Let's start with mental map of SDOF Problems.

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$m\ddot{x} + kx = 0$$

Undamped unforced system

Free vibrations

$$m\ddot{x} + \boxed{c\dot{x}} + kx = 0$$

Damped Unforced Systems

viscous damping

underdamped

overdamped

critically damped

$c\dot{x} = \mu N$ Coulomb damping (dry friction)

$c = \frac{h}{\omega}$ structural damping

$$F(t) = F_0 \sin \omega t$$

$$m\ddot{x} + kx = F_0 \sin(\omega t)$$

Resonance

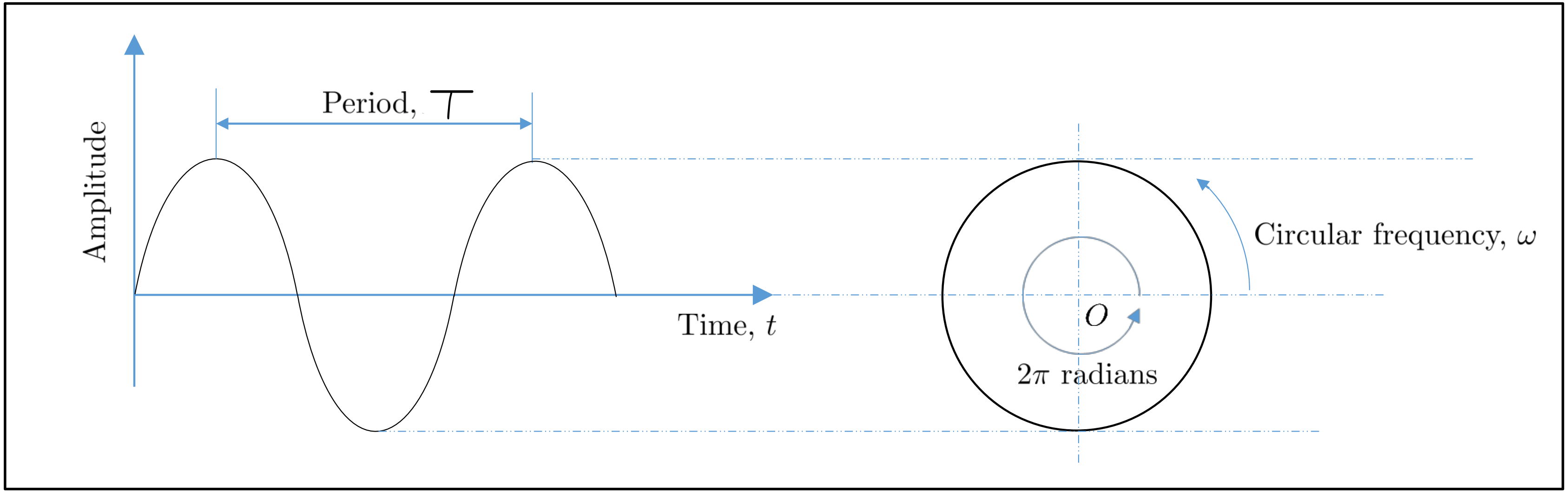
$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$$

Base excitation < absolute / relative

Kinematics of Vibrations

Vibrations usually involve motions which are periodic or cyclic, where motion repeats itself in all respects in a constant interval of time

Characteristics	Symbols	Units	Description
Period	T	seconds	constant interval of time to which the motion repeats
Frequency	$f = 1/T$	Hertz(Hz), one cycle per second	number of cycles completed per unit of time
Circular frequency	$\omega = 2\pi f$	radian per unit time	enables the interpretation of periodic motions by rotating vectors



Free Vibrations

For free vibration, $F(t) = 0$ and the motion of the SDOF system occurs under its own elastic and inertial forces. Previous equation reduces to:

$$\text{Governing Equation}: \quad \ddot{u} + \omega_n^2 u = 0; \quad \text{Solution}: \ u(t) = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

with C_1, C_2 as constants of integration to be evaluated from initial conditions, i.e., displacement and velocity at time $t = 0$:

$$u(0) = u_0; \quad \dot{u}(0) = v_0$$

Evaluating C_1, C_2 from the above conditions gives:

$$u(0) = u_0; C_1 \sin(\omega_n \cdot 0) + C_2 \cos(\omega_n \cdot 0) = u_0 \longrightarrow C_2 = u_0$$
$$\dot{u}(0) = v_0; C_1 \omega_n \cos(\omega_n \cdot 0) - u_0 \omega \sin(\omega_n \cdot 0) = v_0 \longrightarrow C_1 = v_0/\omega_n$$

The undamped free vibration response is then:

$$\boxed{u(t) = \frac{v_0}{\omega_n} \sin(\omega_n t) + u_0 \cos(\omega_n t) \quad or \quad u(t) = R \sin(\omega_n t + \psi)}$$

where

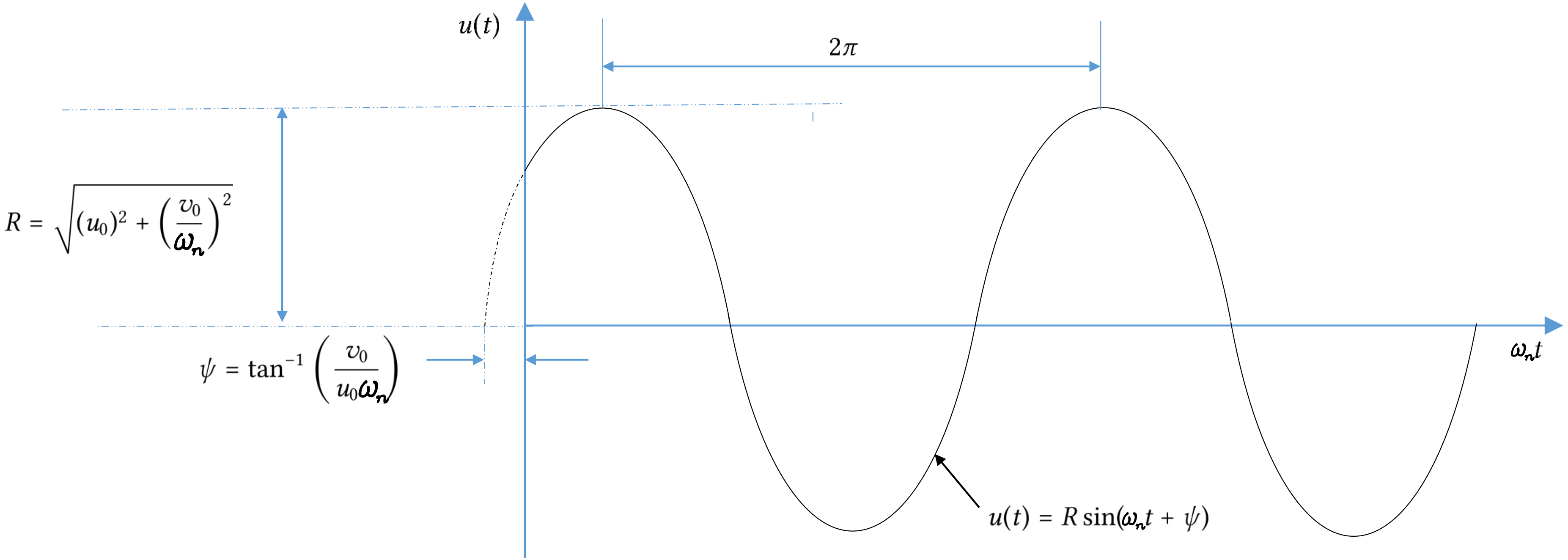
$$\boxed{R = \sqrt{u_0^2 + (v_0/\omega_n)^2} \quad \text{and} \quad \tan(\psi) = \frac{u_0}{v_0/\omega_n}}$$

Harmonic Motion

The solution is harmonic function of time. The motion is symmetric about equilibrium position of the mass m. The quantity ω_n given here represents the system's natural frequency of vibration.

$$u(t) = R\sin(\omega_n t + \psi)$$

with R as the amplitude and ψ as phase angle.

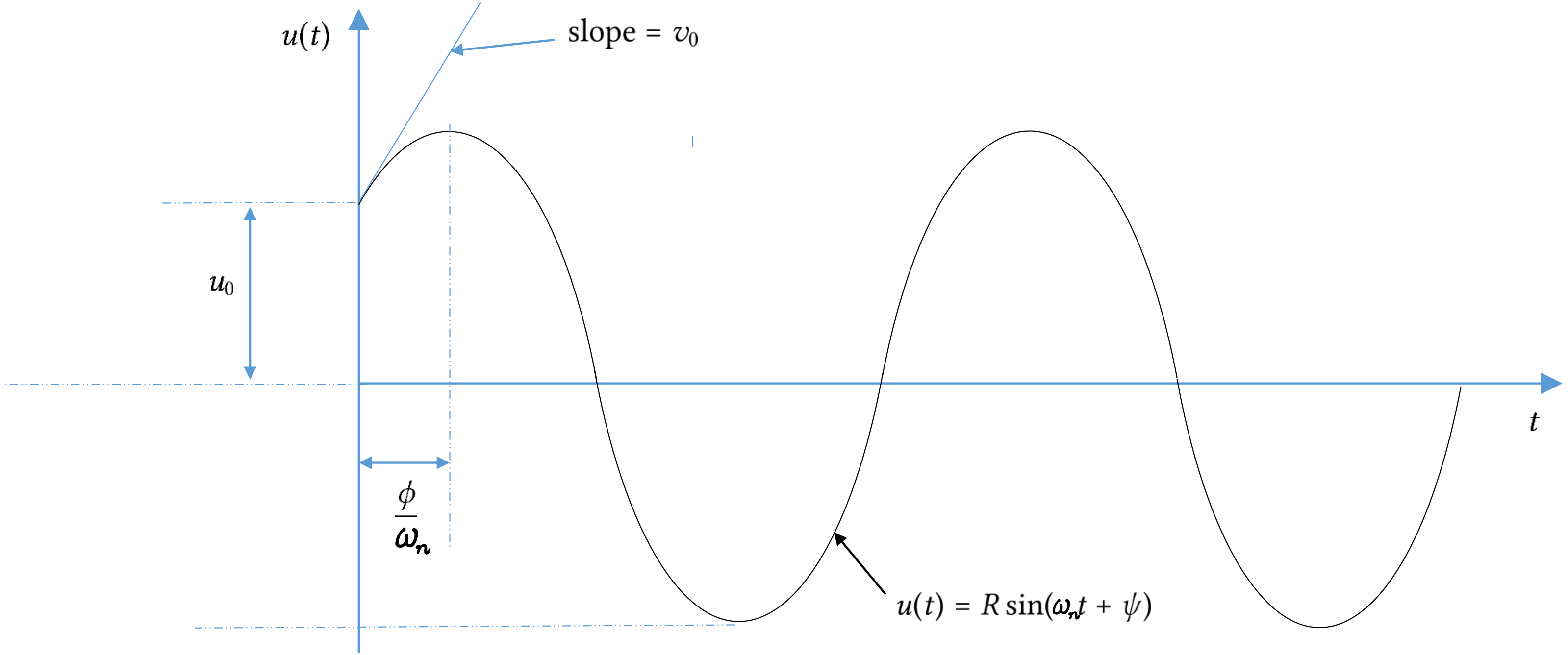


Harmonic Motion

Alternatively, for same harmonic (sinusoidal) vibration, the displacement $u(t)$ has the form:

$$u(t) = R\cos(\omega_n t - \phi)$$

with R as the amplitude and ϕ as phase angle.



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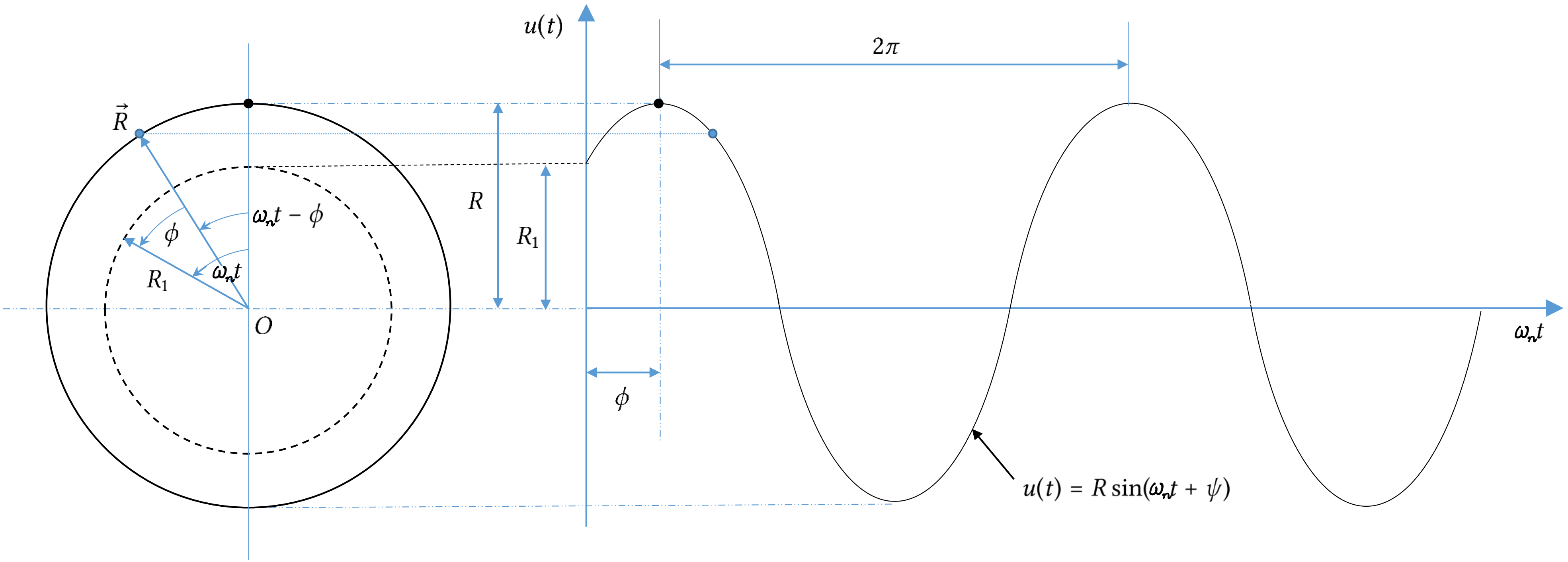
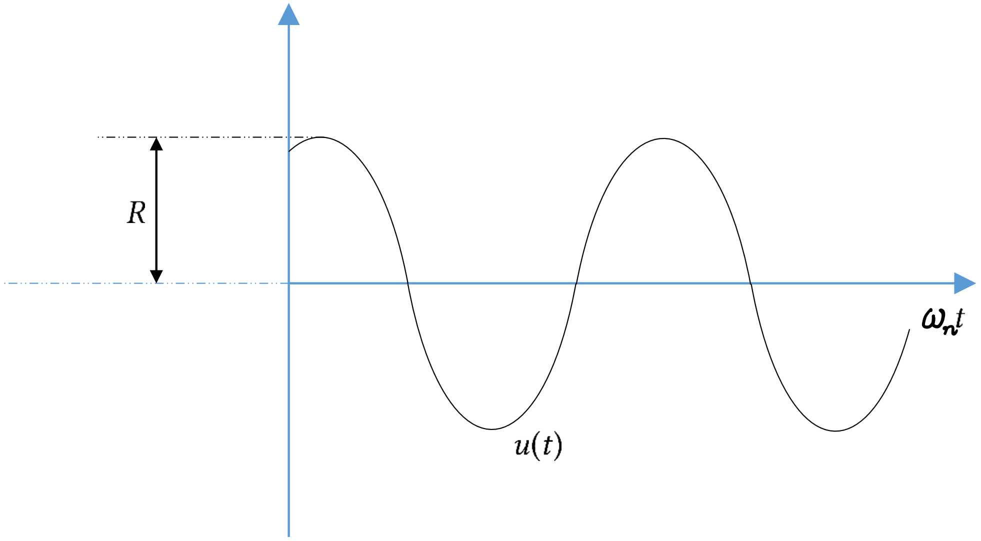


Harmonic Motion

Time derivatives of a harmonic vibration (denoted by superior dots) are also harmonic.

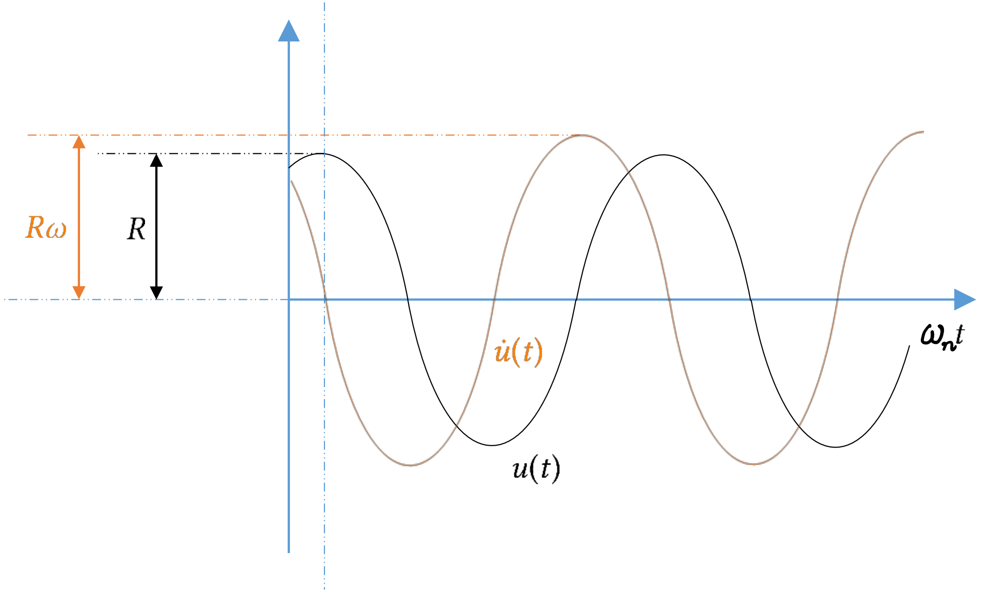
Displacement	u	$R\cos(\omega_n t - \phi)$
Velocity		
Acceleration		



Harmonic Motion

Time derivatives of a harmonic vibration (denoted by superior dots) are also harmonic.

