Example. We say that X is a standard normal random variable if its density is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for all } x \in \mathbb{R}.$$

Exercise: Check that $\phi(x)$ is a density function. that is, $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$. Use $\Gamma(1/2) = \sqrt{\pi}$ where $\Gamma(1/2)$ denotes the Gamma function evaluated at $1/2$.

* Note that $g(x) = x^2$ is not a ~~function which also~~ strictly monotone function. So we can not apply our theorem. But we can approach the problem using the distribution function. Our aim is to derive the density function of X^2 where X is standard normal random variable.

$$\mathbb{P}(X^2 \le x) = \mathbb{P}(-\sqrt{x} \le X \le \sqrt{x})$$

$$= \mathbb{F}(\sqrt{x}) - \mathbb{F}(-\sqrt{x}).$$

Then the density function can be obtained by differentiating $\mathbb{P}(X^2 \le x)$ with respect to x which leads to:

$$\frac{d}{dx}\left[\mathbb{F}(\sqrt{x}) - \mathbb{F}(-\sqrt{x})\right] = \frac{d}{dx}\left[\int_{-\infty}^{\sqrt{x}} du \frac{1}{\sqrt{2\pi}} e^{-u^2/2} - \int_{-\infty}^{-\sqrt{x}} du \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\right]$$

$$= \frac{1}{\sqrt{x}} \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} + \frac{1}{2} \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-x/2}$$

$$= \frac{1}{\sqrt{2\pi x}} e^{-x/2} \quad \text{for every } x > 0.$$

$$f_{X^2}(x) = \begin{cases} 0 & \text{if } x \leq 0. \\ \dfrac{1}{\sqrt{2\pi x}} e^{-x/2} & \text{for } x > 0. \end{cases}$$

Example.

Let X be a random variable ~~such that~~ with probability density given by.

$$f(x) = \begin{cases} \dfrac{2x}{\pi^2} & \text{if } 0 < x < \pi \\ 0 & \text{otherwise.} \end{cases}$$

Define $Y = \sin X$. and we are interested in deriving the density function of Y.

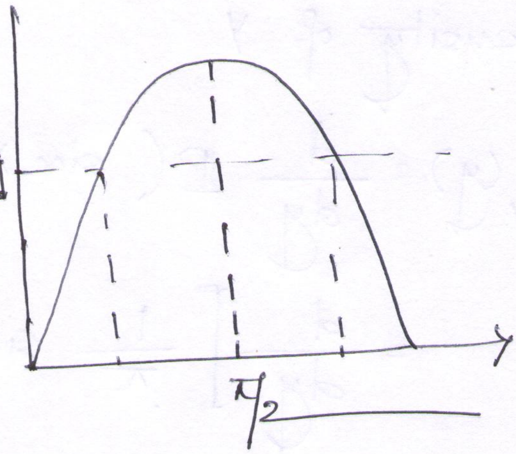
$$\frac{dy}{dx} = \cos x.$$

Note that $\dfrac{dy}{dx} < 0$ if $x > \pi/2$

$\dfrac{dy}{dx} > 0$ if $x < \dfrac{\pi}{2}$

$\dfrac{dy}{dx} = 0$ if $x = \pi/2$.

Again we can see that $y = \sin x$ doesnot have unique inverse as it is not strictly monotone on $(0, \pi)$.



We shall use the distribution function technique.

$$\mathbb{P}(Y \le y) = \mathbb{P}(\sin X \le y)$$

$$= \mathbb{P}\left(\{\sin X \le y\} \cap \left(\{0 < X \le \pi/2\} \cup \{\pi/2 < X < \pi\}\right)\right)$$

$$= \mathbb{P}\left(\{\sin X \le y\} \cap \{0 < X \le \pi/2\}\right)$$
$$+ \mathbb{P}\left(\{\sin X \le y\} \cap \{\pi/2 < X < \pi\}\right)$$

$$= \mathbb{P}\left(\{X \le \sin^{-1} y\} \cap \{0 < X \le \pi/2\}\right)$$
$$+ \mathbb{P}\left(\{X \le \sin^{-1} y\} \cap \{\pi/2 < X < \pi\}\right)$$

