Thin Airfoil Theory

Aerodynamics

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Thin airfoil theory – Introduction

We have made arguments in favour of several simplifications of the problem of external attached flow over slender bodies at high Reynolds numbers:

- The flow is steady
- The viscous region (boundary layer and wake) is very thin
- The entire outer flow is irrotational (i.e., potential)
- For the prediction of lift and pitching moment, we only need to consider the outer flow

Now, we restrict our attention to

- Two-dimensional flow; i.e., flow over an airfoil section representing a wing of uniform section of infinite span
- Low-speed, essentially incompressible, flow

Thin airfoil theory – Restriction

Hoping to gain deep insight into the problem by sacrificing some generality of application, we develop an analytical (instead of computational) theory by further restricting ourselves to

- Thin airfoils
- With small camber
- At low angles of attack

Then, the airfoil disturbs or perturbs the freestream MINIMALLY

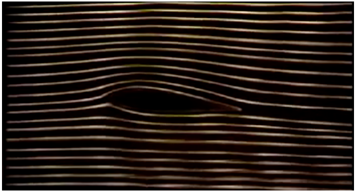
Applying potential flow theory

Recall observations made in 2D potential flow theory to be used here:

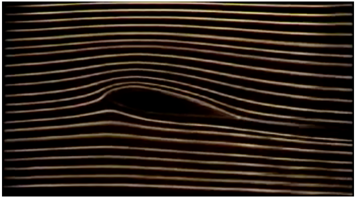
- Freestream approaching an airfoil is readily simulated by 'uniform flow'
- There is no lift without circulation around the airfoil
  - Think of Kutta-Joukowsky theorem
- Only way to simulate circulation in potential flow theory is to have 'irrotational line vortices' (distributed within boundary of solid body)
  - Think of rotating circular cylinder flow simulation
- Displacement effect on streamlines around an immersed body (having a closed curve for a boundary) can be simulated by arraying 'line sources/sinks' of zero net strength (within boundary of body)
  - Think of Rankine oval simulation
  - Think of circular cylinder flow simulation

Kutta condition – Setup

Before we discuss thin airfoil theory, we establish a shortcoming of all potential flow theories for airfoils, and understand how it is overcome



At a small angle of attack

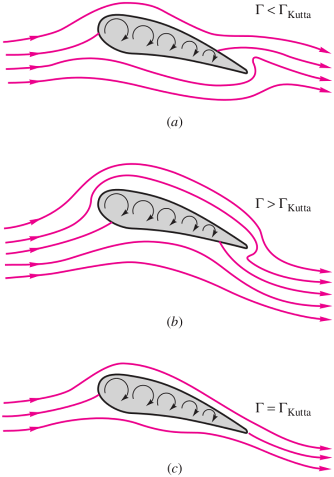


At a higher angle of attack

Flow leaving the trailing edge smoothly in two stills from Prof. Babinsky's video.

https://www.youtube.com/watch?v=6UlsArvbTeo

Kutta condition



By itself, potential flow theory cannot determine the unique amount of circulation required to simulate a flow

All the three solutions here are valid per potential flow theory

But only the last one is physical, as dictated by viscous effects not accounted in potential flow

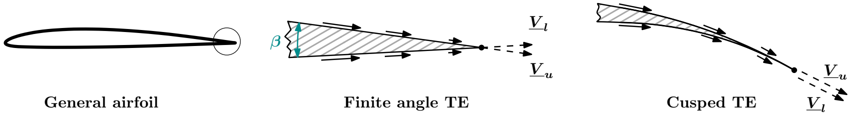
(Viscous) flow will separate if it is to turn around the sharp trailing edge, as required in the first two solutions

Flow must smoothly leave the sharp trailing edge of an airfoil – Kutta

Incorporates essential viscous effect in potential theory

White, F. M., *Fluid Mechanics*, McGraw-Hill, 2009

Implication of Kutta condition



Setup for deriving mathematical statement of Kutta condition.

For finite-angle (wedge) TE, two streamlines apparently intersect

- This is impossible unless TE is a stagnation point

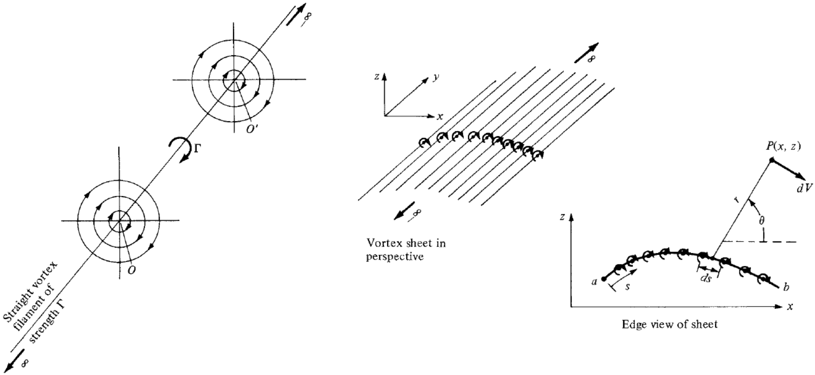
For cusped TE, upper and lower streamlines are parallel at TE

- Static pressure must be same on both streamlines; else they must shift
- Since prevailing assumptions support Bernoulli's principle, sameness of pressure implies sameness of velocity magnitudes too

$$\textbf{Kutta condition:} \quad \begin{cases} \text{Wedge TE:} & V_u = V_l = 0 \quad \text{at TE,} \\ \text{Cusped TE:} & \underline{V}_u = \underline{V}_l \quad \text{at TE.} \end{cases}$$

Two-Dimensional Source and Vortex Sheets

Extension of line singularity to sheet singularity



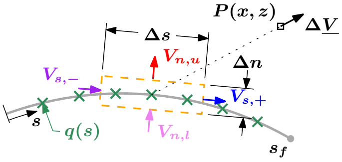
Irrotational vortex singularities: (Left) line vortex, and (Right) vortex sheet

Similarly, source sheet can be thought to be formed from line sources

These are needed in two-dimensional airfoil theory

Anderson, Jr., J. D., *Fundamentals of Aerodynamics*, McGraw-Hill, 2011

Source sheet



Source sheet of density $q(s)$

Source density $q(s)$ is volume flow rate generated per unit length of sheet (along s) per unit span (along y)

Consider small rectangular C.V. locally aligned with sheet

Volume flow rate per unit span due to enclosed sheet is $q(s)\Delta s$

As no other source in C.V., this must equal net outflow rate from C.S.

