AE 308: Control Theory
AE 775: System Modelling, Dynamics & Control
Lecture 21: Lead-Lag Compensator Design

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Lag Compensator - Introduction

Design Procedure

- Set the gain, K, to the value that satisfies the steady-state error specification and plot the Bode magnitude and phase diagrams for this value of gain
- Find the frequency where the phase margin is 5 to 12 greater than the phase margin that yields the desired transient response. This step compensates for the fact that the phase of the lag compensator may still contribute anywhere from -5^0 to -12^0 of phase at the phase-margin frequency

Lag Compensator - Introduction

Design Procedure

- Select a lag compensator whose magnitude response yields a composite Bode magnitude diagram that goes through 0 dB at the frequency found in Step 2 as follows:
    - Draw the compensator's high-frequency asymptote to yield 0 dB for the compensated system at the frequency found in above step. Thus, if the gain at the frequency found in Step 2 is $20\log K_{PM}$, then the compensator's high-frequency asymptote will be set at $-20\log K_{PM}$
    - select the upper break frequency to be 1 decade below the frequency found in Step 2
    - select the low-frequency asymptote to be at 0 dB
    - connect the compensator's high and low-frequency asymptotes with a -20 dB/decade line to locate the lower break frequency
- Reset the system gain, K, to compensate for any attenuation in the lag network in order to keep the static error constant the same as that found in earlier steps

Lag Compensator - Example

Example: A system is defined by the following transfer function,

$$G(s) = \frac{58390}{s(s+100)(s+36)}$$

- With the help of Bode diagrams design a lag compensator to yield a tenfold improvement in steady-state error while keeping the percent overshoot around 9.5%.

Lag Compensator - Example

Solution:

- Initially K_v is given by,

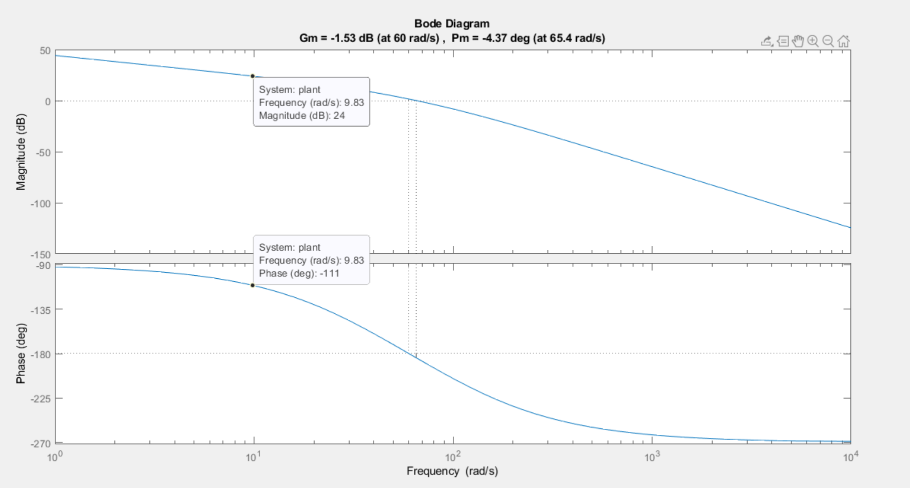
$$K_v = \lim_{s \to 0} sG(s) = \frac{58390}{3600} = 16.21$$

- For a tenfold improvement in steady-state error, K_v must increase by a factor of 10 i.e. 162.1, therefore new transfer function becomes,

$$G^{'}(s) = \frac{583900}{s(s+100)(s+36)}$$

Lag Compensator - Example

Bode plot of $G'(s)$



Lag Compensator - Example

- We know,

$$\text{Percentage overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$9.5\% \text{ overshoot} \implies \zeta = 0.6$$

- We also know,

$$\text{Phase margin} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$

$$\zeta = 0.6 \implies \text{PM} = 59.2^\circ$$

- The phase margin required for a 9.5% overshoot is found to be 59.2°.

Lag Compensator - Example

- We increase this value of phase margin by 10° to 69.2° in order to compensate for the phase angle contribution of the lag compensator.
- Now find the frequency where the phase margin is 69.2°.
- This frequency occurs at a phase angle of,

$$-180^\circ + 69.2^\circ = -110.8^\circ$$

  which is 9.8 rad/sec.
- At this frequency, the magnitude plot must go through 0 dB.
- The magnitude at 9.8 rad/s is now $+24$ dB. Thus, the lag compensator must provide -24 dB attenuation at 9.8 rad/s.

Lag Compensator - Example

Compensator Design

- First draw the high-frequency asymptote at -24 dB
- Arbitrarily select the higher break frequency to be about one decade below the phase-margin frequency, or 0.98 rad/s.
- Starting at the intersection of this frequency with the lag compensator's high-frequency asymptote, draw a -20 dB/decade line until 0 dB is reached.
- The lower break frequency is found to be 0.062 rad/s.
- Hence, the lag compensator's transfer function is,

$$G_c(s) = \frac{0.063(s + 0.98)}{s + 0.062}$$

Here, gain of the compensator is 0.063 to yield a dc gain of unity.

