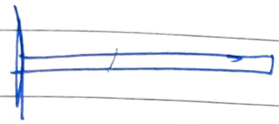
* Segue into 1- and 2-DoF systems

- designers tool
- Abstraction - reality → model
  approximation of cont sys.



make 1 DoF

$$w(x,t) = f(x)\,g(t)$$

$$T = \int_0^L \frac{1}{2} m \left(\frac{dw}{dt}\right)^2 = \int_0^L \frac{1}{2} m \left(\dot{g}(t)\right)^2 \left(f(x)\right)^2 dx$$

$$T = \frac{1}{2} \tilde{m} \left(\dot{g}(t)\right)^2$$ — eqv velocity

↳ equivalent mass

equ KE

$$V = \int_0^L \frac{1}{2} EI \left(\frac{d^2 w}{dx^2}\right)^2 dx$$

$$= \int_0^L \frac{1}{2} EI \left(g^2(t)\right) \left(f''(x)\right)^2 dx$$

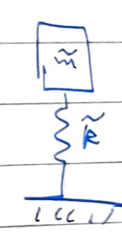
$$= \frac{1}{2} EI\, g^2(t) \int_0^L \left(f''(x)\right)^2 dx$$

$$V = \frac{1}{2} \tilde{k}\, g^2(t)$$ → eqv disp

↳ eqv spring const

eqv PE

now,



we can do this

quantitative + qualitative

dede

reality → model → approx → soln. ⇐ engg

deterministic   1D 2D   soln method

→ we deal w/ time invariant system

→ for 2DoF

$$y(x,t) = a_1 f_1(x) q_1(t) + a_2 f_2(x) q_2(t)$$

Subsn in T & V: for string

$$T = \frac{1}{2}\int_0^L m\,\dot{y}^2\,dx$$

$$= \frac{1}{2} m \int_0^L \left[A_1 \dot{q}_1^2(t) + A_2 \dot{q}_2^2(t) + A_3 \dot{q}_1 \dot{q}_2\right]$$

$$= \frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2\end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} \approx \frac{1}{2} M v^2$$

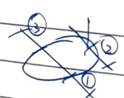
generic form of 2DoF system

general expn $E = \frac{1}{2}\left[\quad\right]_{1\times n}\left[\quad\right]_{n\times n}\left[\quad\right]_{n\times 1}$

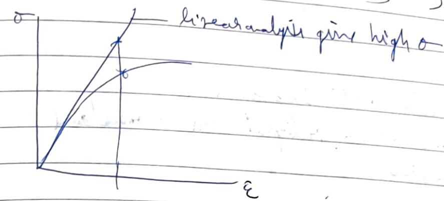
Domain $[f(x)] \in [0, L]$

global shape function

breaking of parts is Ritz methods



$$V = \frac{1}{2}\{q_1\ q_2\}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix}\begin{bmatrix} q_1 \\ q_2\end{bmatrix}$$



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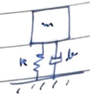
→ 1 DoF system



- formulate the 1DoF problem
- solve said problem, ~~interpret and understand~~
- interpret/understand

→ general form of 1DoF

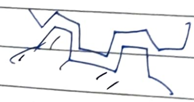
No damping

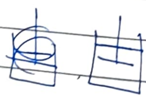
- $m\ddot{x} + kx = F$
- where x, F are f^n of t
- m, k may be f^n of t
- m, k const unless specified otherwise

Damping (Estimate)

1) viscous : solid - fluid friction
2) Friction : solid - solid external (contact)
3) Material : solid - solid (internal to materials)

how damping affects soln

gears  & friction

dampers  viscous

leaf springs

→ damped metal plates.

Most of the times we use viscous models

deal w/ "small" levels / amounts of damping

much nicer mathematically.

→ Equivalent viscous damping

$m\ddot{x} + c\dot{x} + kx = F$ $\quad m, c, k = \text{const}$

$x = \underset{F=0}{\text{homogeneous}} + \underset{F=F}{\text{particular}}$

$m\ddot{x} + c\dot{x} + kx = 0$

initial cond^n $x(0) = x_0$ $\quad \dot{x}(0) = \dot{x}_0 = v_0$

Laplace ✓ state space ✓ general form of sol^n $x(t) = A_0 e^{st}$

$(ms^2 + cs + k)A_0 e^{st} = 0$

non trivial sol^n ⇒ $ms^2 + cs + k = 0$

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$c = 0$ ⇒ undamped case

$c^2 > 4km$ real purely → not arise usually

$c^2 = 4km$ degenerate roots → purely real

$c^2 < 4km$ complex roots. → also difficult

↳ most oscillatory part comes here

↳ we the most interested here.