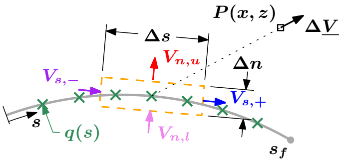
By taking $\Delta n \to 0$ to focus on immediate vicinity of sheet, we conclude

For a general (curved) source sheet: $\boxed{V_{n,u}(s) - V_{n,l}(s) = q(s)}$

Source sheet creates difference in local sheet-normal velocity immediately above and below it, the difference being equalling the local source density

Actual velocity magnitudes are determined by entire flow field

Velocity field associated with a source sheet



$\Delta \underline{V}(x,z)$: velocity at (x,z) due to infinitesimal length Δs of sheet shown

Infinitesimal sheet section is line source!

Recalling its associated velocity field,

$$\Delta \underline{V}(x,z) = \frac{q(s)\Delta s}{2\pi} \frac{(x - x(s))\underline{\hat{i}} + (z - z(s))\underline{\hat{k}}}{(x - x(s))^2 + (z - z(s))^2}$$

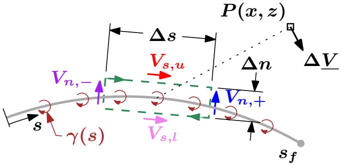
$(x(s), z(s))$ are Cartesian coordinates corresponding to sheet coordinate s

This is singular at sheet; we should use previous result for that

Velocity due to sheet: $$\boxed{\underline{V}(x,z) = \int_0^{s_f} \frac{q(s)}{2\pi} \frac{(x - x(s))\underline{\hat{i}} + (z - z(s))\underline{\hat{k}}}{(x - x(s))^2 + (z - z(s))^2} ds}$$

Doesn't account for any other causes (e.g., other sources, vortices, streams)

Vortex sheet



Vortex sheet of density $\gamma(s)$

Circulation density $\gamma(s)$ is circulation due to unit length of sheet (along s)

By convention, it's positive clockwise

Consider small rectangular closed curve with sides locally aligned with sheet

Circulation from enclosed sheet: $\gamma(s)\Delta s$

As no other vortex in curve, this must equal circulation around it

By taking $\Delta n \to 0$ to focus on immediate vicinity of sheet, we conclude

For a general (curved) vortex sheet: $\boxed{V_{s,u}(s) - V_{s,l}(s) = \gamma(s)}$

Vortex sheet creates difference in local sheet-tangent velocity immediately above and below it, the difference being equalling local circulation density

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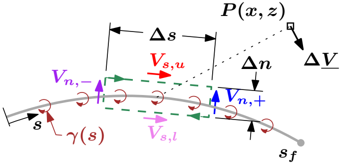
Velocity field associated with a vortex sheet



$\Delta \underline{V}(x,z)$: velocity at (x,z) due to infinitesimal length Δs of sheet shown

Infinitesimal sheet section is line vortex!

Recalling its associated velocity field,

$$\Delta \underline{V}(x,z) = \frac{\gamma(s)\Delta s}{2\pi} \frac{(z - z(s))\underline{\hat{i}} - (x - x(s))\underline{\hat{k}}}{(x - x(s))^2 + (z - z(s))^2}$$

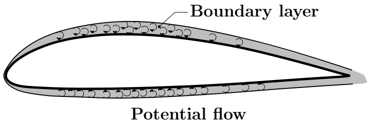
Again singular at sheet; should use previous result for that

Velocity due to sheet: $$\boxed{\underline{V}(x,z) = \int_0^{s_f} \frac{\gamma(s)}{2\pi} \frac{(z - z(s))\underline{\hat{i}} - (x - x(s))\underline{\hat{k}}}{(x - x(s))^2 + (z - z(s))^2} ds}$$

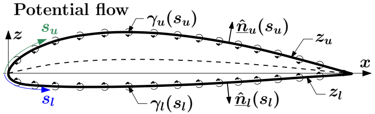
Doesn't account for any other causes (e.g., other vortices, sources, streams)

Hierarchy of Models for 2D Airfoil Flow

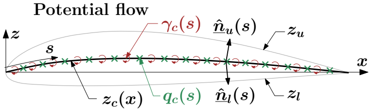
Sequence of model simplifications



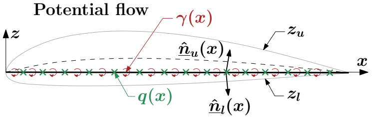
Vorticity concentrated in thin boundary layer; potential flow everywhere else



Vorticity concentrated in a sheet on the airfoil surface; potential flow outside



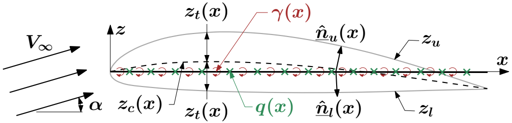
Vorticity and source concentrated in a sheet on airfoil camber line; flow tangency applied at camber line but w.r.t. corresponding $\underline{\hat{n}}_u$ & $\underline{\hat{n}}_l$



Vorticity and source concentrated in a sheet on the airfoil chord; flow tangency applied at chord line but w.r.t. corresponding $\underline{\hat{n}}_u$ & $\underline{\hat{n}}_l$

