random variable.

Exercise. Prove this theorem, when X is a discrete random variable.

Remark.

The theorem can be used to construct a random variable which does not have moments of certain orders. It says if we control the tails we control the moments.

Theorem-5 (Sufficient condition for all moments to exist)

Let X be a random variable such that

$$\lim_{x\to\infty} \frac{\mathbb{P}(|X| > cx)}{\mathbb{P}(|X| > x)} = 0 \quad \text{for any } c > 1.$$

Then X possesses moments of all orders.

Exercise Why we did not mention about the tail when $c \le 1$. ? Think about it.

Proof. For any small enough positive number $\epsilon > 0$, we can choose a number x_0 large enough so that

$$\frac{\mathbb{P}(|X| > cx)}{\mathbb{P}(|X| > x)} < \epsilon \quad \text{for all } x \geq x_0. \longrightarrow (***)$$

where $c(\gamma)$ is some fixed real number. Choose x_1 large enough such that

$$\mathbb{P}(|X| \geq x) < \epsilon \quad \text{for all } x \geq x_1. \longrightarrow ***_2$$

(Think about why such x_1 exists.)

Define $N = \max(x_0, x_1)$. Then for any positive integer r,

$$\frac{\mathbb{P}(|X| \geq e^r x)}{\mathbb{P}(|X| \geq x)} = \prod_{p=1}^{r} \frac{\mathbb{P}(|X| \geq e^p x)}{\mathbb{P}(|X| \geq e^{p-1} x)} \leq \epsilon^r \longrightarrow ***_3$$

for $x \geq N$. Thus for $x \geq N$ we have from $(***_2)$ and $(***_3)$

$$\mathbb{P}(|X| \geq e^r x) \leq \epsilon^{r+1}. \longrightarrow ***_4$$

$$\boxed{\begin{array}{l}\text{We shall show that all the integral moments exist} \\ \mathbb{E}(|X|^n) < \infty \text{ for every } n \in \mathbb{N}.\end{array}}$$

We can use then Theorem-2 to conclude that all moments of X exist.

Fix an integer n and note that

$$\mathbb{E}(|X|^n) = n \int_0^N x^{n-1}\, \mathbb{P}(|X| \geq x)\, dx + n \int_N^\infty x^{n-1}\, \mathbb{P}(|X| \geq x)\, dx.$$

$$\longrightarrow ***_5$$

Note further that

$$\int_N^\infty x^{n-1}\, \mathbb{P}(|X| \geq x)\, dx = \sum_{r=1}^{\infty} \int_{e^{r-1}N}^{e^r N} x^{n-1}\, \mathbb{P}(|X| \geq x)\, dx$$

$$\underset{***_4}{\leq} \sum_{r=1}^{\infty} \frac{1}{n}\, \epsilon^{r+1} \left[x^n\right]_{e^{r-1}N}^{e^r N}$$

$$\leq \frac{N^n}{n} \epsilon \sum_{r=1}^{\infty} (\epsilon e^n)^r \longrightarrow ***_6$$

We can now choose ϵ such that $\epsilon e^n < 1$. Then the ~~infinite~~ series in $***_6$ converges and the series equals.

$$\frac{N^n \epsilon}{n} \frac{\epsilon e^n}{1 - \epsilon e^n} \longrightarrow ***_7$$

Using the ~~decompot~~ decomposition obtained in $***_5$ combined with the bounds obtained in $***_6$ and $***_7$, we have $\mathbb{E}(|X|^n)$. Hence, proof ~~the~~ of the theorem follows.

<u>Theorem-5</u> Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable and $h_i : \mathbb{R} \longrightarrow \mathbb{R}$ are <u>measurable maps</u>. for $1 \leq i \leq n$. Assume further that

$$\mathbb{E}(|h_i(X)|) < \infty \text{ for all } i = 1, 2, \ldots n.$$

Then we have

$$\mathbb{E}\left(\sum_{i=1}^{n} h_i(X)\right) = \sum_{i=1}^{n} \mathbb{E}[h_i(X)].$$

<u>Definition</u>

Let $k \in \mathbb{N}$ and c be a ~~positive~~ constant. If

$$\mathbb{E}(|X - c|^k) < \infty.$$

Then we call $\mathbb{E}[(X-c)^k]$ to be the moment of order k about the point c. If we take $c = \mu = \mathbb{E}(X)$, ~~then~~ (when $\mathbb{E}(|X|) < \infty$ that is, $\mathbb{E}(X)$ exists), we call

$$\mu_k = \mathbb{E}[(X-\mu)^k]$$

the central moment of order k or the moment of order k about the mean whenever

$$\mathbb{E}(|X-\mu|^k) < \infty.$$

Definition

If $\mathbb{E}(X^2) < \infty$, we call $\mathbb{E}([X-\mu]^2)$ is called the variance of X and we write

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2].$$

The quantity $\sigma = \sqrt{\mathrm{Var}(X)}$ is called standard deviation of X.

Definition.

We say a random variable X to be degenerate at a real number c if

$$\mathbb{P}(X=c) = 1 \quad \text{for some } c < \infty.$$

Theorem - 6 $\mathrm{Var}(X) = 0$ if and only if X is degenerate at $\mathbb{E}(X)$.

Proof- Left as an exercise.

Theorem-7. $\mathrm{Var}(X) < \mathbb{E}[(X-c)^2]$ for any $c \neq \mathbb{E}(X)$.

