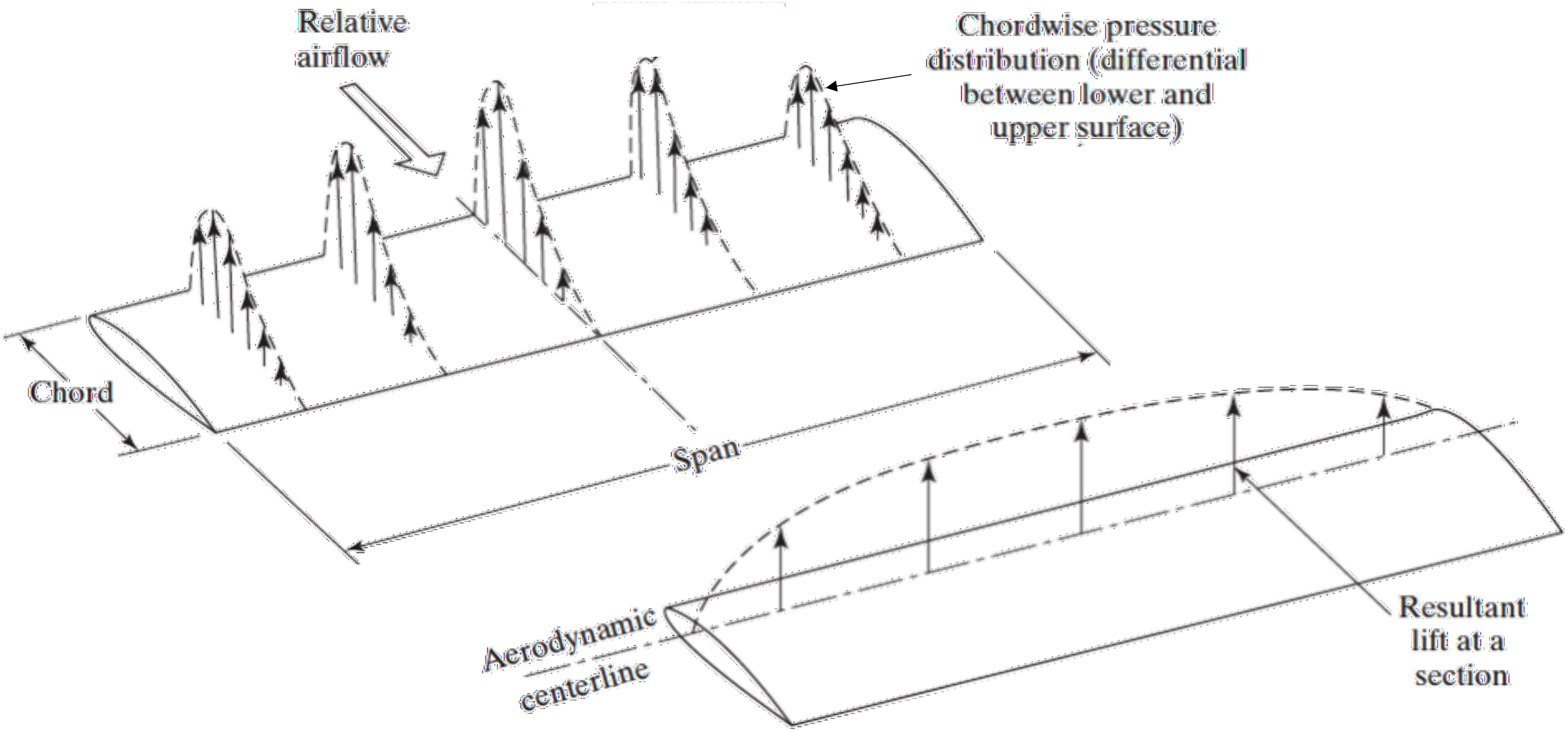
Finite Wing Aerodynamics

Aniruddha Sinha

Introduction

Finite Wing Aerodynamics

From (infinite-span wing) airfoil to finite wing



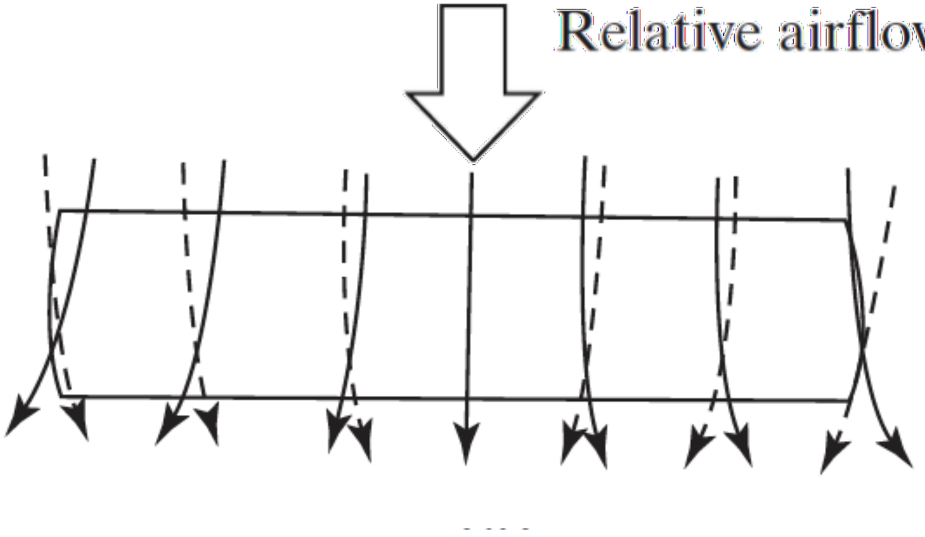
Bertin & Cummings, 2013

Trailing vortex system of a finite wing

Relative airflow

Leading edge

Trailing edge

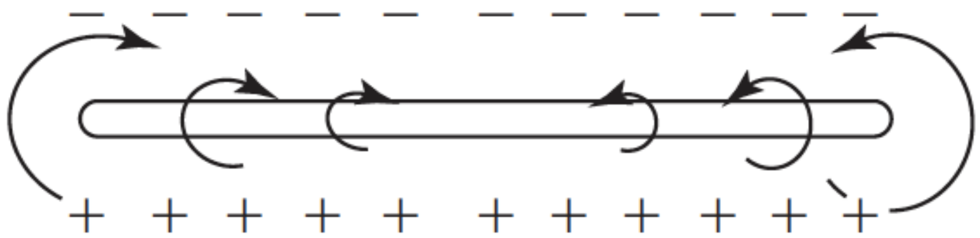


Upper surface flow (inboard)

Lower surface flow (outboard)

Bottom view

Relatively low pressure on upper surface



Tip vortex

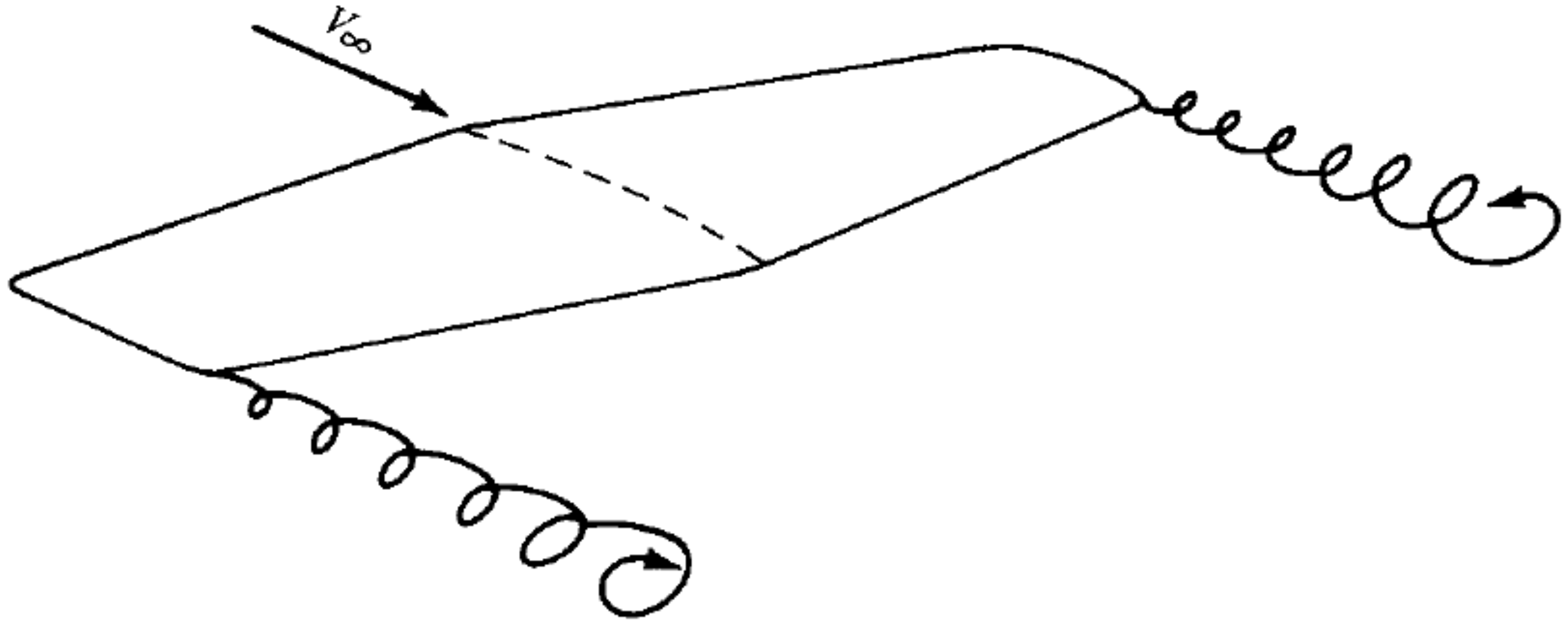
Rear view

Relatively high pressure on lower surface

Bertin & Cummings, 2013

Tip vortices

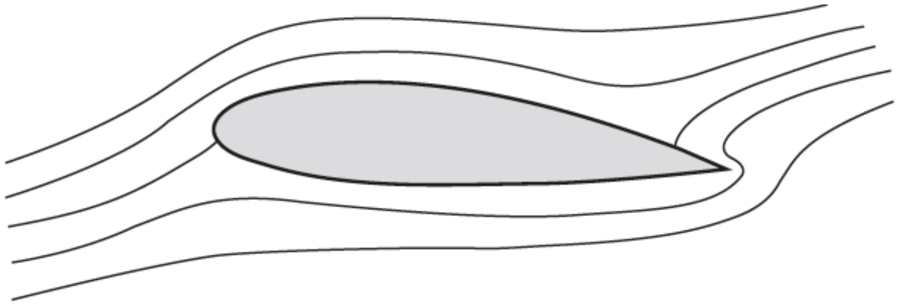
Tip vortices are strongest



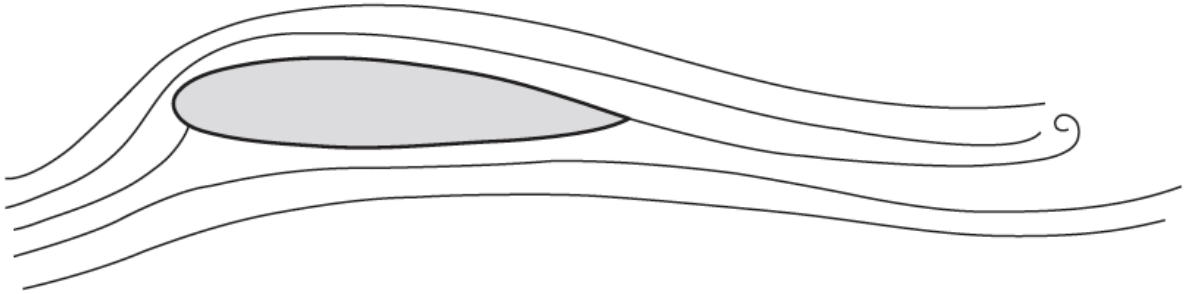
But, they are just a part of overall vortex system that trails the entire TE

Anderson, 2011

Starting vortex (also applies to airfoils)



Streamlines when flow starts (i.e., circulation is zero). Stagnation point is located on upper surface



Streamlines with full circulation, with stagnation point at TE. The initial eddy (starting vortex) is soon left far behind, and becomes negligible to flight

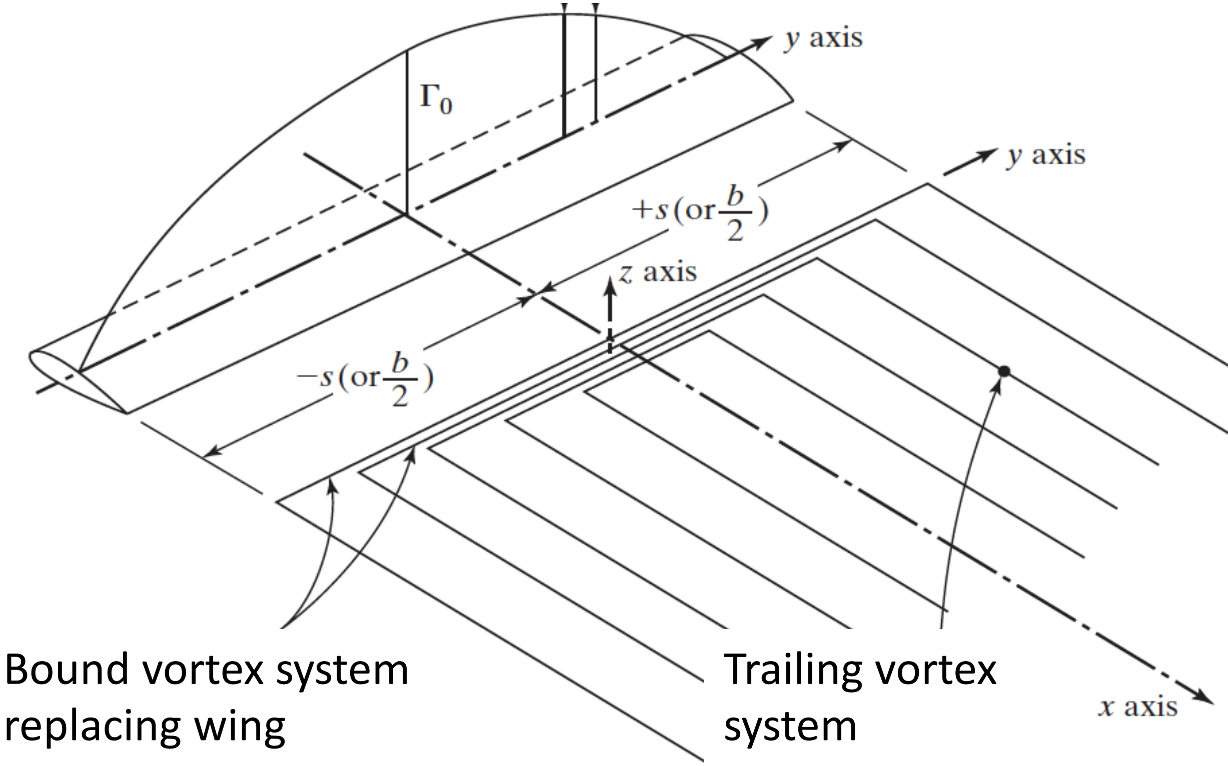
Similar 'starting vortices' are formed every time the lift (i.e., the circulation) changes, as during diving, landing, etc.

