AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 7: Time Response

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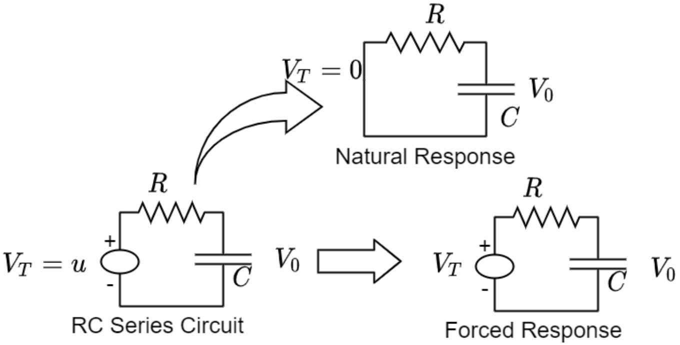
Introduction

- Once the mathematical representation of the system is obtained, the system is analyzed for its transient and steady state response.
- The output response of the system is sum of natural and forced response.



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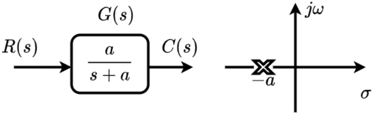
First Order Systems

First Order Systems



- A first order system without zeros can be described as shown in the figure.
- The output of the system is given as

$$C(s) = R(s)G(s).$$

- If the input is step, then Laplace transform of the input is

$$R(s) = \frac{1}{s}.$$

- The step response of the system is given by

$$C(s) = \frac{a}{s(s+a)} = \frac{A}{s} + \frac{B}{(s+a)}.$$

First Order Systems

First Order Systems

- Simplifying the $C(s)$ expression,

$$a = A(s+a) + Bs \implies A = 1, \quad B = -1$$

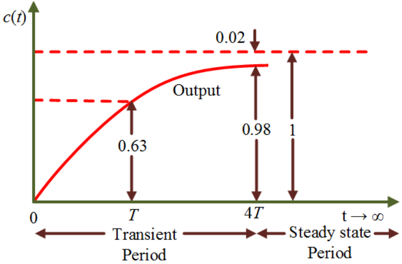
- Taking inverse Laplace transform of $C(s)$, the step response is given by

$$c(t) = c_f + c_n \implies 1 - e^{-at}$$

- The input pole at the origin generated force response $c_f = 1$.
- The system pole at $-a$ generated natural response $c_n = e^{-at}$.

First Order Systems - Time Constant

Time Constant

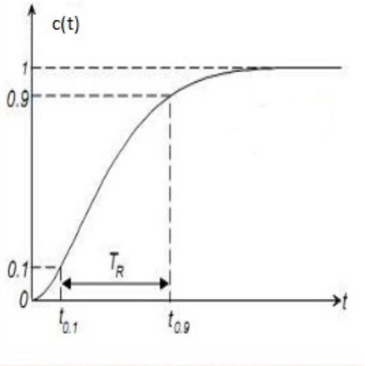


- $\frac{1}{a}$ is called **Time Constant** of a system.
- Time constant is defined as time taken by e^{-at} to decay to its 36.8% of its initial value.

- Alternatively, from $c(t)$, time constant is the time it takes for the step response to rise 63.2% of its final value.
- The time constant can be evaluated from the pole plot. Pole is located at reciprocal of time constant.
- Further the pole from origin, faster the transient response.

First Order Systems - Rise Time

Rise Time



- **Rise Time** is defined as time taken by system to go from 10% to 90% of its final value.
- The time taken by system to reach 10% of its final value is

$$c(t) = 0.1 = 1 - e^{-at_1} \implies t_1 = \frac{0.11}{a}$$

- The time taken by system to reach 90% of its final value is

$$c(t) = 0.9 = 1 - e^{-at_2} \implies t_2 = \frac{2.33}{a}$$

First Order Systems - Rise Time and Setting Time

Rise Time

- Rise time is given as

$$T_r = t_2 - t_1 = \frac{2.2}{a}$$

Settling Time

- Settling Time is defined as time taken by system to reach and stay within 2% or 5% of its final value.
- Letting $c(t) = 0.98$ and finding the corresponding t, we have

$$\begin{aligned} 0.98 =& 1 - e^{-aT_s} \\ e^{-aT_s} =& 0.02 \\ T_s =& \frac{4}{a} \end{aligned}$$

