Aerospace Engineering Department, IIT Bombay
AE 308 & AE 775 - Control Theory
Tutorial 3 Solution

Q1

Comment whether the response of the given systems subjected to a unit step input is overdamped/underdamped/undamped/critically damped.

1. $\dfrac{25}{s^2 + 12s + 25}$
2. $\dfrac{100}{s^2 + 7s + 100}$
3. $\dfrac{49}{s^2 + 14s + 49}$
4. $\dfrac{121}{s^2 + 121}$
5. $\dfrac{64}{s^2 + 8s + 64}$

Solution:

Comparing with $G(s) = \dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$ and calculating ζ.

1. $w_n = 5$, $\zeta = 1.2$ overdamped.
2. $w_n = 10$, $\zeta = 0.35$ underdamped.
3. $w_n = 7$, $\zeta = 1$ critically damped.
4. $w_n = 11$, $\zeta = 0$ undamped.
5. $w_n = 8$, $\zeta = 0.5$ underdamped.

Q2

Find the steady-state error due to a ramp input for a type two system.

Solution:

The steady-state error (e_{ss}) for a ramp input is given by

$$e_{ss} = \frac{1}{K_v} \qquad \text{where} \quad K_v = \lim_{s \to 0} sG(s)$$

A standard type two system can be described by

$$G(s) = \frac{(s+a)(s+b)\cdots}{s^2(s+p)(s+q)\cdots}$$

In this case

$$\begin{aligned} K_v &= \lim_{s \to 0} \; s\frac{(s+a)(s+b)\cdots}{s^2(s+p)(s+q)\cdots} \\ &= \lim_{s \to 0} \; \frac{(s+a)(s+b)\cdots}{s(s+p)(s+q)\cdots} \\ &= \infty \end{aligned}$$

Then

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

Q3

The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{K}{s(s+1)}$$

If the value of gain K is such that the system is critically damped, find the location of the closed-loop poles of the system.

Solution:

The forward path and feedback path transfer functions here are as follows

$$G(s) = \frac{K}{s(s+1)} \quad ; \quad H(s) = 1$$

The closed-loop transfer function can then be given by

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{K}{s(s+1) + K} = \frac{K}{s^2 + s + K}$$

Since the system is critically damped, $\zeta = 1$. The characteristic equation for a standard second order system is given by

$$s^2 + 2\zeta\omega_n s + {\omega_n}^2 = 0$$

Comparing with $s^2 + s + K$, we get

$$2\zeta\omega_n = 1 \Rightarrow \omega_n = \frac{1}{2}$$

But ω_n is also equal to $\sqrt{K}$. Thus, $K = \frac{1}{4}$.
The location of the closed-loop poles of the system can then be given by the roots of the following equation

$$s^2 + s + \frac{1}{4} = 0$$

which are $-0.5, -0.5$.

Q4

The open-loop transfer function of a unity feedback system is given by

$$G(s) = \frac{10}{s+1}$$

Find the steady-state error due to a unit step input signal.

Solution:

The steady-state error due to an input signal $r(t)$ is given by

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + G(s)H(s)} R(s)$$

where $R(s)$ is the Laplace transform of $r(t)$, $H(s) = 1$ since this is a unity feedback system and $G(s)$ is the open-loop transfer function (or forward path transfer function) of the system.

In this case, $r(t) = u(t)$. Then $R(s) = \frac{1}{s}$. Then

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} \; s \frac{1}{1 + \frac{10}{s+1}.1} . \frac{1}{s} \\ &= \lim_{s \to 0} \; \frac{s+1}{s+11} \\ &= \frac{1}{11} \end{aligned}$$

Q5

Consider a system with the following forward path and feedback path transfer functions

$$G(s) = \frac{20}{s^2} \qquad ; \qquad H(s) = (s+5)$$

respectively. For a unit step input, find the steady-state output of the system.

Solution:

The steady-state error due to an input signal $r(t)$ is given by

$$e_{ss} = \lim_{s \to 0} \; sE(s) = \lim_{s \to 0} \; s \frac{1}{1 + G(s)H(s)} R(s)$$

Here, $R(s) = \frac{1}{s}$. The closed-loop transfer function is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

where $C(s)$ is the output response. Then, the steady-state output response of the system can be given by

$$\begin{aligned}
C_{ss} &= \lim_{s\to 0}\ sC(s) \\
&= \lim_{s\to 0}\ sT(s)R(s) \\
&= \lim_{s\to 0}\ s\frac{G(s)}{1+G(s)H(s)}R(s) \\
&= \lim_{s\to 0}\ s\frac{20/s^2}{1+(20/s^2)(s+5)}\frac{1}{s} \\
&= \lim_{s\to 0}\ \frac{20}{s^2+20s+100} \\
&= \frac{1}{5}
\end{aligned}$$

Q6

A system has a damping ratio of 0.5, a natural frequency of 100 rad/s, and a dc gain of 1. Find the response of the system to a unit step input.