Houghton, 2013

Bound vortex system

The conceptual vortex system that can replace the wing in all its effect on the surrounding flow, and be used for predicting lift & moments

- Either have wing in model, or replace it with bound vortex system
- Continuation of vortex sheet concept used in 2D airfoils
- Unlike the starting vortex and the trailing vortex system that are physical, the bound vortex system is a modelling artifice



Bertin & Cummings, 2013

Inviscid Vortex Kinematics in 3D

Finite Wing Aerodynamics

Vortex filament and vortex tube

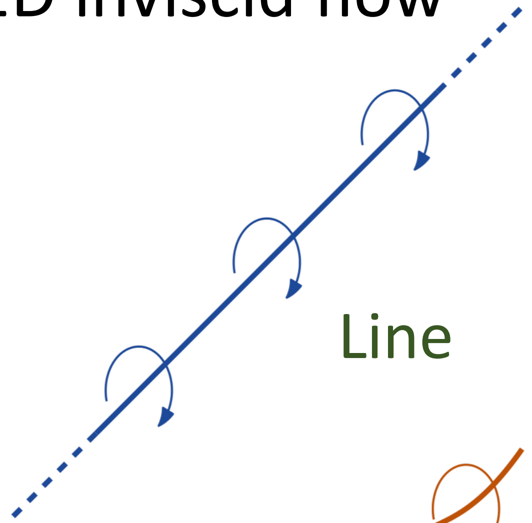
Earlier considered a straight line irrotational vortex in 2D inviscid flow

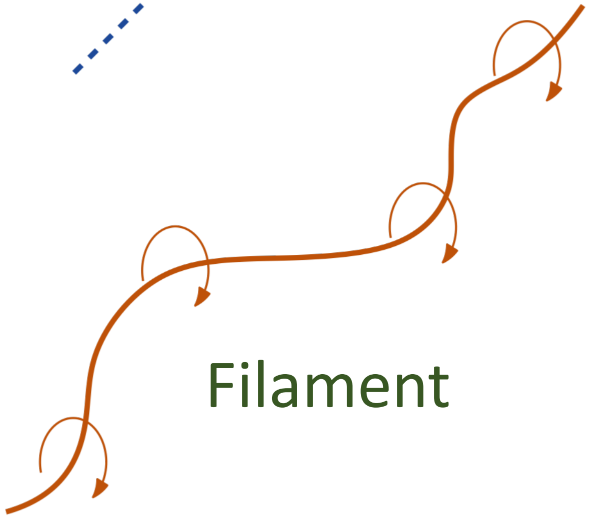
- This was a doubly-infinite line

Vortex filament is its natural extension to 3D flows

- Singularity at its infinitesimally thin ‘core’
- Irrotational everywhere else

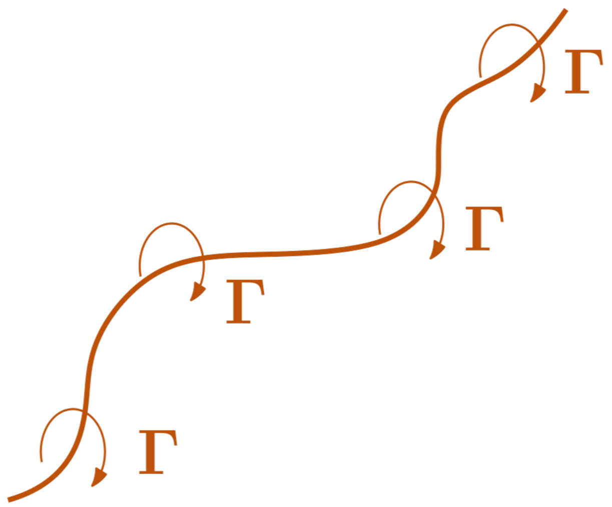
‘Vortex tube’ is a collection of filaments, it has a finite core but is irrotational elsewhere





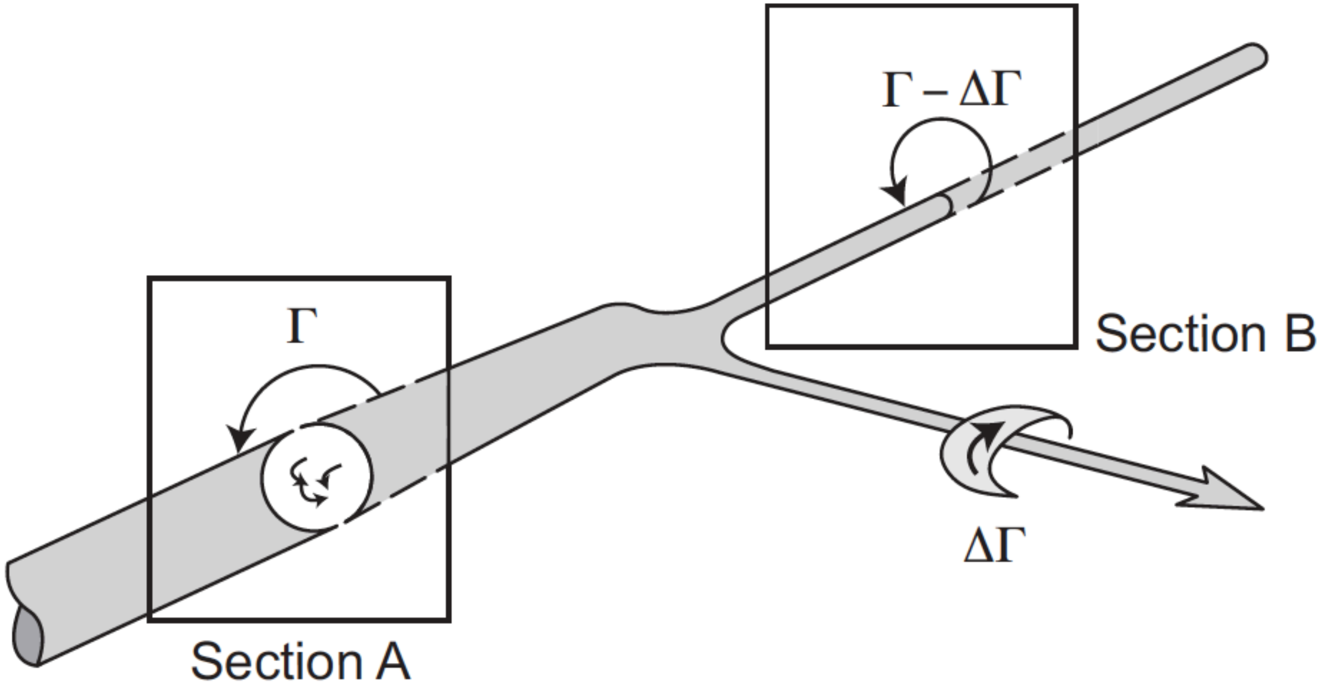
Helmholtz' law of vortex motion in inviscid flows: 1

Vortex filament strength (circulation) is constant along its length



Corollary of Helmholtz' 1st law

If strength of a vortex tube changes between two sections, a secondary vortex tube of the differential strength must join/leave the primary one



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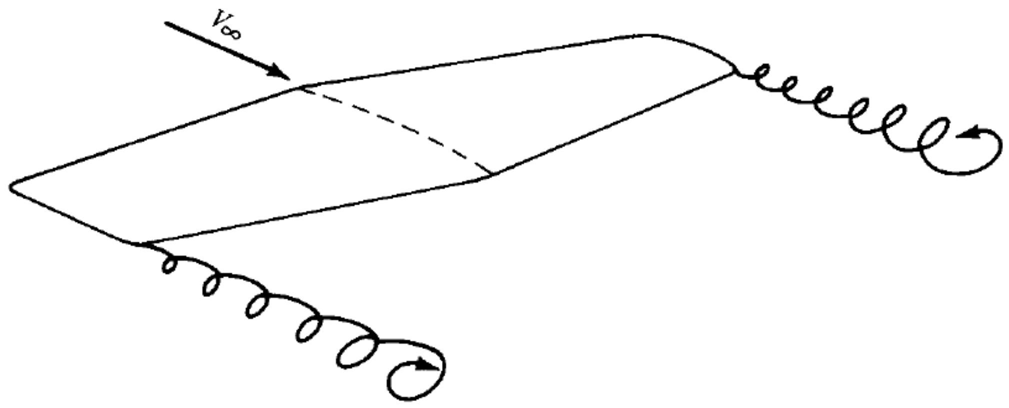
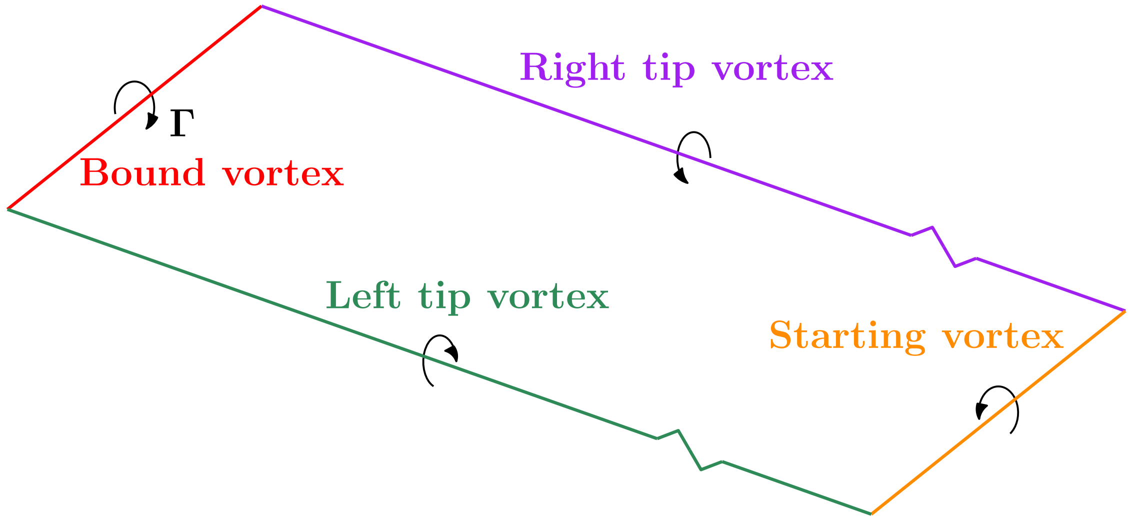
Helmholtz' law of vortex motion in inviscid flows: 2

Vortex filament cannot end in the fluid

- It must extend to domain boundaries (infinity or solid surface), or
- It must form a closed path





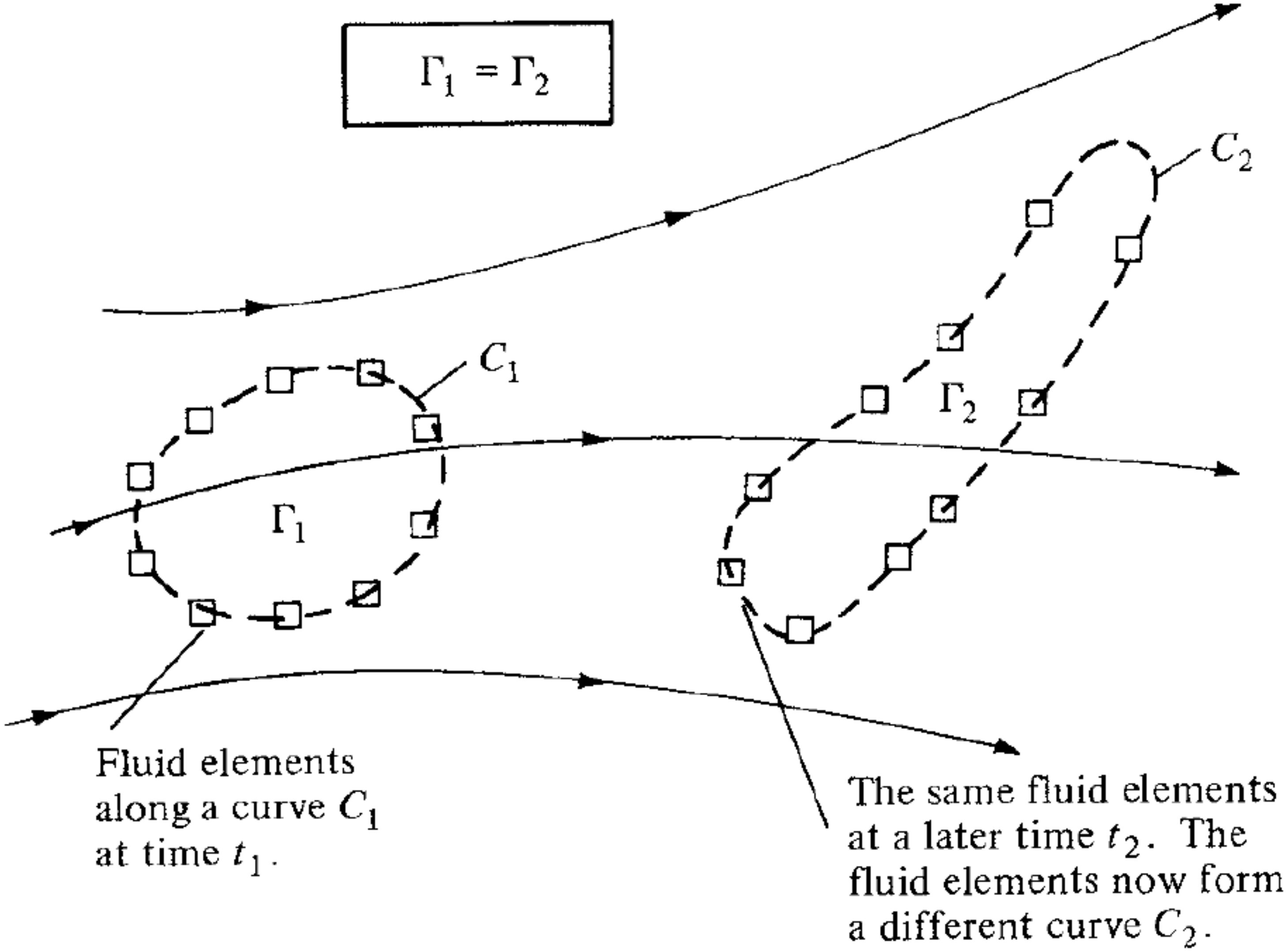
Anderson, 2011

Helmholtz’ law of vortex motion in inviscid flows: 3

Initially irrotational flow remains irrotational in the absence of external rotational effects like viscosity, baroclinic torque, shocks

Helmholtz' law of vortex motion in inviscid flows: 4

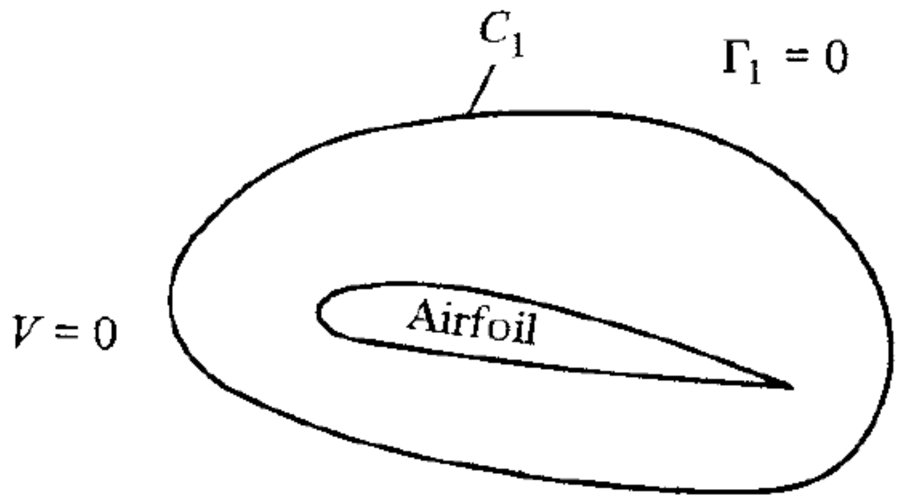
In the absence of rotational forces, circulation around material path remains invariant



