AE 308: Control Theory

AE 775: System Modelling, Dynamics & Control

Lecture 6: Laplace Transform and Transfer Function

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Laplace Transform - Introduction

Definition

- Laplace transform is defined as

$$\boxed{\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt}$$

where,

$s = \sigma + j\omega$

$F(s)$ is called the **Laplace transform of** $f(t)$

- Inverse Laplace transform is defined as

$$\boxed{\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = f(t)u(t)}$$

Laplace Transform - Introduction

Multiplication by a constant

- Let k be a constant and $F(s)$ be the Laplace transform of $f(t)$. Then

$$\mathscr{L}\left[kf(t)\right] = kF(s)$$

Sum and Difference

- Let $F_1(s)$ and $F_2(s)$ be the Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively. Then

$$\mathscr{L}\left[f_1(t) \pm f_2(t)\right] = F_1(s) \pm F_2(s)$$

Laplace Transform - Introduction

Differentiation

- Let $F(s)$ be the Laplace transform of $f(t)$, and $f(0)$ is the limit of $f(t)$ as t approaches to 0. The Laplace transform of time derivative of $f(t)$ is

$$\mathscr{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

- In general, for higher order derivative of $f(t)$,

$$\mathscr{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ \ldots \ f^{n-1}(0)$$

Laplace Transform - Introduction

Integration

- The Laplace transform of the first integral of $f(t)$ with respect to t is the Laplace transform of $f(t)$ divided by s, that is,

$$\mathscr{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s}$$

- For n^{th} order integration

$$\mathscr{L}\left[\int_0^{t_n}\int_0^{t_{n-1}} \dots \int_0^{t_1} f(t)d\tau dt_1 dt_2 \, \dots \, dt_{n-1}\right] = \frac{F(s)}{s^n}$$

Laplace Transform - Introduction

Shift in Time

- The Laplace transform of $f(t)$ delayed by time T is equal to the Laplace transform of $f(t)$ multiplied by e^{-Ts}, that is

$$\mathscr{L}\left[f(t-T)u_s(t-T)\right] = e^{-Ts}F(s)$$

where $u_s(t-T)$ denotes the unit-step function that is shifted in time to the right by T.

Laplace Transform - Theorems

Initial Value Theorem

- Let the Laplace transform of $f(t)$ be $F(s)$, then

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s),$$

if the limit exists.

Final Value Theorem

- Let the Laplace transform of $f(t)$ be $F(s)$, then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s),$$

if the limit exists.

Laplace Transform - Common Signals

Laplace transform of some common signals

Signal $u(t)$	Laplace transform $U(s)$	Signal $u(t)$	Laplace transform $U(s)$
$S(t)$ [unit step]	$\frac{1}{s}$	$\delta(t)$ [impulse]	1
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\cos(at)$	$\frac{s}{s^2+a^2}$
$e^{-\alpha t}\sin(at)$	$\frac{a}{(s+\alpha)^2+a^2}$	$e^{-\alpha t}\cos(at)$	$\frac{s+\alpha}{(s+\alpha)^2+a^2}$

Figure: *Source - "Feedback Systems - An Introduction for Scientists and Engineers", by K. J. Åström and R. M. Murray*

Laplace Transform - Standard Results

Laplace transform of some standard functions

No.	function	LT
1	$\delta(t)$	1
2	$1(t)$	$1/s$
3	t	$1/s^2$
4	t^2	$2!/s^3$
5	t^3	$3!/s^4$
6	t^m	$m!/s^{m+1}$

No.	function	LT
7	e^{-at}	$\frac{a}{s+a}$
8	te^{-at}	$\frac{1}{(s+a)^2}$
9	$\frac{1}{2!}t^2e^{-at}$	$\frac{1}{(s+a)^3}$
10	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	$\frac{1}{(s+a)^m}$

Laplace Transform - Standard Results

Laplace transform of some standard functions

No.	function	LT
11	$1 - e^{-at}$	$\dfrac{a}{s(s+a)}$
12	$\frac{1}{a}(at - 1 + e^{-at})$	$\dfrac{a}{s^2(s+a)}$
13	$e^{-at} - e^{-bt}$	$\dfrac{b-a}{(s+a)(s+b)}$
14	$(1 - at)e^{-at}$	$\dfrac{s}{(s+a)^2 s}$