Lag Compensator - Example

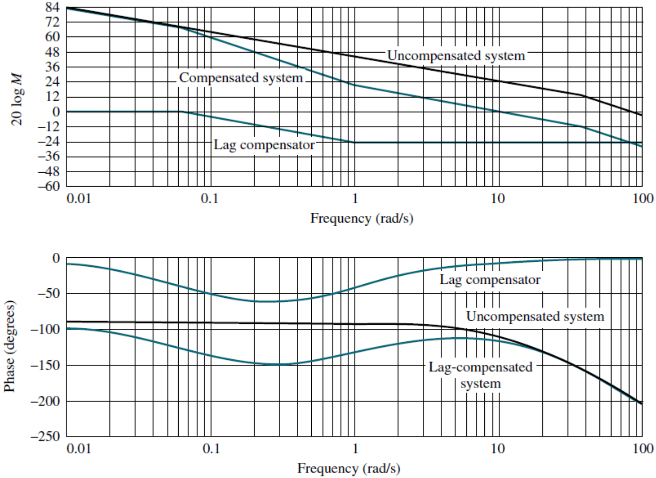
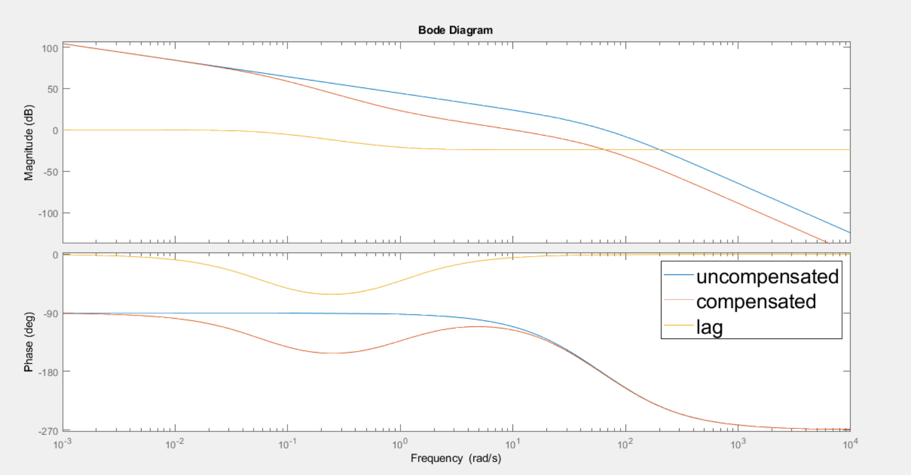