Displacement	u	$R\cos(\omega_n t - \phi)$
Velocity	$\dot{u}$	$-R\omega_n \sin(\omega_n t - \phi) = R\omega_n \cos(\omega_n t - \phi + \pi/2)$
Acceleration		



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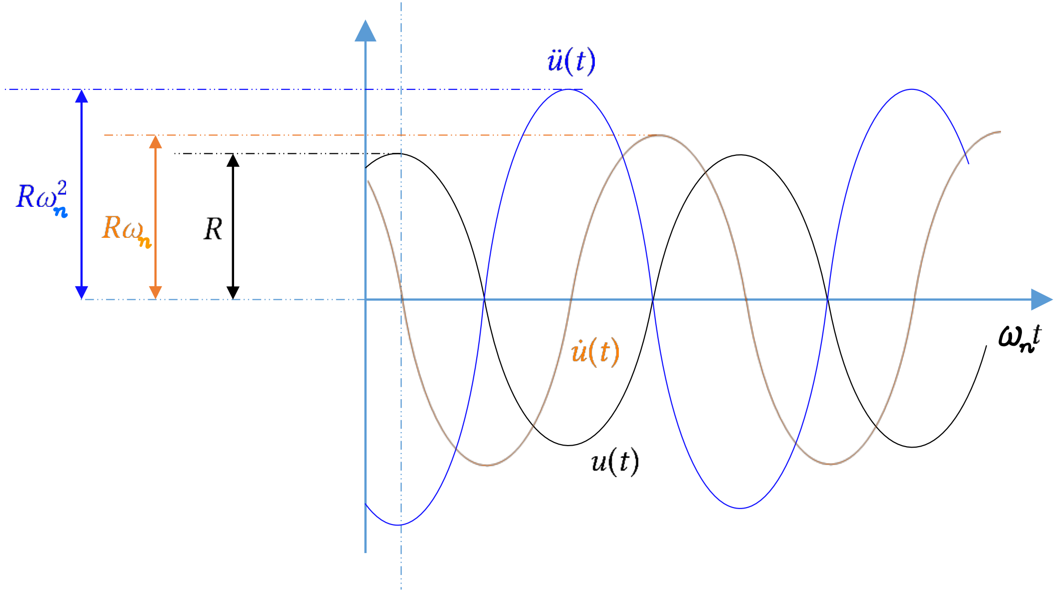
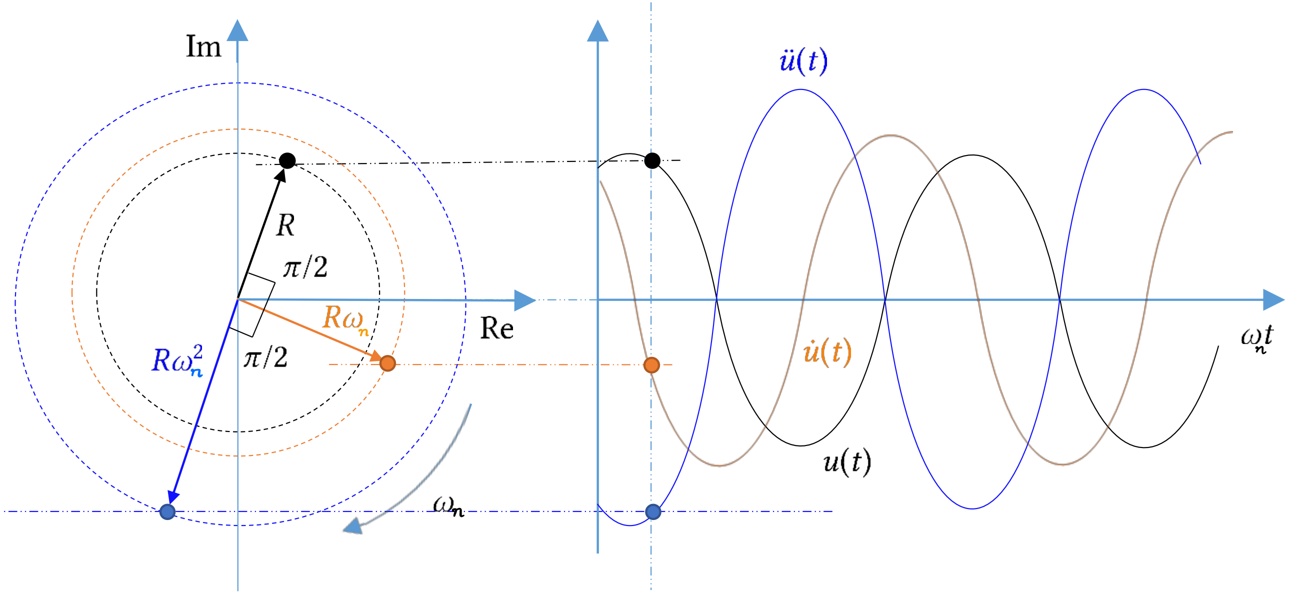
Velocity	$\dot{u}$	$-R\omega_n \sin(\omega_n t - \phi) = R\omega_n \cos(\omega_n t - \phi + \pi/2)$
Acceleration	$\ddot{u}$	$-R\omega_n^2 \cos(\omega_n t - \phi) = R\omega_n^2 \cos(\omega_n t - \phi + \pi)$



Harmonic Motion

By Euler's formula, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ where $i = \sqrt{-1}$

Displacement	u	$R\cos(\omega_n t - \phi)$	$Re^{i(\omega_n t - \phi)}$
Velocity	$\dot{u}$	$-R\omega_n \sin(\omega_n t - \phi) = R\omega_n \cos(\omega_n t - \phi + \pi/2)$	$R(i\omega_n)e^{i(\omega_n t - \phi)} = R\omega_n e^{i(\omega_n t - \phi)} e^{i\pi/2}$
Acceleration	$\ddot{u}$	$-R\omega_n^2 \cos(\omega_n t - \phi) = R\omega_n^2 \cos(\omega_n t - \phi + \pi)$	$R(i\omega_n)^2 e^{i(\omega_n t - \phi)} = R\omega_n^2 e^{i(\omega_n t - \phi)} e^{\pi}$



Unforced Damped Systems

Damped motions in Single Degree of Freedom (SDOF) Systems

General damping mechanisms are acknowledged to be useful models, i.e.,

1. Coulomb damping (dry friction) (Already done in class)
2. Viscous damping
3. Structural or solid damping (hysteresis)

Viscously Damped Free Vibrations

Consider viscously damped motions with initial conditions of $u(0) = u_0$ and $\dot{u}(0) = v_0$. The solution to previous equation with $F(t) = 0$ can be obtained by assuming its form to be $u(t) = e^{st}$ where the values of s are the roots to indicial equation:

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0 \quad \rightarrow \quad s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Hence, the homogeneous solution has the form:

$$u(t) = C_1 e^{s_1} t + C_2 e^{s_2} t$$

where constants of integration C_1 and C_2 are to be evaluated from initial conditions. Note that positive or negative nature of the radical leads to different types of motion. Let us first introduce the concept of critical damping $c = c_c$, which is the value of c by the vanishing of the radical in above equation.

$$c_c = 2\sqrt{km} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

where ω_n is the frequency of the undamped SDOF system. Let ζ denote the ratio of the viscous damping coefficient to its critical value, i.e., $\zeta = c/c_c$. Then, the roots s_1, s_2 can be rewritten as:

$$s_{1,2} = \omega_n\left\{\zeta \pm \sqrt{\zeta^2 - 1}\right\}$$

Viscously Damped Free Vibrations

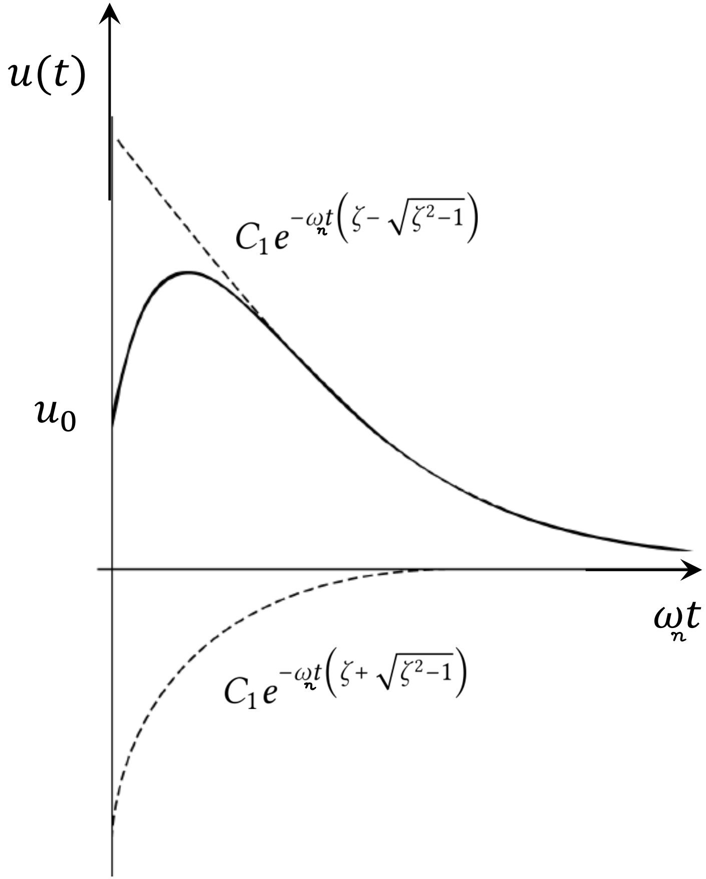
Three cases can be distinguished.

Case 1: Damping Greater than Critical $\zeta > 1$.
The radical is positive. Hence, the motion is aperiodic and the displacement consists of two decaying exponential functions:

$$u(t) = C_1 e^{-\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right)} = C_2 e^{-\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right)}$$

where C_1 and C_2 in the light of the initial conditions are:

$$C_1 = \frac{\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right)u_0 + v_0}{2\omega_n\sqrt{\zeta^2-1}}; \quad C_2 = \frac{\omega_n\left(\zeta - \sqrt{\zeta^2-1}\right)u_0 + v_0}{2\omega_n\sqrt{\zeta^2-1}}$$



Viscously Damped Free Vibrations

Case 2: Damping less than Critical $\zeta < 1$.
The radical is negative and the motion in this case is oscillatory and solution form can be rewritten in either of the following two forms:

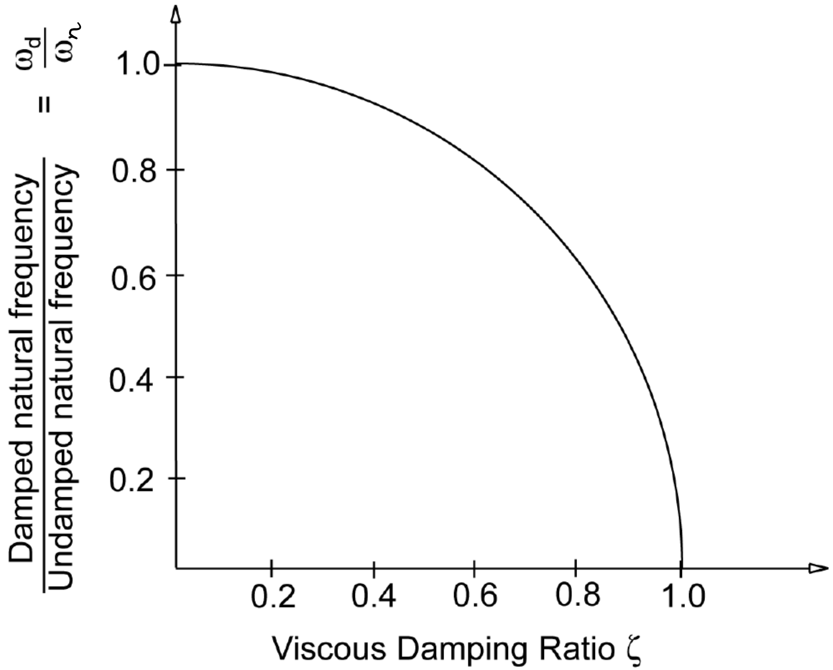
$$u(t) = e^{-\omega_n \zeta t} \left\{ C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t} \right\}$$

or

$$u(t) = e^{-\omega_n \zeta t} \left\{ C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t) \right\}$$

where $\boxed{\omega_d = \omega_n \sqrt{1 - \zeta^2}}$ and C_1 and C_2 are two alternate constants of integration.

The dependence of ω_d on ζ can be seen in Figure, where for values of $\zeta < 0.5$, there is only a small difference between ω_n and ω_d.



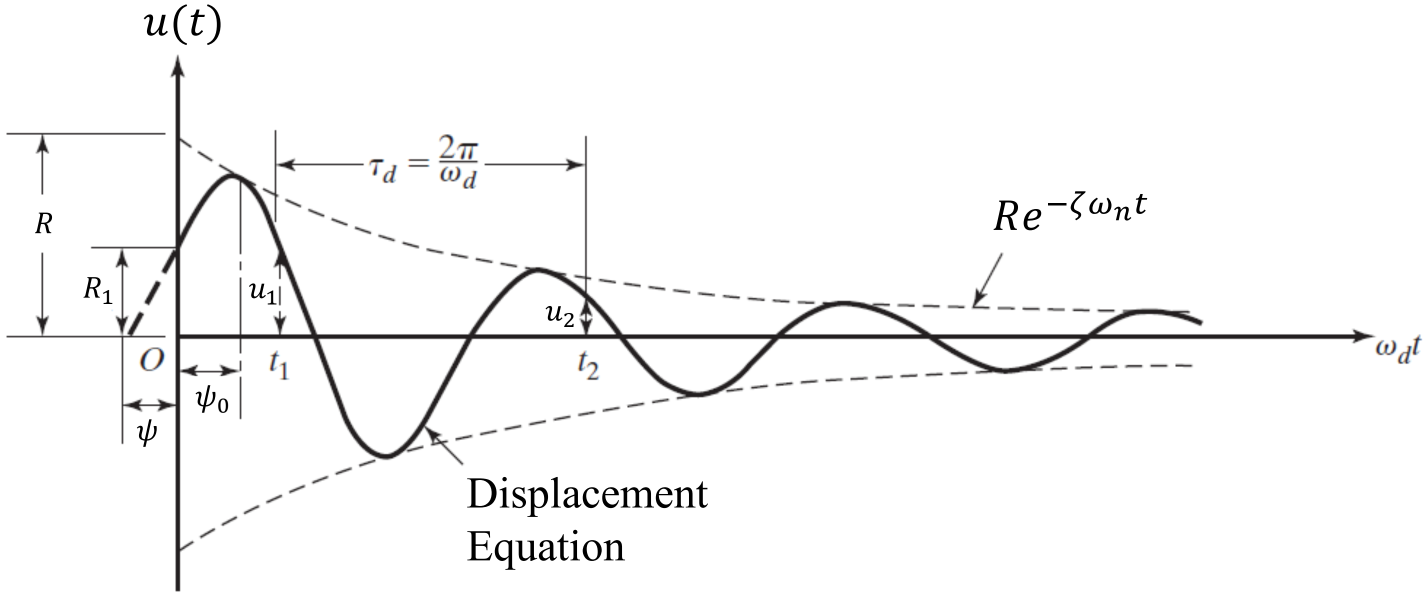
Viscously Damped Free Vibrations

By considering the initial conditions, solution becomes:

$$u(t) = e^{-\omega_n \zeta t}\left[u_0 \cos(\omega_d t) + \left(\frac{v_0 + \omega_n \zeta u_0}{\omega_d}\right)\sin(\omega_d t)\right] \quad or \quad Re^{\omega_n \zeta t}\sin(\omega_d t + \psi)$$

where

$$R = \sqrt{\frac{(v_0/\omega_n)^2 + 2u_0(v_0/\omega_n)\zeta + u_0^2}{1-\zeta^2}}; \quad \cot(\psi) = \frac{(v_0/\omega_n) + u_0\zeta}{u_0\sqrt{1-\zeta^2}}$$



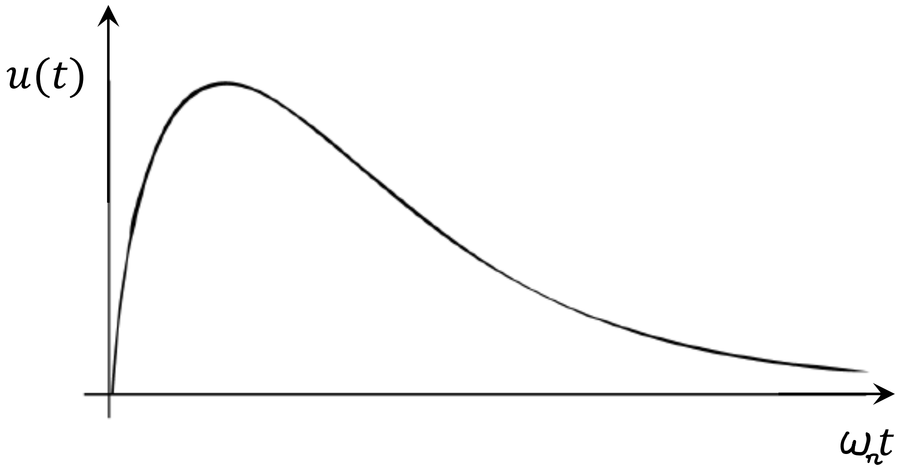
Viscously Damped Free Vibrations

Case 3: Critical Damping $\zeta = 1$
When $\zeta = 1$, there is a transition from periodic to aperiodic motions. The indicial equation in this situation has repeated roots. Therefore, another independent homogeneous solution must be found. The solution form for this case has the form