$$= \mathbb{P}\left(\{X \le \sin^{-1} y\}\right) + \mathbb{P}\left(\pi - \sin^{-1} y < X < \pi\right)$$

$$= \frac{2}{\pi^2} \int_0^{\sin^{-1} y} dx\,(x) + \frac{2}{\pi^2} \int_{\pi - \sin^{-1} y}^{\pi} dx\,(x)$$

$$= \frac{1}{\pi^2} (\sin^{-1} y)^2 + \frac{1}{\pi^2}\left[\pi^2 - (\pi - \sin^{-1} y)^2\right]$$

$$= \frac{1}{\pi^2}\left[\cancel{(\sin^{-1} y)^2} + \cancel{\pi^2} - \cancel{\pi^2} - \cancel{(\sin^{-1} y)^2} + 2\pi \sin^{-1} y\right]$$

$$= \frac{1}{\pi} \sin^{-1} y$$

Density of Y

$$f_Y(y) = \frac{d}{dy} \mathbb{P}(\sin X \le y)$$

$$= \frac{d}{dy}\left[\frac{1}{\pi}\sin^{-1} y\right] = \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} \quad 0<y<1.$$

$$f_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}} & \text{if } 0<y<1 \\ 0 & \text{if } y \geq 1. \end{cases}$$

* Note that we decomposed the range or values taken by X into two parts $\{0 < X \le \pi/2\}$ and $\{\pi/2 < X < \pi\}$. Note that $\sin x : (0, \pi/2] \to (0,1]$ is a ~~monoto~~ strictly monotone continuous and differentiable function. The same is true for $\sin x : (\pi/2, \pi) \to (0,1)$.

Q<u>n</u>. Can we extend or generalize the formula for the probability density function of $g(x)$ when g is not monotone but can be decomposed into monotone functions?

The answer of this question is provided in the next theorem.

Theorem

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a continuous random variable with probability density function f.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function which is differentiable w.r.t. x at all the points. Assume that $g'(x)$ is continuous and nonzero at all but a finite number of values of x. Then for every real number y,

a) There exists a positive number $n = n(y)$ (may depend on the value y), and real numbers

$$x_1(y),\ x_2(y), \ldots\ldots\ldots\ x_n(y)$$

such that

$$g(x_k(y)) = y \quad \text{and} \quad g'(x_k(y)) \neq 0, \quad k = 1, \ldots n.$$

b) or there does not exist any x such that

$g(x) = y$, $g'(x) = 0$, in which case we write

$n(y) = 0$.

Then $Y = g(X)$ is a continuous random variable with probability density function given by

$$f_Y(y) = \begin{cases} \sum_{k=1}^{n} f(x_k(y))\, |g'[x_k(y)]|^{-1} & \text{if } n > 0. \\ 0 & \text{if } n = 0. \end{cases}$$

Proof. We skip the proof right now as it is tedious. But the proof use similar idea as in the last example.

Remark. The theorem works when there are finitely many inverses for each $x \in \mathbb{R}$.

Question: Can we deal with the maps which have countably many inverses? The answer is 'YES'. But we have to approach the problem through distribution function.

Here is how we can do this. Suppose $g: \mathbb{R} \to \mathbb{R}$ has countably many inverses. This means, we can decompose $\mathbb{R}$ into at ~~coutable Bor~~ most countably many Borel subsets say $(A_k : k \geq 1)$ such that

$$g\big|_{A_k} \cancel{\text{is continuous, differentiable}} : A_k \to \mathbb{R}$$

has unique inverse. Then

$$\mathbb{P}(g(X) \leq y) = \mathbb{P}\left(\{g(X) \leq y\} \cap \{\bigcup_{k=1}^{\infty} \{X \in A_k\}\}\right)$$

$$= \mathbb{P}\left(\bigcup_{k=1}^{\infty} \left(\{g(X) \leq y\} \cap \{X \in A_k\}\right)\right)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}\left(\{g(X) \leq y\} \cap \{X \in A_k\}\right)$$

(using countable additivity property of probability and the fact that $(A_k : k \geq 1)$ is a disjoint collection of Borel subsets).