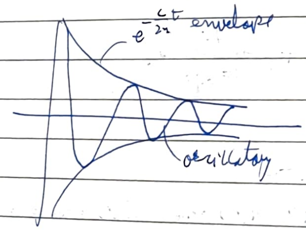
$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

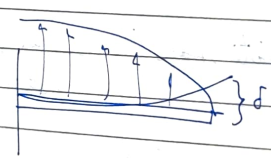
$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$ → natural ω for undamped sys.

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= e^{-c/2m\, t}\left[A_1 e^{-i(\ )t} + A_2 e^{-i(\ )t}\right]$$

exponentially decaying   oscillatory nature (purely)





$e^{i(\omega)t}$ → frequency

freq of the damped system is

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{m}\right)^2}$$

$$\omega_d = \left(\sqrt{1 - \left(\frac{c}{2m}\right)^2 \left(\frac{m}{k}\right)}\right)\sqrt{\frac{k}{m}}$$

$$\omega_d = \omega_n(\qquad)$$

$$\omega_d < \omega_n$$

$$\frac{c^2}{4m^2}\frac{m}{k} = \frac{c^2}{4km}$$

define ~~ex~~ $\xi^2 = \frac{c^2}{4km}$

$\xi = \sqrt{\frac{c^2}{4km}} = \frac{c}{2\sqrt{km}}$ damping ratio

$$\therefore \omega_d = \omega_n\left(\sqrt{1 - \xi^2}\right)$$

need to find good ref for value of ~~damp~~ damping.

$$\frac{c^2}{4m^2} - \frac{k}{m} = 0$$

$$\boxed{c^2 = 4km}$$

critically damping.

$c^2 - 4km$ > 0 non oscillatory; = 0 demarkation pt.; < 0 oscillatory

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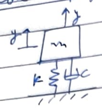
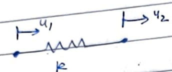
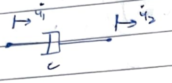
→ Viscous damping



$m\ddot{y} + c\dot{y} + ky = 0$



$k(u_2 - u_1)$



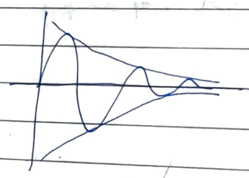
$c(\dot{u}_2 - \dot{u}_1)$

$$y(t) = e^{-c/2m\, t}\left(A_1 e^{i\sqrt{\frac{k}{m}}\sqrt{1-\zeta^2}\,t} + A_2 e^{-i\sqrt{\frac{k}{m}}\sqrt{1-\zeta^2}\,t}\right)$$

$$\zeta^2 = \frac{c^2}{4km}$$

$\omega_n = \sqrt{\frac{k}{m}}$ $\quad \omega_d = \omega_n\sqrt{1-\zeta^2}$

$$= e^{-\zeta\omega_n t}\left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right]$$



$\zeta < 1$

$\zeta = 1$ converges faster

$\zeta > 1$ gradual exponential decay

→ in practice, damping is interp. as fraction of critical damping.

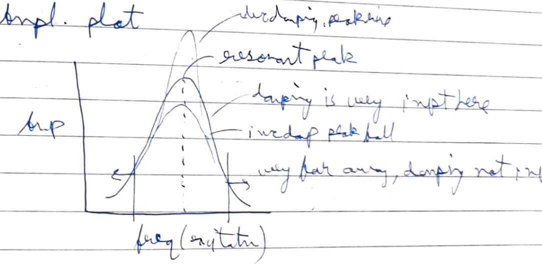
→ in practice, easiest to obtain equivalent viscous damping.

estimate damping as a purely viscous damper.

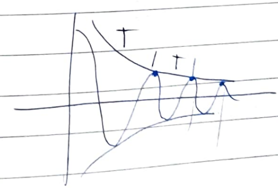
- free vibration — easy but more approx
- forced vibration — not so easy, requires specific equipment & knowledge of the system but more accurate.

half power, bandwidth, resonance peak, ...

