easy to check that

$t \mapsto \phi(t)$ is a continuous function.

and $|\phi(t)| \leq 1$ for any $t \in \mathbb{R}$.

The characteristic function plays an important role in probability theory. But detailed study of characteristic function involves ~~de~~ deep results from complex analysis. So we postpone the ~~the~~ study here.

Moment problems

~~There are few problems~~ We have seen that the moment generating function uniquely characterizes the law or distribution of the random variable when exists on a neighbourhood containing the origin. We have seen that sometimes all the moments exist but the moment generating function may not exist. So the following questions come naturally in mind.

i) If all the moments (integral moments are enough) of a random variable X with distribution function F exist, do the sequence of moments $(\mu_k : k \geq 1)$ uniquely characterizes the distribution function F?

ii) If we are given a sequence of real numbers $(a_k : k \geq 1)$, does there exist a random variable (can we construct a random variable) X such that

$$a_k = \mathbb{E}(X^k) \quad \text{for every } k \geq 1 \text{ ?}$$

iii) If $\mathbb{E}(|X|^n) < \infty$ for all $n \geq 1$, then does the moment generating function of X exist?

The answer is 'NO' to each of the above assumptions in general. But if we put restrictions on the law of the random variable X, then we may expect some affirmative answers.

Theorem

Let $(m_k : k \geq 1)$ be the moment sequence of a random variable X that is, $m_k = \mathbb{E}(X^k)$ for all $k \geq 1$. Then $(m_k : k \geq 1)$ uniquely characterizes the distribution function F of X if

$$\sum_{k=1}^{\infty} \frac{m_k}{k!} s^k$$

converges absolutely for some $s > 0$.

Theorem

Suppose there exists $(a,b) \subset \mathbb{R}$ such that $\mathbb{P}(X \in (a,b))$. Then $(\mathbb{E}(X^k) : k \geq 1)$ characterizes uniquely the distribution function of X.

Theorem (Carleman's sufficient condition).

~~Suppose that~~ ~~$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$~~ ~~be a random~~

Let $(m_k : k \geq 1)$ be a sequence of moments. Then there exists only one distribution function F (that is, only one random variable $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$) such that $m_k = \mathbb{E}(X^k)$ if

$$\sum_{k \geq 1}^{\infty} (m_{2k})^{-1/2k} = \infty.$$

~~If the random variable~~

Theorem

If $\lim_{n \to \infty} \frac{1}{2n} (m_{2n})^{1/2n}$ is finite then the moment sequence $(m_k : k \geq 1)$ uniquely characterizes the distribution of a random variable X such that

$$m_k = \mathbb{E}(X^k).$$

* We shall not prove these theorems in this course and rarely use them.

Lecture - X. Classification of random vectors through their distribution functions.

Probability space - $(\Omega, \mathcal{F}, \mathbb{P})$.

A Description of $\mathcal{B}(\mathbb{R}^n)$.

Consider the collection $\left((-\infty, a_1] \times (-\infty, a_2] \times \cdots \times (-\infty, a_n] : (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \right) = \left(\{ \underline{x} \in \mathbb{R}^n : x_i \le a_i \text{ for } i = 1, 2, \ldots, n \} : (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \right) = \mathcal{R}^n$ (n-dimensional ~~rect~~ rectangles). Each element of $\mathcal{B}(\mathbb{R}^n)$ can be thought of as the countable unions, intersections of $\mathcal{R}^n$ or their complements. Roughly speakings $\mathcal{R}^n$ turns out to be the generator of $\mathcal{B}(\mathbb{R}^n)$.

* This description is rough and there are other precise and equivalent descriptions.

Definition of a random vector.

A map $(X_1, X_2, \ldots, X_n) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is called a random vector if for every $A \in \mathcal{B}(\mathbb{R}^n)$

$$(X_1, \ldots, X_n)^{-1}(A) = \{ \omega \in \Omega : (X_1(\omega), X_2(\omega), \ldots, X_n(\omega)) \in A \} \in \mathcal{F}.$$

Remark If $(X_1, X_2, \ldots, X_n)$ be a random vector then $X_i : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable. Why??

Consider any subset $A \in \mathcal{B}(\mathbb{R})$. We show that

$X_1 : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable.