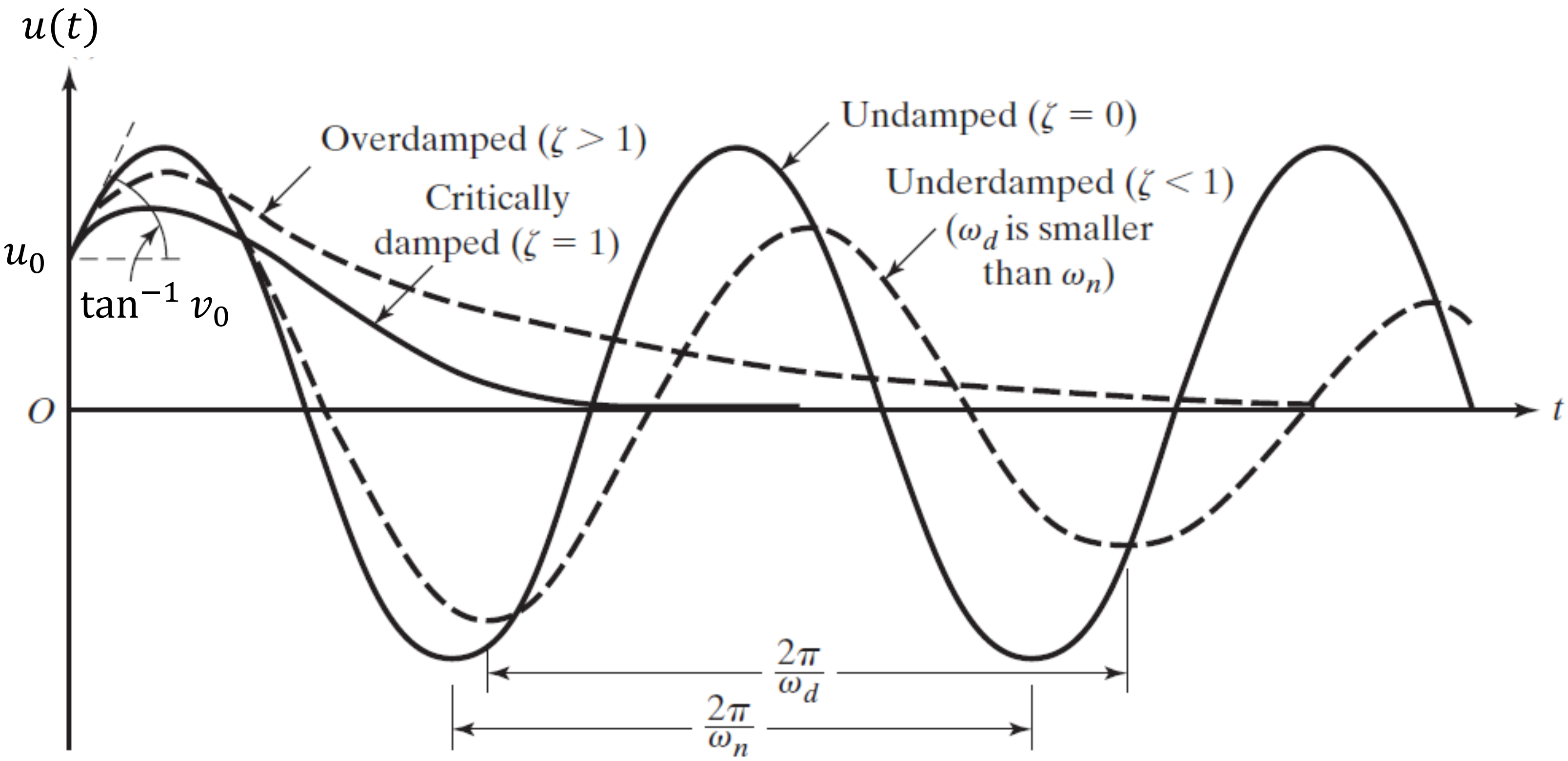
$$u(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

or with the conditions in place,

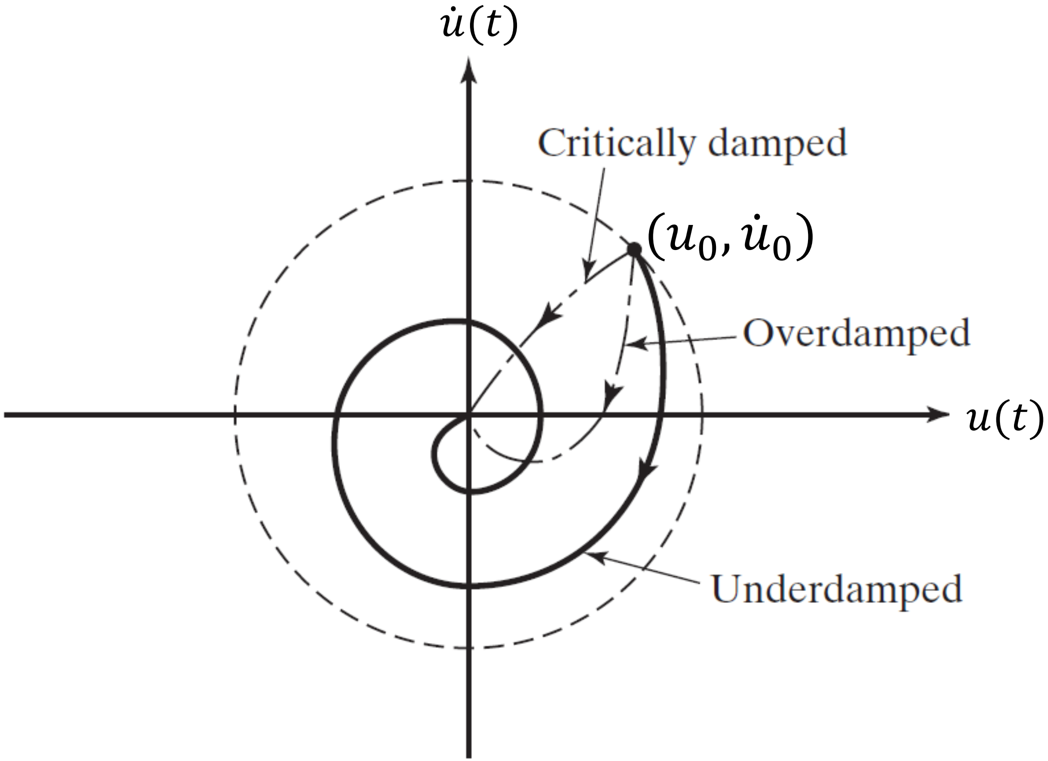
$$\boxed{u(t) = [u_0 + (v_0/\omega_n) + u_0\omega_n t] \cdot e^{-\omega_n t}}$$



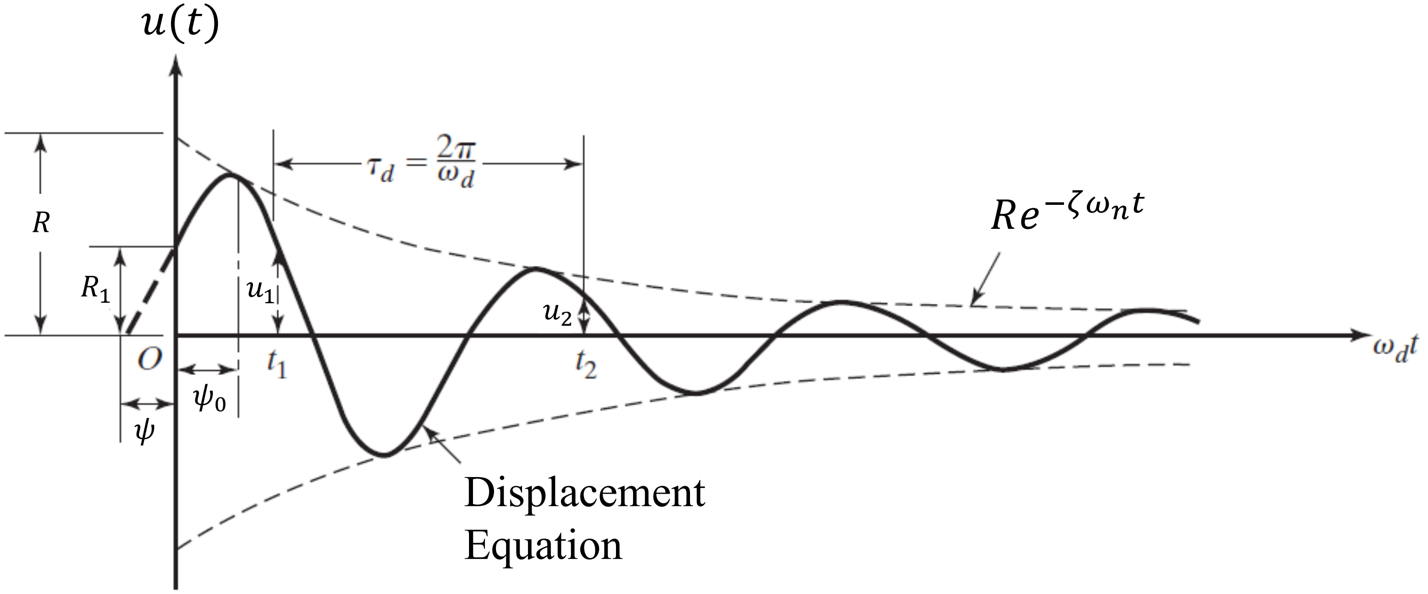
Viscously Damped Free Vibrations



Viscously Damped Free Vibrations



Logarithmic Decrement



Logarithmic Decrement

The amount of damping can be expressed by the logarithmic decrement δ defined as the natural logarithm of the ratio of two successive maximum amplitudes. Referring to Figure, $u_{(i)}\text{max}$ and $u_{(i+1)}\text{max}$ represent two successive maximums separated by one period $\tau_d = 2\pi/\omega_d$. Assume that the times for these maximums are t_1 and $t_1 + \tau_d$. Then the ratio of two successive maxima is

$$u_{(i)}\text{max}/u_{(i+1)max} = Re^{-\omega_n\zeta t_1}/Re^{-\omega_n\zeta(t_1+\tau_d)} = e^{\omega_n\zeta\tau_d}$$

and

$$\delta = \ln\left(e^{\omega_n\zeta\tau_d}\right) = \omega_n\zeta\tau_d \quad \text{or} \quad \boxed{\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

For small values of ζ, Equation can be linearized.

$$\delta = 2\pi\zeta$$

The error incurred by this linearization can be seen in Figure, which shows that for $\zeta < 0.2$, there is almost no difference.

Logarithmic Decrement

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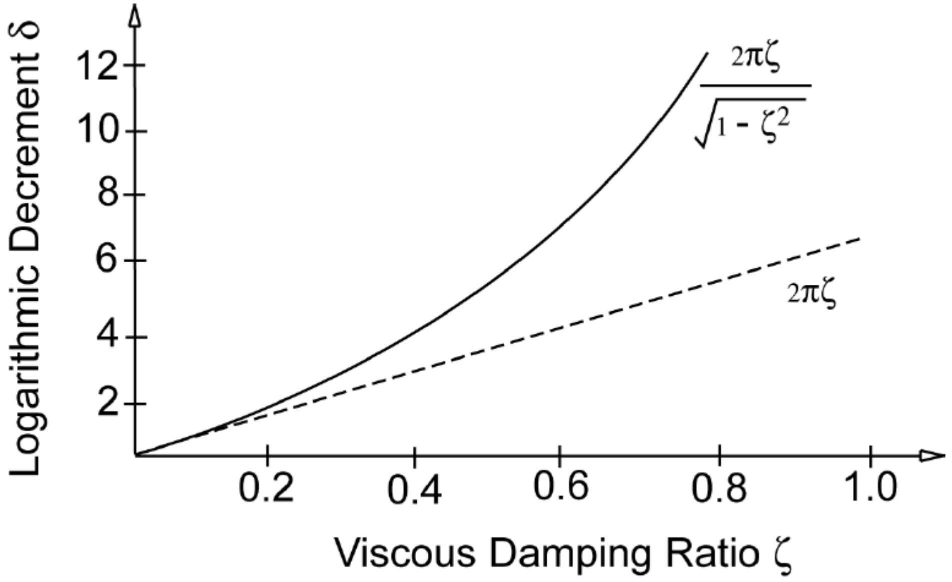
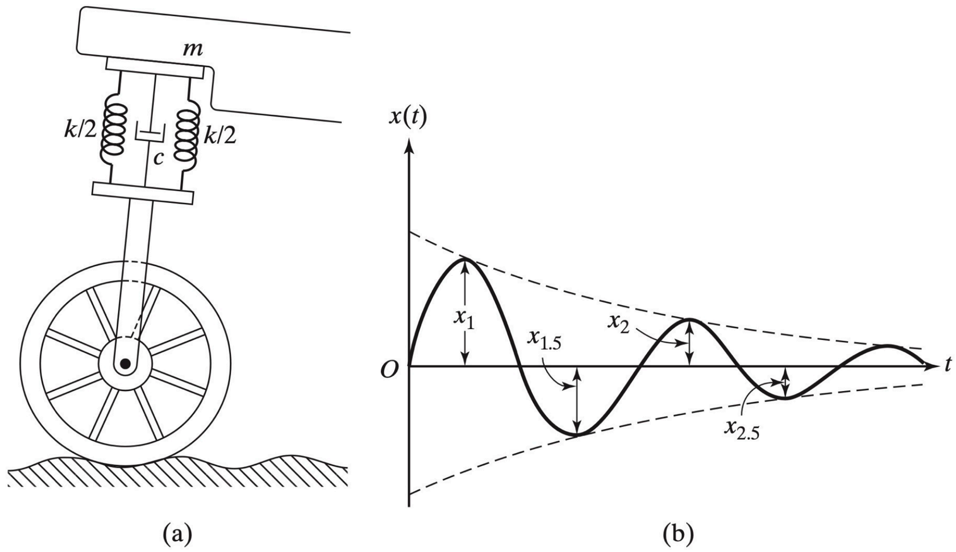


Question A
Design an underdamped shock absorber of motorcycle

Data 1: When the shock absorber is subjected to an initial vertical velocity (v_0) due to a Gun mounted, the resulting displacement-time curve is to be as indicated in Figure (b).



Data 2: Damped period of vibration is to be 2s and amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e. $x_{1.5} = x_1/4$).

Data 3: Equivalent mass of BatPod = 200 kg.

Find the necessary stiffness and damping constants of shock absorber.
Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

(A)

$x_{1.5} = x_1/4$

$x_2 = x_{1.5}/4 = x_1/16$

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln 16 = 2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$4\pi^2\zeta^2 = 2.7726^2(1-\zeta^2)$$
$$(4\pi^2\zeta^2 + 2.7726^2)\zeta^2 = 2.7726^2$$
$$\zeta^2 = \frac{2.7726^2}{(4\pi^2 + 2.7726^2)} = 0.16298$$

$$\zeta = 0.4037$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 2 \text{ sec}$$

$$\omega_n = 3.4338 \text{ rad/s}$$

Critical damping constant $c_c = 2m\omega_n$

$= 2 \times 200 \times 3.4338$ N-s/m

$= 1373.54$ Ns/m

Damping constant $c = \zeta c_c = 554.4981$ Ns/m

Stiffness $k = m\omega_n^2 = 2358.2652$ N/m

Response

$$x(t) = e^{-\zeta\omega_n t}\left(\overset{0}{\cancel{x_0}}\cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n \overset{0}{\cancel{x_0}}}{\omega_d}\sin\omega_d t\right)$$

$$x(t) = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

$$\dot{x}(t) = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin\omega_d t + \omega_d\cos\omega_d t\right)$$

The displacement of the mass will attain its maximum value at time t_1

$$x(t)_{max} \quad \text{when} \quad \dot{x}(t) = 0$$

$$-\zeta\omega_n \sin\omega_d t_1 + \omega_d \cos\omega_d t_1 = 0$$

$$\tan\omega_d t_1 = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$t_1 = 0.367 \text{ sec}$$

Maximum displacement at $t = t_1$ in terms of v_0 (250mm)

$$x(t_1) = \frac{v_0}{\omega_d} e^{-\zeta\omega_n t_1} \sin\omega_d t_1$$

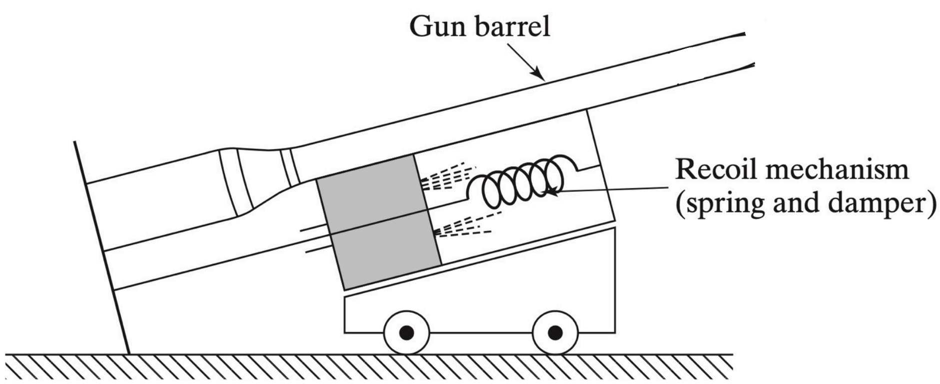
$$v_0 = \frac{250 \times 10^{-3} \times \pi}{e^{-\zeta\omega_n t_1} \sin\omega_d t_1} = 1.429 \text{ m/s}$$

Question B. Critically damped recoil mechanism for built in gun. When the gun is fired, high pressure gas accelerate the bullets inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the direction opposite that of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the recoil mechanism. In a particular case, the gun barrel and the recoil mechanism have a mass of 50 kg with a recoil spring of stiffness 1000 N/m. The gun recoils 0.04 m upon firing. Find

(1) the critical damping coefficient of the damper,

(2) the initial recoil velocity of the gun, and

(3) the time taken by the gun to return to a position 0.01 m from its initial position.



