given by

$$\mathbb{P}(\{a_1, a_2, \dots a_k\}) = \left(\frac{1}{6}\right)^k \text{ for every } k \in \mathbb{N}.$$

We can check that

$\mathbb{P}$ is a probability on $(\Omega^k, 2^{\Omega^k})$ for every $k \geq 1$

(This is how probabilities on infinite sequences are defied which we shall learn in the second half of the course)

Here, we are to check that X which is the first time 6 occurs in the sequence, is a random variable.

This experiments are commonly called

"Infinite sequence of independent trials".

We shall properly address them later.

If we ignore the technical details, then we have already computed

$$\mathbb{P}(X=k) = \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^{k-1} \text{ for } k = 1, 2, \dots$$

From here, we can check that

$(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P})$ is a probability space

and X is the identity map on this space.

<u>Example - 3</u> (Real valued random variables).

Suppose that we are noting the height of a person ~~ever~~ around us. Then the measurement takes any positive real value and the measurement is different for different persons. We can model this using real-valued random variable with $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$. and a probability on it. To be more specific, we need to know more about ~~an~~ random variables and probability measures induced by them.

Let X be a real valued random variable that is,

$X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \tilde{\mathbb{P}})$. such that

$\tilde{\mathbb{P}}(B) = \mathbb{P}(\{\omega : X(\omega) \in B\})$ for every $B \in \mathcal{B}(\mathbb{R})$.

Then the probability $\tilde{\mathbb{P}}$ is called <u>probability distribution</u> of the random variable X or the <u>probability induced</u> by the random variable X.

Sometimes <u>probability distribution</u> of X also called <u>distribution</u> of the random variable X.

As the set functions are difficult to handle, we look for a point function, which uniquely specify the probability distribution of X. For this, we can again turn to our favourite Borel sets $(-\infty, a)$ for a $\in \mathbb{R}$. We can define a point function $\mathbb{F} : \mathbb{R} \to [0,1]$ such that $\mathbb{F}(a)$ $\mathbb{P}(X \in (-\infty, a))$. for every $a \in \mathbb{R}$. As $(-\infty, a)$ is the generator of the Borel sets, it turns out that $\mathbb{P}(X \in B)$ can be obtained from the function $\mathbb{F}$ using continuity property of probability (recall that any Borel set can be obtained ~~from~~ as a limit of sequence of sets $\{(-\infty, a_n) : n \geq 1\}$ chosen appropriately.

<u>Definition (Distribution function)</u>

A function $\mathbb{F} : \mathbb{R} \to [0,1]$ is called a distribution function if.

i) $\mathbb{F}(-\infty) = 0$, $\mathbb{F}(\infty) = 1$

ii) $x \longmapsto \mathbb{F}(x)$ is non-decreasing.

iii) $x \longmapsto \mathbb{F}(x)$ is right continuous.

[Recall that $\mathbb{F}$ is right continuous at x if $\mathbb{F}(x) = \mathbb{F}(x+)$ that is, $\mathbb{F}(x) = \lim_{h \downarrow 0} \mathbb{F}(x+h)$. Here $h \downarrow 0$ means h decreases to 0 and we are approximating $\mathbb{F}$ at x from the values of it at right side of x].

Note : $\mathbb{F}(-\infty) = \lim_{t \downarrow -\infty} \mathbb{F}(t)$ as $\mathbb{F}$ is assumed to be right-continuous

As $\mathbb{F}$ is non-decreasing and $0 = \mathbb{F}(-\infty) < \mathbb{F}(\infty) = 1$, it follows that both of $\mathbb{F}(x-) = \lim_{h \uparrow 0} \mathbb{F}(x+h)$ and $\mathbb{F}(x+) = \lim_{h \downarrow 0} \mathbb{F}(x+h)$ exist and satisfies

$$\mathbb{F}(x-) \leq \mathbb{F}(x+) = \mathbb{F}(x) \qquad \text{for all } x \in \mathbb{R}.$$

right-continuity

We say x is a ~~disc~~ <u>point of discontinuity</u> of $\mathbb{F}$ if

$$\mathbb{F}(x) - \mathbb{F}(x-) > 0.$$

Sometimes, x is also referred to as the <u>jump point</u> of $\mathbb{F}$.

<u>Theorem</u> The set of discontinuity points of $\mathbb{F}$ is almost-countable.