→ freq ampl. plot



$$y(t) = e^{-\zeta\omega_n t}\left[A_1\cos\omega_d t + A_2\sin\omega_d t\right]$$



→ how to obtain estimate of ζ?

rate of change at peaks → ζ info get

$e^{-\zeta\omega_n t}\cos(\omega_d t+\phi)$ for given initial co, A ϕ ...

y_1

y_2

$\vdots$

y_n

$$\frac{y_1}{y_2}=\frac{e^{-\zeta\omega_n t}}{e^{-\zeta\omega_n 2t}}$$

for only peaks $\cos = 1$

$$\frac{y_1}{y_n}=\frac{e^{-\zeta\omega_n T}}{e^{-\zeta\omega_n nT}}=e^{\zeta(\omega_n(n-1)T)}$$

take log get estimate

T = period of oscillation of damped system

If freq of damped system is ω_d

$$T=\frac{2\pi}{\omega_d}=\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\zeta\omega_n(n-1)T=\zeta\omega_n(n-1)\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}=\frac{2\pi(n-1)\zeta}{\sqrt{1-\zeta^2}}$$

take $\log_e$ or ln of response

$$\ln\left(\frac{y_j}{y_{j+n}}\right)=\frac{2\pi(n-1)\zeta}{\sqrt{1-\zeta^2}}$$

j ⇒ peak of cycle j

for small ζ,

$$\frac{1}{\sqrt{1-\zeta^2}}=(1-\zeta^2)^{-1/2}=1+\frac{\zeta^2}{2}$$

or $\zeta<<1$ $\quad 1-\zeta^2\approx 1$

$$\zeta=\frac{\delta}{2\pi(n-1)} \quad \text{where } \delta=\ln\left(\frac{y_j}{y_{j+n}}\right)$$

this approach called, logarithmic damping

→ assignment

1DOF system

M, K given to sim

estimate ζ or C.

Hands on.

doable using simple constructs

estimate ζ, prove it is correct.

If there is a deviation from expectation, discuss

7

6/10/22

→ friction damping
→ Forced response
→ 17th oct quiz #2 (either on 17th or 31st)
→ classes end on 8th Nov

→ prev lecture
- viscous damping

dashpots

- friction

wedge scratch the surface

- material damping
- no symbol, but

seen in books.

→ Friction damping

friction b/w 'm' & the surface. force is F_d
fricti(F_d) always opposes motion $x(t)$

$\dot{x}(t)$ ✓

$\ddot{x}(t)$

$F \propto -\dot{x}$? ✗

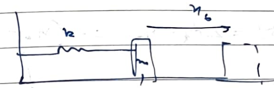
$F_d = \mu_k N$ normal rxn, coeff of fric^n

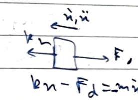
dynamic/kinetic friction

$$F_d = \mu_k N \frac{(-\dot{x})}{|\dot{x}|}$$ signum function

- note, signum f^n is not linear over the domain

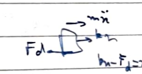
- split the problem into (2) pieces based on the sign of $\dot{x}$





$kx - F_d = m\ddot{x}$

initial cond^n $x(0) = x_0$ $\dot{x}(0) = 0$



$m\ddot{x} + kx - F_d = 0$ $\dot{x} < 0$ moving left

$m\ddot{x} + kx = F_d$ —(1)

once $\dot{x} \geq 0$, moving right

$m\ddot{x} + kx + F_d\,\mathrm{signum}(\dot{x})$

$m\ddot{x} + kx + F_d = 0$

$m\ddot{x} + kx = -F_d$ —(2)

$$\ddot{x} + \frac{k}{m}x = \frac{F_d}{m}\cdot\frac{k}{k}$$

$\ddot{x} + \omega_n^2 x = \omega_n^2 \cdot f_d$ nondimensionalise let $f_d = \frac{F_d}{k}$

↳ make chart using this

$\frac{kx}{k} = x_{eff}$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + f_d$$

$$x(0) = x_0 = A + f_d$$

$$\dot{x}(0) = 0 = -A\omega_n \cdot 0 + B\omega_n$$

$$A = x_0 - f_d, \quad B = 0$$

$$\boxed{\begin{aligned} x(t) &= (x_0 - f_d)\cos\omega_n t + f_d \\ \dot{x}(t) &= (f_d - x_0)\,\omega_n \sin\omega_n t \\ t &= \cancel{\pi\omega_n}\ \frac{\pi}{\omega_n} \end{aligned}}$$

first half cycle.