The same line of arguments work for any X_i with $i \neq 1$.

~~Note that~~ $\cancel{A = \bigcup_{n=1}^{\infty} \left(A \times (-\infty, n] \times (-\infty, n] \times \cdots \times (-\infty, n]\right)}$

and so, $\cancel{A \in \mathcal{B}(\mathbb{R}^n)}$. Note that

$$A \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \bigcup_{n=1}^{\infty} \left(A \times (-\infty, n] \times (-\infty, n] \times \cdots \times (-\infty, n]\right)$$

and so, $A \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \in \mathcal{B}(\mathbb{R}^n)$.

Observe that

$$\{\omega : X_1(\omega) \in A\} = \{\omega \in \Omega : X_1(\omega) \in A, \text{ and } X_i(\omega) \in \mathbb{R} \text{ for } i \geq 2\} = \{\omega \in \Omega : (X_1, X_2, \ldots X_n)(\omega) \in A \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}\}$$

$$= \{\omega \in \Omega : (X_1(\omega), X_2(\omega), \ldots, X_n(\omega)) \in A \times \mathbb{R} \times \cdots \times \mathbb{R}\}$$

$$\in \mathcal{F}$$

as $(X_1, X_2, \ldots X_n)$ is a random vector.

Theorem-1.

Let $(X_1, X_2, \ldots X_n) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a random vector and $g : \mathbb{R}^n \to \mathbb{R}^k$ be a continuous function. Then $g(X_1, X_2, \ldots, X_n) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ is a random vector.

__Proof.__ Proof is beyond the scope of this course and therefore, ommitted.

__Notation.__

$$\mathbb{P}_{X_1, X_2, \ldots, X_n}(A_1 \times A_2 \times \cdots \times A_n)$$
$$= \mathbb{P}\left(\{\omega \in \Omega : X_1(\omega) \in A_1, X_2(\omega) \in A_2, \ldots, X_n(\omega) \in A_n\}\right)$$
$$= \mathbb{P}\left(\bigcap_{i=1}^{n} \{\omega \in \Omega : X_i(\omega) \in A_i\}\right)$$

for every $A_1 \times A_2 \times \cdots \times A_n \in \mathcal{B}(\mathbb{R}^n)$.

Probability on $(\Omega, \mathcal{F})$ leads us to a probability $\mathbb{P}_{X_1, X_2, \ldots, X_n}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ through the random vector $(X_1, X_2, \ldots, X_n)$.

> We call $\mathbb{P}_{X_1, X_2, \ldots, X_n}$ the probability measure induced by the random vector.

__Distribution function of a random vector.__

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$
$$= \mathbb{P}_{X_1, X_2, \ldots, X_n}\left((-\infty, x_1] \times (-\infty, x_2] \times \cdots \times (-\infty, x_n]\right)$$

By the properties of probability

$F_{X_1, X_2, \ldots X_n} : \mathbb{R}^n \longrightarrow [0,1]$.

Theorem

A function $F : \mathbb{R}^n \longrightarrow [0,1]$ is joint probability distribution fuction of some n-dimensional random vector if and only if the following conditions hold.

i) For any pair $\underline{x} = (x_1, x_2, \ldots x_n)$ and $\underline{y} = (y_1, y_2, \ldots y_n)$, such that $x_i \le y_i$ for all $i = 1, 2, \ldots, n$. Then

$$F(x_1, x_2, x_3, \ldots, x_n) \le F(y_1, y_2, y_3, \ldots, y_n).$$

ii) $$F(-\infty, x_2, \ldots x_n) = F(x_1, -\infty, x_3, \ldots x_n)$$
$$= F(x_1, x_2, -\infty, \ldots x_n) = \ldots$$
$$= F(x_1, x_2, \ldots x_{n-1}, -\infty) = 0.$$

iii) $F(\infty, \infty, \ldots, \infty) = 1$.

