We have derived that

$$f_Y(y) = \frac{1}{\theta\sqrt{1-y^2}} \frac{1}{1-e^{-2\theta\pi}} \left(e^{-\theta\pi + \theta \sin^{-1} y} - e^{-\theta \sin^{-1} y} \right)$$

for $0 < y < 1$.

The probability density function of the random variable Y can be obtained. for the region

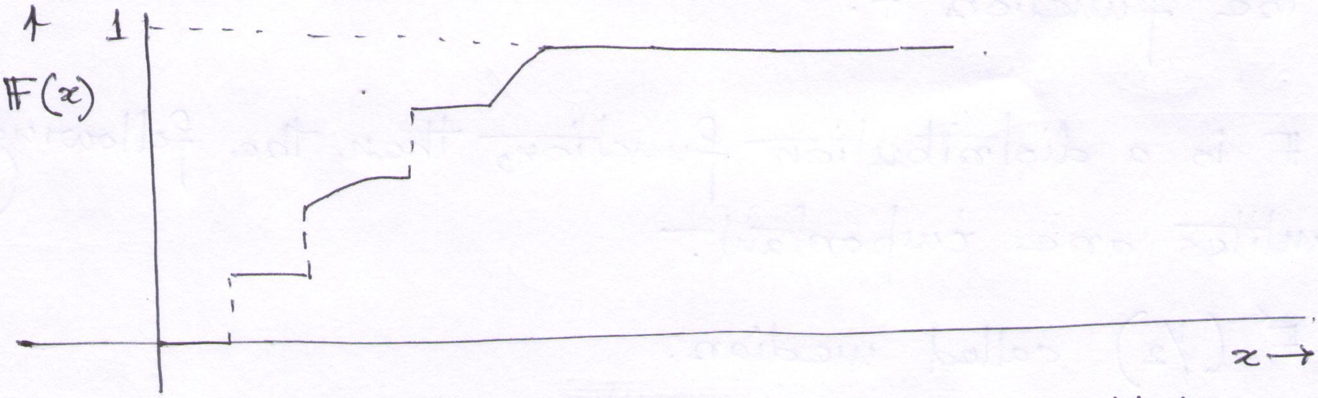
$y \in (-1, 0)$

in a similar way.

$\boxed{\text{I am leaving this as a homework!}}$

Lecture - VIII : Quantiles.

Here we shall mention some of the important parameters (functions) associated to a distribution function. We do not need to assume the random variable X to be discrete or continuous. So the distribution function may not be continuous.



A generic distribution function and its plot.

Definition : Let $\mathbb{F}$ be a distribution function. A real number x is said to be the p^{th} quantile if
$$x = \inf\{u : \mathbb{F}(u) \geq p\}. \quad \text{for every } p \in (0,1).$$
and the p^{th} quantile ($(100p)^{th}$ percentile) is denoted by $\mathbb{F}^{\leftarrow}(p)$.

Remark Note that $\inf\{u : \mathbb{F}(u) \geq p\}$ exists as $\mathbb{F}$ is right continuous. Note that $\mathbb{F}^{\leftarrow}(p)$ is unique.

Remark It may happen that $\mathbb{F}(\mathbb{F}^{\leftarrow}(p)) \geq p$.

Exercise Find out a ~~distrib~~ distribution function $\mathbb{F}$ such that $\mathbb{F}(\mathbb{F}^{\leftarrow}(\frac{1}{2})) > \frac{1}{2}$.

Remark. Suppose that $\mathbb{F} : \mathbb{R} \to [0,1]$ is continuous, then $\mathbb{F}(\mathbb{F}^{\leftarrow}(p)) = p$. If additionally, $\mathbb{F} : \mathbb{R} \to [0,1]$ is monotonically increasing, then $\mathbb{F}^{\leftarrow}(x) = \mathbb{F}^{-1}(x)$ for all $x \in (0,1)$, where $\mathbb{F}^{-1}$ is the inverse image of the function $\mathbb{F}$.