Figure: *Source -"Control Systems Engineering" by Norman S. Nise*

Lag Compensator - Example

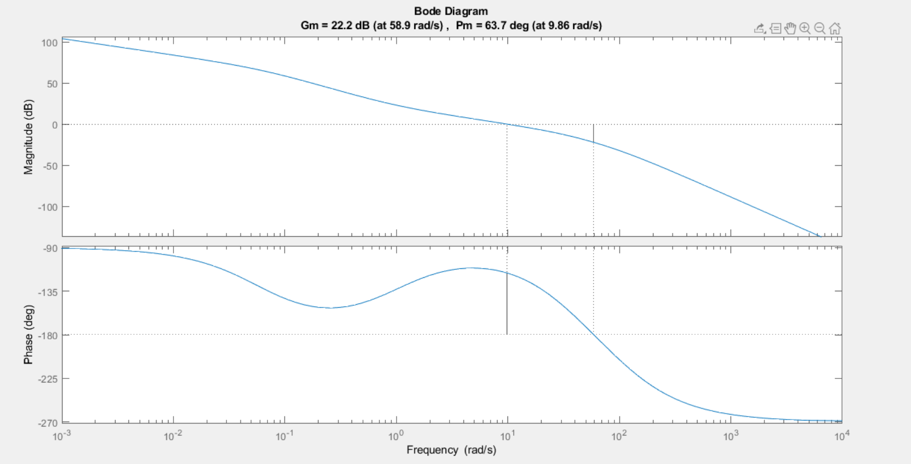


Lag Compensator - Example

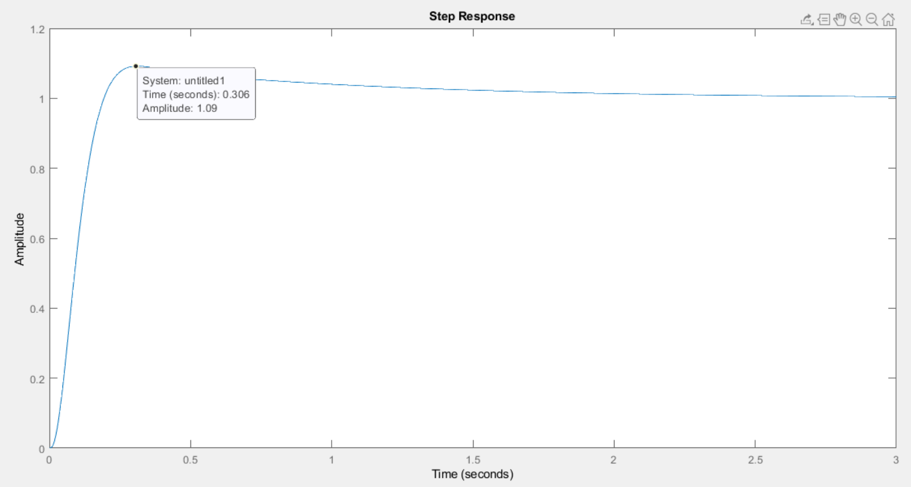
- Compensated transfer function becomes,

$$G(s)G_c(s) = \frac{36786(s+0.98)}{(s(s+36)(s+100)(s+0.062))}$$

Lag Compensator - Example



Lag Compensator - Example



Lag Compensator - Example

Parameter	Proposed specification	Lag compensated value
K_v	162.2	161.5
Percent overshoot	9.5	10 (approx.)

Lead Compensator - Introduction

Lead Compensator Frequency Response

- Lead compensator, $G_c(s) = [1/\beta][(s + 1/T)/(s + 1/\beta T)]$

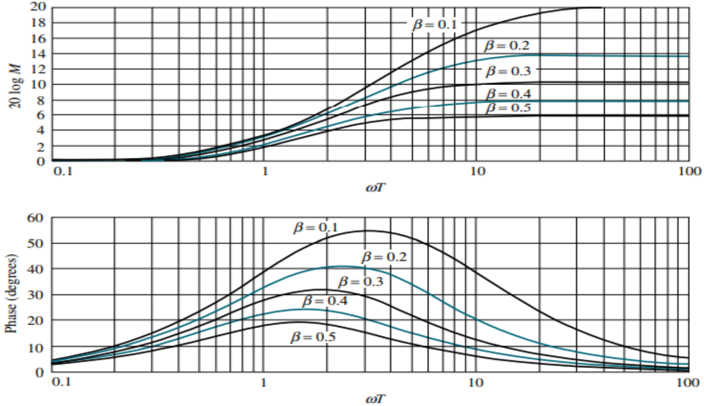


Figure: *Source -"Control Systems Engineering" by Norman S. Nise*

Lead Compensator - Introduction

Lead Compensator Frequency Response

- For various values of β, where $\beta < 1$. Notice that the peaks of the phase curve vary in maximum angle and in the frequency at which the maximum occurs.
- The dc gain of the compensator is set to unity with the coefficient $1/\beta$, in order not to change the dc gain designed for the static error constant when the compensator is inserted into the system.
- In order to design a lead compensator and change both the phase margin and phase-margin frequency, it is helpful to have an analytical expression for the maximum value of phase and the frequency at which the maximum value of phase occurs

Lead Compensator - Introduction

Lead Compensator Frequency Response

- Consider a lead compensator of form

$$G_c(s) = \frac{1}{\beta}\frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\beta T}}$$

- The phase angle of the lead compensator, ϕ_c is

$$\phi_c = \tan^{-1}\omega T - \tan^{-1}\omega\beta T$$

- Differentiate with respect to ω we get

$$\frac{d\phi_c}{d\omega} = \frac{T}{1 + (\omega T)^2} - \frac{\beta T}{1 + (\omega\beta T)^2}$$

Lead Compensator - Introduction

Lead Compensator Frequency Response

- Setting above equation to zero, we get

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$

- Rearranging the terms, we get

$$G_c(j\omega_{max}) = \frac{1}{\beta}\frac{j\omega_{max} + \dfrac{1}{T}}{j\omega_{max} + \dfrac{1}{\beta T}}$$

- Making use of $\tan(\phi_1 - \phi_2) = (\tan\phi_1 - \tan\phi_2)/(1 + \tan\phi_1 \tan\phi_2)$

$$\phi_{max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta}$$

Lead Compensator - Introduction

Lead Compensator Frequency Response

- Compensator's magnitude at ϕ_{max}

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}}$$

Design Procedure

- Find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement
- Since the lead compensator has negligible effect at low frequencies, set the gain, K, of the uncompensated system to the value that satisfies the steadystate error requirement
- Plot the Bode magnitude and phase diagrams for this value of gain and determine the uncompensated system's phase margin

Design Procedure

- Find the phase margin to meet the damping ratio or percent overshoot requirement. Then evaluate the additional phase contribution required from the compensator
- Determine the value of β (see equation for ϕ_{max}) from the lead compensator's required phase contribution
- Determine the compensator's magnitude at the peak of the phase curve (see equation for $|G_c(j\omega_{max})|$)
- Determine the new phase-margin frequency by finding where the uncompensated system's magnitude curve is the negative of the lead compensator's magnitude at the peak of the compensator's phase curve
- Design the lead compensator's break frequencies, (see equation for ω_{max}) to find T and the break frequencies

