This leads us to the joint density function of $(Y_1, Y_2, \ldots, Y_m)$.

Consequence

There is a general version of theorem.

The transformation / mapping $g = (g_1, g_2, \ldots, g_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ does not need to be one-to-one on the whole domain $\mathbb{R}^n$.

In that case, we partition (divide it into disjoint subsets) $\mathbb{R}^n$ such that in each partition g is one-to-one. The partition must be finite.

Example $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that ~~a~~ $g(x) = x^2$.

Then g is not one-to-one. But if we divide $\mathbb{R}$ into two disjoint halves $(-\infty, 0]$, and $(0, \infty)$.

Then $g : (-\infty, 0] \rightarrow \mathbb{R}$ and $g : (0, \infty) \longrightarrow \mathbb{R}$ is one-to-one and ~~into~~ ~~inver~~ so invertible.

Let $A_1, A_2, \ldots, A_k$ be a partion of $\mathbb{R}^n$ that is, $A_i \cap A_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{k} A_i = \mathbb{R}^n$. Further assume that the restriction of g on A_i is one-to-one and so invertible for every $i = 1, 2, \ldots, n$. Let ~~g denote the~~ $g^{(i)}$ denote the restriction of g on A_i.

Recall $\underline{g} = (g_1, g_2, \ldots g_n)$.

$$\underline{g}^{(i)} = (g_1^{(i)}, g_2^{(i)}, \ldots, g_n^{(i)}) \text{ for } 1 \le i \le n.$$

Define $\underline{h}^{(i)}$ to be the inverse of $\underline{g}^{(i)}$. Then for each $i = 1, 2, \ldots n$, we can compute

$$J^{(i)} = \begin{vmatrix} \frac{\partial h_1^{(i)}}{\partial y_1} & \frac{\partial h_1^{(i)}}{\partial y_2} & \cdots & \frac{\partial h_1^{(i)}}{\partial y_n} \\ \vdots & & & \\ \frac{\partial h_n^{(i)}}{\partial y_1} & \frac{\partial h_n^{(i)}}{\partial y_2} & \cdots & \frac{\partial h_n^{(i)}}{\partial y_n} \end{vmatrix}$$

It is important to have $\det(J^{(i)}) \neq 0$ for all $i = 1, 2 \ldots n$.

Suppose that the random vector $\underline{X} = (X_1, X_2, \ldots X_n)$ has joint density $f(x_1, \ldots x_n)$. Then the joint density of $(Y_1, Y_2, \ldots, Y_n) = \underline{g}(X_1, \ldots X_n)$
$= (g_1(X_1, X_2, \ldots X_n), g_2(X_1, X_2, \ldots X_n), \ldots g_n(X_1, X_2, \ldots X_n))$
is given by

$$w(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{k} |J_i| \, f(h_1^{(i)}(y_1, y_2, \ldots y_n), h_2^{(i)}(y_1, y_2, \ldots y_n), \ldots, h_n^{(i)}(y_1, y_2, \ldots y_n)).$$
$$\mathbb{1}[(y_1, y_2, \ldots, y_n) \in \underline{g}(A_i)].$$