Laplace Transform - Standard Results

Laplace transform of some standard functions

No.	function	LT
15	$1 - e^{-at}(1 + at)$	$\dfrac{a^2}{s(s+a)^2}$
16	$be^{-bt} - ae^{-at}$	$\dfrac{(b-a)s}{(s+a)(s+b)}$
17	$\sin at$	$\dfrac{a}{s^2+a^2}$
18	$\cos at$	$\dfrac{s}{s^2+a^2}$

Laplace Transform - Standard Results

Laplace transform of some standard functions

No.	function	LT
19	$e^{-at}\cos at$	$\dfrac{s+a}{(s+a)^2+b^2}$
20	$e^{-at}\sin at$	$\dfrac{b}{(s+a)^2+b^2}$
21	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$	$\dfrac{a^2+b^2}{s\left[(s+a)^2+b^2\right]}$

Laplace Transform - Features

Properties of Laplace transform

1. The homogeneous equation and the particular integral of the solution of the differential equation are obtained in **one operation**.
2. The Laplace transform converts the differential equation into an **algebraic equation in s-domain**. It is then possible to manipulate the algebraic equation by simple algebraic rules to obtain the solution in the s-domain. The final solution is obtained by taking the inverse Laplace transform.

Limitation

- Laplace transform is a linear operation and hence it is applicable only in the context of LTI systems.

Laplace Transform - Example

2^{nd} order LTI system

- The system is given by

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 r(t)$$

- Take Laplace transform of each term

$$\left[s^2 Y(s) - \dot{y}(0) - sy(0)\right] + 2\zeta\omega_n \left[sY(s) - y(0)\right] + \omega_n^2 Y(s) = \omega_n^2 R(s)$$

- After rearranging

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \left[\dot{y}(0) + (s + 2\zeta\omega_n)y(0)\right] + \omega_n^2 R(s)$$

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Transfer Function - Introduction

Definition

- **Transfer Function** is defined as ratio of the Laplace transforms of output and input for a system **under zero initial condition**.
- Transfer Function $G(s) = \left.\dfrac{\mathscr{L}(\text{output})}{\mathscr{L}(\text{input})}\right|_{\text{zero initial condition}}$
- LTI representation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_0 \frac{d^m u}{dt^m} + \ldots + b_m u$$

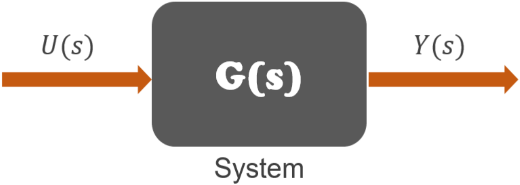
- Take Laplace Transform of term by term

$$(a_0 s^n + \ \ldots \ + a_{n-1} s + a_n) Y(s) = (b_0 s^m + b_1 s^{m-1} + \ \ldots \ + b_{m-1} s + b_m) U(s)$$

Transfer Function - Introduction

(cont...)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ \dots \ + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ \dots \ + a_{n-1} s + a_n}$$



Transfer Function - Features

- Transfer function is the s-domain unit impulse response

$$Y(s) = G(s).U(s) \rightarrow \text{for } U(s) = \delta(s) = 1, Y(s) = G(s)$$

- In general, $G(s)$ is represented in polynomial and factored form

$$\begin{aligned} G(S) &= K\frac{s^m + b_1 s^{n-1} + \ \dots \ + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ \dots \ + a_{n-1} + a_n} \\ &= K\frac{(s - z_1)(s - z_2) \ \dots \ (s - z_m)}{s^k(s - p_{k+1})(s - p_{k+2}) \ \dots \ (s - p_n)} \end{aligned}$$

where

- p_i's $\rightarrow$ poles, roots of the denominator polynomial
- z_j's $\rightarrow$ zeros, roots of the numerator polynomial
- K $\rightarrow$ gain parameter
- k $\rightarrow$ system type

Transfer Function - Example

2^{nd} order LTI system

- We already established the relation

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = [\dot{y}(0) + (s + 2\zeta\omega_n)y(0)] + \omega_n^2 R(s)$$

- We know zero initial condition is must for Transfer function

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \omega_n^2 R(s)$$

- Applying definition of TF

$$G(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Transfer Function - Example

2^{nd} order LTI system (cont...)

