Lecture 6: Two-Dimensional Panel Methods

Aerodynamics

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We wish to solve for the incompressible potential flow around a two-dimensional (2D) body immersed in a freestream of speed V_∞. For repeated calculations involving the same body geometry but different operating conditions (i.e., angle of attack α), it is useful to keep the body position unchanged in some suitably-defined coordinate frame, say (x, z). Thus, the freestream makes the angle α with the positive x–axis.

We provide detailed derivations for two different panel methods: the linear vortex panel (as described in the textbook of Kuethe and Chow), and the original method of Hess and Smith involving constant source panels and uniform vortex panels.

1 Linear vortex panel method

This section details the derivation of equations arising in the application of two-dimensional (2D) linear vortex panels for modelling the potential flow over a 2D body, following the textbook of Kuethe and Chow. Figure 1 shows the setup of the problem.

The 2D body (e.g., airfoil section) is divided into m (possibly unequal) planar panels. The panels are numbered clockwise starting from the trailing edge (assuming that the leading edge is to the left of the trailing edge), and the last panel ends at the trailing edge again. The intersection of two adjacent panels is called a node. Thus, panel 1 extends from node 1 (the trailing edge) to node 2, panel 2 extends from node 2 to node 3, and so on. In general,

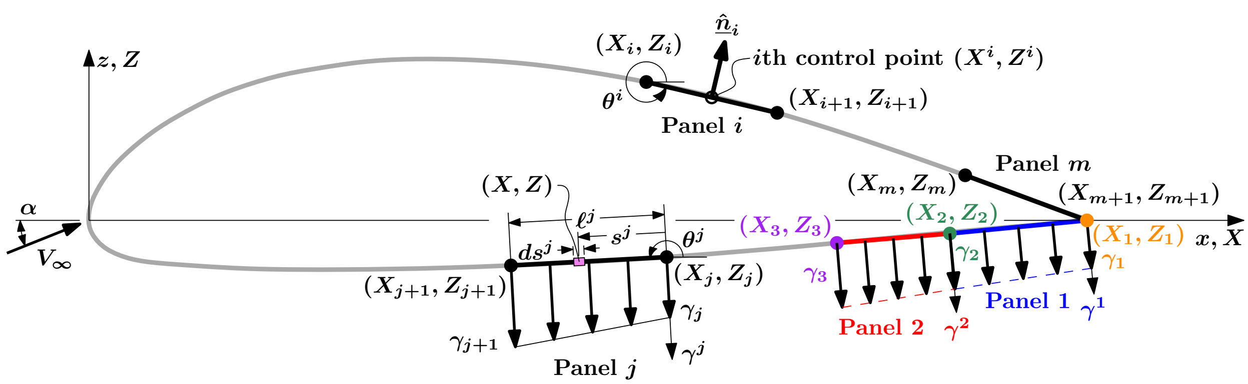

Figure 1: Replacement of an airfoil by vortex panels of linearly varying circulation density.

panel j extends from node j to node $(j+1)$. Note that there are two coincident nodes at the trailing edge, viz. node 1 and node $(m+1)$. An algorithm for choosing these nodes on the airfoil surface is described in Appendix A.

The circulation density (circulation per unit length) γ on each panel varies linearly from one end node of the panel to the other, and is continuous across the node between two adjacent panels (see figure 1). Note that this does not hold true for the trailing edge where the Kutta condition is imposed (see later). Thus, there are $(m+1)$ unknown circulation density values at the respective nodes.

The condition that the airfoil is a streamline of the flow is to be met approximately by applying the condition of zero normal velocity component at "control points" specified as the mid-points of the panels.

The Cartesian coordinates of an arbitrary point in the flow are denoted by (x, z), whereas the Cartesian coordinates of a point on the body surface (actually the panels) will be denoted by (X, Z); both are with respect to the same coordinate frame. Node indices will be specified in subscripts, whereas panel indices will be given in superscripts. Thus, the coordinates of the jth node are denoted by (X_j, Z_j). Further, let the length of the jth panel be ℓ^j, and the angle that it makes with the positive x axis be θ^j. That is,

$$\ell^j := \sqrt{(X_{j+1}-X_j)^2 + (Z_{j+1}-Z_j)^2}, \qquad \theta^j := \tan^{-1}\frac{Z_{j+1}-Z_j}{X_{j+1}-X_j}, \qquad j \in [1, m]. \tag{1}$$

Let us introduce the fractional length variable along the jth panel, s^j, such that

$$\left.\begin{array}{l} X(s^j) = X_j + s^j(X_{j+1}-X_j) = X_j + s^j\ell^j\cos\theta^j \\ Z(s^j) = Z_j + s^j(Z_{j+1}-Z_j) = Z_j + s^j\ell^j\sin\theta^j \end{array}\right\}, \qquad 0 \le s^j \le 1. \tag{2}$$

Corresponding to this notation, let the circulation density at the jth node be γ_j. The linearly varying circulation density on the jth panel is then

$$\gamma^j(s^j) = (1-s^j)\gamma_j + s^j\gamma_{j+1}, \qquad 0 \le s^j \le 1. \tag{3}$$

The velocity at point (x, z) due to a vortex of clock-wise circulation Γ at the origin is

$$\underline{V}(x, z) = \frac{\Gamma}{2\pi}\frac{z\hat{\underline{i}} - x\hat{\underline{k}}}{x^2+z^2}.$$

Now consider an infinitesimal element of the jth panel of length $\ell^j ds^j$ at $(X(s^j), Z(s^j))$ where the circulation density is $\gamma^j(s^j)$. Thus, the infinitesimal velocity at point (x, z) due to the infinitesimal vortex of circulation $\gamma^j(s^j)\ell^j ds^j$ is

$$d\underline{V}(x, z) = \frac{\gamma^j(s^j)}{2\pi}\frac{(z-Z(s^j))\hat{\underline{i}} - (x-X(s^j))\hat{\underline{k}}}{(x-X(s^j))^2 + (z-Z(s^j))^2}\ell^j ds^j. \tag{4}$$

The velocity at any point due solely to the freestream is $V_\infty(\cos\alpha\hat{\underline{i}} + \sin\alpha\hat{\underline{k}})$. Thus, the net velocity at point (x, z) due to the freestream and all the vortex panels is

$$\underline{V}(x, z) = V_\infty(\cos\alpha\hat{\underline{i}} + \sin\alpha\hat{\underline{k}})$$

$$+\sum_{j=1}^{m}\int_0^1 \frac{(1-s^j)\gamma_j + s^j\gamma_{j+1}}{2\pi}\frac{(z-Z_j-s^j\ell^j\sin\theta^j)\hat{\underline{i}}-(x-X_j-s^j\ell^j\cos\theta^j)\hat{\underline{k}}}{(x-X_j-s^j\ell^j\cos\theta^j)^2+(z-Z_j-s^j\ell^j\sin\theta^j)^2}\ell^j ds^j. \quad (5)$$

