Governing Equations in Aerodynamics

Aerodynamics

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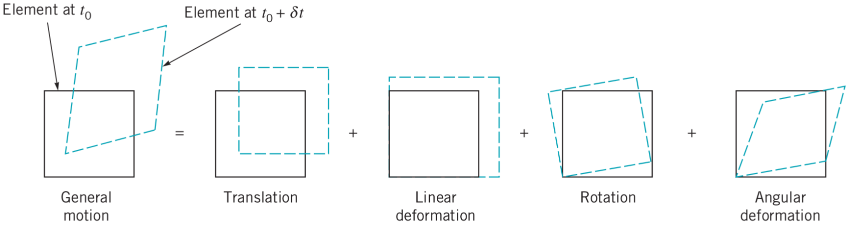
Review of General Governing Equations

Fluid kinematics

The kinematics of a fluid particle can be decomposed into:

- Translational – accounted for by velocity vector field
- Dilatational (linear deformation or change of volume) – accounted for by divergence of velocity, $\sigma := \nabla \cdot \underline{V}$
- Rotational – accounted for by the curl of velocity
- Shear (angular deformation) – accounted for by rate of strain tensor

$$\underline{\underline{\mathscr{S}}} := \frac{1}{2}\left[\nabla \underline{V} + (\nabla \underline{V})^{\mathsf{T}}\right]$$



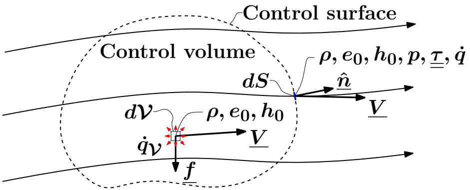
Munson, Young, Okiishi & Huebsch, Fundamentals of Fluid Mechanics, 2009

Conservation laws and Reynolds transport theorem

The conservation laws (e.g., mass, momentum and energy) are originally formulated for a system – a collection of (fluid) particles of unique identity

For flows, it is difficult and usually irrelevant to study such systems

Much more useful to identify a control volume (C.V.) in flow domain, and formulate conservation laws for corresponding flow quantities within it



Reynolds transport theorem transforms governing eqns. from system to C.V.

We will consider fixed control volumes only

Adapted from: Drela, Flight vehicle aerodynamics, MIT Press, 2014

Mass conservation equation

Integral form

$$\iiint \frac{\partial \rho}{\partial t} d\mathcal{V} + \oiint \rho \underline{V} \cdot \underline{\hat{n}} dS = 0$$

Differential form – conservative

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

Differential form – convective

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \underline{V} \cdot \nabla \right) \rho = -\rho\sigma, \qquad \sigma := \nabla \cdot \underline{V}$$

Momentum conservation – Preliminaries

Pressure

Surface stress tensor on a fluid element is decomposed into an isotropic (independent of direction) part, viz. pressure, p

Deviatoric stress tensor

For Newtonian fluids, remaining (deviatoric) stress tensor $\underline{\underline{\tau}}$ is proportional to strain rate tensor as

$$\underline{\underline{\tau}} = \mu \left[2\underline{\underline{\mathscr{S}}} - \frac{2}{3}\sigma \underline{\underline{I}} \right]$$

Body force

Body force per unit mass is $\underline{f}$ (think of g); it is assumed to be conservative (i.e., it is the gradient of a potential field, and hence curl-free)

Momentum conservation equation – Newton's 2nd law

Integral form

$$\iiint \frac{\partial(\rho \underline{V})}{\partial t} d\mathcal{V} + \oiint \rho(\underline{V} \cdot \underline{\hat{n}})\underline{V} dS = -\oiint p\underline{\hat{n}} dS + \oiint \underline{\underline{\tau}} \cdot \underline{\hat{n}} dS + \iiint \rho \underline{f} d\mathcal{V}$$

Differential form – conservative

$$\frac{\partial(\rho \underline{V})}{\partial t} + \nabla \cdot (\rho \underline{V}\,\underline{V}^{\mathsf{T}}) = -\nabla p + \nabla \cdot \underline{\underline{\tau}} + \rho \underline{f}$$

Differential form – convective

$$\rho \frac{\mathrm{D}\underline{V}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \underline{\underline{\tau}} + \rho \underline{f}$$

