Multiple levels of abstraction to scale up/down modeling complexity

Level	Description	Remarks
1	Convert principal elements of system to 3-D continuous entities with distributed mass and stiffness.	• Practically infinite degree of freedom system. • No viable solution is most situations.
2	Convert the continuous system into a discrete system using appropriate lumping procedures. Convert distributed quantities (mass, stiffness, forces, etc.) to equivalent concentrated quantities at select locations.	• Finite but large number of degrees of freedom. • Matrix solution can be obtained but expensive.
3	Use condensation techniques to neglect specific terms and express some degrees of freedom in terms of others.	• Not a trivial step. Requires expertise. • Smaller matrices but captures essence of model in previous step.
4	Reduce complexity further via kinematic assumptions. Example, assuming certain structural elements cannot deform (i.e., rigid).	• Improves ease of modeling. • Kinematic assumptions likely to introduce inconsistencies at a theoretical level.
5	Reduce complexity further by retaining only principal motions and eliminating everything else.	• Physical insight limited to specific settings. • Hand calculations and exact solutions possible.

Basic abstractions used in the design and analysis of vibratory systems

Representations of components can be categorized based on forces or energy

Force generated depends on

Energy characteristics

Components using which basic abstractions are created

Abstraction of components can be categorized based on forces or energy

Force generated depends on

Position

Acceleration

Velocity

Store-and-release

Dissipative

Energy characteristics

Store-and-release components usually attempt to restore the system back to its previous stable equilibrium configuration.

Dissipative components reduce the overall energy of the system via irreversible processes.

Basic abstractions

Abstraction of components can be categorized based on forces or energy

Force generated depends on

Position

Acceleration

Velocity

Mass & stiffness

Store-and-release

Dissipative

Damping

Energy characteristics

Store-and-release components usually attempt to restore the system back to its previous stable equilibrium configuration.

Dissipative components reduce the overall energy of the system via irreversible processes.

Commonly used idealizations

Material particle (isolates mass)

- Localized object or entity that has properties such as mass, volume, velocity, acceleration, etc.
- Size can range from sub-atomic to macroscopic depending on the purpose of representation
- May have negligible dimensions but possess inertia in both translation and rotation.

Body

- Collection of material particles that are connected to each other.
- **Rigid** when the connections do not allow relative motion between inter-connected particles. Suitable when deformation of an object is negligible compared to its overall motion.
- **Flexible** when a limited amount of relative motion between particles is allowed.

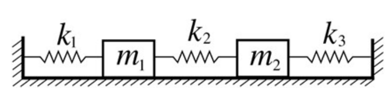
Massless spring (elastic constraint) and damper

- Isolates a stiffness or dissipative property
- Suitable when the mass of component is negligible compared to mass of other components

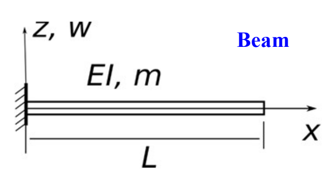
Sample abstractions for mass and stiffness (damping is not shown)

	Discrete (localized)	Continuous (distributed)
A	Mass & Stiffness	
B		Mass & Stiffness
C	Mass	Stiffness
D	Stiffness	Mass

Discrete masses attached by springs



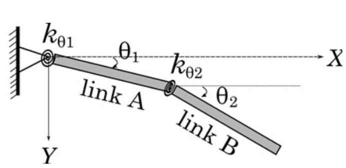
A



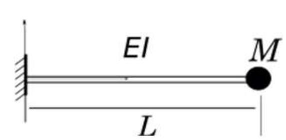
Beam

B

D



Rigid links attached by torsional springs



C

Mass attached to a very light (massless) beam

Spring Elements

- *Linear* spring is a type of mechanical link that is generally assumed to have negligible mass and damping
- *Spring force* is given by

$$F = kx$$

F = spring force,
k = spring stiffness or spring constant, and
x = deformation (displacement of one end with respect to the other)

