AE 707: Tutorial 5. Thin Airfoil Theory

1. Solve the thickness problem for a NACA 4-digit airfoil.

   Solution:

   We will take two approaches to solving the thickness problem. The first will be a generic approach that will apply to all airfoil shapes. The second will be specific to the NACA 4-digit airfoil family, for which an analytical function exists for describing the thickness distribution.

   Thickness problem solution for generic airfoil: Airfoils having rounded or sharp leading edges and sharp trailing edges have their thickness distribution function (or data, if no analytical function exists) going to zero at $x = 0$ and c. For them, as suggested in Drela's textbook, we can expand the thickness distribution function $z_t(x)$ in a sine series in the trigonometric coordinate θ_0, where $x = 0.5c(1 - \cos\theta_0)$:

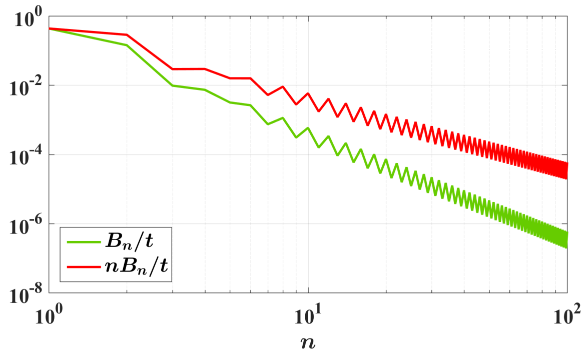
$$\frac{z_t(x/c)}{c} = \frac{z_t(\theta_0)}{c} = \sum_{n=1}^{\infty} B_n \sin n\theta_0, \qquad B_n = \frac{2}{\pi}\int_0^{\pi} \frac{z_t(\theta_0)}{c} \sin n\theta_0 d\theta_0.$$

   One can verify that each term of the sine series indeed vanishes at the leading and trailing edges (i.e., $\theta_0 = 0$ and π).

   For the thickness problem, one is interested in the thickness derivative:

$$\frac{dz_t}{dx}(\theta_0) = \frac{d(z_t/c)}{d(x/c)}(\theta_0) = \frac{d\theta_0}{d(x/c)}\,\frac{d(z_t/c)}{d\theta_0}(\theta_0) = \frac{2}{\sin\theta_0}\sum_{n=1}^{\infty} nB_n \cos n\theta_0.$$

   We note that for airfoils with rounded leading edges, whose thickness derivative becomes infinite at $\theta_0 = 0$, we may require many terms for adequately representation of the leading edge. This is not a severe limitation since the thickness problem solution should not be trusted thereat owing to the invalidity of the 'small-perturbation' assumption near the stagnation point. The following figure shows that the sine series coefficients decay rapidly, with the 11th one, even after weighting by 11, being less than 100th of the first one in magnitude.

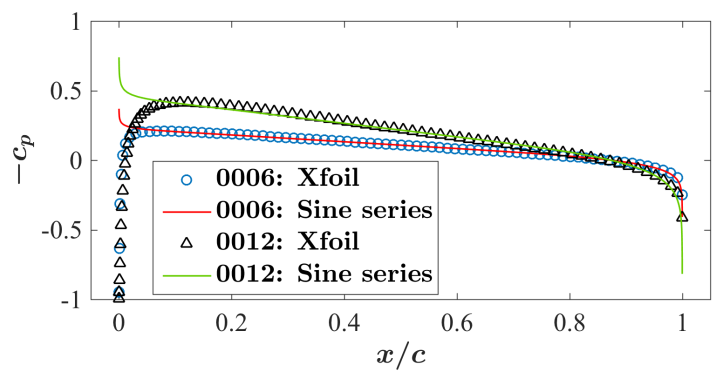


First 100 sine series coefficients of the NACA 4-digit airfoil family's thickness distribution, normalized by the maximum fractional thickness t.

Substituting this in the solution of the thickness problem, and recalling that $c_{p,u}(x) = c_{p,l}(x) = c_p(x)$, we have

$$\begin{aligned}
c_p(x) &= -\frac{2}{\pi}\int_0^c \frac{dz_t}{d\zeta}(\zeta)\frac{d\zeta}{x-\zeta} = -\frac{2}{\pi}\int_0^\pi \frac{dz_t}{d\zeta}(\theta)\frac{0.5c\sin\theta d\theta}{0.5c(1-\cos\theta_0)-0.5c(1-\cos\theta)} \\
&= -\frac{4}{\pi}\sum_{n=1}^{\infty} nB_n \int_0^\pi \frac{\cos n\theta}{\cos\theta-\cos\theta_0}d\theta \\
&= -\frac{4}{\pi}\sum_{n=1}^{\infty} nB_n \left(\frac{\pi\sin n\theta_0}{\sin\theta_0}\right) \qquad \text{[recognizing the above as the Glauert's integral]} \\
&= -\frac{4}{\sin\theta_0}\sum_{n=1}^{\infty} nB_n \sin n\theta_0.
\end{aligned}$$

We can readily evaluate the above sum for a finite number of terms, to obtain the common c_p on both surfaces of the airfoil, barring the stagnation zones near the leading and trailing edges.



