Revision from last class

Harmonically forced system

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x(t) = e^{-\omega_n \zeta t} \{ C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t) \} + \frac{F_0}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi)$$

$$r = \frac{\omega}{\omega_n}$$

Use initial conditions

$$\tan \phi = \frac{2\zeta r}{1-r^2}$$

$$\frac{c}{c_c} = \zeta \qquad c_c = 2\sqrt{km}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

If $F(t) = 0$

$$x(t) = e^{-\omega_n \zeta t} \left[u_0 \cos(\omega_d t) + \left(\frac{v_0 + \omega_n \zeta u_0}{\omega_d} \right) \sin(\omega_d t) \right]$$

when
$x(t=0) = u_0$
$\dot{x}(t=0) = v_0$

Base excitation problems

y : Base excitation

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x-y) = 0$$

Relative motion $\quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad z = x - y$

Absolute motion $\quad m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$

Nonharmonically Forced Vibration

We have seen that periodic forces of any general waveform can be represented by Fourier series as a superposition of harmonic components of various frequencies. The response of a linear system is then found by superposing the harmonic response to each of the exciting forces. When the exciting force $F(t)$ is nonperiodic, such as that due to the blast from an explosion, a different method of calculating the response is required. Various methods can be used to find the response of the system to an arbitrary excitation. Some of these methods are as follows:

1. Representing the excitation by a Fourier integral.
2. Using the method of convolution integral.
3. Using the method of Laplace transforms.

Fourier Series method for periodic excitations

Response under a general periodic force

If the forcing function is periodic, we can use the Fourier series and the principle of superposition to get the response. The Fourier series states that a periodic function can be represented as a series of sines and cosines:

$$F(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots$$

$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + b_j \sin j\omega t$$

$$a_j = \frac{2}{T} \int_0^T F(t) \cos(j\omega t)\, dt \quad , \quad j = 0, 1, 2, \ldots$$

$$b_j = \frac{2}{T} \int_0^T F(t) \sin(j\omega t)\, dt \quad , \quad j = 1, 2, 3, \ldots$$

where $T = 2\pi/\omega$ is the period. Now the equation of motion can be written as:

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + b_j \sin j\omega t$$

Using the principle of superposition, the steady-sate solution of this equation is the sum of the steady-state solutions of:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t$$

The particular solution of the 1st equation is:

$$x_p(t) = \frac{a_0}{2k}$$

The particular solutions of the 2nd and 3rd equations are:

$$x_p(t) = \frac{a_j/k}{\sqrt{\left(1-j^2r^2\right)^2 + \left(2\zeta jr\right)^2}} \cos\left(j\omega t - \phi_j\right)$$

$$x_p(t) = \frac{b_j/k}{\sqrt{\left(1-j^2r^2\right)^2 + \left(2\zeta jr\right)^2}} \sin\left(j\omega t - \phi_j\right)$$

$$\phi_j = \tan^{-1}\left(\frac{2\zeta jr}{1-j^2r^2}\right)$$

$$r = \frac{\omega}{\omega_n}$$

Then add up all the sums to get the complete steady-state solution as:

$$x_p(t) = \frac{a_o}{2k} + \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{\left(1-j^2r^2\right)^2 + \left(2\zeta jr\right)^2}} \cos\left(j\omega t - \phi_j\right) + \sum_{j=1}^{\infty} \frac{b_j/k}{\sqrt{\left(1-j^2r^2\right)^2 + \left(2\zeta jr\right)^2}} \sin\left(j\omega t - \phi_j\right)$$

Observe that if $j\omega = \omega_n$, the amplitude will be significantly large, especially for small j and ζ. Further, as j becomes large, the amplitude becomes smaller and the corresponding terms tend to zero. How many terms do you need to include?

Nonharmonic Periodic Motion

If $f(t)$ is continous, the series converges uniformly. At a discontinuity in $f(t)$ say at t_0, it converges to the mean value of the function from both the left, $f(t_0^-)$, and the right, $f(t_0^+)$, i.e.,

$$f(t_0) \rightarrow \frac{1}{2}[f(t_0^+) + f(t_0^-)]$$

Some simplification in the coefficients, a_n and b_n, is possible by decomposing $f(t)$ into two functions, $f_e(t)$ and $f_o(t)$, even anf off about $t = 0$,

$$f(t) = f_e(t) + f_o(t)$$

The sine series is represents $f_o(t)$, and the cosine series $f_e(t)$, The term $a_0/2$ represents the mean value of $f_e(t)$ over the periodic interval τ_p.

An even function satisfies the relation, $f(-t) = f(t)$, and the Fourier series expansion of $f(t)$ contains only cosine terms, i.e. $f(t) = f_e(t)$

Similiarly, an odd function satisfies the relation, $f(-t) = -f(t)$, and the Fourier series expansion of $f(t)$ contains only sine terms, i.e. $f(t) = f_o(t)$

Nonharmonic Periodic Motion

One target at a shooting gallery moves vertically by means of a cam rotating at a constant angular speed. The resulting motion in time is given by a sawtooth curve:

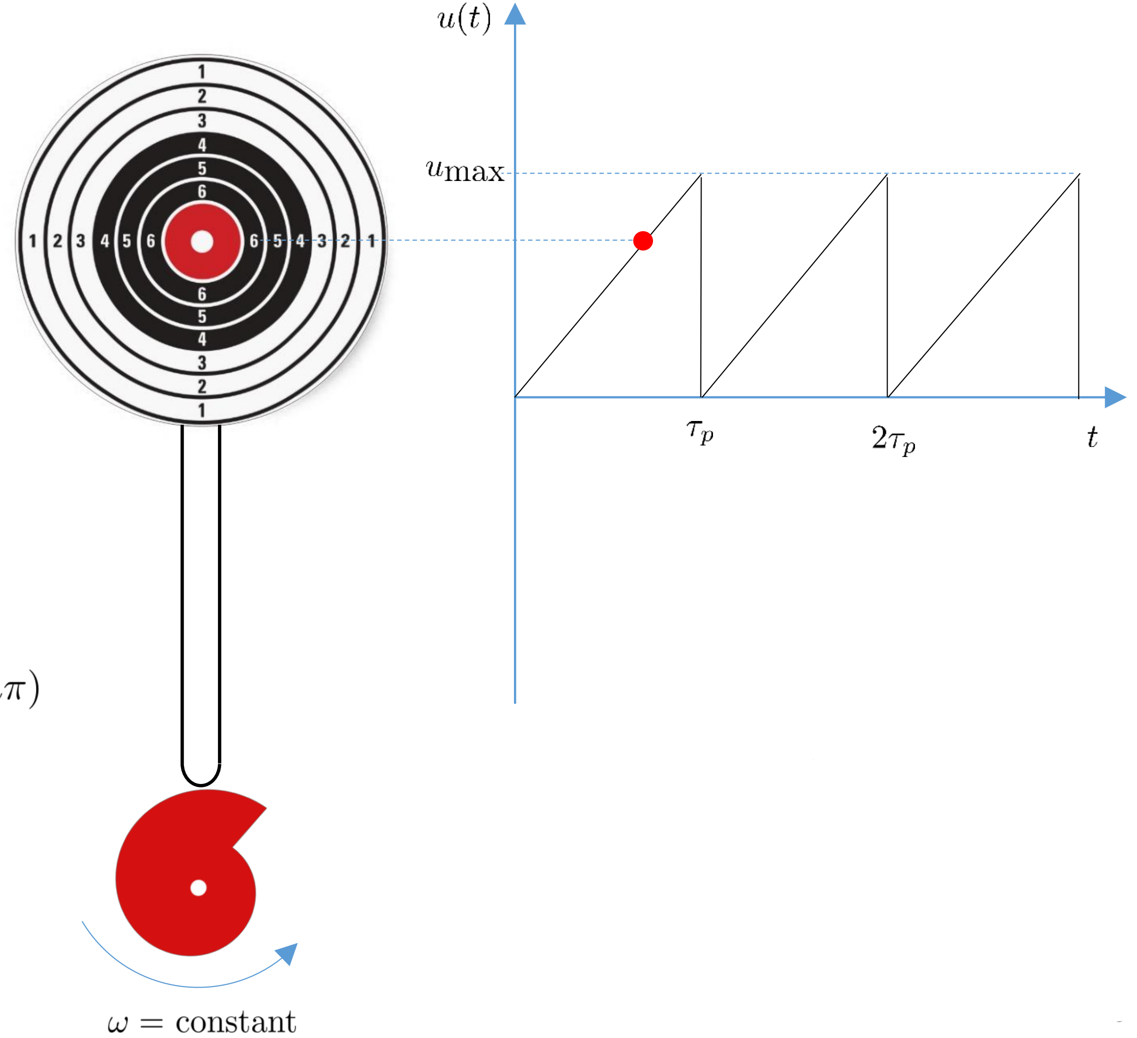
$$u(t) = u_{\max}(t/\tau_p); \quad 0 \leq t \leq \tau_p$$

