AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 5: Test Signals and Convolution

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Standard Test Signals

Introduction

- In general, input, $u(t)$, is not fully known ahead of time.
- It is, therefore, difficult to express the actual input as an expression.
- This has given rise to test signals, which provide a way of characterizing the behaviour during design.
- These test signals are simplified forms of the realistic inputs.
- In control analysis and design, impulse, step, ramp, parabolic are treated as test signals, as these are able to excite the relevant dynamical features.

Standard Test Signals - Impulse

Impulse Function

The signal imitates sudden shock characteristics.

$$u(t) = \delta(t) = \begin{Bmatrix} A : t = 0 \\ 0 : t \neq 0 \end{Bmatrix}$$

When $A = 1$, then it is called unit impulse.

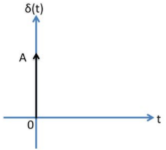

Figure: *Source - http://engineering.electrical-equipment.org/*

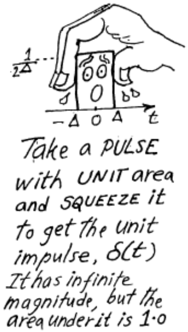

Figure: Source - "Cartoon Tour Of Control Theory" by S. M. Joshi

Standard Test Signals - Impulse

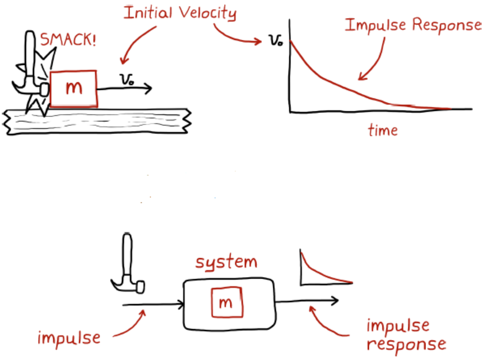

Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

Standard Test Signals - Step

Step Function

The signal imitates sudden change characteristics.

$$u(t) = \begin{cases} A : t \geq 0 \\ 0 : t < 0 \end{cases}$$

When $A = 1$, then it is called unit step.

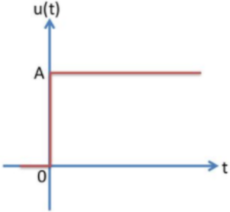

Figure: *Source - http://engineering.electrical-equipment.org/*

Standard Test Signals- Ramp

Ramp Function

The signal imitates constant velocity characteristics.

$$u(t) = \begin{cases} At : t \geq 0 \\ 0 : t < 0 \end{cases}$$

When $A = 1$, then it is called unit ramp.

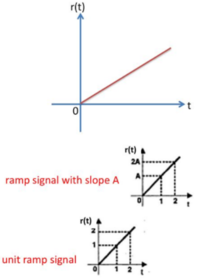

Figure: *Source - http://engineering.electrical-equipment.org/*

Standard Test Signals- Parabolic

Parabolic Function

The signal imitates constant acceleration characteristics.

$$u(t) = \begin{cases} \frac{At^2}{2} : t \geq 0 \\ 0 : t < 0 \end{cases}$$

When $A = 1$, then it is called unit parabola.

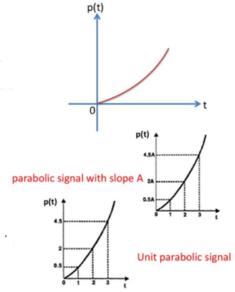

Figure: *Source - http://engineering.electrical-equipment.org/*

Standard Test Signals

Relation between standard Test Signals

- Impulse
- Step
- Ramp
- Parabolic

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ $\int$ $\int$

$\frac{d}{dt}$ $\frac{d}{dt}$ $\frac{d}{dt}$

Figure: *Source - "Feedback Control Systems" by Imtiaz Hussain*

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Time Response I

- In dealing with LTI, principle of superposition is invoked in order to simplify the solution procedure.Thus, total response is a sum of natural and forced.
- This philosophy is further extended by decomposing the general n^{th} order system into a number of 1^{st} and 2^{nd} order systems, whose responses are added to get full response.
- As a consequence, solution methodologies give a lot of importance to 1^{st} and 2^{nd} order system responses, which are also part of responses of even higher order systems.
- While we can obtain 1^{st} and 2^{nd} order responses through assumed functions, as most systems are of higher order and experience complex inputs, we need a generic procedure.

Time Response II

- As integrating factor for a general input, is not feasible, an alternative strategy, which uses impulse response is employed.

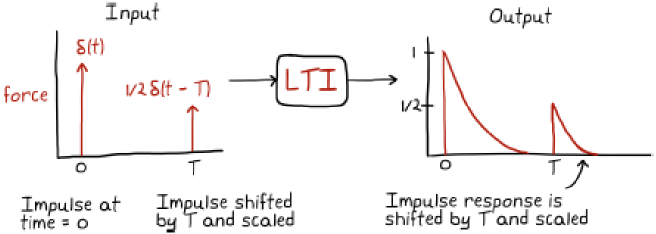

Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

Time Response III

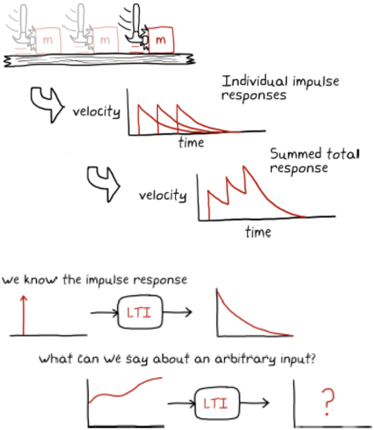

Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

Convolution

- **Convolution** is such a technique, based on the concept of assembling a large number of impulse responses to arrive at response to general inputs.