Energy conservation – Preliminaries

Ideal gas law – equation of state

Valid for most flows of interest in aerodynamics:

$$p = \rho R T, \qquad R = 287.04\ \mathrm{J/kg \cdot K} \quad \text{(for air)}$$

Specific (static) internal energy and specific (static) enthalpy

For calorically perfect gases (as is usual in aerodynamic flows), internal energy per unit mass of the fluid e, and its related quantity, enthalpy h, are

$$e = C_v T, \qquad h = e + p/\rho = C_p T,$$

where sp. heats at constant volume (C_v) and pressure (C_p) are constants.

$$\gamma = C_p/C_v = 1.4, \qquad C_p = \frac{\gamma}{\gamma - 1} R = 1004.6\ \mathrm{J/kg \cdot K} \quad \text{(for air)}$$

Energy conservation – Preliminaries (Contd.)

Specific total internal energy and specific total enthalpy

'Total' energy/enthalpy quantities are defined as their static counterparts plus specific kinetic energy $V^2/2$, where $V := |\underline{V}|$ is velocity magnitude

$$e_0 := e + V^2/2, \qquad h_0 := h + V^2/2$$

Fourier's law of heat conduction

Conductive heat flux vector $\underline{\dot{q}}$ in the flow is governed by Fourier's law through the heat conductivity κ:

$$\underline{\dot{q}} = -\kappa \nabla T$$

Volumetric heat and work sources

$\dot{q}_{\mathcal{V}}$: Rate of heat addition per unit volume due to source inside flow domain

In aerodynamics, we neglect all sources of work within the fluid volume

Energy conservation eqn. – 1st law of thermodynamics

Integral form in terms of total energy

$$\iiint \frac{\partial(\rho e_0)}{\partial t} d\mathcal{V} + \oiint \rho(\underline{V} \cdot \underline{\hat{n}}) e_0 \, dS = -\oiint p\underline{V} \cdot \underline{\hat{n}} dS + \oiint \underline{V} \cdot \underline{\underline{\tau}} \cdot \underline{\hat{n}} dS$$
$$+ \iiint \rho \underline{f} \cdot \underline{V} d\mathcal{V} + \iiint \dot{q}_{\mathcal{V}} d\mathcal{V} - \oiint \underline{\dot{q}} \cdot \underline{\hat{n}} dS$$

Differential form – conservative

$$\frac{\partial(\rho e_0)}{\partial t} + \nabla \cdot (\rho \underline{V} e_0) = -\nabla \cdot (p\underline{V}) + \nabla \cdot (\underline{\underline{\tau}} \cdot \underline{V}) + \rho \underline{f} \cdot \underline{V} + \dot{q}_{\mathcal{V}} - \nabla \cdot \underline{\dot{q}}$$

Differential form – convective

$$\rho \frac{\mathrm{D} e_0}{\mathrm{D} t} = -\nabla \cdot (p\underline{V}) + \nabla \cdot (\underline{\underline{\tau}} \cdot \underline{V}) + \rho \underline{f} \cdot \underline{V} + \dot{q}_{\mathcal{V}} - \nabla \cdot \underline{\dot{q}}$$

Energy conservation eqn. – Enthalpy form

Integral form in terms of total enthalpy

$$\iiint \frac{\partial(\rho h_0 - p)}{\partial t} d\mathcal{V} + \oiint \rho(\underline{V} \cdot \underline{\hat{n}}) h_0 dS = \oiint \underline{V} \cdot \underline{\underline{\tau}} \cdot \underline{\hat{n}} dS + \iiint \rho \underline{f} \cdot \underline{V} d\mathcal{V} + \iiint \dot{q}_{\mathcal{V}} d\mathcal{V} - \oiint \underline{\dot{q}} \cdot \underline{\hat{n}} dS$$