Solution:

Given $\zeta = 0.5$, $w_n = 100$, $K = 1$. The step response of a second order system is

$$c(t) = 1 - \frac{e^{-\zeta w_n t}}{\sqrt{1-\zeta^2}}\cos\left(w_d t - \phi\right) \quad ; \quad \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Putting the given values we get

$$c(t) = 1 - \frac{2e^{-50t}}{\sqrt{3}}\cos\left(86.6t - 30^\circ\right)$$

Q7

Given the system shown in Figure 1, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input of torque T(t).

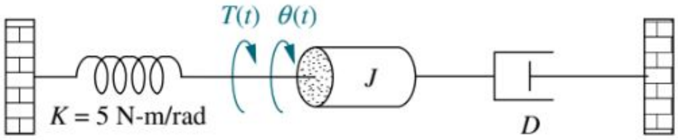


Figure 1: Rotational Mechanical System

Solution:

The given system is represented by

$$J\ddot{\theta} + D\dot{\theta} + K\theta = T.$$

The transfer function representation is

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Ds + K}$$

Converting in standard form $\dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2}$

$$\frac{\Theta(s)}{T(s)} = \frac{\frac{1}{J}}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

Comparing, $w_n = \sqrt{\dfrac{K}{J}}$, $2\zeta w_n = \dfrac{D}{J}$. Given, $T_s = 2 = \dfrac{4}{\zeta w_n}$, OS 20% $\implies e^{-\pi cot\phi} = 0.2$, where $\zeta = cos\phi \implies \zeta = 0.456$. Given $K = 5$, we get $D = 1.04$ and $J = 0.26$.

Q8

Reduce the system shown in Figure 2 to a single transfer function, $T(s) = C(s)/R(s)$.

Solution:

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s) + G_3(s)}{1 + G_2(s)G_4(s) + H(s)G_1(s)G_2(s) + H(s)G_3(s)}$$

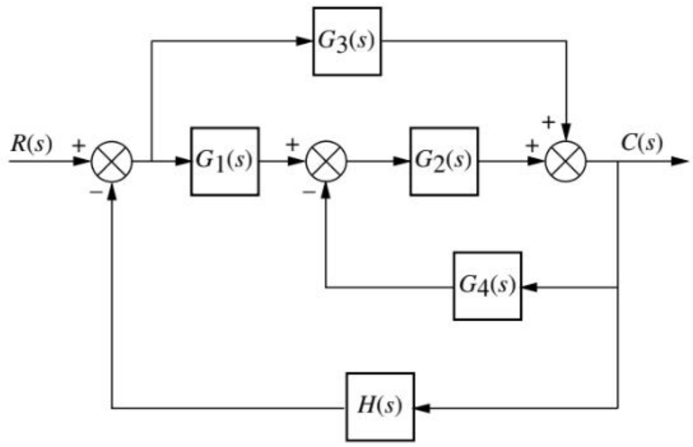


Figure 2: Block diagram

Q9

For the system shown in Figure 3, find the poles and zeros of the closed-loop transfer function, $T(s) = C(s)/R(s)$.

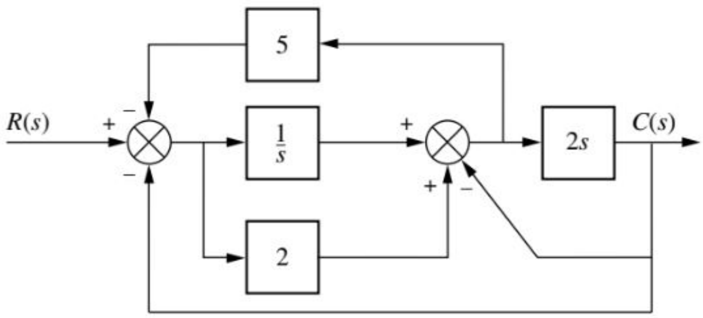


Figure 3: Block diagram

Solution:

$$\frac{C(s)}{R(s)} = \frac{2s}{3s+5}$$

Q10

For each of the following transfer functions, write the general form of the step response:

$$(1)\quad G(s) = \frac{400}{s^2 + 16s + 400} \qquad\qquad (2)\quad G(s) = \frac{196}{s^2 + 14s + 196}$$