Design Procedure

- Reset the system gain to compensate for the lead compensator's gain
- Check the bandwidth to be sure the speed requirement in Step 1 has been met
- Simulate to be sure all requirements are met
- Redesign if necessary to meet requirements

Lead Compensator - Example

Example: A system is defined by the following transfer function,

$$G(s) = \frac{100K}{s(s+100)(s+36)}$$

- Design a lead compensator to yield a 20% overshoot and $K_v = 40$, with a peak time of 0.1 second.

Lead Compensator - Example

- We know,

$$\text{Percentage overshoot} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$20\% \text{ overshoot} \implies \zeta = 0.456$$

- We also know,

$$\text{Phase margin} = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$

$$\zeta = 0.456 \implies \text{PM} = 48.1^\circ$$

- We also know,

$$\text{Bandwidth} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\zeta = 0.456,\ T_p = 0.1 \implies \text{BW} = 46.6 \text{ rad/s}$$

Lead Compensator - Example

- The specified K_v can be achieved by,

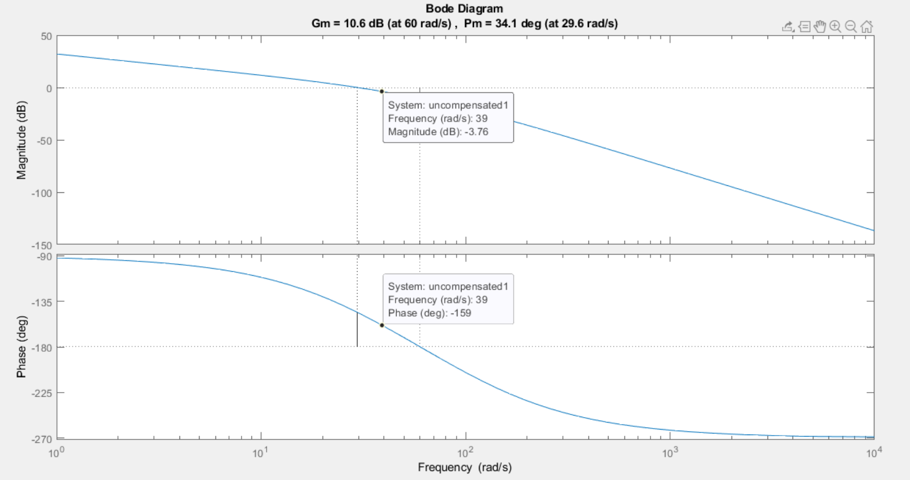
$$K_v = \lim_{s \to 0} sG(s)$$

$$\implies \frac{100K}{3600} = 40, \quad K = 1440$$

$$G^{'}(s) = \frac{144000}{s(s+100)(s+36)}$$

Lead Compensator - Example

Bode plot of $G'(s)$



Lead Compensator - Example

- The uncompensated system with $K = 1440$ has a phase margin of 34°.
- To increase the phase margin, we insert a lead network that adds enough phase to yield a 48.1°. phase margin.
- Since we know that the lead network will also increase the GCO frequency, we add a correction factor to compensate for the lower uncompensated system's phase angle at this higher phase-margin frequency. Since we do not know the higher phase-margin frequency, we assume a correction factor of 10°.
- Thus, the total phase contribution required from the compensator is $48.1^\circ - 34^\circ + 10^\circ = 24.1^\circ$.
- In summary, our compensated system should have a phase margin of 48.1° with a bandwidth of 46.6 rad/s.
- If the system's characteristics are not acceptable after the design, then a redesign with a different correction factor may be necessary