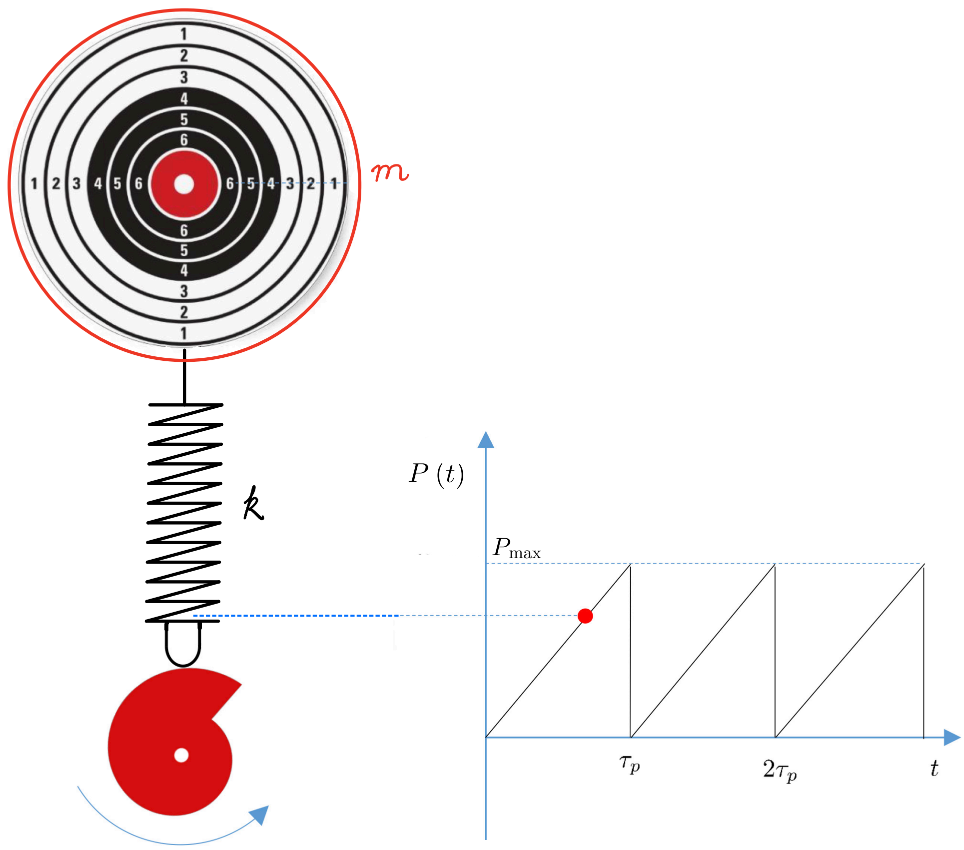
Note that this sawtooth curve,after a vertical translation of $u_{\max}/2$, is odd realtive to $t = 0$. Hence, only a_0 and b_n coefficients will be present.

$$a_0 = \frac{2}{\tau_p}\int_0^{\tau_p} u_{\max}(t/\tau_p)dt = u_{\max}$$

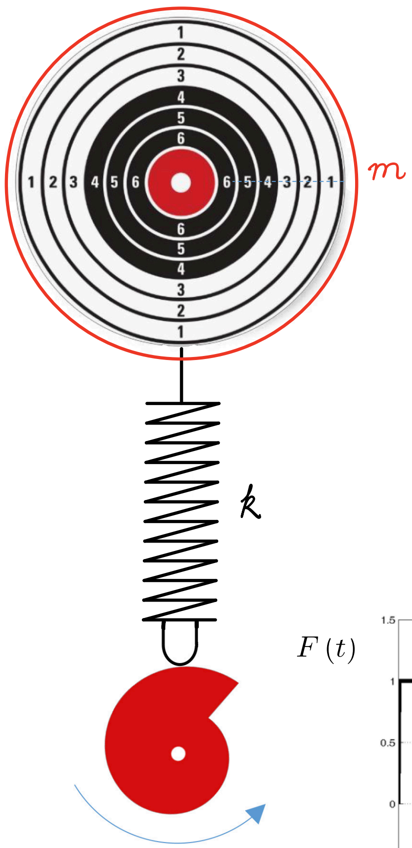
$$b_n = \frac{2}{\tau_p}\int_0^{\tau_p} u_{\max}(t/\tau_p)\sin(2\pi nt/\tau_p)dt = -(u_{\max}/n\pi)$$

Application of Fourier series extends beyond the representation of periodic motions and is used to describe other phenomenon such as cyclic forces, stresses, environmental processes, etc.

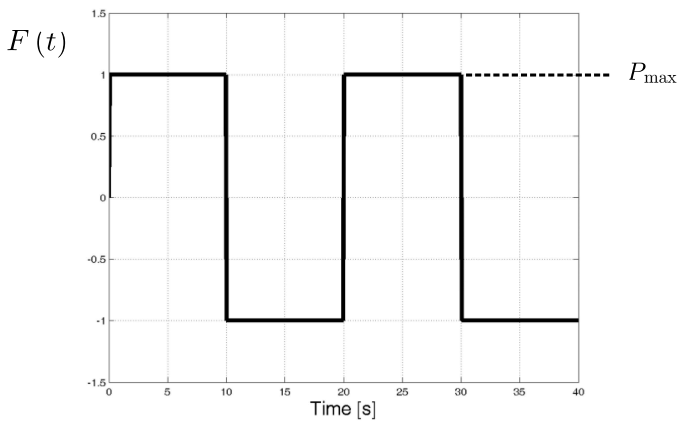




m
k
P (t)
$P_{\max}$
τ_p
$2\tau_p$
t



m
k



$F(t)$
$P_{\max}$
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
40
Time [s]

Solution: here we have:

$$T = 20 \quad , \quad \omega = 2\pi/T = 2\pi/20 = \pi/10$$

and the forcing function is given by:

$$F(t) = 1 \qquad 0 \le t \le 10$$

$$F(t) = -1 \quad 10 \le t \le 20$$

The Fourier series of the forcing function is given by:

$$F(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + ... + b_1 \sin \omega t + b_2 \sin 2\omega t + ...$$

To get the constants, we have:

$$a_0 = \frac{2}{T}\int_0^T F(t)\,dt = \frac{2}{20}\left[\int_0^{10} dt - \int_{10}^{20} dt\right] = 0.1\left[10 - (20-10)\right] = 0$$

$$a_j = \frac{2}{T}\int_0^T F(t)\cos(j\omega t)\,dt$$

$$= \frac{2}{20}\left[\int_0^{10} \cos\left(j\frac{\pi}{10}t\right)dt - \int_{10}^{20} \cos\left(j\frac{\pi}{10}t\right)dt\right]$$

$$= 0.1\left[\frac{10}{j\pi}\sin j\frac{\pi}{10}t\Big|_0^{10} - \frac{10}{j\pi}\sin j\frac{\pi}{10}t\Big|_{10}^{20}\right] = 0$$

i.e. all cosine terms vanish

$$b_j = \frac{2}{T}\int_0^T F(t)\sin(j\omega t)\,dt$$

$$= \frac{2}{20}\left[\int_0^{10} \sin\left(j\frac{\pi}{10}t\right)dt - \int_{10}^{20} \sin\left(j\frac{\pi}{10}t\right)dt\right]$$

$$= 0.1\left[\frac{-10}{j\pi}\cos j\frac{\pi}{10}t\Big|_0^{10} - \frac{-10}{j\pi}\cos j\frac{\pi}{10}t\Big|_{10}^{20}\right]$$

$$= 0.1\left[\frac{-10}{j\pi}(\cos j\pi - 1) + \frac{10}{j\pi}(\cos 2j\pi - \cos j\pi)\right]$$