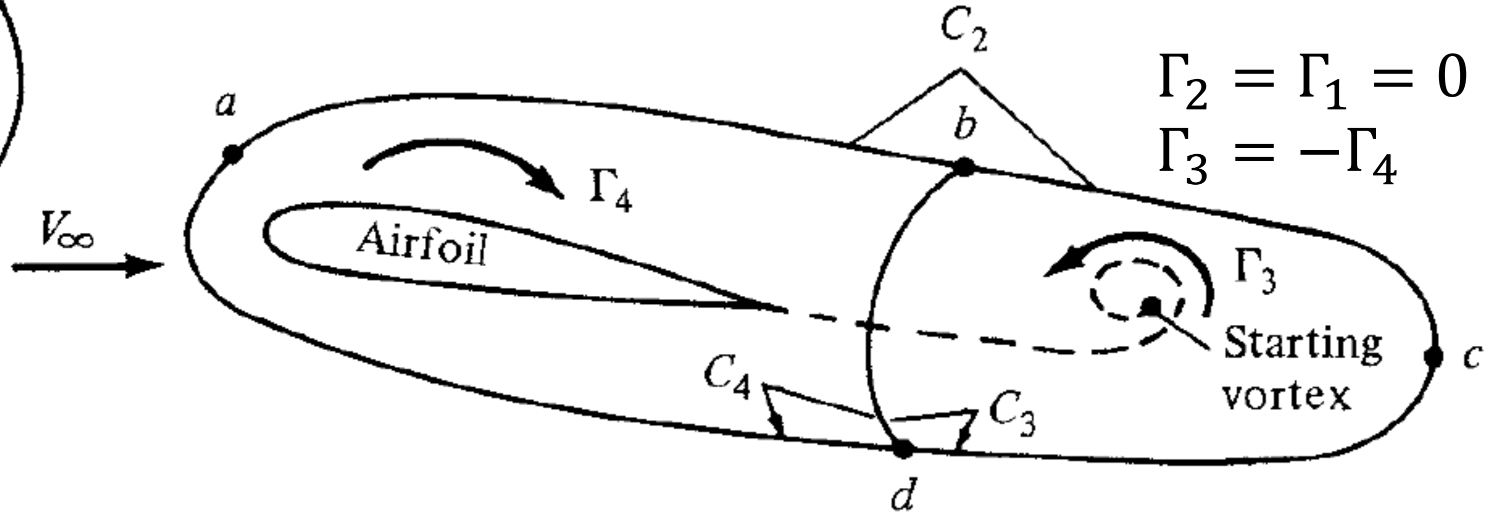
Anderson, 2011

Corollary of Helmholtz' 4th law

Starting vortex circulation is equal and opposite to bound vortex (or airfoil/wing) circulation



Fluid at rest relative to airfoil



$$\Gamma_2 = \Gamma_1 = 0$$
$$\Gamma_3 = -\Gamma_4$$

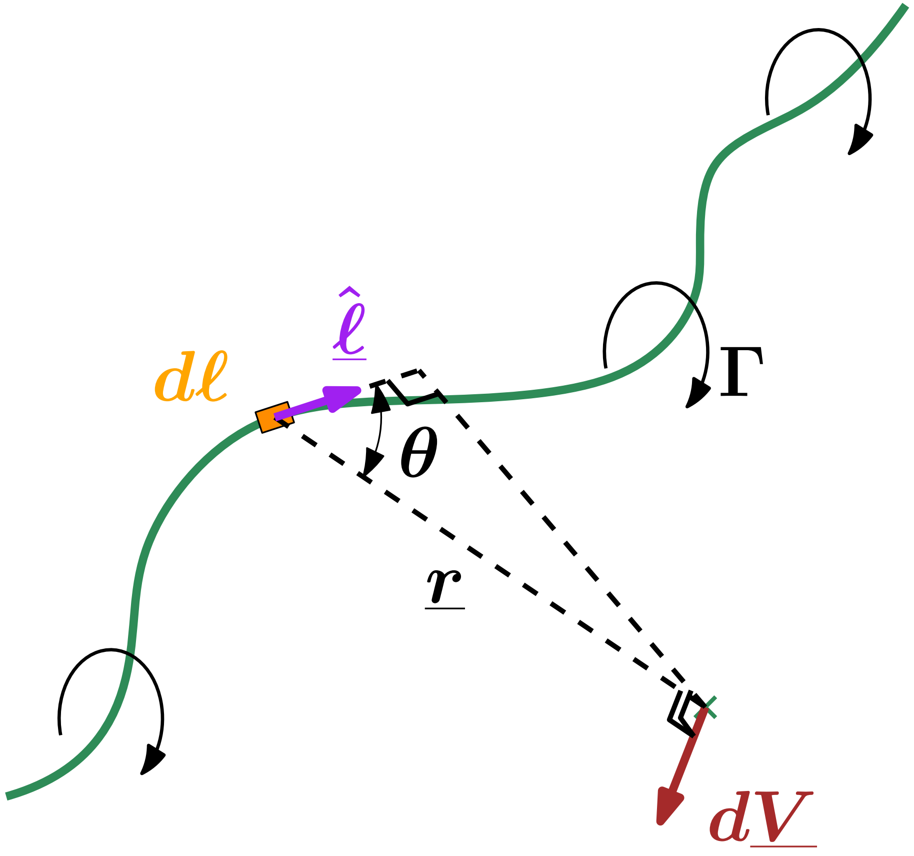
Picture some moments after flow has started

Anderson, 2011

Biot-Savart law – Velocity due to vortex filament

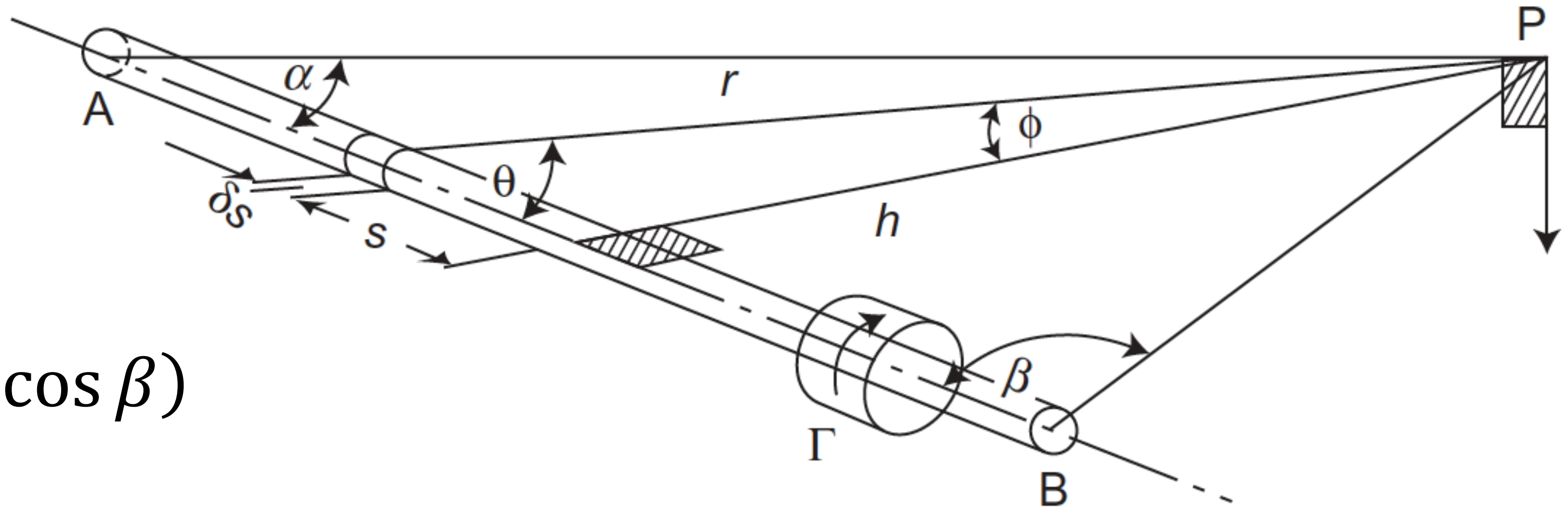
Velocity field associated with a vortex filament of circulation (strength) Γ is given by Biot-Savart law as

$$d\underline{V} = \frac{\Gamma}{4\pi}\frac{d\underline{\ell}\times\underline{r}}{|\underline{r}|^3} = \frac{\Gamma}{4\pi}\frac{d\ell\sin\theta}{r^2}$$



Special case – Straight filament of finite length

Of particular relevance for our finite wing theory



$$V = \frac{\Gamma}{4\pi h}(\cos\alpha + \cos\beta)$$

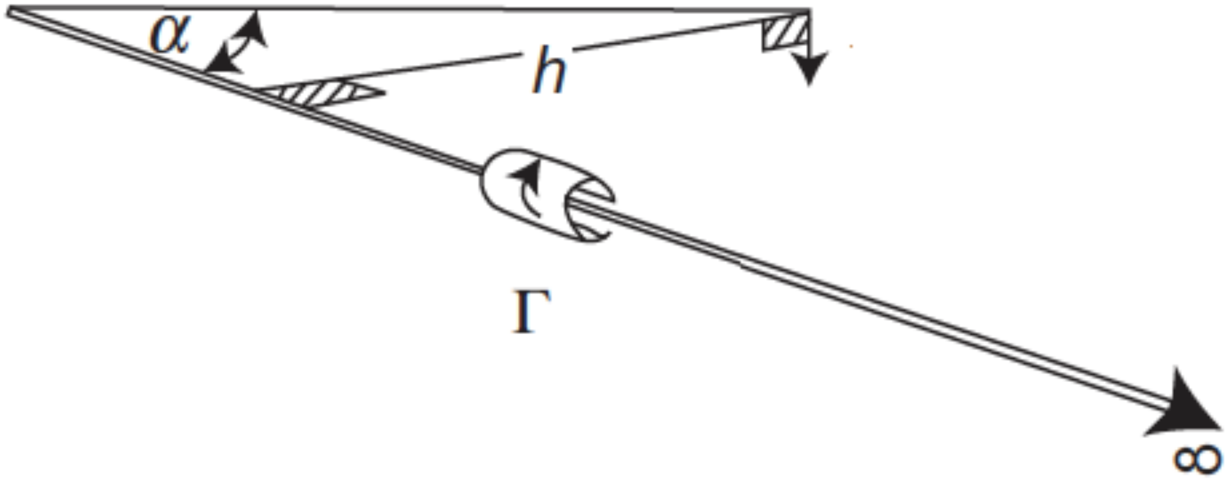
For doubly-infinite (2D) filament, this reduces to (as before)

$$V = \frac{\Gamma}{4\pi h}(\cos 0 + \cos 0) = \frac{\Gamma}{2\pi h}$$

Houghton, 2013

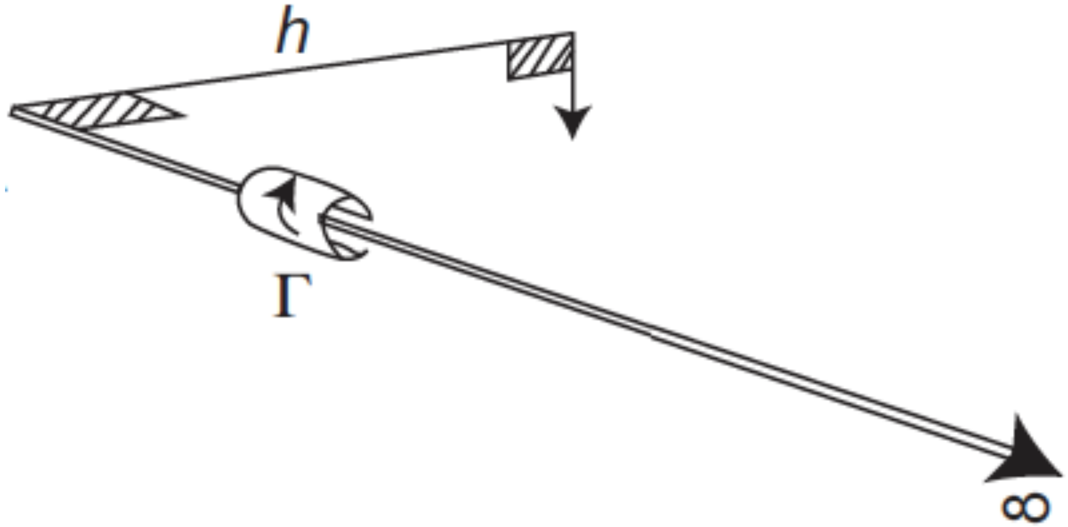
Special case – Straight semi-infinite filament

$$V = \frac{\Gamma}{4\pi h}(\cos\alpha + \cos(\beta = 0))$$
$$= \frac{\Gamma}{4\pi h}(\cos\alpha + 1)$$



For point in normal plane thru end:

$$V = \frac{\Gamma}{4\pi h}\left(\cos\frac{\pi}{2} + \cos 0\right) = \frac{\Gamma}{4\pi h}$$



As expected, this is half the velocity due to a doubly-infinite filament

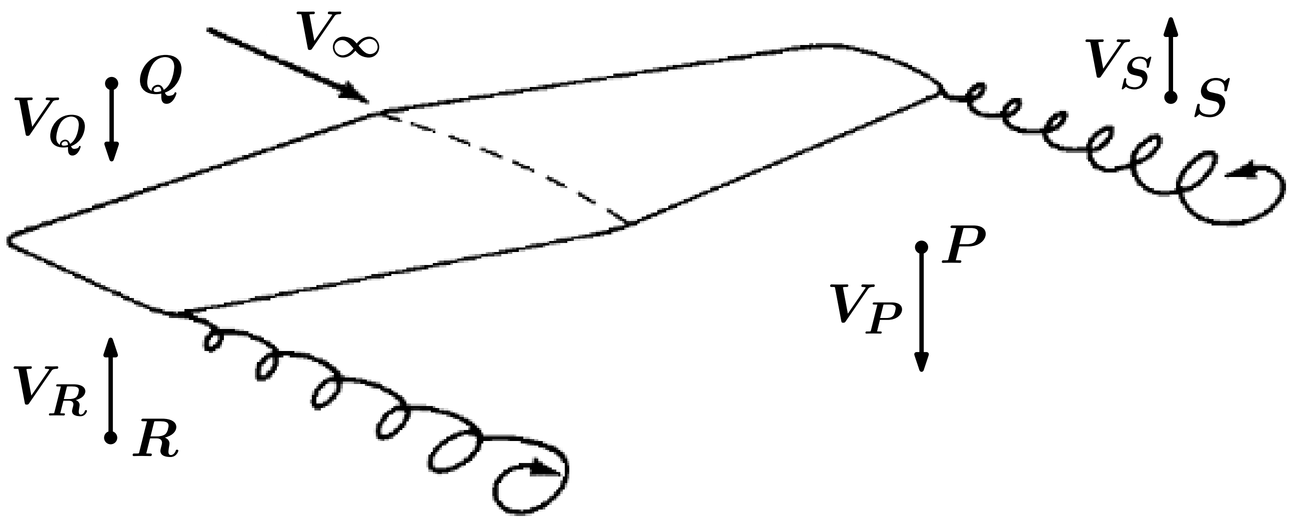
Houghton, 2013

Downwash, Induced Drag and Lift Reduction

Finite Wing Aerodynamics

Downwash

Sense of tip vortex pair is such that they induce a downward velocity (called downwash) at all points like *P* and *Q* in the plane of the wing within its span, both fore & aft of it

