Aerospace Engineering Department, IIT Bombay
AE 308 & AE 775 - Control Theory
Quiz 1 Solution

Q.1

Write state-space model for the translational mechanical system shown in Fig.1, where $x_1(t)$ and $x_2(t)$ are displacement of the masses M_1 and M_2 respectively, K

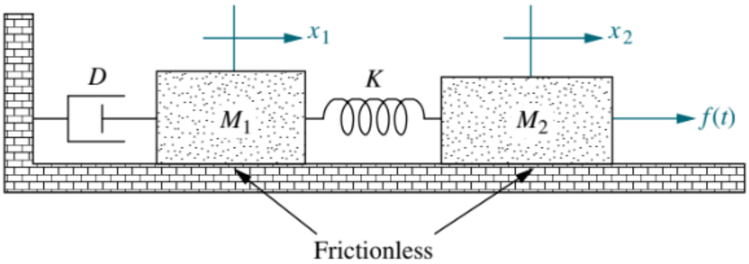

Figure 1: Spring Mass Damper System

is spring constant, D is damping coefficient, and $f(t)$ is the external force applied on the system. The outputs are $x_1(t)$ and $x_2(t)$.

Solution:

$$\begin{aligned} M_1\frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + K(x_1 - x_2) =& 0, \\ M_2\frac{d^2x_2}{dt^2} + K(x_2 - x_1) =& f(t). \end{aligned} \tag{1}$$

Choosing x_1, x_2, $x_3 = \dfrac{dx_1}{dt}$ and $x_4 = \dfrac{dx_2}{dt}$ as state-variables of the system. Rewrting (1) in terms of chosen state-variables.

$$
\begin{aligned}
\dot{x}_1 &= x_3, \\
\dot{x}_3 &= -\frac{K}{M_1}x_1 + \frac{K}{M_1}x_2 - \frac{D}{M_1}x_3, \\
\dot{x}_2 &= x_4, \\
\dot{x}_4 &= \frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t).
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M_1} & -\frac{D}{M_1} & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & -\frac{K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} f(t), \\
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix}.
\end{aligned} \tag{3}
$$

Q.2

Linearize the nonlinear equation

$$z = x^2 + 8xy + 3y^2$$

in the region defined by $2 \leq x \leq 4, \quad 10 \leq y \leq 12$.

Solution:

Consider

$$f(x, y) = z = x^2 + 8xy + 3y^2$$

Then

$$z = f(x, y) = f(\bar{x}, \bar{y}) + \left[\frac{\partial f}{\partial x}(x - \bar{x}) + \frac{\partial f}{\partial y}(y - \bar{y})\right]_{x=\bar{x},\ y=\bar{y}} + \cdots$$

where we choose $x = 3, y = 11$.

Neglecting the higher-order terms, we get

$$z - \bar{z} = K_1(x - \bar{x}) + K_2(y - \bar{y})$$

where

$$K_1 = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}, y=\bar{y}} = 2\bar{x} + 8\bar{y} = 2 \times 3 + 8 \times 11 = 94$$

$$K_2 = \left.\frac{\partial f}{\partial y}\right|_{x=\bar{x}, y=\bar{y}} = 8\bar{x} + 6\bar{y} = 8 \times 3 + 6 \times 11 = 90$$

$$\bar{z} = \bar{x}^2 + 8\bar{x}\bar{y} + 3\bar{y}^2 = 3^2 + 8 \times 3 \times 11 + 3 \times 11^2 = 636$$

Thus

$$z - 636 = 94(x - 3) + 90(y - 11)$$

Hence a linear approximation of the given nonlinear equation near the operating point is

$$z - 94x - 90y + 636 = 0$$

Q.3

Verify whether each of the following functions is linear or nonlinear and also show reason(s).

1. $ln(x(t))$
2. $\int_{-\infty}^{t} x(\tau)d\tau$

Solution:

1. Nonlinear, since it doesn't satisfy homogeneity and superposition principle.
2. Linear, since it satisfies both homogeneity and superposition principle.

Q.4

Verify whether each of the following functions is time-variant or time-invariant and also show reason(s).

1. $x(3t)$
2. $cos(5t).x(t)$

Solution:

1. Time-variant, since a time delay of the input doesn't equate to a time delay of the output.

2. Time-variant, since a time delay of the input doesn't equate to a time delay of the output.

Q.5

Consider a second order system represented by:

$$\ddot{y} + 3\dot{y} + 2y = \dot{x} + 5x$$

where $y(t)$ is the output and $x(t)$ is the input of the system.

1. Find the transfer function of the system.

2. Find the poles and zeros of the system.

Solution

1. Taking laplace transform of the given system (initial conditions are assumed to be zero while deriving transfer function).

$$\begin{aligned} s^2Y(s) + 3sY(s) + 2Y(s) &= sX(s) + 5X(s) \\ (s^2 + 3s + 2)Y(s) &= (s + 5)X(s) \\ \frac{Y(s)}{X(s)} &= \frac{s + 5}{s^2 + 3s + 2} \end{aligned}$$

2.
   - Roots of polynomial $s^2 + 3s + 2 = 0$ are $s = -1, -2$. So the poles of the system are -1 and -2.
   - Root of polynomial $s + 5 = 0$ is $s = -5$. So the only zero of the system is -5.