AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 15: Polar Plot

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Introduction - Bode Drawbacks

Drawbacks

- Bode plot consists of two graphics, which need to be interpreted together.
- However, in some cases, there is a need to see complete frequency response in a single graphic and Nyquist plot addresses this need.
- It is a plot of Imag$[G(j\omega)]$ versus Re$[G(j\omega)]$ in 2-D complex plane as ω varies from $-\infty$ to $+\infty$, where unity feedback is considered.
- Polar plot is the plot from 0 to ∞ and hence is a subset of the Nyquist plot.

Introduction

- The Polar plot is a plot, drawn between the magnitude and the phase angle of $G(j\omega)$ by varying ω from 0 to ∞

$$G(j\omega) = |G(j\omega)|\angle G(j\omega)$$

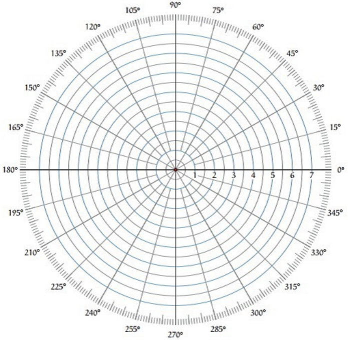

Figure: *Source - "https://www.tutorialspoint.com/"*

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Polar Plot - Construction

Drawing Polar Plots

- Substitute, $s = j\omega$ in the open loop transfer function
- Write the expressions for magnitude and the phase of $G(j\omega)$
- Find the starting magnitude and the phase of $G(j\omega)$ by substituting $\omega = 0$.
- The polar plot starts with this magnitude and the phase angle.
- Find the ending magnitude and the phase of $G(j\omega)$ by substituting $\omega = \infty$.
- The polar plot ends with this magnitude and the phase angle.
- Check whether the polar plot intersects the real axis, by making the imaginary term of $G(j\omega)$ equal to zero and find the value(s) of ω.

Polar Plot - Construction

Drawing Polar Plots Contd..

- Check whether the polar plot intersects the imaginary axis, by making real term of $G(j\omega)$ equal to zero and find the value(s) of ω
- For drawing polar plot more clearly, find the magnitude and phase of $G(j\omega)$ by considering the other value(s) of ω

Polar Plot - Example

Example: Consider the open loop transfer function with unity feedback and draw its polar plot

$$G(s) = \frac{1}{1 + 2s}$$

Polar Plot - Example

Example: Consider the open loop transfer function with unity feedback and draw its polar plot

$$G(s) = \frac{1}{1+2s}$$

Solution:

- Substitute, $s = j\omega$ in the open loop transfer function,

$$G(j\omega) = \frac{1}{1+2j\omega} = \frac{1}{1+4\omega^2} + j\frac{-2\omega}{1+4\omega^2}$$

- The magnitude of the open loop transfer function is,

$$|G(j\omega)| = M = \frac{1}{\sqrt{4\omega^2+1}}$$

- The phase angle of the open loop transfer function is,

$$\angle G(j\omega) = \phi = -\tan^{-1} 2\omega$$

Polar Plot - Example

- The start of plot i.e. $\omega = 0$,

$$M = \frac{1}{\sqrt{1+0}} = 1, \qquad \phi = -\tan^{-1} 0 = 0$$

- The end of plot i.e. $\omega = \infty$,

$$M = \frac{1}{\sqrt{1+\infty}} = 0, \qquad \phi = -\tan^{-1} \infty = -90^\circ$$

- Where the plot crosses the real axis, i.e. $\text{Imag}(G(j\omega)) = 0$,

$$\frac{-2\omega}{1+4\omega^2} = 0 \implies \omega = 0, \infty$$

- Where the plot crosses the imaginary axis, i.e. $\text{Re}(G(j\omega)) = 0$,

$$\frac{1}{1+4\omega^2} = 0 \implies \omega = \infty$$

Polar Plot - Example

- At $\omega = 0.5$,

$$M = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}, \qquad \phi = -\tan^{-1} 1 = -45^\circ$$

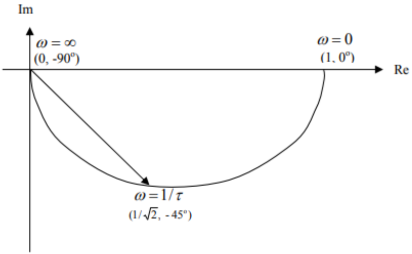

Figure: *Lecture Notes - "Analysis and Synthesis of Linear Control System", University of Saskatchewan*

Polar Plot - Example

Example: Obtain the polar plot of the following transfer function with unity feedback

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} \quad \text{for } \zeta > 0$$

Polar Plot - Example

Example: Obtain the polar plot of the following transfer function with unity feedback

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} \quad \text{for } \zeta > 0$$

Solution:

- On simplyfying,

$$G(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2\zeta\left(j\frac{\omega}{\omega_n}\right)}$$

- Magnitude of system is given by,

$$|G(j\omega)| = M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\left(\frac{\zeta\omega}{\omega_n}\right)^2}}$$

Polar Plot - Example

- Phase angle of system is given by,

$$\angle G(j\omega) = \phi = -\tan^{-1} \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

- At $\omega = 0$,

$$M = 1, \quad \phi = 0^\circ$$

- At $\omega = \infty$,

$$M = 0, \quad \phi = -180^\circ$$

- At $\omega = \omega_n$,

$$M = \frac{1}{2\zeta}, \quad \phi = -90^\circ$$

Polar Plot - Example

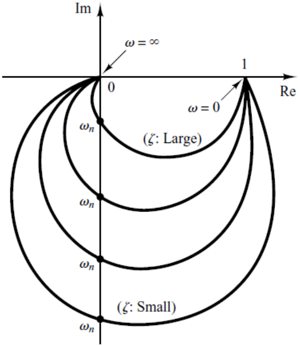

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Polar Plot - Observations

What do you observe?