The normal component of velocity must vanish on each panel at its control point (mid point). Recalling that the ith panel makes the angle θ^i with the x–axis, the unit normal to the ith panel is given by

$$\hat{\underline{n}}^i = -\sin\theta^i\hat{\underline{i}} + \cos\theta^i\hat{\underline{k}}. \quad (6)$$

Also, the coordinates of the mid-point of the ith panel are

$$X^i = \frac{X_i + X_{i+1}}{2}, \qquad Z^i = \frac{Z_i + Z_{i+1}}{2}. \quad (7)$$

Thus, the set of m constraints that must be satisfied by the vorticity distribution on the body is

$$\underline{V}(X^i, Z^i)\cdot\hat{\underline{n}}^i = 0, \qquad \forall i \in [1, m]. \quad (8)$$

For notational convenience, let us introduce the normalized circulation variable $\gamma_j' := \gamma_j/2\pi V_\infty$. Then, noting that the normal component of freestream velocity on the ith panel is $V_\infty(\cos\alpha\hat{\underline{i}}+\sin\alpha\hat{\underline{k}})\cdot\hat{\underline{n}}^i = -V_\infty\sin(\theta^i-\alpha)$, we have the following form of the set of constraint equations

$$\sum_{j=1}^{m}\left(C^{n1}_{i,j}\gamma_j' + C^{n2}_{i,j}\gamma_{j+1}'\right) = \sin(\theta^i-\alpha), \qquad \forall i \in [1, m]. \quad (9)$$

Here, $C^{n1}_{i,j}$ and $C^{n2}_{i,j}$ are the two 'normal influence coefficients' representing the influence of the jth vortex panel on the normal velocity component at the ith control point; expressions for these are determined below.

1.1 Influence coefficients – normal velocity

From the foregoing, the expressions for the normal influence coefficients are

$$\begin{aligned}
C^{n2}_{i,j} &:= -\int_0^1 s\frac{(Z^i-Z_j-s\ell^j\sin\theta^j)\sin\theta^i + (X^j-X_j-s\ell^j\cos\theta^j)\cos\theta^i}{(X^i-X_j-s\ell^j\cos\theta^j)^2+(Z^i-Z_j-s\ell^j\sin\theta^j)^2}\ell^j ds,\\
C^{n1}_{i,j} &:= -\int_0^1 (1-s)\frac{(Z^i-Z_j-s\ell^j\sin\theta^j)\sin\theta^i + (X^j-X_j-s\ell^j\cos\theta^j)\cos\theta^i}{(X^i-X_j-s\ell^j\cos\theta^j)^2+(Z^i-Z_j-s\ell^j\sin\theta^j)^2}\ell^j ds\\
&= -\int_0^1 \frac{(Z^i-Z_j-s\ell^j\sin\theta^j)\sin\theta^i + (X^j-X_j-s\ell^j\cos\theta^j)\cos\theta^i}{(X^i-X_j-s\ell^j\cos\theta^j)^2+(Z^i-Z_j-s\ell^j\sin\theta^j)^2}\ell^j ds - C^{n2}_{i,j}.
\end{aligned}$$

In the above, we have replaced the dummy variable s^j with s for notational convenience.

The denominator of both integrands are simplified as

$$\begin{aligned}
&(X^i-X_j-s\ell^j\cos\theta^j)^2+(Z^i-Z_j-s\ell^j\sin\theta^j)^2\\
&\quad = s^2(\ell^j)^2 + 2s\ell^j\underbrace{\left\{-(X^i-X_j)\cos\theta^j-(Z^i-Z_j)\sin\theta^j\right\}}_{=:A} + \underbrace{(X^i-X_j)^2+(Z^i-Z_j)^2}_{=:B}.
\end{aligned}$$

The numerator factor of both integrands are also simplified to

$$\begin{aligned}
&(Z^i - Z_j - s\ell^j \sin\theta^j)\sin\theta^i + (X^j - X_j - s\ell^j \cos\theta^j)\cos\theta^i \\
&\quad = \underbrace{(X^i - X_j)\cos\theta^i + (Z^i - Z_j)\sin\theta^i}_{=:T} - s\ell^j \underbrace{\cos(\theta^i - \theta^j)}_{=:D}.
\end{aligned}$$

Thus, the expressions become

$$\begin{aligned}
C^{n2}_{i,j} &= \int_0^1 \frac{s\ell^j D - T}{s^2(\ell^j)^2 + 2s\ell^j A + B}\ell^j s ds, \\
C^{n1}_{i,j} &= \int_0^1 \frac{s\ell^j D - T}{s^2(\ell^j)^2 + 2s\ell^j A + B}\ell^j ds - C^{n2}_{i,j}.
\end{aligned}$$

1.1.1 Simplified expression for $C^{n2}_{i,j}$:

We simplify the polynomial fraction in the integrand of $C^{n2}_{i,j}$ as

$$\begin{aligned}
\frac{s^2(\ell^j)^2 D - s\ell^j T}{s^2(\ell^j)^2 + 2s\ell^j A + B} &= \frac{[s^2(\ell^j)^2 + 2s\ell^j A + B]D - (2s\ell^j A + B)D - s\ell^j T}{s^2(\ell^j)^2 + 2s\ell^j A + B} \\
&= D - \frac{s\ell^j(2AD + T) + BD}{s^2(\ell^j)^2 + 2s\ell^j A + B}.
\end{aligned}$$

The integral of the latter term may be found in a general form as follows:

$$\begin{aligned}
\int \frac{ex + f}{ax^2 + bx + c}dx &= \frac{e}{2a}\int \frac{2ax + b}{ax^2 + bx + c}dx + \frac{2af - eb}{2a^2}\int \frac{1}{(x + b/2a)^2 + (c/a - b^2/4a^2)}dx \\
&= \frac{e}{2a}\ln\left|ax^2 + bx + c\right| + \frac{2af - eb}{2a^2}\int \frac{1}{(x + b/2a)^2 + (c/a - b^2/4a^2)}dx.
\end{aligned}$$

The latter integral is in the standard form $\int dx/(x^2 + d^2) = d^{-1}\tan^{-1}(x/d)$. Thus, the final result is

$$\int \frac{ex + f}{ax^2 + bx + c}dx = \frac{e}{2a}\ln\left|ax^2 + bx + c\right| + \frac{2af - eb}{a\sqrt{4ac - b^2}}\tan^{-1}\frac{2ax + b}{\sqrt{4ac - b^2}}.$$

