AE 330 Rocket Propulsion
Multi-stage Rockets

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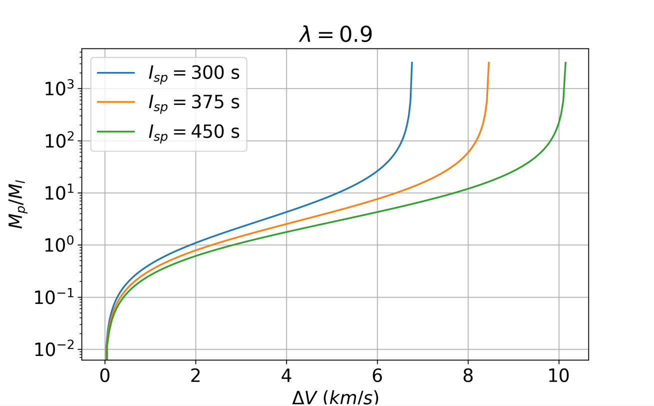
Single-Stage Rocket

Rocket Equation: ΔV

$$\Delta V = u_{eq} \ln\left(\frac{M_o}{M_f}\right) \equiv u_{eq} \ln\left(\mathsf{MR}\right) \quad \text{or} \quad \mathsf{MR} = \exp\left(\frac{\Delta V}{u_{eq}}\right)$$



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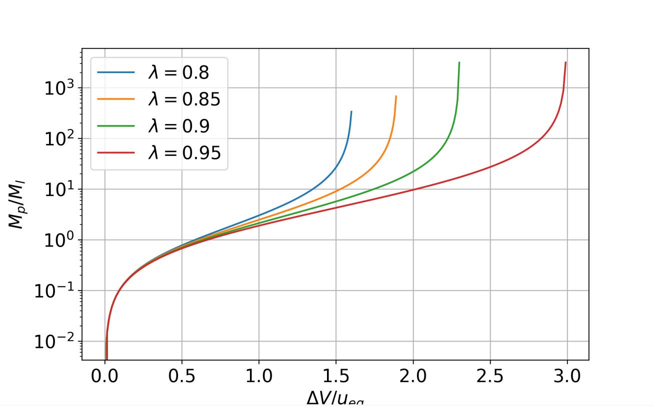

Continuous Ejection

Eject inert mass along with propellant

$$\lambda = \frac{M_p}{M_p + M_i} \implies \dot{m} = \dot{m}_p + \dot{m}_i = \frac{\dot{m}_p}{\lambda}$$

$$M\frac{dv}{dt} = \dot{m}_p u_{eq} = \left(\dot{m}_p + \dot{m}_i\right)\lambda u_{eq} = -\frac{dM}{dt}\lambda u_{eq}$$

$$\Delta V_\infty = \lambda u_{eq} \ln\left(\frac{M_o}{M_l}\right)$$

For the masses considered for the earlier Vikas engine

$M_i = 1$ ton, $M_p = 22$ tons, $M_i = 2.45$ tons

$$\Delta V_\infty = \lambda u_{eq} \ln\left(\frac{M_o}{M_l}\right) = 8,572\ m/s$$

Launch mass to Payload mass ratio

$$MR_\infty \equiv \frac{M_o}{M_l} = \exp\left(\frac{\Delta V}{\lambda u_{eq}}\right)$$

$$1 + \frac{M_p + M_i}{M_l} \equiv 1 + \frac{M_p}{\lambda M_l} = MR_\infty$$

$$\frac{M_p}{M_l} = \lambda\left(\exp\left(\frac{\Delta V}{\lambda u_{eq}}\right) - 1\right)$$

$$\frac{M_i}{M_l} = (1-\lambda)\left(\exp\left(\frac{\Delta V}{\lambda u_{eq}}\right) - 1\right)$$