If j is odd,

$$b_j = 0.1\left[\frac{-10}{j\pi}(-2)+\frac{10}{j\pi}(2)\right]=\frac{4}{j\pi}$$

If j is even,

$$b_j = 0.1\left[\frac{-10}{j\pi}(0)+\frac{10}{j\pi}(0)\right]=0$$

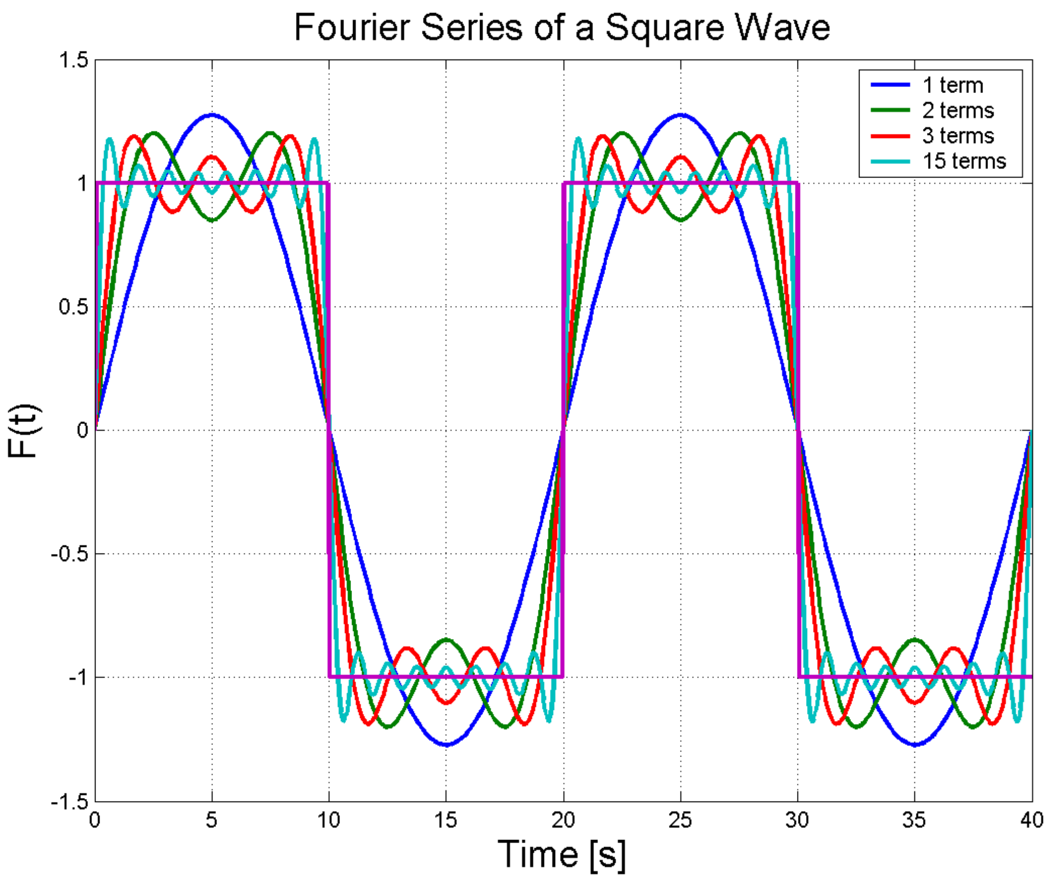
i.e. all even terms vanish.

In this way, the force can be represented by a Fourier series as:

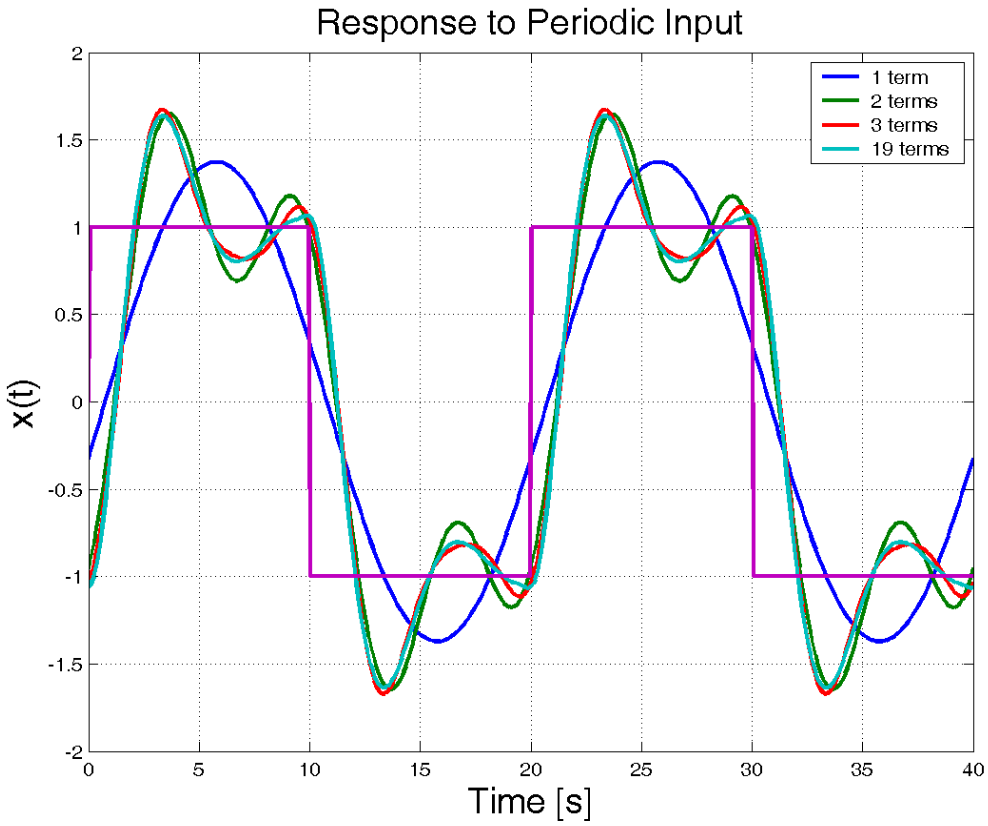
$$F(t)=b_1\sin\omega_1 t+b_3\sin\omega_3 t+b_5\sin\omega_5 t+\cdots=\sum_{j=1}^{\infty}b_j\sin(j\omega t) \quad , \quad j=1,3,5,7,\ldots$$

$$=\frac{4}{\pi}\sin\frac{\pi}{10}t+\frac{4}{3\pi}\sin\frac{3\pi}{10}t+\frac{4}{5\pi}\sin\frac{5\pi}{10}t+\cdots=\sum_{j=1}^{\infty}\frac{4}{j\pi}\sin\left(j\frac{\pi}{10}t\right) \quad , \quad j=1,3,5,7,\ldots$$

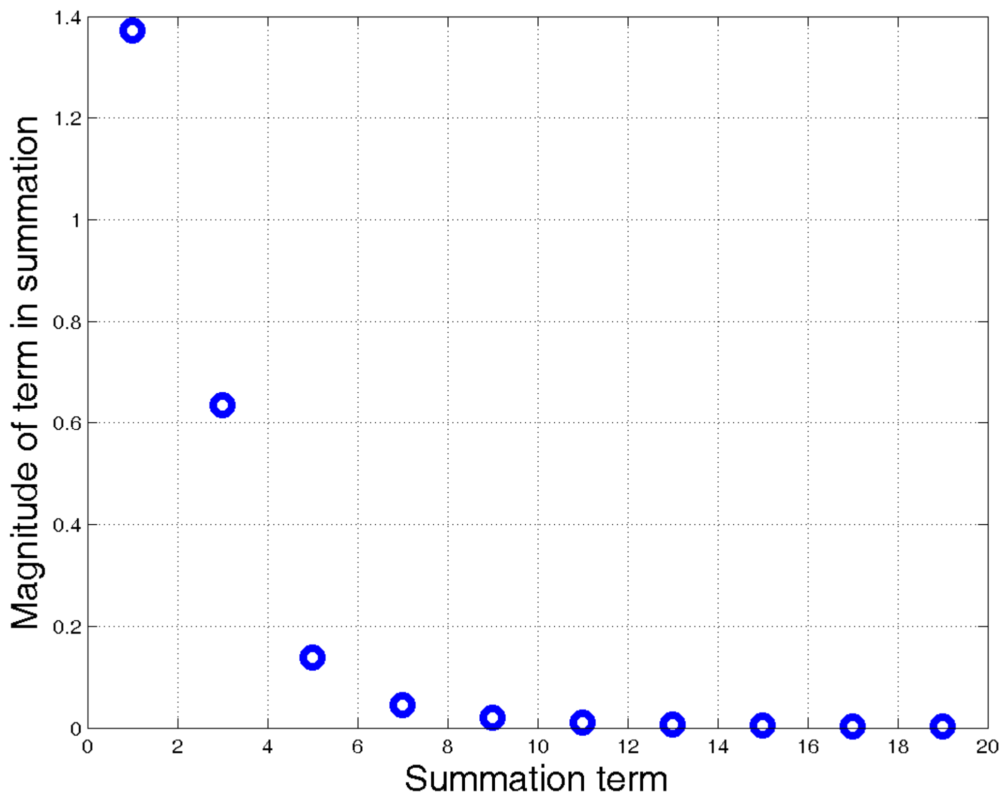
or graphically as:



The system response is shown below:



The contribution of each term in the summation is determined from:



Nonharmonically Forced Vibration

Consider the case of a forcing function which is periodic but nonharmonic such as that shown here. The response of the system to this loading function can be treated by superposing a series of harmonic responses. It involves the expansion of $P(t)$ in a Fourier series:

$$P(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

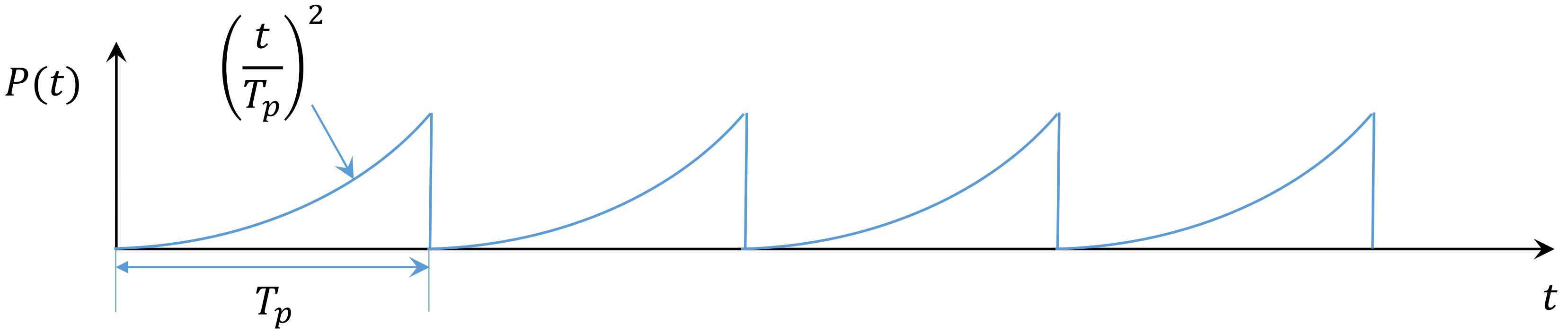
where T_p is the period of the nonharmonically forcing functions. The coefficients a_i and b_i have the form:

$$a_0 = \frac{2}{T_p}\int_0^{T_p} P(t)dt; \quad a_n = \frac{2}{T_p}\int_0^{T_p} P(t)\cos(p_n t)dt; \quad b_n = \frac{2}{T_p}\int_0^{T_p} P(t)\sin(p_n t)dt$$

where $n = 1, 2, ...$ and $(p_n = 2\pi n/T_p)$. The response $u(t)$ consists of a series of forced harmonic motions for each term in the series, i.e., $a_0/2, a_n\cos(p_n t)$ and $b_n \sin(p_n t)$.

Nonharmonically Forced Vibration

Consider a SDOF system with stiffness k and mass m. Let a forcing function be parabolic over a period T_p as shown in the figure below. Determine the response of this system. Assume at rest initial conditions.



The Fourier coefficients for $P(t) = (t/T_p)^2$ are:

$$a_0 = \frac{2}{T_p}\int_0^{T_p}\left(\frac{t}{T_p}\right)^2 dt = \frac{2}{3}\left(\frac{t}{T_p}\right)^3\Bigg|_0^{T_p} = \frac{2}{3}$$

$$a_n = \frac{2}{T_p}\int_0^{T_p}\left(\frac{t}{T_p}\right)^2 \cos(p_n t)dt$$

$$= \frac{1}{4\pi^3 n^3}\left[2p_n t\cos(p_n t) + \left\{(p_n t)^2 - 2\right\}\sin(p_n t)\right]\Big|_o^{T_p} = \frac{2}{\pi^2 n^2}$$

$$b_n = \frac{2}{T_p}\int_0^{T_p}\left(\frac{t}{T_p}\right)^2 \sin(p_n t)dt$$

$$= \frac{1}{4\pi^3 n^3}\left[2p_n t\sin(p_n t) + \left\{(p_n t)^2 - 2\right\}\cos(p_n t)\right]\Big|_o^{T_p} = \frac{1}{\pi n}$$

Therefore,

$$P(t) = \frac{1}{3} + \sum_{n=1}^{N}\frac{1}{\pi^2 n^2}\cos\left(\frac{2\pi n t}{T_p}\right) - \sum_{n=1}^{N}\frac{1}{n\pi}\sin\left(\frac{2\pi n t}{T_p}\right)$$

Nonharmonically Forced Vibration

The governing equation for this problem has the form:

$$\ddot{u} + \omega^2 u = \frac{\omega^2}{k} P(t)$$

The complete solution consists of a homogeneous part and a particular solution, which must be determined for each term in the expansion of $P(t)$.

$$\begin{aligned} u(t) &= u_p(t) + u_h(t) = C_1 \sin(\omega t) + C_2 \cos(\omega t) \\ &= \frac{1}{3k} + \sum_{n=1}^{N} \frac{1}{\pi^2 n^2 k} \left[\frac{1}{1-\alpha_n^2}\right] \cos\left(\frac{2\pi n t}{T_p}\right) - \sum_{n=1}^{N} \frac{1}{\pi n k} \left[\frac{1}{1-\alpha_n^2}\right] \sin\left(\frac{2\pi n t}{T_p}\right) \end{aligned}$$

where $\alpha_n = p_n/\omega = 2\pi n/(T_p\omega)$.Evaluating the constants of integration C1 and C2 at rest initial conditions yields:

$$C_1 = \frac{2}{T_p k \omega} \sum_{n=1}^{N} \frac{1}{1-\alpha_n^2}; \quad C_2 = -\frac{1}{3k} \left[1 + \sum_{n=1}^{N} \frac{3}{\pi^2 n^2 (1-\alpha_n^2)}\right]$$

Duhamel's integral approach for arbitrary excitations

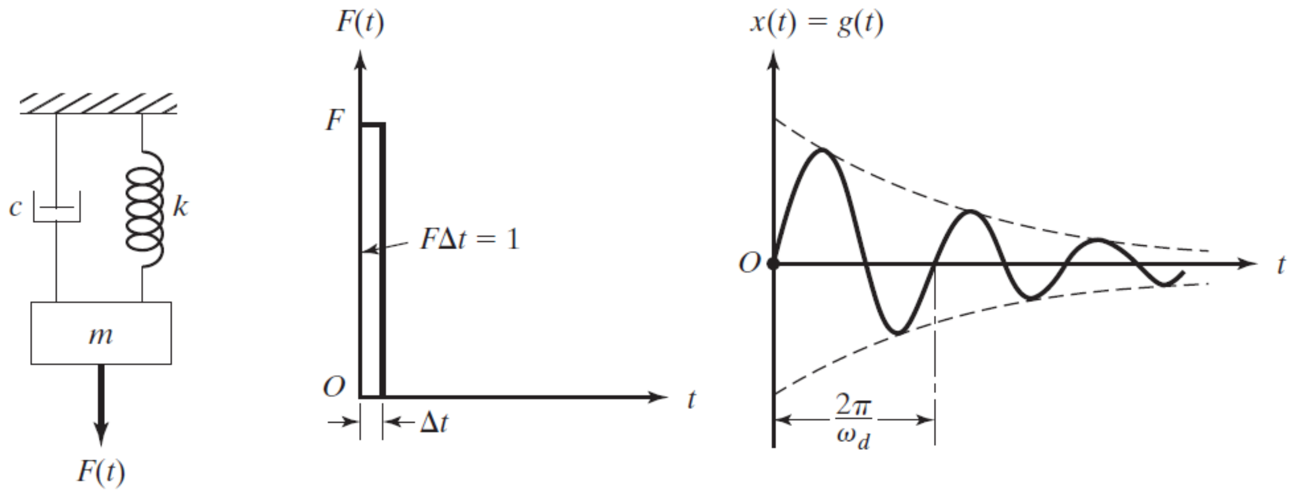
Response to Impulse

If the mass is at rest before the unit impulse is applied ($x = \dot{x} = 0$ for $t < 0$ or at $t = 0^-$), we obtain, from the impulse-momentum relation,

$$\text{Impulse} = \hat{f} = 1 = m\dot{x}(t = 0) - m\dot{x}(t = 0^-) = m\dot{x}_0$$

Thus the initial conditions are given by

$$x(t = 0) = x_0 = 0$$
$$\dot{x}(t = 0) = \dot{x}_0 = \frac{1}{m}$$



Response to Impulse

Consider a viscously damped spring-mass system subjected to a unit impulse at $t = 0$, as shown in Figs. 4.6(a) and (b). For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

is given by Eq. (2.72a) as

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

where

$$\zeta = \frac{c}{2m\omega_n}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Hence,

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

Response to Impulse

Previous slide equation gives the response of a single-degree-of-freedom system to a unit impulse, which is also known as the impulse response function, denoted by $g(t)$. The function $g(t)$, Eq. (4.25), is shown in Fig. 4.6(c).

