SI427: Assignment 1
Group 4

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Problem 5.

Suppose that X is a standard normal random variable that is, X has p.d.f.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{1}$$

for all $x \in \mathbb{R}$.

Then find the probability density function of the following random variables

1. $$Y_1 = e^X \tag{2}$$

   As X is a standard normal random variable which is continuous. We take function $g : \mathbb{R} \to \mathbb{R}^+$, $g(x) = e^x$, so that $Y_1 = g(X)$. $g(x)$ is a differentiable and monotonically increasing function for all $x \in \mathbb{R}$ (derivative $= e^x > 0$). Thus, $g(x)$ will have an unique inverse $g^{-1}(y)$, $\forall\ y > 0$,

   $$g^{-1}(y) = \ln y. \tag{3}$$

   So, we can directly use the result derived in class[1], that the probability density function of Y_1 is,

   $$h(y) = \phi\left(g^{-1}(y)\right) \cdot \left| \frac{d}{dy}\left(g^{-1}(y)\right) \right| \tag{4}$$

for $y \in (min\{g(-\infty), g(\infty)\}, max\{g(-\infty), g(\infty)\})$ i.e. $y \in (0, \infty)$ in this case.

$$
\begin{aligned}
h(y) &= \phi\left(g^{-1}(y)\right) \cdot \left| \frac{d}{dy}\left(g^{-1}(y)\right) \right| && (5)\\
&= \phi(\ln y) \cdot \left| \frac{d}{dy}(\ln y) \right| && (6)\\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln y)^2}{2}} \cdot \left| \frac{1}{y} \right| && (7)\\
&= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{(\ln y)^2}{2}}}{y} && \text{As } y > 0 \quad (8)\\
&= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}\ln y - 1} && e^{a \ln b} = b^a \quad (9)
\end{aligned}
$$

So, finally, the Probability Density Function of the random variable given by e^X is given by,

$$
\boxed{h(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}\ln y - 1} & \text{if } y > 0 \end{cases}} \tag{10}
$$

2.

$$
Y_2 = 2X^2 + 1 \tag{11}
$$

Let $g : \mathbb{R} \to \mathbb{R}$,be a function st $g(x) = 2x^2 + 1$. Then $Y_2 = g(X)$. g does not have a unique inverse over its domain, as $g(x) = g(-x)$.

We can break $(-\infty, \infty)$ into $(-\infty, 0) \cup [0, \infty)$ such that $g^{-1}(y)$ has atmost one value in each of $(-\infty, 0)$ and $[0, \infty)$. We can then use the result derived for the general case in class directly.

However we shall approach this problem from the distributions. Let $\mathbb{F}_X$ and $\mathbb{F}_Y$ be the corresponding distribution functions for X and Y_2.

[1]We shall go through CDF method in the second part.

$$
\begin{aligned}
\mathbb{F}_Y(y) &= \mathbb{P}(2x^2+1 \le y) = \mathbb{P}_X(\{x : 2x^2+1 \le y\}) && (12)\\
&= \mathbb{P}_X\left(\left\{x : -\sqrt{\frac{y-1}{2}} \le x \le \sqrt{\frac{y-1}{2}}\right\}\right) && \forall y \ge 1. \\
& && (13)\\
&= \mathbb{P}_X\left(\left\{x : x \le \sqrt{\frac{y-1}{2}}\right\}\right) - \mathbb{P}_X\left(\left\{x : x < -\sqrt{\frac{y-1}{2}}\right\}\right) && (14)\\
&= \mathbb{F}_X\left(\sqrt{\frac{y-1}{2}}\right) - \lim_{n\to\infty} \mathbb{F}_X\left(-\sqrt{\frac{y-1}{2}} - \frac{1}{n}\right) && (15)\\
&= \mathbb{F}_X\left(\sqrt{\frac{y-1}{2}}\right) - \mathbb{F}_X\left(-\sqrt{\frac{y-1}{2}}\right). && \text{As } \mathbb{F}_X \text{ is continuous.}\\
& && (16)
\end{aligned}
$$