iv) For any $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ such that $\epsilon_i \downarrow 0$ for $1 \le i \le n$ (or $\max\limits_{1 \le i \le n} \epsilon_i \downarrow 0$),

$$\lim_{\underline{\epsilon} \downarrow 0} F(\underline{x} + \underline{\epsilon}) = \lim_{\epsilon_i \downarrow 0 \,\forall\, i = 1, 2 \ldots n} F(x_1 + \epsilon_1, x_2 + \epsilon_2, \ldots, x_n + \epsilon_n)$$

$$= \lim_{\max\limits_{1 \le i \le n} \epsilon_i \downarrow 0} F(x_1 + \epsilon_1, x_2 + \epsilon_2, \ldots, x_n + \epsilon_n)$$

$$= F(\underline{x}) = F(x_1, x_2, \ldots x_n).$$

This property is called right continuity of the function F.

v) For every $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\delta_i > 0$ for $i = 1, 2, \ldots, n$, then the following inequality

$$F(x_1+\delta_1, x_2+\delta_2, \ldots, x_n+\delta_n)$$

$$-\sum_{i=1}^{n} F(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1}, \ldots, x_n+\delta_n)$$

$$+\sum_{i=1}^{n} \sum_{j=1, j>i}^{n} F(x_1+\delta_1, \ldots, x_{i-1}+\delta_{i-1}, x_i, x_{i+1}+\delta_{i+1}, \ldots,$$

$$x_{j-1}+\delta_{j-1}, x_j, x_{j+1}+\delta_{j+1}, x_{j+2}+\delta_{j+2}, \ldots, x_n+\delta_n)$$

$$+ \cdots$$

$$+(-1)^n F(x_1, x_2, \ldots, x_n) \geq 0.$$

Proof. We can do it for two-dimensional random vectors or distribution function. Then, most probably, the method of induction can be used to generalize it.

* We shall only prove that distribution function of a random vector satisfies that property.

* If a ~~new~~ function $F : \mathbb{R}^n \to [0,1]$ satisfies the properties mentioned in the theorem, then a random vector $(X_1, X_2, \ldots, X_n)$ can

can be constructed such that for every $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i \le x_i\}\right) = \mathbb{F}(x_1, x_2, \ldots, x_n).$$

But this construction is not in your syllabus and hence is skipped.

Let us consider a two-dimensional random vector $(X, Y) : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

Define

$$\mathbb{F}(x, y) = \mathbb{P}(X \le x, Y \le y) = \mathbb{P}\left(\{X \le x\} \cap \{Y \le y\}\right)$$

for every $(x, y) \in \mathbb{R}^2$. Consider another pair (x', y'). Then it follows ~~that~~ from the monotonicity property of probability that

$$\begin{aligned}
\mathbb{P}(X \le x, Y \le y) &= \mathbb{P}\left(\{X \le x\} \cap \{Y \le y\}\right) \\
&= \mathbb{P}\left((X, Y) \in (-\infty, x] \times (-\infty, y]\right) \\
&\le \mathbb{P}\left((X, Y) \in (-\infty, x'] \times (-\infty, y']\right) \\
&= \mathbb{P}(X \le x', Y \le y').
\end{aligned}$$

if $x' \ge x$ and $y' \ge y$. Hence, the first property follows.

<u>Exercise</u> – Derive right continuity of $\mathbb{F}$ as a consequence of continuity of probability.

Hint. – Check the derivation/proof of right-continuity of distribution function of a <u>random variable</u>.

Note that

$$\mathbb{F}(\infty,\infty) = \mathbb{P}\left((X,Y) \in \mathbb{R}\times\mathbb{R}\right)$$

$$= \mathbb{P}\left(\bigcup_{n=1}^{\infty} \{(X,Y) \in (-\infty,n]\times(-\infty,n]\}\right)$$

As $\left((-\infty,n]\times(-\infty,n] : n\geq 1\right)$ is an increasing sequence of sets, it follows from continuity of probability that

$$\mathbb{P}\left(\bigcup_{n\geq 1}^{\infty} \{ (X,Y) \in (-\infty,n]\times(-\infty,n] \}\right)$$

$$= \lim_{n\to\infty} \mathbb{P}\left((X,Y) \in (-\infty,n]\times(-\infty,n]\right)$$

$$= \lim_{n\to\infty} \mathbb{F}(n,n).$$

Note that by definition of random vector, it follows that $\mathbb{P}((X,Y) \in \mathbb{R}\times\mathbb{R}) = 1$. which implies that

$$\lim_{n\to\infty} \mathbb{F}(n,n) = 1.$$

- <u>Generalization</u> Note that the argument only uses the fact that (n,n)