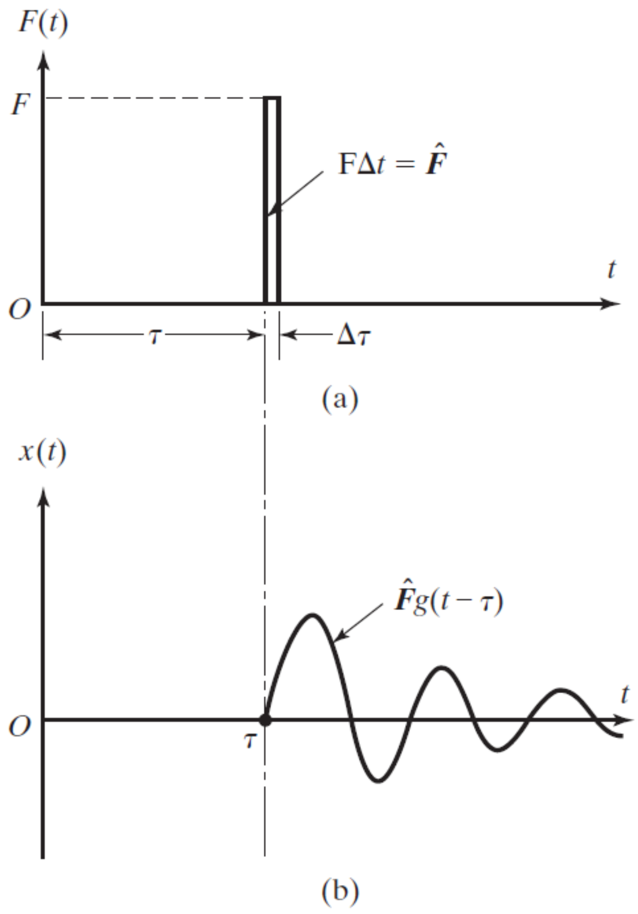
If the magnitude of the impulse is F instead of unity, the initial velocity x_0 is F/m and the response of the system becomes

$$x(t) = \frac{\mathrm{F}e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t = \hat{F} g(t)$$

If the impulse F is applied at an arbitrary time $t = \tau$, as shown in Fig. 4.7(a), it will change the velocity at $t = \tau$ by an amount F/m. Assuming that $x = 0$ until the impulse is applied, the displacement x at any subsequent time t, caused by a change in the velocity at time τ, is given by Eq. (4.26) with t replaced by the time elapsed after the application of the impulse-that is, $t - \tau$. Thus we obtain

$$x(t) = \hat{F} g(t - \tau)$$

This is shown in Figure



Response to an impulse

the principle of impulse and momentum states that impulse equals change in momentum:

$$Impulse = F\Delta t = m(v_2 - v_1)$$

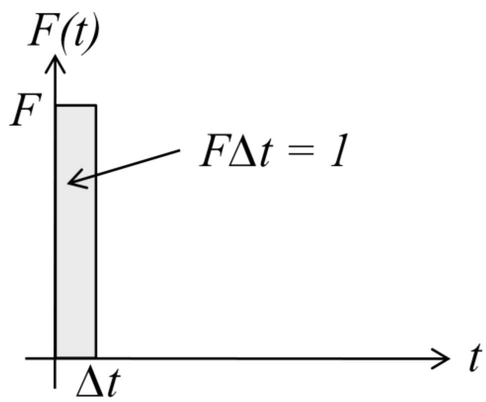
or:

$$\underline{F} = \int_t^{t+\Delta t} F dt = m\dot{x}_2 - m\dot{x}_1$$

A unit impulse is defined as:

$$\underline{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F dt = 1$$

Now consider the response of an undamped system to a unit impulse. Recall that the free vibration response is given by:



$$x(t) = x_0 cos\omega_n t + \frac{\dot{x}_0}{\omega_n} sin\omega_n t$$

If the mass starts from rest, we can get the velocity just after impulse as:

$$\dot{x}_0 = \frac{\underline{f}}{m} = \frac{1}{m}$$

and the response becomes:

$$x(t) = \frac{1}{m\omega_n} sin\omega_n t$$

for a non-unit impulse, the response is:

$$x(t) = \frac{\underline{F}}{m\omega_n} sin\omega_n t$$

For an underdamped system, recall that the free response was gven by:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} sin\omega_d t\right]$$

For a unit impulse, the response for zero initial conditions is:

$$x(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} sin\omega_d t = g(t)$$

where $g(t)$ is known as the *impulse response function*. For a non-unit impulse, the response is:

$$x(t) = \frac{\underline{F}e^{-\zeta\omega_n t}}{m\omega_d} sin\omega_d t = \underline{F}g(t)$$

If the impulse occurs at a delayed time $t = \tau$, then

$$x(t) = \frac{\underline{F}e^{-\zeta\omega_n (t-\tau)}}{m\omega_d} sin\omega_d (t-\tau) = \underline{F}g(t-\tau)$$

If two impulses occur at two different times, then their responses will superimpose.

Example: For a system having $m = 1$ kg; $c = 0.5$ kg/s; $k = 4$ N/m; F = 2 Ns obtain the response when two impulses are applied 5 seconds apart.

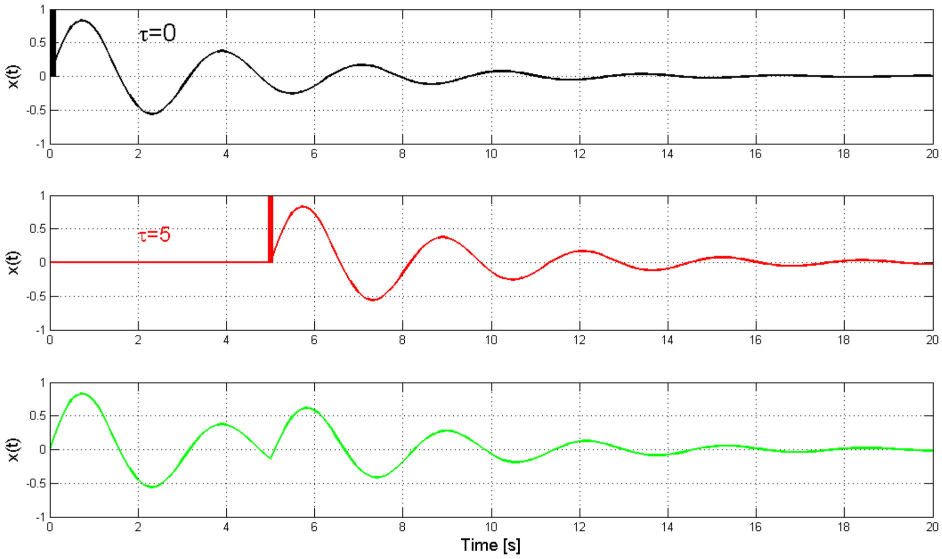
Solution: here we have $\omega_n = 2\frac{\text{rad}}{\text{s}}, \zeta = 0.125, \omega_d = 1.984\frac{\text{rad}}{\text{s}}$ so the solutions become:

$$x_1(t) = \frac{2e^{-0.25t}}{1.984} \sin 1.984t \qquad t > 0$$

$$x_2(t) = \frac{2e^{-0.25(t-\tau)}}{1.984} \sin[1.984(t-\tau)] \qquad t > 5$$

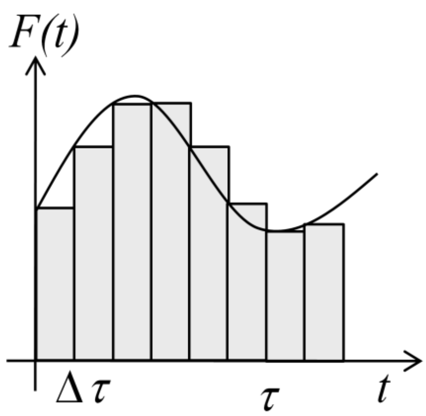
And the total response is:

$$x(t) = \begin{cases} \dfrac{2e^{-0.25t}}{1.984}\sin 1.984t & 0 < t < 5 \\ \dfrac{2e^{-0.25t}}{1.984}\sin 1.984t + \dfrac{2e^{-0.25(t-\tau)}}{1.984}\sin[1.984(t-\tau)] & 5 < t < 20 \end{cases}$$