Thin Airfoil Theory Setup

Problem setup



Origin of coordinates system placed at LE

Freestream makes angle-of-attack α with $x-$axis

Usually, $x-$axis is along chord, but TE need not be on $x-$axis

- Think of a deflected flap

Functions $z_c(x)$, $z_t(x)$, $z_u(x)$ and $z_l(x)$ can be suitably redefined

- $z_c(x = c)$ need not be 0
- z_t is measured normal to $x-$axis and not to chord, but $z_t(x = c) = 0$
- $z_u(x) = z_c(x) + z_t(x)$ and $z_l(x) = z_c(x) - z_t(x)$ as before

Setup – Perturbation velocity

Write overall velocity as freestream velocity $\underline{V}_\infty$ plus 'perturbation' velocity $(u'\hat{\underline{i}} + w'\hat{\underline{k}})$ due to source and vortex sheets modelling airfoil

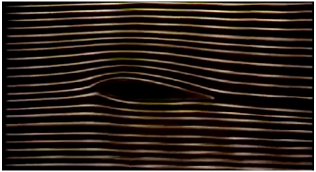
With abuse of notation, u' & w' will be written as u & w here

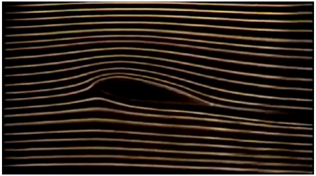
$$\underline{V} = (V_\infty \cos\alpha + u)\hat{\underline{i}} + (V_\infty \sin\alpha + w)\hat{\underline{k}} \approx (V_\infty + u)\hat{\underline{i}} + (V_\infty \alpha + w)\hat{\underline{k}},$$

where small AoA is used to write $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$

Airfoil being thin and at small α will perturb freestream by small amount

$$\boxed{|u| \ll V_\infty, \qquad |w| \ll V_\infty}$$





https://www.youtube.com/watch?v=6UlsArvbTeo

Setup – Boundary conditions: Far-field decay

To determine unique strengths of each of the ingredients - viz.

- uniform flow $\underline{U}$,
- vortex sheet circulation density $\gamma(x)$, and
- source sheet source density $q(x)$,

we must satisfy the applicable boundary conditions as discussed now

Perturbation decay in the far field

The flow should relax to the freestream far enough away from the airfoil

Finite vortex and source sheet (placed on airfoil chord) induce velocities that decay inversely with distance from them

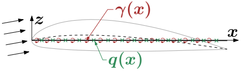
So, in far field, only uniform stream will survive

$$\implies \underline{U} = V_\infty \left(\underline{\hat{i}} + \alpha \underline{\hat{k}} \right)$$

Setup – Boundary conditions: Kutta condition

Kutta condition: $\underline{V}_u = \underline{V}_l$ at trailing edge, for both wedge and cusped TE's

In context of thin airfoil theory, $\underline{V}(c, 0^+) = \underline{V}(c, 0^-)$



$$
\begin{aligned}
&(V_\infty + u(c, 0^+))\hat{\underline{i}} + (V_\infty \alpha + w(c, 0^+))\hat{\underline{k}} \\
&\quad = (V_\infty + u(c, 0^-))\hat{\underline{i}} + (V_\infty \alpha + w(c, 0^-))\hat{\underline{k}}
\end{aligned}
$$

$$\implies u(c, 0^+) = u(c, 0^-) \qquad \text{and} \qquad w(c, 0^+) = w(c, 0^-)$$

Source sheet on chord causes local $z-$velocity difference across it

- So, $w(c, 0^+) = w(c, 0^-) \implies q(c) = 0$
- Just restates that TE is sharp; so no flow is displaced by airfoil there

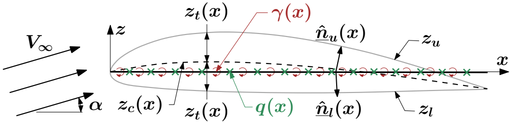
Chord vortex sheet causes local u difference across it; $u(c, 0^+) = u(c, 0^-)$

$$\implies \boxed{\text{Kutta condition for thin airfoil theory:} \qquad \gamma(c)\,(\equiv \gamma(TE)) = 0}$$

That Kutta condition constrains vortex sheet alone isn't surprising, as it only determines uniquely the *circulation* around the airfoil

No-penetration condition

Flow must not penetrate airfoil surface



In the context of thin airfoil theory, we require

$$\underline{V}(x, 0^+) \cdot \underline{\hat{n}}_u(x) = 0, \quad \text{and} \quad \underline{V}(x, 0^-) \cdot \underline{\hat{n}}_l(x) = 0, \quad x \in (0, c)$$

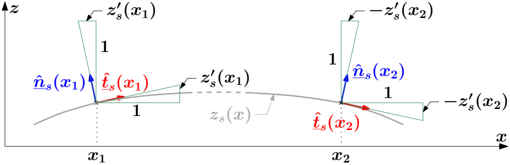
N.B.: Conditions are imposed on chord line in thin airfoil theory

Denoting upper (resp. lower) surface quantities by $+$ (resp. $-$) superscripts,

$$\underline{V}(x, 0^\pm) \cdot \underline{\hat{n}}^\pm(x) = 0, \quad x \in (0, c)$$

Surface normals – An aside

Need surface normals $\underline{\hat{n}}_u$ & $\underline{\hat{n}}_l$ in terms of surface functions z_u & z_l



Unit normal to surface $z_s(x)$: $\underline{\hat{n}}_s(x) = \dfrac{-z_s'(x)\underline{\hat{i}} + \underline{\hat{k}}}{\sqrt{(z_s'(x))^2 + 1}} \approx -z_s'(x)\underline{\hat{i}} + \underline{\hat{k}}$

Recalling $z_u = z_c + z_t$ and $z_l = z_c - z_t$,

Unit normal on upper surface $z_u(x)$: $\underline{\hat{n}}_u(x) = -\left\{z_c'(x) + z_t'(x)\right\}\underline{\hat{i}} + \underline{\hat{k}}$,

Unit normal on lower surface $z_l(x)$: $\underline{\hat{n}}_l(x) = -\left\{z_c'(x) - z_t'(x)\right\}\underline{\hat{i}} + \underline{\hat{k}}$,

Or, unit normals on surfaces $z^{\pm}(x)$: $\underline{\hat{n}}^{\pm}(x) = -\left\{z_c'(x) \pm z_t'(x)\right\}\underline{\hat{i}} + \underline{\hat{k}}$

Setup – Boundary Conditions: On airfoil body (contd.)