Differential form – conservative

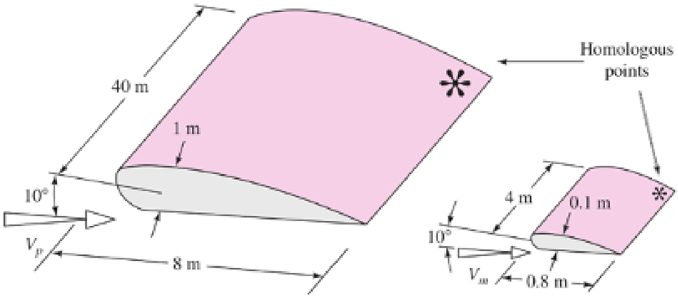
$$\frac{\partial(\rho h_0 - p)}{\partial t} + \nabla \cdot (\rho \underline{V} h_0) = \nabla \cdot (\underline{\underline{\tau}} \cdot \underline{V}) + \rho \underline{f} \cdot \underline{V} + \dot{q}_{\mathcal{V}} - \nabla \cdot \underline{\dot{q}}$$

Differential form – convective

$$\rho \frac{\mathrm{D} h_0}{\mathrm{D} t} = \frac{\partial p}{\partial t} + \nabla \cdot (\underline{\underline{\tau}} \cdot \underline{V}) + \rho \underline{f} \cdot \underline{V} + \dot{q}_{\mathcal{V}} - \nabla \cdot \underline{\dot{q}}$$

Non-dimensional Governing Equations

Many flows appear quite different but actually share deep similarities, as long as one renormalizes all flow variables and geometries by analogous reference quantities



Reference quantities or scales serve as natural units for reporting and comparing results across similar flows

White, Fluid Mechanics, 2009

Specific choices of reference quantities in aerodynamics

Physical scales of aerodynamic flows. Scales in bottom block are relevant only for compressible flows. Units: length l, time t, mass m, temperature θ.

Scale		Units	Typical specific choice	
ℓ_{ref}	length	l	c	airfoil chord
V_{ref}	velocity	l/t	V_∞	freestream speed
ρ_{ref}	density	m/l^3	ρ_∞	freestream density
μ_{ref}	viscosity	m/lt	μ_∞	freestream viscosity
a_{ref}	speed of sound	l/t	a_∞	freestream speed of sound
k_{ref}	heat conductivity	$ml/t^3\theta$	k_∞	freestream value
c_p	heat capacity	$l^2/t^2\theta$	c_p	freestream value ($\sim$ constant)
γ	ratio of specific heats	—	γ	freestream value ($\sim$ constant)

Absolute reference pressure and temperature (p_{ref}, T_{ref}) have been omitted as, for ideal gases, they are effectively redundant; specifically

$$p_{\mathrm{ref}} \equiv \rho_{\mathrm{ref}} V_{\mathrm{ref}}^2, \qquad T_{\mathrm{ref}} \equiv a_{\mathrm{ref}}^2 / C_p$$

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Non-dimensionalization

Coordinates and field variables are converted from their standard units to their dimensionless natural-unit counterparts (denoted by an overbar $\overline{(\cdot)}$) using their reference scales

$$
\begin{aligned}
(\overline{x},\overline{y},\overline{z},\overline{r}) &= (x,y,z,r)/\ell_{\mathrm{ref}} & \overline{t} &= tV_{\mathrm{ref}}/\ell_{\mathrm{ref}} & \overline{\nabla} &= \nabla\ell_{\mathrm{ref}} \\
\overline{\underline{V}} &= \underline{V}/V_{\mathrm{ref}} & \overline{\rho} &= \rho/\rho_{\mathrm{ref}} & \overline{p} &= p/(\rho_{\mathrm{ref}}V_{\mathrm{ref}}^2) \\
\overline{h}_0 &= h_0/a_{\mathrm{ref}}^2 & \overline{\mu} &= \mu/\mu_{\mathrm{ref}} & \overline{\kappa} &= \kappa/\kappa_{\mathrm{ref}}
\end{aligned}
$$

Parameter			Common name
Re_{ref}	$\equiv$	$\rho_{\mathrm{ref}}V_{\mathrm{ref}}\ell_{\mathrm{ref}}/\mu_{\mathrm{ref}}$	Reynolds number
M_{ref}	$\equiv$	$V_{\mathrm{ref}}/a_{\mathrm{ref}}$	Mach number
Pr_{ref}	$\equiv$	$c_p\,\mu_{\mathrm{ref}}/k_{\mathrm{ref}}$	Prandtl number
γ			ratio of specific heats

Reference non-dimensional parameters

Non-dimensional governing equations – convective form

Omitting body forces and volumetric heat sources (that are irrelevant in typical aerodynamic flows),