If $\mathbb{F}$ is a distribution function, then the following quantites are important.

1. $\mathbb{F}^{\leftarrow}(1/2)$ called median.
2. $\mathbb{F}^{\leftarrow}(1/4)$ $(1/4)^{\text{th}}$ quantile.
3. $\mathbb{F}^{\leftarrow}(3/4)$ $(3/4)^{\text{th}}$ quantile.

Remark. Median is considered to be a measure of central tendency.

Exercise Find out a distribution function $\mathbb{F}$ such that $\mathbb{E}(|X|) = \infty$ and hence, mean can not be used as a measure of central tendency. In such cases median turns out to be really important.

Remark Let X be a random variable with distribution function $\mathbb{F}$, then $\frac{1}{2}[\mathbb{F}^{\leftarrow}(3/4) - \mathbb{F}^{\leftarrow}(1/4)]$ is considered to be a measure of dispersion.

Lecture VII : Moments, and generating functions

> * For this lecture, we shall assume that the random variables are either discrete or continuous. unless mentioned otherwise.

Here the aim is to understand the distribution function of a random variables. Recall that random variables are introduced to analyze and study random objects or random experiments using the properties of real numbers. Here, the important quantity the law or distribution or probability $\mathbb{P}_X$ induced by the random variable X. We then proved that the set function $\mathbb{P}_X$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is uniquely characterized by the distribution function $\mathbb{F}$ of X. So to understand the random experiment better, we must try to understand the distribution function $\mathbb{F}$.

* Sometimes, it is not so important to understand the distribution function $\mathbb{F}$ completely, some partial knowledge turns out to be enough.

For example, consider the coin tossing experiment. The experiment is completely specified if we know the probability of obtaining 'HEAD'. (or probability of 'TAIL'). One may be interested to know whether

"the coin is biased or unbiased that is,

~~$\mathbb{P}$ (HEAD) ≠ $\mathbb{P}$ (TAI~~

$\mathbb{P}(\text{HEAD}) \neq \mathbb{P}(\text{TAIL})$ or $\mathbb{P}(\text{HEAD}) = \mathbb{P}(\text{TAIL})$.

To solve this type of problems, some "functions" associated to the distribution function F. which may or may not offer a full knowledge of the distribution function.

Moments of a distribution function.

* Let X be a random variable of discrete type with probability mass function $(p_k : k \geq 1)$ such that $\mathbb{P}_X(\{x_k\}) = \mathbb{P}(X = x_k) = p_k$ for every $k \geq 1$.

If $\sum_{k=1}^{\infty} |x_k| \, p_k < \infty$, we say that the expected value of random variable X exists and defined to be

$$\mu_X = \sum_{k=1}^{\infty} x_k \, p_k = \mathbb{E}(X)$$

Sometimes, it may happen that $\sum_{k=1}^{\infty} x_k p_k < \infty$ but $\sum_{k=1}^{\infty} |x_k| \, p_k$ does not. In that case, we say that

$\mathbb{E}(X)$ does not exist.

The expected value of X is also called <u>mean</u> of the random variable X.

* If X is of continuous type and has probability density function f, we say that $\mathbb{E}(X)$ exists if.

$$\int_{\mathbb{R}} |x| f(x)\, dx < \infty$$

and define

$$\mathbb{E}(X) = \int_{\mathbb{R}} x f(x)\, dx$$

* Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. We can then consider a function (measurable) $h : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and define

$$\mathbb{E}(h(X)) = \sum_{k \geq 1}^{\infty} h(x_k)\, p_k$$ if. X is a discrete random variable with probability mass function $(p_k : k \geq 1)$ and

$$\sum_{k \geq 1}^{\infty} |h(x_k)|\, p_k < \infty.$$

$$\mathbb{E}(h(X)) = \int_{\mathbb{R}} f(x)\, h(x)\, dx$$ if X is a continuous random variable with probability density function f. and $\int |h(x)| f(x)\, dx < \infty$.