Recall: $\underline{V}(x, 0^{\pm})\cdot\underline{\hat{n}}^{\pm}(x) = 0,\ \ x \in (0, c);\quad \underline{\hat{n}}^{\pm}(x) = -\left\{z_c'(x) \pm z_t'(x)\right\}\underline{\hat{i}} + \underline{\hat{k}}$

With $\underline{V} = (V_\infty + u)\underline{\hat{i}} + (V_\infty\alpha + w)\underline{\hat{k}}$, for all $x \in (0, c)$, we have

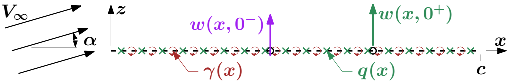
$$\left[(V_\infty + u(x, 0^{\pm}))\underline{\hat{i}} + (V_\infty\alpha + w(x, 0^{\pm}))\underline{\hat{k}}\right] \cdot \left[-(z_c'(x) \pm z_t'(x))\underline{\hat{i}} + \underline{\hat{k}}\right] = 0$$

$$\implies -(V_\infty + u(x, 0^{\pm}))(z_c'(x) \pm z_t'(x)) + V_\infty\alpha + w(x, 0^{\pm}) = 0$$

Recalling that $|u| \ll V_\infty$, we have the final expression of the body b.c.

$$\frac{w(x, 0^{\pm})}{V_\infty} = -\alpha + z_c'(x) \pm z_t'(x), \qquad x \in (0, c)$$

Discussion of thin airfoil theory setup



Thin airfoil problem is to find source & vortex sheet $q(x)$ & $\gamma(x)$ s.t.

$$w(x,0^\pm)/V_\infty = -\alpha + z_c'(x) \pm z_t'(x), \quad x \in (0,c); \qquad \gamma(c) = 0$$

Only planar source sheet can cause equal and opposite normal velocity across it

Only planar vortex sheet has the same normal velocity across it

Agrees with experimental results on lift and Kutta-Joukowsky theorem

This can be conveniently divided into three independent sub-problems explicitly separating effect of the three geometric aspects of airfoils:

AoA problem: Find $\gamma(x)$ s.t. $w(x,0^\pm)/V_\infty = -\alpha,\ x \in (0,c);\ \ \gamma(c) = 0$
Camber problem: Find $\gamma(x)$ s.t. $w(x,0^\pm)/V_\infty = z_c'(x),\ x \in (0,c);\ \ \gamma(c) = 0$
Thickness problem: Find $q(x)$ s.t. $w(x,0^\pm)/V_\infty = \pm z_t'(x),\ x \in (0,c)$

Pressure coefficient in thin airfoil theory

Derive approximate c_p expression consistent with prevailing assumptions

Bernoulli's principle applies, so that local pressure is related to local velocity:

$$p_\infty + \rho_\infty \frac{V_\infty^2}{2} = p + \rho_\infty \frac{V^2}{2} = p + \rho_\infty \frac{(V_\infty \cos\alpha + u)^2 + (V_\infty \sin\alpha + w)^2}{2}$$

$$\implies c_p = \frac{p - p_\infty}{0.5\rho_\infty V_\infty^2} = -\frac{0.5\rho_\infty \left\{2V_\infty u \cos\alpha + 2V_\infty w \sin\alpha + u^2 + w^2\right\}}{0.5\rho_\infty V_\infty^2}$$

But in thin airfoil theory

- $\alpha \ll 1$ (in radians!), so that $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$,
- $|u| \ll V_\infty$, and
- $|w| \ll V_\infty$

In thin airfoil theory: $\boxed{c_p \approx -\frac{2u}{V_\infty}}$

Angle-of-Attack Problem of Thin Airfoil Theory

Angle-of-attack problem setup

Angle-of-attack problem

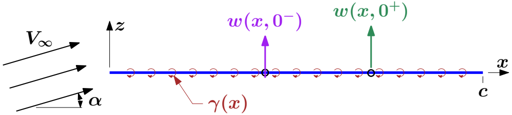
Consider a flat plate of length c immersed in a stream of speed V_∞ at an angle α to the plate.

Take the origin at the leading edge of the plate, and the $x-$axis along it.

Then, find the circulation density $\gamma(x)$ of a vortex sheet on the $x-$axis simulating the effect of the plate on the stream, such that

$$w(x,0) = -V_\infty \alpha \qquad \forall x \in (0,c), \qquad \text{and} \qquad \gamma(c) = 0,$$

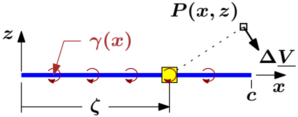
where w is the $z-$velocity *perturbation*



Restatement of angle-of-attack problem

Recall velocity due to curved vortex sheet:

$$\underline{V}(x,z) = \int_0^{s_f} \frac{\gamma(s)}{2\pi} \frac{(z - z(s))\underline{\hat{i}} - (x - x(s))\underline{\hat{k}}}{(x - x(s))^2 + (z - z(s))^2} ds$$



Specializing to the planar vortex sheet on the $x-$axis between 0 and c,

$$\underline{V}(x,z) = \int_0^{c} \frac{\gamma(\zeta)}{2\pi} \frac{z\underline{\hat{i}} - (x - \zeta)\underline{\hat{k}}}{(x - \zeta)^2 + z^2} d\zeta \quad \Longrightarrow \quad w(x,0) = -\frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)}{x - \zeta} d\zeta$$

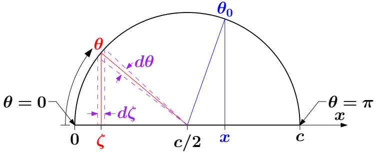
Evidently, $w(x,0^+) = w(x,0^-)$ as expected for a vortex sheet