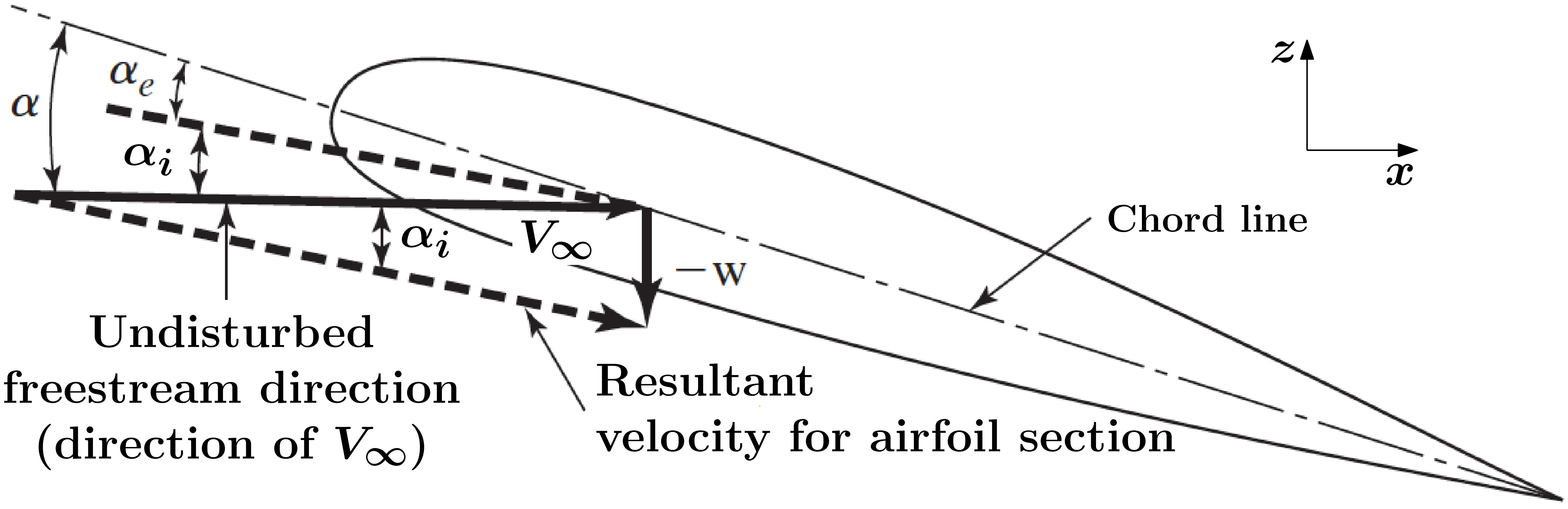


It will be shown that the velocity directions shown above are preserved even when the entire trailing vortex system is considered

Induced and effective angles of attack

Due to downwash, effective flow direction changes on wing

- (Section-averaged) induced AoA α_i
- Effective (or aerodynamic) AoA $\alpha_e (= \alpha - \alpha_i) \leq$ (geometric) AoA α



Adapted from Bertin & Cummings, 2013

Induced drag and effective lift

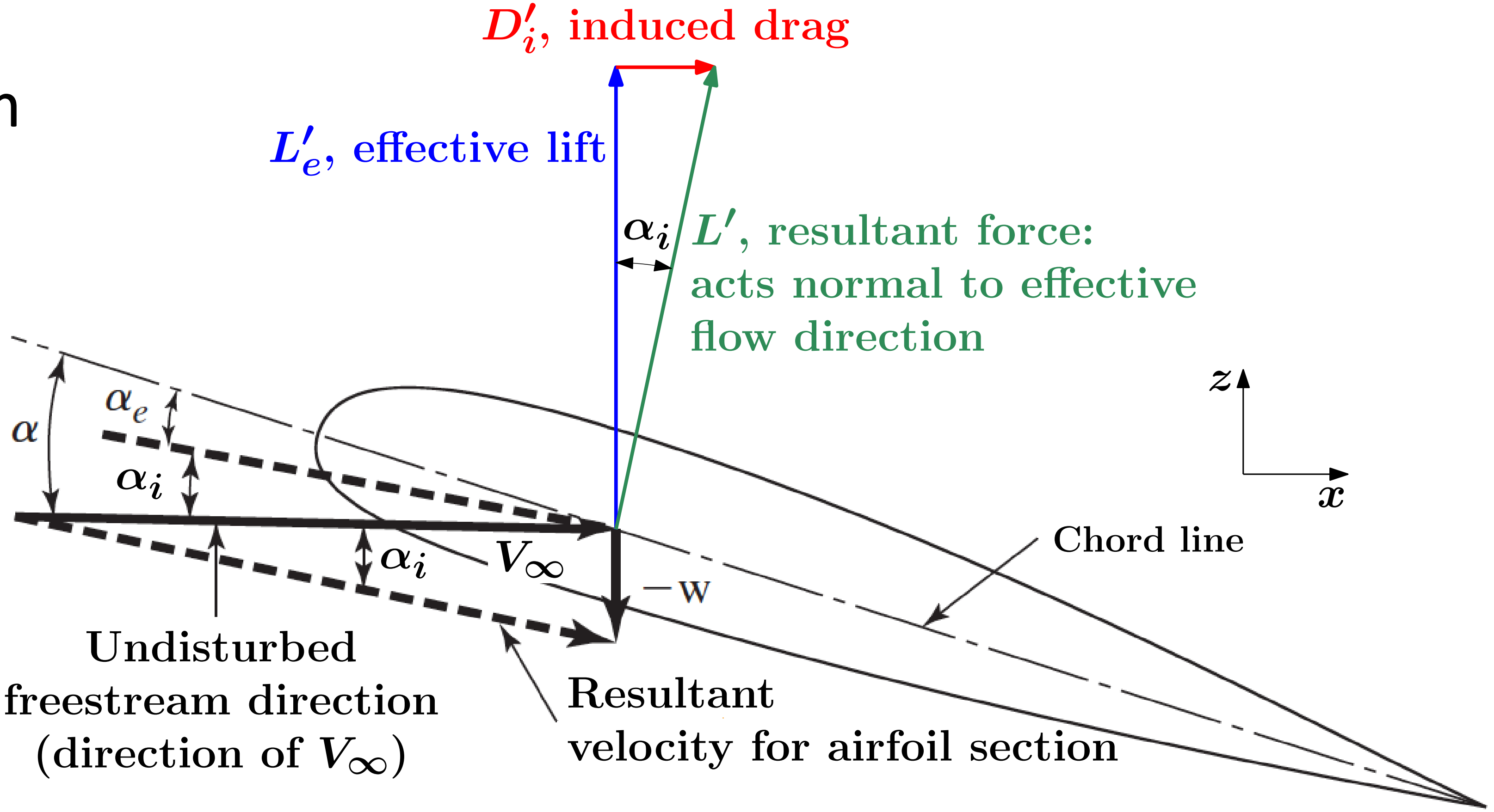
Kutta-Joukowski theorem: resultant sectional force is normal to effective flow in inviscid (potential) flow

- Induces a drag in the undisturbed freestream direction,

$$D_i' = L' \sin \alpha_i$$

- Reduces effective lift normal to the undisturbed freestream direction,

$$L_e' = L' \cos \alpha_i$$



Adapted from Bertin & Cummings, 2013

Two other ways of interpreting induced drag

1. The 3D flow produced by the trailing vortex system alters the pressure distribution on the finite wing, such that a net force is generated in the direction of the freestream
2. The trailing vortex system is carrying downstream some energy (roiling the air); this energy is 'wasted', and hence constitutes drag

N.B. the drag is induced only when there is some lift

- Hence it is 'lift-induced drag'

Downwash, and all its detrimental effects, vanish at zero lift

Thus, $\alpha_{L=0}$ is same for airfoil & wing

Hierarchy of Wing Theories

Finite Wing Aerodynamics

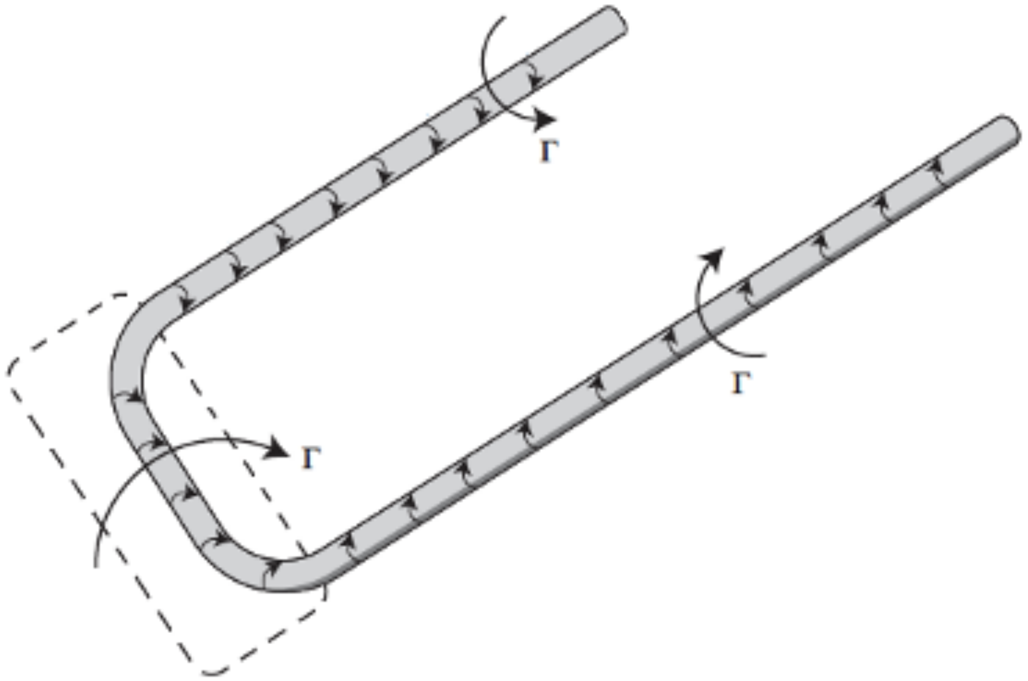
Horseshoe vortex

Essential aspect of a finite wing model is a horseshoe vortex having

- A finite-length bound filament of strength Γ aligned with wing span
- Two semi-infinite filaments of the same strength & consistent sense, trailing downstream from either end of the bound vortex

N.B. horseshoe vortex satisfies all vortex kinematics laws

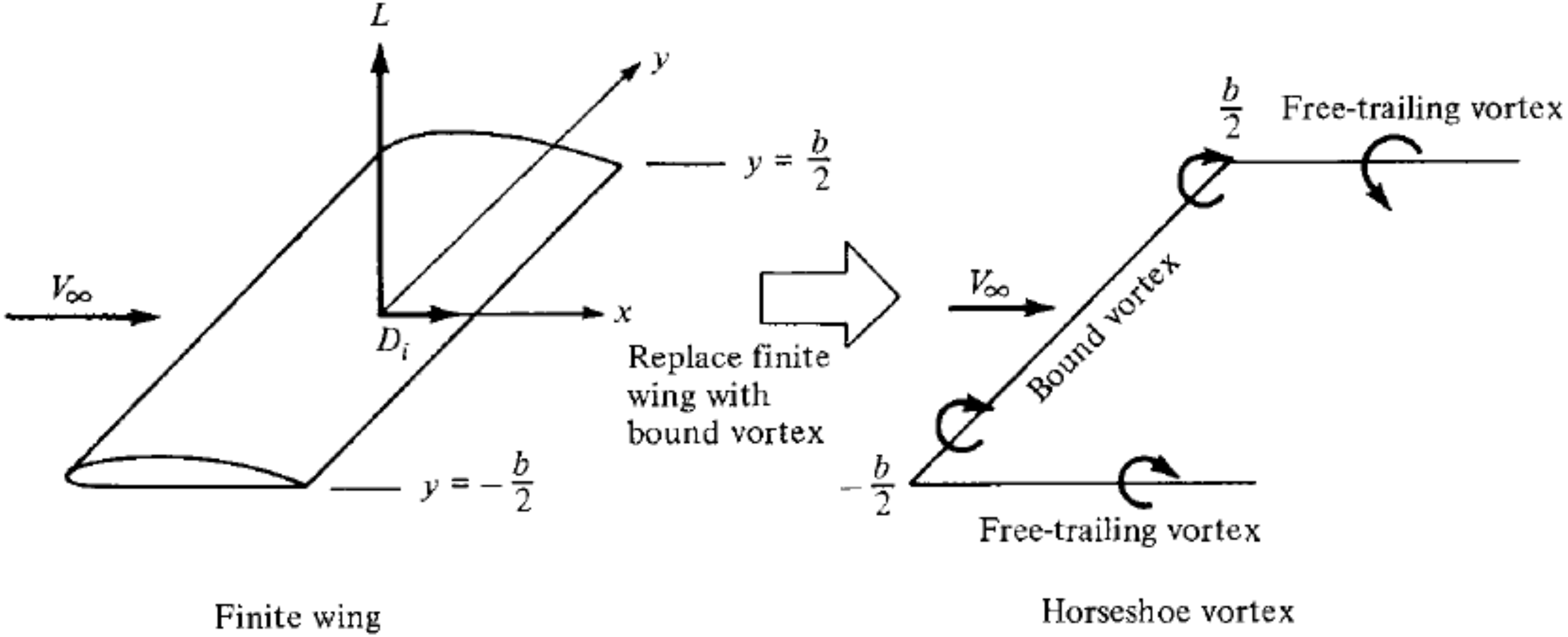
It is the building block of most wing theories