Thus, our integral evaluates to

$$\begin{aligned}
C^{n2}_{i,j} = D &- \frac{(2AD + T)\ell^j}{2(\ell^j)^2}\left[\ln\left|s^2(\ell^j)^2 + 2s\ell^j A + B\right|\right]_0^1 \\
&- \frac{2(\ell^j)^2 BD - \ell^j(2AD + T)(2\ell^j A)}{(\ell^j)^2\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\left[\tan^{-1}\frac{2(\ell^j)^2 s + 2\ell^j A}{\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\right]_0^1.
\end{aligned}$$

The term with the logarithm simplifies to

$$\left[\ln\left|s^2(\ell^j)^2 + 2s\ell^j A + B\right|\right]_0^1 = \ln\left|\frac{(\ell^j)^2 + 2\ell^j A + B}{B}\right| = \ln\left|1 + \frac{(\ell^j)^2 + 2\ell^j A}{B}\right| =: F.$$

For its coefficient, we have

$$\begin{aligned}
-2AD - T &= -2\left\{-(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j\right\}\cos(\theta^i - \theta^j) \\
&\quad - \left\{(X^i - X_j)\cos\theta^i + (Z^i - Z_j)\sin\theta^i\right\} \\
&= (X^i - X_j)\left\{2\cos\theta^j\cos(\theta^i - \theta^j) - \cos\theta^i\right\} \\
&\quad + (Z^i - Z_j)\left\{2\sin\theta^j\cos(\theta^i - \theta^j) - \sin\theta^i\right\} \\
&= (X^i - X_j)\left\{\cos(\theta^j + (\theta^i - \theta^j)) + \cos(\theta^j - (\theta^i - \theta^j)) - \cos\theta^i\right\} \\
&\quad + (Z^i - Z_j)\left\{\sin(\theta^j + (\theta^i - \theta^j)) + \sin(\theta^j - (\theta^i - \theta^j)) - \sin\theta^i\right\} \\
&= (X^i - X_j)\cos(\theta^i - 2\theta^j) - (Z^i - Z_j)\sin(\theta^i - 2\theta^j) =: Q.
\end{aligned}$$

Thus, the whole term with the logarithm is $0.5QF/\ell^j$.

To simplify the arctan term, note that there are two square root terms both involving the factor $B - A^2$. This factor simplifies as

$$\begin{aligned}
B - A^2 &= (X^i - X_j)^2 + (Z^i - Z_j)^2 - \left\{-(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j\right\}^2 \\
&= (X^i - X_j)^2\sin^2\theta^j + (Z^i - Z_j)^2\cos^2\theta^j - 2(X^i - X_j)(Z^i - Z_j)\sin\theta^j\cos\theta^j \\
&= \left[(X^i - X_j)\sin\theta^j - (Z^i - Z_j)\cos\theta^j\right]^2 =: E^2.
\end{aligned}$$

Thus, it is a perfect square. The arctan factor evaluates as

$$\begin{aligned}
\left[\tan^{-1}\frac{2(\ell^j)^2 s + 2\ell^j A}{\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\right]_0^1 &= \left[\tan^{-1}\frac{\ell^j s + A}{\sqrt{B - A^2}}\right]_0^1 = \tan^{-1}\frac{\ell^j + A}{E} - \tan^{-1}\frac{A}{E} \\
&= \tan^{-1}\frac{\dfrac{\ell^j + A}{E} - \dfrac{A}{E}}{1 + \dfrac{\ell^j + A}{E}\dfrac{A}{E}} = \tan^{-1}\frac{E\ell^j}{E^2 + A^2 + A\ell^j} = \tan^{-1}\frac{E\ell^j}{B + A\ell^j} =: G.
\end{aligned}$$

For its coefficient, we have

$$\frac{2(\ell^j)^2\left[BD - (2AD + T)A\right]}{(\ell^j)^2(2\ell^j E)} = \frac{(B - A^2)D + A(-AD - T)}{\ell^j E} = \frac{DE^2 + A(-AD - T)}{\ell^j E}.$$

But,

$$\begin{aligned}
-AD - T &= -\left\{-(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j\right\}\cos(\theta^i - \theta^j) \\
&\quad - \left\{(X^i - X_j)\cos\theta^i + (Z^i - Z_j)\sin\theta^i\right\} \\
&= (X^i - X_j)\left\{\cos\theta^j\cos(\theta^i - \theta^j) - \cos\theta^i\right\} + (Z^i - Z_j)\left\{\sin\theta^j\cos(\theta^i - \theta^j) - \sin\theta^i\right\} \\
&= (X^i - X_j)\left\{\cos\theta^j\cos(\theta^i - \theta^j) - \cos(\theta^j + (\theta^i - \theta^j))\right\} \\
&\quad + (Z^i - Z_j)\left\{\sin\theta^j\cos(\theta^i - \theta^j) - \sin(\theta^j + (\theta^i - \theta^j))\right\} \\
&= (X^i - X_j)\left\{\cos\theta^j\cos(\theta^i - \theta^j) - \cos\theta^j\cos(\theta^i - \theta^j) + \sin\theta^j\sin(\theta^i - \theta^j)\right\} \\
&\quad + (Z^i - Z_j)\left\{\sin\theta^j\cos(\theta^i - \theta^j) - \sin\theta^j\cos(\theta^i - \theta^j) - \cos\theta^j\sin(\theta^i - \theta^j)\right\} \\
&= \left\{(X^i - X_j)\sin\theta^j - (Z^i - Z_j)\cos\theta^j\right\}\underbrace{\sin(\theta^i - \theta^j)}_{=:C} = EC.
\end{aligned}$$

Thus, the overall arctan term is $-(AC + DE)G/\ell^j$.

In summary, we have $C^{n2}_{i,j} = D + 0.5QF/\ell^j - (AC + DE)G/\ell^j$.