Mass: $\dfrac{\partial\overline{\rho}}{\partial\overline{t}} + \underline{\overline{V}} \cdot \overline{\nabla}\,\overline{\rho} + \overline{\rho}\overline{\nabla} \cdot \underline{\overline{V}} = 0$

Momentum: $\overline{\rho}\dfrac{\partial\underline{\overline{V}}}{\partial\overline{t}} + \overline{\rho}\underline{\overline{V}} \cdot \overline{\nabla}\,\underline{\overline{V}} = -\overline{\nabla}\overline{p} + \dfrac{1}{Re_{\text{ref}}}\overline{\nabla} \cdot \underline{\underline{\overline{\tau}}}$

Energy: $\overline{\rho}\dfrac{\partial\overline{h}_0}{\partial\overline{t}} + \overline{\rho}\underline{\overline{V}} \cdot \overline{\nabla}\,\overline{h}_0 = M_{\text{ref}}^2\dfrac{\partial\overline{p}}{\partial\overline{t}} + \dfrac{M_{\text{ref}}^2}{Re_{\text{ref}}}\overline{\nabla} \cdot \left(\underline{\underline{\overline{\tau}}} \cdot \underline{\overline{V}}\right) - \dfrac{1}{Re_{\text{ref}}Pr_{\text{ref}}}\overline{\nabla} \cdot \underline{\dot{\overline{q}}}$

Eqn. of state: $\overline{p} = \dfrac{\gamma - 1}{\gamma}\overline{\rho}\left(\dfrac{\overline{h}_0}{M_{\text{ref}}^2} - \dfrac{1}{2}\left|\underline{\overline{V}}\right|^2\right)$

Amplifies the fact that, as long as the reference Reynolds number, Mach number, Prandtl number and specific heat ratio are all maintained same, the non-dimensional flow quantities in geometrically similar flows are identical

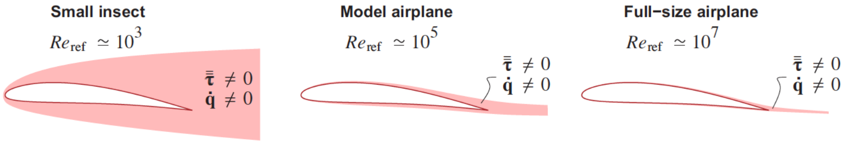
High Reynolds number aerodynamic flows

Typical aerodynamic flows have very high reference Reynolds numbers

Viscous and heat conduction terms, being scaled by $1/Re_{\text{ref}}$ must be negligible over most of the flow field

Exception occurs very close to the body where velocity and temperature gradients are very high

Viscous and heat conduction effects are confined to very thin boundary layers and wakes, collectively called shear layers or viscous regions



Typical aerodynamic flows have relatively very thin viscous regions

The entire outer flow is inviscid, which tremendously simplifies modelling

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Adiabatic Flows

Adiabatic flows

$$\text{Energy eqn:} \quad \rho \frac{\mathrm{D} h_0}{\mathrm{D} t} = \frac{\partial p}{\partial t} + \nabla \cdot (\underline{\underline{\tau}} \cdot \underline{V}) + \rho \underline{f} \cdot \underline{V} + \dot{q}_{\mathcal{V}} - \nabla \cdot \underline{\dot{q}}$$

For typical aerodynamic flows, we have

- Steady flow: $\partial/\partial t = 0$
- No volume heating: $\dot{q}_{\mathcal{V}} = 0$
- Negligible volume work: $\underline{f} \cdot \underline{V} \approx 0$
- Negligible viscous stress outside of viscous layers: $\underline{\underline{\tau}} \approx 0$
- Negligible heat conduction outside of viscous layers: $\underline{\dot{q}} \approx 0$

Under these circumstances, convective enthalpy equation reduces to

$$\frac{\mathrm{D} h_0}{\mathrm{D} t} = 0 \qquad \Longrightarrow h_0 = \text{ constant } = h_{0_\infty} \quad \text{along pathlines}$$

This gives interpretation of enthalpy in terms of heat content of flow, which is constant under adiabatic conditions

Isentropic Flows

Entropy transport

Equation governing transport (or conservation) of entropy, s, can be derived from the other transport (i.e., convective) equations