Rocket Equation: ΔV

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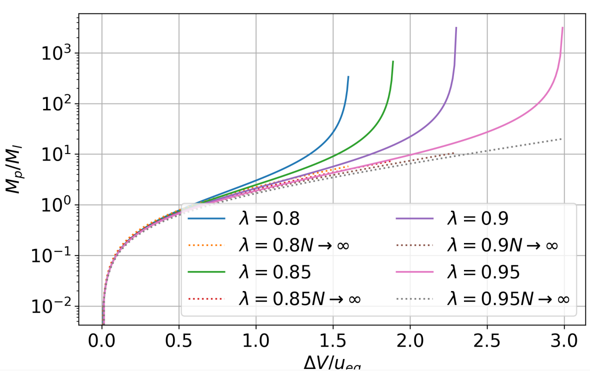
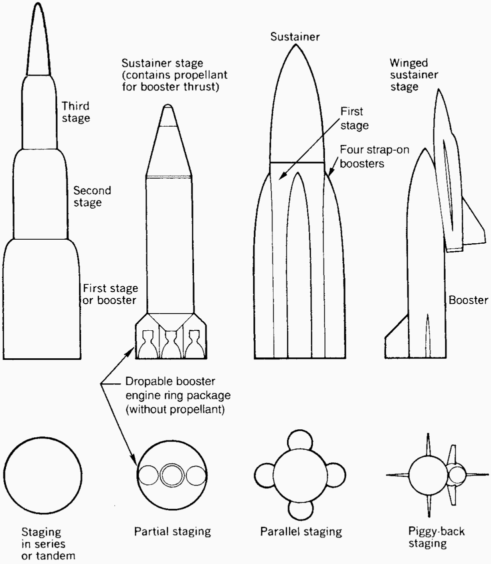
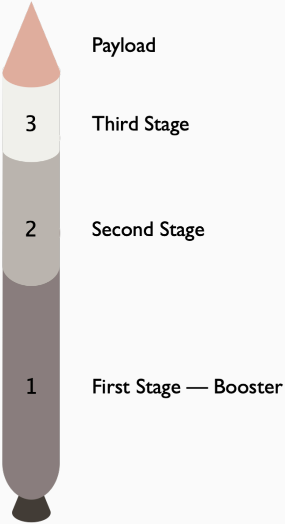


Multistage Rocket

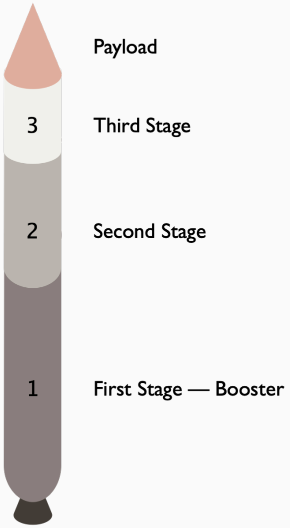
Launch Vehicle Staging

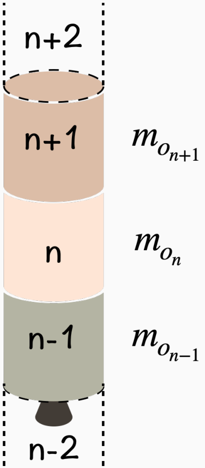


Multistage Rocket



Multistage Rocket





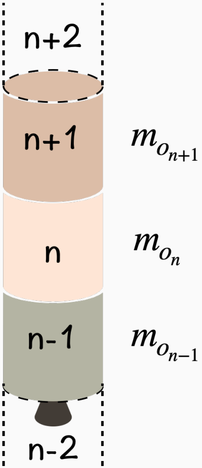
Multistage Rocket

	Mass
Propellant	m_{p_n}
Inert	m_{i_n}
Stage	m_{o_n}

Stage Mass: $m_{o_n} = m_{p_n} + m_{i_n}$

Structural Factor: $\epsilon_n = \dfrac{m_{f_n}}{m_{o_n}} \equiv \dfrac{m_{i_n}}{m_{i_n} + m_{p_n}}$

Propellant Fraction: $\lambda_n = \dfrac{m_{p_n}}{m_{i_n} + m_{p_n}} \equiv 1 - \epsilon_n$



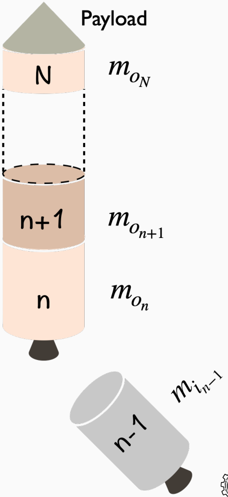
For $n^{th}-$stage

Total Mass: $M_{o_n} = \sum_{j=n}^{N} m_{o_j}$

Payload Mass: $M_{l_n} \equiv M_{o_{n+1}} = \sum_{j=n+1}^{N} m_{o_j}$

Stage Payload Factor: $\beta_n = \frac{M_{l_n}}{M_{o_n}}$

Final Mass: $M_{f_n} = m_{i_n} + M_{l_n}$



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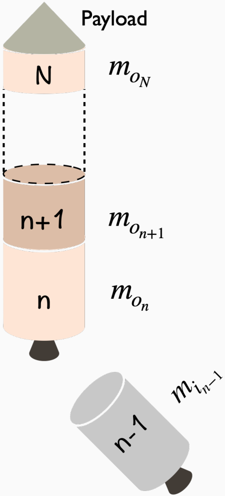
$$\Delta V_n = u_{eq_n} \ln\left(\frac{M_{o_n}}{M_{f_n}}\right)$$
$$= -u_{eq_n} \ln\left(\epsilon_n + (1-\epsilon_n)\beta_n\right)$$