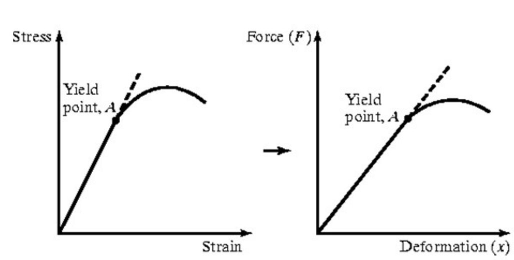
- *Work done (U)* in deforming a spring or the strain (potential) energy is given by:

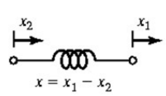
$$U = \frac{1}{2}kx^2$$

Spring elements

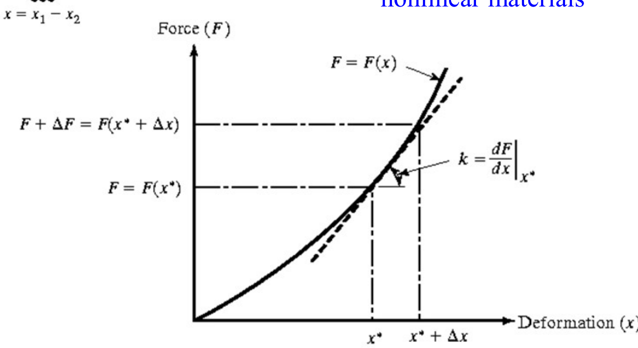
- When an incremental force *ΔF* is added to F:

$$F + \Delta F = F(x^* + \Delta x)\ F(x^*) + \left.\frac{dF}{dx}\right|_{x^*}(\Delta x) \quad + \frac{1}{2!}\left.\frac{d^2F}{dx^2}\right|_{x^*}(\Delta x)^2 + \ldots$$





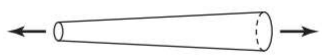
Tangent stiffness is used for nonlinear materials



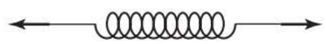
Identification of stiffness for linear response is relatively straightforward.

Equivalent spring stiffness is based on the work done by the applied force

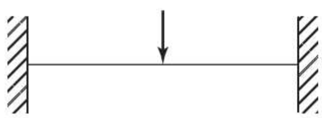
Rod under axial load
(l = length, A = cross sectional area) $\quad k_{eq} = \dfrac{EA}{l}$



Tapered rod under axial load
(D, d = end diameters) $\quad k_{eq} = \dfrac{\pi EDd}{4l}$



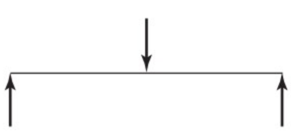
Helical spring under axial load
(d = wire diameter,
D = mean coil diameter,
n = number of active turns) $\quad k_{eq} = \dfrac{Gd^4}{8nD^3}$



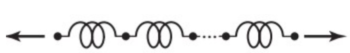
Fixed-fixed beam with
load at the middle $\quad k_{eq} = \dfrac{192EI}{l^3}$

Equivalent spring stiffness

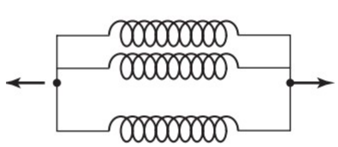
Cantilever beam with end load $\quad k_{eq} = \dfrac{3EI}{l^3}$



Simply supported beam with load
at the middle $\quad k_{eq} = \dfrac{48EI}{l^3}$

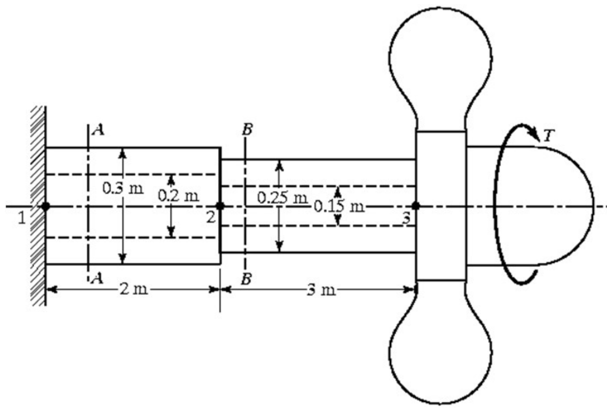


Springs in series $\quad \dfrac{1}{k_{eq}} = \dfrac{1}{k_1} + \dfrac{1}{k_2} + \cdots + \dfrac{1}{k_n}$



Springs in parallel $\quad k_{eq} = k_1 + k_2 + \cdots + k_n$

Torsional Spring Constant of a Propeller Shaft



Determine the torsional spring constant of the speed propeller shaft shown in Fig. 1.25.