Proof- Let $(a, b]$ be a finite interval with finitely many point of discontinuities say,

$$a < x_1 < x_2 < \cdots < x_n \leq b.$$

Then

$$\mathbb{F}(a) \leq \mathbb{F}(x_1-) < \mathbb{F}(x_1) < \cdots \leq \mathbb{F}(x_n-) < \mathbb{F}(x_n) \leq \mathbb{F}(b).$$

and

$$\mathbb{F}(b) - \mathbb{F}(a) \geq \sum_{k=1}^{n} (\mathbb{F}(x_k) - \mathbb{F}(x_k-)).$$

Fix a large enough integer N. We want ~~done~~ to derive an upper bound to ~~a~~ the number of discontinuities of $\{x_k : \mathbb{F}(x_k) - \mathbb{F}(x_k-) \geq \frac{1}{N}\}$. If we apply this with the fact that $\mathbb{F}(b) - \mathbb{F}(a) \leq \mathbb{F}(\infty) - \mathbb{F}(-\infty) = 1$, then we see

$$\left|\left\{x_k : \mathbb{F}(x_k) - \mathbb{F}(x_k-) \geq \frac{1}{N}\right\}\right| \frac{1}{N} \leq 1.$$

$$\left|\left\{x_k : \mathbb{F}(x_k) - \mathbb{F}(x_k-) \geq \frac{1}{N}\right\}\right| \leq N.$$

Consider the finite interval $(-n, n]$ and then we can see that

$$\mathbb{R} = \bigcup_{n\geq 1}^{\infty} (-n, n].$$

On each interval $(-n, n]$, there can be atmost finitely many (N) point of ~~disconti~~ discontinuities with height of jump at ~~the~~ least N. So if we take union ~~of~~ over N and n, then we can see that there are only countably many such points. Hence, the theorem is proved.

<u>Theorem</u> Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ be a random variable. such that

$\mathbb{P}_X(B) = \mathbb{P}(\{\omega : X(\omega) \in B\}) = \mathbb{P}(X^{-1}(B))$ for every $B \in \mathcal{B}(\mathbb{R})$. Then $x \longmapsto \mathbb{P}_X((-\infty, x]) = \mathbb{P}(\{\omega : X(\omega) \leq x\})$

is a distribution function.

Proof. Let us denote $\mathbb{F}(x) = \mathbb{P}_X((-\infty, x])$
$= \mathbb{P}(\{\omega : X(\omega) \leq x\}) = \mathbb{P}(X \leq x)$. for every $x \in \mathbb{R}$.
We have to show that $\mathbb{F}$ is a distribution function.

i) $\mathbb{F}$ is non-decreasing.

Let $x_1 < x_2$. Then $(-\infty, x_1] \subset (-\infty, x_2]$ and by monotonicity property of probability, we have

$$\mathbb{F}(x_1) = \mathbb{P}_X((-\infty, x_1]) \leq \mathbb{P}_X((-\infty, x_2]) \leq \mathbb{F}(x_2).$$

ii) $\mathbb{F}$ is right continuous.

Consider a sequence $(x_n : n \geq 1)$ such that $x_n \downarrow x$ that is, $x_1 > x_2 > x_3 > \ldots x_n > \ldots > x$. Define

$$A_k = \{\omega : X(\omega) \in (x, x_k]\} \text{ for every } k \geq 1.$$

It is immediate to check that $A_k \in \mathcal{F}$ and $A_{k+1} \subseteq A_k$ that is, $(A_k : k \geq 1)$ is sequence of non-increasing events. Hence,

$$\lim_{n \to \infty} A_n = \bigcap_{n \geq 1}^{\infty} A_n = \phi.$$

We now use continuity (from above) property of probability to conclude that

$$\lim_{n \to \infty} \mathbb{P}(A_n) = 0. \longrightarrow (*14)$$

Note that

$$\mathbb{P}(A_n) = \mathbb{P}(\{\omega : x < X(\omega) \le x_n\})$$
$$= \mathbb{P}(\{\omega : X(\omega) \le x_n\}) - \mathbb{P}(\{\omega : X(\omega) \le x\})$$
$$= \mathbb{F}(x_n) - \mathbb{F}(x). \longrightarrow (*15)$$

Now, the claim follows from $(*14)$ and $(*15)$.

iii) $\mathbb{F}(-\infty) = 0$.

Consider a sequence $(x_n : n \geq 1)$ such that $x_n \downarrow -\infty$.

Then it follows that

$$\{\omega : X(\omega) \le x_n\} \supseteq \{\omega : X(\omega) \le x_{n+1}\} \text{ for all } n \geq 1$$

and $\lim_{n\to\infty} \{\omega : X(\omega) \le x_n\} = \bigcap_{n=1}^{\infty} \{\omega : X(\omega) \le x_n\} = \phi$.

We can use again the continuity of probability (from above) to conclude that

$$\therefore \lim_{n\to\infty} \mathbb{F}(x_n) = \lim_{n\to\infty} \mathbb{P}(\{\omega : X(\omega) \le x_n\}) = 0.$$

A similar argument.

Consider a sequence $(x_n : n \geq 1)$ such that $x_n \uparrow \infty$.