(B)

undamped natural freq

$$\omega_n = \sqrt{\frac{k}{m}} = 4.4721 \text{ rad/s}$$

Critical damping coefficient

$$C_c = 2m\omega_n = 2 \times 50 \times 4.4721 \text{ Ns/m}$$
$$= 447.21 \text{ Ns/m}$$

Response of critically damped system

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

$$C_1 = x_0$$
$$C_2 = \dot{x}_0 + \omega_n x_0$$

The time t_1 at which $x(t)$ reaches maximum value can be obtained using

$$\dot{x}(t) = 0$$

$$C_2 e^{-\omega_n t} - \omega_n (C_1 + C_2 t) e^{-\omega_n t} = 0$$

$$t_1 = \left(\frac{1}{\omega_n} - \frac{C_1}{C_2}\right)$$

In case of $x_0 = 0 \Rightarrow C_1 = 0 \quad t_1 = \frac{1}{\omega_n}$

$x_{max} = 0.04m$

$$x(t = t_1) = C_2 t_1 e^{-\omega_n t_1} = \frac{\dot{x}_0}{e\omega_n}$$

$$\dot{x}_0 = x_{max}\,\omega_n e = 0.4862 \text{ m/s}$$

If t_2 denotes the time taken by gun to return to position 0.01m from its initial position

$$0.01 = C_2 t_2 e^{-\omega_n t_2}$$

$$0.01 = 0.4862\, t_2\, e^{-4.472 t_2}$$

$$t_2 = 0.02277 \text{ sec}$$

Undamped Systems under Harmonic force

Undamped Harmonically Forced Vibration $F(t) = F_0 \sin(\omega t)$

Consider a harmonically forcing function of magnitude F_0 and frequency ω. The equation of motion in this case takes the form:

$$\ddot{x} + \omega_n^2 x = \frac{F_0 \omega_n^2}{k} \sin(\omega t)$$

The complete solution consists of the homogeneous solution and a particular integral. The homogeneous solution has the same form as previous. The solution form is:

$$u(t) = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t) + \frac{F_0}{k} \left\{ \frac{1}{1 - (\omega/\omega_n)^2} \right\} \sin(\omega t)$$

Assuming quiescent initial conditions, i.e., $x(0) = 0; \dot{x}(0) = 0$, for evaluation of the integration constants, the solution form is:

$$x(t) = \frac{F_0}{k} \left\{ \frac{1}{1 - r^2} \right\} \{\sin(\omega t) - r \sin(\omega_n t)\}$$

with $r = \omega/\omega_n$.

Undamped Harmonically Forced Vibration $F(t) = F_0 \sin(\omega t)$

The terms in this solution have the following significance ascribed to them.

- Response to an equivalent static load F_0.

$$x_{st} = F_0/k$$

- Magnification factor giving increase of $x(t)$ over x_{st} due to harmonic time dependence.

$$M = \frac{1}{1 - r^2}$$

- Dynamic Load Factor

$$D(t) = \frac{x(t)}{x_{st}} = M\left\{\sin(\omega t) - r\sin(\omega_n t)\right\}$$

- Steady state response due to harmonic load.

$$\sin(\omega t)$$

- "Transient response," a free vibration term needed to satisfy quiescent initial conditions.

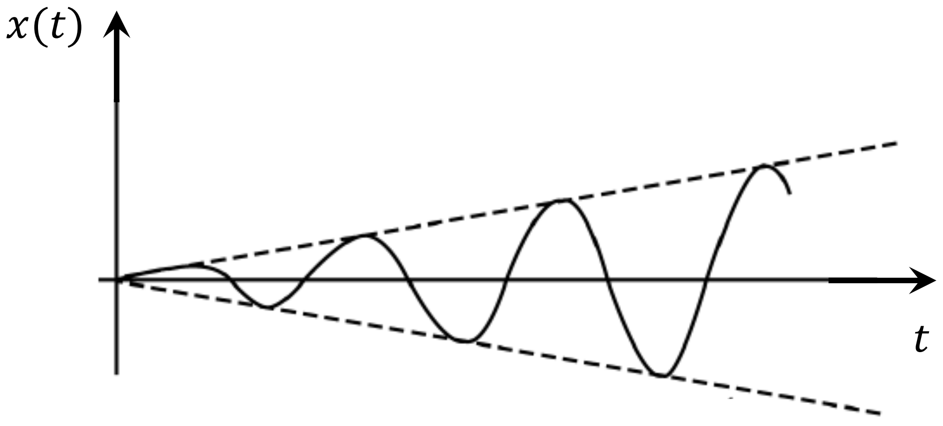
$$rsin(\omega_n t)$$

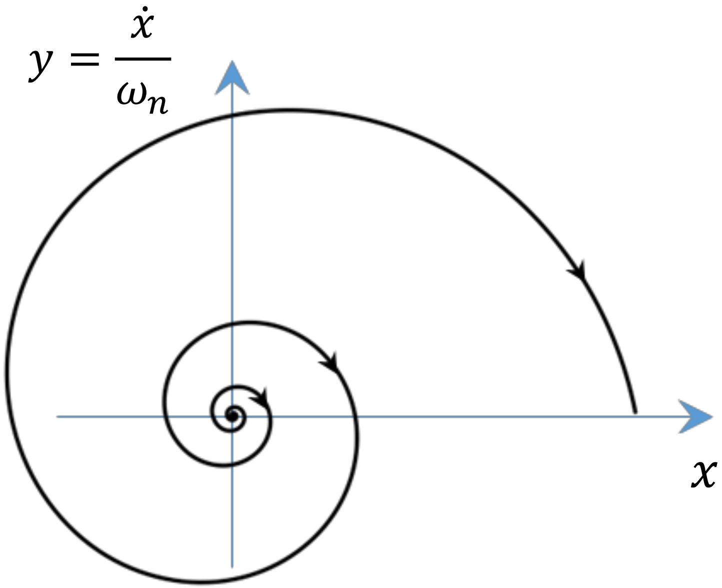
Resonance

When the forcing function frequency ω is the same as the natural frequency ω_n of the system (i.e., $r \longrightarrow 1$) we have a **resonance** condition. The dynamic load factor $D(t)$ is indeterminate i.e., $D(t) = 0/0$ and L'Hospital's rule must be applied to determine the motion of the system. Differentiating the dynamic load factor $D(t)$ with respect to r yields:

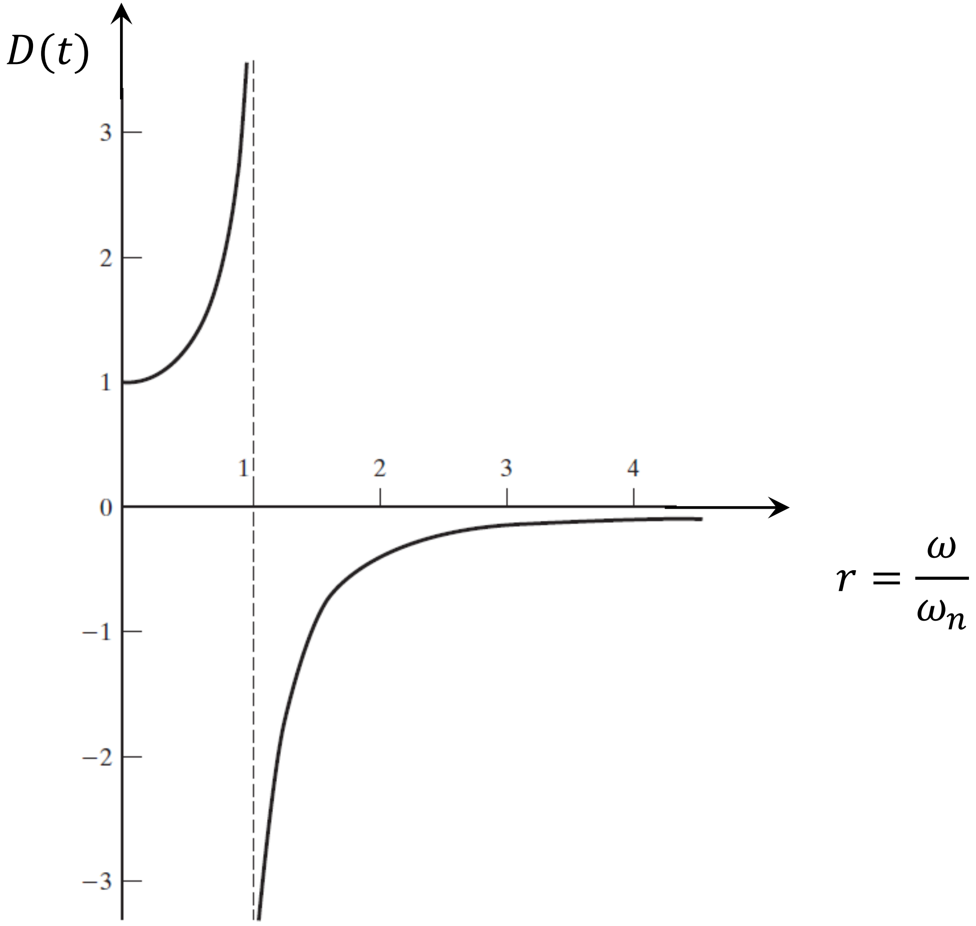
$$\lim_{r\longrightarrow 1} \frac{\frac{d}{dr}\{\sin(r\omega_n t) - r\sin(\omega_n t)\}}{\frac{d}{dr}(1-r^2)} = \lim_{r\longrightarrow 1} \frac{\omega_n t\cos(r\omega_n t) - \sin(\omega_n t)}{-2r} = -\frac{1}{2}\{\omega_n t\cos(\omega_n t) - \sin(\omega_n t)\}$$

A plot of this versus time is shown in this figure. Note that the response is not periodic any more but grows with time.





Resonance



Viscously damped Systems under Harmonic force

Harmonic forcing Function

Consider the underdamped viscous motion of a SDOF system in the presence of a harmonic input $F(t) = F_0 \sin(\omega t)$: Equation of motion takes the form:

$$\ddot{x} + \frac{c}{m}\dot{x} + \omega_n^2 x = \frac{F_0 \omega_n^2}{m} \sin(\omega t)$$

Its solution consists of a homogeneous solution and a particular integral in the form:

$$\boxed{x(t) = e^{-\omega_n \zeta t}\left\{C_1 \sin \omega_d t + C_2 \cos(\omega_d t)\right\} + \frac{F_0}{k}\frac{1}{\sqrt{\{1-r^2\}^2 + \{2\zeta r\}^2}} \sin(\omega t - \phi)}$$

where $r = \omega/\omega_n$ and $\tan(\phi) = 2\zeta r/(1-r^2)$. Observe that the homogeneous solution will die out in time, thus it is called the transient solution. It serves only to meet the prescribed initial conditions. In this case, the magnification factor R_d is:

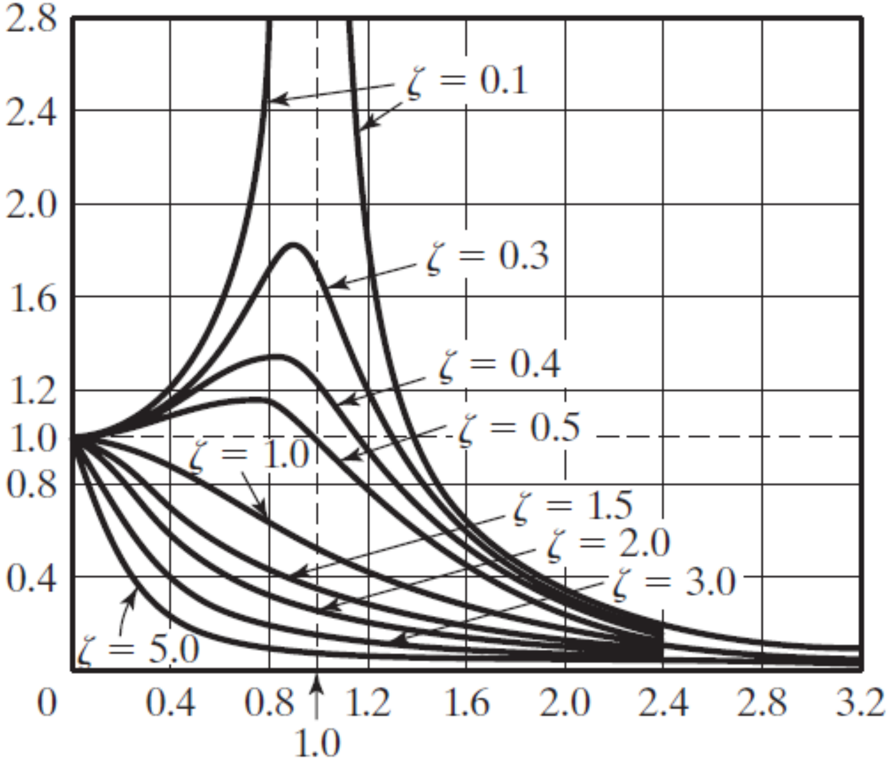
$$\boxed{R_d = \frac{1}{\sqrt{\{1-r^2\}^2 + \{2\zeta r\}^2}}}$$

which reduces to the previous case for undamped systems when $\zeta = 0$. The dynamic load factor or response factor in this case is:

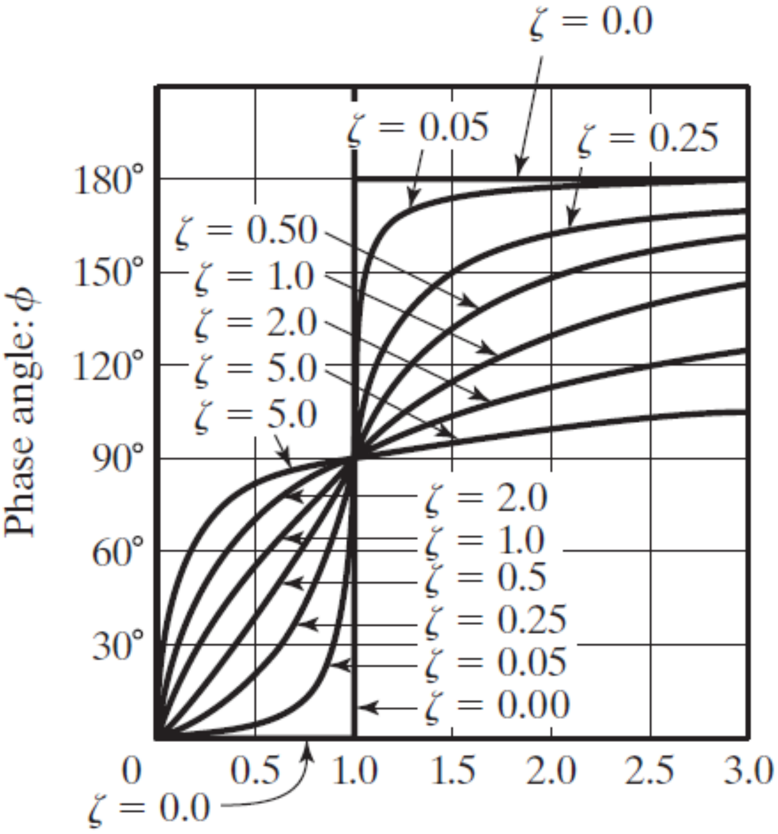
$$D(t) = \frac{x(t)}{x_{st}} = \frac{\sin(\omega t - \phi)}{\sqrt{\{1-r^2\}^2 + \{2\zeta r\}^2}} = R_d \sin(\omega t - \phi)$$