1.1.2 Simplified expression for $C^{n1}_{i,j}$:

We note that the polynomial fraction in the integrand appearing in the first term of $C^{n1}_{i,j}$ is already in the standard form as described above. Thus,

$$\begin{aligned}\int_0^1 \frac{s(\ell^j)^2 D - \ell^j T}{s^2(\ell^j)^2 + 2s\ell^j A + B} ds &= \frac{(\ell^j)^2 D}{2(\ell^j)^2}\left[\ln\left|s^2(\ell^j)^2 + 2s\ell^j A + B\right|\right]_0^1 \\ &\quad - \frac{2(\ell^j)^2\ell^j T + (\ell^j)^2 D(2\ell^j A)}{(\ell^j)^2\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\left[\tan^{-1}\frac{2(\ell^j)^2 s + 2\ell^j A}{\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\right]_0^1 \\ &= 0.5DF - \frac{AD+T}{E}G = 0.5DF + CG, \quad [\because AD + T = -EC].\end{aligned}$$

In summary, we confirm the expressions given by Kuethe and Chow:

$$C^{n1}_{i,j} = 0.5DF + CG - C^{n2}_{i,j}, \tag{10a}$$
$$C^{n2}_{i,j} = D + 0.5QF/\ell^j - (AC + DE)G/\ell^j, \tag{10b}$$

with (noting the alternate expressions for the arctan and log terms from Appendix B)

$$\begin{aligned}
A &= -(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j, \\
B &= (X^i - X_j)^2 + (Z^i - Z_j)^2, \\
C &= \sin(\theta^i - \theta^j), \\
D &= \cos(\theta^i - \theta^j), \\
E &= (X^i - X_j)\sin\theta^j - (Z^i - Z_j)\cos\theta^j, \\
F &= \ln\left|1 + \frac{(\ell^j)^2 + 2\ell^j A}{B}\right| = 2\ln\left|\frac{r_{i,j+1}}{r_{i,j}}\right|, \\
G &= \tan^{-1}\frac{E\ell^j}{B + A\ell^j} = \tan^{-1}\frac{Z^i - Z_j}{X^i - X_j} - \tan^{-1}\frac{Z^i - Z_{j+1}}{X^i - X_{j+1}}, \\
P &= (X^i - X_j)\sin(\theta^i - 2\theta^j) + (Z^i - Z_j)\cos(\theta^i - 2\theta^j), \\
Q &= (X^i - X_j)\cos(\theta^i - 2\theta^j) - (Z^i - Z_j)\sin(\theta^i - 2\theta^j).
\end{aligned}$$

Following Kuethe and Chow, we have also defined the coefficient P in anticipation of developments to follow.

It should be noted that the integrands have a singularity when $i = j$; however, this is an integrable singularity. Note that $X^i = X_i + 0.5\ell^i\cos\theta^i$ and $Z^i = Z_i + 0.5\ell^i\sin\theta^i$. Thus, the expressions for the coefficients simplify as follows when $i = j$ (we also use results from Appendix B to evaluate the arctan and log terms in this case):

$$\begin{aligned}
A &= -0.5\ell^i\cos\theta^i\cos\theta^i - 0.5\ell^i\sin\theta^i\sin\theta^i = -0.5\ell^i, \\
B &= (0.5\ell^i\cos\theta^i)^2 + (0.5\ell^i\sin\theta^i)^2 = 0.25(\ell^i)^2, \\
C &= \sin(\theta^i - \theta^i) = 0, \\
D &= \cos(\theta^i - \theta^i) = 1,
\end{aligned}$$

$$
\begin{aligned}
E &= 0.5\ell^i \cos\theta^i \sin\theta^i - 0.5\ell^i \sin\theta^i \cos\theta^i = 0, \\
F &= \ln\left|1 + \frac{(\ell^i)^2 + 2\ell^i A}{B}\right| = \ln\left|1 + \frac{(\ell^i)^2 - (\ell^i)^2}{0.25(\ell^i)^2}\right| = 0, \\
G &= \tan^{-1}\frac{E\ell^i}{B + A\ell^i} = \tan^{-1}\frac{0}{0.25(\ell^i)^2 - 0.5(\ell^i)^2} = -\pi, \\
P &= 0.5\ell^i \cos\theta^i \sin(\theta^i - 2\theta^i) + 0.5\ell^i \sin\theta^i \cos(\theta^i - 2\theta^i) = 0, \\
Q &= 0.5\ell^i \cos\theta^i \cos(\theta^i - 2\theta^i) - 0.5\ell^i \sin\theta^i \sin(\theta^i - 2\theta^i) = 0.5\ell^i.
\end{aligned}
$$

Thus, we have

$$
C^{n1}_{i,i} = -1, \qquad C^{n2}_{i,i} = 1. \tag{11}
$$

1.2 Kutta condition

The final constraint is the Kutta condition to ensure smooth flow at the trailing edge. We use the form of the Kutta condition which requires the vorticity to vanish at the trailing edge. In the present notation, this becomes

$$
\gamma'_1 + \gamma'_{m+1} = 0. \tag{12}
$$

1.3 Solving for the vortex panels' densities

It will be recalled from eqn (9) that we have m linear equations in the $m+1$ unknowns $\left\{\gamma'_j\right\}_{j=1}^{m+1}$; these of course arose from having to ensure that the normal component of velocity vanish at the mid-point of each of the m panels. The Kutta condition completes the set of $m+1$ equations to determine the $m+1$ unknowns.

Following Kuethe and Chow, we may rewrite these in a more convenient form that brings out the linear nature of the problem:

$$
\underline{\underline{W}}\,\underline{\eta} = \underline{\zeta}, \tag{13}
$$

where the vector of unknowns is $\underline{\eta} = \left[\gamma'_1, \gamma'_2, \ldots, \gamma'_m, \gamma'_{m+1}\right]^{\mathrm{T}}$. For $i \in [1, m]$:

$$
\zeta_i = \sin(\theta^i - \alpha); \qquad W_{i,1} = C^{n1}_{i,1}; \qquad W_{i,m+1} = C^{n2}_{i,m}; \qquad W_{i,j} = C^{n2}_{i,j-1} + C^{n1}_{i,j}, \quad j \in [2, m].
$$

And, for $i = m+1$:

$$
\zeta_{m+1} = 0; \qquad W_{m+1,1} = 1; \qquad W_{m+1,m+1} = 1; \qquad W_{m+1,j} = 0, \quad j \in [2, m].
$$

1.4 Computation of surface pressure coefficient

Once the vortex panel densities have been determined above, we can proceed with the calculation of the surface pressure coefficient. In the incompressible irrotational flow assumed, Bernoulli's equation can be applied between any two points in the flow. In particular, we can relate the surface pressure p (actually the pressure anywhere in the flow) to the surface velocity $\underline{V}$ (actually the corresponding velocity) as

$$p = p_\infty + \frac{1}{2}\rho_\infty V_\infty^2 - \frac{1}{2}\rho_\infty V^2, \tag{14}$$

where V denotes the magnitude of the velocity $\underline{V}$. Then, noting that the only remaining component of velocity at the control point of the ith panel is tangential (denoted V_t^i), the pressure coefficient at this control point is

$$C_p^i = 1 - \left(\frac{V_t^i}{V_\infty}\right)^2. \tag{15}$$

Thus, our next task is to find the tangential component of velocity at a control point.