Total ΔV

$$\Delta V_n = u_{eq_n} \ln\left(\frac{M_{o_n}}{M_{f_n}}\right) = -u_{eq_n} \ln\left(\epsilon_n + (1-\epsilon_n)\,\beta_n\right)$$

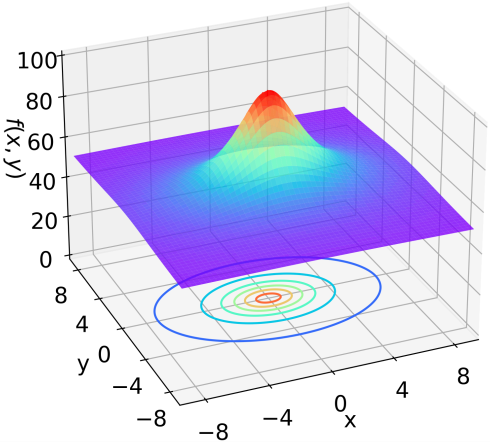
$$\begin{aligned}\Delta V &= \sum_{n=1}^{N} \Delta V_n \\ &= -\sum_{n=1}^{N} u_{eq_n} \ln\left(\epsilon_n + (1-\epsilon_n)\,\beta_n\right)\end{aligned}$$

For a given M_l and $M_p = \sum_{n=1}^{N} m_{p_n}$ (and hence, $M_o \equiv M_{o_1}$):

For a given M_l/M_o, when is ΔV maximum?

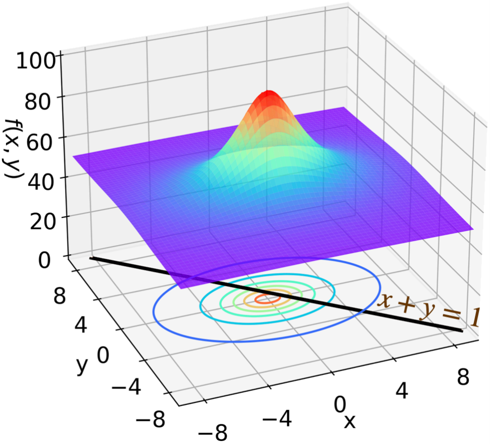
Optimisation with Constraints

Function in 2D-space



$$f(x, y) = 50 + \frac{10}{0.2 + \left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2}$$ (max. at origin)

Along a straight line: $x + y = 1$



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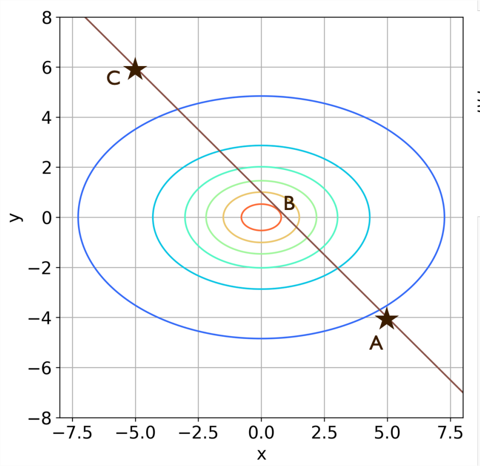
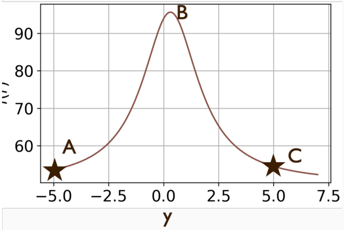
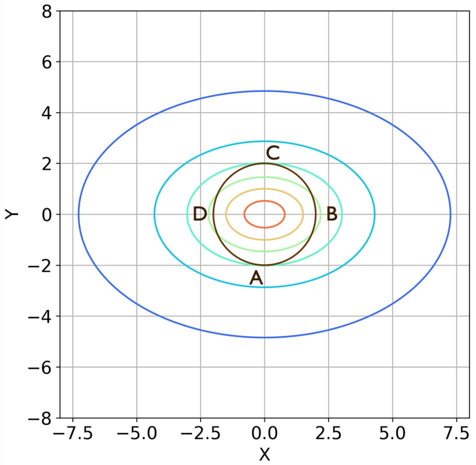
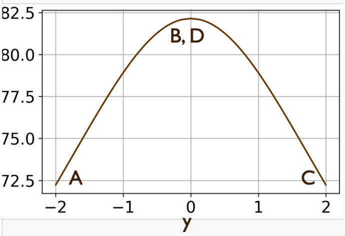