Harmonic forcing Function

R_d

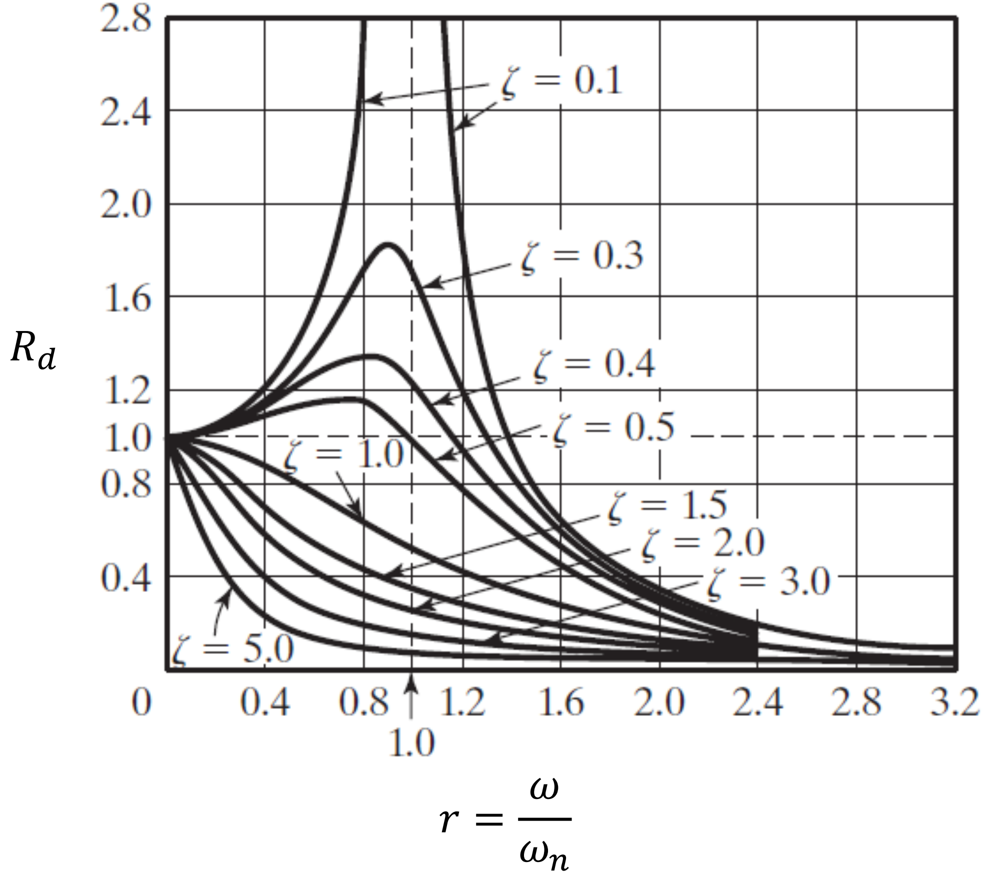


$$r = \frac{\omega}{\omega_n}$$



$$r = \frac{\omega}{\omega_n}$$

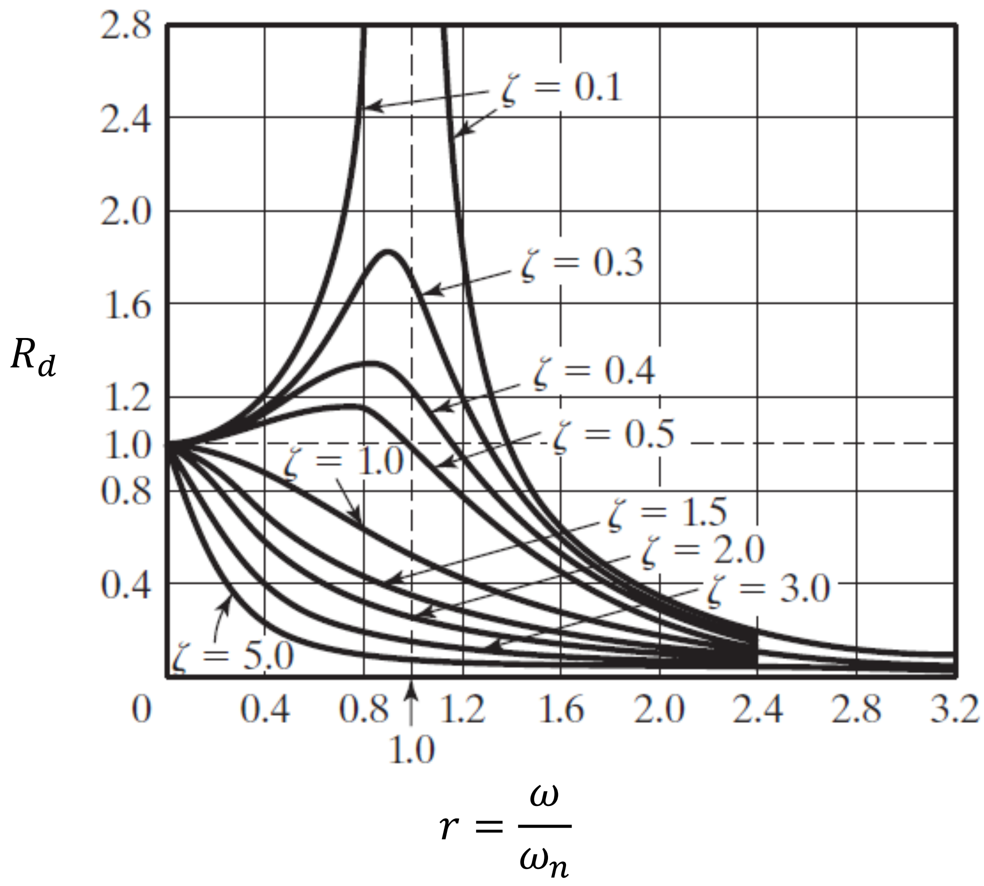
Harmonic forcing Function



The following characteristics of the magnification factor can be noted from Figure:

- For an undamped system ($\zeta = 0$), $R_d \longrightarrow \infty$ as $r \longrightarrow 1$
- Any amount of damping ($\zeta > 0$) reduces the magnification factor (R_d) for all values of the forcing frequency.
- For any specified value of r, a higher value of damping reduces the value of R_d.

Harmonic forcing Function



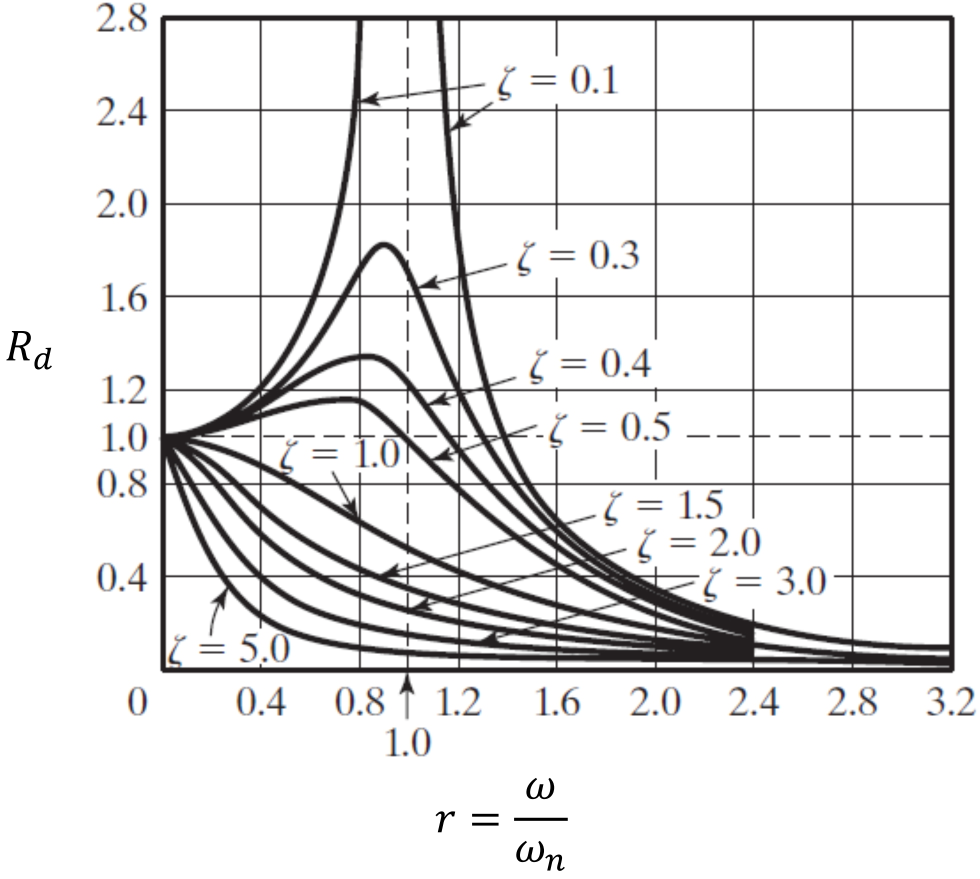
The following characteristics of the magnification factor can be noted from Figure:

- For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum value of R_d occurs when $r = \sqrt{1 - 2\zeta^2}$ or $\omega = \omega_n\sqrt{1 - 2\zeta^2}$ which can be seen to be lower than the undamped natural frequency ω_n and the damped natural frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$
- For $\zeta = \frac{1}{\sqrt{2}}$,

$$\frac{dR_d}{dr} = 0$$

  when $r = 0$. For $\zeta > \frac{1}{\sqrt{2}}$, the graph of R_d monotonically decreases with increasing values of r.

Harmonic forcing Function



The following characteristics of the magnification factor can be noted from Figure:

- The maximum value of X (when $r = \sqrt{1 - 2\zeta^2}$) is given by

$$\left(\frac{X}{\delta_{st}}\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

and the value of X at $\omega = \omega_n$ by

$\delta_{st} = \frac{F_0}{k}$

$$\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$$

This equation can be used for the experimental determination of the measure of damping present in the system. If the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

Motion Involving Base Input

Certain problems received their forced inputs through excitation of the support rather than direct application on the mass of the system. An easily identifiable problem is that due to an earthquake. To treat this case, consider the free body diagram shown in Figure here. Let $u(t)$ denote the absolute motion of the mass (i.e., with respect to a fixed inertial framework), $u_r(t)$ be the relative motion and $u_b(t)$ denote the base displacement history. The relative displacement $u_r(t)$ is merely the difference between $u(t)$ and $u_b(t)$. Two formulations are possible.

1. Equation of Motion in terms of $u(t)$: In this case, the spring force is given by $k(u-u_b)$ and the equation of motion is:

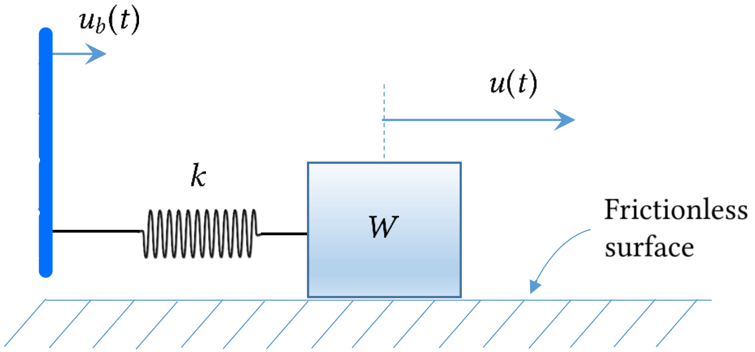
$$m\ddot{u} + k(u-u_b) = 0 \quad or \quad \ddot{u} + \omega^2 u = \frac{ku_b}{m} = \frac{P_u\text{eff}}{m}$$

where P_{eff} is the *effective dynamic load* given in terms of the *displacement history of the base*

2. Equation of Motion in terms of $u_r(t)$: Using the relative displacement, the equation of motion has the form:

$$m(\ddot{u} + \ddot{u}_b) + ku_r = 0 \quad \text{or} \quad \ddot{u}_r + \omega^2 u_r = -\ddot{u}_b = \frac{P_{\bar{u}}\text{eff}}{m}$$

where $P_{\bar{u}\text{eff}}$ is the *effective dynamic load* given in terms of the *acceleration history of the base.*



Base Motion

The equation of motion of the system is:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

Relative Motion

Sometimes we are concerned with the relative motion of the mass with respect to the base. (Example: accelerometer and the velocity meter). In this case, we can define $z=x-y$. Our equation of motion becomes:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

If the periodic input is in the form

$$y = Y \sin \omega t$$

the equation of motion becomes:

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = \underbrace{\boxed{m\omega^2 Y}}_{P_0} \sin \omega t$$

which is identical to the previous case (harmonic force), with F_0 replaced by P_0. The steady state solution can be written as:

$$z(t) = Z \sin(\omega t - \phi)$$

where

$$\frac{Z}{Z_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

But

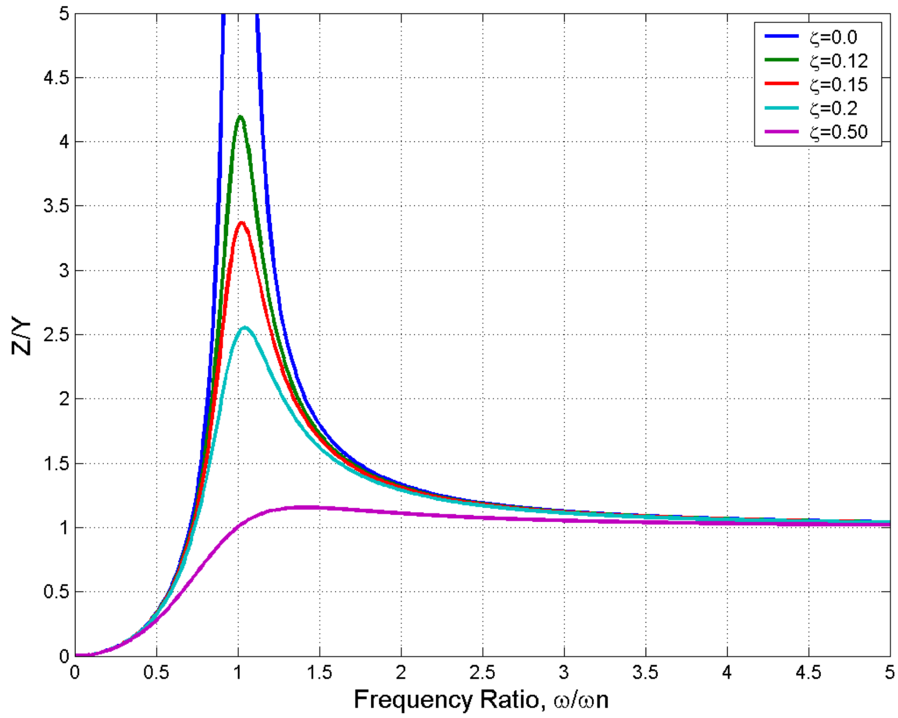
$$Z_0 = \frac{P_0}{k} = \frac{m\omega^2 Y}{k} = Yr^2$$

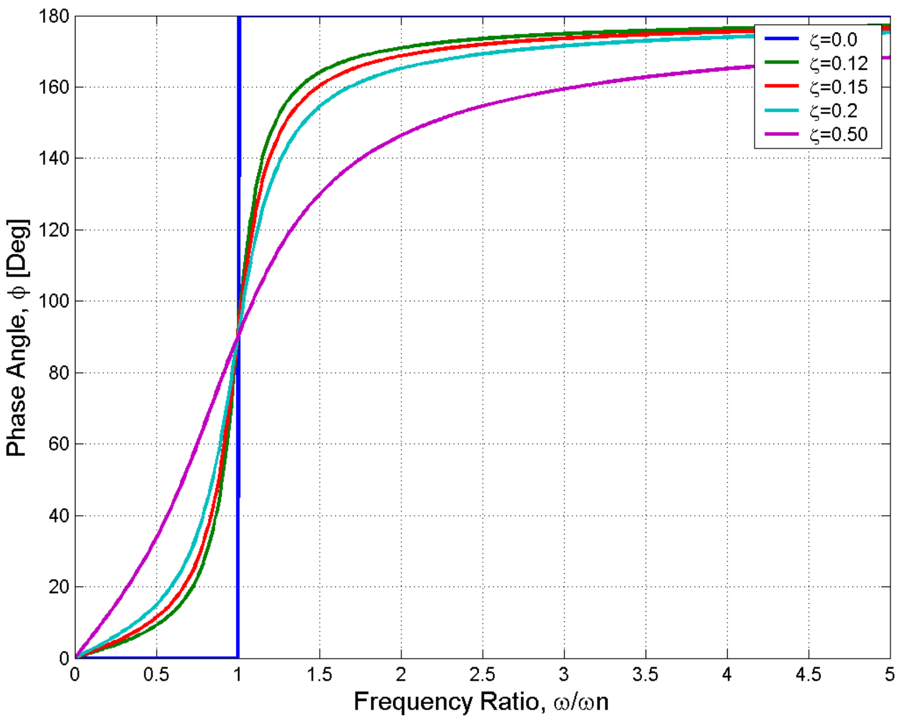
hence the solution is:

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$$





Application: vibration measuring instruments

For negligible damping ($\zeta << 1$), we have

$$\frac{Z}{Y} = \frac{r^2}{1-r^2}$$

Case 1: $r << 1 \quad \Rightarrow \quad Z = Yr^2 = Y\frac{\omega^2}{\omega_n^2}$

At "low" frequencies, Z is proportional to $\omega^2 Y$, i.e. proportional to acceleration. In this range the accelerometer works. It must have a high ω_n such that ω/ω_n is small .

Case 2: $r >> 1 \quad \Rightarrow \quad Z = Y$

At "high" frequencies, the ratio between Z and Y is one, i.e. Z is equal to the displacement. In this range the vibrometer works. It must have a low ω_n such that ω/ω_n is high.