We have already found an expression for the velocity vector at an arbitrary point in the flow in eqn (5). Note that the unit tangent to the ith panel is given by

$$\underline{\hat{t}}^i = \cos\theta^i \underline{\hat{i}} + \sin\theta^i \underline{\hat{k}}. \tag{16}$$

Thus, we have the tangential component of velocity at the control point of the ith panel as

$$\begin{aligned} V_t^i = V_\infty \cos(\theta^i - \alpha) + \frac{1}{2\pi}\sum_{j=1}^{m}\int_0^1 &\left[(1-s^j)\gamma_j + s^j\gamma_{j+1}\right] \\ &\times \frac{(Z^i - Z_j - s^j\ell^j\sin\theta^j)\cos\theta^i - (X^i - X_j - s^j\ell^j\cos\theta^j)\sin\theta^i}{(X^i - X_j - s^j\ell^j\cos\theta^j)^2 + (Z^i - Z_j - s^j\ell^j\sin\theta^j)^2}\ell^j ds^j. \end{aligned}$$

This can be further rewritten as

$$\frac{V_t^i}{V_\infty} = \cos(\theta^i - \alpha) + \sum_{j=1}^{m}\left(C_{i,j}^{t1}\gamma_j' + C_{i,j}^{t2}\gamma_{j+1}'\right), \qquad \forall i \in [1, m]. \tag{17}$$

Here, $C_{i,j}^{t1}$ and $C_{i,j}^{t2}$ are the two 'tangential influence coefficients' representing the influence of the jth vortex panel on the tangential velocity component at the ith control point; expressions for these are determined below.

1.5 Influence coefficients – tangential velocity

From the foregoing, the expressions for the tangential influence coefficients are

$$\begin{aligned} C_{i,j}^{t2} &:= \int_0^1 s\frac{(Z^i - Z_j - s\ell^j\sin\theta^j)\cos\theta^i - (X^j - X_j - s\ell^j\cos\theta^j)\sin\theta^i}{(X^i - X_j - s\ell^j\cos\theta^j)^2 + (Z^i - Z_j - s\ell^j\sin\theta^j)^2}\ell^j ds, \\ C_{i,j}^{t1} &:= \int_0^1 (1-s)\frac{(Z^i - Z_j - s\ell^j\sin\theta^j)\cos\theta^i - (X^j - X_j - s\ell^j\cos\theta^j)\sin\theta^i}{(X^i - X_j - s\ell^j\cos\theta^j)^2 + (Z^i - Z_j - s\ell^j\sin\theta^j)^2}\ell^j ds \\ &= \int_0^1 \frac{(Z^i - Z_j - s\ell^j\sin\theta^j)\cos\theta^i - (X^j - X_j - s\ell^j\cos\theta^j)\sin\theta^i}{(X^i - X_j - s\ell^j\cos\theta^j)^2 + (Z^i - Z_j - s\ell^j\sin\theta^j)^2}\ell^j ds - C_{i,j}^{t2}. \end{aligned}$$

In the above, we have replaced the dummy variable s^j with s for notational convenience.

These integrals are very similar to those appearing in the normal influence coefficients. In fact, following the developments in the latter case, the denominator of both integrands are written as $s^2(\ell^j)^2 + 2s\ell^j A + B$. The common numerator factors in both integrands are simplified as

$$\begin{aligned}(Z^i - Z_j - s\ell^j \sin\theta^j)\cos\theta^i - (X^j - X_j - s\ell^j \cos\theta^j)\sin\theta^i \\ = \underbrace{(Z^i - Z_j)\cos\theta^i - (X^i - X_j)\sin\theta^i}_{=:R} + s\ell^j \underbrace{\sin(\theta^i - \theta^j)}_{=C}.\end{aligned}$$

Thus, the expressions become

$$\begin{aligned} C^{t2}_{i,j} &= \int_0^1 \frac{s\ell^j C + R}{s^2(\ell^j)^2 + 2s\ell^j A + B}\ell^j s ds, \\ C^{t1}_{i,j} &= \int_0^1 \frac{s\ell^j C + R}{s^2(\ell^j)^2 + 2s\ell^j A + B}\ell^j ds - C^{t2}_{i,j}. \end{aligned}$$

1.5.1 Simplified expression for $C^{t2}_{i,j}$:

We simplify the polynomial fraction in the integrand of $C^{t2}_{i,j}$ as

$$\begin{aligned} \frac{s^2(\ell^j)^2 C + s\ell^j R}{s^2(\ell^j)^2 + 2s\ell^j A + B} &= \frac{[s^2(\ell^j)^2 + 2s\ell^j A + B]C - (2s\ell^j A + B)C + s\ell^j R}{s^2(\ell^j)^2 + 2s\ell^j A + B} \\ &= C - \frac{s\ell^j(2AC - R) + BC}{s^2(\ell^j)^2 + 2s\ell^j A + B}. \end{aligned}$$

The integral of this form has been established previously for the normal influence coefficients; thus we have

$$\begin{aligned} C^{t2}_{i,j} = C &- \frac{(2AC - R)\ell^j}{2(\ell^j)^2}\left[\ln\left|s^2(\ell^j)^2 + 2s\ell^j A + B\right|\right]_0^1 \\ &- \frac{2(\ell^j)^2 BC - \ell^j(2AC - R)(2\ell^j A)}{(\ell^j)^2\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\left[\tan^{-1}\frac{2(\ell^j)^2 s + 2\ell^j A}{\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\right]_0^1 \\ = C &+ 0.5(R - 2AC)F/\ell^j + [A(AC - R)/E - CE]G/\ell^j. \end{aligned}$$

Here we have used the simplifications obtained earlier in deriving the normal influence coefficients.

The two terms requiring further simplification are $(R - 2AC)$ and $(AC - R)$; we have:

$$\begin{aligned} R - 2AC = &\left\{(Z^i - Z_j)\cos\theta^i - (X^i - X_j)\sin\theta^i\right\} \\ &- 2\left\{-(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j\right\}\sin(\theta^i - \theta^j) \\ = &(X^i - X_j)\left\{2\cos\theta^j\sin(\theta^i - \theta^j) - \sin\theta^i\right\} + (Z^i - Z_j)\left\{2\sin\theta^j\sin(\theta^i - \theta^j) + \cos\theta^i\right\} \\ = &(X^i - X_j)\left\{\sin(\theta^j + (\theta^i - \theta^j)) - \sin(\theta^j - (\theta^i - \theta^j)) - \sin\theta^i\right\} \\ &+ (Z^i - Z_j)\left\{-\cos(\theta^j + (\theta^i - \theta^j)) + \cos(\theta^j - (\theta^i - \theta^j)) + \cos\theta^i\right\} \end{aligned}$$