Lead Compensator - Example

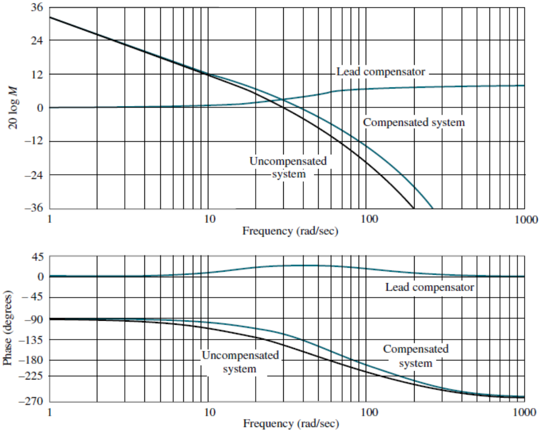


Figure: *Source -"Control Systems Engineering" by Norman S. Nise*

Lead Compensator - Example

- We know,

$$\phi_{max} = \sin^{-1} \frac{1-\beta}{1+\beta}$$

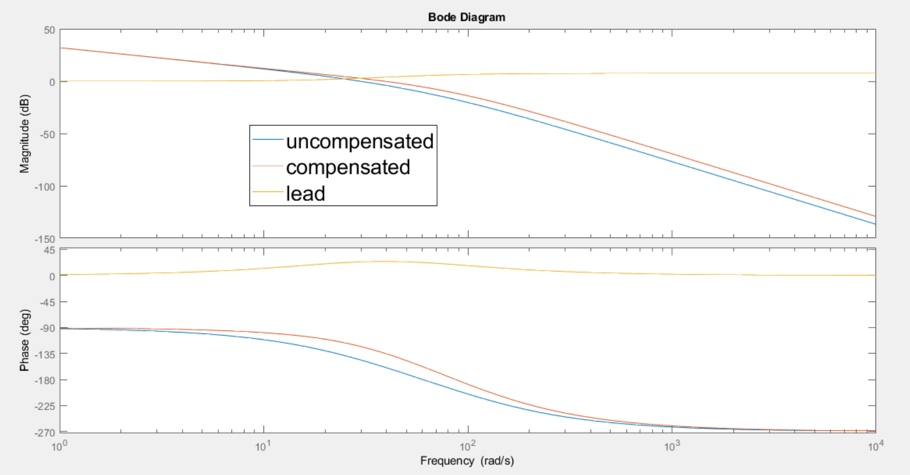
$$\phi_{max} = 24.1^\circ \implies \beta = 0.42$$

- Also,

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}} = 1.5432$$
$$= 3.76 \text{ dB}$$

- If we select ω_{max} to be GCO, the uncompensated system's magnitude at this frequency must be -3.76 dB to yield a 0 dB crossover at ω_{max} for the compensated system.
- The uncompensated system passes through -3.76 dB at $\omega_{max} = 39$ rad/s. This frequency is thus the new GCO.

Lead Compensator - Example



Lead Compensator - Example

- We know,

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$

$$\implies 1/T = 25.3, \quad 1/\beta T = 60.2$$

- Hence the compensator is given by

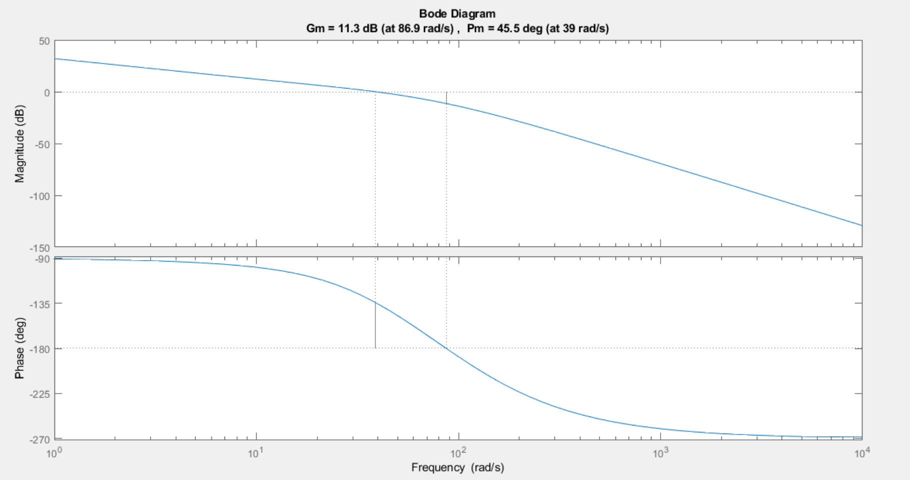
$$G_c(s) = \frac{1}{\beta} \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = 2.38 \frac{s + 25.3}{s + 60.2}$$

  Here, 2.38 is the gain required to keep the dc gain of the compensator at unity.

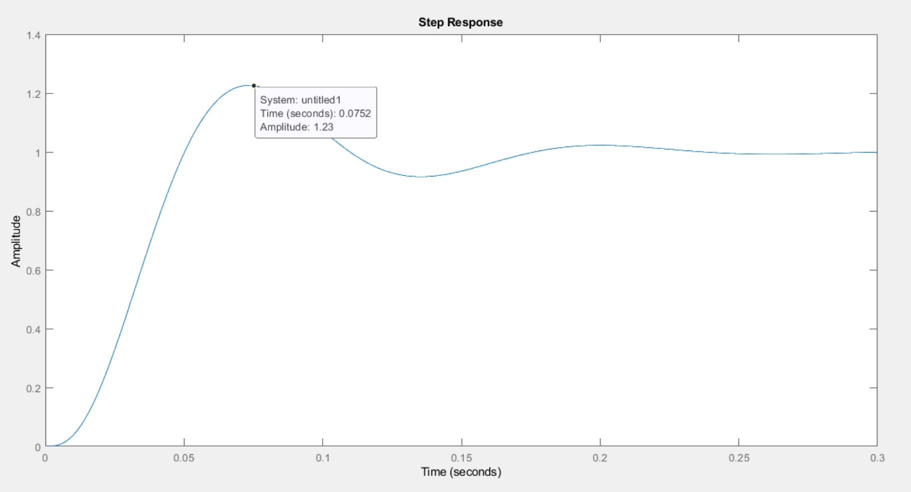
- Thus the final compensated open loop transfer function is given by