First Order Systems

Question:

- Find the output response, settling and rise time of the following system

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow C(s)$$

First Order Systems

Question:

- Find the output response, settling and rise time of the following system

$$R(s) = \frac{1}{s} \longrightarrow \boxed{\frac{1}{s+1}} \longrightarrow C(s)$$

- The output response:

$$c(t) = 1 - e^{-t}.$$

- Settling time:

$$T_s = 4s.$$

- Rise time:

$$T_r = 2.2s.$$

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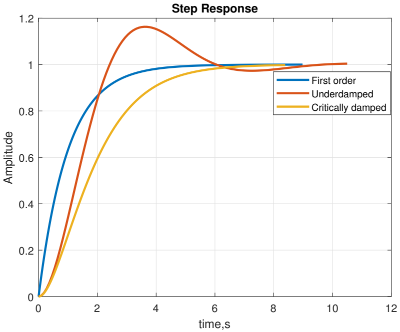
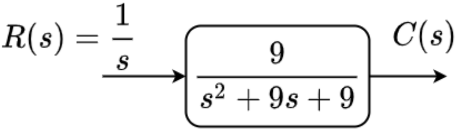
Second Order Systems - Introduction

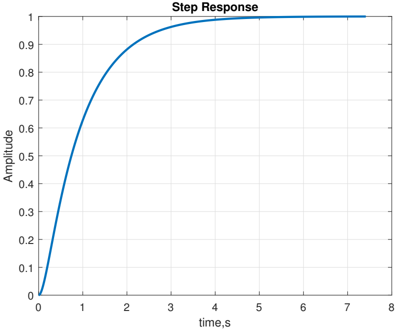


- Second order system exhibits a wide range of response that has to be addressed and analyzed.
- Varying first order system's parameter changes the speed of the system.
- Varying second order systm's parameter will change the form of the response.

Second Order Systems - Types of Response

Overdamped Response





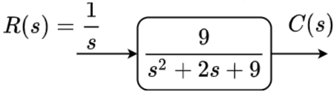
- $C(s)$ is given by

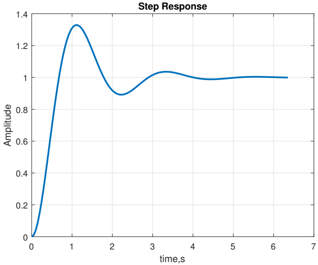
$$C(s) = \frac{9}{s(s + 7.854)(s + 1.146)}$$

- Pole at the origin generates a constant forced response.
- System poles generate a natural response whose frequency is equal to pole location.

Second Order Systems - Types of Response

Underdamped Response





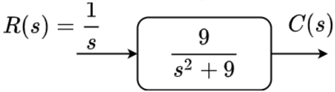
- $C(s)$ is given as

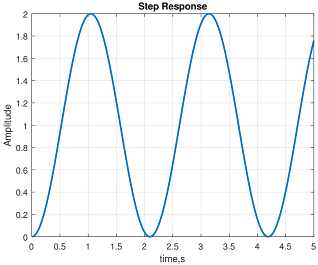
$$C(s) = \frac{9}{s(s+1+\sqrt{8}i)(s+1-\sqrt{8}i)}$$

- Real part of the poles is responsible for exponential decay frequency of signal's amplitude
- Imaginary part of the poles is responsible for frequency of the signal oscillation.

Second Order Systems - Types of Response

Undamped Response





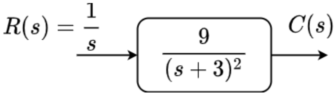
- $C(s)$ is given as

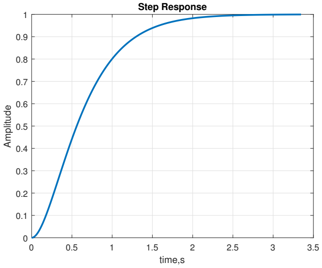
$$C(s) = \frac{9}{s(s^2+9)}$$

- The pole at the origin generates a constant forced response.
- Two system poles on imaginary axis at $\pm 3j$ generate a sinusoidal natural response.

Second Order Systems - Types of Response

Critically Damped Response





- $C(s)$ is given by

$$C(s) = \frac{9}{s(s+3)^2}$$

- Pole at the origin generates a constant forced response.
- Two poles on real axis at -3 generate a natural response consisting of exponential, whose frequency is equal to location to poles.