- We need to consider the segments 12 and 23 of the shaft as springs in combination.
- The torque induced at any cross section of the shaft (such as *AA* or *BB*) can be seen to be equal to the torque applied at the propeller, *T*.
- Hence, the elasticities (springs) corresponding to the two segments 12 and 23 are to be considered as series springs.

Torsional Spring Constant of a Propeller Shaft

The spring constants of segments 12 and 23 of the shaft (k_{t12} and k_{t23}) are given by

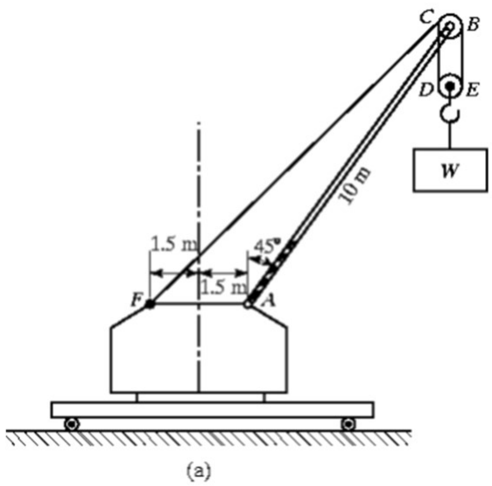
$$k_{t_{12}} = \frac{GJ_{12}}{l_{12}} = \frac{G\pi(D_{12}^4 - d_{12}^4)}{32 l_{12}} = \frac{(80\times10^9)\pi(0.3^4 - 0.2^4)}{32(2)} = 25.5255 \times 10^6 \text{N}-\text{m/rad}$$

$$k_{t_{23}} = \frac{GJ_{23}}{l_{23}} = \frac{G\pi(D_{23}^4 - d_{23}^4)}{32 l_{23}} = \frac{(80\times10^9)\pi(0.25^4 - 0.15^4)}{32(3)} = 8.9012 \times 10^6 \text{N}-\text{m/rad}$$

Since the springs are in series,

$$k_{t_{eq}} = \frac{k_{t_{12}} k_{t_{23}}}{k_{t_{12}} + k_{t_{23}}} = \frac{(25.5255\times10^6)(8.9012\times10^6)}{(25.5255\times10^6 + 8.9012\times10^6)} = 6.5997 \times 10^6 \text{N}-\text{m/rad}$$

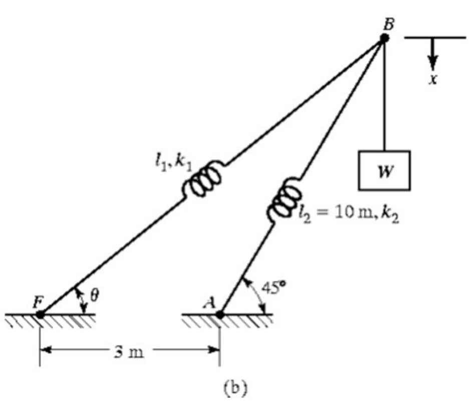
Equivalent stiffness of a crane



- The boom *AB* of crane is a uniform steel bar of length 10 m and x-section area of 2,500 mm^2.
- A weight *W* is suspended while the crane is stationary.
- The steel cable *CDEBF* has x-sectional area of 100 mm^2.
- Neglect effect of cable *CDEB*
- Find equivalent spring constant of system in the vertical direction.

