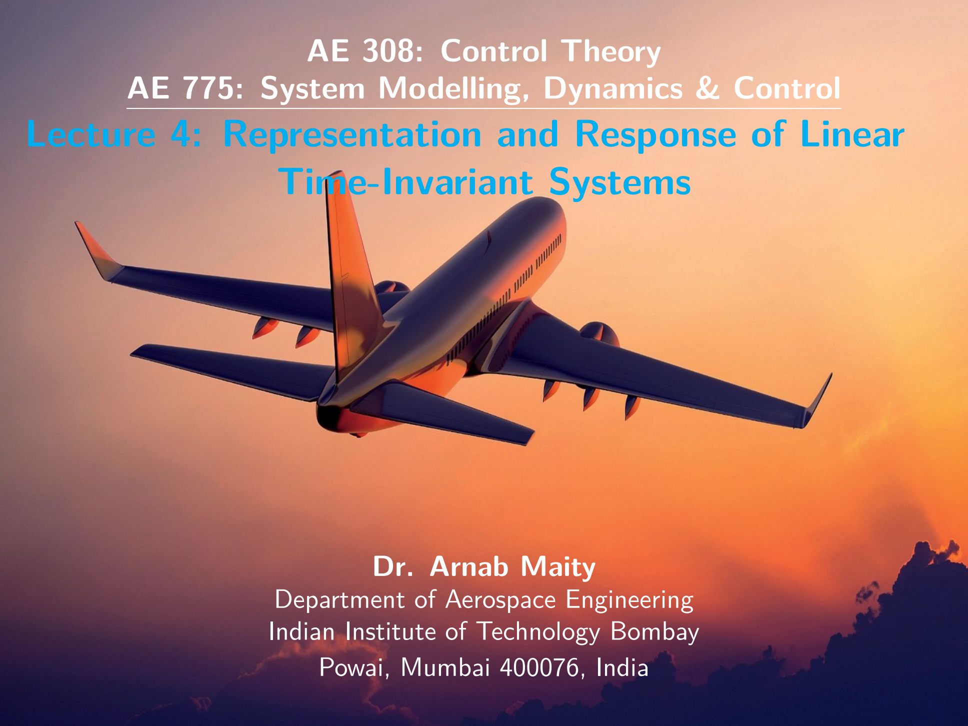
AE 308: Control Theory
AE 775: System Modelling, Dynamics & Control

Lecture 4: Representation and Response of Linear Time-Invariant Systems



Dr. Arnab Maity
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

Introduction - Responses

Analysis and Design Methodology

- All control tasks $\rightarrow$ stated in terms of achieving desired system performance
- These requirements are translated into desired system behavior (or response) under operating conditions
- This create a need to use system response for solving control problem
- Response $\rightarrow$ the output behavior of a system with respect to time for a given input

Introduction - Responses

Desired Performance

- I want to complete race in x seconds (In control term - settling time should be below x seconds)

Terms: Settling Time and Damping Ratio

Settling Time

- Settling time is the time required for a response to become steady.
- It is defined as the time required by the response to reach and steady within specified range of 2% or 5%.

Damping Ratio

- The damping ratio is a dimensionless measure describing how oscillations in a system decay after a disturbance.

Introduction - Responses

Response

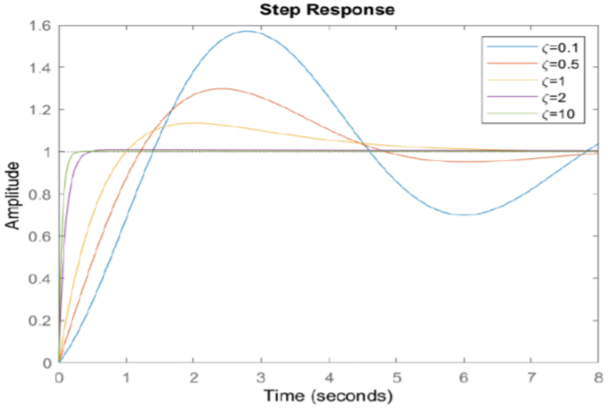


Figure: Step responses for different damping ratios (suspension system)

Introduction - Responses

Desired performance and the response need not to be same

- I wanted to score 100 out of 100 in AE 308/AE 775, but I scored 99.