First we use a Gibb's thermodynamic relation between changes in entropy and those of other thermodynamic quantities experienced by a system

$$Tds = de + pd(1/\rho) = d(e_0 - V^2/2) - \rho^{-2} p d\rho$$

Rate of change experienced by fluid element (system) is material derivative:

$$T\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\mathrm{D}e_0}{\mathrm{D}t} - \underline{V} \cdot \frac{\mathrm{D}\underline{V}}{\mathrm{D}t} - \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}$$

Combining ρ^{-1} times energy equation, etc., we have

Entropy transport

$$T\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{\rho}\left[(\underline{\underline{\tau}} \cdot \nabla) \cdot \underline{V} + \dot{q}_{\mathcal{V}} - \nabla \cdot \underline{\dot{q}}\right]$$

Isentropic flows

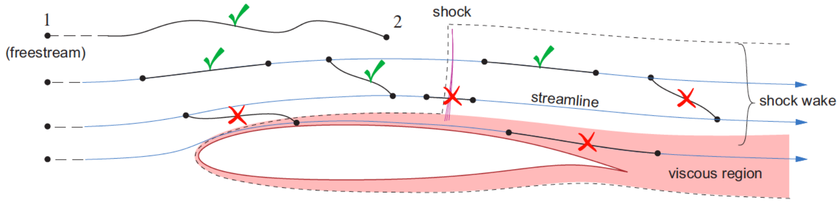
Entropy transport: $$T\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{1}{\rho}\left[(\underline{\underline{\tau}}\cdot\nabla)\cdot\underline{V} + \dot{q}_{\mathcal{V}} - \nabla\cdot\underline{\dot{q}}\right]$$

Whenever all three terms on RHS are negligible, we have $\mathrm{D}s/\mathrm{D}t = 0$; i.e.,

$$s = \text{constant, for each fluid element}$$

Flow regions which are both inviscid and adiabatic must also be isentropic

This is typical situation outside viscous layers & shocks, & w/o combustion



Valid and invalid paths for applying isentropic relation between 2 points

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Low Speed and Incompressible Flows

Density changes, Mach no. and incompressible flow

Estimates of typical changes $\Delta(\,)$ of flow quantities along pathlines:

From ideal gas law:	$\gamma\Delta p \sim (\gamma-1)(h\Delta\rho + \rho\Delta h)$
From momentum eqn.:	$\Delta p \sim -\rho V \Delta V$
From total enthalpy defn.:	$\Delta h \sim \Delta h_0 - V\Delta V$
From Mach no. defn.:	$V^2/h = (\gamma-1)M^2$
Finally, eliminating Δp and Δh:	$\boxed{\dfrac{\Delta\rho}{\rho} \sim -M^2\dfrac{\Delta V}{V} - \dfrac{\Delta h_0}{h}}$

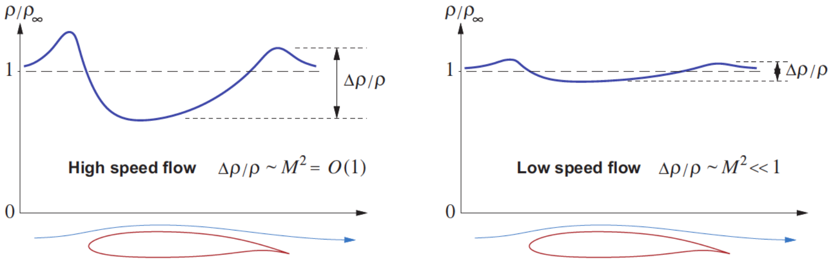
Low speed flow is one with negligibly small Mach no. everywhere ($M^2 \ll 1$)

If, in addition, the flow is also adiabatic (i.e., $\Delta h_0 \sim 0$), then

$$\frac{\Delta\rho}{\rho} \ll 1, \qquad \text{or} \quad \rho \approx \text{constant along pathline}$$

which constitutes an incompressible flow

Typical fractional density variations in aerodynamic flows



In typical aerodynamic flows, where freestream density is uniform, incompressibility (i.e., $\rho \approx$ constant along pathline) reduces to the more general statement that density is constant everywhere

$$\text{Incompressible aerodynamic flows:} \qquad \rho \approx \text{constant} = \rho_\infty$$