Equivalent spring stiffness of a crane

A vertical displacement x at B will cause the spring k_2 (boom) to deform by $x_2 = x \cos 45^\circ$ and the spring k_1 (cable) to deform by an amount $x_1 = x \cos (90^\circ - \theta)$. Length of cable FB, l_1 is as shown.



$$l_1^2 = 3^2 + 10^2 - 2(3)(10)\cos 135^\circ = 151.426$$
$$So \quad l_1 = 12.3055\text{m}$$

- The angle θ satisfies the relation:

$$l_1^2 + 3^2 - 2(l_1)(3)\cos\theta = 10^2$$
$$\cos\theta = 0.8184$$
$$\therefore \theta = 35.0736^\circ$$

Equivalent spring stiffness of a crane

The total potential energy (U):

$$U = \frac{1}{2}k_1(x\cos 45°)^2 + \frac{1}{2}k_2[x\cos(90° - \theta)]^2$$

The equivalent stiffness coefficients for the individual elements

$$k_1 = \frac{A_1E_1}{l_1} = \frac{(100 \times 10^{-6})(207 \times 10^9)}{12.0355} = 1.6822 \times 10^6 \mathrm{N/m}$$

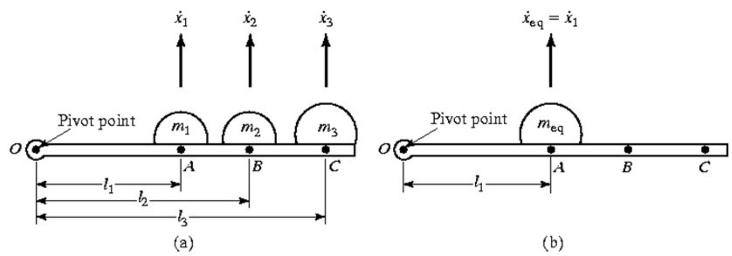
$$k_2 = \frac{A_2E_2}{l_2} = \frac{(2500 \times 10^{-6})(207 \times 10^9)}{10} = 5.1750 \times 10^7 \mathrm{N/m}$$

Equivalent spring stiffness of a crane

- Potential Energy of the equivalent spring is: $U_{eq} = \frac{1}{2}k_{eq}x^2$
- By setting $U = U_{eq}$: $k_{\mathrm{eq}} = 26.4304 \times 10^6 \mathrm{N/m}$

Equivalent Mass

- Consider an example of masses in translation that are connected by a rigid bar



  - Velocities of masses can be expressed as:

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1 \qquad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1$$

Equivalent Mass

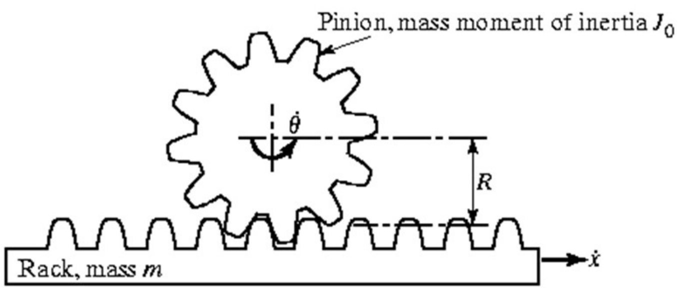
- By equating the kinetic energy of the system:

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{\text{eq}}\dot{x}_{\text{eq}}^2$$

$$m_{\text{eq}} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3$$

Equivalent Mass

- Translational and Rotational Masses Coupled Together.
- Pinion rotates due to translation of the rack.



m_{eq} = single equivalent translational mass
$\dot{x}$ = translational velocity
$\dot{\theta}$ = rotational velocity
J_0 = mass moment of inertia
J_{eq} = single equivalent rotational mass

Equivalent Mass

- Kinetic energy of the two masses is given by $T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$
- Kinetic energy of the equivalent mass is given by: $T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^{2}$
- Note $\dot{\theta} = \frac{\dot{x}}{R}$ and $\dot{x}_{\text{eq}} = \dot{x}$
- Equating T_{eq} & T gives equivalent translation mass as $m_{\text{eq}} = m + \frac{J_0}{R^2}$