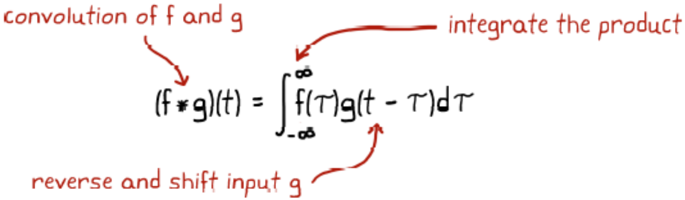

Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

Convolution

Definition

$$y(t) = \int_0^t h(t-\tau)u(\tau) = \int_0^t h(\tau)u(t-\tau), \quad t \geq 0$$

where,

$$h(t) \longrightarrow \text{Impulse Response of System}$$
$$u(t) \longrightarrow \text{Input to the System}$$

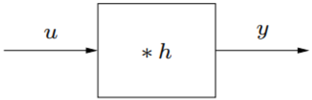

Figure: *Convolution Block Diagram*

Convolution

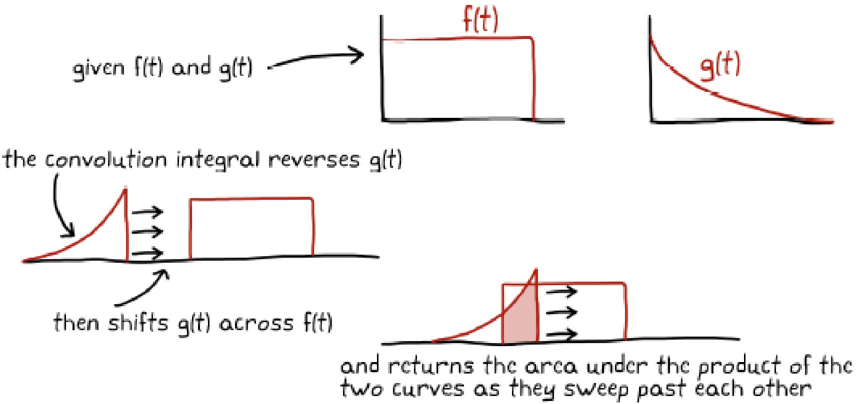

Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

Convolution - Graphical Representation

$$y(t) = \int_0^t h(t-\tau)u(\tau)$$

To obtain $y(t)$:

- Flip impulse response $h(\tau)$ backwards in time (yields $h(-\tau)$)
- Drag to the right over t (yields $h(t-\tau)$)
- Multiply point-wise by u (yields $u(\tau)h(t-\tau)$)
- Integrate over τ to get $y(t)$

Convolution - Graphical Representation

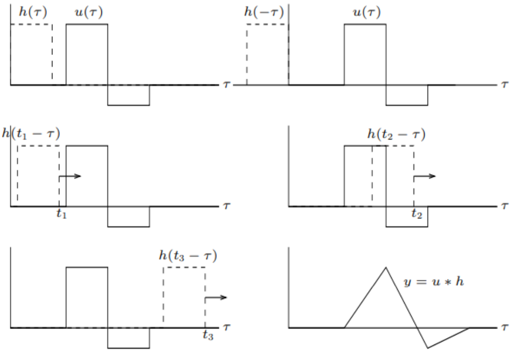

Figure: *Source - "Introduction to Signals & Systems" by Stephen P. Boyd*

Convolution - Property

For piecewise continuous functions f, g, h:

1. Commutativity:
$$f * g = g * f$$
2. Associativity:
$$f * (g * h) = (f * g) * h$$
3. Distributivity:
$$f * (g + h) = f * g + f * h$$
4. Neutral Element:
$$f * 0 = 0$$
5. Identity Element:
$$f * \delta = f$$

Convolution - Example

Example: Consider a 1^{st} **order system** subjected to unit step input

$$u(t) = 1, \quad g(t) = \frac{1}{T}e^{-t/T}$$

Solution:

Step Response will be given by,

$$c_{step}(t) = \int_0^t g(t-\tau)u(\tau)$$

Substituting values,

$$c_{step}(t) = \int_0^t g(t-\tau)1d\tau = \frac{1}{T}\int_0^t e^{-t/T}e^{\tau/T}d\tau = e^{-t/T}\int_0^t e^{\tau/T}d\tau$$

On Solving,

$$c_{step}(t) = e^{-t/T}\frac{1}{T}\left[\ Te^{\tau/T}\right]_0^t = e^{-t/T}\left[e^{t/T} - 1\right] = 1 - e^{-t/T}$$

Convolution - Limitations

- Convolution approach is generally feasible only for 1^{st} or 2^{nd} order systems and also only for simple inputs.
- We see that as order increases or input function becomes complex, integration process becomes tedious.
- Further, as we need to generate responses repeatedly during the design phase, it is necessary to have a strategy that can do the task quickly for all systems and inputs.
- **Laplace transform** and transfer function based techniques are part of such a solution strategy.

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