Polar Plot - Observations

Observations

- For the overdamped case, as ζ increases well beyond unity, the $G(j\omega)$ locus approaches a semicircle.
- This may be seen from the fact that, for a heavily damped system, the characteristic roots are real, and one is much smaller than the other.
- Since,for sufficiently large ζ, the effect of the larger root (larger in the absolute value) on the response becomes very small, the system behaves like a first-order one.

Polar Plot - Example

Example: Obtain the polar plot of the following transfer function with unity feedback

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

Polar Plot - Example

Example: Obtain the polar plot of the following transfer function with unity feedback

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

Solution:

- Magnitude of system is given by,

$$|G(j\omega)| = M = \frac{1}{\sqrt{1 + T^2\omega^2}}$$

- Phase angle of system is given by,

$$\angle G(j\omega) = \phi = -\omega L - \tan^{-1}\omega T$$

Polar Plot - Example

- The magnitude decreases from unity monotonically and the phase angle also decreases monotonically and indefinitely, the polar plot of the given transfer function is a spiral

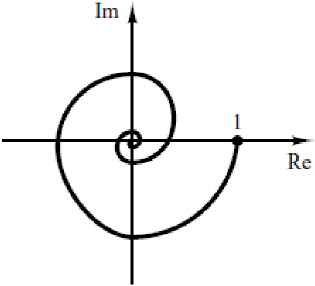

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Polar Plot - Features

Features

- Type 0 Systems
  - The starting point of the polar plot (which corresponds to $\omega = 0$) is finite and is on the positive real axis
  - The terminal point, which corresponds to $\omega = \infty$, is at the origin, and the curve is tangent to one of the axes.

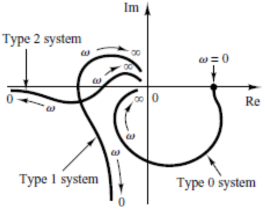

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Polar Plot - Features

Features

- Type 1 Systems
  - The $j\omega$ term in the denominator contributes -90° to the total phase angle of $G(j\omega)$ for $0 \leq \omega \leq \infty$
  - At $\omega = 0$, the magnitude of $G(j\omega)$ is infinity, and the phase angle becomes -90°.
  - At $\omega = \infty$, the magnitude becomes zero, and the curve converges to the origin and is tangent to one of the axes.

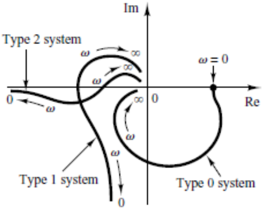

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Polar Plot - Features

Features

- Type 2 Systems
  - The $(j\omega)^2$ term in the denominator contributes -180° to the total phase angle of $G(j\omega)$ for $0 \leq \omega \leq \infty$
  - At $\omega = 0$, the magnitude of $G(j\omega)$ is infinity, and the phase angle becomes -180°.
  - At $\omega = \infty$, the magnitude becomes zero, and the curve is tangent to one of the axes.

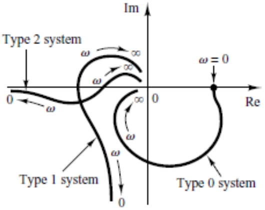

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Polar Plot - Example

Polar Plots of Simple Transfer Functions

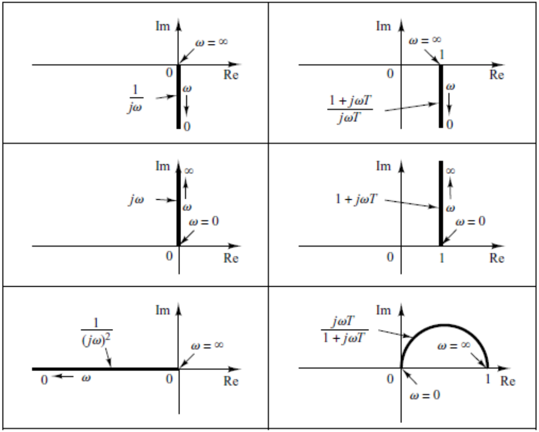

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

Polar Plot - Example

Polar Plots of Simple Transfer Functions

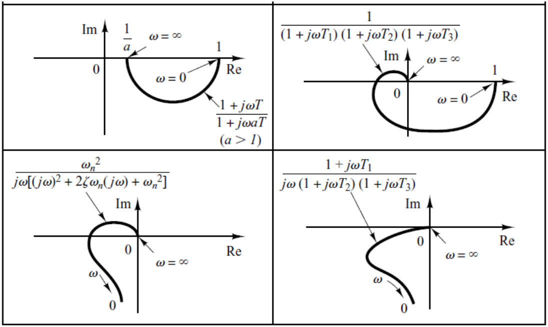

Figure: *Source -"Modern Control Engineering" by Katsuhiko Ogata*

References I

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