AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 12: Control Elements

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Control Element Specification

- We see that basic mathematical operations, e.g. scaling (K), integration $(1/s)$ and differentiation (s), are able to achieve the desired performance attributes.
- Therefore, we can set up closed loop control systems by incorporating these mathematical actions in the feedback control structure.
- However, we need to understand their implications so that we can link these to complex control objectives.

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P Control Action

Proportional Controller

- Produces an output, which is proportional to error signal, i.e. difference between reference signal and output

$$u(t) \propto e(t), \quad e(t) = r(t) - c(t)$$

- Using proportionality constant(K_p),

$$u(t) = K_p e(t)$$

- Apply Laplace Transform,

$$U(s) = K_p E(s)$$

- Therefore,

$$\frac{U(s)}{E(s)} = K_p$$

Thus, the transfer function of proportional controller is K_p.

P Control Action

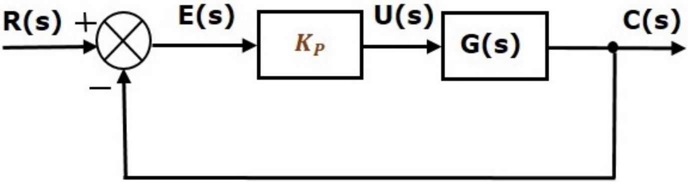


Figure: *"https://www.tutorialspoint.com"*

Figure: *Source - Ashok Joshi: "System Modeling Dynamics and Control", Lecture Notes, IIT Bombay, Mumbai, 2019."*

P Control Action - Features

Features

- P control remains active for non-zero values of $e(t)$. Further, higher error, leads to larger action.
- It is also seen that if K_p is higher, same error results in larger control action, causing tighter control.
- P control is the simplest and hence, is common in most situations as it has the ability to achieve the objectives.

P Control Action - Example

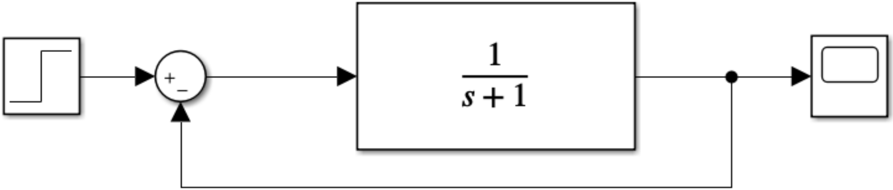
Example: A system is defined by the following transfer function,

$$G(s) = \frac{1}{s+1}$$

- Find the closed loop unity feedback transfer function.
- Unit step is applied as input to this system, calculate the settling time and steady state error for this system.
- If a P controller with $K_P = 5$ is cascaded with system, find the new settling time and steady state error.

P Control Action - Example

Solution:



- Closed loop transfer function is given by,

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1+G(s)} = \frac{1}{s+2}$$

P Control Action - Example

- For step input,

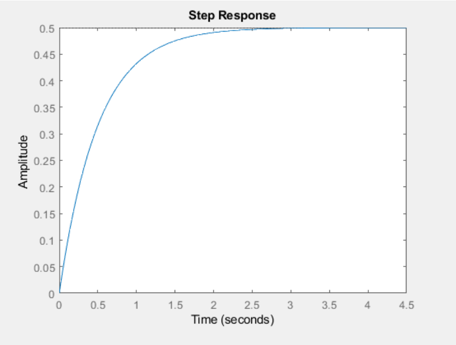
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{1}{s+2}\right)$$