Absolute Motion

If we are concerned with the absolute motion, we can write the equation of motion as:

$$m\ddot{x}+c\dot{x}+kx=ky+c\dot{y}$$

Here it becomes more convenient to assume the base motion having the form :

$$y=Ye^{i\omega t}$$

We seek a solution in the form:

$$x=Xe^{i(\omega t-\phi)}$$

Substituting into the equation of motion yields:

$$\left(-m\omega^2+c\omega i+k\right)Xe^{i(\omega t-\phi)}=\left[k+i\omega c\right]Ye^{i\omega t}$$

from which

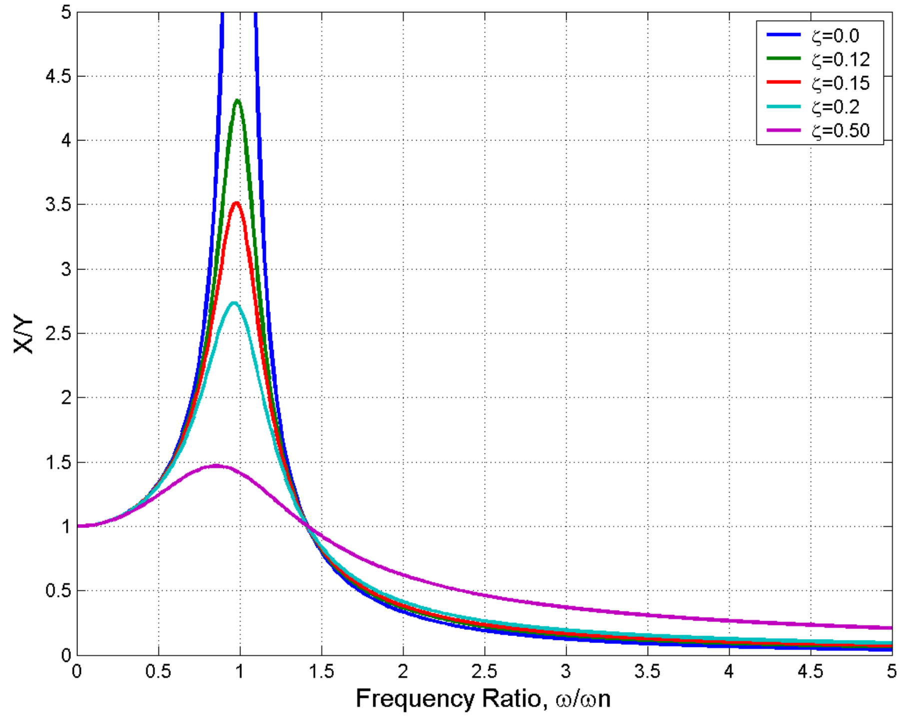
$$\frac{X}{Y}e^{-i\phi}=\frac{k+i\omega c}{\left(k-m\omega^2\right)+i\omega c}$$

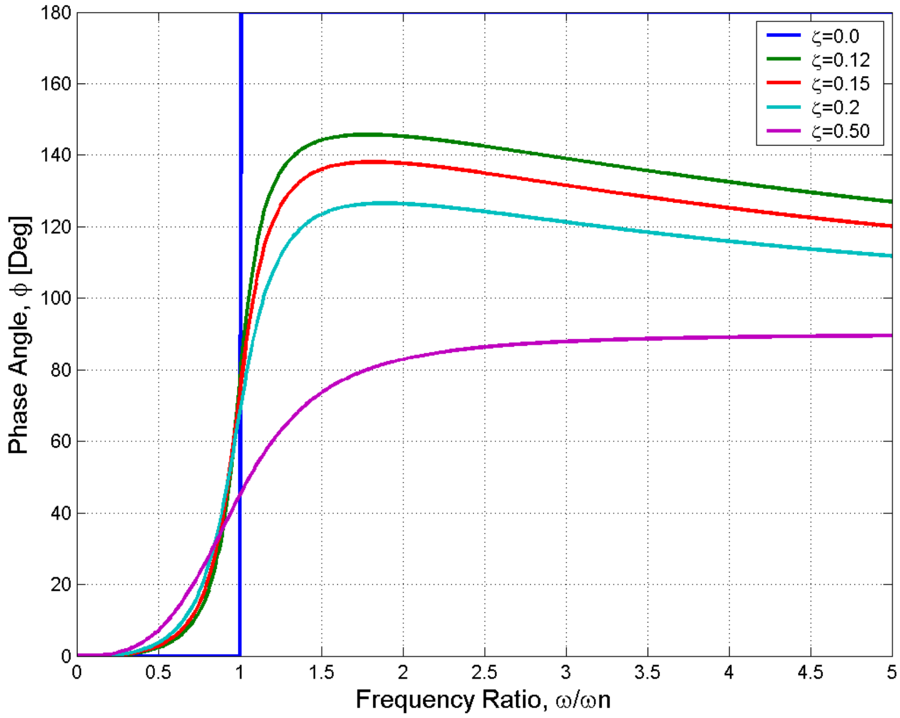
hence

$$T_d = \left|\frac{X}{Y}\right|=\frac{\sqrt{1+\left(2\zeta r\right)^2}}{\sqrt{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}}$$

and

$$\phi=\tan^{-1}\left(\frac{2\zeta r^3}{1-r^2+\left(2\zeta r\right)^2}\right)$$





Note: the motion transmitted is less than 1 for $r > \sqrt{2}$. Hence for vibration isolation, we must have $\omega/\omega_n > \sqrt{2}$, i.e. ω_n must be small compared to ω.

The above solution can also be obtained if we assume a periodic input of

$$y = Y\sin\omega t$$

the equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = kY\sin\omega t + c\omega Y\cos\omega t$$

$$m\ddot{x} + c\dot{x} + kx = A\sin(\omega t - \alpha)$$

where

$$A = Y\sqrt{k^2 + (c\omega)^2}, \alpha = \tan^{-1}\left(\frac{c\omega}{k}\right) \text{ (see page 2)}$$

But the above equation is in the same format now as the forced vibration one, with an amplitude of A instead of F_0 and an additional phase α. So we can find (see page 2)

$$x = X\sin(\omega t - \alpha - \phi_1) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}\sin(\omega t - \alpha - \phi_1)$$

$$\phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

We can write the ratio of the amplitudes as

$$\frac{X}{Y} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

using the same definition for ζ and r as before, we can also write the total solution as

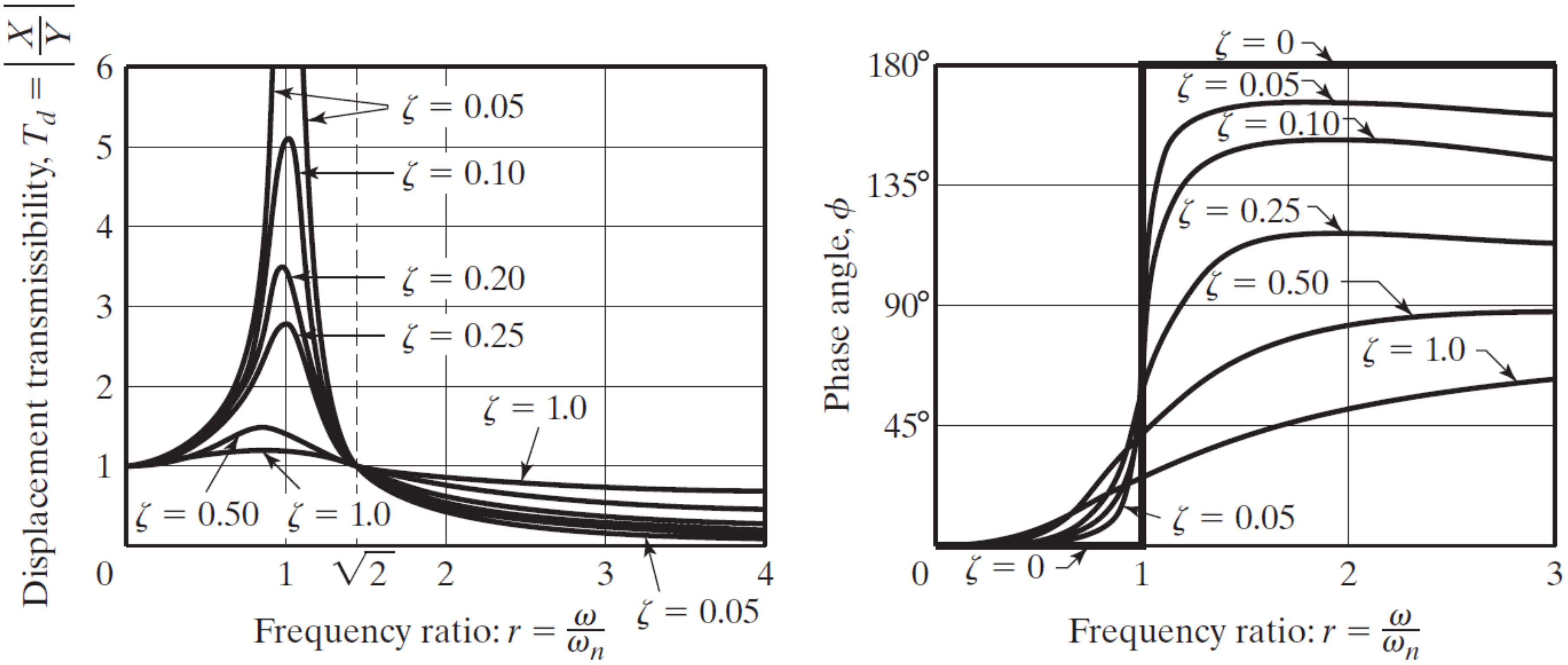
$$x_p(t) = X\sin(\omega t - \phi)$$

where

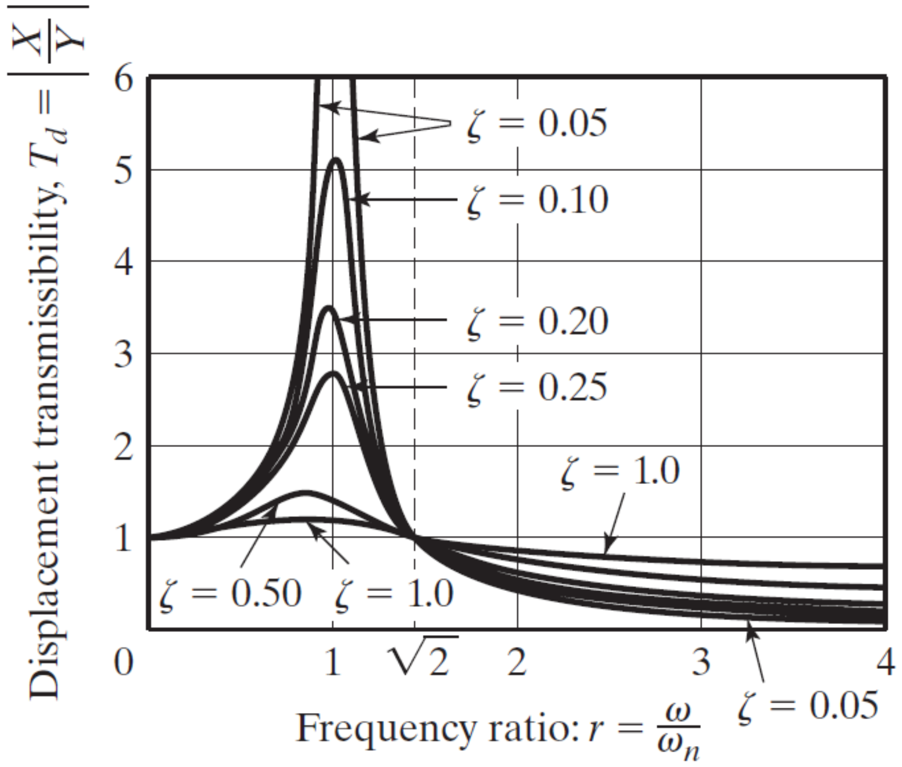
$$\phi = \tan^{-1}\left(\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right) = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

which is the same solution obtained using complex analysis.

Motion Involving Base Input



Motion Involving Base Input



The following aspects of displacement transmissibility, $T_d = \frac{X}{Y}$, can be noted from Figure:

- The value of T_d is unity at $r = 0$ and close to unity for small values of r.
- For an undamped system ($\zeta = 0$), $T_d \longrightarrow \infty$ at resowance ($r = 1$).
- The value of T_d is less than unity ($T_d < 1$) for values of $r > \sqrt{2}$ (for any amount of damping ζ).
- The value of T_d is unity for all values of ζ at $r = \sqrt{2}$.
- For $r < \sqrt{2}$, smaller damping ratios lead to larger values of T_d. On the other hand, for $r > \sqrt{2}$, smaller values of damping ratio lead to smaller values of T_d.
- The displacement transmissibility, T_d, attains a maximum for $0 < \zeta < 1$ at the frequency ratio $r = r_m < 1$ given by

$$r_m = \frac{1}{2\zeta}\left[\sqrt{1 + 8\zeta^2} - 1\right]^{1/2}$$

Force Transmitted

To find the force on the base during base motion, we have:

$$F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}$$

but x is known from before, so F is given by:

$$F = m\omega^2 X \sin(\omega t - \phi) = F_T \sin(\omega t - \phi)$$

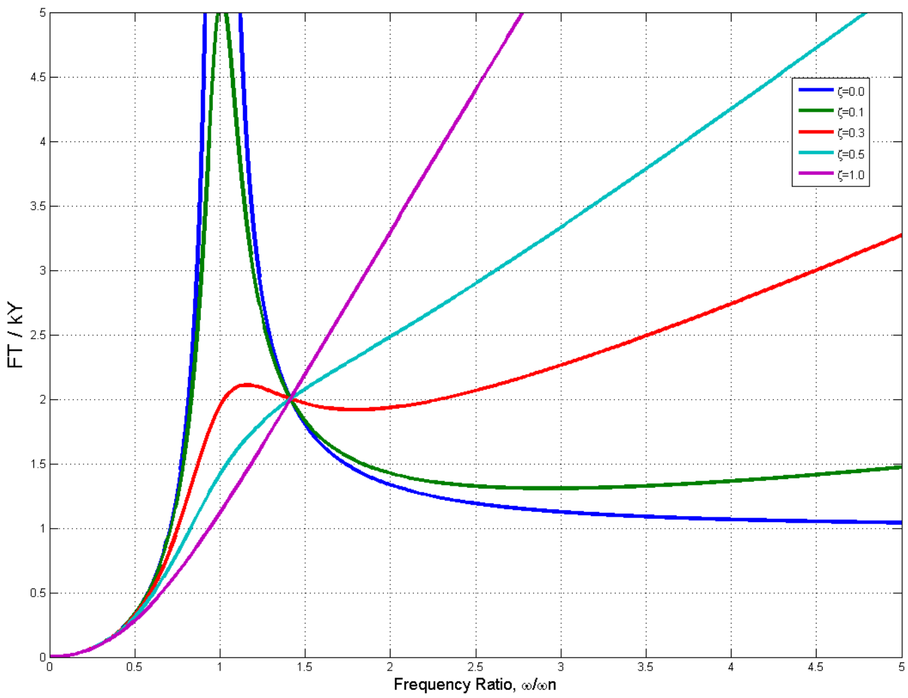
where F_T is the amplitude or maximum value of the force transmitted to the base. Hence:

$$F_T = m\omega^2 Y \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

from which we get:

$$\frac{F_T}{kY} = r^2 \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$$

In this relationship, kY represents the force on the mass if it remained stationary, while the base moved with only the spring attached. This ratio is called the *force transmissibility*.



The force transmitted can also be calculated for the case of a harmonic force

$$F_T = kx + c\dot{x} = kx + i\omega cx$$

so the force amplitude is:

$$|F_T| = \sqrt{(kX)^2 + (\omega cX)^2}$$

Thus, the *transmissibility* or *transmission ratio* of the isolator (T_r) can be calculated to be:

$$T_r = \frac{F_T}{F_0} = \sqrt{\frac{k^2 + \omega^2 c^2}{\left(k - m\omega^2\right)^2 + \omega^2 c^2}} = \sqrt{\frac{1 + (2\zeta r)^2}{\left(1 - r^2\right)^2 + (2\zeta r)^2}}$$

This has the same amplitude as the base motion ratio found earlier. If T_r is less than one, then the system behaves like a vibration isolator, i.e. the ground receives less force than the input force.

Using Laplace Transforms

Using Laplace Transforms

The equation of motion of the system is given by

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

By taking the Laplace transforms of both sides of this Eq., we obtain

$$m\mathscr{L}[\ddot{x}(t)] + c\mathscr{L}[\dot{x}(t)] + k\mathscr{L}[x(t)] = \mathscr{L}[f(t)]$$

or

$$m\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + [sX(s) - x(0)] + kX(s) = F(s)$$

Equation can be rewritten as

$$\left(ms^2 + cs + k\right)X(s) - [msx(0) + m\dot{x}(0) + sx(0)] = F(s)$$

where $X(s) = \mathscr{L}[x(t)]$ and $F(s) = \mathscr{L}[f(t)]$. The transfer function of the system can be obtained from this, by setting $x(0) = \dot{x}(0) = 0$, as

$$T(s) = \left.\frac{\mathscr{L}\,[\text{output}\,]}{\mathscr{L}[\text{ input }]}\right|_{\text{zero initial conditions}} = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

Using Laplace Transforms

$$X(s) = \frac{F(s)}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} + \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} x(0) + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \dot{x}(0)$$

The complete response of the system can be found by taking inverse Laplace transforms of each term on the right-hand side of Equation. For convenience, we define the following functions with the subscripts i and s denoting the input and system, respectively:

$$F_i(s) = F(s)$$
$$F_s(s) = \frac{1}{m(s^2+2\zeta\omega_n s+\omega_n^2)}$$

We note that the inverse Laplace transform of $F_i(s)$ will be equal to the known forcing function

$$f_i(t) = F_0 \cos \omega t$$

and the inverse Laplace transform of $F_s(s)$ is given by

$$f_s(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

where.