Then, in incompressible flows, mass conservation equation becomes

$$\mathrm{D}\rho/\mathrm{D}t + \rho\nabla\cdot\underline{V} = 0 \qquad \Longrightarrow \boxed{\nabla\cdot\underline{V} = 0}$$

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Viscous diffusion term in incompressible flows

The $\Delta h_0 \sim 0$ (or $\Delta h_0/h \ll 1$) adiabatic condition in low speed ($M^2 \ll 1$) flows implies

$$\frac{\Delta h}{h} = \frac{\Delta h_0}{h} - (\gamma - 1)M^2 \frac{\Delta V}{V} \ll 1$$

I.e., $h \approx$ constant in low speed adiabatic flows; i.e., they are isothermal too

Dynamic viscosity μ is primarily a function of temperature, so that

$$\text{Incompressible aerodynamic flows:} \qquad \mu \approx \text{constant} = \mu_\infty$$

This, along with incompressible continuity eqn. ($\nabla \cdot \underline{V} = 0$), allows great simplification in the viscous momentum term

$$\nabla \cdot \underline{\underline{\tau}} = \nabla \cdot \left[\mu \left\{ \nabla \underline{V} + (\nabla \underline{V})^{\mathsf{T}} - \frac{2}{3} \sigma \underline{\underline{I}} \right\} \right] = \mu \nabla^2 \underline{V}$$

since $\nabla \cdot (\nabla \underline{V})^{\mathsf{T}} = \nabla (\nabla \cdot \underline{V})$

Governing equations for incompressible flows

Collecting the previous results, we can write the governing equations for incompressible (i.e., low speed adiabatic) flows

Differential form – convective

Mass conservation: $\nabla \cdot \underline{V} = 0$

Momentum conservation: $\dfrac{\mathrm{D}\underline{V}}{\mathrm{D}t} = -\dfrac{\nabla p}{\rho} + \underline{f} + \nu \nabla^2 \underline{V}$

Here $\nu := \mu/\rho$ is the kinematic viscosity

The energy equation and equation of state are decoupled from the above equations, and are not needed any longer

Vorticity Transport and Irrotationality

Vorticity is a measure of the rotationality in a flow

Vorticity is mathematically the curl of velocity, $\underline{\omega} = \nabla \times \underline{V}$

Irrotational flows have zero vorticity

Rotational flow is analogous to merry-go-round; irrotational to Ferris wheel

Cengel & Cimbala, Fluid Mechanics and Its Applications, McGraw-Hill, 2006

Vorticity transport equation

Vector calculus identities for any two vector fields $\underline{a}$ and $\underline{b}$:

$$\nabla(\underline{a} \cdot \underline{b}) = \underline{a} \cdot \nabla \underline{b} + \underline{b} \cdot \nabla \underline{a} + \underline{a} \times (\nabla \times \underline{b}) + \underline{b} \times (\nabla \times \underline{a}),$$
$$\nabla \times (\underline{a} \times \underline{b}) = \underline{a} \nabla \cdot \underline{b} - \underline{a} \cdot \nabla \underline{b} - \underline{b} \nabla \cdot \underline{a} + \underline{b} \cdot \nabla \underline{a}$$

Taking curl of momentum equation, using above identities and continuity:

Helmholtz vorticity transport equation

General: $$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\underline{\omega}}{\rho}\right) = \underbrace{\frac{\underline{\omega}}{\rho} \cdot \nabla \underline{V}}_{\text{Vortex tilting/stretching}} + \underbrace{\frac{\nabla \rho \times \nabla p}{\rho^3}}_{\text{Baroclinic source}} + \underbrace{\frac{1}{\rho} \nabla \times \left(\frac{\nabla \cdot \underline{\underline{\tau}}}{\rho}\right)}_{\text{Viscous source}}$$

Incompressible: $$\frac{\mathrm{D}\underline{\omega}}{\mathrm{D}t} = \underline{\omega} \cdot \nabla \underline{V} + \nu \nabla^2 \underline{\omega}$$

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Helmholtz vorticity eqn: $\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\underline{\omega}}{\rho}\right) = \frac{\underline{\omega}}{\rho}\cdot\nabla\underline{V} + \frac{\nabla\rho\times\nabla p}{\rho^3} + \frac{1}{\rho}\nabla\times\left(\frac{\nabla\cdot\underline{\underline{\tau}}}{\rho}\right)$