Solution:

1. $w_n = 20$, $\zeta = 0.4$

   $$c(t) = a + be^{-8t}cos(18.33t - 66.42°)$$

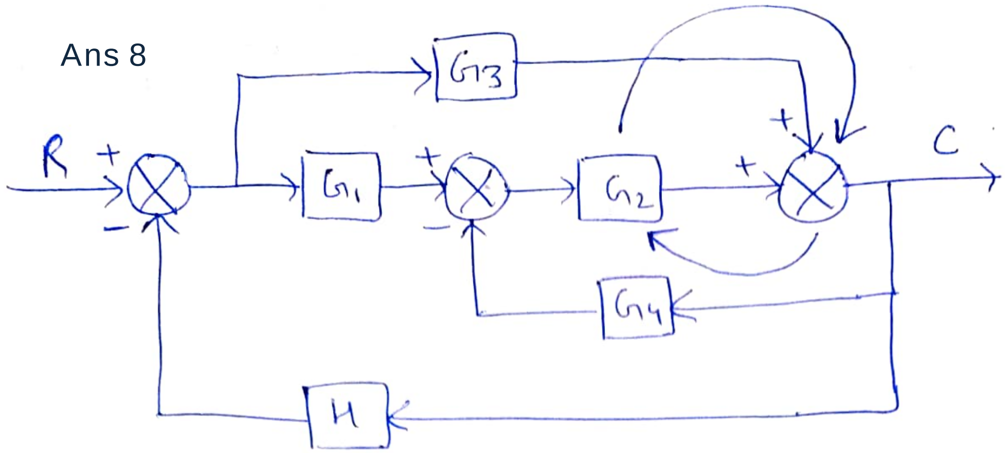
   where a, b are real numbers.

2. $w_n = 14$, $\zeta = 0.5$

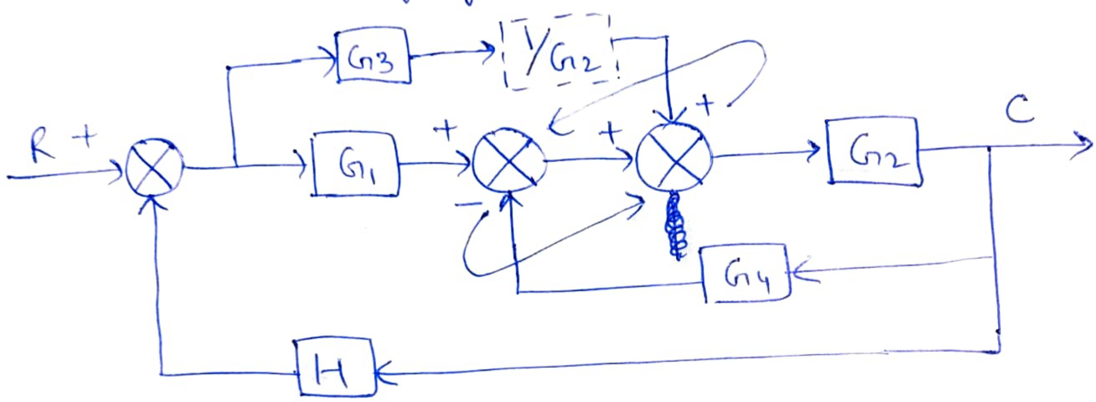
   $$c(t) = a + be^{-7t}cos(12.12t - 60°)$$

   where a, b are real numbers.