$\dot{x}(t)$ reverses direction

This solⁿ valid for $t \in [0, \pi/\omega_n]$

$$x(\pi/\omega_n) = f_d - x_0 + f_d = 2f_d - x_0 = -(x_0 - 2f_d)$$

$$\dot{x}(\pi/\omega_n) = 0$$

amplitude dec on other side due to fricⁿ.

next we

$$m\ddot{x} + kx = -F_d$$

$$\frac{\pi}{\omega_n} \quad x(0) = -(x_0 - 2f_d)$$

$$\frac{\pi}{\omega_n} \quad \dot{x}(0) = 0$$

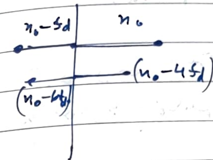
$$x(t) = (x_0 - 3f_d)\cos\omega_n t - f_d$$

again at $2\pi/\omega_n$ change of directⁿ.

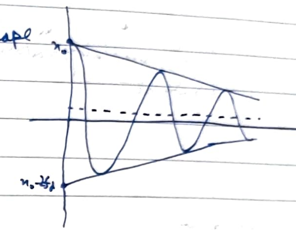
undamped system sgs at $\frac{\pi}{\omega_n}, \frac{2\pi}{\omega_n} \dots$ reverses

friction damping does not affect frequency of oscillation

→ amplitude



linear envelope



this cause difficulty when we approx using viscous.

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots$$

$$e^{-x} = 1 - x + O(x^2)$$

$$x \ll 1$$

expontial fit as linear

$$x = \zeta\omega_n t$$

$\therefore x$ changes, progressive loss of accuracy

10/10/22

→ Forced vibration

- nature/characteristic of force (static, dynamic < periodic / nonperiodic) — perpetual loads, finite duration; also cont. or discont.
- duration of load — (wrt T period)
- type of response — (decay, oscillatory w/ or w/o decay, divergent)

load = 1/10 T period → short
load = n T period → long

→ gonna do
- Fourier analysis/series
- Laplace
- convolution integral

→ Forced vibration

- Periodic forcing : trigonometric / harmonic — fourier based methods
- non-periodic : long duration, short duration etc. — convolution integral, fourier also work

→ Damped system subject to harmonic loading

$$m\ddot{x} + c\dot{x} + kx = F(t)$$
$$= F_0 \cos\Omega t$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos\Omega t$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{c}{m}\sqrt{\frac{m}{k}}\,\frac{1}{2} = \frac{1}{2}\,\frac{c}{m}\,\frac{1}{\omega_n}$$

$$\boxed{c = 2\zeta m\omega_n}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\cos\Omega t$$

$$\frac{F_0}{m} = \frac{F_0}{k}\,\frac{k}{m} = \frac{F_0\,\omega_n^2}{k}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{k}\,\omega_n^2\cos\Omega t$$

$x(t)$ = homogeneous + particular

$$= e^{-\zeta\omega_n t}\left[A\cos\omega_d t + B\sin\omega_d t\right] + x_p(t)$$

$x_p(t)$ →
will be of the form
$\alpha \cos \Omega t + \beta \sin \Omega t$

~~$-\Omega^2 x_p(t) + -\Omega\alpha \sin\Omega t + \beta\Omega \cos\Omega t$~~

$-\Omega^2 x_p(t) + (\alpha\Omega \sin\Omega t + \beta\Omega\cos\Omega t)\, 2\zeta\omega_n + \omega_n^2 x = \frac{F_0}{k}\omega_n^2 \cos\Omega t$

group all 'sin' term together
all 'cos' term together & equate.

~~$x_p(t) = \frac{A_0}{1-\left(\frac{\Omega}{\omega_n}\right)^2}$~~

$$x_p(t) = \frac{F_0}{k}\cdot\frac{1}{G}\left[\frac{2\zeta\Omega}{\omega_n}\sin\Omega t + \left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)\cos\Omega t\right]$$

where $G = \left(1-\left(\frac{\Omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\Omega}{\omega_n}\right)^2$

- we dont care about homogen system (damped) coz it $\to 0$ as $t\to\infty$.
keep looking at sys. for long t, particular dominates.

- damped system $x_h(t) \to 0$ as $t\to\infty$ only $x_p(t)$ remains
- response depends only on forcing, not on initial cond? (long time record).

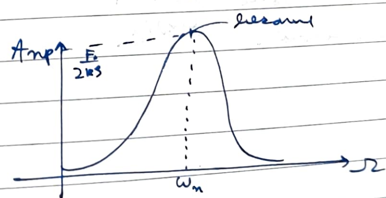
$\frac{F_0}{k}$ = eqv static disp. $= x_s$

$x_p(t) = \frac{(F_0/k)}{\sqrt{G}} \sin(\Omega t - \phi)$
delay due to damping
scaled static response

- what happens when $\Omega \to \omega_n$
$G = 4\zeta^2$
for undamped $G = 0$
$x_p(t) \to \infty$ for undamped sys.