Houghton, 2013

1: Single horseshoe vortex

Singe horseshoe vortex replaces the wing and trailing vortex system

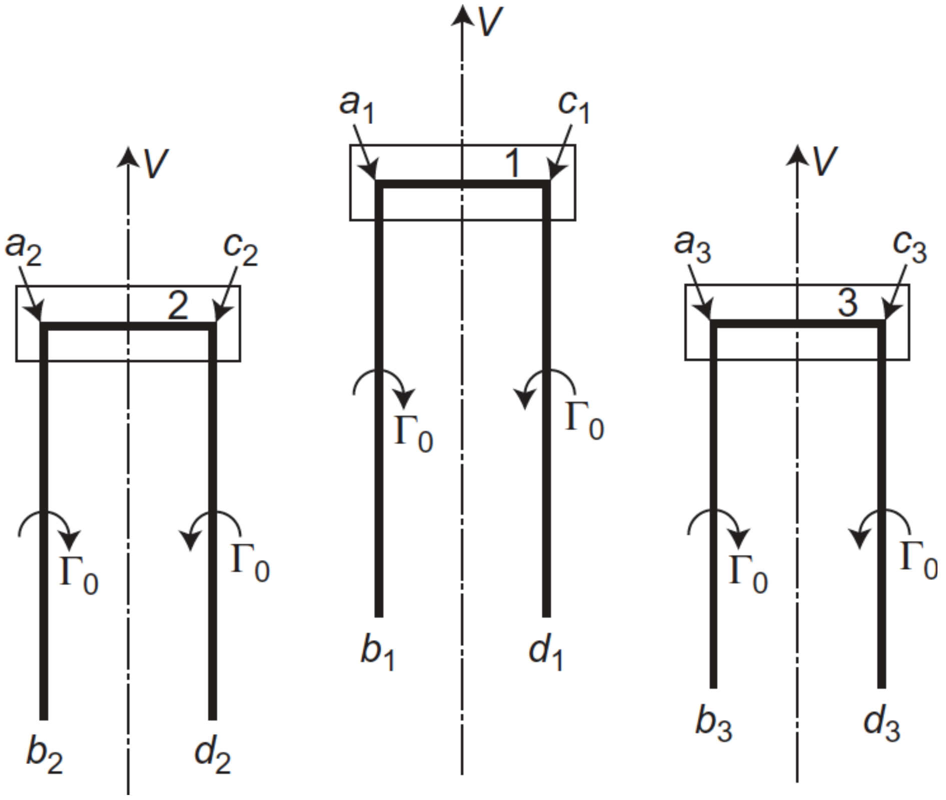


N.B. Separate theory needed to determine strength of the vortex itself

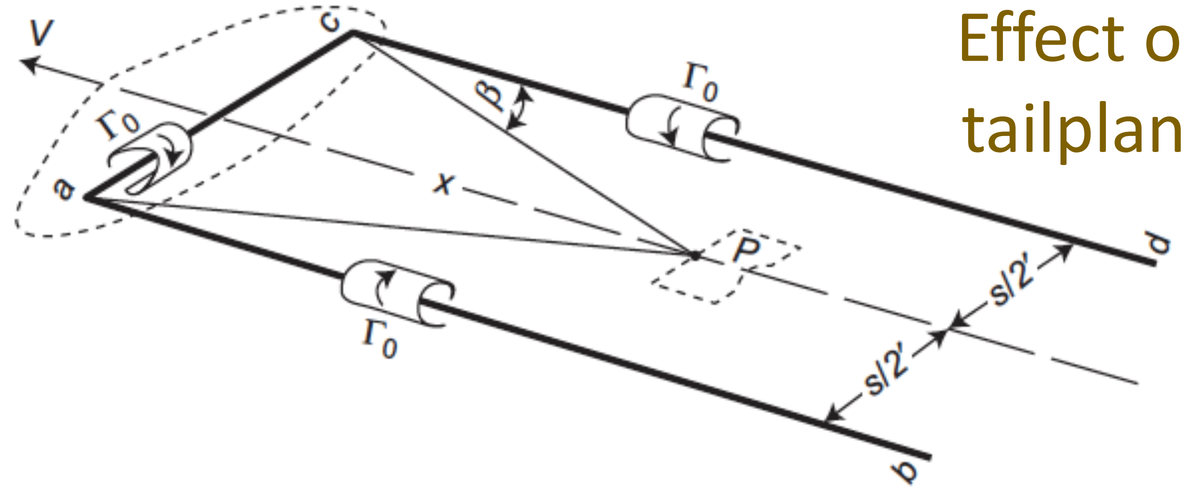
Anderson, 2011

1: Single horseshoe vortex – Utility

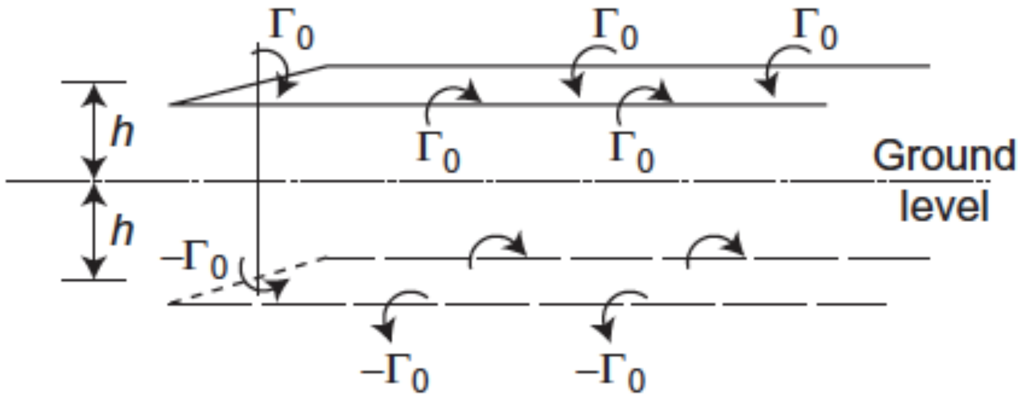
Useful for a rough estimate of effect of wing at distant points in flow



Formation flying effects



Effect on tailplane



Ground effects

Houghton, 2013

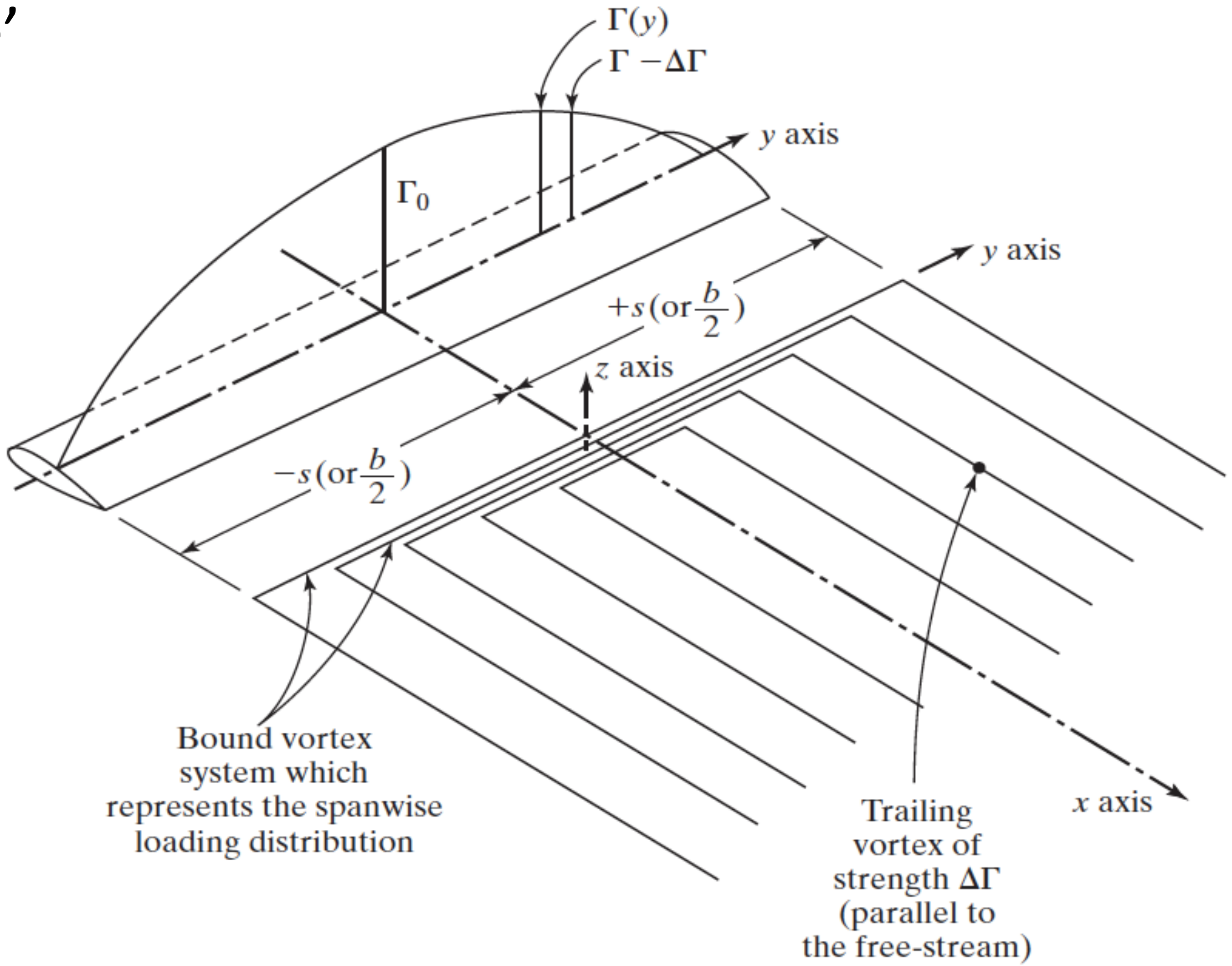
2: Prandtl's lifting line theory

Wing modelled as a 'lifting line'

- Maybe locus of centers of pressure at all sections

Horseshoe vortices, each of infinitesimal strength and varying span, are arrayed symmetrically with

- Their bound vortices coinciding with lifting line
- Their infinite tails forming parts of trailing vortex sheet



Bertin & Cummings, 2013

2: Prandtl's lifting line theory – Utility

Very useful for estimating effect of flow and wing parameters on load distribution (i.e., distribution of sectional lift)

Works well for

- Wings of large aspect ratio (> 6)
- Wing with gentle sweep
- Thin wings
- Operating at moderate AoAs

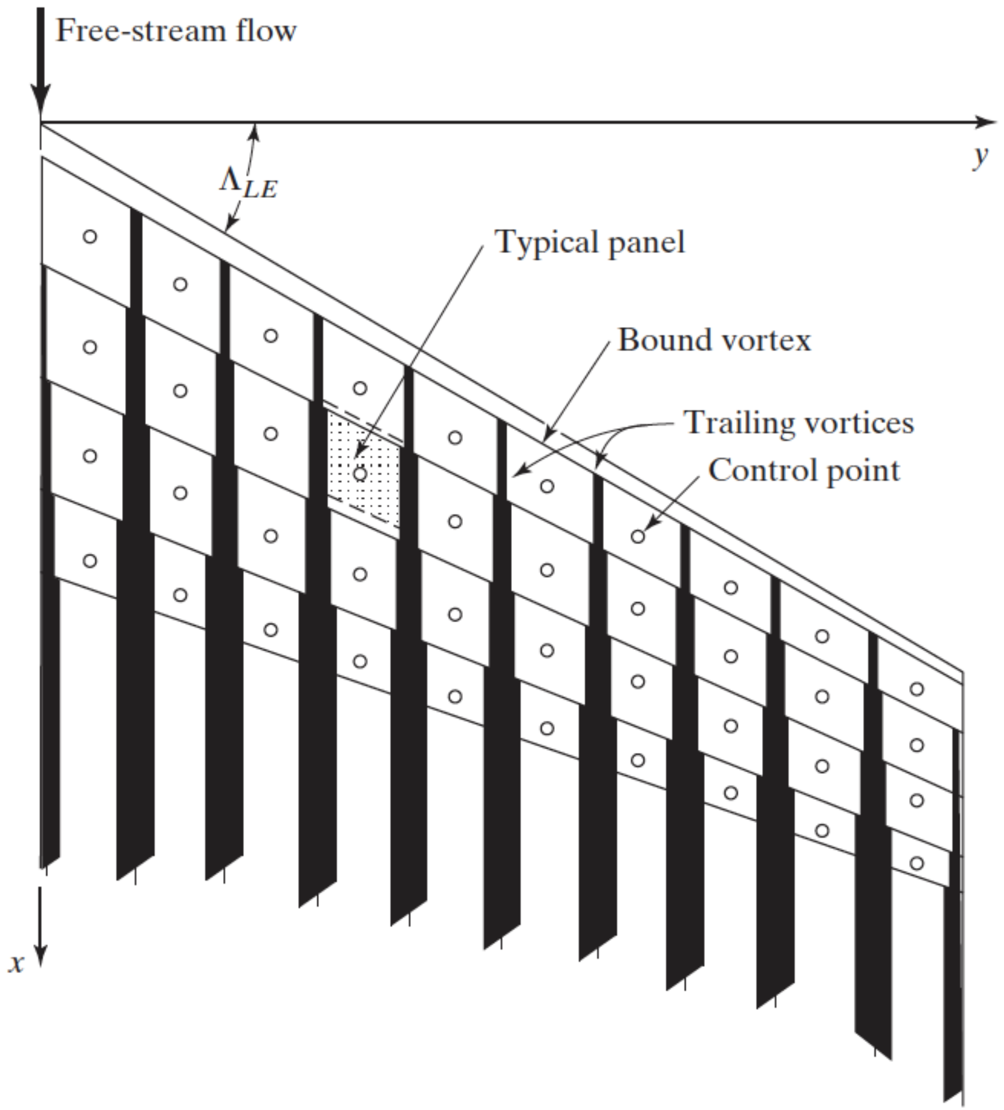
We will have much more to say about this theory soon

Bertin & Cummings, 2013

3: Lifting surface (vortex lattice) theory

Wing is modelled as a surface on which a grid of horseshoe vortices is superimposed

- Velocities induced by each horseshoe vortices at a specified control point are calculated from Biot-Savart rule
- Total contribution from all vortices and freestream at each control point should satisfy no-thru flow constraint
- Yields set of linear algebraic equations
- Extension of 2D panel method but on flat ‘lifting surface’ and not on wing surface

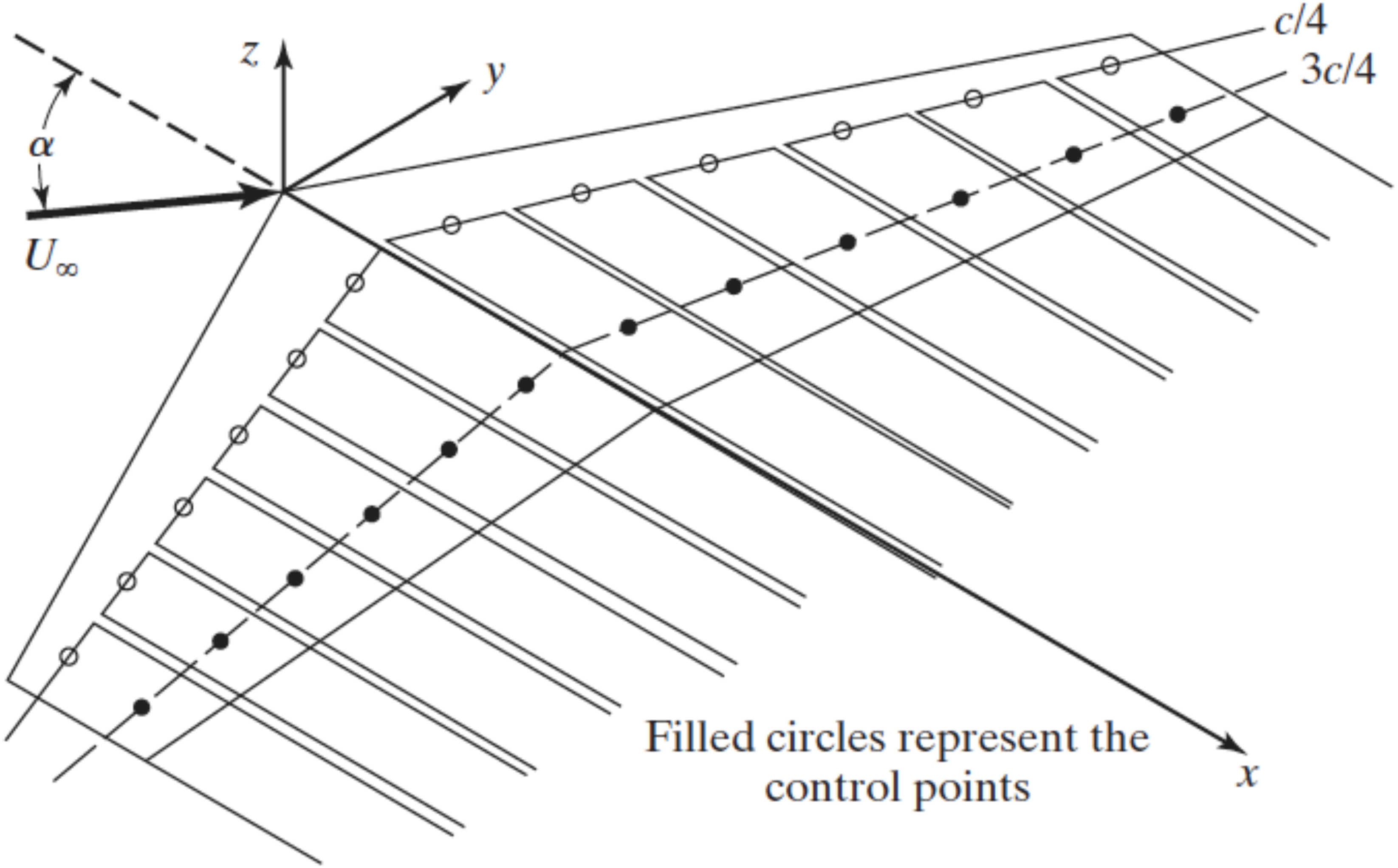


Bertin & Cummings, 2013

3: Lifting surface theory – Swept wing example

Very simple example, with one panel in chordwise direction but multiple panels in spanwise direction

