dv_2}{dt} + \dfrac{v_2}{R_2}$	$v_1 = L\dfrac{di_L}{dt} + v_2$

Mathematical Model - Fluid System

Water Tank 1

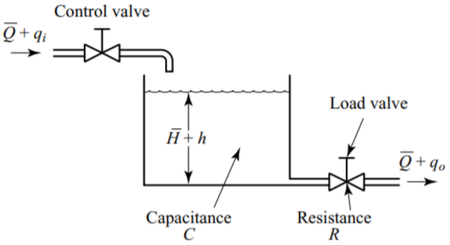


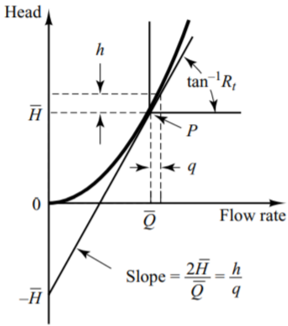
- Assuming incompressible flow

$$\dot{h} = \frac{1}{A_\rho}(w_{in} - w_{out})$$

Mathematical Model - Fluid System

Water Tank 2





Mathematical Model - Fluid System

Water Tank 2 (cont...)

- Resistance of Liquid level system

$$R = \frac{\text{change in level difference}, m}{\text{change in flow rate}, m^3/sec} = \frac{dH}{dQ}$$

- The steady state flow rate (for turbulent flow) is given by

$$Q = K\sqrt{H}$$

- The resistance R_t for turbulent flow is

$$R_t = \frac{2H}{Q} \rightarrow Q = \frac{2H}{R_t}$$

Mathematical Model - Fluid System

Water Tank 2 (cont...)

- Capacitance of Liquid level system

$$C = \frac{\text{change in liquid stored}, m^3}{\text{change in head}, m} = \frac{(q_i - q_o)dt}{dh}$$

- We see that,

$$Cdh = (q_i - q_0)dt$$

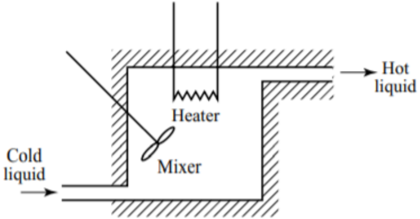
- From definition of resistance

$$q_0 = \frac{2h}{R}$$

- The differential equation for this system for a constant value of R becomes

$$RC\frac{dh}{dt} + h = Rq_i$$

Mathematical Model - Thermal System



- Conductive or convective heat transfer coefficient

$$q = K\Delta\theta$$

where,

$$\begin{aligned} q &= \text{heat flow rate, } kcal/sec \\ \Delta\theta &= \text{temperature difference, } {}^0C \\ K &= \text{coefficient, } kcal/sec\,{}^0C \end{aligned}$$

Mathematical Model - Thermal System

- The coefficient K is given by

$$K = \frac{kA}{\Delta x}, \text{ for conducition}$$
$$= HA, \text{ for convection}$$

where,

$$k = \text{thermal conductivity, kcal/m } sec\,^0C$$
$$A = \text{area normal to heat flow, } m^2$$
$$\Delta x = \text{thickness of conductor, } m$$
$$H = \text{convection coefficient, } kcal/m^2 sec\,^0C$$

Mathematical Model - Thermal System

- Assume that the temperature of the inflowing liquid is kept constant.
- Let the heat input rate to the system (heat supplied by the heater) is changed from Q_0 to $Q_0 + q_i$.
- The heat outflow rate then changes gradually to $Q_0 + q_o$.
- The temperature of the outflow liquid also be changed from θ_0 to $\theta_0 + \theta$.
- Hence, the change in temperature is θ and the change in output heat flow rate is q_o.
- As per definition of the thermal resistance,

$$R = \frac{\theta}{q_0}.$$

Mathematical Model - Thermal System

- Thermal resistance R is defined by

$$R = \frac{\text{change in temperature difference,}^0 C}{\text{change in heat flow rate,} \, kcal/sec} = \frac{\Delta\theta}{q} = \frac{\Delta\theta}{K\Delta\theta}$$

$$R = 1/K$$

- Thermal capacitance C is defined by

$$C = \frac{\text{change in heat stored,} \, kcal}{\text{change in temperature,}^0 C} = \frac{mc\Delta\theta}{\Delta\theta} = \frac{(q_i - q_o)dt}{d\theta}$$

$$C = mc$$

where,

$m =$ mass of substance considered, Kg
$c =$ specific heat of substance, $kcal/Kg\,^0C$