Let the pdf of Y_2 be $f(Y_2)$.

$$
\begin{aligned}
f(y) &= \frac{d}{dy}\mathbb{F}_Y(y) = \frac{d}{dy}\mathbb{F}_X\left(\sqrt{\frac{y-1}{2}}\right) - \frac{d}{dy}\mathbb{F}_X\left(-\sqrt{\frac{y-1}{2}}\right) && (17)\\
&= \left[\frac{d}{dy}\sqrt{\frac{y-1}{2}}\right]\left(\phi\left(\sqrt{\frac{y-1}{2}}\right) + \phi\left(-\sqrt{\frac{y-1}{2}}\right)\right) && \text{Chain rule} \quad (18)\\
&= \frac{1}{2\sqrt{2(y-1)}}\left(\frac{2}{\sqrt{2\pi}}e^{-\frac{y-1}{4}}\right) && \forall y > 1 \quad (19)\\
&= \frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{y-1}}e^{-\frac{y-1}{4}}. && (20)
\end{aligned}
$$

Thus $f(y)$ is

$$
\boxed{f(y) = \begin{cases} 0 & \text{if } y \le 1 \\ \frac{1}{\sqrt{4\pi}}\frac{1}{\sqrt{y-1}}e^{-\frac{y-1}{4}} & \text{if } y > 1 \end{cases}} \tag{21}
$$

3.

$$
Y_3 = g(X) = \begin{cases} 1 & \text{if } X > 0 \\ \frac{1}{2} & \text{if } X = 0 \\ -1 & \text{if } X < 0 \end{cases} \tag{22}
$$

Y_3 takes only 3 distinct values, hence it is a discrete random variable.

The inverse relation on g, $g^{-1}(Y_3)$ is:

$$
g^{-1}(Y_3) = \begin{cases} (-\infty, 0) & \text{if } Y_3 = -1 \\ \{0\} & \text{if } Y_3 = \frac{1}{2} \\ (0, \infty) & \text{if } Y_3 = 1 \\ \{\} & \text{otherwise} \end{cases} \tag{23}
$$

We shall calculate the probability *mass* function $p(y)$.

$$p(y) = \mathbb{P}_{Y_3}(y) = \mathbb{P}_X\left(g^{-1}(y)\right). \tag{24}$$

Thus,

$$\begin{aligned}
p(-1) &= \mathbb{P}_X\left((-\infty, 0)\right) && (25)\\
&= \int_{-\infty}^{0} \phi(t)dt && (26)\\
&= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt && (27)\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt && \text{As } \phi \text{ is even.} \quad (28)\\
&= \frac{1}{2}. && \text{Integral is 1} \quad (29)
\end{aligned}$$

$$\begin{aligned}
p(1) &= \mathbb{P}_X\left((0, \infty)\right) && (30)\\
&= \int_{0}^{\infty} \phi(t)dt && (31)\\
&= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt && (32)\\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt && \text{As } \phi \text{ is even.} \quad (33)\\
&= \frac{1}{2}. && \text{Integral is 1} \quad (34)
\end{aligned}$$

$$\begin{aligned}
p(0) &= \mathbb{P}_X\left(\{0\}\right) && (35)\\
&= 0 && (36)
\end{aligned}$$

As probability at a point for continuous distribution is zero.

And $\forall y \notin \left\{-1, \frac{1}{2}, 1\right\}$

$$\begin{aligned}
p(y) &= \mathbb{P}_X\left(\{\}\right) && (37)\\
&= 0 && (38)
\end{aligned}$$

So, the final probability density function of Y_3 comes out to be,

$$\boxed{p(y) = \begin{cases} \frac{1}{2} & \text{if } y = -1 \\ \frac{1}{2} & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}} \tag{39}$$