Response to an Arbitrary Input

The input force is viewed as a series of impulses. The response at time t due to an impulse at time τ is:



$$x(t) = F(\tau)\Delta\tau g(t-\tau)$$

The total response at time t is the sum of all responses:

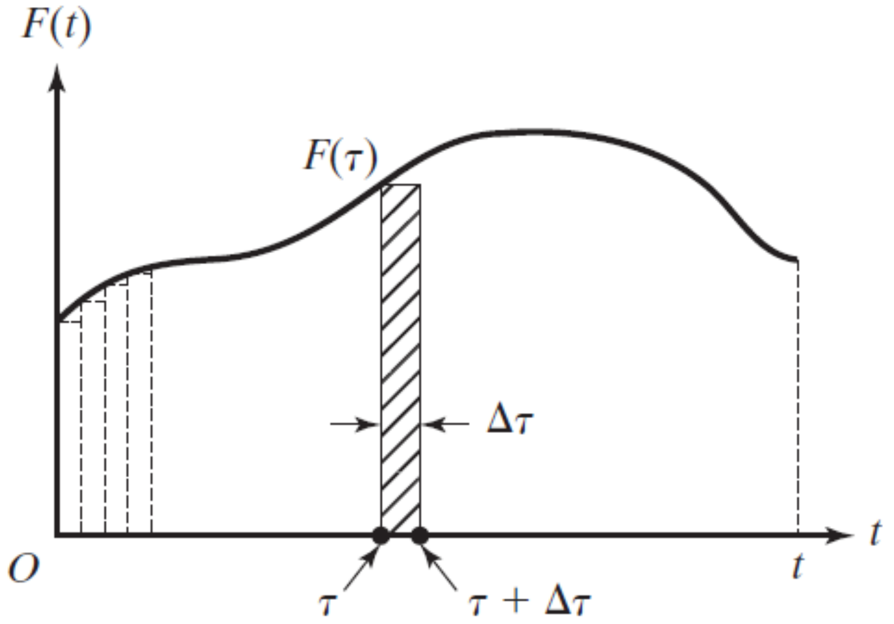
Response to General Forcing Condition

We consider the response of the system under an arbitrary external force $F(t)$, shown in Fig. 4.9. This force may be assumed to be made up of a series of impulses of varying magnitude. Assuming that at time τ, the force $F(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $F(\tau)\Delta\tau$. At any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at t due to this impulse alone is given by Eq. (4.27) with $\hat{\mathrm{F}} = F(\tau)\Delta\tau$:

$$\Delta x(t) = F(\tau)\Delta\tau g(t - \tau)$$

The total response at time t can be found by summing all the responses due to the elementary impulses acting at all times τ :

$$x(t) \simeq \sum F(\tau)g(t - \tau)\Delta\tau$$



$$x(t) = \sum F(\tau)g(t-\tau)\Delta\tau$$

Hence

$$x(t) = \int_0^t F(\tau)g(t-\tau)d\tau$$

For an underdamped system:

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)} sin\omega_d(t-\tau)\, d\tau$$

Note: this does not consider initial conditions. This type of formula is called the *convolution integral* or the *Duhamel integral*. For base excitation, the resulting response is

$$z(t) = -\frac{1}{\omega_d}\int_0^t \ddot{y}(\tau)e^{-\zeta\omega_n(t-\tau)}\sin(\omega_d(t-\tau))d\tau$$

Example: determine the response of a spring-mass-damper system due to the application of a force

Here we have

$F(t) = F_0$

so the response is obtained from

$$x(t) = \frac{1}{m\omega_d}\int_0^t F_0 e^{-\zeta\omega_n(t-\tau)}\sin\left[\omega_d(t-\tau)\right]d\tau$$

Therefore the solution is:

$$x(t)=\frac{-F_0}{m\omega_d}\left[\frac{-\omega_d+e^{-\zeta\omega_n t}\omega_d\cos\omega_d t+e^{-\zeta\omega_n t}\zeta\omega_n\sin\omega_d t}{\zeta^2\omega_n^2+\omega_d^2}\right]$$

which can be put in the form:

$$x(t)=\frac{-F_0}{m\omega_d}\left[\frac{-\omega_n\sqrt{1-\zeta^2}+e^{-\zeta\omega_n t}\omega_n\sqrt{1-\zeta^2}\cos\omega_d t+e^{-\zeta\omega_n t}\zeta\omega_n\sin\omega_d t}{\zeta^2\omega_n^2+\omega_n^2\left(1-\zeta^2\right)}\right]$$

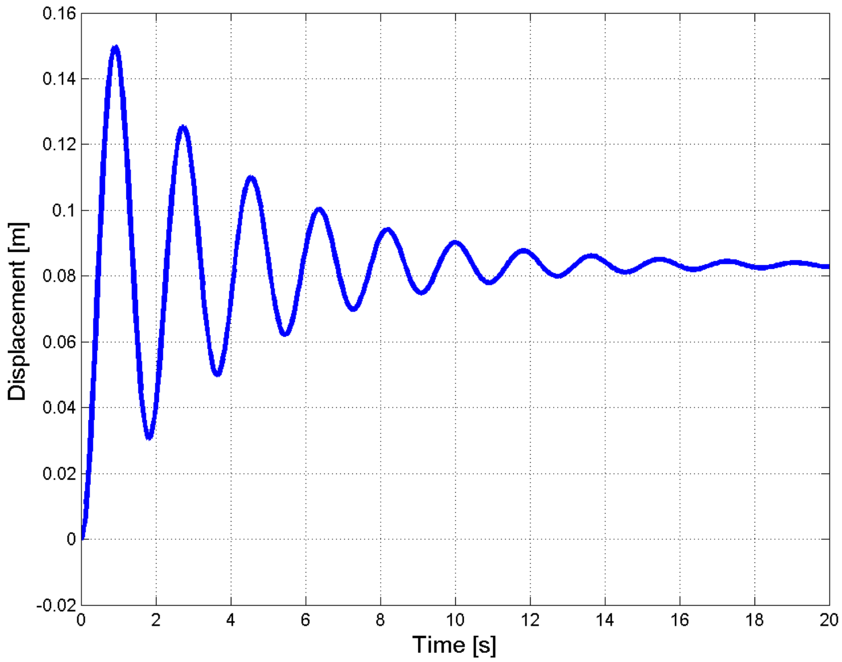
$$=\frac{-F_0}{m\omega_d\omega_n}\left[-\sqrt{1-\zeta^2}+e^{-\zeta\omega_n t}\cos\left(\omega_d t-\phi\right)\right]$$

$$=\frac{F_0}{k}\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t-\phi\right)\right]$$

where

$$\phi=\tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

The response is shown below. Notice how it converges to $\frac{F_0}{k}$.