Mathematical Model - Thermal System

- The heat balance equation for this system is

$$C d\theta = (q_i - q_o) dt.$$

- R can be obtained as

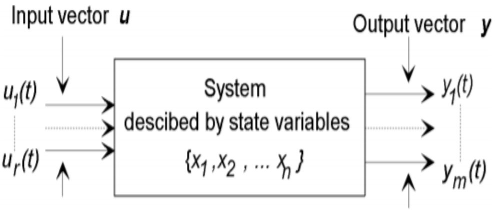
$$R = \frac{\theta}{q_0}.$$

- Rearranging the equations gives

$$RC \frac{d\theta}{dt} + \theta = R q_i.$$

Mathematical Model - State Space

- **State** - Minimum set of variables, known as **state variables**, that fully describe the system and its response to any given set of inputs



Mathematical Model - State Space

State Space Representation Principles

1. Identify the states of the system such as
    - position
    - velocity
    - inductor current
    - capacitor voltage
2. Use physics to find $\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}$
3. Organize as

$$\frac{d\boldsymbol{x}}{dt} = f(\boldsymbol{x}, \boldsymbol{u})$$

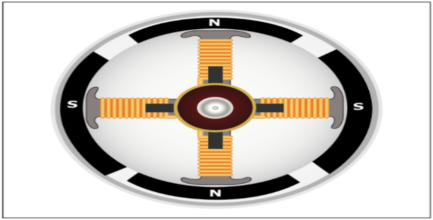
where,

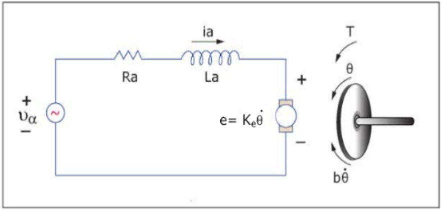
$\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ is the state vector

$\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^T$ is the control input vector

Mathematical Model - State Space

DC Motor





Mathematical Model - State Space

DC Motor (cont...)

- The states are:

$$x_1 = \theta - \text{motor angle}$$
$$x_1 = \dot{\theta} - \text{motor angular velocity}$$
$$x_3 = i_a - \text{armature current}$$

- Equations of motion:

$$\dot{x_1} = \frac{d\theta}{dt} = \dot{\theta} = x_2$$
$$\dot{x_2} = \frac{d\ddot{\theta}_1}{dt} = \ddot{\theta}$$

- Balancing the torque from the free body diagram:

$$J\ddot{\theta} = -b\dot{\theta} + T$$
$$-b\dot{\theta} = \text{viscous drag on rotor}$$
$$T = \text{torque due to current} = K_t i_a$$

Mathematical Model - State Space

DC Motor (cont...)

- So, the following can be written as

$$\ddot{\theta} = -\frac{b}{J}\dot{\theta} + \frac{K_t}{J} i_a$$
$$\dot{x_2} = -\frac{b}{J} x_2 + \frac{K_t}{J} x_3$$

- The power supplied to the motor is

$$P = i_a e = T\dot{\theta} = K_t i_a \dot{\theta} \implies e = K_t \dot{\theta}$$

- Now we can find di_a/dt using the Kirchhoff's voltage law

$$\begin{aligned} \frac{di_a}{dt} &= \frac{1}{L}(v_a - i_a R_a - e) \\ &= \frac{1}{L}(v_a - i_a R_a - K_t \dot{\theta}) \end{aligned}$$

Mathematical Model - State Space

DC Motor (cont...)

- Therefore,

$$\dot{x_3} = -\frac{k_t}{L}x_2 - \frac{R_a}{L} + \frac{1}{L}v_a$$

- The state space form is represented by

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K_t}{J} \\ 0 & -\frac{-K_t}{L} & -\frac{-R_a}{L} \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L} \end{pmatrix} v_a$$

References I

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