Equivalent Mass

Equivalent rotational mass:

Note $\dot{\theta}_{\text{eq}} = \dot{\theta}$ and $\dot{x} = \dot{\theta}R$

$$\frac{1}{2}J_{\text{eq}}\dot{\theta}^2 = \frac{1}{2}m\left(\dot{\theta}R\right)^2 + \frac{1}{2}J_0\dot{\theta}^2$$

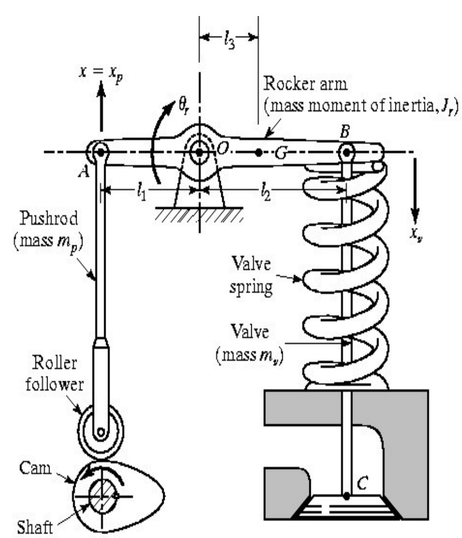
$$J_{\text{eq}} = J_0 + mR^2$$

Equivalent Mass

A cam-follower mechanism is used to convert the rotary motion of a shaft into the oscillating or reciprocating motion of a valve.

The follower system consists of a pushrod of mass m_p, a rocker arm of mass m_r, and mass moment of inertia J_r about its C.G., a valve of mass m_v, and a valve spring of negligible mass.

Find the equivalent mass (m_{eq}) of this cam-follower system by assuming the location of m_{eq} as (i) pt A and (ii) pt C.



Equivalent Mass

The kinetic energy of the system *(T)* is: $T = \frac{1}{2} m_p \dot{x}_p^2 + \frac{1}{2} m_v \dot{x}_v^2 + \frac{1}{2} J_r \dot{\theta}_r^2 + \frac{1}{2} m_r \dot{x}_r^2$

If m_{eq} denotes equivalent mass placed at pt *A* with $\dot{x}_{\text{eq}} = \dot{x}$

The kinetic energy equivalent mass system T_{eq} is: $T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2$

By equating T and T_{eq}, and noting that $\dot{x}_p = \dot{x}, \ \dot{x}_v = \frac{\dot{x} l_2}{l_1}, \ \dot{x}_r = \frac{\dot{x} l_3}{l_1}$, and $\dot{\theta}_r = \frac{\dot{x}}{l_1}$

$$m_{\text{eq}} = m_p + \frac{J_r}{l_1^2} + m_v \frac{l_2^2}{l_1^2} + m_r \frac{l_3^2}{l_1^2}$$

Equivalent Mass

Designer's choice: If equivalent mass is located at point C, then we obtain

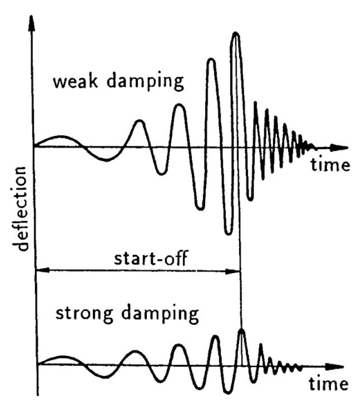
$$T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{x}_{\text{eq}}^2 = \frac{1}{2} m_{\text{eq}} \dot{x}_v^2$$

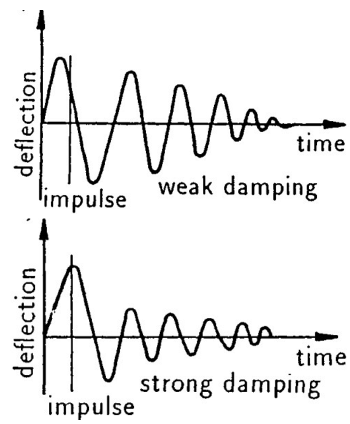
$$m_{\text{eq}} = m_v + \frac{J_r}{l_2^2} + m_p \left(\frac{l_1}{l_2}\right)^2 + m_r \left(\frac{l_3^2}{l_1^2}\right)^2$$

Damping is critical in the design of vibratory systems

- In most systems, damping is small and is usually neglected when determining response.
- Situations in which damping becomes crucial include resonance and impulse response.