Response to General Forcing Condition

Letting $\Delta\tau \longrightarrow 0$ and replacing the summation by integration, we obtain

$$x(t) = \int_0^t F(\tau)g(t-\tau)d\tau$$

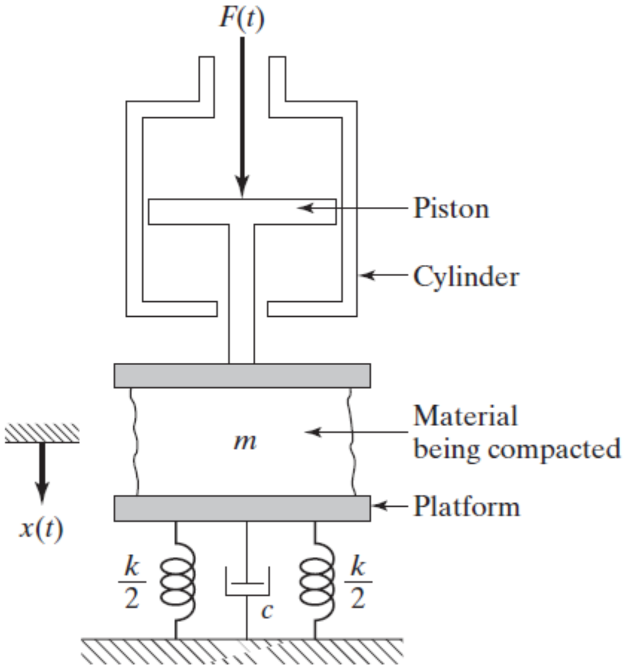
By substituting Eq. (4.25) into Eq. (4.30), we obtain

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau$$

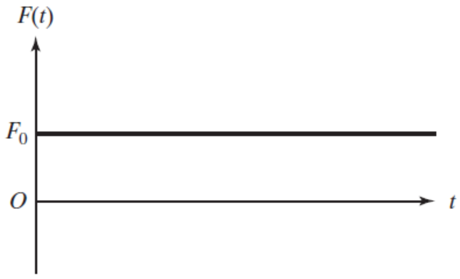
which represents the response of an underdamped single-degree-of-freedom system to the arbitrary excitation $F(t)$. Note that Eq. (4.31) does not consider the effect of initial conditions of the system, because the mass is assumed to be at rest before the application of the impulse, as implied by Eqs. (4.25) and (4.28). The integral in Eq. (4.30) or Eq. (4.31) is called the convolution or Duhamel integral. In many cases the function F(t) has a form that permits an explicit integration of Eq. (4.31). If such integration is not possible, we can evaluate numerically without much difficulty, as illustrated in Section 4.9 and in Chapter 11. An elementary discussion of the Duhamel integral in vibration analysis is given in reference [4.6].

Step Force on a Compacting Machine

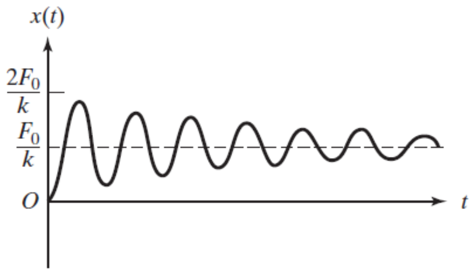
Question: A compacting machine, modeled as a single-degree-of-freedom system, is shown in Fig. (a) The force acting on the mass m (m includes the masses of the piston, the platform, and the material being compacted) due to a sudden application of the pressure can be idealized as a step force, as shown in Fig. (b) . Determine the response of the system.



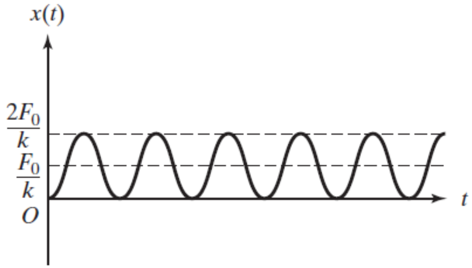
(a)



(b)



(c)



(d)

Step Force on a Compacting Machine

Solution: Since the compacting machine is modeled as a mass-spring-damper system, the problem is to find the response of a damped single-degree-of-freedom system subjected to a step force. By noting that $F(t) = F_0$, we can write Eq. as

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_d}\int_0^{l} e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)d\tau \\ &= \frac{F_0}{m\omega_d}\left[e^{-\zeta\omega_n(t-\tau)}\left\{\frac{\zeta\omega_n \sin \omega_d(t-\tau) + \omega_d \cos \omega_d(t-\tau)}{(\zeta\omega_n)^2 + (\omega_d)^2}\right\}\right]_{\tau=0}^{t} \\ &= \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_d t - \phi\right)\right] \end{aligned}$$

where

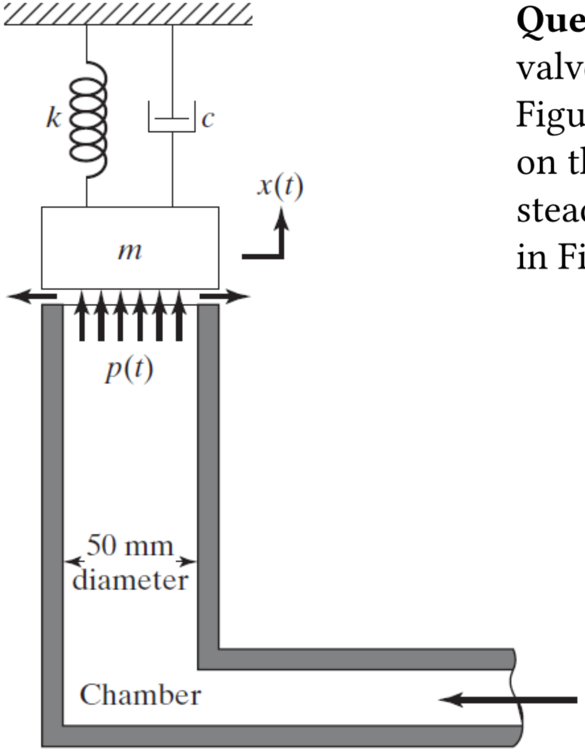
$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

This response is shown in Fig. (c) . If the system is undamped ($\zeta = 0$ and $\omega_d = \omega_n$). Eq. reduces to

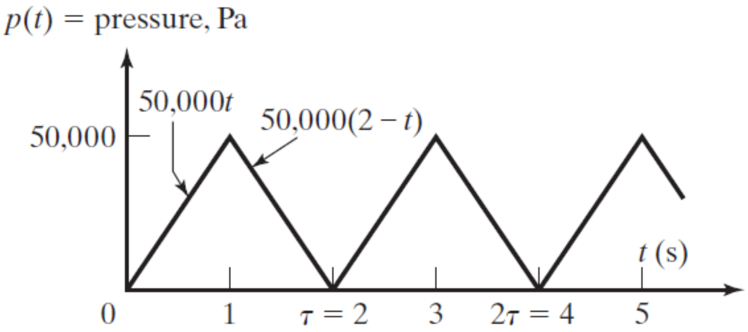
$$x(t) = \frac{F_0}{k}\left[1 - \cos \omega_n t\right]$$

Equation is shown graphically in Fig. (d). It can be seen that if the load is instantaneously applied to an undamped system, a maximum displacement of twice the static displacement will be attained - that is, $x_{\max} = 2F_0/k$

Periodic Vibration of a Hydraulic Valve



Question: In the study of vibrations of valves used in hydraulic control systems, the valve and its elastic stem are modeled as a damped spring-mass system, as shown in Figure. In addition to the spring force and damping force, there is a fluid-pressure force on the valve that changes with the amount of opening or closing of the valve. Find the steady-state response of the valve when the pressure in the chamber varies as indicated in Figure. Assume k = 2500 N/m, c = 10 N − s/m, and m = 0.25 kg.



Periodic Vibration of a Hydraulic Valve

Solution: The valve can be considered as a mass connected to a spring and a damper on one side and subjected to a forcing function $F(t)$ on the other side. The forcing function can be expressed as

$$F(t) = Ap(t)$$

where A is the cross-sectional area of the chamber, given by

$$A = \frac{\pi(50)^2}{4} = 625\pi\text{mm}^2 = 0.000625\pi\text{m}^2$$

and $p(t)$ is the pressure acting on the valve at any instant t. Since $p(t)$ is periodic with period $\tau = 2$ seconds and A is a constant, $F(t)$ is also a periodic function of period $\tau = 2$ seconds. The frequency of the forcing function is $\omega = (2\pi/\tau) = \pi\text{rad/s}.F(t)$ can be expressed in a Fourier serios as

$$F(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots$$

where a_j and b_j are Fourier coefficients.