Then it follows that

$$\{\omega : X(\omega) \le x_n\} \subseteq \{\omega : X(\omega) \le x_{n+1}\} \ \forall \ n \geq 1$$

and $\lim_{n\to\infty} \{\omega : X(\omega) \le x_n\} = \bigcup_{n=1}^{\infty} \{\omega : X(\omega) \le x_n\} = \Omega$.

We can now use continuity (from below)

property of probability to conclude

$$\lim_{n\to\infty} \mathbb{F}(x_n) = \lim_{n\to\infty} \mathbb{P}\left(\{\omega : X(\omega) \leq x_n\}\right) = 1.$$ $\blacksquare$

The next result says that the distribution function uniquely characterizes a probability on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ induced by a random variable.

Theorem

Given a probability $\mathbb{P}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exists a distribution function $\mathbb{F} : \mathbb{R} \to [0,1]$ satisfying

$$\mathbb{P}((-\infty, x]) = \mathbb{F}(x) \text{ for every } x \in \mathbb{R}$$

and conversely, given a distribution function $\mathbb{F} : \mathbb{R} \to [0,1]$, there exists ~~an~~ a unique probability $\mathbb{P}'$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mathbb{F}(x) = \mathbb{P}'((-\infty, x])$ for every $x \in \mathbb{R}$.

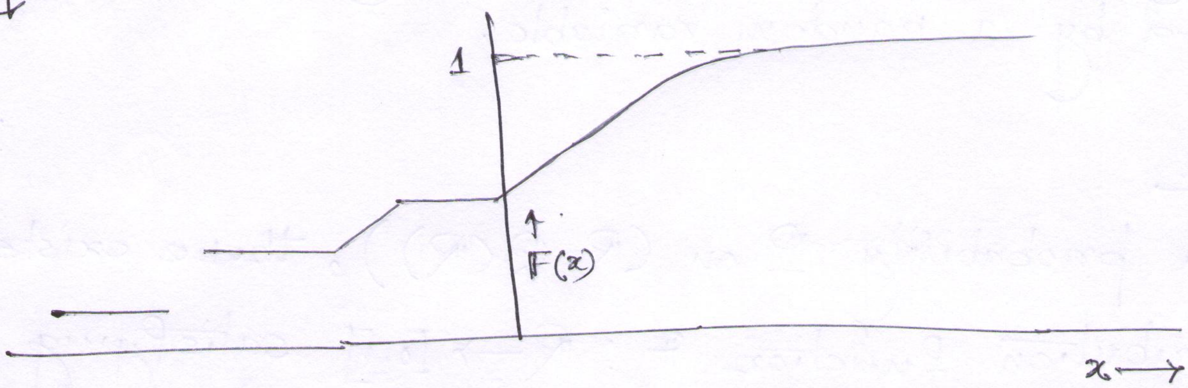
The proof is beyond the scope of this course and so we keep it for an advanced course.

Theorem

Given a distribution function $\mathbb{F} : \mathbb{R} \to [0,1]$, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ such that $\mathbb{F}$ is the distribution of the random variable X.

The theorem says that working with a random variable is equivalent to work with a distribution function which is deterministic.

A typical example of distribution function looks like



Exercise

1. Suppose that X is a random variable which counts the number of heads occurred in 10 independent toss of a coin. Write down the distribution function associated to it.

2. Check which of the following functions are distribution functions.

a) $F(x) = \begin{cases} 0 & \text{if } x < 0. \\ x & \text{if } 0 \leq x < \frac{1}{2} \\ 1 & \text{if } x \geq \frac{1}{2}. \end{cases}$

b) $F(x) = \frac{1}{\pi} \tan^{-1} x, \quad x \in \mathbb{R}.$

c) $F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - \frac{1}{x} & \text{if } x > 1. \end{cases}$

d) $F(x) = \begin{cases} 1-e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$

3. For every distribution function F in Problem 2, compute the following probabilities

a) $P(X \geq \frac{1}{4})$, $P(\frac{1}{3} < X \leq \frac{3}{8})$

b) $P(-\infty < X < 2)$

where X is the random variable with distribution function F.

4. Let F_1 and F_2 be two distribution functions. Consider $\alpha \in [0,1]$. Then show that $\alpha F_1 + (1-\alpha) F_2$ is a distribution function where

$(\alpha F_1 + (1-\alpha) F_2)(x) = \alpha F_1(x) + (1-\alpha) F_2(x) \; \forall x \in \mathbb{R}.$

5. Examine if the function F defined below can be regarded as a distribution function or not.

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0. \\ x^2/2 & \text{if } x \in (0,1]. \\ \frac{1}{2} + \frac{1}{3}(x-1)^3 & \text{if } x \in (1,2]. \\ \frac{6}{7} + \frac{1}{7}(x-2)^4 & \text{if } 2 < x \leq 3. \\ 1 & \text{if } x \geq 3 \end{cases}$$