Pressure coefficient predicted by the sine-series method in thin airfoil theory for NACA 0006 and NACA 0012 airfoils at $\alpha = 0$, compared with results from inviscid calculations using XFoil.

Integration of specific analytical thickness function: For the NACA 4-digit airfoil family, we know the thickness distribution function:

$$\frac{z_t(x/c)}{c} = 5t\left[0.2969\sqrt{\frac{x}{c}} - 0.1260\frac{x}{c} - 0.3516\left(\frac{x}{c}\right)^2 + 0.2843\left(\frac{x}{c}\right)^3 - 0.1036\left(\frac{x}{c}\right)^4\right].$$

Its requisite derivative is

$$\frac{dz_t}{dx} = 5t\left[\frac{0.2969}{2\sqrt{x/c}} - 0.1260 - 2\times 0.3516\frac{x}{c} + 3\times 0.2843\left(\frac{x}{c}\right)^2 - 4\times 0.1036\left(\frac{x}{c}\right)^3\right].$$

This can be used in the integral for the pressure coefficient:

$$c_p(x) = -\frac{2}{\pi}\int_0^c \frac{dz_t}{d\zeta}(\zeta)\frac{d\zeta}{x-\zeta}.$$

Denoting $X := x/c$ and $\xi := \zeta/c$, the expression is

$$c_p(X) = -\frac{10t}{\pi}\int_0^1 \left(\frac{0.14845}{\sqrt{\xi}} - 0.1260 - 0.7032\xi + 0.8529\xi^2 - 0.4144\xi^3\right)\frac{d\xi}{X-\xi}.$$

We evaluate this integral term by term, noting that all the terms are improper integrals, and thus must be evaluated as Cauchy's principal values.

The crux of the first term is

$$\begin{aligned}
I_0(X) &:= \int_0^1 \frac{d\xi}{(X-\xi)\sqrt{\xi}} = \int_0^1 \frac{2d\eta}{X-\eta^2} \qquad [\text{with } \sqrt{\xi} =: \eta, \implies d\eta = 0.5d\xi/\sqrt{\xi}] \\
&= \frac{1}{\sqrt{X}}\int_0^1 \left(\frac{1}{\eta+\sqrt{X}} - \frac{1}{\eta-\sqrt{X}}\right)d\eta \\
&= \frac{1}{\sqrt{X}}\int_0^1 \frac{1}{\eta+\sqrt{X}}d\eta - \frac{1}{\sqrt{X}}\lim_{\epsilon\to 0}\left[\int_0^{\sqrt{X}-\epsilon}\frac{d\eta}{\eta-\sqrt{X}} + \int_{\sqrt{X}+\epsilon}^1 \frac{d\eta}{\eta-\sqrt{X}}\right] \\
&= \frac{1}{\sqrt{X}}\ln\left|\frac{1+\sqrt{X}}{\sqrt{X}}\right| - \frac{1}{\sqrt{X}}\lim_{\epsilon\to 0}\left[\ln\left|\frac{\sqrt{X}-\epsilon-\sqrt{X}}{-\sqrt{X}}\right| + \ln\left|\frac{1-\sqrt{X}}{\sqrt{X}+\epsilon-\sqrt{X}}\right|\right] \\
&= \frac{1}{\sqrt{X}}\ln\left|\frac{1+\sqrt{X}}{\sqrt{X}}\right| - \frac{1}{\sqrt{X}}\lim_{\epsilon\to 0}\left[\ln\left|\frac{1-\sqrt{X}}{\sqrt{X}}\right| + \ln\left|\frac{-\epsilon}{\epsilon}\right|\right] \\
&= \frac{1}{\sqrt{X}}\ln\left|\frac{1+\sqrt{X}}{1-\sqrt{X}}\right|.
\end{aligned}$$

Note that this term grows unbounded at $x \to c$ but vanishes at $x = 0$.