- By taking inverse Laplace,

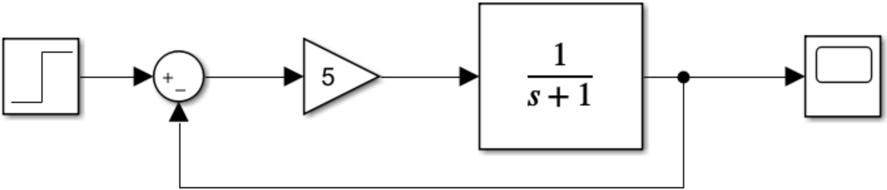
$$c(t) = 0.5(1 - e^{-2t})$$

- Time constant $= 1/2$
- Settling time $= 4 \times 0.5 = 2$
- Steady state error,

$$e_{ss} = 1 - 0.5 = 0.5$$



P Control Action - Example



- Closed loop transfer function is given by,

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1+G(s)} = \frac{5}{s+6}$$

P Control Action - Example

- For step input,

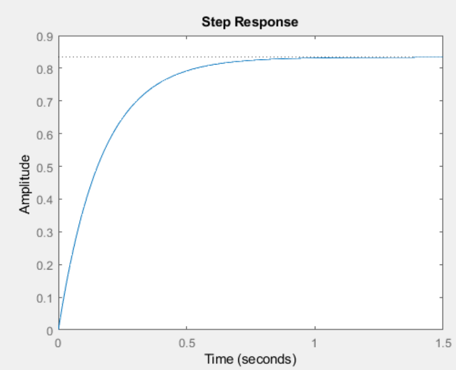
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{5}{s+6}\right)$$

- By taking inverse Laplace,

$$c(t) = 0.83(1 - e^{-6t})$$

- Time constant $= 1/6$
- Settling time $= 4 \times 6 = 0.66$
- Steady state error,

$$e_{ss} = 1 - 0.83 = 0.17$$



By cascading P controller, both settling time and steady state error decrease.

P Control Action

Any other method to calculate e_{ss} directly?

P Control Action - Example

- Using **final value theorm** to find steady state value,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{5}{s+6}\right)$$

$$c(\infty) = 0.83$$

Therefore,

$$e_{ss} = 1 - 0.83 = 0.17$$

P Control Action - Advantages and Limitations

Advantages

- The proportional controller helps in reducing the steady-state error.
- The response of system can be made faster with the help of these controllers.

Limitations

- If K_P is large, system may oscillate.
- Even if the system is stable, it may take a long time to settle to its final output value or exhibit large overshoots.
- It may not have sufficient tolerance to perturbations or disturbances.

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I Control Action

Integral Controller

- Produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t)$$

- Apply Laplace Transform,

$$U(s) = \frac{K_I E(s)}{s}$$

- Therefore,

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Thus, the transfer function of integral controller is $\frac{K_I}{s}$.

I Control Action

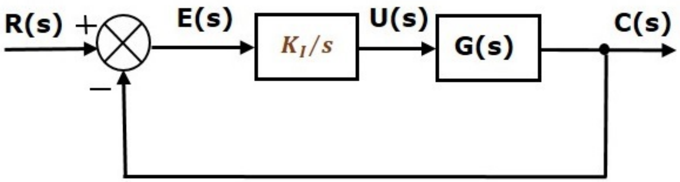


Figure: *"https://www.tutorialspoint.com""*

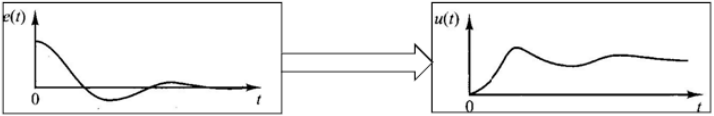


Figure: *Source - Ashok Joshi: "System Modeling Dynamics and Control", Lecture Notes, IIT Bombay, Mumbai, 2019."*

I Control Action - Features

Features

- In this case, the control action continues till the accumulated error remains non-zero.
- Control action continues long after the instantaneous error has gone to zero and can be used to make it more appropriate for tracking task.
- It also takes a long time for the controller to go to zero.
- Proportional and integral controllers are used together.