Ans 8

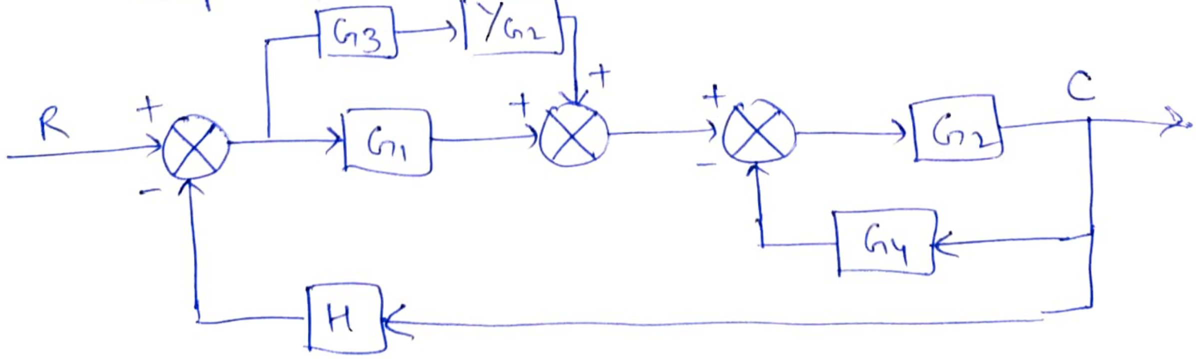


Step 1: Exchanging G_2 block and junction.

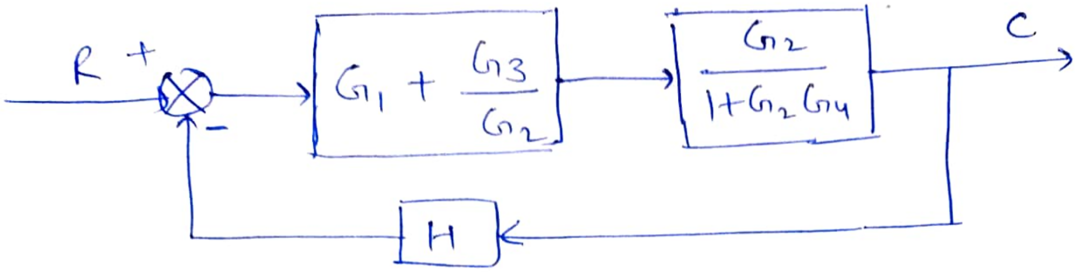


$\frac{1}{G_2}$ is introduced in path containing G_3 because of the exchange.

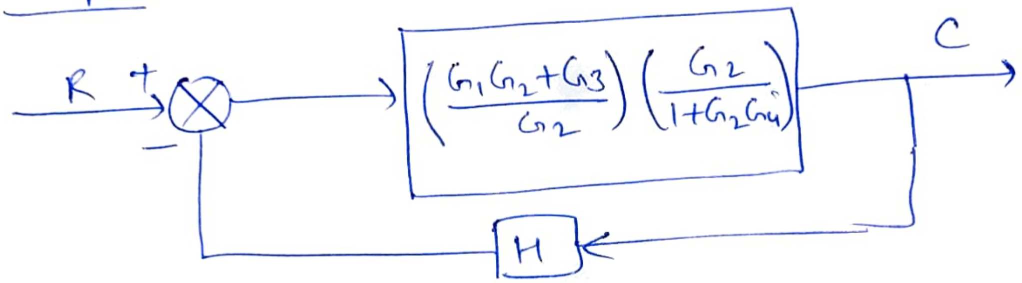
Step 2: Interchange the two junctions, we are doing this to create parallel paths and a feedback loop.



Step 3 :- Solving parallel path and feedback loop



Step 4 :

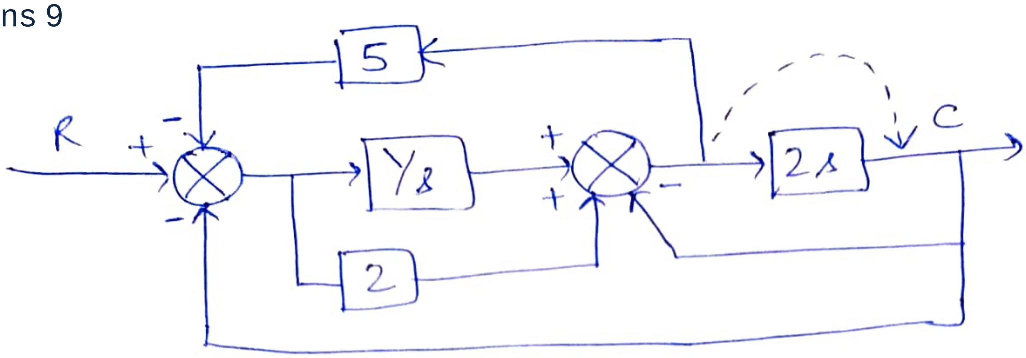


Step 5 :