$$= \sum_{k \geq 1}^{\infty} \mathbb{P}\left(g|_{A_k}(X) \leq y \right).$$

__Remark.__ If $g|_{A_k} : A_k \to \mathbb{R}$ is a continuous, differentiable function such that $g|'_{A_k}(x) > 0$ ~~on~~ for every $x \in A_k$ or $g|'_{A_k}(x) < 0$ for every $x \in A_k$, then we can differentiate

$$\mathbb{P}\left(g|_{A_k}(X) \leq y \right) \text{ with respect to } y \text{ for every } k \geq 1.$$

to get the density

$$\sum_{k \geq 1}^{\infty} \frac{d}{dy}\left[\mathbb{P}\left(g|_{A_k}(X) \leq y \right) \right].$$

(differentiation is only allowed when the series after differentiation converges uniformly)

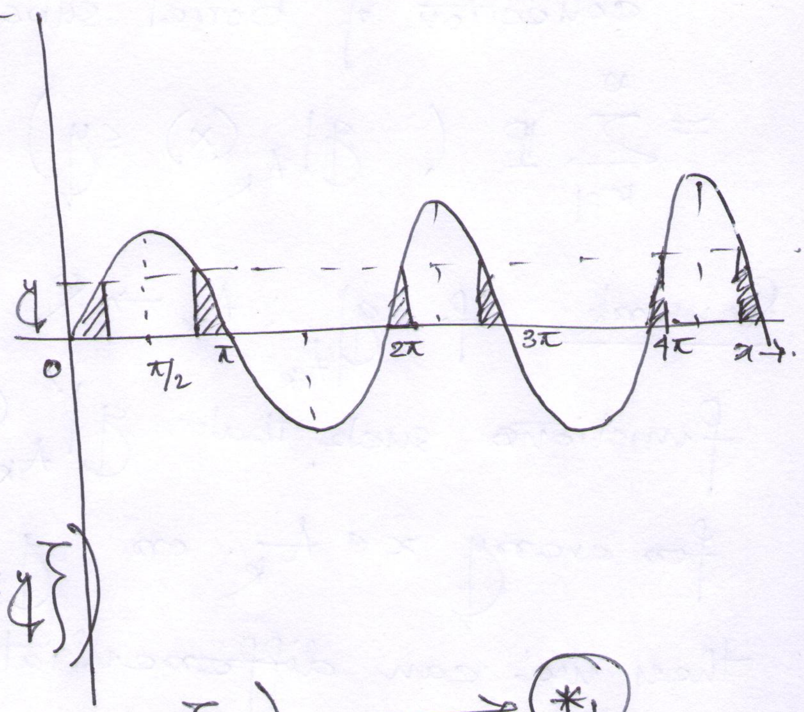
where $g|'_{A_k}(x) = \frac{d}{dx} g|_{A_k}(x) \quad \forall x \in A_k.$

__Example.__ Let X be a random variable with probability density function.

$$f(x) = \begin{cases} \theta e^{-\theta x} & \text{if } x > 0, \quad \theta > 0. \\ 0 & \text{if } x \leq 0. \end{cases}$$

We are interested in the formula for density function of $Y = \sin X$.

$\sin x : (0,\infty) \to (-1,1)$.



Choose $y \in (0,1)$.

Then we have

$\mathbb{P}(\sin X \le y)$.

$= \mathbb{P}(\{\sin X \le 0\} \cup \{0 < \sin X \le y\})$

$= \mathbb{P}(\sin X \le 0) + \mathbb{P}(\{0 < \sin X < y\}) \longrightarrow *_1$

To understand $\{0 < \sin X \le y\}$, we decompose it into disjoint subsets.

$$\{0 < X < \pi\} \cup \{2\pi < X < 3\pi\} \cup \{4\pi < X < 5\pi\}$$

$$\cup \{6\pi < X < 7\pi\} \cup \cdots < \{2n\pi < X < (2n+1)\pi\}$$

$$= \bigcup_{k=0}^{\infty} \{2k\pi < X < (2k+1)\pi\}.$$

Then we can write down the probability

$$\mathbb{P}\left(\{0 < \sin X \le y\} \cap \left\{\bigcup_{k=0}^{\infty}\{2k\pi < X < (2k+1)\pi\}\right\}\right)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(\{0 < \sin X \le y\} \cap \{2k\pi < X < (2k+1)\pi\})$$