Along a circle: $x^2 + y^2 = 4$





Maximum/Minimum along a curve

- We considered two curves (straight line and circle)
- We can find the peak location by plotting the function along the curve
- If s is the coordinate along the curve or interest, we can verify that $\partial f/\partial s = 0$ at the maximum/minimum location
- Can be tedious for more number of variables $\rightarrow$ higher dimensions

Maximum/Minimum along a curve

These curves restrict the domain of interest

So, we call them constraint functions or constraints

Our constraint functions were:

$$g_l(x, y) = x + y - 1$$
$$g_c(x, y) = x^2 + y^2 - 4$$

The constraints were $g_l(x, y) = 0$ and $g_c(x, y) = 0$

The function that is to be minimised is called the objective function or merit function. We denote it by $f(x, y)$

At a constrained maxima/minima

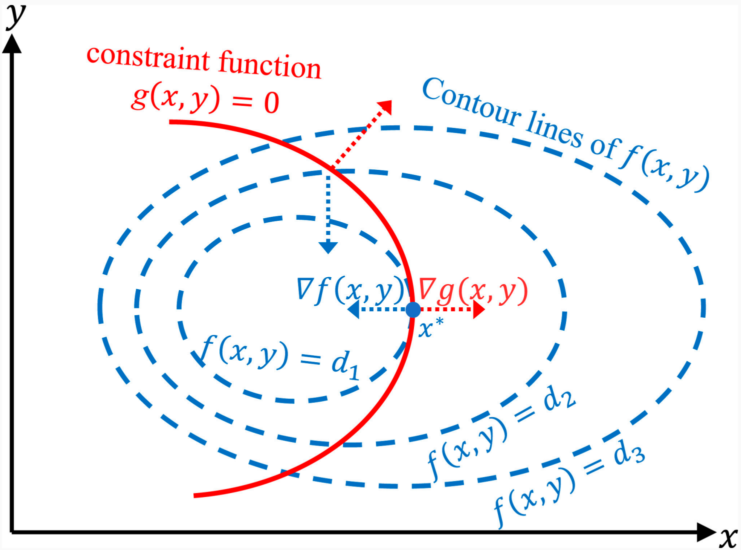


image from Vadlamani et al., PNAS 2020

At a constrained maxima/minima

$$\nabla f(x,y) \propto \nabla g(x,y) \implies \nabla f(x,y) = \alpha \nabla g(x,y)$$

$$\nabla \{f(x,y) - \alpha g(x,y)\} = 0$$

So, we can minimise $f - \alpha g$ instead of $f(x,y)$

α is called Lagrange Multiplier

Optimum mass distribution

Staging

Our objective function was

$$f\left(u_{eq_n}, \epsilon_n, \beta_n\right) \equiv \Delta V = \sum_{n=1}^{N} \Delta V_n$$

$$= -\sum_{n=1}^{N} u_{eq_n} \ln\left(\epsilon_n + \left(1-\epsilon_n\right)\beta_n\right)$$

Our constraint function is the vehicle payload fraction:

$$\frac{M_l}{M_o} = \frac{M_{l_1}}{M_o}\frac{M_{l_2}}{M_{o_3}}\cdots\frac{M_l}{M_{o_N}} \equiv \prod_{n=1}^{N} \beta_n$$

$$g\left(\beta_n\right) = \ln\frac{M_l}{M_o} = \sum_{n=1}^{N} \ln\beta_n$$

Function to be minimised

$$f - \alpha g = -\sum_{n=1}^{N} u_{eq_n} \ln\left(\epsilon_n + \left(1 - \epsilon_n\right)\beta_n\right) - \alpha \sum_{n=1}^{N} \ln \beta_n$$

f and g depend on β_n

$$\frac{\partial}{\partial \beta_n}\left(f - \alpha g\right) = 0$$

$$-u_{eq_n} \frac{1 - \epsilon_n}{\epsilon_n + \left(1 - \epsilon_n\right)\beta_n} = \frac{\alpha}{\beta_n} \implies \beta_n = -\frac{\alpha}{\left(\alpha + u_{eq_n}\right)} \frac{\epsilon_n}{\left(1 - \epsilon_n\right)}$$