Important note.

Improper integral is to be understood in the following sense

$$\int_{-\infty}^{\infty} x f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b x f(x)\,dx .$$

Here a may not be equal to b.

Note that according to Cauchy's principal value theorem

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b x f(x)\,dx = \lim_{a\to\infty} \int_{-a}^{a} x f(x)\,dx$$

$$\lim_{\substack{a\to-\infty \\ b\to\infty}} \int_a^b x f(x)\,dx < \infty .$$

Here we need to be extra careful.

Remarks / properties of expectation.

i) If $A \in \mathcal{B}(\mathbb{R})$, then we can take

$$h(X) = \mathbb{1}(X \in A).$$

$$h(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A . \end{cases}$$

It is immediate that h is measurable $f^{\underline{n}}$.

and therefore. $\mathbb{E}(h(X))$ is welldefined (why??).

$$\mathbb{E}(h(X)) = \mathbb{P}(X \in A) = \mathbb{P}_X(A).$$

2. We can talk about $X_+ = \max(X, 0)$.

Think about $\mathbb{E}(X_+)$

3. If $a \in \mathbb{R}$, $b \in \mathbb{R}$, then $\mathbb{E}(aX+b) = a\mathbb{E}(X)+b$.

4. If X is a bounded random variable that is, $\mathbb{P}(|X| < M) = 1$ for some $M > 0$. Then $\mathbb{E}(X)$ exists.

5. If $\mathbb{P}(X > 0) = 1$, the $\mathbb{E}(X)$ exists (~~we~~ allows. $\mathbb{E}(X) = \infty$). X (this is not our convention).

Theorem 1 Let $X: (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Consider a measurable function $g: (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then $g(X) = g \circ X$: $(\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable.

i) If X is of discrete type, then

$$\mathbb{E}(g(X)) = \sum_{j=1}^{\infty} g(x_j)\, \mathbb{P}(X = x_j).$$

if $\sum_{j=1}^{\infty} |g(x_j)|\, \mathbb{P}(X = x_j) < \infty.$

iii) If X is a continuous random variable and $g(X)$ is also a continuous random variable, then

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

if $\int_{-\infty}^{\infty} |g(x)|\, f(x) < \infty$.

<u>Remark</u> X is continuous does not imply that $g(X)$ is continuos. Think about $X+$.

<u>Proof</u> – Left as an exercise. Try to use change of variable formula for sum and Riemann integral.

$\boxed{\text{Think about a probabilistic proof.}}$

<u>Moments.</u>

$\mathbb{E}(X^n)$ – n^{th} moment of the random variable X about origin.

n^{th} RAW MOMENT.

$\mathbb{E}(|X|^\alpha)$ – α^{th} absolute moment of the random variable X.

fractional moment if $\alpha \in (0,1)$.

Moment inequalities.

Theorem - 2 If the moment of ~~order~~ order t exists for a random variable X, then moments of order $0<s<t$ exist.

Proof. There are two versions. One version of the proof is for discrete random variables and the other version is for the continuous random variables.

Proof when X is continuous.

$$\mathbb{E}(|X|^s) = \int |x|^s f(x)\,dx$$

$$= \int_{|x|^s \le M} |x|^s f(x)\,dx + \int_{|x|^s > M} |x|^s f(x)\,dx$$

$$\le M \int_{|x|^s \le M} f(x)\,dx + \int_{|x|^s > M} |x|^t f(x)\,dx$$

$$= M\,\mathbb{P}(|X|^s \le M) + \int |x|^t f(x)\,dx$$

$$= M\,\mathbb{P}(|X|^s \le M) + \mathbb{E}(|X|^t) < \infty.$$

where M is chosen such that $|x|^s < |x|^t$ when $|x|^s > M$. In particular, we can choose $M=1$.

Exercise Write down the proof when X is a discrete random variable.