$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$$

Using Laplace Transforms

The inverse Laplace transform of the first term on the right-hand side of Equation can be expressed as:

$$\mathscr{L}^{-1} F_i(s) F_s(s) = \int_{r=0}^{t} f_i(\tau) f_s(t-\tau) d\tau = \frac{1}{m\omega_d} \int_{r=0}^{t} f(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau) d\tau$$

The inverse Laplace transform of the coefficient of $x(0)$ in Eq. in previous slide yields

$$\mathscr{L}^{-1} \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_1)$$

where

$$\phi_1 = \tan^{-1} \frac{\zeta\omega_n}{\omega_d} = \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}}$$

The inverse Laplace transform of the coefficient of $\dot{x}(0)$ can be obtained by multiplying $f_s(t)$ by m so that

$$\mathscr{L}^{-1} \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

Using Laplace Transforms

Thus, the complete response of the system, using the responses given on the right-hand sides, can be expressed as

$$
\begin{aligned}
x(t) = & \frac{1}{m\omega_d} \int_{\tau=0}^{t} f(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau) d\tau \\
& + \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \cos\left(\omega_d t - \phi_1\right) + \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t
\end{aligned}
$$

Noting that the inverse Laplace transform of the product function can also be expressed as

$$
\mathscr{L}^{-1} F_i(s) F_s(s) = \int_{\tau=0}^{t} f_i(t-\tau) f_s(\tau) d\tau = \frac{1}{m\omega_d} \int_{\tau=0}^{t} f(t-\tau) e^{-\zeta\omega_n \tau} \sin \omega_d \tau d\tau
$$

the complete response of the system can also be expressed as

$$
\begin{aligned}
x(t) = & \frac{1}{m\omega_d} \int_{\tau=0}^{t} f(t-\tau) e^{-\zeta\omega_n \tau} \sin \omega_d \tau d\tau \\
& + \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \cos\left(\omega_d t - \phi_1\right) + \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t
\end{aligned}
$$

Using Laplace Transforms

Question: Find the steady-state response of a damped single-degree-of-freedom system subjected to a harmonic force $f(t) = F_0 \cos \omega t$ using Laplace transform.

Solution: The Laplace transform of Equation leads to the relation (with zero initial conditions for steady-state response in following equation:

$$X(s) = \frac{F(s)}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

The Laplace transform of the input $f(t) = F_0 \cos \omega t$ is given by $F(s) = F_0 \frac{s}{s^2+\omega^2}$. Thus Eq. previous equation becomes

$$X(s) = \frac{F_0}{m} \frac{s}{\left(s^2 + \omega^2\right)} \frac{1}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

where the relations $\omega_n = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2\sqrt{mk}}$ have been used. By expressing the right-hand side as

$$F(s) = \frac{F_0}{m}\left(\frac{a_1 s + a_2}{s^2 + \omega^2} + \frac{a_3 s + a_4}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

Using Laplace Transforms

The constants a_1, a_2, a_3, and a_4 can be identified as

$$a_1 = \frac{\omega_n^2 - \omega^2}{(2\zeta\omega_n)^2\,\omega^2 + (\omega_n^2 - \omega^2)^2}$$

$$a_2 = \frac{2\zeta\omega_n\omega^2}{(2\zeta\omega_n)^2\,\omega^2 + (\omega_n^2 - \omega^2)^2}$$

$$a_3 = -\frac{\omega_n^2 - \omega^2}{(2\zeta\omega_n)^2\,\omega^2 + (\omega_n^2 - \omega^2)^2}$$

$$a_4 = -\frac{\omega_n^2 - \omega^2}{(2\zeta\omega_n)^2\,\omega^2 + (\omega_n^2 - \omega^2)^2}$$

Thus, $X(s)$ can be expressed as

$$\begin{aligned} X(s) = \frac{F_0}{m}\frac{1}{(2\zeta\omega_n)^2\,\omega^2 + (\omega_n^2 - \omega^2)^2}\Bigg[\left(\omega_n^2 - \omega^2\right)\left(\frac{s}{s^2 + \omega^2}\right) + (2\zeta\omega_n\omega)\left(\frac{\omega}{s^2 + \omega^2}\right) \\ - \left(\omega_n^2 - \omega^2\right)\left(\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) - (2\zeta\omega_n)\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\Bigg] \end{aligned}$$

Using Laplace Transforms

Using the previous relations, the response of the system can be expressed as

$$x(t) = \frac{F_0}{m} \frac{1}{(2\zeta\omega_n)^2 + (\omega_n^2 - \omega^2)^2} \Bigg[\left(\omega_n^2 - \omega^2\right) \cos \omega t + 2\zeta\omega_n\omega \sin \omega t$$
$$+ \frac{\left(\omega_n^2 - \omega^2\right)}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t - \phi \right)$$
$$- \frac{\left(2\zeta\omega_n^2\right)}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t \right) \Bigg]$$

where

$$\phi = \tan^{-1}\left(\frac{1-\zeta^2}{\zeta} \right)$$

It can be observed that as $t \longrightarrow \infty$, the terms involving $e^{-\zeta\omega_n t}$ approach zero.

Using Laplace Transforms

Thus the steady-state response of the system can be expressed as

$$x(t) = \frac{F_0}{m} \frac{1}{(2\zeta\omega_n)^2 + (\omega_n^2 - \omega^2)^2} \left[\left(\omega_n^2 - \omega^2 \right) \cos \omega t + 2\zeta\omega_n\omega \sin \omega t \right]$$

which can be simplified as

$$x(t) = \frac{F_0}{\sqrt{c^2\omega^2 + (k - m\omega^2)^2}} \cos(\omega t - \phi)$$

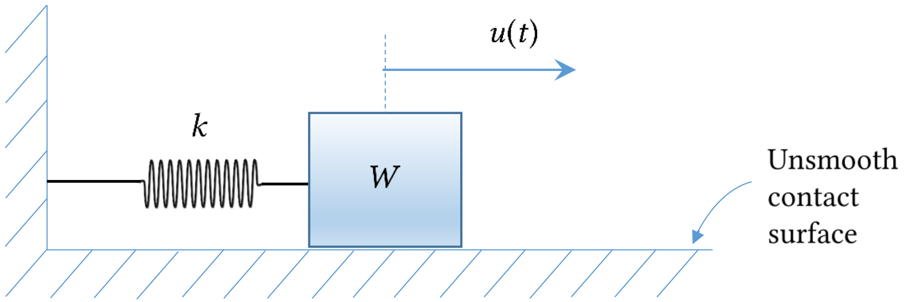
Coulomb Damping

This frictional force always acts in a direction opposite to the motion, and its magnitude is proportional to the normal force N between the contact surfaces:

$$F_d = \mu N$$

where μ is the coefficient of kinetic friction.

Consider a single-degree-of-freedom system with dry friction as shown in Figure.



Coulomb Damping

Since the friction force varies with the direction of velocity, we need to consider two cases, as indicated in Figure:

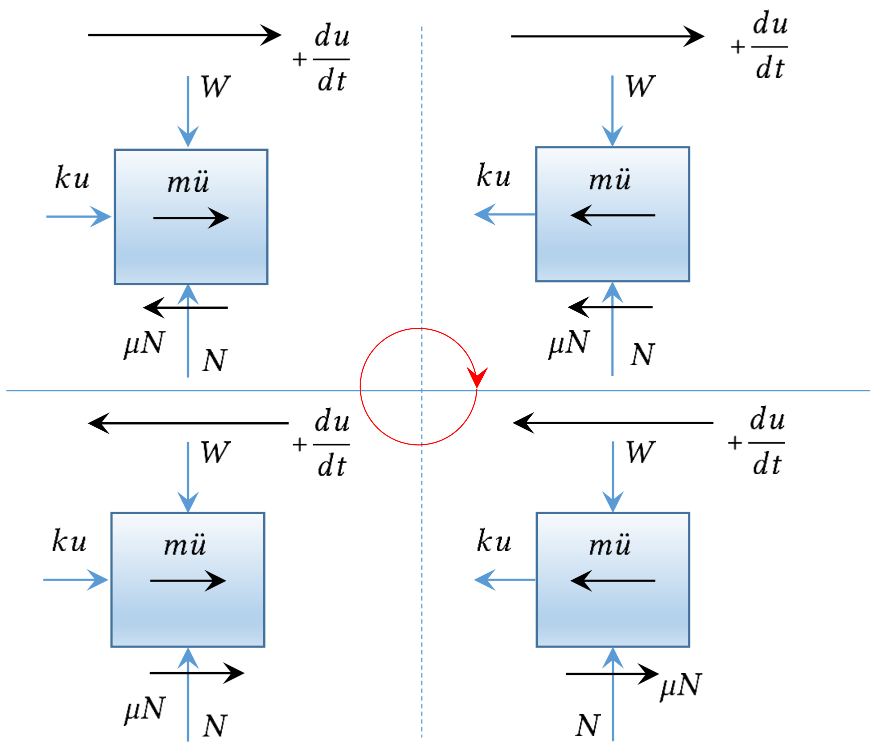
1. When u is positive and du/dt is positive or when u is negative and du/dt is positive(i.e., for the half cycle during which the mass moves from left to right), the equation of motion can be obtained using Newton's second law:

$$m\ddot{u} + \mu N + ku = 0$$

2. When u is positive and du/dt is negative or when u is negative and du/dt is negative (i.e., for the half cycle during which the mass moves from right to left), the equation of motion can be derived as:

$$m\ddot{u} - \mu N + ku = 0$$

These equations of motion can be solved analytically if we break the time axis into segments separated by $\dot{u} = 0$ (i.e., time intervals with different directions of motion)



Coulomb Damping

To find the solution using this procedure, let us assume the initial conditions as: $u(t=0)=u_0; \quad \dot{u}(t=0)=0$

$$\text{Equation of Motion:}\;\; m\ddot{u}+\mu mg+ku=0; \qquad \text{Solution:}\;\; u(t)=A_1\cos(\omega t)+A_2\sin(\omega t)+\frac{\mu mg}{k}\cdots\cdots\text{System 1}$$

$$\text{Equation of Motion:}\;\; m\ddot{u}-\mu mg+ku=0; \qquad \text{Solution:}\;\; u(t)=A_1\cos(\omega t)+A_2\sin(\omega t)-\frac{\mu mg}{k}\cdots\cdots\text{System 2}$$

where $\omega=\sqrt{k/m}$ is the frequency of vibration and A_1, A_2, A_3 and A_4 are constants whose values depend on the initial conditions of their corresponding half cycles. The system starts with zero velocity and displacement u_0 at $t=0$. Since $u=u_0$ at $t=0$, the motion starts from right to left. Let $u_0, u_1, u_2, \ldots$ denote the amplitudes of motion at successive half cycles. Using system 2 and initial conditions, we can evaluate the constants A_3 and A_4 :

$$A_3=u_0-\frac{\mu N}{k}, \quad A_4=0$$

Thus system 2 solution becomes

$$u(t)=\left(u_0-\frac{\mu N}{k}\right)\cos\omega t+\frac{\mu N}{k} \tag{1}$$

This solution is valid for half the cycle only for $0\le t\le \pi/\omega$. When $t=\pi/\omega$, the mass will be at its extreme left position and its displacement from equilibrium position can be found from equation (1):

$$-u_1=u\left(t=\frac{\pi}{\omega}\right)=\left(u_0-\frac{\mu N}{k}\right)\cos\pi+\frac{\mu N}{k}=-\left(u_0-\frac{2\mu N}{k}\right)$$

Coulomb Damping

Since the motion started with a displacement of $u = u_0$ and, in a half cycle, the value of u became $-[u_0 - (2\mu N/k)]$, the reduction in magnitude of u in time π/ω is $2\mu N/k$. In the second half cycle, the mass moves from left to right, so system 1 can be used. The initial conditions for this half cycle are

$$u(t=0) = \text{ value of } u \text{ at } t = \frac{\pi}{\omega} \text{ in Equation (1) } = -\left(u_0 - \frac{2\mu N}{k}\right)$$

and

$$\dot{u}(t=0) = \text{ value of } \dot{x} \text{ at } t = \frac{\pi}{\omega_n} \text{ in Equation (1) } = \left\{ \text{value of } -\omega\left(x_0 - \frac{\mu N}{k}\right) \sin \omega t \text{ at } t = \frac{\pi}{\omega} \right\} = 0$$

Thus the constants become

$$-A_1 = -u_0 + \frac{3\mu N}{k}, \quad A_2 = 0$$

so that Equation can be written as

$$u(t) = \left(u_0 - \frac{3\mu N}{k}\right) \cos \omega t - \frac{\mu N}{k} \tag{2}$$

This equation is valid only for the second half cycle-that is, for $\pi/\omega \leq t \leq 2\pi/\omega$.

Coulomb Damping

At the end of this half cycle the value of $u(t)$ is

$$u_2 = u\left(t = \frac{\pi}{\omega}\right) \text{ in Equation (2) } = u_0 - \frac{4\mu N}{k}$$

and

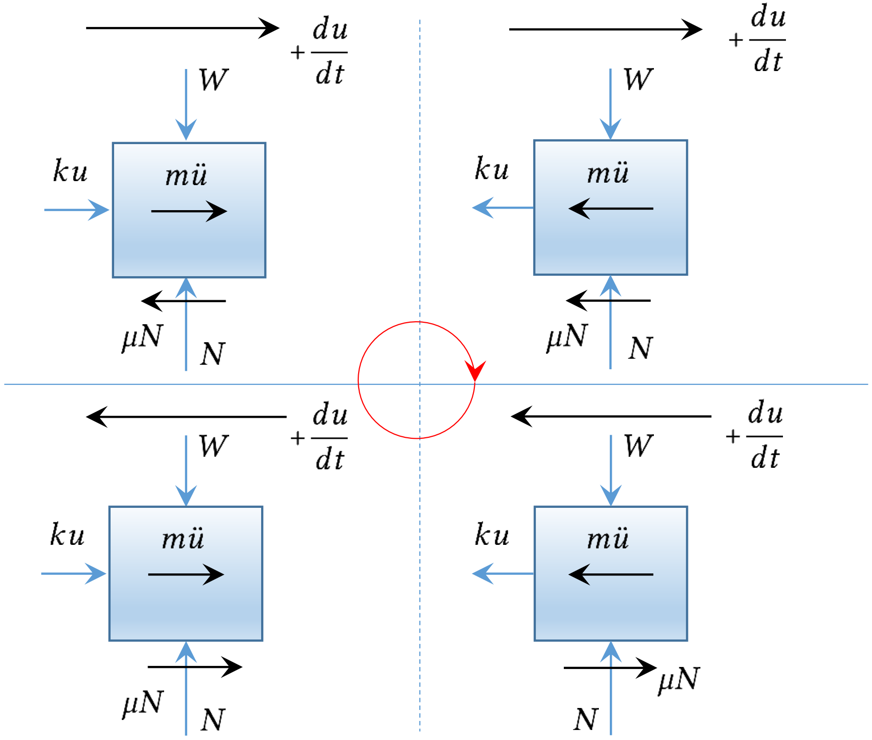
$$\dot{u}\left(t = \frac{\pi}{\omega}\right) \text{ in Equation (2)} = 0$$

These become the initial conditions for the third half cycle, and the procedure can be continued until the motion stops. The motion stops when $u_n \leq \mu N/k$, since the restoring force exerted by the spring (ku) will then be less than the friction force μN. Thus the number of half cycles (r) that elapse before the motion ceases is given by

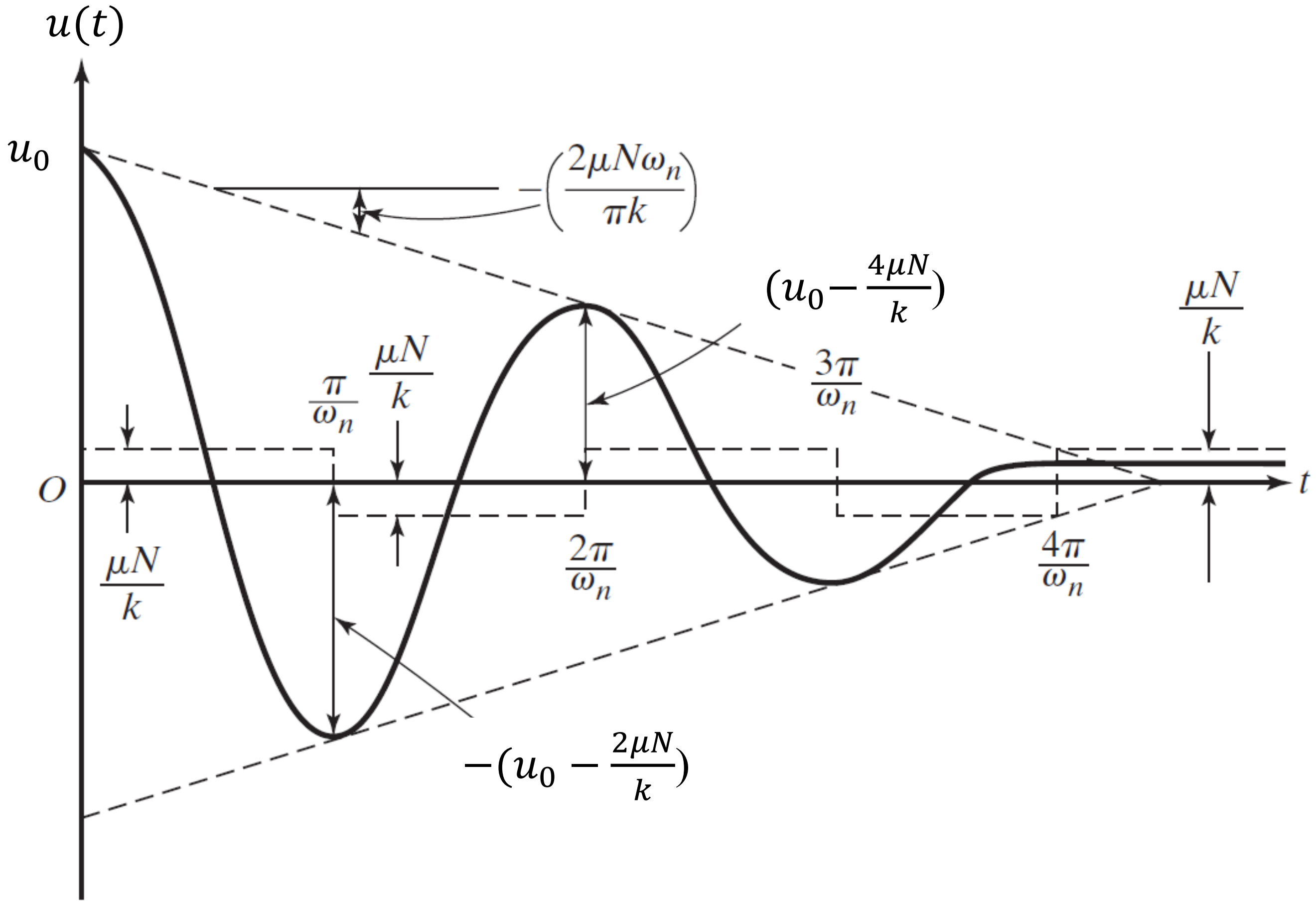
$$u_0 - r\frac{2\mu N}{k} \leq \frac{\mu N}{k}$$

