$$\frac{\text{volume of sphere of radius } r}{\text{volume of sphere of radius } R} = \frac{r^3}{R^3}.$$

$\mathbb{P}$(a point does not lie on ~~or~~ ~~is~~ inside a sphere of radius r) $= 1 - \dfrac{r^3}{R^3}$.

So the required probability is

$\mathbb{P}$(n independently chosen points do not ~~lie~~ lie on or inside a sphere of radius r)

$$= \left(1 - \frac{r^3}{R^3}\right)^n.$$

Question 2: What happens if Ω is an unbounded region (positive half of real line, first quadrant of the higher dimensional Euclidean space). ?

Definitely our definition of geometric probability does not extend. Why ?

Question 3: Can we unify these concepts together? Is ~~it~~ it possible to give one definition of probability which covers all the aforementioned examples and more general situation ?

Probability theory begins!

Axiomatic definition of probability due to Kolmogorov.

Sample space $- \Omega$

Collection of events $- \mathcal{F}$

A set function $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$.

'Set function' is a function which ~~maps on~~ assigns value to ~~a set on~~ every set in ~~domain~~ its domain.

If $|\Omega| < \infty$, then we may choose $\mathcal{F} = 2^{\Omega}$.

But if $|\Omega| = \infty$, then 2^{Ω} is still well-defined but contains too many sets. Some of these are very difficult to realize and not so important in the analysis. So we throw away the 'bad sets' and focus on 'good sets' (well-behaved sets).

$\boxed{\sigma\text{-field}}$ A σ-field on Ω is defined to be a collection $\mathcal{F}$ of subsets of Ω such that

i) $\phi \in \mathcal{F}$, $\Omega \in \mathcal{F}$.

ii) $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

iii) if $A_i \in \mathcal{F}$ for $i = 1, 2, \ldots \infty$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

$\boxed{\text{Exercise 2.1}}$ Let $\mathcal{F}$ be a σ-field on Ω. Show that if $(A_i : i \geq 1) \in \mathcal{F}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Exercise 2.D Show that ~~there are no~~ the σ-field on Ω is not unique unless Ω is singletone.

* For any sample space Ω, there exists at least on σ-field which is $\{\phi, \Omega\}$. We call it ~~trivid~~ trivial σ-field.

* For any set Ω, 2^{Ω} is also a σ-field. (CHECK IT). ~~We call~~ It is immediate that 2^{Ω} is the largest σ-field containing Ω whereas the trivial σ-field is the smallest one.

* If Ω is finite, we take $\mathcal{F} = 2^{\Omega}$ without mentioning.

* If $\Omega = \mathbb{N}$, then $\mathcal{F}$ ~~cont~~ contains all the sets in the collection $(\{i\} : i \geq 1)$. This σ-field is also not ~~very~~ different from $2^{\mathbb{N}}$.

* If $\Omega = \mathbb{R}$, the σ-field $\mathcal{F}$ contains all the open intervals $((a,b) : a \in \mathbb{R}, b \in \mathbb{R}, a < b)$.

Question: Does the σ-field generated by open intervals ~~is~~ equal the power set $2^{\mathbb{R}}$?

OR

Does there exist a subset of $\mathbb{R}$ which is not contained in the σ-field generated by open intervals.

To answer the question, we should first note down all the sets which are in the σ-field generated by the open intervals. Then try to guess about a set which may not be contained in the collection.

Unfortunately, the answer to this question is abstract. The construction of such an pathological set is hypothetical (can not write down the subset explicitly) and is based on axiom of choice. Any advanced book on "Measure theory" has the construction of a 'non-measurable' set

The answer turns out that there is a subset of $\mathbb{R}$ which is not contained in the σ-field generated by open intervals. Therefore, the σ-field is different from $2^{\mathbb{R}}$.

σ-field generated by open intervals

It means that $\mathcal{F}$ contains all the open intervals along with ϕ and $\mathbb{R}$. It also contains the other subsets of $\mathbb{R}$ which can be obtained from the open intervals by the operations

i) Complementation ii) countable union

iii) countable intersections.

(c) Let us denote σ-field ~~generated~~ by $\mathcal{B}$ which is generated by the open intervals.

* Fix $x \in \mathbb{R}$. Then note that

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right).$$

that is, singletone $\{x\}$ can be written as a countable intersection of open intervals and therefore,

$$\{x\} \in \mathcal{B} \quad \text{for every } x \in \mathbb{R}.$$

* Fix $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $a < b$. Observe that

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right).$$

Hence, we conclude all the closed intervals are also in $\mathcal{B}$.

* Fix $a \in \mathbb{R}$. Then we can see

$$(-\infty, a) = \bigcup_{n=[a]-1}^{\infty} (-n, a)$$ where $[a]$ denotes the largest integer less than a.

(If a is not an integer $(-\infty, a) = \bigcup_{n=[a]}^{\infty} (-n, a)$).

This means $(-\infty, a) \in \mathcal{B}$ for every $a \in \mathbb{R}$.

* Similarly, you can show that $(a, \infty) \in \mathcal{B}$ for every $a \in \mathbb{R}$.

Exercise 2.3 Let $I = \{x \in \mathbb{R} : x \text{ is irrational}\}$. Show that $I \in \mathcal{B}$.