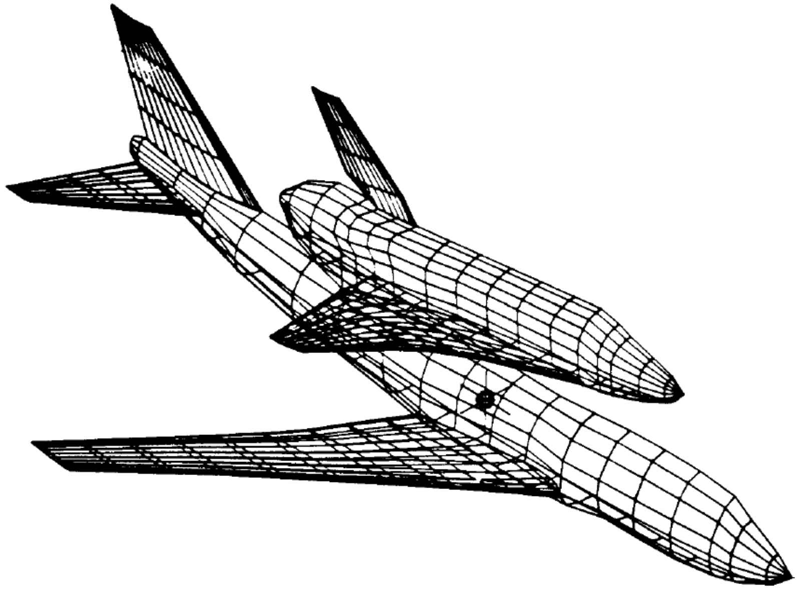
- Bound vortex usually goes thru $c/4$ point
- Control point usually at $3c/4$ point
- See Bertin for reason



Bertin & Cummings, 2013

4: Computational (3D) panel method

Wing/fuselage surface gridded w/ vortex (or doublet) & source panels



Panel distribution on Boeing 747 carrying space shuttle orbiter

Orientation of trailing vortices is solved for instead of being assumed

- In lifting surface theory, trailing vortices are assumed to be aligned with the longitudinal plane of the wing (chordal plane)

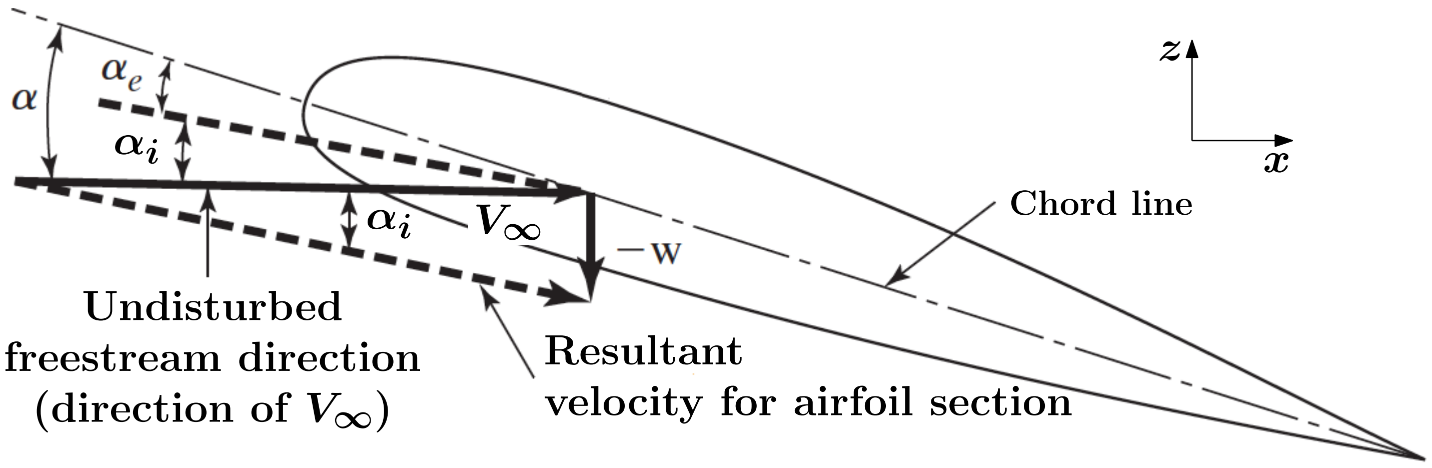
Anderson, 2011

Prandtl’s Lifting Line Theory

Finite Wing Aerodynamics

Assumptions of lifting line theory

1. Aspect ratio is high enough such that spanwise flow is insignificant
    - Each spanwise section of the wing is analyzed as a 2D airfoil, but with approaching flow affected by local (small) downwash
2. Thin airfoil theory assumptions apply at each section
3. Sectional airfoil is modelled as a single vortex (not a vortex sheet) of strength equal to the net circulation around the section



Geometric characteristics of wing

Wing may be twisted; geometric AoA, $\alpha(y)$, at each section may vary

- Recall wash-in and wash-out

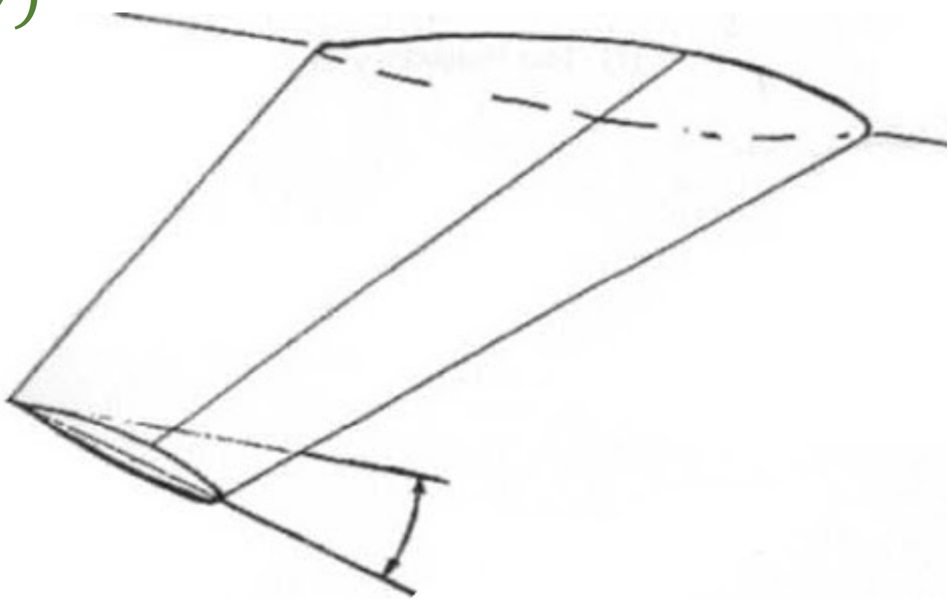
Airfoil shape may vary from one section to another

- In thin airfoil theory, lift slope doesn't change with airfoil shape, but there is variation of zero-lift AoA, $\alpha_{L=0}(y)$

Planform shape will usually not be rectangular (it's generally tapered)

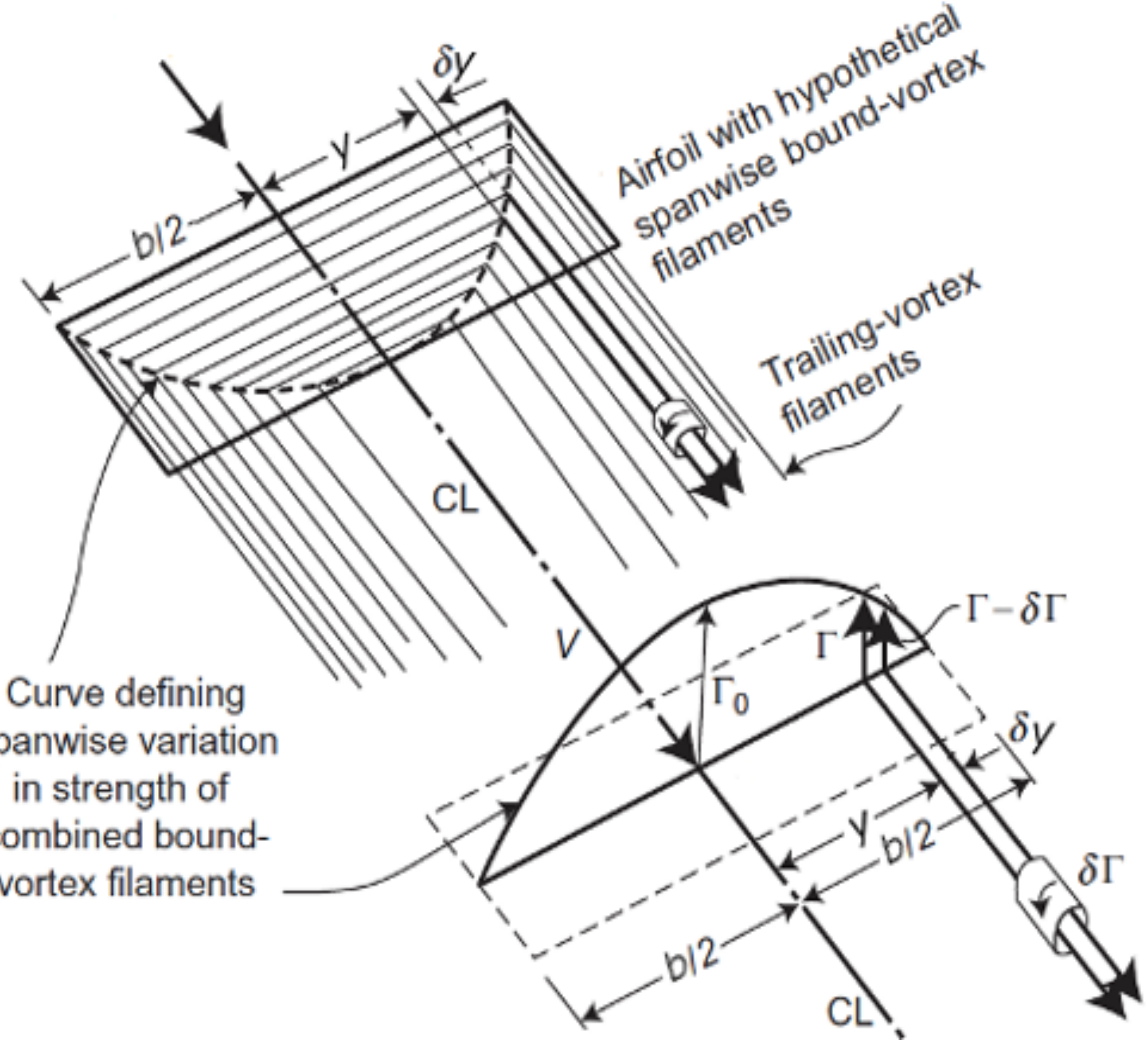
- Chord length, $c(y)$, will vary from one section to another

Wing sweep can't be addressed



Aerospaceweb.org

Lifting line model



Bertin & Cummings, 2013

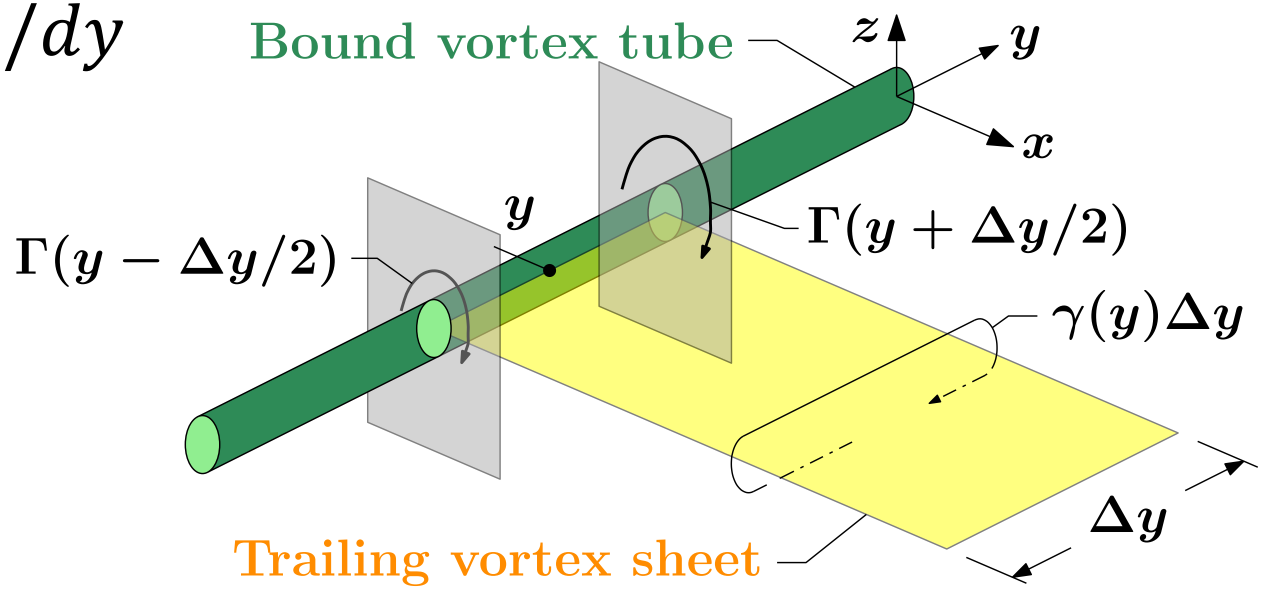
Lifting line circulation & trailing vortex sheet density

Trailing vortex sheet in $z = 0$ plane is semi-infinite, with uniform circulation density in x but variation in y; i.e., γ is function of y alone

In rear view ($y - z$ plane), γ is positive if CW (in A/D sign convention)

By Helmholtz' 1st law: $\gamma(y)\Delta y = \Gamma(y + \Delta y/2) - \Gamma(y - \Delta y/2)$

At limit $\Delta y \to 0$, $\gamma(y) = d\Gamma(y)/dy$



Trailing sheet density implications

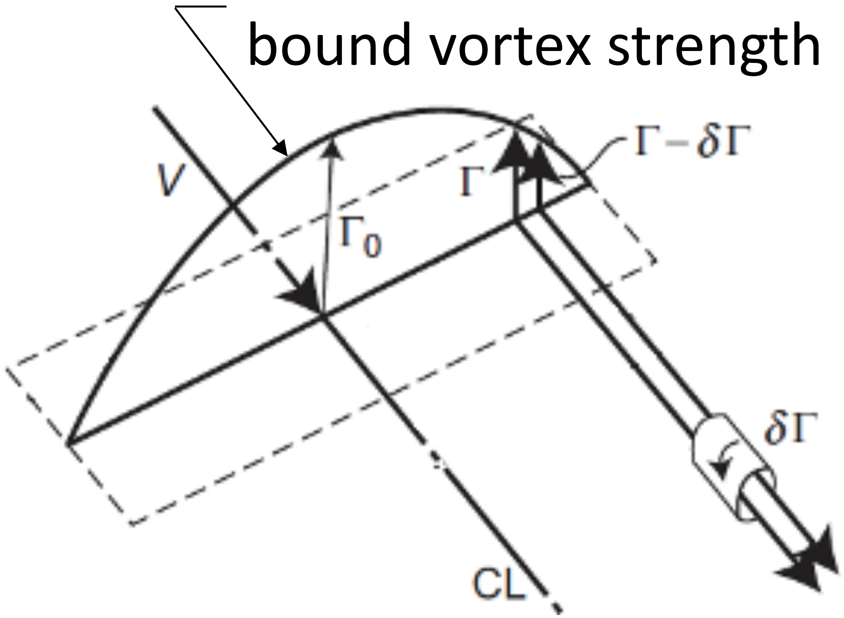
$$\gamma(y) = d\Gamma(y)/dy$$

Trailing sheet circulation density at any spanwise station y equals local spanwise gradient of circulation around bound vortex