General Second Order Systems - Definitions

- Two physical quantities are required to describe the transient response of a second order system.
- These two quantities are called natural frequency and damping ratio.
- Natural Frequency of a second order system is defined as the frequency of oscillation without damping.
- For example, the natural frequency of RLC circuit is oscillation of circuit without resistor.
- Damping Ratio, ζ, is defined as

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}}.$$

General Second Order Systems - Definitions

- Consider a general second order system as

$$G(s) = \frac{b}{s^2 + as + b}.$$

- Without damping, the poles would be on imaginary axis and response would be pure sinusoid. For the poles to be imaginary, $a = 0$, hence,

$$G(s) = \frac{b}{s^2 + b}.$$

- The frequency of oscillation of this system is

$$s = \pm j\sqrt{b}$$

- According to the definition of natural frequency, b, is obtained as

$$\omega_n = \pm j\sqrt{b} \implies b = \omega_n^2$$

General Second Order Systems - Definitions

- Assuming the system is underdamped, the complex roots have a real part equal to $-\frac{a}{2}$.
- The magnitude of real part of complex roots is exponential decay frequency.
- The expression for damping ratio is obtained as

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}} = \frac{|\sigma|}{\omega_n} = \frac{\frac{a}{2}}{\omega_n}$$

$$a = 2\zeta\omega_n$$

- Hence the general second order system is expressed as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Underdamped Systems - Step Response

- The step response of second order underdamped system is given as

$$
\begin{aligned}
C(s) &= \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\
&= \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}
\end{aligned}
$$

- Simplifying the above expression, the step response is obtained as

$$k_1 s^2 + k_1 2\zeta\omega_n s + k_1\omega_n^2 + k_2 s^2 + k_3 s = \omega_n^2$$

- Equating the constant coefficients of L.H.S. and R.H.S., we have

$$k_1\omega_n^2 = \omega_n^2 \implies k_1 = 1$$

Underdamped Systems - Step Response

- Equating coefficients of s^2 of L.H.S and R.H.S., we have

$$k_1 + k_2 = 0 \implies k_2 = -k_1 \implies k_2 = -1$$

- Equating coefficients of s of L.H.S and R.H.S., we have

$$2k_1\zeta\omega_n + k_3 = 0 \implies k_3 = -2\zeta\omega_n$$

- Substituting these coefficients in $C(s)$, we have

$$\begin{aligned} C(s) =& \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ =& \frac{1}{s} - \frac{s + \zeta\omega_n + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

Underdamped Systems - Step Response

- Simplifying the $C(s)$ expression,

$$
\begin{aligned}
C(s) =& \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
=& \frac{1}{s} - \frac{s + \zeta\omega_n + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2} \\
=& \frac{1}{s} - \frac{s + \zeta\omega_n + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1-\zeta^2\right)}
\end{aligned}
$$

- Consider the following expression,

$$
F(s) = \frac{s + \zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2\left(1-\zeta^2\right)}
$$

Underdamped Systems - Step Response

- Let us define $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- Inverse Laplace transform of the above expression is

$$f(t) = e^{-\zeta\omega_n t}\cos\omega_d t$$

- In similar lines,

$$F(s) = \frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$
$$f(t) = e^{-\zeta\omega_n t}\sin\omega_d t$$

- Using the above inverse Laplace transform, step response is obtained as

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\omega_d t$$

Underdamped Systems - Step Response

- Simplifying $c(t)$, we have

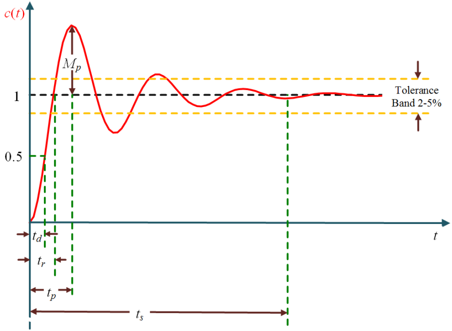
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

- Let $\zeta = \sin\phi$, then $c(t)$ can be simplified as

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\cos\phi\cos\omega_d t + \sin\phi\sin\omega_d t\right]$$

$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)}$$

Underdamped Systems - Definitions



- **Rise Time**, T_r, is defined as time required for system to move from 10% to 90% of its final value
- **Peak Time**, T_p, is defined as time required to reach first peak
- **Settling Time**, T_s, is the time required by system's transient damped oscillations to reach and stay within 2% or 5% of its final value.
- **Percent Overshoot**, $\%OS$, is the amount that system overshoots from steady state value at peak time. It is expressed as percentage of steady state value.