Proof - Left as an exercise.

Definition. (~~Fractional moments~~ Factorial moments)

Fix $k \in \mathbb{N}$. Then, we call

$$\mathbb{E}[X(X-1)(X-2)(X-3)\cdots(X-k+1)]$$

is called the k^{th} factorial moment whenever

$$\mathbb{E}(|X|^k) < \infty.$$

Lecture – IX   Generating functions.

Here we are going to assume that either X is discrete or continuous random variable.

Discrete ~~rando positive~~ random variables and probability generating functions.

Let X be a non-negative integer-valued random variable and

$(p_k : k \geq 0)$ be the p.m.f. of X.

Definition.

The probability generating function of the random variable X is defined to be the following power series

$$\mathcal{P}(s) = \sum_{k=0}^{\infty} p_k s^k$$

where $\mathcal{P}(s)$ converges for $|s| \leq 1$. We shall call it p.g.f. of X.

Remark Since $\mathcal{P}(s)$ converges uniformly and absolutely for $|s| \leq 1$, we can differentiate it for infinitely many times. As $\mathcal{P}(s)$ is a power

$P(s)$ characterizes $(p_k : k \geq 0)$ uniquely. So the law or distribution of ~~the~~ a non-negative discrete random variable X is uniquely specified through its p.g.f.

Remark

We have already noted that $P(s)$ is a power series which is uniformly and absolutely convergent for $|s| \leq 1$. We can differentiate it termwise on the interval $(-1,1)$ and obtain.

$$P^{(k)}(s) = \sum_{n \geq k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\,\mathbb{P}(X=n)\,s^{n-k}$$

where $P^{(k)}(s)$ denotes the k^{th} derivative of P.

Note that $0 \in (-1,1)$ and

$$P^{(k)}(s)\Big|_{s=0} = k!\,\mathbb{P}(X=k).$$

that is,

$$p_k = \mathbb{P}(X=k) = P^{(k)}(s)\Big|_{s=0} \cdot \frac{1}{k!}$$

~~This~~ We can use this formula to obtain the probability mass function of a random variable X from its probability generating function.

Remark Note that $P(s) = \mathbb{E}(s^X)$ for $s \in [-1,1]$.

Definition.

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable (may be discrete or continuous). The moment-generating function (m.g.f.) of X can be defined through the following

$$\mathcal{M}(s) = \mathbb{E}\left(e^{sX}\right)$$

$$= \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} e^{sk} p_k & \text{if } X \text{ is discrete with p.m.f. } (p_k : k \in \mathbb{Z}) \\ \displaystyle\int_{-\infty}^{\infty} e^{sx}\, \mathbb{F}(dx) & \text{if } X \text{ is a continuous random variable with } \mathbb{F}(dx) = f(x)\,dx \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} e^{sk} p_k & \text{if } X \text{ has p.m.f. } (p_k : k \in \mathbb{Z}) \\ \displaystyle\int_{-\infty}^{\infty} e^{sx} f(x)\,dx & \text{if } X \text{ has p.d.f. } f \end{cases}$$

~~whenever~~; for those $s \in \mathbb{R}$ which satisfy

$$\mathbb{E}\left(e^{sX}\right) < \infty.$$

Exercise. Construct a random variable X such that its moment-generating function does not exist.

Theorem

Suppose that $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be either a discrete or a continuous random variable.

If $E(e^{sX})$ exists for $s \in (a,b)$ and $0 \in (a,b)$, then $M_X(s) = \mathbb{E}(e^{sX})$ determines the law $\mathbb{F}$ of X uniquely and $M_X(s)$ is also unique.

__Theorem__ (Moments and moment generating function). If moment generating function $M_X(s)$ of a random variable X exists for $s \in (-s_0, s_0)$ say, $s_0 > 0$, then the derivatives of $M_X(s)$ all orders exist at $s=0$ and can be evaluated under the integral sign that is,

$$M_X^{(k)}(s)\Big|_{s=0} = \mathbb{E}(X^k) \quad \text{for every } k \in \mathbb{N}.$$

> The proof of the theorems are beyond the scope of this course and so we shall use them without the proof.

__Remark__ The requirement that $M_X(s)$ exists in a neighbourhood of zero is very ~~rare~~ restrictive and is not satisfied by many common distributions like Cauchy distribution, Pareto distribution, lognormal distribution.

Characteristic function.

Definition.
Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. The complex-valued function $\phi : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$\phi(t) = \mathbb{E}\left(e^{itX}\right) = \mathbb{E}(\cos tX) + i\, \mathbb{E}(\sin tX), \; t \in \mathbb{R}$$

where $i = \sqrt{-1}$ is the imaginary unit is called the characteristic function of the random variable X.

Clearly,

$$\phi(t) = \sum_{k=-\infty}^{\infty} \left(\cos(t x_k) + i \sin(t x_k) \right) \mathbb{P}(X = x_k)$$

if X is a discrete random variable with p.m.f. $(\mathbb{P}(X = x_k) : k \in \mathbb{Z})$.

$$\phi(t) = \int_{-\infty}^{\infty} f(x) \cos(tx)\, dx + i \int_{-\infty}^{\infty} f(x) \sin(tx)\, dx$$

if X is a continuous random variable with probability density function f.

Remark. Unlike the moment generating function (m.g.f.), the characteristic function always exist and uniquely determines the law or distribution function of the random variable X. It is