In rear view, trailing sheet vorticity is negative (CCW) on starboard side; so $\Gamma(y)$ decreases thereat from root to tip (+ve y)

Gradient is mildest at root, sharpest at tip $\Longrightarrow$ tip vortex is strongest

Spanwise variation of bound vortex strength



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Downwash by trailing vortex sheet at lifting line

Consider infinitesimal strip of trailing vortex sheet at y

- Lifting line is in end plane of this vortex filament

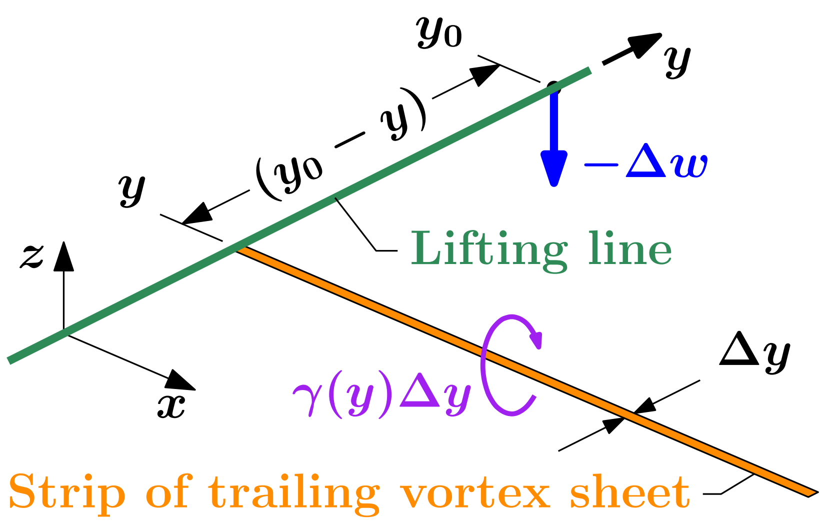
Downwash at y_0 due to this semi-infinite filament from Biot-Savart law:

$$-\Delta w(y_0) = \frac{\gamma(y)\Delta y}{4\pi(y_0 - y)} = \frac{(d\Gamma(y)/dy)\Delta y}{4\pi(y_0 - y)}$$

Total downwash due to trailing vortex sheet:

$$w(y_0) = -\int_{-b/2}^{b/2} \frac{d\Gamma(y)/dy}{4\pi(y_0 - y)} dy$$

This is invariably negative, as argued earlier



Induced AoA at bound vortex on lifting line

There is no self-induced downwash due to bound vortex on lifting line

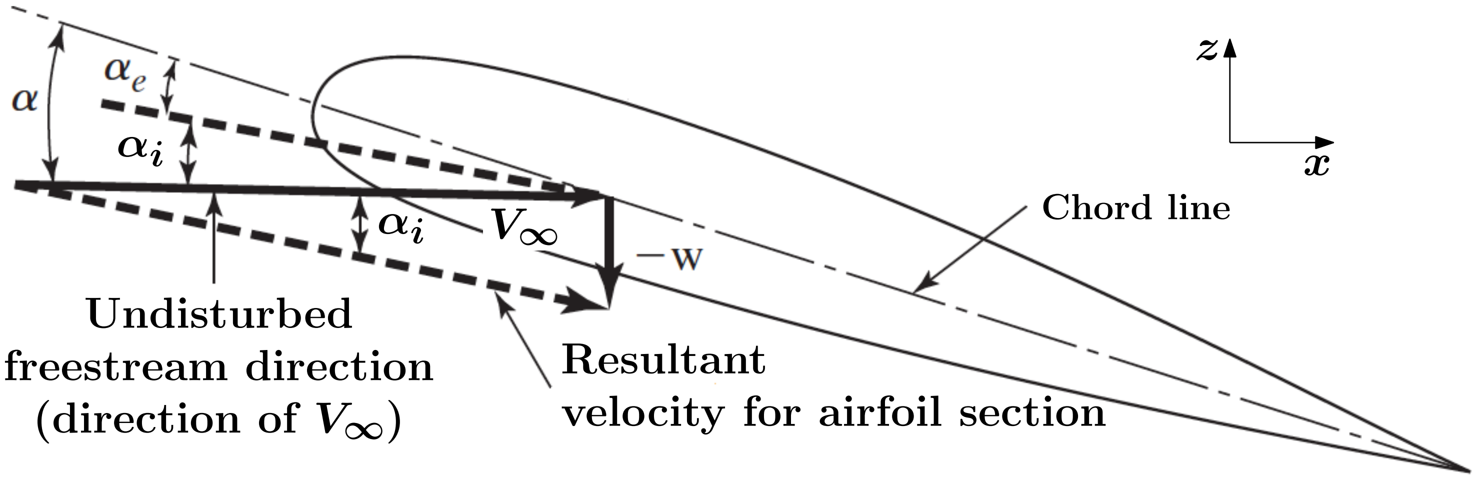
So, all the downwash on lifting line is solely due to trailing vortex sheet

Induced AoA (assumed small) at lifting line at spanwise station y is

$$\alpha_i(y) = \tan^{-1}\left(\frac{-w(y)}{V_\infty}\right) \approx -\frac{w(y)}{V_\infty} = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{d\Gamma(\eta)/d\eta}{(y-\eta)}\, d\eta$$

Effective AoA,

$$\alpha_e(y) = \alpha(y) - \alpha_i(y)$$



Adapted from Anderson, 2011

Lifting line equation

Assume that 2D sectional lift slope $a_0(y)$ & 2D zero-lift AoA $\alpha_{L=0}(y)$ are known from 2D experiments, computations or thin airfoil theory, and are still valid for the wing section due to the high aspect ratio

- Then, sectional lift coefficient is $c_l(y) = a_0(y)[\alpha_e(y) - \alpha_{L=0}(y)]$

But, by Kutta-Joukowsky theorem, $\rho V_\infty \Gamma(y) = 0.5\rho V_\infty^2 c(y) c_l(y)$

Putting everything together, we have the following integro-differential equation for the unknown $\Gamma(y)$, called the lifting-line equation

$$\frac{2\Gamma(y)}{V_\infty a_0(y) c(y)} + \underbrace{\frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{d\Gamma(\eta)/d\eta}{y-\eta} d\eta}_{\alpha_i(y)} = \alpha(y) - \alpha_{L=0}(y)$$

Outcome from lifting line theory

Once $\Gamma(y)$ is found by solving the lifting-line theory, one can find

- Spanwise resultant sectional force distribution, $L'(y) = \rho V_\infty \Gamma(y)$
- Spanwise effective lift (load) distribution,
$$L'_e(y) = L'(y)\cos\alpha_i(y) \approx L'(y) = \rho V_\infty \Gamma(y)$$
- Wing lift coefficient,
$$C_L = \frac{L}{q_\infty S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y) dy$$
- Induced drag coefficient,
$$C_{D,i} = \frac{D_i}{q_\infty S} \approx \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y) dy$$

Special Solution of the Lifting Line Equation

Finite Wing Aerodynamics

Trigonometric coordinate transformation

Lifting line equation in $\eta - y$ variables:

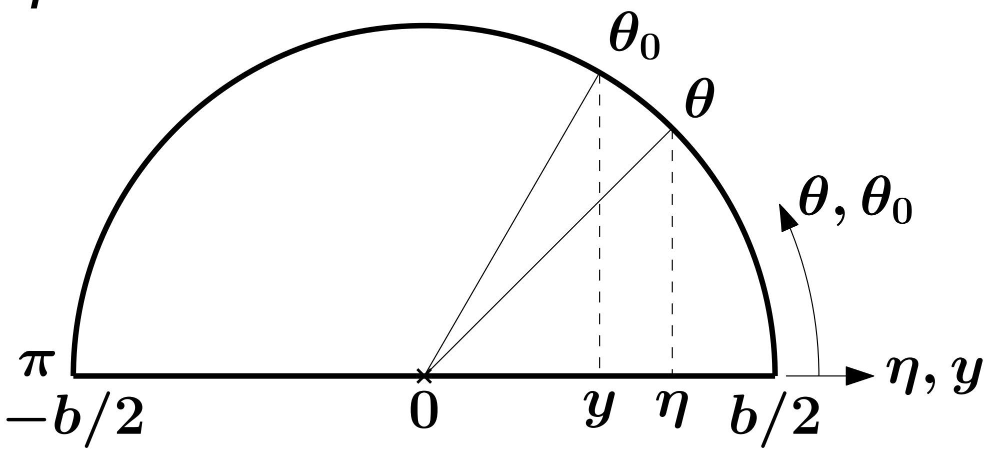
$$\frac{2\Gamma(y)}{V_\infty a_0(y)c(y)} + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{d\Gamma(\eta)/d\eta}{y-\eta}\,d\eta = \alpha(y) - \alpha_{L=0}(y)$$

Use trigonometric change of variables:

$$\eta = \frac{b}{2}\cos\theta\,, \qquad y = \frac{b}{2}\cos\theta_0$$

$$d\eta = -b/2\sin\theta\, d\theta;$$

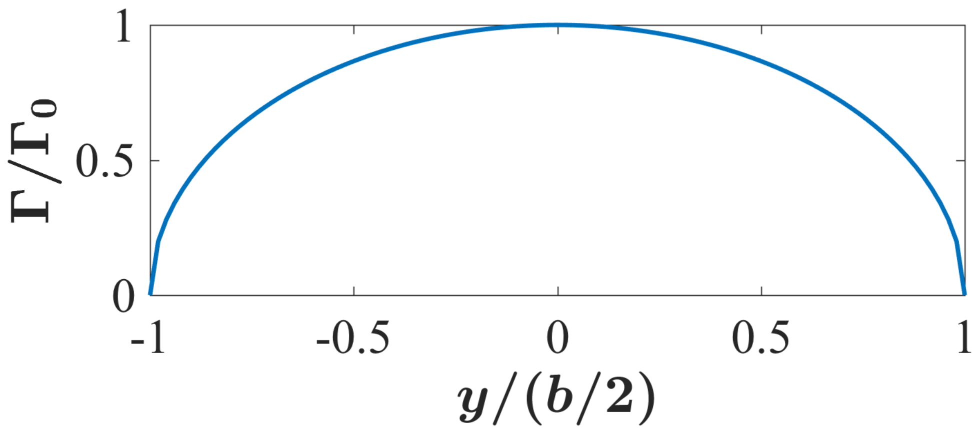
$$\eta = b/2 \Longrightarrow \theta = 0;\ \eta = -b/2 \Longrightarrow \theta = \pi$$



This is very similar to the Glauert transformation in thin airfoil theory

Elliptic load distribution

(Apparently arbitrarily) assume Γ (& hence load distribution) is elliptic



Γ_0 is the bound vortex circulation at center-plane

Eqn. of ellipse: $\left(\frac{\Gamma}{\Gamma_0}\right)^2 + \left(\frac{y}{b/2}\right)^2 = 1 \implies \Gamma = \Gamma_0\sqrt{1-\frac{4y^2}{b^2}} = \Gamma_0 \sin\theta_0$

Then, $\frac{d\Gamma}{dy}(y) = -\frac{\Gamma_0}{b}\frac{4y/b}{\sqrt{1-(y/(b/2))^2}} \implies \frac{d\Gamma}{dy}(\theta_0) = -\frac{\Gamma_0}{b}\frac{2\cos\theta_0}{\sin\theta_0}$

Elliptic load distribution: Downwash & induced AoA

Recall: Downwash, $w(y) = -\dfrac{1}{4\pi}\displaystyle\int_{-b/2}^{b/2} \dfrac{d\Gamma/d\eta(\eta)}{y-\eta}\, d\eta$

With $\eta|y = (b/2)\cos\theta|\theta_0$, $w(\theta_0) = -\dfrac{1}{4\pi}\displaystyle\int_0^{\pi} \dfrac{-d\Gamma/d\eta(\theta)}{\cos\theta - \cos\theta_0}\sin\theta\, d\theta$

With elliptic load distribution, $d\Gamma/d\eta(\theta) = -2(\Gamma_0/b)\cos\theta \,/\, \sin\theta$

Downwash at a section, $w(\theta_0) = \dfrac{-\Gamma_0}{2\pi b}\displaystyle\int_0^{\pi} \dfrac{\cos\theta\, d\theta}{\cos\theta - \cos\theta_0} = -\dfrac{\Gamma_0}{2b}$

Induced AoA, $\alpha_i = -\dfrac{w}{V_\infty} = \dfrac{\Gamma_0}{2bV_\infty}$

Downwash and induced angle of attack are constant along span!

Elliptic load distribution: Possible realization

Assume that wing is untwisted, and has same airfoil at all sections

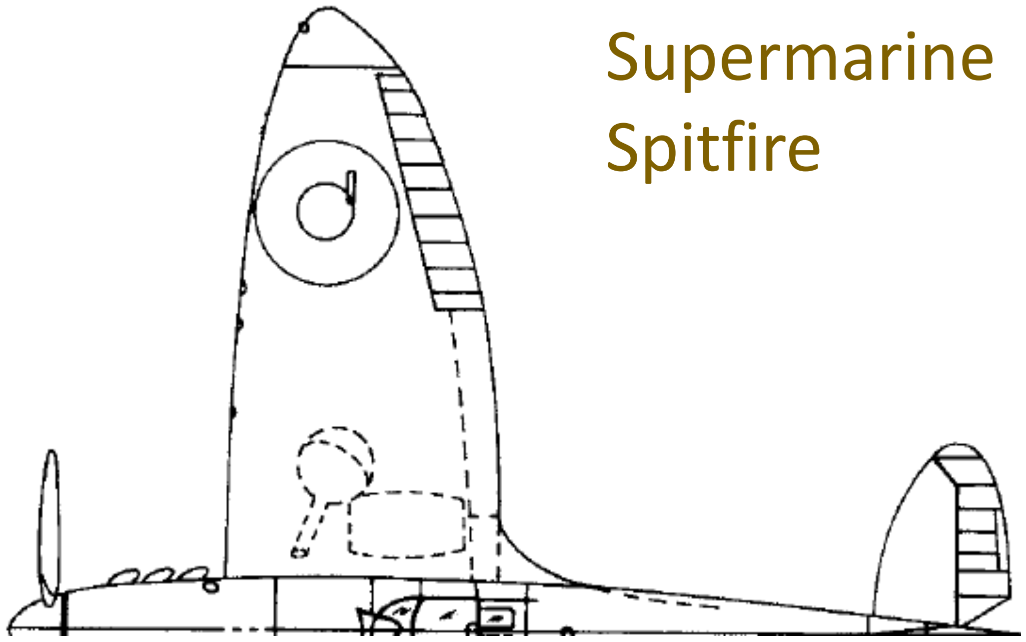
Then, recalling that $\alpha_i = -w/V_\infty = \Gamma_0/2bV_\infty$, the lifting line eqn. gives

$$\frac{2\Gamma(y)}{V_\infty a_0 c(y)} = \alpha_e - \alpha_{L=0} = \alpha - \alpha_i - \alpha_{L=0} = \text{const.}$$