$$
\begin{aligned}
&= (X^i - X_j)\sin(\theta^i - 2\theta^j) + (Z^i - Z_j)\cos(\theta^i - 2\theta^j) =: P. \\
AC - R &= \left\{-(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j\right\}\sin(\theta^i - \theta^j) \\
&\quad - \left\{(Z^i - Z_j)\cos\theta^i - (X^i - X_j)\sin\theta^i\right\} \\
&= (X^i - X_j)\left\{-\cos\theta^j\sin(\theta^i - \theta^j) + \sin\theta^i\right\} - (Z^i - Z_j)\left\{\sin\theta^j\sin(\theta^i - \theta^j) + \cos\theta^i\right\} \\
&= (X^i - X_j)\left\{-\cos\theta^j\sin(\theta^i - \theta^j) + \sin(\theta^j + (\theta^i - \theta^j))\right\} \\
&\quad + (Z^i - Z_j)\left\{\sin\theta^j\sin(\theta^i - \theta^j) + \cos(\theta^j + (\theta^i - \theta^j))\right\} \\
&= (X^i - X_j)\left\{-\cos\theta^j\sin(\theta^i - \theta^j) + \sin\theta^j\cos(\theta^i - \theta^j) + \cos\theta^j\sin(\theta^i - \theta^j)\right\} \\
&\quad + (Z^i - Z_j)\left\{\sin\theta^j\sin(\theta^i - \theta^j) + \cos\theta^j\cos(\theta^i - \theta^j) - \sin\theta^j\sin(\theta^i - \theta^j)\right\} \\
&= \left\{(X^i - X_j)\sin\theta^j - (Z^i - Z_j)\cos\theta^j\right\}\cos(\theta^i - \theta^j) = ED.
\end{aligned}
$$

In summary, we have $C^{t2}_{i,j} = C + 0.5PF/\ell^j + (AD - CE)G/\ell^j$.

1.5.2 Simplified expression for $C^{t1}_{i,j}$:

We note that the polynomial fraction in the integrand appearing in the first term of $C^{t1}_{i,j}$ is already in the standard form as described above. Thus,

$$
\begin{aligned}
&\int_0^1 \frac{s(\ell^j)^2C + \ell^j R}{s^2(\ell^j)^2 + 2s\ell^j A + B} ds \\
&\quad = \frac{(\ell^j)^2 C}{2(\ell^j)^2}\left[\ln\left|s^2(\ell^j)^2 + 2s\ell^j A + B\right|\right]_0^1 \\
&\qquad + \frac{2(\ell^j)^2\ell^j R - (\ell^j)^2 C(2\ell^j A)}{(\ell^j)^2\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\left[\tan^{-1}\frac{2(\ell^j)^2 s + 2\ell^j A}{\sqrt{4(\ell^j)^2 B - (2\ell^j A)^2}}\right]_0^1 \\
&\quad = 0.5CF + \frac{R - AC}{E}G = 0.5CF - DG, \qquad [\because R - AC = -ED].
\end{aligned}
$$

In summary, we confirm the expressions given by Kuethe and Chow:

$$
\begin{aligned}
C^{t1}_{i,j} &= 0.5CF - DG - C^{t2}_{i,j}, && \text{(18a)} \\
C^{t2}_{i,j} &= C + 0.5PF/\ell^j + (AD - CE)G/\ell^j, && \text{(18b)}
\end{aligned}
$$

with the coefficients being as stated before.

For the case of $i = j$, note that the choice of sign convention of θ^i for a panel has ensured that the resulting unit tangent vector is along the direction of the velocity (on the outside surface of the airfoil) associated with clockwise (positive) vorticity on the panel. But, this velocity contribution at the control point is equal to half the circulation density thereat, which is $0.5(\gamma_i + \gamma_{i+1})$. Also, recall that $C^{t1}_{i,i}$ and $C^{t2}_{i,i}$ are coefficients of γ'_i and γ'_{i+1} to the tangential velocity in this direction normalized by V_∞, and that $\gamma'_i = \gamma_i/2\pi V_\infty$. Thus, we have

$$
C^{t1}_{i,i} = \pi/2, \qquad C^{t2}_{i,i} = \pi/2. \tag{19}
$$

This agrees with the specialization of G to $-\pi$ for $i = j$.

2 Hess-Smith method

The original Hess and Smith panel method used constant density source panels (each having a different density); each panel also had constant vorticity, but the vorticity was same in all the panels. Let the source density of the jth panel be q^j, and the uniform circulation density of all panels be τ. We can determine the influence coefficients in this case from the previous results for the linear vortex panel method, owing to the duality of the source and vortex.

As before, we introduce the following normalized densities for notational convenience: $q'^j := q^j/2\pi V_\infty$ and $\tau' := \tau/2\pi V_\infty$. Then, the set of m constraints that must be satisfied by the source and circulation distribution on the body to ensure that the normal velocity vanishes at the control points of each panel, is

$$\sum_{j=1}^{m} \left(\Sigma^n_{i,j} q'^j + \Lambda^n_{i,j} \tau' \right) = \sin(\theta^i - \alpha), \qquad \forall i \in [1, m]. \tag{20}$$

We also note that the tangential velocity at the control points of each panel are given by

$$\frac{V_t^i}{V_\infty} = \cos(\theta^i - \alpha) + \sum_{j=1}^{m} \left(\Sigma^t_{i,j} q'^j + \Lambda^t_{i,j} \tau' \right), \qquad \forall i \in [1, m]. \tag{21}$$

2.1 Kutta condition

As expected, we have $m + 1$ unknowns and m equations, thereby keeping the scope for applying the Kutta condition. In the Hess-Smith approach, the latter is formulated as requiring the two panels meeting at the trailing edge (panels 1 and m) to have the same tangential velocity. Not only must the magnitudes be same, but the sense of both should be the same too; i.e., both should point towards the trailing edge. Now, recall that the sense of the two panels are opposite, as the first panel is directed away from the trailing edge (its starting node) whereas the last panel is directed towards the same (its ending node). Thus, the Kutta condition, in our choice of sign convention of tangential velocities, becomes

$$V_t^m = -V_t^1. \tag{22}$$

2.2 Influence coefficients

The influence of the (constant) circulation density must match the influence of the linear circulation density found in the previous method, if the circulation density in the latter case were same at both nodes of a panel. In other words, we can specialize the previous result to the case where $\gamma'_j = \gamma'_{j+1} = \tau'$. Then, comparison of the two sets of constraint equations reveals that $\Lambda^n_{i,j} = C^{n1}_{i,j} + C^{n2}_{i,j}$, so that

$$\Lambda^n_{i,j} = 0.5DF + CG.$$

Similarly, the corresponding tangential influence coefficients become $\Lambda^t_{i,j} = C^{t1}_{i,j} + C^{t2}_{i,j}$, so that

$$\Lambda^t_{i,j} = 0.5CF - DG.$$

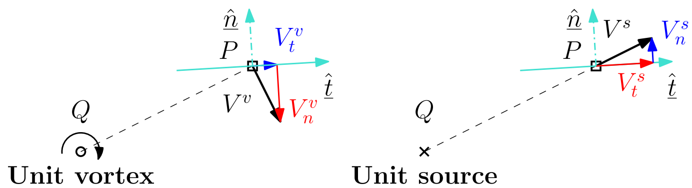

Figure 2: Duality of a line source and a line irrotational vortex.