Taking up the second term,

$$I_1(X) := \int_0^1 \frac{d\xi}{X-\xi} = \lim_{\epsilon\to 0}\left[\int_0^{X-\epsilon}\frac{d\xi}{X-\xi} + \int_{X+\epsilon}^1 \frac{d\xi}{X-\xi}\right]$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\left[|-\ln|\xi - X||_0^{X-\epsilon} + |-\ln|\xi - X||_{X+\epsilon}^1 \right] = \ln\left|\frac{-X}{1-X}\right| + \lim_{\epsilon\to 0}\ln\left|\frac{\epsilon}{-\epsilon}\right| \\
&= \ln\left|\frac{X}{1-X}\right|.
\end{aligned}
$$

This term grows unbounded at both the leading and trailing edges (i.e., $X = 0$ and 1).
Taking up the third term,

$$
I_2(X) := \int_0^1 \frac{\xi}{X-\xi}d\xi = \int_0^1 \frac{X-(X-\xi)}{X-\xi}d\xi = X\int_0^1 \frac{d\xi}{X-\xi} - \int_0^1 d\xi = XI_1 - 1.
$$

As for the fourth term,

$$
\begin{aligned}
I_3(X) &:= \int_0^1 \frac{\xi^2}{X-\xi}d\xi = \int_0^1 \frac{(X-\xi)^2 - X^2 + 2X\xi}{X-\xi}d\xi \\
&= \int_0^1 (X-\xi)d\xi - X^2 I_1 + 2XI_2 = X - \frac{1}{2} - X^2 I_1 + 2X(XI_1 - 1) = X^2 I_1 - X - \frac{1}{2}.
\end{aligned}
$$

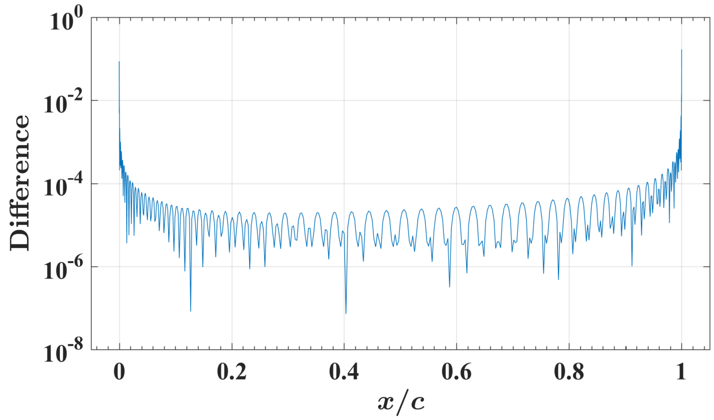
Finally, the fifth term simplifies to

$$
\begin{aligned}
I_4(X) &:= \int_0^1 \frac{\xi^3}{X-\xi}d\xi = \int_0^1 \frac{X^3 - 3X^2\xi + 3X\xi^2 - (X-\xi)^3}{X-\xi}d\xi \\
&= X^3 I_1 - 3X^2 I_2 + 3XI_3 - \int_0^1 (X-\xi)^2 d\xi \\
&= X^3 I_1 - 3X^2(XI_1 - 1) + 3X\left(X^2 I_1 - X - \frac{1}{2}\right) - \left[X^2 - X + \frac{1}{3}\right] \\
&= X^3 I_1 - X^2 - \frac{X}{2} - \frac{1}{3}.
\end{aligned}
$$

Putting all these together, we have the pressure coefficient on either surface of the NACA 4-digit airfoil as

$$
\begin{aligned}
c_p(X) &= \frac{t}{\pi}\left(-1.44845 I_0 + 1.260 I_1 + 7.032 I_2 - 8.529 I_3 + 4.144 I_4\right) \\
&= \frac{t}{\pi}\big[-1.44845 I_0 + 1.260 I_1 + 7.032(XI_1 - 1) - 8.529(X^2 I_1 - X - 0.5) \\
&\qquad\qquad +4.144(X^3 I_1 - X^2 - 0.5X - 0.333)\big] \\
&= t\Bigg[(0.4011 + 2.2384X - 2.7149X^2 + 1.3191X^3)\ln\left|\frac{X}{1-X}\right| \\
&\qquad\qquad -\frac{0.4725}{\sqrt{X}}\ln\left|\frac{1+\sqrt{X}}{1-\sqrt{X}}\right| - (1.3206 - 2.0553X + 1.3191X^2)\Bigg]
\end{aligned}
$$

This is not plotted, since the result is visually indistinguishable from the sine-series solution shown before. Instead, below we show the difference in the two results, which is negligible.



Differences in pressure coefficient predicted by the sine-series method and analytical solution approach.