Optimum mass distribution

Our constraint is now

$$\frac{M_l}{M_o} = \prod_{n=1}^{N} \beta_n = \prod_{n=1}^{N} \frac{-\alpha}{(\alpha + u_{eq_n})} \frac{\epsilon_n}{(1-\epsilon_n)}$$

Once we select the rocket engines for different stages, we know (u_{eq_n}, ϵ_n)

The only unknown now is α, which can be found from the above constraint.

$$\begin{aligned} \Delta V &= -\sum_{n=1}^{N} u_{eq_n} \ln\left(\epsilon_n + (1-\epsilon_n)\beta_n\right) \\ &= \sum_{n=1}^{N} u_{eq_n} \ln\left(\frac{u_{eq_n} + \alpha}{u_{eq_n}\epsilon_n}\right) \end{aligned}$$

Simplification: Use the same engine in all stages

Same engine $\implies u_{eq_n} = u_{eq}$ and $\epsilon_n = \epsilon$

$$\beta_n = -\frac{\alpha}{(\alpha + u_{eq_n})}\frac{\epsilon_n}{(1-\epsilon_n)}$$
$$\implies \beta_n \equiv \beta = -\frac{\alpha}{(\alpha + u_{eq})}\frac{\epsilon}{(1-\epsilon)}$$

Our constraint is

$$\frac{M_l}{M_o} = \prod_{n=1}^{N} \beta_n = \beta^N \implies \beta = \left(\frac{M_l}{M_o}\right)^{1/N}$$

ΔV and Payload Fraction

$$\beta_{opt} = \left(\frac{M_l}{M_o}\right)^{1/N}$$

$$\Delta V = -\sum_{n=1}^{N} u_{eq_n} \ln\left(\epsilon_n + (1-\epsilon_n)\beta_n\right)$$

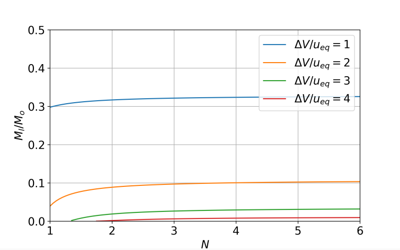
$$\implies \Delta V_{opt} = -N u_{eq} \ln\left(\epsilon + (1-\epsilon)\left(\frac{M_l}{M_o}\right)^{1/N}\right)$$

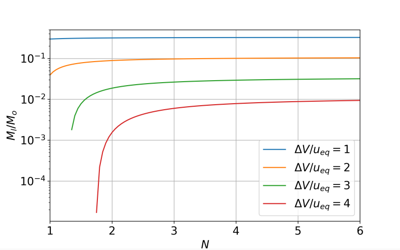
We can express the payload fraction as

$$\frac{M_l}{M_o} = \left(\frac{w^{-1/N} - \epsilon}{1-\epsilon}\right)^{N}$$

where $w = \exp\left(\Delta V / u_{eq}\right)$

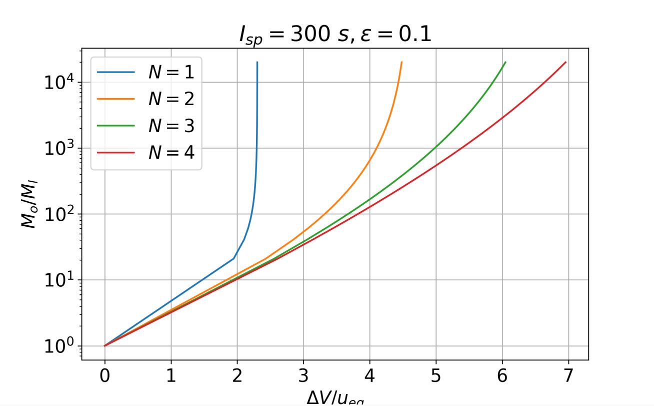
Payload Fraction vs ΔV





No particular advantage beyond 4 stages for most missions (near Earth)

Payload Fraction vs ΔV



Optimum no:of stages depends on ΔV