Response of a structure when passing through resonance

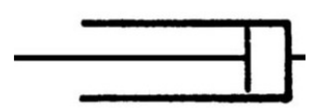




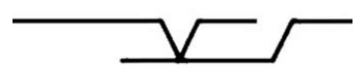
Response of a structure to impulse excitation

Damping

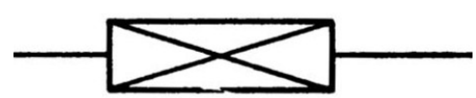
- Viscous damping force is proportional to the velocity of the vibrating body in a fluid medium such as air, water, gas, and oil.



- Coulomb or dry friction damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body between dry surfaces



- In material or solid or hysteretic damping, energy is absorbed or dissipated by material during deformation due to friction between internal planes



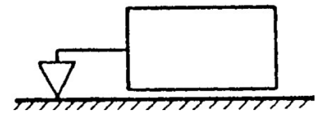
Coulomb or dry friction damping

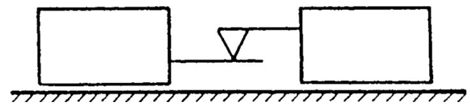
- Usually occurs between poorly lubricated or un-lubricated surfaces in contact.
- Classical representation signifies a slider moving on a surface.
- In most practical analyses, a simplified friction model is used.

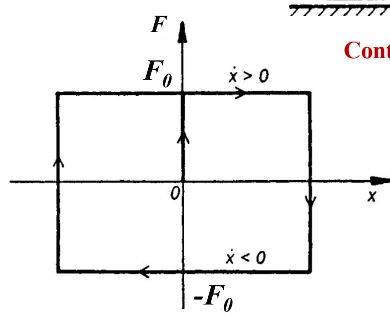
$$F = F_0 \operatorname{sgn}(\dot{x})$$

where

$$F_0 = \mu N$$


Body moving on a surface


Contact with friction between bodies



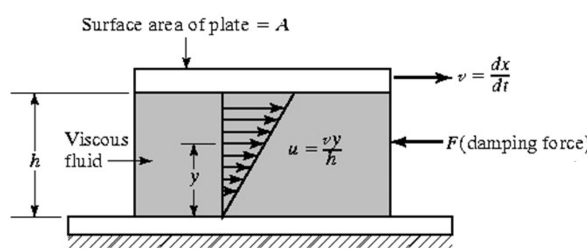
Force as a function of displacement

Damping element: Viscous shear

- *Shear Stress (τ)* developed in the fluid layer at a distance y from the fixed plate is:

$$\tau = \mu \frac{du}{dy}$$

where $du/dy = v/h$ is the velocity gradient.



- *Shear* or *Resisting Force (F)* developed at the bottom surface of the moving plate is:

$$F = \tau A = \mu \frac{Av}{h} = cv$$

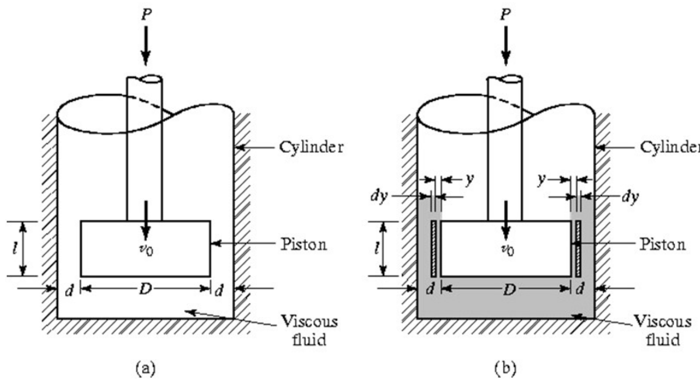
where A is the surface area of the moving plate.