PI Control Action - Example

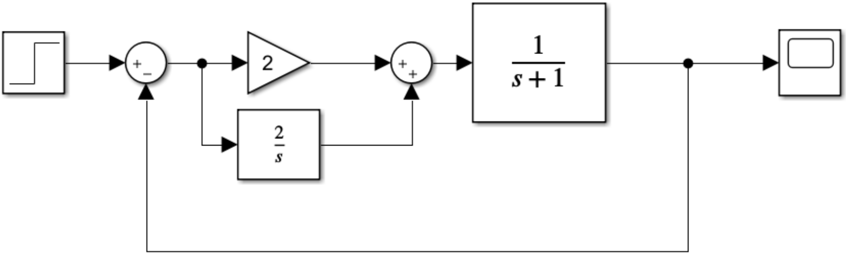
Example: A system is defined by the following transfer function,

$$G(s) = \frac{1}{s+1}$$

- If a PI controller with $K_P = 2$ and $K_I = 2$ is cascaded with the system, find the new settling time and steady state error for unit step input.

PI Control Action - Example

Solution:



- Closed loop transfer function is given by,

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1+G(s)} = \frac{2s+2}{s^2+3s+2}$$

PI Control Action - Example

- For step input,

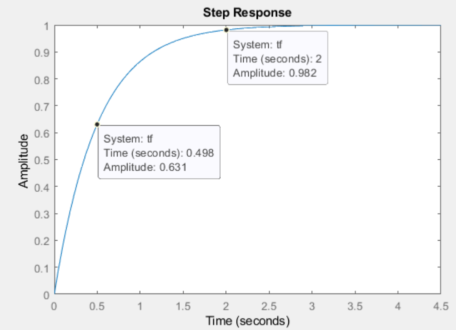
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{2}{s+2}\right)$$

- By taking inverse Laplace,

$$c(t) = 1 - e^{-2t}$$

- Time constant $= 1/2$
- Settling time $= 4 \times 0.5 = 2$
- Steady state error,

$$e_{ss} = 1 - 1 = 0$$



PI Control Action

What do you observe?

PI Control Action - Advantages and Limitations

Advantages

- It eliminates steady state error.

Limitations

- Not much effective in disturbances rejection

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D Control Action

Derivative Controller

- Produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{de(t)}{dt}$$

- Apply Laplace Transform,

$$U(s) = K_D s E(s)$$

- Therefore,

$$\frac{U(s)}{E(s)} = K_D s$$

Thus, the transfer function of derivative controller is $K_D s$.

D Control Action

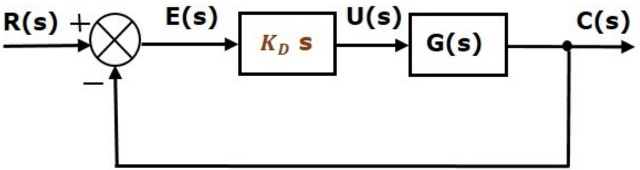


Figure: *"https://www.tutorialspoint.com"*

D Control Action - Features

Features

- In the context of D control, the control action starts, even before the error has time to build-up.
- This is some form of anticipation that the system acquires, which does not allow error to build-up and, has the ability to reach the steady-state faster.
- D control is, therefore, ideal for disturbance rejection.
- However, there can be significantly larger control input at the start, which can also become unbounded in some situations.

PD Control Action - Example

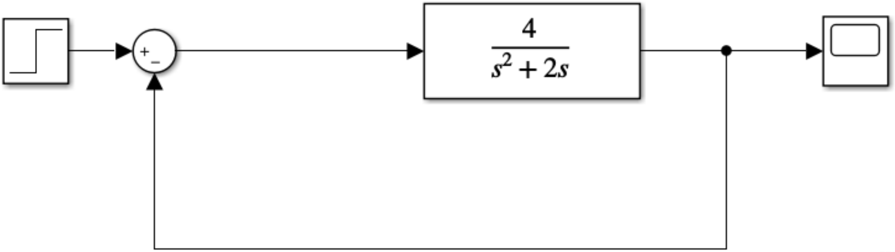
Example: A system is defined by the following transfer function,

$$G(s) = \frac{4}{s^2 + 2s}$$

- Find the closed loop unity feedback transfer function.
- Unit step is applied as input to this system, calculate the percentage overshoot for this system.
- If a PD controller with $K_P = 2$ and $K_D = 1$ is cascaded with system, find the new percentage overshoot for this system.