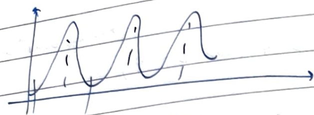
$\Omega \to \omega_n$ condn of resonance
$\Omega = \omega_n$ 'at' pt of resonance

amplitude at resonance $\frac{(F_0/k)}{2\zeta} = x_{p_{max}}$
another way to measure damping.

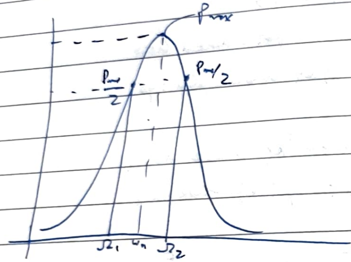


plots must be nondimensionalised
$\Omega \to \frac{\Omega}{\omega_n}$ $\frac{Amp}{stat\ Amp}$ $\frac{F_0/k}{2\zeta} \to \frac{1}{2\zeta}$

for each mod



in approx you get one of them



→ using power reduces error due to integration.
→ half power band width method.

- Integration reduces errors. $f(x)\,\Delta x$
- differentiation inc. errors. $f(x)/\Delta x$

→ work done over a cycle (Ω)

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\Omega t$$

$$x_p(t) = A\sin\Omega t + B\cos\Omega t = A\cos(\Omega t - \phi)$$

- work done by forces in system

inertial, stiffness, damping, applied force

→ multiple terms in the force freq.

$$F(t) = F_1 \cos\Omega_1 t + F_2 \cos\Omega_2 t + \dots$$

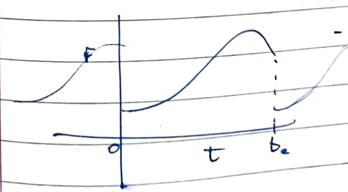
LTI → superpos^n.
or fourier series expan^n.

→ periodic but not harmonic forces

$F(t) = \sum$ fourier series expa^n

- Gibb's phenomenon

→ nonperiodic



- treat the system to being periodic

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→ Methods to table : 1 DoF system.

- periodic loads
  - one term harmonic
  - multiple terms (fourier)
  - general, smooth
  - general non-smooth

- fourier transform / series expn
- can use fourier but gives error (gibbs phenomenon)

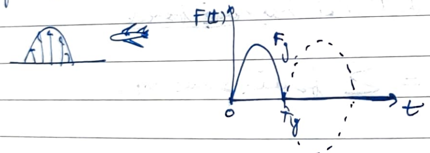
- Nonperiodic loads
  - smooth
  - general

- cheat the system into periodic 1 cycle
- (tricky) still fourier usable. but amount of error maybe unacceptable.

superposition & linear combo for multiple terms.

$x(t) = x_h(t) + x_p(t)$

- gust problem in aerospace



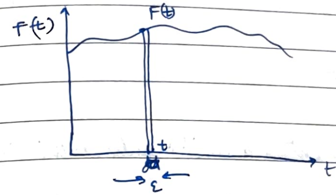
$m\ddot{x} + c\dot{x} + kx = F_g(t) \quad t \in [0, T_g]$

$m\ddot{x} + c\dot{x} + kx = 0$
$t > T_g$

we'll talk about convolution

convolution integral

- unit impulse   Force × time (Ns)
  - change in momentum metric (linear)
  - ~~small~~ time segment "small" compared to duration of force's application.



$F(t).\varepsilon = I(t)$

- force repn as impulse is the central idea.

- Determine response of the system to a unit impulse at some arbitrary time.
- scale to ~~much~~ match impulse due to force (linearity)
- sum over all impulses that make up the force. (superposn)

- define unit impulse based on the dirac delta fn.

$\delta(t-\tau) = 0 \quad t \neq \tau$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = 1$$

- we this don't make sense. ordinated of −∞ for time varying fn?

$\therefore \int_{0}^{\infty} \delta(t-\tau)\,dt = 1$

13/10/

Define $\hat{F} = \int_t^{t+\Delta t} F(t)\,dt$ impulse at $F(t)$ at time 't'.

$= \cancel{\hat{F}\cdot\delta(t)}$

unit impulse when $\hat{F}=1$

Response to unit impulse at time 't' is.