$$G_c(s)G(s) = \frac{342600(s + 25.3)}{s(s + 36)(s + 100)(s + 60.2)}$$

Lead Compensator - Example



Lead Compensator - Example



Lead Compensator - Example

Parameter	Proposed specification	Lead compensated value
K_v	40	40
Percent overshoot	20	23
Peak time	0.1	0.075

Lead-Lag Compensator - Introduction

Introduction

- Lead-lag compensator has the same action as PID controller and is normally employed when plant type and relative degree $(n-m)$ are to be preserved.
- However, these require five parameters to be adjusted, as against three in case of PID, as shown below

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)$$

where, $\beta, \gamma > 1$

- Lead-Lag compensator can be tuned better as there are more parameters for adjustment.

Lead-Lag Compensator - Bode Plot

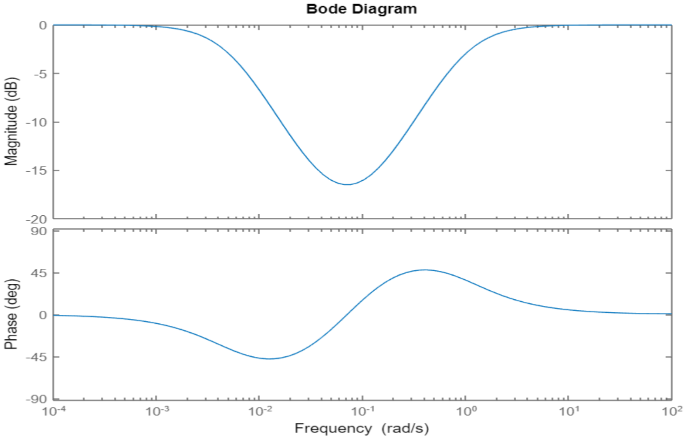
Introduction

- Design of Lag–Lead compensator in frequency domain is a broad extension of lag and lead design strategies
- This is possible as role of lead part is to increase the phase margin and bandwidth, while that of lag part is to maintain low frequency gain
- In this context, it should be noted that it is a good design philosophy to keep additional phase margin from the compensator $\approx 50^0$
- Given below is the typical bode plot of a lead-lag compensator
- We see that it behaves like a lag compensator in low frequency range and as a lead compensator in high frequency range

Lead-Lag Compensator - Bode Plot

Introduction

- Generic bode plot



Lead-Lag Compensator - Bode Plot

Design Procedure

- Consider a lead-lag compensator of the form

$$G_{lead}(s)G_{lag}(s) = \left(\frac{s+\dfrac{1}{T_1}}{s+\dfrac{\gamma}{T_1}}\right)\left(\frac{s+\dfrac{1}{T_2}}{s+\dfrac{1}{\gamma T_2}}\right)$$

  where, $\gamma > 1$.
- The first in parentheses produces the lead compensation, and the second term in parentheses produces the lag compensation.
- Now onwards, we consider the same type of compensator.

Lead-Lag Compensator - Bode Plot

Design Procedure

- Using a second-order approximation, find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement
- Set the gain, K, to the value required by the steady-state error specification
- Plot the Bode magnitude and phase diagrams for this value of gain
- Using a second-order approximation, calculate the phase margin to meet the damping ratio or percent overshoot requirement
- Select a new phase-margin frequency near ω_{BW}
- At the new phase-margin frequency, determine the additional amount of phase lead required to meet the phase-margin requirement. Add a small contribution that will be required after the addition of the lag compensator

Lead-Lag Compensator - Bode Plot

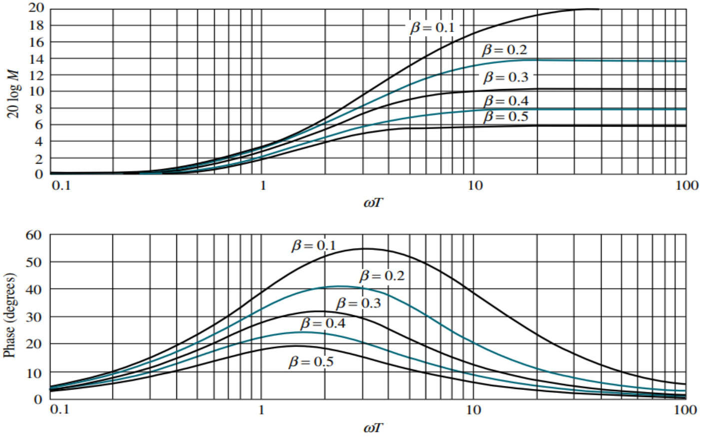
Design Procedure

- Design the lag compensator by selecting the higher break frequency one decade below the new phase-margin frequency. The design of the lag compensator is not critical, and any design for the proper phase margin will be relegated to the lead compensator. The lag compensator simply provides stabilization of the system with the gain required for the steady-state error specification. Find the value of γ from the lead compensator's requirements. Using the phase required from the lead compensator, we can find the value of $\gamma = 1/\beta$ using following figure. This value, along with the previously found lag's upper break frequency, allows us to find the lag's lower break frequency