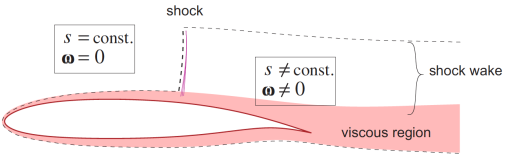
Simplifies greatly for most aerodynamic flows:

- Upstream (freestream) flow is uniform, and hence irrotational
- Viscous stresses are negligible outside of thin viscous layers and shocks

This implies $\mathrm{D}(\underline{\omega}/\rho)/\mathrm{D}t = 0 \implies \underline{\omega} = \underline{0}$

Initial irrotationality persists downstream outside viscous layer, shock wakes

These are the same requirements as those for isentropy



Isentropic flow regions are also irrotational, and vice versa

Drela, Flight vehicle aerodynamics, MIT Press, 2014

Bernoulli's Equation

Momentum conservation equation in irrotational flows

General momentum eqn.: $$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\frac{\nabla p}{\rho} + \frac{\nabla \cdot \underline{\underline{\tau}}}{\rho} + \underline{f}$$

From vector calculus, if $\underline{V}$ is curl-free, then there exists ϕ s.t. $\underline{V} = \nabla\phi$

Vector calculus identity: $$\nabla(\underline{a} \cdot \underline{b}) = \underline{a} \cdot \nabla \underline{b} + \underline{b} \cdot \nabla \underline{a} + \underline{a} \times (\nabla \times \underline{b}) + \underline{b} \times (\nabla \times \underline{a})$$

Setting $\underline{a} = \underline{b} = \underline{V}$ in above yields $\underline{V} \cdot \nabla \underline{V} = \nabla(V^2/2) - \underline{V} \times \underline{\omega}$

Gravitational (the most common) body force is $\underline{f} = -g\underline{\hat{k}} = -\nabla(gz)$

Inviscidity being a pre-condition for irrotationality, mom. eqn. becomes

Irrotational (inviscid) momentum eqn.: $$\nabla\left(\frac{\partial \phi}{\partial t}\right) + \nabla\left(\frac{V^2}{2}\right) = -\frac{\nabla p}{\rho} - \nabla(gz)$$

Incompressible, irrotational mom. eqn. – Bernoulli's eqn.

Irrotational (inviscid) momentum eqn.: $\nabla \left(\frac{\partial \phi}{\partial t} \right) + \nabla \left(\frac{V^2}{2} \right) = -\frac{\nabla p}{\rho} - \nabla (gz)$

Specializing to incompressible flows, integration between any two points in the flow yields the familiar *unsteady* Bernoulli's equation

Incompressible, irrotational mom. eqn.: $\boxed{\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gz = \text{constant}}$

This is unlike the usual constraint of applying Bernoulli's equation between two points on a streamline, which is the case for rotational flows

In aerodynamic flows, (buoyancy) effect of gravity is usually negligible

Else, we notationally redefine pressure by separating hydrostatic ($\rho g z$) part

Compressible 'Bernoulli' equation

Irrotational (inviscid) momentum eqn.: $\nabla\left(\frac{\partial\phi}{\partial t}\right)+\nabla\left(\frac{V^2}{2}\right)=-\frac{\nabla p}{\rho}-\nabla(gz)$

In case of compressible flows, we need to relate density to pressure prior to integration of irrotational momentum equation

But, conditions for irrotationality are also those for isentropy; so we can use $p/\rho^\gamma = \text{constant} = p_\infty/\rho_\infty^\gamma$, say, so that

$$\frac{\nabla p}{\rho}=\frac{\nabla p}{\rho_\infty (p/p_\infty)^{1/\gamma}}=\frac{p_\infty^{1/\gamma}}{\rho_\infty}\frac{\gamma}{\gamma-1}\nabla p^{\frac{\gamma-1}{\gamma}}=\frac{a_\infty^2}{\gamma-1}\nabla\left\{\left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}}\right\}$$

This can be integrated again between any two points in the flow

Compressible 'Bernoulli' eqn.: $\boxed{\frac{\partial\phi}{\partial t}+\frac{V^2}{2}+\frac{a_\infty^2}{\gamma-1}\left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}}+gz=\text{const}}$