PD Control Action - Example

Solution:



- Closed loop transfer function is given by

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1+G(s)} = \frac{4}{s^2+2s+4}$$

PD Control Action - Example

- From general second order system analysis,

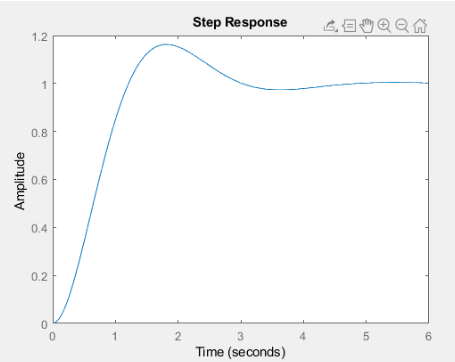
$$\frac{C(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Therefore,

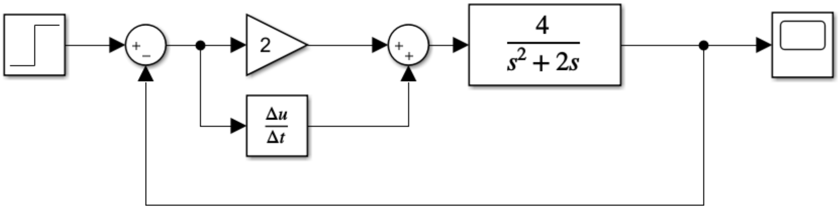
$$\omega_n = 2, \zeta = 0.5$$

- We know percentage overshoot is given by

$$\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 16.3\%$$



PD Control Action - Example



- Closed loop transfer function is given by

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1 + G(s)} = \frac{4(s+2)}{s^2 + 6s + 8}$$

PD Control Action - Example

- For step input,

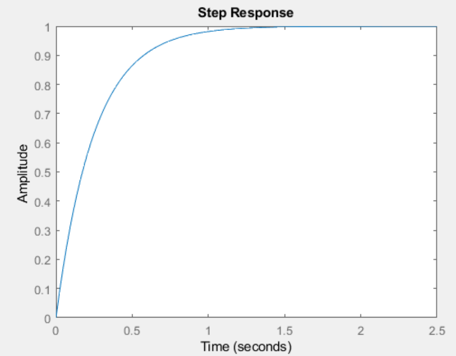
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{4}{s+4}\right)$$

- By taking inverse Laplace,

$$c(t) = 1 - e^{-4t}$$

- We can see their is no overshoot, therefore

$$\%OS = 0\%$$



PD Control Action - Advantages and Limitations

Advantages

- It can reduce the overshoot of a proportional controller response, because PD controller takes into account the rate of change in error.
- It can improve the system tolerance to external disturbances.

Limitations

- It does not improve the steady-state error in general.
- It amplifies the noise signals produced in the system.

PID Control Action - Example

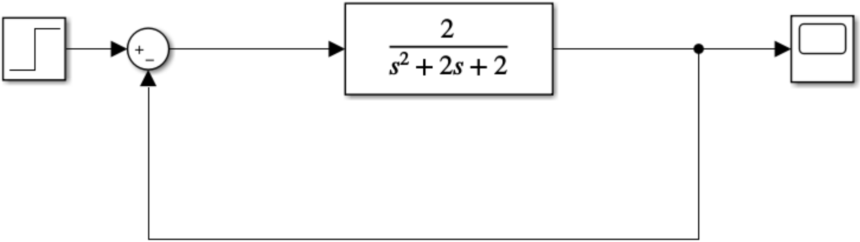
Example: A system is defined by the following transfer function,

$$G(s) = \frac{2}{s^2 + 2s + 2}$$

- Find the closed loop unity feedback transfer function.
- Unit step is applied as input to this system, calculate the percentage overshoot for this system.
- If a PID controller with $K_P = 10$, $K_I = 10$ and $K_D = 5$ is cascaded with system, what will happen to steady state error and overshoot?