Introduction - LTI

What are LTI systems

- LTI $\rightarrow$ **Linear Time Invariant**
- In terms of mathematical model $\rightarrow$ those systems that can be described by linear differential equations having constant coefficients.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + ... + a_n y = b_0 \frac{d^m u}{dt^m} + ... + b_m u$$

where,

$y =$ output

$u =$ input

$a_1 \cdots a_n, b_0 \cdots b_m =$ **constants**

- $n =$ highest degree of derivative of $y =$ **order of system**

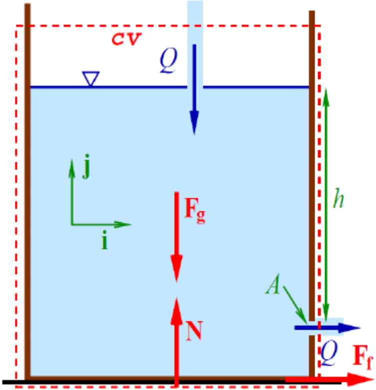
Introduction - LTI

Linearity and Time Invariance

- Linear
  - Satisfies superposition and homogeneity $\rightarrow$ **addition, scaling**
  - Coefficients are independent of output, input, and their derivatives
- Time invariant
  - Delayed input produces the same response with time delay
  - Coefficients independent of t

Introduction - LTI

Non-LTI System - Example



Introduction - LTI

Superposition Example

- Show that superposition holds

$$\dot{y} + ky = u$$

- Solution:
  - Let

$$u = \alpha_1 u_1 + \alpha_2 u_2$$
$$y = \alpha_1 y_1 + \alpha_2 y_2$$

  - Then

$$\dot{y} = \alpha_1 \dot{y_1} + \alpha_2 \dot{y_2}$$

  - After substituting

$$\alpha_1 \dot{y_1} + \alpha_2 \dot{y_2} + k(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 u_1 + \alpha_2 u_2$$

  - After rearranging

$$\alpha_1(\dot{y_1} + ky_1 - u_1) + \alpha_2(\dot{y_2} + ky_2 - u_2) = 0$$

  - Superposition holds.

Introduction - LTI

Time Invariance - Example

- Consider

$$\dot{y}_1(t) + k(t)y_1(t) = u_1(t)\dot{y}_2(t) + k(t)y_2(t) \qquad = u_1(t-\tau)$$

where, τ is time shift.

- Assume that $y_2(t) = y_1(t-\tau)$

$$\frac{y_1(t-\tau)}{dt} + k(t)y_1(t-\tau) = u_1(t-\tau)$$

- Let $t - \tau = \eta$

$$\frac{y_1(\eta)}{d\eta} + k(\eta+\tau)y_1(\eta) = u_1(\eta)$$

- Hence, for system to be time invariant, k **must be constant.**

LTI System Response - Natural

Natural Response

- Natural Response $\rightarrow$ Response with input $= 0$
- Determined by the initial conditions

$$y(0), \frac{dy}{dt}(0), \frac{d^2y}{dt^2}(0), \ \ldots \ , \frac{d^{n-1}y}{dt^{n-1}}(0)$$

LTI system response - Natural

Natural Response Example

- Let system be

$$\frac{dy}{dt} + a_1 y = 0$$

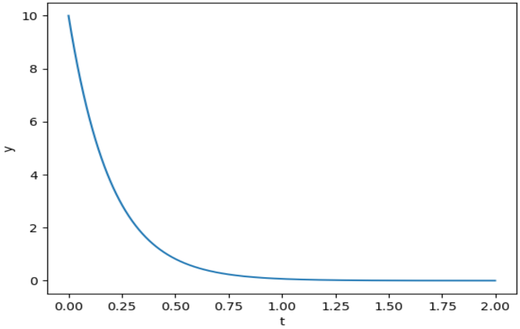
and $y(0) = y_0$

- Solution is

$$y(t) = y_0 e^{-a_1 t}$$

LTI System Response - Natural

Natural Response - Example (cont...)



- Parameters: $y_0 = 10$, $a_1 = 1$

LTI System Response - Forced

Forced Response - Example

- Let a system be

$$\frac{dy}{dt} + a_1 y = u(t)$$

- Solution is given by

$$y(t) = e^{-a_1 t}\left(\int u(\tau)e^{a_1\tau}d\tau\right)$$

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