Lead-Lag Compensator - Bode Plot

Design Procedure

- Bode plot of lead compensator $G_c(s) = [1/\beta][(s + 1/T)/(s + 1/\beta T)]$



Lead-Lag Compensator - Bode Plot

Design Procedure

- Design the lead compensator. Using the value of γ from the lag compensator design and the value assumed for the new phase-margin frequency, find the lower and upper break frequency for the lead compensator and solving for T
- Check the bandwidth to be sure the speed requirement in first step has been met
- Redesign if phase-margin or transient specifications are not met, as shown by analysis or simulation.

Lead-Lag Compensator - Bode Plot

Example

- Given a unity feedback system, where

$$G(s) = \frac{K}{s(s+1)(s+4)}$$

design a passive lead-lag compensator using Bode diagrams to yield a 13.25% overshoot, a peak time of 2 seconds, and $K_v = 12$.

Lead-Lag Compensator - Bode Plot

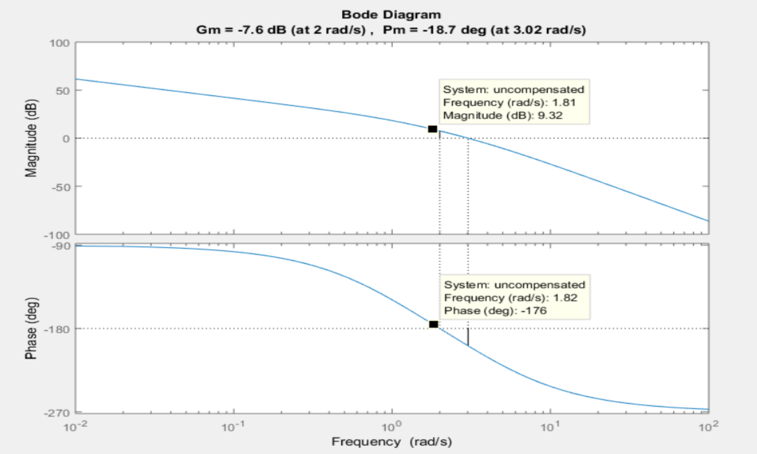
Example - Solution

- Bandwidth required for a 2-second peak time is 2.29 rad/s.
- In order to meet the steady-state error requirement, $K_v = 12$, the value of K is 48.
- The Bode plots for the uncompensated system with $K = 48$ are shown in below figure. We can see that the system is unstable.
- The required phase margin to yield a 13.25% overshoot is 55^0.

Lead-Lag Compensator - Bode Plot

Example - Solution

- Bode of uncompensated system



Lead-Lag Compensator - Bode Plot

Example - Solution

- Let us select $\omega = 1.8$ rad/s as the new phase-margin frequency.
- At this frequency, the uncompensated phase is -176^0 and would require, if we add a -5^0 contribution from the lag compensator, a 56^0 contribution from the lead portion of the compensator.

Lead-Lag Compensator - Bode Plot

Example - Solution

- The design of the lag compensator is next. The lag compensator allows us to keep the gain of 48 required for $K_v = 12$ and not have to lower the gain to stabilize the system.
- As long as the lag compensator stabilizes the system, the design parameters are not critical since the phase margin will be designed with the lead compensator.
- Thus, choose the lag compensator so that its phase response will have minimal effect at the new phase-margin frequency. Let us choose the lag compensator's higher break frequency to be 1 decade below the new phase-margin frequency, at 0.18 rad/s.

Lead-Lag Compensator - Bode Plot

Example - Solution

- Since we need to add 56^0 of phase shift with the lead compensator at $\omega = 1.8$ rad/s, we estimate from following figure that, if $\gamma = 10.6$ (since $\gamma = 1/\beta,\ \beta = 0.094$), we can obtain about 56^0 of phase shift from the lead compensator
- Thus with $\gamma = 10.6$ and a new phase-margin frequency of $\omega = 1.8$ rad/s, the transfer function of the lag compensator is