Using this in problem statement $w(x,0) = -V_\infty \alpha$ for $x \in (0,c)$ with $\gamma(c) = 0$, we have the *integral problem*

$$\boxed{\text{Find } \gamma(x) \text{ s.t.:} \quad \frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)}{x-\zeta} d\zeta = V_\infty \alpha \ \ \forall x \in (0,c) \quad \text{subject to } \gamma(c) = 0}$$

The problem was elegantly solved by Glauert; we follow his steps

AoA problem solution step 1: Coordinate transform



$$\zeta = \frac{c}{2}(1 - \cos\theta), \quad d\zeta = \frac{c}{2}\sin\theta \; d\theta$$

$$\zeta = 0 \rightarrow \theta = 0, \quad \zeta = c \rightarrow \theta = \pi$$

$$x = \frac{c}{2}(1 - \cos\theta_0)$$

Since transformation is invertible, functions of ζ (or x) can be expressed in terms of θ (resp. θ_0)

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)}{x - \zeta} d\zeta = V_\infty \alpha \quad \forall x \in (0, c), \quad \text{subject to } \gamma(c) = 0$$

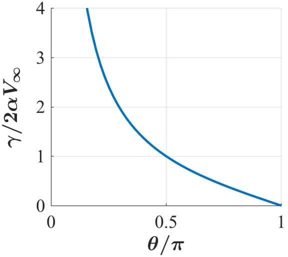
$$\implies \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{\cos\theta - \cos\theta_0} d\theta = V_\infty \alpha \quad \forall \theta_0 \in (0, \pi), \quad \text{subject to } \gamma(\pi) = 0$$

Apparently, this integral equation is no more easier to solve!

AoA problem solution step 2: Claimed solution

It is claimed that

$$\boxed{\text{AoA problem solution:} \quad \gamma(\theta) = 2V_\infty \alpha \frac{1+\cos\theta}{\sin\theta} = 2V_\infty \alpha \cot\frac{\theta}{2}, \quad \theta \in (0,\pi)}$$



N.B.: Vortex sheet has zero strength at TE ($\theta = \pi$) satisfying Kutta condn.

It has infinite strength at LE ($\theta = 0$)!

AoA problem solution step 3: Proof using Glauert integral

$$\text{Claimed solution:} \quad \gamma(\theta) = 2V_\infty\alpha\frac{1+\cos\theta}{\sin\theta}, \quad \forall\theta\in(0,\pi)$$

To see that this is a solution, we need the following result (due to Glauert)

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\theta_0}d\theta = \frac{\pi\sin n\theta_0}{\sin\theta_0}$$

With this,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{\cos\theta-\cos\theta_0}d\theta &= \frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{1+\cos\theta}{\cos\theta-\cos\theta_0}d\theta \\
&= \frac{V_\infty\alpha}{\pi}\int_0^\pi \frac{\cos(0\times\theta)+\cos(1\times\theta)}{\cos\theta-\cos\theta_0}d\theta \\
&= \frac{V_\infty\alpha}{\pi}\ \frac{\pi}{\sin\theta_0}\left[\sin(0\times\theta_0)+\sin(1\times\theta_0)\right] = V_\infty\alpha. \qquad \text{Q.E.D.}
\end{aligned}$$

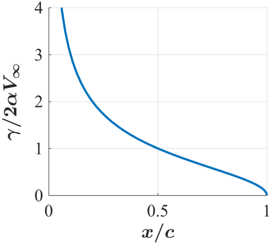
Thus it indeed satisfies the AoA problem

AoA problem solution in physical coordinates

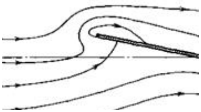
Recalling that $x = 0.5c(1 - \cos\theta)$ (as ζ is dummy argument, it's replaced by x), the vortex sheet circulation density is

$$\frac{\gamma(\theta)}{2V_\infty\alpha} = \frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+(1-2x/c)}{2x/c}}$$

$$\implies \frac{\gamma(x)}{2V_\infty\alpha} = \sqrt{\frac{c-x}{x}}$$



N.B.: $\gamma(x=0) \to \infty$ as model attempts to simulate the condition of the stagnation point being on the pressure surface of the flat plate, and the flow having to thus curl around the sharp leading edge

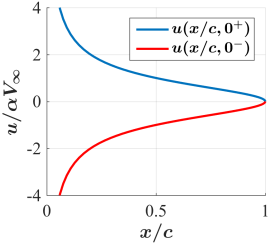


This singularity is however weak, in the sense that the integrated results for c_l and c_m are still valid

Perturbation velocity and c_p in angle-of-attack problem

In case of planar vortex sheet, $u(x, 0^+) = -u(x, 0^-) = \gamma(x)/2$; so that

$$u(x, 0^\pm) = \pm V_\infty \alpha \sqrt{\frac{c - x}{x}} \quad \forall x \in (0, c)$$



N.B.: $x-$component of perturbation velocity is equal and opposite on the upper and lower surfaces, with +ve values above

N.B.: Overall velocity parallel to the plate is $V_\infty + u$

Pressure coefficient in thin airfoil theory is $c_p = -2u/V_\infty$, so above is also a plot of $-c_p/2\alpha$

Indeed, we predict suction on upper surface and equal pressure on lower one; distribution is symmetric

Area enclosed by c_p plot (i.e., c_l) is directly proportional to α!