Periodic Vibration of a Hydraulic Valve

Since the function $F(t)$ is given by

$$F(t) = \begin{cases} 50,000\text{At} & \text{for } 0 \leq 1 \leqslant \frac{\tau}{2} \\ 50,000A(2-t) & \text{for } \frac{\tau}{2} \leqslant t \leqslant \tau \end{cases}$$

The Fourier coefficients a_j and b_j can be computed with the help of:

$$a_0 = \frac{2}{2}\left[\int_0^1 50.000Atdt + \int_1^2 50,000A(2-t)dt\right] = 50,000 \text{ A}$$

$$a_1 = \frac{2}{2}\left[\int_0^1 50,000At\cos\pi tdt + \int_1^2 50,000A(2-t)\cos\pi tdt\right] = \frac{2\times 10^5 A}{\pi^2}$$

$$b_1 = \frac{2}{2}\left[\int_0^1 50.000At\sin\pi tdt + \int_1^2 50.000A(2-t)\sin\pi tdt\right] = 0$$

$$a_2 = \frac{2}{2}\left[\int_0^1 50,000At\cos 2\pi tdt + \int_1^2 50.000A(2-t)\cos 2\pi tdt\right] = 0$$

$$b_2 = \frac{2}{2}\left[\int_0^1 50.000At\sin 2\pi tdt + \int_1^2 50,000A(2-t)\sin 2\pi tdt\right] = 0$$

Periodic Vibration of a Hydraulic Valve

$$a_3 = \frac{2}{2}\left[\int_0^1 50,000At\cos 3\pi t dt + \int_1^2 50.000A(2-t)\cos 3\pi t dt\right] = \frac{2\times 10^5 \text{ A}}{9\pi^2}$$

$$b_3 = \frac{2}{2}\left[\int_0^1 50,000At\sin 3\pi t dt + \int_1^2 50,000A(2-t)\sin 3\pi t dt\right] = 0$$

Likewise, we can obtain

$$a_4 = a_6 = \cdots = b_4 = b_5 = b_6 = \cdots = 0.$$

By considering only the first three harmonics, the forcing function can be approximated:

$$F(t) = 25,000A - \frac{2\times 10^5 A}{\pi^2}\cos\omega t - \frac{2\times 10^5 A}{9\pi^2}\cos 3\omega t$$

The steady-state response of the valve to the forcing function of Fq. (E. 12) can be expressed as

$$x_p(t) = \frac{25,0004}{k} - \frac{\left(2\times 10^5 A/\left(k\pi^2\right)\right)}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}\cos(\omega t-\phi_1) - \frac{\left(2\times 10^5 A/\left(9k\pi^2\right)\right)}{\sqrt{\left(1-9r^2\right)^2+(6\xi r)^2}}\cos(3\omega t-\phi_3)$$

Periodic Vibration of a Hydraulic Valve

The natural frequency of the valve is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{0.25}} = 100\text{rad/s}$$

and the forcing frequency ω by

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi\text{rad/s}$$

Thus the frequency ratio can be obtaincd:

$$r = \frac{\omega}{\omega_n} = \frac{\pi}{100} = 0.031416$$

and the damping ratio:

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{10.0}{2(0.25)(100)} = 0.2$$

Periodic Vibration of a Hydraulic Valve

The phase angles ϕ_1 and ϕ_3 can be computed as follows:

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) = \tan^{-1}\left(\frac{2 \times 0.2 \times 0.031416}{1 - 0.031416^2}\right) = 0.0125664\text{rad}$$

and

$$\phi_3 = \tan^{-1}\left(\frac{6gr}{1 - 9r^2}\right) = \tan^{-1}\left(\frac{6 \times 0.2 \times 0.031416}{1 - 9(0.031416)^2}\right) = 0.0380483\text{rad}$$

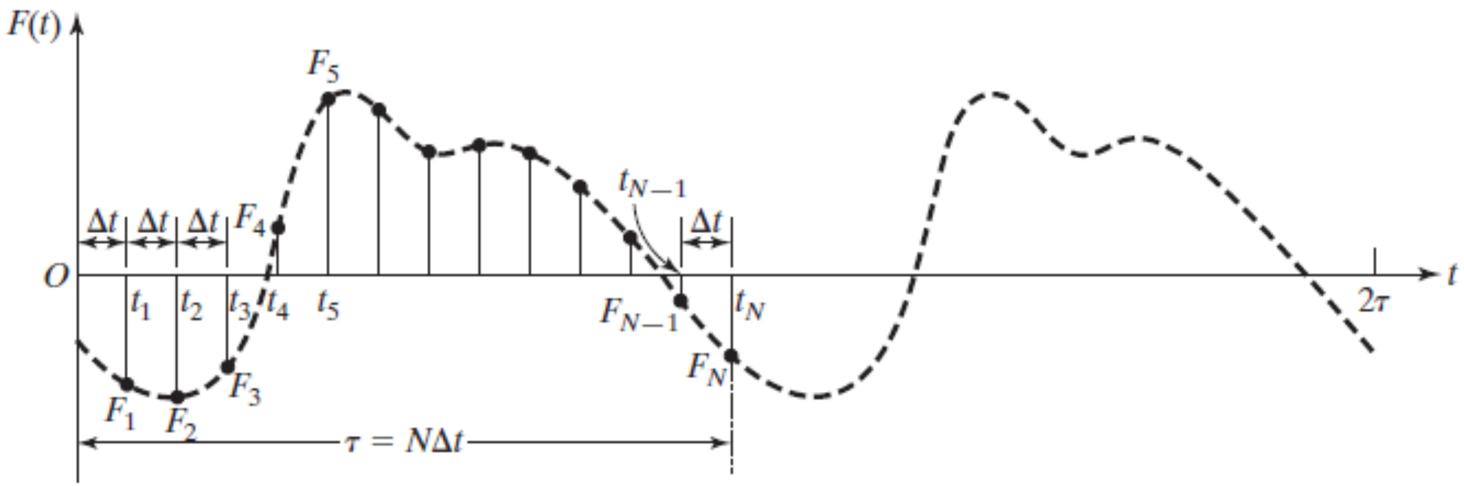
The solution can be written as

$$x_p(t) = 0.019635 - 0.015930\cos(\pi t - 0.0125664) - 0.0017828\cos(3mt - 0.0380483)\text{m}$$

Nonharmonically Forced Vibration

Sometimes, the value of $F(t)$ may be available only at a number of discrete points $t_1, t_2, \ldots, t_N$. In all these cases, it is possible to find the Fourier coefficients by using a numerical integration procedure. If $F_1, F_2, \ldots, F_N$ denote the values of $F(t)$ at $t_1, t_2, \ldots, t_N$, respectively, where N denotes an even number of equidistant points in one time period $\tau (\tau = N\Delta t)$, as shown in Figure, gives

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} F_i$$
$$a_j = \frac{2}{N}\sum_{i=1}^{N} F_i \cos\frac{2j\pi t_i}{\tau}, \quad j = 1, 2, \ldots$$
$$b_j = \frac{2}{N}\sum_{i=1}^{N} F_i \sin\frac{2j\pi t_i}{\tau}, \quad j = 1, 2, \ldots$$



Steady-State Vibration of a Hydraulic Valve

Question: Find the steady-state response of the valve if the pressure fluctuations in the chamber are found to be periodic. The values of pressure measured at 0.01-second intervals in one cycle are given as:

Time (t_i) (seconds)	$p_i = p(t_i)$
0	0
0.01	20
0.02	34
0.03	42
0.04	49
0.05	53
0.06	70
0.07	60
0.08	36
0.09	22
0.10	16
0.11	7
0.12	0

Solution:Since the pressure fluctuations on the valve are periodic, the Fourier analysis of the given data of pressures in a cycle gives

$$\begin{aligned} p(t) =& 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t \\ &+ 1416.7 \cos 104.72t + 3608.3 \sin 104.72t \\ &- 5833.3 \cos 157.08t - 2333.3 \sin 157.08t + \ldots \mathrm{N/m^2} \end{aligned}$$

Other quantities needed for the computation are

$$\begin{aligned} \omega &= \frac{2\pi}{\tau} = \frac{2\pi}{0.12} = 52.36\mathrm{rad/s} \\ \omega_n &= 100\mathrm{rad/s} \\ r &= \frac{\omega}{\omega_n} = 0.5236 \end{aligned}$$

Steady-State Vibration of a Hydraulic Valve

$$\zeta = 0.2$$
$$A = 0.000625\pi \mathrm{m}^2$$
$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\times0.2\times0.5236}{1-0.5236^2}\right) = 16.1^\circ$$
$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}\left(\frac{4\times0.2\times0.5236}{1-4\times0.5236^2}\right) = -77.01^\circ$$
$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}\left(\frac{6\times0.2\times0.5236}{1-9\times0.5236^2}\right) = -23.18^\circ$$

The steady-state response of the valve can be expressed as:

$$x_p(t) = \frac{34083.3A}{k} - \frac{(26996.0A/k)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(52.36t-\phi_1) + \frac{(8309.7A/k)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(52.36t-\phi_1)$$

$$+\frac{(1416.7A/k)}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\cos(104.72t-\phi_2) + \frac{(3608.3A/k)}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\sin(104.72t-\phi_2) - \frac{(5833.3A/k)}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\cos(157.08t-\phi_3)$$

$$+\frac{(2333.3A/k)}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\sin(157.08t-\phi_3)$$