$\lim_{\Delta t\to 0} \hat{F} = \hat{f}\cdot\delta(t)$

suppose we want to obtain a unit impulse at time 't'.

$\hat{f} = \lim_{\Delta t\to 0}\int_t^{t+\Delta t} F\,dt$ ← this $=1$ needed.

as $\Delta t\to 0$, $\int_t^{t+\Delta t} F(t)\,dt = 1$ yields a dirac delta f^n.

$m\ddot{x} + c\dot{x} + kx = \delta(t-\tau)$

└ this is incorrect way to write.

instead $\int_t^{t+\Delta t} (m\ddot{x} + c\dot{x} + kx = \delta(t-\tau))\,dt$

impulse causes change in momentum

$\int_t^{t+\Delta t} m\ddot{x} = m\dot{x}(t+\Delta t) - m\cancel{\dot{x}(t)}^{\,0 \text{ for rest}}$

$\int_t^{t+\Delta t} c\dot{x}\,dt = c\cancel{x(t+\Delta t)}^{\,0\,(\text{taylor series})} - c\cancel{x(t)}^{\,0 \text{ for rest}} = 0$ based on arguments of expansion.

$\int_t^{t+\Delta t} kx\,dt = 0$ based on arguments of expan.

$\lim_{\Delta t\to 0} m\dot{x}(t+\Delta t) = \underbrace{\int_t^{t+\Delta t} F(t)\,dt}_{\text{unit impulse}} = 1$

$m\dot{x}(t^+) = 1$

$\dot{x}(t^+) = \frac{1}{m}$

* impulse at time 't' produces an incremental vel. of $1/m$ to a system at rest.
* what is response of the sys to the unit impulse.

$m\ddot{x} + c\dot{x} + kx = 0$ will be solved after the time of i...

$x(0) = 0$

$\dot{x}(0) = 1/m$

unit impulse applied at $t=0$

will change later to unit impulse applied at time ...

$x(t) = e^{-\zeta\omega_n t}[A\cos\omega_d t + B\sin\omega_d t)$

$A = 0$

$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t} B\sin\omega_d t + \omega_d e^{-\zeta\omega_n t} B\cos\omega_d t$

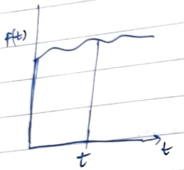
$= \cancel{-\zeta\omega_n 0}\ \omega_d B\cancel{\cos\omega_d t} = 1/m$

$B = 1/\omega_d m$

13

$A=0,\ B = \frac{1}{m\omega_d}$

$$x(t) = \frac{e^{-\zeta\omega_n t}\sin\omega_d t}{m\omega_d}$$



unit impulse applied at $t=\tau$, system at rest prior to impulse. $t \to t-\tau$

$x(t-\tau)$

$\downarrow$

$$x(t-\tau) = \frac{e^{-\zeta\omega_n(t-\tau)}\sin(\omega_d(t-\tau))}{m\omega_d}$$

If $\hat{F}$ is the mag^n of impulse due to force at time $t-\tau$. Scale up $x(t-\tau)$ using $\hat{F}$

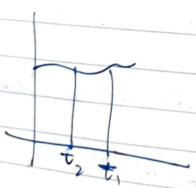
$x(t-\tau)$ is unit impulse response at time τ.

$g(t-\tau)$ formal def^n.

scale up by $\hat{F}(\tau)$

$\hat{F}(\tau)\, g(t-\tau)$

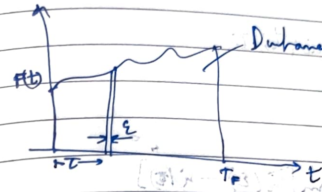
$$\int_0^t \hat{F}(\tau)\, g(t-\tau)\, d\tau$$ total response after



response would be $R(t_1) + R(t_2)$ ....

17/10/2

→ second attempt at convolution.



Duhamel's integral

Idea: apply $F(t)$ as a sequence of impulses

1) obtain response to a unit impulse at some arbitrary time $\tau \in [0, T_f]$
2) scale response for impulse due to $F(t)$
3) Response to force is sumation of responses to series of impulses.

~~Dat~~

$\hat{F} = F(\tau)\varepsilon$ — impulse of interest, impulse from τ to $\tau+\varepsilon$

assumptions
- system at rest prior to impulse.
- $x(t)$ can't be discontinuous.