that is,

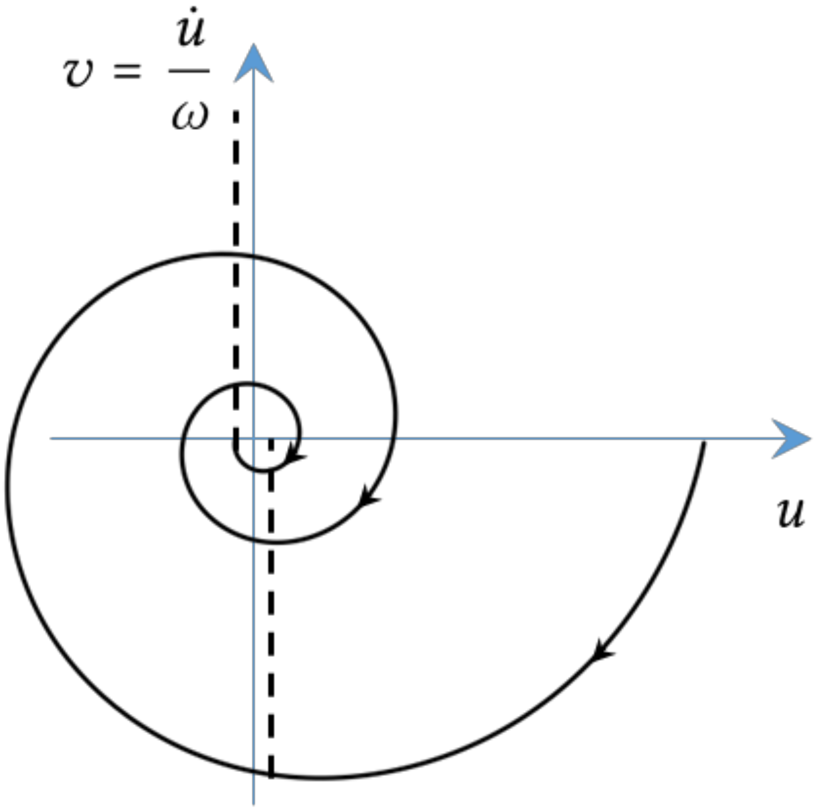
$$r \geq \left\{\frac{u_0 - \frac{\mu N}{k}}{\frac{2\mu N}{k}}\right\}$$



Coulomb Damping



Coulomb Damping



Coulomb Damping

Both equations can be expressed as a single equation:

$$m\ddot{u} + \mu mg \ \mathrm{sgn}(\dot{u}) + ku = 0$$

where sgn(v) is called the signum function, whose value is defined as 1 for $v > 0$, -1 for $v < 0$, and 0 for $v = 0$.

In each successive cycle, the amplitude of motion is reduced by the amount $4\mu N/k$, so the amplitudes at the end of any two consecutive cycles are related:

$$u_m = u_{m-1} - \frac{4\mu N}{k}$$

As the amplitude is reduced by an amount $4\mu N/k$ in one cycle (i.e., in time $2\pi/\omega$), the slope of the enveloping straight lines (shown dotted) is

$$-\left[\frac{\left(\frac{4\mu N}{k}\right)}{\left(\frac{4\pi}{\omega}\right)}\right] = -\left(\frac{2\mu N\omega}{\pi k}\right)$$

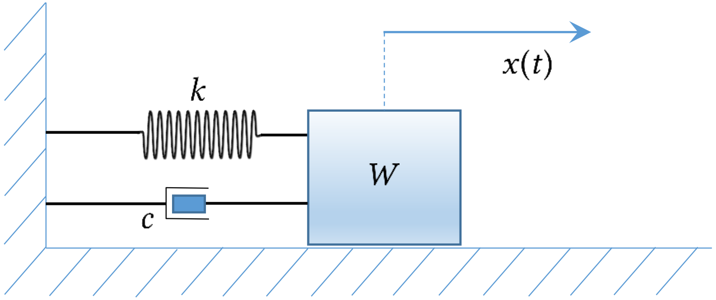
Energy Dissipation

Consider the spring-viscous-damper arrangement shown in Figure. For this system, the force F needed to cause a displacement $x(t)$ is given by

$$F = kx + c\dot{x}$$

For a harmonic motion of frequency ω and amplitude X,

$$x(t) = X \sin \omega t$$



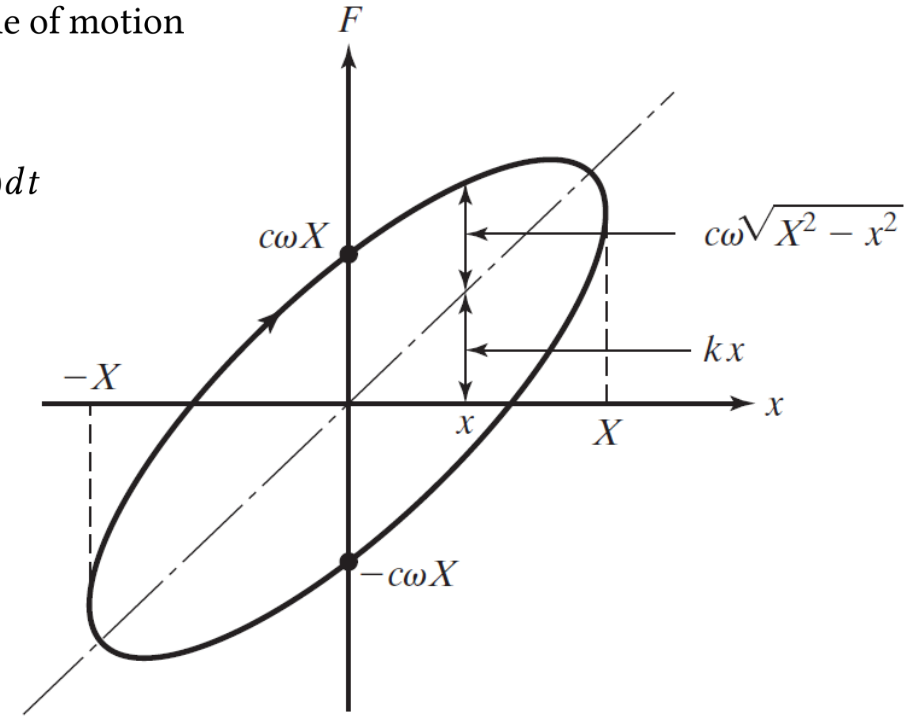
Both equations yield

$$\begin{aligned} F(t) &= kX \sin \omega t + cX\omega \cos \omega t \\ &= kx \pm c\omega\sqrt{X^2 - (X \sin \omega t)^2} \\ &= kx \pm c\omega\sqrt{X^2 - x^2} \end{aligned}$$

Energy Dissipation

When F versus x is plotted, it represents a closed loop, as shown in Figure. The area of the loop denotes the energy dissipated by the damper in a cycle of motion and is given by

$$\Delta W = \oint F dx = \int_0^{2\pi/\omega} (kX \sin \omega t + cX\omega \cos \omega t)(\omega X \cos \omega t) dt$$
$$= \pi \omega c X^2$$



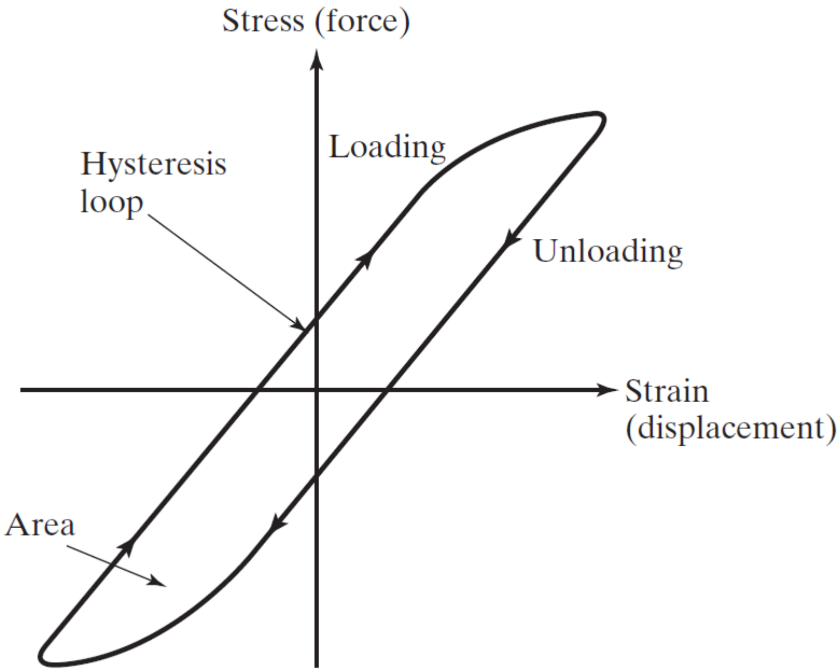
Structural Damping

The damping coefficient c is assumed to be inversely proportional to the frequency as

$$c = \frac{h}{\omega}$$

where h is called the hysteresis damping constant. Hence

$$\Delta W = \pi h X^2$$



Structural Damping

Complex Stiffness: For a general harmonic motion, $x = Xe^{i\omega t}$, the force is given by

$$F = kXe^{i\omega t} + c\omega iXe^{i\omega t} = (k + i\omega c)x$$

Similarly, if a spring and a hysteresis damper are connected in parallel, as shown in Figure, the force-displacement relation can be expressed as

$$F = (k + ih)x$$

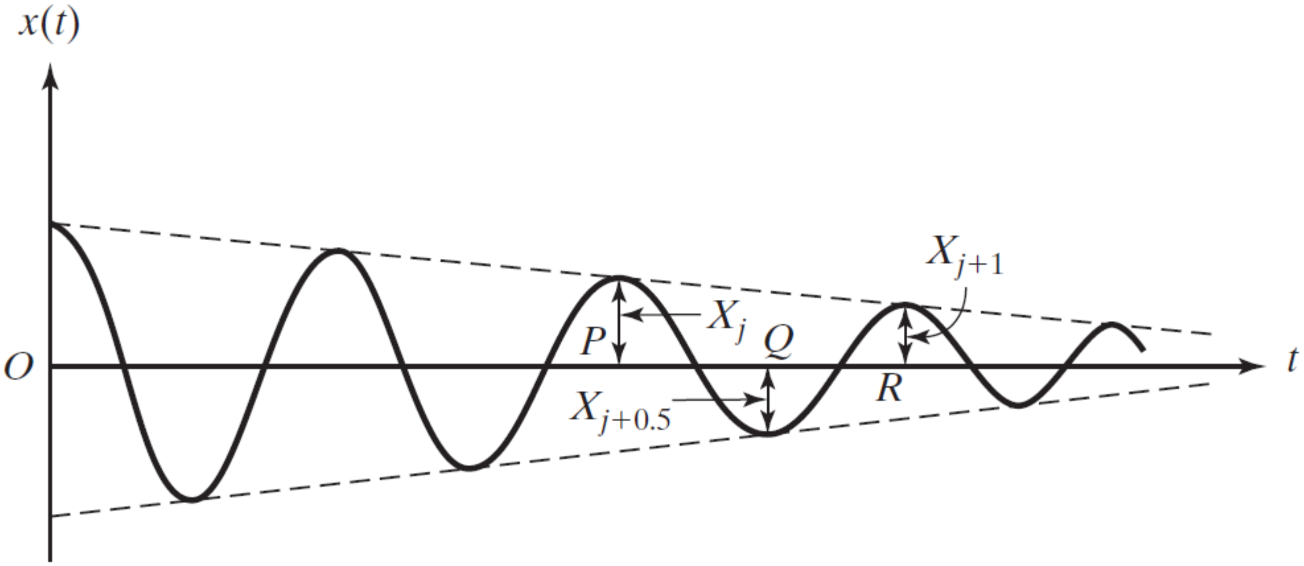
where

$$k + ih = k\left(1 + i\frac{h}{k}\right) = k(1 + i\beta)$$

is called the complex stiffness of the system and $\beta = h/k$ is a constant indicating a dimensionless measure of damping. Response of the System. In terms of β, the energy loss per cycle can be expressed as

$$\Delta W = \pi k \beta X^2$$

Structural Damping



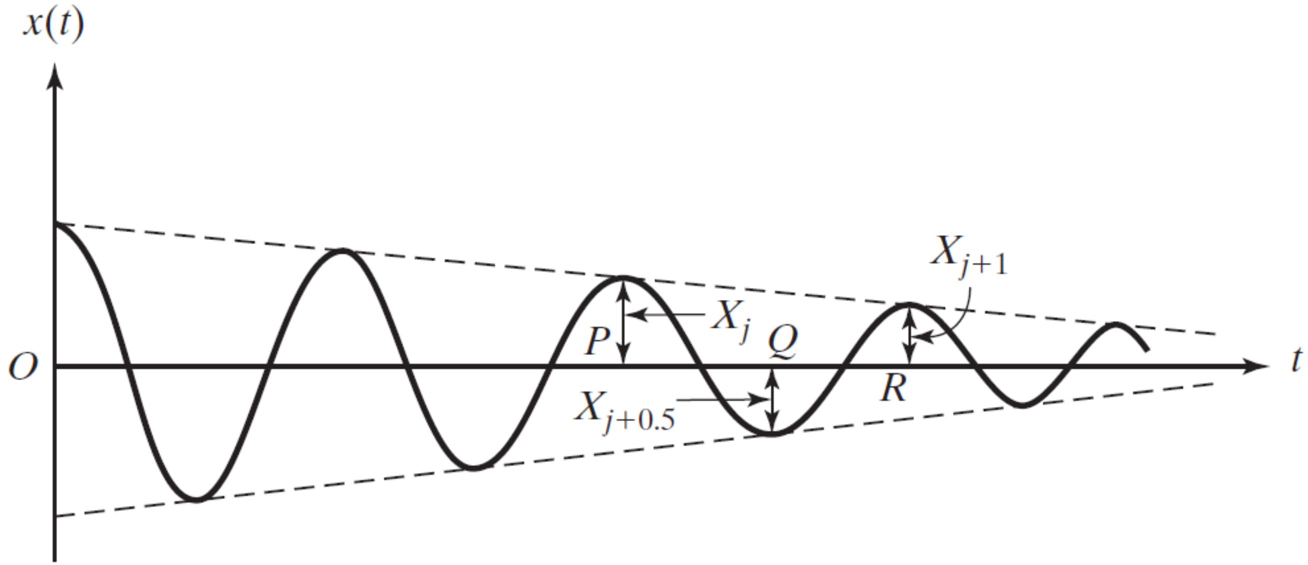
The energies at points P and Q (separated by half a cycle) in Figure are related as

$$\frac{kX_j^2}{2} - \frac{\pi k\beta X_j^2}{4} - \frac{\pi k\beta X_{j+0.5}^2}{4} = \frac{kX_{j+0.5}^2}{2}$$

or

$$\frac{X_j}{X_{j+0.5}} = \sqrt{\frac{2+\pi\beta}{2-\pi\beta}}$$

Structural Damping



Similarly, the energies at points Q and R give

$$\frac{X_{j+0.5}}{X_{j+1}} = \sqrt{\frac{2+\pi\beta}{2-\pi\beta}}$$

Multiplication of Eqs. (2.151) and (2.152) gives

$$\frac{X_j}{X_{j+1}} = \frac{2+\pi\beta}{2-\pi\beta} = \frac{2-\pi\beta+2\pi\beta}{2-\pi\beta} \simeq 1+\pi\beta = \text{ constant}$$

Structural Damping

The hysteresis logarithmic decrement can be defined as

$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) = \ln(1 + \pi\beta) \simeq \pi\beta$$

Since the motion is assumed to be approximately harmonic, the corresponding frequency is defined as:

$$\omega = \sqrt{\frac{k}{m}}$$

The equivalent viscous damping ratio ζ_{eq} can be found by equating the relation for the logarithmic decrement δ :

$$\delta \simeq 2\pi\zeta_{\mathrm{eq}} \simeq \pi\beta = \frac{\pi h}{k}$$

$$\zeta_{\mathrm{eq}} = \frac{\beta}{2} = \frac{h}{2k}$$

Thus the equivalent damping constant c_{eq} is given by

$$c_{\mathrm{cq}} = c_c \cdot \zeta_{\mathrm{eq}} = 2\sqrt{mk} \cdot \frac{\beta}{2} = \beta\sqrt{mk} = \frac{\beta k}{\omega} = \frac{h}{\omega}$$