Underdamped Systems - Evaluation of T_p

Peak Time, T_p

- Peak time is obtained by differentiating step response $c(t)$ and finding first zero crossing after $t = 0$
- The task is simplified by differentiating $c(t)$ in frequency domain

$$\begin{aligned}
\mathcal{L}(\dot{c}(t)) = sC(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2\left(1-\zeta^2\right)} \\
\dot{c}(t) &= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t)
\end{aligned}$$

- Peak time is obtained as follows $\dot{c}(t)\big|_{t=T_p} = 0$

Underdamped Systems - Evaluation of T_p

Peak Time, T_p

- Hence, T_p is obtained as

$$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_p} \sin(\omega_d T_p) = 0$$
$$\sin(\omega_d T_p) = 0$$
$$\omega_d T_p = n\pi$$

- As we are interested in first peak, $n = 1$,

$$\boxed{T_p = \frac{\pi}{\omega_d}}$$

Underdamped Systems - Evaluation of $\%OS$

Percentage Overshoot $\%OS$

- Percentage Overshoot is defined as

$$\%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

- c_{max} is $c(t)$ evaluated at $t = T_p$. c_{max} is given as

$$\begin{aligned} c_{\text{max}} =& 1 - e^{-\zeta\omega_n T_P}\left(\cos\omega_d T_P - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_p\right) \\ =& 1 - e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\left(\cos\pi - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right) = 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \end{aligned}$$

- Final value is $c_{\text{final}} = 1$. Hence $\%OS$ is given as

$$\%OS = \left(1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} - 1\right) \times 100$$

Underdamped Systems- Evaluation of $\%OS$ & T_s

Percentage Overshoot, $\%OS$

- Percentage Overshoot, $\%OS$, is given as

$$\boxed{\%OS = 100 \times e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}}$$

Settling Time, T_s

- T_s is corresponding time for which $c(t)$ stays within $\pm 2\%$ of steady state value.
- T_s is obtained as

$$0.98 = 1 - \frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}} \cos(\omega_d T_s - \phi)$$

- Assume $\cos(\omega_d T_s - \phi) = 1$ at settling time

Underdamped Systems - Evaluation of T_s

Evaluation of T_s

- Simplifying the previous equation, we obtain

$$\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}} = 0.02$$

$$T_s = -ln\left(0.02\frac{\sqrt{1-\zeta^2}}{\zeta\omega_n}\right)$$

- Varying ζ from 0 to 1, the T_s can be approximated as

$$\boxed{T_s = \frac{4}{\zeta\omega_n}}$$

Underdamped Systems - Evaluation of T_r

Evaluation of T_r

- Approximate T_r is obtained by letting $c(t)|_{t=T_r} = 1$

$$c(T_r) = 1 = 1 - e^{-\zeta\omega_n T_r}\left[\cos\omega_d T_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_r\right]$$

- Assuming $e^{-\zeta\omega_n T_r} \neq 0$, we have

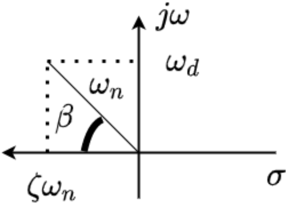
$$\cos\omega_d T_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d T_r = 0$$

$$\tan\omega_d T_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\tan\omega_d T_r = -\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}$$

Underdamped Systems - Evaluation of T_r

Evaluation of T_r



- Consider ω_d expression,

$$\omega_d^2 = \omega_n^2(1-\zeta^2)$$
$$\omega_d^2 + \omega_n^2\zeta^2 = \omega_n^2$$

- From the figure, define β as

$$\tan\beta = \frac{\omega_d}{\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}$$

- Using β expression,

$$\tan\omega_d T_r = -\tan\beta \implies \omega_d T_r = \pi - \beta$$

$$\boxed{T_r = \frac{\pi-\beta}{\omega_d}}$$

Second Order Systems - Types of Response

Types of Response	***Natural Response***	***Damping Ratio***
Overdamped	$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$	$\zeta > 1$
Underdamped	$c(t) = A e^{-\sigma t} \cos(\omega_d t + \phi)$	$0 < \zeta < 1$
Undamped	$c(t) = A \cos(\omega_n t - \phi)$	$\zeta = 0$
Critically Damped	$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$	$\zeta = 1$