Sectional lift coefficient in angle-of-attack problem

The sectional lift coefficient can be found from Kutta-Joukowsky theorem

For this, we start by calculating the total circulation generated by the flat plate at the angle α to the freestream (recalling $x = 0.5c(1 - \cos\theta_0)$)

$$\Gamma := \int_0^c \gamma(x)dx = \frac{c}{2}\int_0^\pi \gamma(\theta_0)\sin\theta_0 d\theta_0 = cV_\infty\alpha\int_0^\pi \frac{1+\cos\theta_0}{\sin\theta_0}\sin\theta_0 d\theta_0$$
$$= cV_\infty\alpha\left|\theta_0 + \sin\theta_0\right|_0^\pi = \pi cV_\infty\alpha$$

Then, using Kutta-Joukowsky theorem ($L' = \rho_\infty V_\infty \Gamma$), we have

$$c_l = \frac{L'}{0.5\rho_\infty V_\infty^2 c} = \frac{\rho_\infty V_\infty \Gamma}{0.5\rho_\infty V_\infty^2 c} = \frac{\rho_\infty V_\infty}{0.5\rho_\infty V_\infty^2 c}(\pi c V_\infty \alpha)$$

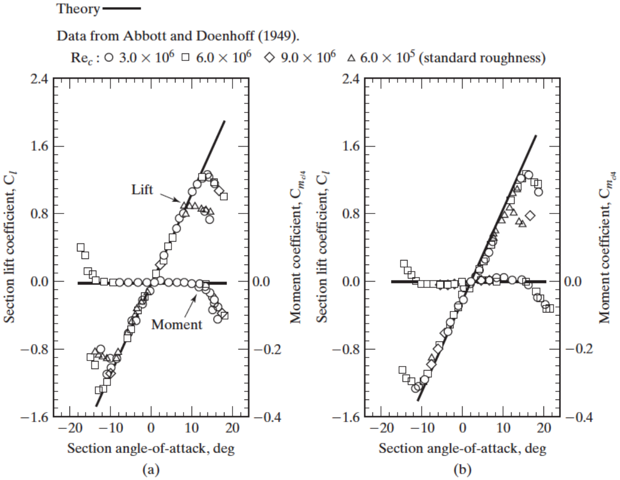
We arrive at the most important result of thin airfoil theory

$$\boxed{\text{Flat plate at angle } \alpha \text{ to freestream:} \quad c_l = 2\pi\alpha}$$

Since the angle of attack doesn't appear anywhere else in the theory,

$$\boxed{\text{Lift slope for all thin airfoils:} \quad \frac{\partial c_l}{\partial \alpha} = 2\pi = 0.11 \text{ per degree}}$$

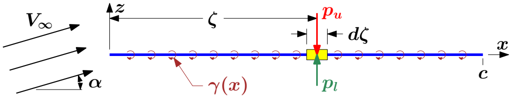
Angle-of-attack solution: Experimental validation



Comparison of the aerodynamic coefficients calculated using thin-airfoil theory for symmetric airfoils: (a) NACA 0009 wing section; (b) NACA 0012-64 wing section

Bertin and Cummings (2013)

Sectional pitching moment coefficient in AoA problem



The sectional pitching moment coefficient about the leading edge is

$$c_{m,LE} = \frac{M'_{LE}}{0.5\rho_\infty V_\infty^2 c^2} = \int_0^c \frac{\zeta(p_u - p_l)d\zeta}{0.5\rho_\infty V_\infty^2 c^2} = \int_0^c \frac{\zeta}{c}\left[c_{p,u}(\zeta) - c_{p,l}(\zeta)\right]\frac{d\zeta}{c}$$

$$\approx \int_0^c \frac{\zeta}{c}\left[\left(-\frac{\gamma(\zeta)}{V_\infty}\right) - \frac{\gamma(\zeta)}{V_\infty}\right]\frac{d\zeta}{c} = -2\int_0^c \frac{\zeta}{c}\frac{\gamma(\zeta)}{V_\infty}\frac{d\zeta}{c}$$

$$= -2\int_0^\pi \frac{1-\cos\theta}{2}\left(2\alpha\frac{1+\cos\theta}{\sin\theta}\right)\left(\frac{\sin\theta\ d\theta}{2}\right) \qquad \left[\because \zeta = \frac{c}{2}(1-\cos\theta)\right]$$

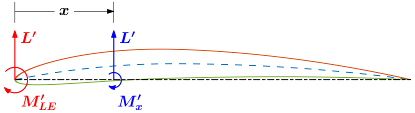
$$= -\alpha\int_0^\pi \sin^2\theta\ d\theta = -\frac{\alpha}{2}\int_0^\pi (1-\cos 2\theta)d\theta = -\frac{\alpha}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi = -\frac{\pi\alpha}{2}$$

As thickness doesn't contribute to pitching moment per thin airfoil theory,

For thin symmetric airfoils: $\boxed{c_{m,LE} = -\frac{\pi\alpha}{2}}$

Center of pressure in angle-of-attack problem

Recall: center of pressure is the point about which pitching moment is zero



Two equivalent force-moment systems in red and blue

To find the center of pressure, we equate the moment of sectional lift acting thru x_{cp} about LE to the sectional pitching moment about LE; i.e.,

$$L' x_{cp} = M'_{LE} \qquad \Longrightarrow \; x_{cp} = -\frac{M'_{LE}}{L'} = -\frac{c_{m,LE} \;\rho_\infty V_\infty^2 c^2/2}{c_l \;\rho_\infty V_\infty^2 c/2} = -\frac{c_{m,LE}}{c_l} c$$

For thin airfoils, for a particular α, we use $c_l = 2\pi\alpha$ and $c_{m,LE} = -\pi\alpha/2$ to find $x_{cp} = c/4$

Since thickness doesn't affect x_{cp}, we conclude

$$\boxed{\text{For thin symmetric airfoils:} \quad x_{cp} = \frac{c}{4} \quad (\text{independent of } \alpha)}$$

Aerodynamic center in thin airfoil theory

Recall: aerodynamic center is the point about which pitching moment independent of α

We have found that the center of pressure of thin symmetric airfoils is at $x_{cp} = c/4$, irrespective of α

Thus, this coincides with the aerodynamic center for thin symmetric airfoils:

$$\boxed{\text{For thin symmetric airfoils:} \quad x_{ac} = \frac{c}{4} \quad \text{and} \quad c_{m,ac} = c_{m,cp} = 0}$$

As AoA doesn't appear anywhere else in thin airfoil theory, we conclude

$$\boxed{\text{For all thin airfoils:} \quad x_{ac} = \frac{c}{4}}$$

N.B.: $c_{m,ac} \neq 0$ for cambered airfoils

Camber Problem of Thin Airfoil Theory

Camber problem setup

Camber problem

A 2D curved plate has chord (of length c) aligned with stream of speed V_∞.