To obtain the influence coefficients for a source panel, consider figure 2. A unit line irrotational vortex is placed at point Q, and we are interested in the corresponding normal and tangential velocity at point P of a panel. Alternately, a unit line source may be placed at the same point Q, and we would like to know its influence on the two velocity components at the same point P. Note that both singularities produce the same magnitude of velocity, and their respective directions are as shown. The well-known duality of a line source with a line irrotational vortex means that $V_n^s = V_t^v$ and $V_t^s = -V_n^v$; this is also seen in figure 2. For the influence coefficients, this means that $\Sigma_{i,j}^n = \Lambda_{i,j}^t$ and $\Sigma_{i,j}^t = -\Lambda_{i,j}^n$.

Summarizing the above, the Hess-Smith panel method's influence coefficients are as follows:

$$\Sigma_{i,j}^n = \Lambda_{i,j}^t = 0.5CF - DG, \qquad \Sigma_{i,j}^t = -\Lambda_{i,j}^n = -0.5DF - CG, \qquad i \neq j, \tag{23}$$

$$\Sigma_{i,i}^n = \Lambda_{i,i}^t = \pi, \qquad \Sigma_{i,i}^t = -\Lambda_{i,i}^n = 0. \tag{24}$$

2.3 Solving for the panels' source and circulation densities

We may rewrite the $m+1$ constraint equations in a more convenient form that brings out the linear nature of the problem:

$$\underline{\underline{Z}}\,\underline{\xi} = \underline{\chi}, \tag{25}$$

where the vector of unknowns is $\underline{\xi} = [q'^1, q'^2, \ldots, q'^{m-1}, q'^m, \tau']^{\mathrm{T}}$. The first m equations encode the normal velocity constraints, so that

$$Z_{i,j} = \Sigma_{i,j}^n, \quad \forall i, j \in [1, m]; \qquad Z_{i,m+1} = \sum_{j=1}^{m} \Lambda_{i,j}^n, \quad \forall i \in [1, m];$$

$$\chi_i = \sin(\theta^i - \alpha), \quad \forall i \in [1, m].$$

The last equation encodes the Kutta condition; recall that it read as $V_t^1 + V_t^m = 0$. Using the expressions for the tangential velocities, we can rewrite this as

$$\sum_{j=1}^{m} \left(\Sigma_{1,j}^t q'^j + \Lambda_{1,j}^t \tau'\right) + \cos(\theta^1 - \alpha) + \sum_{j=1}^{m} \left(\Sigma_{m,j}^t q'^j + \Lambda_{m,j}^t \tau'\right) + \cos(\theta^m - \alpha) = 0.$$

Thus, the entries of the last row of the matrix equation are

$$Z_{m+1,j} = \Sigma_{1,j}^t + \Sigma_{m,j}^t, \quad \forall j \in [1, m]; \qquad Z_{m+1,m+1} = \sum_{j=1}^{m} (\Lambda_{1,j}^t + \Lambda_{m,j}^t);$$

$$\chi_{m+1} = -\cos(\theta^1 - \alpha) - \cos(\theta^m - \alpha).$$

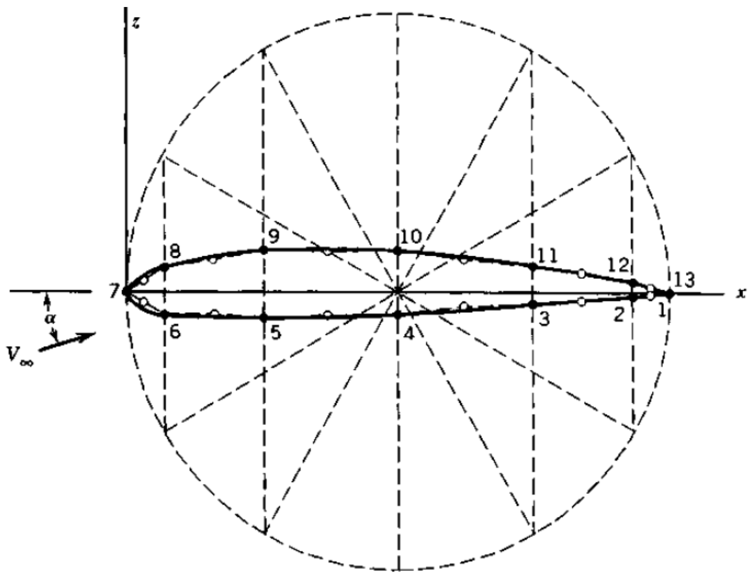

Figure 3: Determination of panel boundary points on an airfoil. Hollow circles represent control points at the centers of the panels. Adapted from the textbook of Kuethe and Chow.

APPENDIX

A Panel boundary point selection

The curvature of an airfoil is maximum at the leading edge. Changes in surface circulation density is typically greatest near the leading edge, and also near the trailing edge. Thus, we should have smaller panels near these ends. Given a fixed number of panels that we wish to use, their approximately optimal distribution can be found from the following heuristic approach, given in the textbook of Kuethe and Chow.

With reference to figure 3, let us say that we wish to discretize an airfoil's surface with 12 panels. We start by drawing a circle of diameter equal to the airfoil's chord that just envelops the airfoil as shown. It is naturally centered at the mid chord. Next, we divide the circle into 12 equal arcs, starting from the trailing edge, going clockwise. Then, the projection of these arcs' end points on the chord line gives us the x coordinates of the panels' end points. The corresponding z coordinates are found by projecting these x coordinates on to the lower and upper surfaces of the airfoil, parallel to the z-axis. This way, we automatically obtain smaller (and more numerous) panels near the leading and trailing edges.

B Alternate expressions for the arctan and log terms

An elegant geometric interpretation can be obtained for the arctan and log terms in the influence coefficients. We will refer to figure 4, where the end nodes of the jth panel are $P_j \equiv (X_j, Z_j)$ and $P_{j+1} \equiv (X_{j+1}, Z_{j+1})$, and the ith control point (on the ith panel) is $P^i \equiv (X^i, X^i)$. The point $P_{i,j}$ is the foot of the perpendicular dropped from P_{j+1} on to the line joining P_j and P^i.