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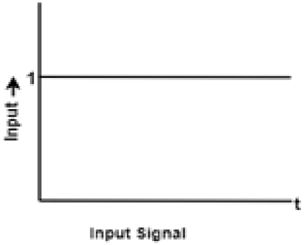
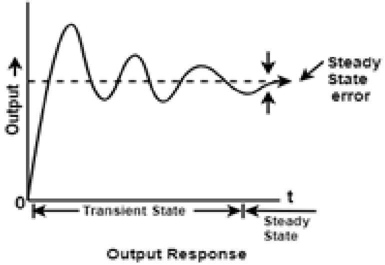
Steady State Errors - Introduction

- **Steady State Error** is defined as difference between input and output as $t \to \infty$

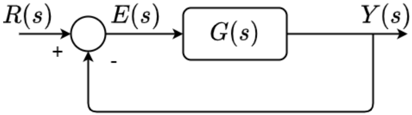




- The steady state errors we study here are the errors that arise from the configuration of system and applied input.

Steady State Errors - Unity Feedback Systems

Steady State Errors in Terms of $G(s)$



- Since the feedback $H(s) = 1$, it is called unity feedback system.
- As it is unity feedback, the error is directly difference between output and input.
- Apply final value theorem to obtain steady state error.

- The error is given as

$$E = R - Y$$
$$E = R - EG$$
$$E = \frac{R}{1+G}$$

Steady State Errors - Unity Feedback Systems

Steady State Errors in Terms of $G(s)$

- Apply final value theorem to obtain steady state error,

$$\boxed{e(\infty) = \lim_{s \to 0} \frac{sR}{1+G}}$$

- Let us use three test signals and evaluate its effect on the steady state error using above equation.

  Step Input

- The steady state error for step input is

$$\begin{aligned} e(\infty) &= \lim_{s \to 0} \frac{s\frac{1}{s}}{1+G} \\ &= \frac{1}{1+\lim\limits_{s \to 0} G} \end{aligned}$$

Steady State Errors - Unity Feedback Systems

Position Constant, K_p

- The term $\lim_{s\to 0} G$ is DC gain of forward transfer function and it is called position constant, K_p.

$$\boxed{K_p = \lim_{s\to 0} G}$$

- In summary, for a step input to unity feedback system, the steady state error will go to zero, if $K_p = \infty$.

Steady State Errors - Unity Feedback Systems

Ramp Input

- Consider ramp input to unity feedback system ($R(s) = \frac{1}{s^2}$), then error is obtained as

$$E(s) = \frac{1}{s^2(1+G)}$$

- Using final value theorem, steady state error is obtained as

$$e(\infty) = \lim_{s\to 0} \frac{1}{s(1+G)} = \frac{1}{\lim_{s\to 0} sG}$$

- The term $\lim_{s\to 0} sG$ is called velocity constant, K_v

$$\boxed{K_v = \lim_{s\to 0} sG}$$

- To have zero steady state error, K_v has to be infinity

Steady State Errors - Unity Feedback Systems

Parabolic Input

- Consider parabolic input to unity feedback system $(R(s) = \frac{1}{s^3})$, then error is obtained as

$$E(s) = \frac{1}{s^3(1+G)}$$

- Using final value theorem, steady state error is obtained as

$$e(\infty) = \lim_{s\to 0} \frac{1}{s^2(1+G)} = \frac{1}{\lim\limits_{s\to 0} s^2 G}$$

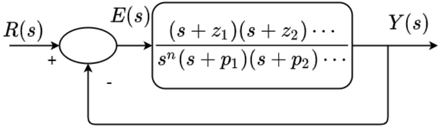
- The term $\lim\limits_{s^2\to 0} sG$ is called acceleration constant, K_v

$$\boxed{K_a = \lim_{s\to 0} s^2 G}$$

- To have zero steady state error, K_a has to be infinity

Steady State Errors - Unity Feedback Systems

Type of a System



- Consider a following Unity feedback system, where z are zeros and p are poles.
- Type of the system is value of n in the denominator.

- If $n = 0$, $n = 1$, $n = 2$, corresponding type of system is said to be Type 0, Type 1, Type 2.

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