$\dot{x}, \ddot{x}$ can be discontinuous.

$m\ddot{x} + c\dot{x} + kx = F(t)$ ↳ $F(t)$ width ε

$$\int_\tau^{\tau+\varepsilon} (m\ddot{x} + c\dot{x} + kx) = F(t)$$

$$\int_\tau^{\tau+\varepsilon} m\ddot{x}\varepsilon = F(\tau)\varepsilon$$

ε is same as δt, dt

$$\int_t^{t+\epsilon} m\ddot{x}(t)\,dt$$

sys at rest prior to τ

$$\tilde{t} = t - \tau$$

$$\int_0^{\epsilon} m\ddot{x}(\tilde{t})\,d\tilde{t} = m[\dot{x}(\epsilon) - \cancel{\dot{x}(0)}]$$

$$= \cancel{m\dot{x}(t+\epsilon)}$$

$$= m\dot{x}(\epsilon), \quad \tau = 0$$

∴ response prior to impulse is 0.

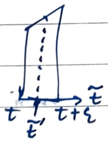
$$m\dot{x}(\epsilon) = \int_0^{\epsilon} F(\tilde{t})\,d\tilde{t} = F(0)\,\epsilon$$

↳ simplified.

$$\int_0^{\epsilon} c\dot{x}(\tilde{t})\,d\tilde{t} = c[x(\epsilon) - \cancel{x(0)}]$$

$$\lim_{\epsilon \to 0} c\,x(\epsilon) \to 0$$

$$\int_0^{\epsilon} kx(\tilde{t})\,d\tilde{t} = kx(\tilde{t}')\Delta\tilde{t}$$



$$\lim_{\epsilon \to 0} kx(\tilde{t}')\underbrace{\Delta\tilde{t}}_{\epsilon} \to 0$$

$$m\dot{x}(\epsilon) = \int_0^{\epsilon} F(\tilde{t})\,d\tilde{t}$$

$$\lim_{\epsilon \to 0} m\dot{x}(\epsilon) = \int F(\tilde{t})\,\delta(0) \qquad \int_0^{\epsilon} F(\tilde{t})\,d\tilde{t} = 1$$

$$= F(0)$$

$$m\dot{x}(0^+) = 1$$

$$\dot{x}(0^+) = 1/m$$

$$m\ddot{x}(\tilde{t}) + c\dot{x}(\tilde{t}) + kx(\tilde{t}) = 0$$

$$x(0) = 0$$

$$\dot{x}(0^+) = 1/m$$

$$x(\tilde{t}) = \frac{1}{m\omega_d} e^{-\zeta\omega_n\tilde{t}} \sin\omega_d\tilde{t} = g(\tilde{t}) \text{ called}$$

↳ response to unit impulse at $\tilde{t} = 0$

response to impulse of mag^n $F(0)\,\epsilon$

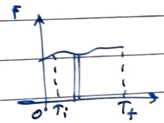
scale, ~~$x(\tilde{t}) = F$~~

$$x(\tilde{t}) \cdot F(0) \cdot \epsilon$$

$$x(\tilde{t}) \cdot F(0) \cdot \Delta t$$

$$g(\tilde{t}) \cdot F(0)\Delta t \qquad \tilde{t} \geq 0$$

summation over all impulses



$$\sum_{\text{all impulses}} g(\tilde{t})\,F(0)\,\Delta t$$

$$\int_0^{T_f} g(t-\tau)\,F(\tau)\,d\tau \quad \text{or} \quad \int_{T_i}^{T_f} g(t-T)\,F(\tau)\,d\tau$$

total response for any force.

Most basic req is $F(t)$ to be integrable $\int F(\tau)\epsilon$

→ Random signals
- noise
- earth quake
- shocks
- turbulence

randomized distribution of freq. & amplitudes

'No bias' or 'unknown bias'

such loads known as transient load.

→ Applications
- response calculations
  - ICS
  - Forcing fvcn
    - gust
    - shocks
    - steady state loads (long duration)

20/10/22

$m\ddot{x} + c\dot{x} + ~~kx~~ kx = F(t)$

$F(t) = F_0 \sin\omega t$

$m\ddot{x} + kx = 0$ free vibration.