The σ-~~fie~~ field ~~B~~ $\mathcal{B}$ is called Borel σ-field on real line.

We have seen that we can define a σ-field on a given sample space Ω. Usually, the σ-field will be denoted by $\mathcal{F}$.

Event – Let Ω be a sample space and $\mathcal{F}$ be a σ-field on Ω. Then the elements in $\mathcal{F}$ are called events.

* Thanks to the definition of σ-field. Events enjoy the following properties.

i) Events are closed under complementation.

ii) Events are closed under countable unions.

iii) Events are closed under countable intersections

Axiomatic definition of probability.

Let Ω be a sample space and $\mathcal{F}$ be a σ-field on it. A Probability is a non-negative countably additive set function that is, $\mathbb{P} : \mathcal{F} \to [0,1]$ such that

i) $\mathbb{P}(A) = \mathbb{P}(\phi) = 0$ for any $A \in \mathcal{F}$ and $\mathbb{P}(\Omega) = 1$.

iii) If $(E_i : i \geq 1)$ be a sequence of disjoint sets, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

(countable additivity).

We call $(\Omega, \mathcal{F}, \mathbb{P})$ a <u>probability space</u>.

<u>Ω – sample space</u>

$\mathcal{F}$ – σ-field or "collection of events".

$\mathbb{P}$ – a set function called probability.

On a sample space Ω, the σ-field and probabilities are not unique unless Ω is singleton.

So it is customary to specify $\Omega, \mathcal{F}, \mathbb{P}$ for every problem.

<u>Some properties of probability on $(\Omega, \mathcal{F})$.</u>

i) <u>monotonicity</u>: If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
for every $A \in \mathcal{F}$ and $B \in \mathcal{F}$.

ii) subadditivity: If $A_i \in \mathcal{F}$ for $i \geq 1$, ~~then~~ and $A \in \mathcal{F}$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$, then

$$\mathbb{P}(A) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

(Caution! $\sum_{i\geq 1}^{\infty} \mathbb{P}(A_i)$ may not be finite).

Limits of a sequence of sets.

i) Consider a sequence $(E_i : i \geq 1)$ of sets such that $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_i \subseteq E_{i+1} \subseteq \cdots$, then we call the sequence $(E_i : i \geq 1)$ an ~~increasing~~ non-decreasing sequence of sets.

If $(E_i : i \geq 1)$ is an ~~increasing~~ non-decreasing sequence of events on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (that is, $E_i \in \mathcal{F}$ for every $i \geq 1$), then

$$\lim_{n\to\infty} A_n = \bigcup_{i\geq 1}^{\infty} A_i \in \mathcal{F}.$$

that is, limit of the sequence is also an event.

ii) Similarly, ~~one~~ one can define a ~~decreasing~~ non-increasing sequence of sets. Show that limit of a ~~decreasing~~ non-increasing sequence of events is an event.

For a non-increasing sequence $(E_i : i \geq 1)$, we use

$$E_n \downarrow E = \bigcap_{i\geq 1}^{\infty} E_i$$

and a non-decreasing sequence $(E_i : i \geq 1)$, we have

$$E_n \uparrow E = \bigcup_{i\geq 1}^{\infty} E_i.$$

Limit supremum

Let $(A_n : n \geq 1)$ be a sequence of sets. The set of all points $\omega \in \Omega$ that belong to A_n for infinitely many values of n is known as the limit superior of the sequence and is denoted by

$$\limsup_{n\to\infty} A_n \quad \text{or} \quad \overline{\lim_{n\to\infty}} A_n .$$

Observe that

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m .$$

Limit infimum

The set of all points that belong to A_n for all but a finite number of values of n is known as the limit inferior of the sequence $(A_n : n \geq 1)$. We denote the limit infimum by

$$\liminf_{n\to\infty} A_n \quad \text{or} \quad \underline{\lim_{n\to\infty}} A_n$$

Observe that

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{n=m}^{\infty} A_m .$$

$\boxed{\text{Exercise 2.4}}$ Show that $\limsup_{n\to\infty} A_n \supseteq \liminf_{n\to\infty} A_n$.

Note on limit supremum:

We have used the ~~f~~ phrase that $\limsup_{n\to\infty} A_n$ contains those elements which are the elements of A_n for infinitely many values of n.

This means that $\omega \in \limsup_{n\to\infty} A_n$ if and only if there exists a ~~sequence~~ subsequence $(A_{n_k} : k \geq 1)$ such that $\omega \in A_{n_k}$ for all $k \geq 1$.

$\boxed{\text{Example 3}}$ Let $A_{2m} = \{2\}$ for all $m \geq 1$ and

$A_{2m+1} = \{1\}$ for all $m = 0, 1, 2, \ldots$ Then

$\therefore \limsup_{n\to\infty} A_n = \{1, 2\}$.

Note on limit infimum:

We have said that limit infimum of a sequence of sets contains those elements which are contained in all ~~oo~~ of the sets except a finitely many of them.

This means $\omega \in \liminf_{n\to\infty} A_n$ if and only if there exists a finite integer N_ω (depends on ω and is not uniform for all elements in $\liminf_{n\to\infty} A_n$) such that $\omega \in A_n$ for all $n = N_\omega, N_\omega + 1, \ldots$