- Try to factorize the denominator

$$G(s) = \frac{\omega_n^2}{\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]} = \frac{\omega_n^2}{\left[\left(s + \zeta\omega_n\right)^2 + \omega_d^2\right]}$$

where $\omega_d^2 = \omega_n^2(1 - \zeta^2)$

- We get

$$G(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + j\omega_d\right)\left(s + \zeta\omega_n - j\omega_d\right)}$$

LTI System Response - Transfer Function

Transfer function for generating responses

- Transfer functions are used to generate time responses of LTI systems based on the principle of superposition.
- This involves
  1. decomposing $Y(s)$ into its characteristic components
  2. mapping these components to their time domain counterparts
- Decomposition is based on the premise that any complex LTI system can be synthesized as a linear combination of 1^{st} and 2^{nd} order terms.
- Partial fraction is standard method for decomposing.

LTI System Response - Transfer Function

Partial Fraction

- Partial fraction decomposition uses method of residues to decompose a n^{th} order fraction into a set of n, 1^{st} order fractions.
- Consider n^{th} order system, along with its decomposed form, as given below.

$$G(s) = K \frac{(s - z_1)(s - z_2)\ \ldots\ (s - z_m)}{s^k (s - p_{k+1})(s - p_{k+2})\ \ldots\ (s - p_n)}$$

$$G(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \ \ldots\ + \frac{A_n}{s - p_n}$$

where, $A_1, \ldots, A_n$ are called **Residuals**.

LTI System Response - Transfer Function

Partial Fraction (cont...)

- The residues represent the contributions of each of the factors to the total response and are obtained as follows
  - Distinct poles

$$A_i = \left[(s-p_i)Y(s)\right]\big|_{s=p_i}$$

  - Multiple poles at same location

$$G(s) = K\frac{(s-z_1)(s-z_2)\ \dots\ (s-z_m)}{(s-p_1)^k(s-p_{k+1}\ \dots\ +(s-p_n))}$$

$$Y(s) = \frac{A_1}{s-p_1} + \frac{A_2}{(s-p_1)^2} + \ \dots\ + \frac{A_k}{(s-p_1)^k} + \ \dots\ + \frac{A_i}{s-p_i}$$

$$A_j = \frac{1}{(k-j)!}\frac{d^{k-j}}{ds^{k-j}}\left[(s-p_1)^kY(s)\right]\Big|_{s=p_1}$$

where, $j=1,\ \dots\ k$; $i=k+1,\ \dots\ n$

LTI System Response - Transfer Function

Partial Fraction Example - Distinct Roots

- Consider a TF and obtain unit impulse response

$$G(s) = \frac{(s+3)}{(s+1)(s+2)}$$

Hence, $p_1 = -1$ and $p_2 = -2$

- We know for unit impulse response $Y(s) = G(s)$

$$Y(s) = \frac{(s+3)}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

- Solution for finding the residuals

$$A_1 = [(s+1)Y(s)] = \left[\frac{(s+3)}{(s+2)}\right]\Bigg|_{s=-1} = 2$$

$$A_2 = [(s+2)Y(s)] = \left[\frac{(s+3)}{(s+1)}\right]\Bigg|_{s=-2} = -1$$

LTI System Response - Transfer Function

Partial Fraction Example - Distinct Roots (cont...)

- Factorized form becomes

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+2}$$

- Take the inverse Laplace transformation

$$y(t) = \mathscr{L}^{-1}\left[Y(s)\right] = \mathscr{L}^{-1}\left[\frac{2}{s+1}\right] - \mathscr{L}^{-1}\left[\frac{1}{s+2}\right]$$

.

- Thus, the impulse response of given system is

$$\boldsymbol{y(t) = 2e^{-t} - e^{-2t}}$$

LTI System Response - Transfer Function

Partial Fraction Example - Repeated Roots

- Obtain the unit impulse response of given system

$$G(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

Hence, $p_1 = -1, -1, -1.$

- We know for unit impulse response $Y(s) = G(s)$

$$Y(s) = \frac{s^2 + 2s + 3}{(s+1)^3} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

LTI System R2esponse - Transfer Function

Partial Fraction Example - Repeated Roots (cont...)

- Solution for finding the residuals

$$b_3 = \left[(s+1)^3 Y(s)\right]\Big|_{s=-1} = 2$$

$$b_2 = \frac{d}{ds}\left[(s+1)^3 Y(s)\right]\Big|_{s=-1} = 0$$

$$b_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3 Y(s)\right]\Big|_{s=-1} = 1$$

- Factorized form becomes

$$Y(s) = \frac{1}{s+1} + \frac{2}{(s+1)^3}$$

- Thus, the impulse response of given system is

$$(1+t^2)e^{-t}$$

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