$\longrightarrow *_2$

$$= \sum_{k=0}^{\infty} \mathbb{P}\left(\{ 2k\pi < X < 2k\pi + \sin^{-1}y \} \cup \{ (2k+1)\pi - \sin^{-1}y < X < (2k+1)\pi \} \right)$$

$$= \sum_{k=0}^{\infty} \left[\mathbb{P}\left(2k\pi < X < 2k\pi + \sin^{-1}y \right) + \mathbb{P}\left((2k+1)\pi - \sin^{-1}y < X < (2k+1)\pi \right) \right] \longrightarrow *_3$$

Here $\sin^{-1}y$ denotes the principle value of $\sin^{-1}y$.

Note that

$$\mathbb{P}\left(2k\pi < X < 2k\pi + \sin^{-1}y \right) = \theta \int_{2k\pi}^{2k\pi + \sin^{-1}y} dx \, e^{-\theta x}$$

$$= \left[-e^{-\theta x} \right]_{2k\pi}^{2k\pi + \sin^{-1}y} = e^{-2k\theta\pi} - e^{-2k\theta\pi - \theta \sin^{-1}y}.$$

$$\mathbb{P}\left((2k+1)\pi - \sin^{-1}y < X < (2k+1)\pi \right)$$

$$= e^{-\theta(2k+1)\pi + \theta \sin^{-1}y} - e^{-\theta(2k+1)\pi} \longrightarrow *_4$$

Combining $*_3$ and $*_4$, we get following expression for $*_2$

$$\mathbb{P}\left(\{ 0 < \sin X \le y \} \right)$$

$$= \sum_{k=0}^{\infty} \left[e^{-2k\theta\pi} - e^{-2k\theta\pi - \theta \sin^{-1}y} + e^{-\theta(2k+1)\pi + \theta \sin^{-1}y} - e^{-\theta(2k+1)\pi} \right].$$

$$= \sum_{k=0}^{\infty} \left[e^{-2k\theta\pi} - e^{-\theta(2k+1)\pi} \right] + \sum_{k=0}^{\infty} e^{-2k\theta\pi} \left(e^{\theta \sin^{-1}y - \theta\pi} - e^{-\theta \sin^{-1}y} \right). \longrightarrow *_5$$

Combining *1 and *5

$$\mathbb{P}(\sin X \le y)$$

$$= \mathbb{P}(\sin X \le 0) + \sum_{k=0}^{\infty} e^{-2k\theta\pi}\left(1 - e^{-\theta\pi}\right) \longrightarrow *6$$

$$\underbrace{+ \sum_{k=0}^{\infty} e^{-2k\theta\pi}\left(e^{-\theta(\pi - \sin^{-1} y)} - e^{-\theta \sin^{-1} y}\right)}_{\text{function of } y}$$

You can easily check that the series are convergent as

$$\sum_{k=0}^{\infty} e^{-2k\theta\pi} = \frac{1}{1 - e^{-2\theta\pi}} < \infty .$$

We can derive the density of $\sin X$ by ~~differen~~ differentiating both the sides of *6 w.r.t y.

$$f_Y(y) = \frac{d}{dy}\left(\sum_{k=0}^{\infty} e^{-2k\theta\pi}\left(e^{-\theta\pi + \theta \sin^{-1} y} - e^{-\theta \sin^{-1} y}\right)\right)$$

$$= \frac{1}{1 - e^{-2\theta\pi}} \frac{d}{dy}\left(e^{-\theta\pi + \theta\sin^{-1} y} - e^{-\theta \sin^{-1} y}\right)$$

$$= \frac{1}{\theta\sqrt{1-y^2}} \frac{1}{1 - e^{-2\theta\pi}} \cdot \left(e^{-\theta\pi + \theta\sin^{-1} y} - e^{-\theta\sin^{-1} y}\right)$$