- The damping constant is $c = \dfrac{\mu A}{h}$

Damping element: Piston in a cylinder

Damping constant of a piston moving in a viscous medium

- Use the shear stress equation for viscous fluid flow and the rate of fluid flow equation.
- Piston diameter D and length l, moving with velocity v_0 in a cylinder filled with a liquid of viscosity μ.
- Clearance between the piston and the cylinder wall be d.



Damping element: Piston in a cylinder

- At a distance y from the moving surface, velocity and shear stress are v and τ
- At a distance $(y + dy)$ the velocity and shear stress be $(v - dv)$ and $(\tau + d\tau)$, respectively
- The negative sign for dv shows that the velocity decreases as we move toward the cylinder wall.
- The viscous force on this annular ring is equal to $F = \pi D l d\tau = \pi D l \frac{d\tau}{dy} dy$
- But the shear stress is given by $\tau = -\mu \frac{dv}{dy}$
- Combining the above equations, we get $F = -\pi D l dy \mu \frac{d^2 v}{dy^2}$

Damping element: Piston in a cylinder

The force on the piston will cause a pressure difference on the ends of the element, given by

$$p = \frac{P}{\left(\frac{\pi D^2}{4}\right)} = \frac{4P}{\pi D^2}$$

Thus the pressure force on the end of the element is $p(\pi D dy) = \frac{4P}{D} dy$

Where (πDdy) denotes the annular area between y and (y + dy).

Assuming uniform mean velocity in the direction of motion of the fluid, the forces due to the viscous drag and the pressure must be equal. Thus

$$\frac{d^2v}{dy^2} = -\frac{4P}{\pi D^2 l\mu}$$

Damping element: Piston in a cylinder

Integrating this equation twice and using the boundary conditions $v = -v_0$ at $y = 0$ and $v = 0$ at $y = d$, we obtain

$$v = \frac{2P}{\pi D^2 l\mu}(yd - y^2) - v_0\left(1 - \frac{y}{d}\right)$$

The rate of flow through the clearance space can be obtained by integrating the rate of flow through an element between the limits $y = 0$ and $y = d$:

$$Q = \int_0^d v\pi D dy = \pi D\left[\frac{2Pd^3}{6\pi D^2 l\mu} - \frac{1}{2}v_0 d\right]$$

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Damping element: Piston in a cylinder

The volume of the liquid flowing through the clearance space per second must be equal to the volume per second displaced by the piston.

Hence the velocity of the piston will be equal to this rate of flow divided by the piston area. This gives

$$v_0 = \frac{Q}{\left(\frac{\pi}{4}D^2\right)}$$

Substituting the above into the previous expression for Q, we obtain

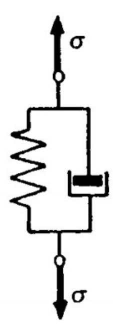
$$P = \left[\frac{3\pi D^3 l\left(1 + \frac{2d}{D}\right)}{4d^3}\right]\mu v_0$$

The force as $P = cv_0$. The damping constant c can be found as

$$c = \mu\left[\frac{3\pi D^3 l}{4d^3}\left(1 + \frac{2d}{D}\right)\right]$$

Viscoelastic material models using spring-damper combinations

Kelvin-Voigt model



- Rheological equation of state is $\sigma = E\varepsilon + \eta\dot{\varepsilon}$
- The response is $\varepsilon = \frac{\sigma_0}{E} + \left(\varepsilon_0 - \frac{\sigma_0}{E}\right)e^{-\alpha t}$
- Where ε_0 is the initial strain and $\alpha = \frac{E}{\eta}$
- Model includes strain relaxation that is inherent in viscoelastic materials.
- Predicts viscoelastic response to stress and creep accurately. Inaccurate for relaxation.
- Organic polymers, rubber, and wood

Viscoelastic material models using spring-damper combinations

Maxwell model

- Rheological equation of state is $\dot{\varepsilon} = \frac{\sigma}{E} + \frac{\dot{\sigma}}{\eta}$
- The response is $\sigma = \sigma_0 e^{-\frac{t}{\tau}}$
  - Where σ_0 is the initial stress and $\tau = \frac{\eta}{E}$
  - Model predicts exponential stress decay that is inherent in viscoelastic materials.
  - Inaccurate for creep.
  - Thermoplastic polymers and metals close to melting temperature and fresh concrete.