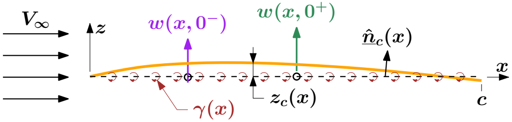
With origin at LE, curve deviation from $x-$axis (taken along chord) is $z_c(x)$.

Then, find circulation density $\gamma(x)$ of vortex sheet on $x-$axis such that

$$w(x,0) = V_\infty z_c'(x) \qquad \forall x \in (0,c), \qquad \text{and} \qquad \gamma(c) = 0,$$

where w is the $z-$velocity *perturbation*.

N.B.: The plate's trailing edge need not lie on the $x-$axis.



Restatement of camber problem

Following the development of the angle-of-attack problem, we arrive at the analogous integral problem for $\gamma(x)$:

$$\boxed{\gamma(x) \text{ s.t. } \frac{1}{2\pi}\int_0^c \frac{\gamma(\zeta)}{x-\zeta}d\zeta = -V_\infty z_c'(x) \quad \forall x \in (0,c) \quad \text{subject to } \gamma(c) = 0}$$

Applying the aerodynamic coordinate transformation, the problem becomes

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{\cos\theta - \cos\theta_0}d\theta = -V_\infty \frac{dz_c}{dx}(\theta_0) \quad \forall \theta_0 \in (0,\pi) \quad \text{subject to } \gamma(\pi) = 0$$

N.B. dz_c/dx is to be evaluated at transformed θ_0; this is not same as $z_c'(\theta_0)$

Solution of camber problem: Step 1

Let us write the camber derivative function as

$$\frac{dz_c}{dx}(\theta_0) = \underbrace{\frac{1}{\pi}\int_0^{\pi} \frac{dz_c}{dx}(\theta)d\theta}_{\text{Average}} + \underbrace{\left[\frac{dz_c}{dx}(\theta_0) - \frac{1}{\pi}\int_0^{\pi} \frac{dz_c}{dx}(\theta)d\theta\right]}_{\text{Fluctuation}}$$

As problem is linear, solution $\gamma(\theta)$ can also be separated into a contribution from the (constant) average part of z_c' and another from its (zero-mean) fluctuating part

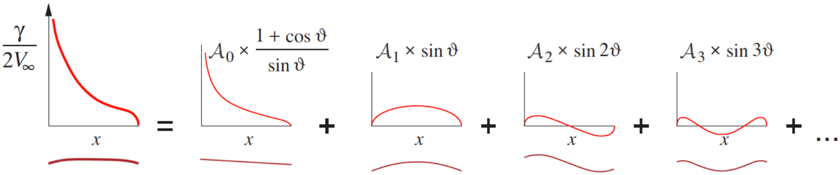
Solution of first problem is already known from the angle-of-attack problem, where the right hand side was indeed constant

- Its solution was $\gamma_\alpha(\theta) = 2V_\infty\alpha(1+\cos\theta)/\sin\theta$

Then, it is proposed that camber problem solution is of the form

$$\gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right)$$

Solution of camber problem: Step 2



Proposed solution form: $$\gamma(\theta) = 2V_\infty \left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right)$$

- V_∞ factor yields dimensionless coefficients; 2 factor follows convention
- Fourier series expansion is allowed since $\gamma(\theta)$ (a) must be well-behaved other than the abstracted LE singularity, and (b) is defined in $(0, \pi)$
- Cosine terms are omitted as they would directly violate Kutta condition
- Sine terms (along with first term) automatically satisfy Kutta condition

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Camber problem solution: Step 3

Put assumed solution form in no-penetration b.c. LHS

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta}{\cos\theta-\cos\theta_0}d\theta = \frac{V_\infty}{\pi}\int_0^\pi \left[A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin n\theta\right]\frac{\sin\theta d\theta}{\cos\theta-\cos\theta_0}$$

First term $\int_0^\pi (1+\cos\theta)/(\cos\theta-\cos\theta_0)d\theta=\pi$, known from AoA problem

For each integral in second series term, use another Glauert integral result:

$$\int_0^\pi \frac{\sin n\theta\sin\theta}{\cos\theta-\cos\theta_0}d\theta = -\pi\cos n\theta_0$$

Putting everything together, the no-penetration b.c. is

$$V_\infty\left(A_0-\sum_{n=1}^{\infty}A_n\cos n\theta_0\right) = -V_\infty\frac{dz_c}{dx}(\theta_0) \qquad \forall\theta_0\in(0,\pi)$$

$z_c'(\theta_0)$ must be expanded in Fourier cosine series and matched term-by-term

Camber problem solution: Step 4

Recall that cosine transform of $g(\theta)$ is defined as

$$g(\theta) = B_0 + \sum_{m=1}^{\infty} B_m \cos m\theta, \quad B_0 = \frac{1}{\pi}\int_0^{\pi} g(\theta) d\theta, \quad B_m = \frac{2}{\pi}\int_0^{\pi} g(\theta) \cos m\theta \; d\theta$$

Comparing terms of

$$V_\infty \left(A_0 - \sum_{n=1}^{\infty} A_n \cos n\theta_0 \right) = -V_\infty \frac{dz_c}{dx}(\theta_0) \qquad \forall \theta_0 \in (0, \pi)$$

we have the expressions for the coefficients

$$\boxed{\text{Camber problem soln:} \quad \frac{\gamma(\theta)}{V_\infty} = 2\left(A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right) \quad \forall \theta \in (0, \pi)}$$

$$\boxed{A_0 = -\frac{1}{\pi}\int_0^{\pi} \frac{dz_c}{dx}(\theta_0) d\theta_0, \quad A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz_c}{dx}(\theta_0) \cos n\theta_0 \; d\theta_0}$$