$c\dot{x} = \mu N$, coulomb damping
$c = h/\omega$ structural damping

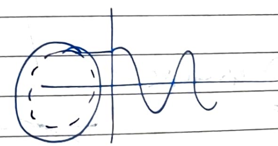
only thing to excite the system is initial cond^n in free vibra^n

$u(t) = \frac{v_0}{\omega_n} \sin\omega_n t + u_0 \cos\omega_n t$

$~~R=~~ \quad u(t) = R\sin(\omega t + \psi)$

$R = \sqrt{u_0^2 + \frac{v_0^2}{\omega_n^2}} \qquad \psi = \tan^{-1}\frac{v_0}{u_0\omega_n}$

$\omega_n = \sqrt{\frac{k}{m}}$

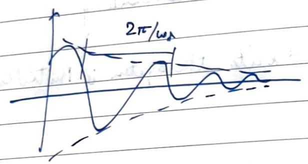


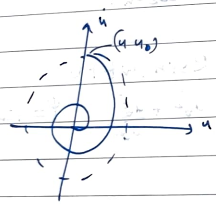
critically damped → spider-man 2 train scene.

$$u(t) = e^{-\zeta\omega t}\left\{u_0\cos\omega_d t + \left(\frac{v_0+\omega\zeta u_0}{\omega_d}\right)\sin\omega_d t\right\} = Re^{-\omega\zeta t}\sin(\omega_d t+\psi)$$

$$R = \sqrt{\frac{(v_0/\omega)^2 + 2u_0(v_0/\omega)\zeta + u_0^2}{(1-\zeta^2)}}$$

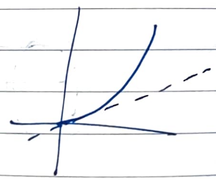
$$\cot\psi = \frac{v_0/\omega + u_0\zeta}{u_0\sqrt{1-\zeta^2}}$$





$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$ linearise

$$\delta = 2\pi\zeta$$



$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$ critically damped

$C_1 = x_0$

$C_2 = v_0 + x_0\omega_n$

$$F(t) = \frac{a_0}{2} + a_1\cos\omega t + a_2\cos 2\omega t \ldots + b_1\sin\omega t + b_2\sin 2\omega t \ldots$$

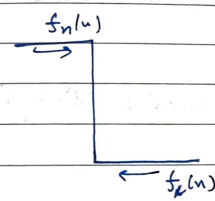
$$= \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j\cos j\omega t + b_j\sin j\omega t$$

$$a_j = \frac{2}{T}\int_0^T F(t)$$

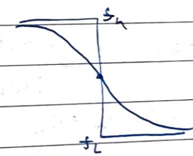
$$\sum_{n=1}^{\infty} F_n(x) = f(x)$$ Fourier convergence

valid when $f(x)$ smooth & cont.

$$\lim_{n\to\infty} |F_n(x) - f(x)| < \varepsilon$$



$$\lim_{n\to\infty} F_n(x) = \frac{1}{2}(f_L + f_R)$$



Gibbs oscillations

i) every 'x' where f is discont.

ii) magnitude of error – size of discont

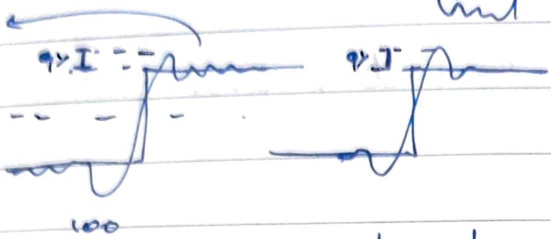
iii) gibbs – — const

$$\lim_{n\to\infty}\int_0^T \frac{f(t)\,dt}{T} = 0.089\ldots$$

overprediction

9% of error at discont are

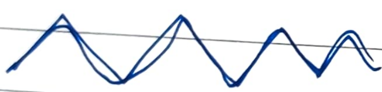
$n = 100 \longrightarrow 200$ region of oscill^n decreased
but mag^n is still 9%

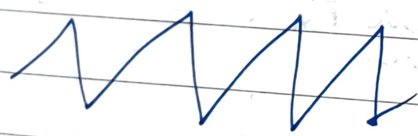


in aerospace, shocks have ⌐ type of discont.

gibbs oscill^n → high freq comp.
↓
use lowpass filter
↓
remove this oscill^n

gibbs osc only occurs when f is discont & nondiff


⇒ fourier series can be used


⇒ fourier series gibbs' oscill^n

$$x(t) = g(t) = \frac{e^{-\zeta \omega_n t}}{m \omega_d} \sin \omega_d t$$

$x_0 = 0$ $\quad \dot{x}_0 = 1/m$

Damping – viscous
Forcing – harmonic, nonharmonic (periodic) const general forcing