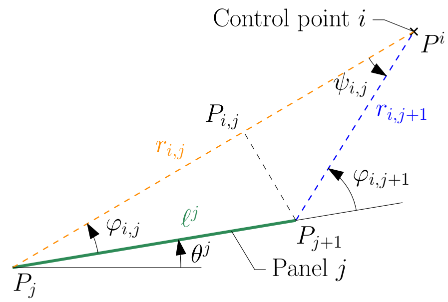

Figure 4: The jth panel and the ith control point.

B.1 The arctan term

Recall that the arctan term is $G = \tan^{-1} E\ell^j\big/(B + A\ell^j)$. Let us denote the distance from node j to control point i by $r_{i,j}$. Then the numerator factor is

$$
\begin{aligned}
E &= (X^i - X_j)\sin\theta^j - (Z^i - Z_j)\cos\theta^j = r_{i,j}\cos(\theta^j + \varphi_{i,j})\sin\theta^j - r_{i,j}\sin(\theta^j + \varphi_{i,j})\cos\theta^j \\
&= -r_{i,j}\sin\varphi_{i,j}.
\end{aligned}
$$

The B term in the denominator is recognized as

$$
B = (X^i - X_j)^2 + (Z^i - Z_j)^2 = r_{i,j}^2.
$$

The A term is seen to be

$$
\begin{aligned}
A &= -(X^i - X_j)\cos\theta^j - (Z^i - Z_j)\sin\theta^j = -r_{i,j}\cos(\theta^j + \varphi_{i,j})\cos\theta^j - r_{i,j}\sin(\theta^j + \varphi_{i,j})\sin\theta^j \\
&= -r_{i,j}\cos\varphi_{i,j}.
\end{aligned}
$$

Bringing these results together,

$$
\begin{aligned}
G &= \tan^{-1}\frac{E\ell^j}{B + A\ell^j} = -\tan^{-1}\frac{r_{i,j}\ell^j\sin\varphi_{i,j}}{r_{i,j}^2 - r_{i,j}\ell^j\cos\varphi_{i,j}} = -\tan^{-1}\frac{\ell^j\sin\varphi_{i,j}}{r_{i,j} - \ell^j\cos\varphi_{i,j}} \\
&= -\tan^{-1}\frac{\overline{P_{j+1}P_{i,j}}}{r_{i,j} - \overline{P_jP_{i,j}}} = -\tan^{-1}\frac{\overline{P_{j+1}P_{i,j}}}{\overline{P^iP_{i,j}}} = -\psi_{i,j},
\end{aligned}
$$

where $\psi_{i,j}$ is the angle subtended by the jth panel at the ith control point. Of course, it can be written as $\psi_{i,j} = \varphi_{i,j+1} - \varphi_{i,j}$. However, we can add θ^j to both angles to obtain the more convenient final expression

$$
G = \tan^{-1}\frac{Z^i - Z_j}{X^i - X_j} - \tan^{-1}\frac{Z^i - Z_{j+1}}{X^i - X_{j+1}}.
$$

A strictly algebraic derivation of the same result follows:

$$
G = \tan^{-1}\frac{E\ell^j}{B + A\ell^j} = \tan^{-1}\frac{(X^i - X_j)\sin\theta^j\ell^j - (Z^i - Z_j)\cos\theta^j\ell^j}{(X^i - X_j)^2 + (Z^i - Z_j)^2 - (X^i - X_j)\cos\theta^j\ell^j - (Z^i - Z_j)\sin\theta^j\ell^j}.
$$

But, note that $\cos\theta^j \ell^j = X_{j+1} - X_j$ and $\sin\theta^j \ell^j = Z_{j+1} - Z_j$. Thus, the denominator simplifies to

$$\begin{aligned}
&(X^i - X_j)^2 + (Z^i - Z_j)^2 - (X^i - X_j)\cos\theta^j \ell^j - (Z^i - Z_j)\sin\theta^j \ell^j \\
&\quad = (X^i - X_j)\left[(X^i - X_j) - (X_{j+1} - X_j)\right] - (Z^i - Z_j)\left[(Z^i - Z_j) - (Z_{j+1} - Z_j)\right] \\
&\quad = (X^i - X_j)(X^i - X_{j+1})\left[1 + \frac{Z^i - Z_{j+1}}{X^i - X_{j+1}}\frac{Z^i - Z_j}{X^i - X_j}\right].
\end{aligned}$$

The above form of the denominator motivates the following simplification of the numerator

$$\begin{aligned}
&(X^i - X_j)(Z_{j+1} - Z_j) - (Z^i - Z_j)(X_{j+1} - X_j) \\
&\quad = (X^i - X_j)\left\{(Z^i - Z_j) - (Z^i - Z_{j+1})\right\} - (Z^i - Z_j)\left\{(X^i - X_j) - (X^i - X_{j+1})\right\} \\
&\quad = -\left[(X^i - X_j)(Z^i - Z_{j+1}) - (Z^i - Z_j)(X^i - X_{j+1})\right] \\
&\quad = -(X^i - X_j)(X^i - X_{j+1})\left[\frac{Z^i - Z_j}{X^i - X_j} - \frac{Z^i - Z_{j+1}}{X^i - X_{j+1}}\right].
\end{aligned}$$

Thus, G is of the form $\tan^{-1}[(\eta - \zeta)/(1 + \eta\zeta)] = \tan^{-1}\eta - \tan^{-1}\zeta$. Thus, we recover the previous expression obtained geometrically.

As an aside, consider the case of $i = j$. Here, we have to remember that the flow is to the left of the panel if one traverses from the jth node to the $j + 1$th node. Thus, the control point i must be brought to the mid point of the jth panel from the left side. This consideration leads to the result that $G = -\pi$ when $i = j$.

B.2 Alternate expression for the log term

Recall that the log term is

$$F = \ln\left|1 + \frac{(\ell^j)^2 + 2\ell^j A}{B}\right| = \ln\left|\frac{r_{i,j}^2 - 2r_{i,j}\ell^j\cos\varphi_{i,j} + (\ell^j)^2}{r_{i,j}^2}\right| = \ln\left|\frac{r_{i,j+1}^2}{r_{i,j}^2}\right| = 2\ln\left|\frac{r_{i,j+1}}{r_{i,j}}\right|.$$

In the penultimate step, we have used the cosine rule of triangles.

As an aside, note that for $i = j$, both distances are same and the term becomes zero.