PID Control Action - Example

Solution:



- Closed loop transfer function is given by,

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1 + G(s)} = \frac{2}{s^2 + 2s + 4}$$

PID Control Action - Example

- From general second order system analysis,

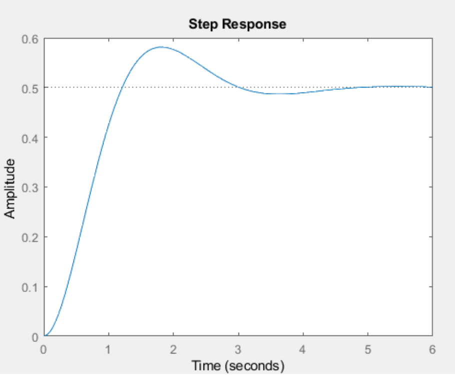
$$\frac{C(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Therefore,

$$\omega_n = 2, \zeta = 0.5$$

- We know percentage overshoot is given by,

$$\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 16.3\%$$



PID Control Action - Example

- Using Final value theorm to find steady state value,

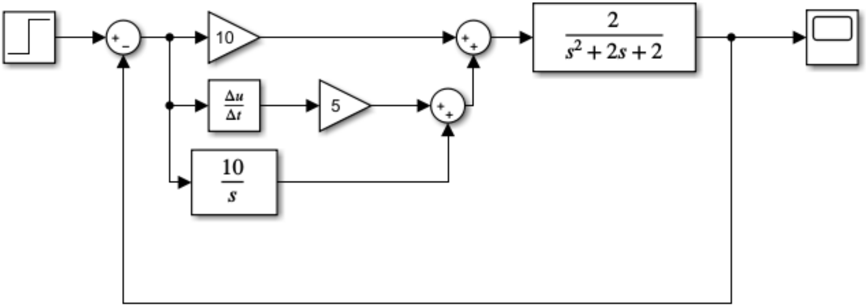
$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{2}{s^2+2s+4}\right)$$

$$c(\infty) = 0.5$$

Therefore,

$$e_{ss} = 1 - 0.5 = 0.5$$

PID Control Action - Example



- Closed loop transfer function is given by,

$$\frac{C(s)}{U(s)} = \frac{G(s)}{1+G(s)} = \frac{10}{s+10}$$

PID Control Action - Example

- For step input,

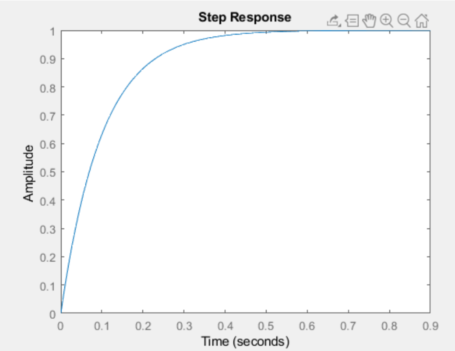
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{10}{s+10}\right)$$

- By taking inverse Laplace,

$$c(t) = 1 - e^{-10t}$$

- We find,

$$\%OS = 0\%, \quad e_{ss} = 0$$



PID Control Action - Advantages and Limitations

Advantages

- It can reduce the overshoot of the system.
- It can improve the system tolerance to external disturbances.
- Steady state error decreases.

Limitations

- Tuning of PID controller parameters is tedious.

PID Control Action - Impact

PID Gain	Percent Overshoot	Settling Time	e_{ss}
Increasing K_P	Increases	Minimal Impact	Decreases
Increasing K_I	Increases	Increases	Reduces
Increasing K_D	Decreases	Decreases	No Impact

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