Lift and pitching moment coefficients in camber problem

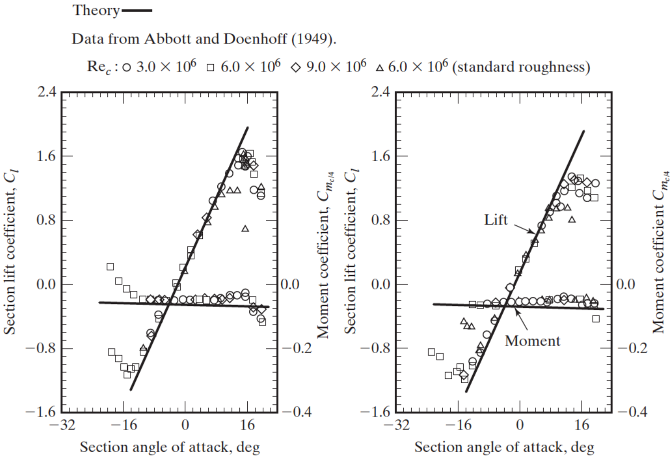
Straightforward math (derivation detailed in notes) shows that

$$\boxed{c_l = (2A_0 + A_1)\,\pi = 2\int_0^{\pi} \frac{dz_c}{dx}(\theta)\,(\cos\theta - 1)\,d\theta}$$

$$\boxed{c_{m,c/4} = c_{m,ac} = \frac{\pi}{4}(A_2 - A_1) = \frac{1}{2}\int_0^{\pi} \frac{dz_c}{dx}(\theta)\,(\cos 2\theta - \cos\theta)\,d\theta}$$

Evidently, only first three cosine series terms of z_c' are relevant for c_l & $c_{m,c/4}$

Camber solution: Experimental validation



(a) NACA 2412 wing section

(b) NACA 2418 wing section

c_l predictions are quite accurate; $c_{m,c/4}$ not so much

In reality, aerodynamic center for these airfoils vary between $0.23c$ and $0.27c$

Bertin and Cummings (2013)

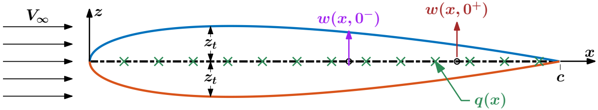
Thickness Problem of Thin Airfoil Theory

Thickness problem setup

Thickness problem

Given a freestream speed of V_∞ aligned with a thin symmetric airfoil of thickness function $z_t(x)$, find the density distribution $q(x)$ of the source sheet on the chord line ($z = 0$) such that the $z-$velocity *perturbation* is

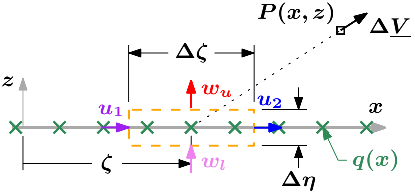
$$w(x, 0^\pm) = \pm V_\infty z_t', \qquad x \in (0, c)$$



Thickness problem – Solution

Note the form of the boundary constraint: $w(x, 0^{\pm}) = \pm V_\infty z_t'$

Recall: local sheet-normal velocity difference across a source sheet equals local source density; i.e., $w(x, 0^+) - w(x, 0^-) = q(x)$



In a *planar* sheet, no other part of sheet contributes to local w; so symmetry dictates that $w(x, 0^+) = -w(x, 0^-) = q(x)/2$

Thickness problem solution: $\boxed{q(x) = 2V_\infty z_t'(x), \qquad x \in (0, c)}$

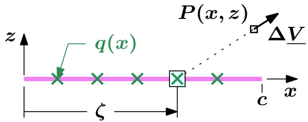
Thickness problem – Pressure coefficient

Find c_p from thickness problem solution: $q(x) = 2V_\infty z_t'(x)$ for $x \in (0, c)$

Recall: (perturbation) velocity associated with a (curved) source sheet is

$$\underline{V}(x,z) = \int_0^{s_f} \frac{q(s)}{2\pi} \frac{(x - x(s))\hat{\underline{i}} + (z - z(s))\hat{\underline{k}}}{(x - x(s))^2 + (z - z(s))^2} ds$$

Specializing to $x-$component of *perturbation* velocity *at* our *planar* sheet,



$$u(x, 0^\pm) = \frac{V_\infty}{\pi} \int_0^c \frac{z_t'(\zeta)}{x - \zeta} d\zeta, \quad c_p(x, 0^\pm) = -\frac{2u(x, 0^\pm)}{V_\infty} = -\frac{2}{\pi} \int_0^c \frac{z_t'(\zeta)}{x - \zeta} d\zeta$$

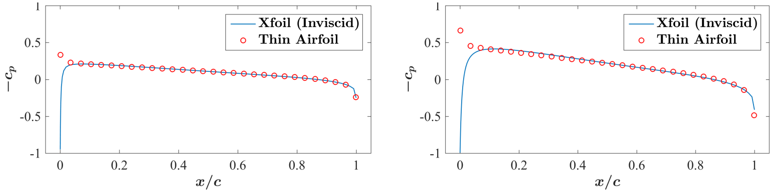
Integrals are *improper*; can still carefully interpret them as finite

Evaluation fails at LE & TE, as $u \ll V_\infty$ is just wrong at stagnation points

Thickness problem – Results

As expected, $c_p(x, 0^\pm) = -2\pi^{-1} \int_0^c z_t'(\zeta)(x - \zeta)^{-1} d\zeta$ same on both surfaces

So, no contribution to lift (as found in experiments)



(a) NACA 0006 at $\alpha = 0$ (b) NACA 0012 at $\alpha = 0$

Encouraging comparison with results from inviscid panel method in XFOIL

- Failure at LE and TE, as expected from previous arguments
- Rounded LE makes matters worse; sharp TE is more benign
- Greater discrepancy for thicker airfoils in maximum thickness region

End of Topic