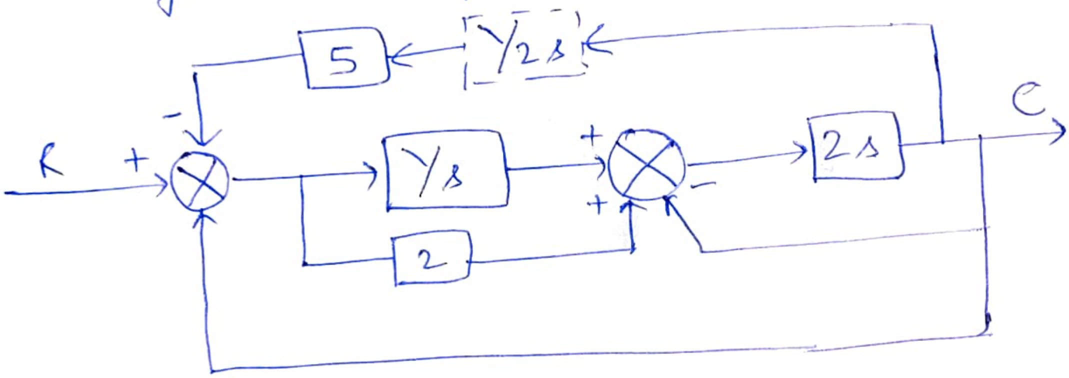
$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2+G_3}{1+G_2G_4}}{1+H\left(\dfrac{G_1G_2+G_3}{1+G_2G_4}\right)}$$

$$= \frac{G_1G_2+G_3}{1+G_2G_4+HG_1G_2+HG_3}$$

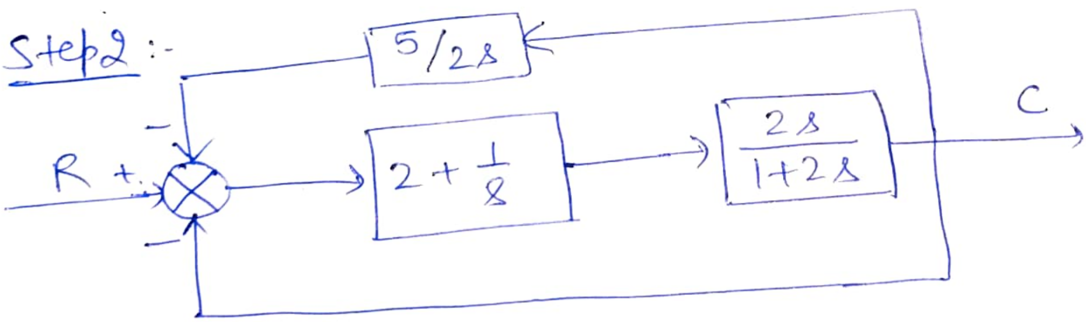
Ans 9



Step 1:- moving take-off point as shown above, this is done to get parallel paths and a feedback loop.



Step 2:-



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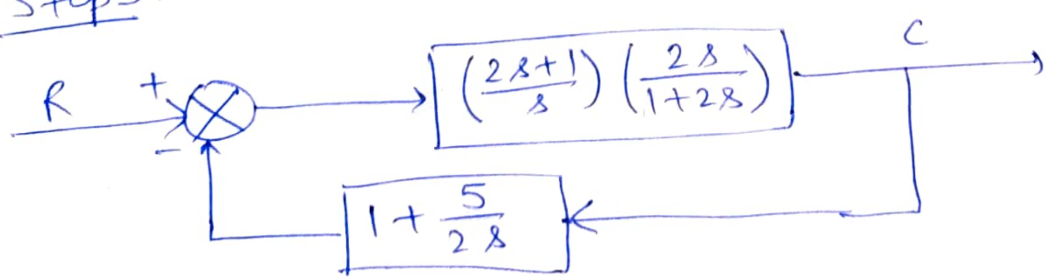

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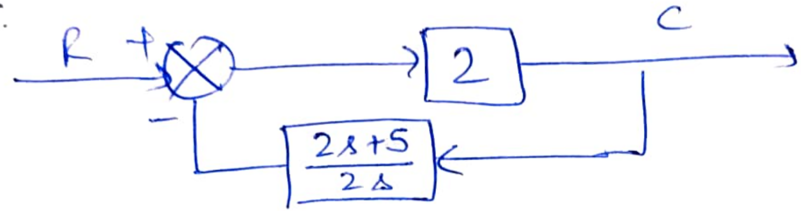

Step 5 :

$$\frac{C(s)}{R(s)} = \frac{2}{1+2\left(\frac{2s+5}{2s}\right)} = \frac{2}{1+\frac{2s+5}{s}}$$

$$= \frac{2s}{3s+5}.$$