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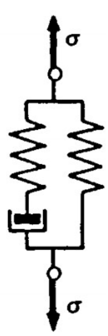
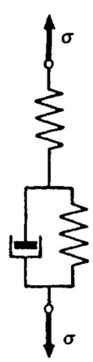
Zener models

(A)



(B)



Simplest models that can be used to represent realistic materials

- Rheological equations of state are

(A) $$\sigma + \frac{\eta}{E_2}\dot{\sigma} = E_1\varepsilon + \frac{\eta(E_1 + E_2)}{E_2}\dot{\varepsilon}$$

(B) $$\sigma + \frac{\eta}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1\eta}{E_1 + E_2}\dot{\varepsilon}$$

- Models can describe both relaxation and creep accurately.
- More accurate than the Maxwell and Kelvin–Voigt models in predicting material responses
- Could produce inaccurate strain under specific loading conditions.

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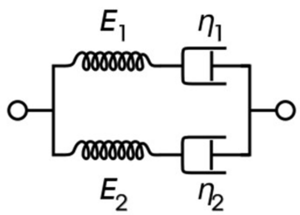
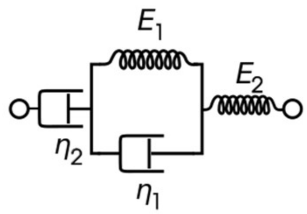
Burgers models

(A)



(B)



- Rheological equations of state are

(A) $$\sigma + \left(\frac{\eta_1}{E_1} + \frac{\eta_2}{E_2}\right)\dot{\sigma} + \frac{\eta_1\eta_2}{E_1E_2}\ddot{\sigma} = (\eta_1 + \eta_2)\,\dot{\varepsilon} + \frac{\eta_1\eta_2\,(E_1 + E_2)}{E_1E_2}\ddot{\varepsilon}$$

(B) $$\sigma + \left(\frac{\eta_1}{E_1} + \frac{\eta_2}{E_1} + \frac{\eta_2}{E_2}\right)\dot{\sigma} + \frac{\eta_1\eta_2}{E_1E_2}\ddot{\sigma} = \eta_2\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_1}\ddot{\varepsilon}$$

- More coefficients to determine
- More accurate than the previous models in predicting material responses.
- May still be inadequate in predicting the material behavior for a wide range of loading conditions.

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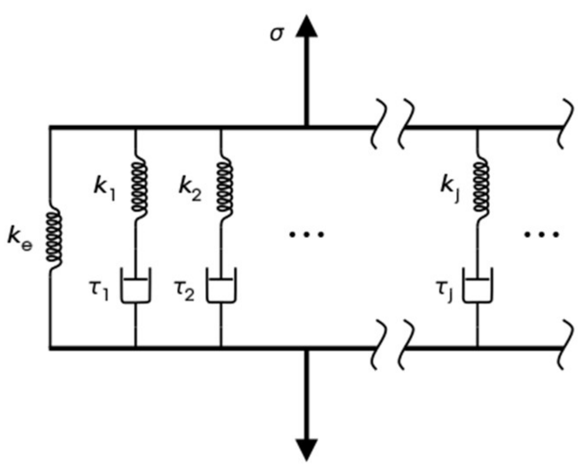
Generalized Maxwell model, also known as the Maxwell-Wiechert or the Wiechert model



- Most general form for a linear viscoelastic material.
- As many springs and dashpots as is necessary.
- Relaxation properties can be distributed over time (i.e. does not occur at specific times)
- Can be used for metals and soft alloys