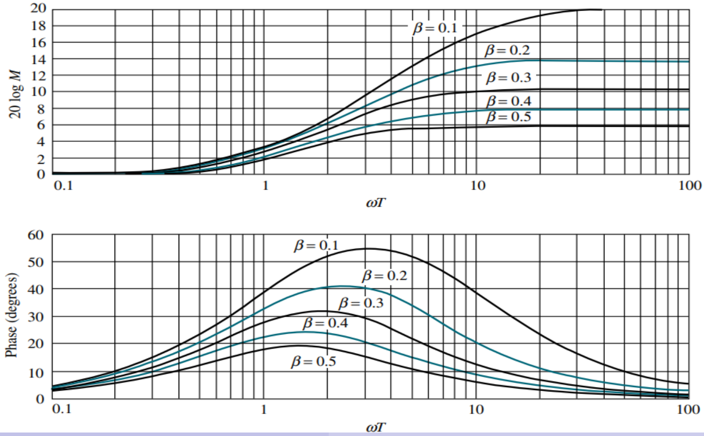
$$G_{lag}(s) = \frac{1}{\gamma}\frac{\left(s + \dfrac{1}{T_2}\right)}{\left(s + \dfrac{1}{\gamma T_2}\right)} = \frac{1}{10.6}\frac{(s + 0.183)}{(s + 0.0172)}$$

- Now we design the lead compensator. Using the values of $\omega_{max} = 1.8$ and $\beta = 0.094$, Following equation yields the lower break, $1/T_1 = 0.56$ rad/s. The higher break is then $1/\beta T_1 = 5.96$ rad/s. The lead compensator is

Lead-Lag Compensator - Bode Plot

Example - Solution

- Frequency response of Lead compensator $G_c(s) = [1/\beta][(s+1/T)/(s+1/\beta T)]$



Lead-Lag Compensator - Bode Plot

Example - Solution

- Now we design the lead compensator. Using the values of $\omega_{max} = 1.8$ and $\beta = 0.094$, Following equation yields the lower break, $1/T_1 = 0.56$ rad/s. The higher break is then $1/\beta T_1 = 5.96$ rad/s.

$$\omega_{max} = \frac{1}{T_1\sqrt{\beta}}$$

- The lead compensator is

$$G_{lead}(s) = \gamma \frac{\left(s + \frac{1}{T_1}\right)}{\left(s + \frac{\gamma}{T_1}\right)} = 10.6 \frac{(s + 0.56)}{s + 5.96}$$

Lead-Lag Compensator - Bode Plot

Example - Solution

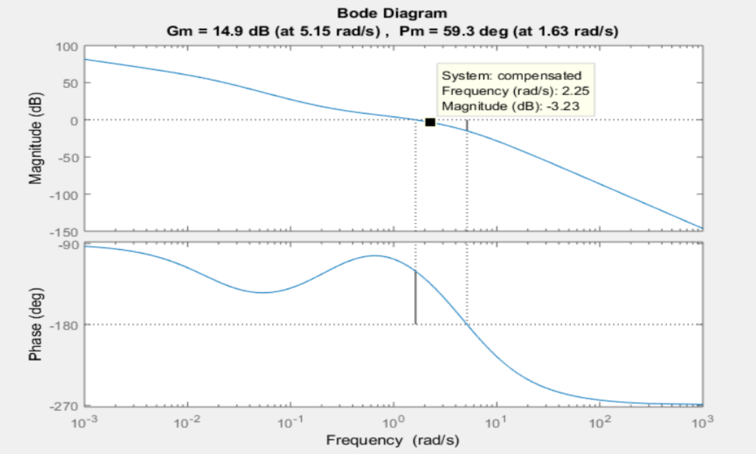
- The lead-lag compensated system's open loop transfer function is

$$G_{lead-lag-comp} = \frac{48(s+0.183)(s+0.56)}{s(s+1)(s+4)(s+0.0172)(s+5.96)}$$

Lead-Lag Compensator - Bode Plot

Example - Verification

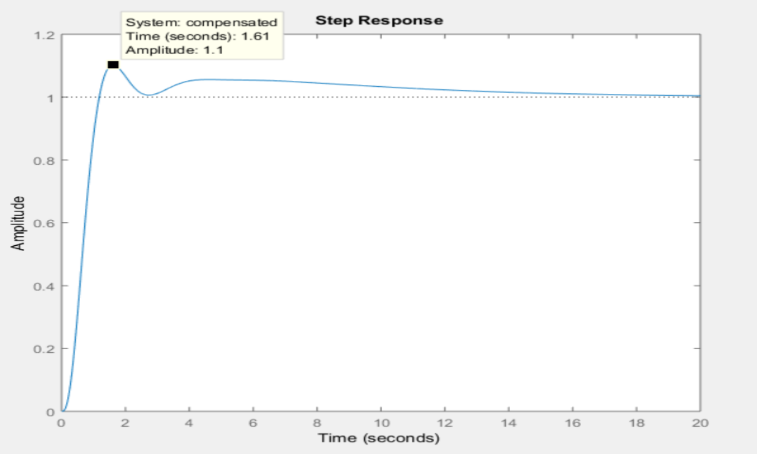
- bode plot of compensated system



Lead-Lag Compensator - Bode Plot

Example - Verification

- step response of compensated system



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