So, one way of realizing an elliptic load distribution is to have an untwisted wing of consistent sectional shape and an elliptic planform with chord variation given by

$$c = c_0\sqrt{1-(y/(b/2))^2}$$

N.B. Quarter-chord points form lifting line



Supermarine Spitfire

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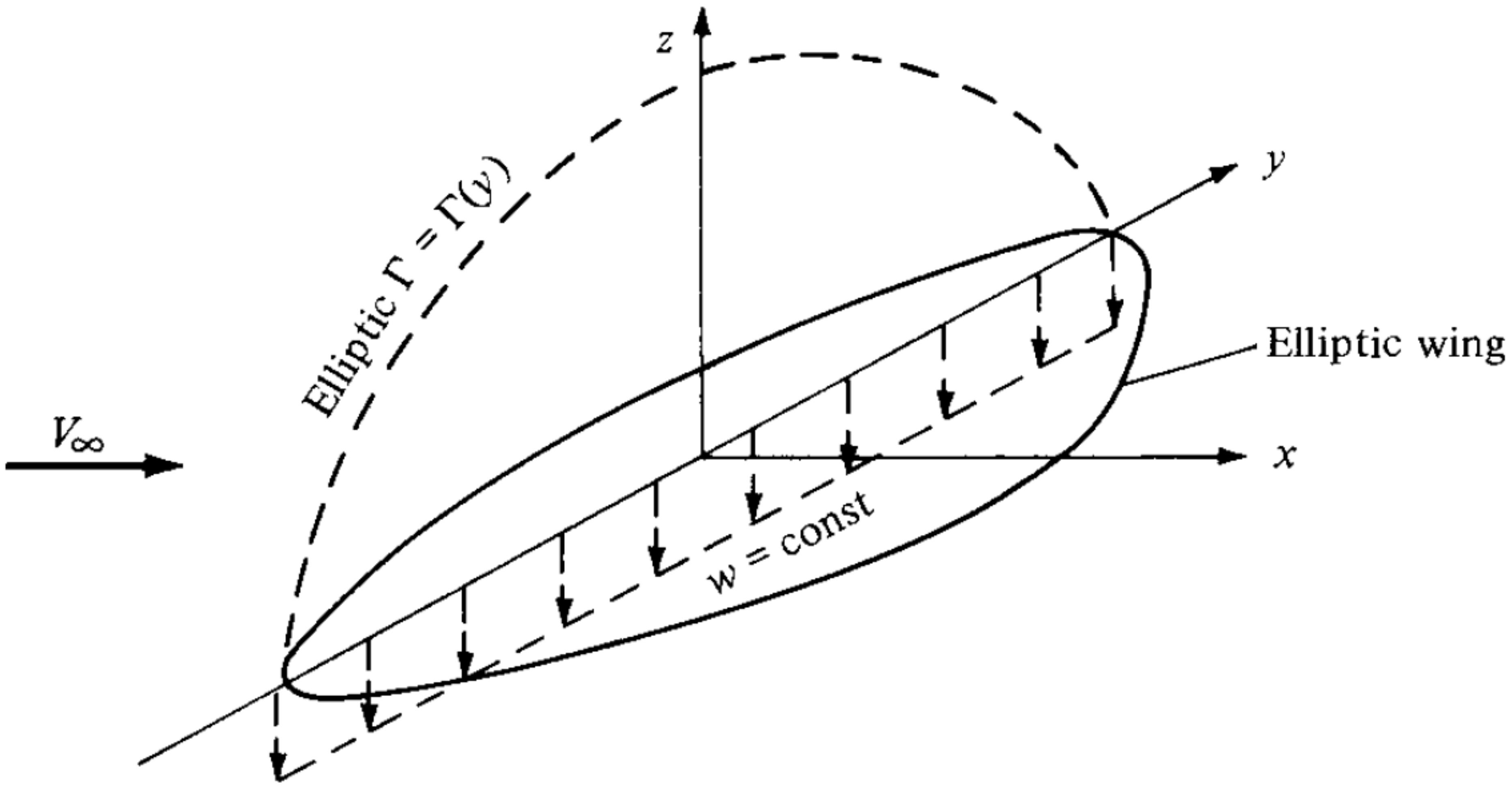
Elliptic lift distribution & planform, const. downwash



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Finding Γ_0 for elliptic wing

Elliptic wing: $\Gamma = \Gamma_0 \sqrt{1 - \left(\frac{y}{b/2}\right)^2}, \quad \alpha_i = \frac{\Gamma_0}{2bV_\infty}, \quad c = c_0 \sqrt{1 - \left(\frac{y}{b/2}\right)^2}$

Lifting line eqn.: $\frac{2\Gamma(y)}{V_\infty a_0 c(y)} = \alpha - \alpha_i - \alpha_{L=0}$

$$\Longrightarrow \frac{2\Gamma_0}{V_\infty (2\pi) c_0} = \alpha - \frac{\Gamma_0}{2bV_\infty} - \alpha_{L=0}, \qquad \text{with } a_0 = 2\pi$$

$$\Longrightarrow \frac{\Gamma_0}{V_\infty} = \frac{\alpha - \alpha_{L=0}}{1/(\pi c_0) + 1/(2b)}$$

Elliptic load distribution: Relating α_i to $\boldsymbol{C_L}$

Lift on wing, given that α_i is small, is

$$L = \rho V_\infty \int_{-b/2}^{b/2} \Gamma dy = \rho V_\infty \Gamma_0 \int_{\pi}^{0} \sin\theta_0 \left(-\frac{b}{2} \sin\theta_0 \, d\theta_0 \right) = \frac{\rho V_\infty \Gamma_0 b \pi}{4}$$

But, $C_L = \dfrac{2L}{\rho V_\infty^2 S}$ $\quad \Longrightarrow \Gamma_0 = \dfrac{2bV_\infty C_L}{\pi \mathrm{AR}}$ $\quad \left[\because \mathrm{AR} = \dfrac{b}{\bar{c}} = \dfrac{b^2}{S} \right]$

Thus, induced AoA, $\alpha_i = \dfrac{\Gamma_0}{2bV_\infty} = \dfrac{C_L}{\pi \mathrm{AR}}$

- Greater is AR, lesser is the 3D effect, and hence lesser is α_i
- Lesser is C_L, lesser is pressure differential across wing, and hence α_i

Elliptic load distribution: Relating $\boldsymbol{C_{D,i}}$ to $\boldsymbol{C_L}$

Induced drag coefficient, $C_{D,i} = \dfrac{D_i}{q_\infty S} = C_L \dfrac{D_i}{L}$

But, since α_i is constant and small, $\dfrac{D_i'}{L'} = \dfrac{D_i}{L} = \tan\alpha_i \approx \alpha_i = \dfrac{C_L}{\pi \mathrm{AR}}$

Thus, induced drag coefficient, $C_{D,i} = C_L \alpha_i = \dfrac{C_L^2}{\pi \mathrm{AR}}$

This substantiates why D_i (and $C_{D,i}$) is called ‘drag due to lift’

- It’s the thrust required by an aircraft to generate the lift
- Since $C_{D,i} \propto C_L^2$, it’s a substantial fraction (~ 0.25) of the total drag at high C_L (as in take-off and landing)
- Another aspect of $C_{D,i}$ is its inverse proportionality to AR

Elliptic load distribution: Wing lift slope, $\boldsymbol{a}$

Consider an untwisted wing (constant α) with same airfoil shape at all sections (constant $\alpha_{L=0}$, a_0 and c_l)

Additionally, if planform is elliptic, then α_i (hence α_e) is also constant

Now, at each section, $c_l = a_0(\alpha_e - \alpha_{L=0})$

Consistently, for the overall wing, $C_L = a_0(\alpha_e - \alpha_{L=0})$

Thus, in this special case, wing lift slope is airfoil lift slope if α_e replaces α

$$C_L = a_0(\alpha - \alpha_i - \alpha_{L=0}) = a_0\left(\alpha - \frac{C_L}{\pi \mathrm{AR}} - \alpha_{L=0}\right) \Rightarrow C_L = \frac{a_0(\alpha - \alpha_{L=0})}{1 + a_0/\pi \mathrm{AR}}$$

So, in this special case, wing lift slope $a \coloneqq \dfrac{dC_L}{d\alpha} = \dfrac{a_0}{1 + a_0/\pi \mathrm{AR}} < a_0$

Lesser is AR, more is the degradation of lift slope

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General Solution of Lifting Line Equation

Finite Wing Aerodynamics

General lift distribution

One can expand the (unknown) $\Gamma(\theta)$ in a sine-series in θ, noting that cosine terms are excluded on grounds that Γ must vanish at wing tips

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{\infty} A_n \sin n\theta \approx 2bV_\infty \sum_{n=1}^{N} A_n \sin n\theta$$

N.B. with elliptic lift distribution $(\Gamma = \Gamma_0 \sin\theta)$, only A_1 is non-trivial

Anderson shows that, by requiring that lifting line equation be satisfied at N discrete points along span, one obtains a set of N linear algebraic equations in the (assumed) N unknown A_n's

General expression for induced drag

Anderson shows that $C_L = A_1 \pi \mathrm{AR}$

Alhough C_L depends only on A_1, one has to solve for all A_n's to find it

Also, Anderson shows that the induced drag coefficient is given by

$$C_{D,i} = \pi \mathrm{AR}\, A_1^2 \left[1 + \underbrace{\sum_{n=2}^{N} \left(\frac{A_n}{A_1}\right)^2}_{\delta > 0}\right] = \frac{C_L^2}{\pi \mathrm{AR}} \underbrace{(1+\delta)}_{e^{-1},\, e<1} = \frac{C_L^2}{\pi e \mathrm{AR}}$$

Recall that for elliptic load distribution, $C_{D,i} = C_L^2/(\pi \mathrm{AR})$

This justifies our initial interest in this special case; it's the most efficient

e is called the span efficiency factor

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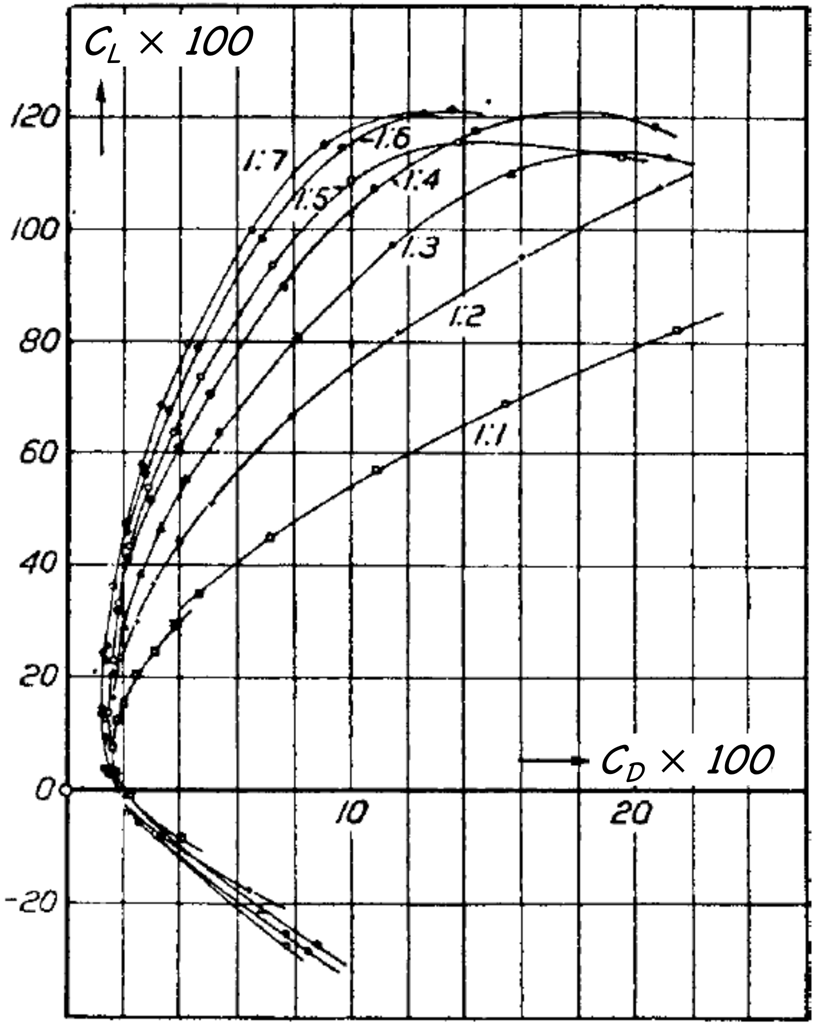
Prandtl's classic rectangular wing data for 7 AR's



Prandtl, L., Applications of modern hydrodynamics to aeronautics, NACA Report No. 116, 1921

Prandtl's classic drag polar data – Explained

For rectangular wing, total drag ≈ infinite-wing (parasitic) drag (c_d) + finite-wing lift-induced drag ($C_{D.i}$)

$$C_D \approx c_d + C_{D.i} = c_d + C_L^2/(\pi e \mathrm{AR})$$

Total (measured) drag (as a function of lift) at two AR's:

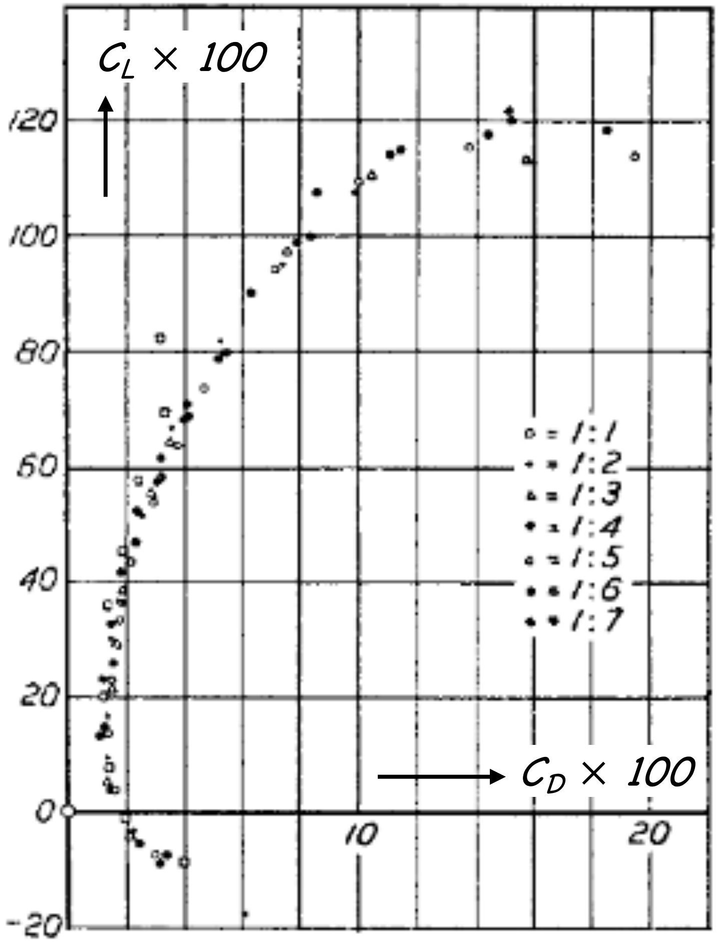
$$C_{D,1}(C_L) = c_d + \frac{C_L^2}{\pi e \mathrm{AR}_1}, \qquad C_{D,2}(C_L) = c_d + \frac{C_L^2}{\pi e \mathrm{AR}_2}$$

$$\therefore C_{D,2}(C_L) = C_{D,1}(C_L) + \frac{C_L^2}{\pi e}\left(\frac{1}{\mathrm{AR}_2} - \frac{1}{\mathrm{AR}_1}\right)$$

In particular, one can find *e* from AR = 5 (say).

Then, for any AR, Prandtl scaled drag polar to AR = 5:

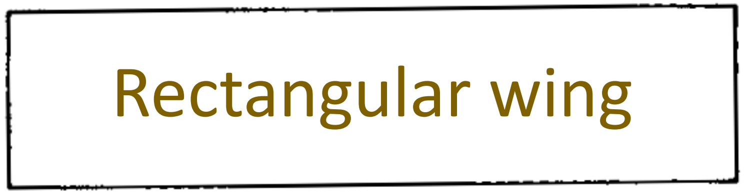
$$C_{D,\mathrm{AR}=5}(C_L) = C_D\,(C_L) + \frac{C_L^2}{\pi e}\left(\frac{1}{5} - \frac{1}{\mathrm{AR}}\right)$$



Previous data scaled to AR = 5

Prandtl, L., Applications of modern hydrodynamics to aeronautics, NACA Report No. 116, 1921

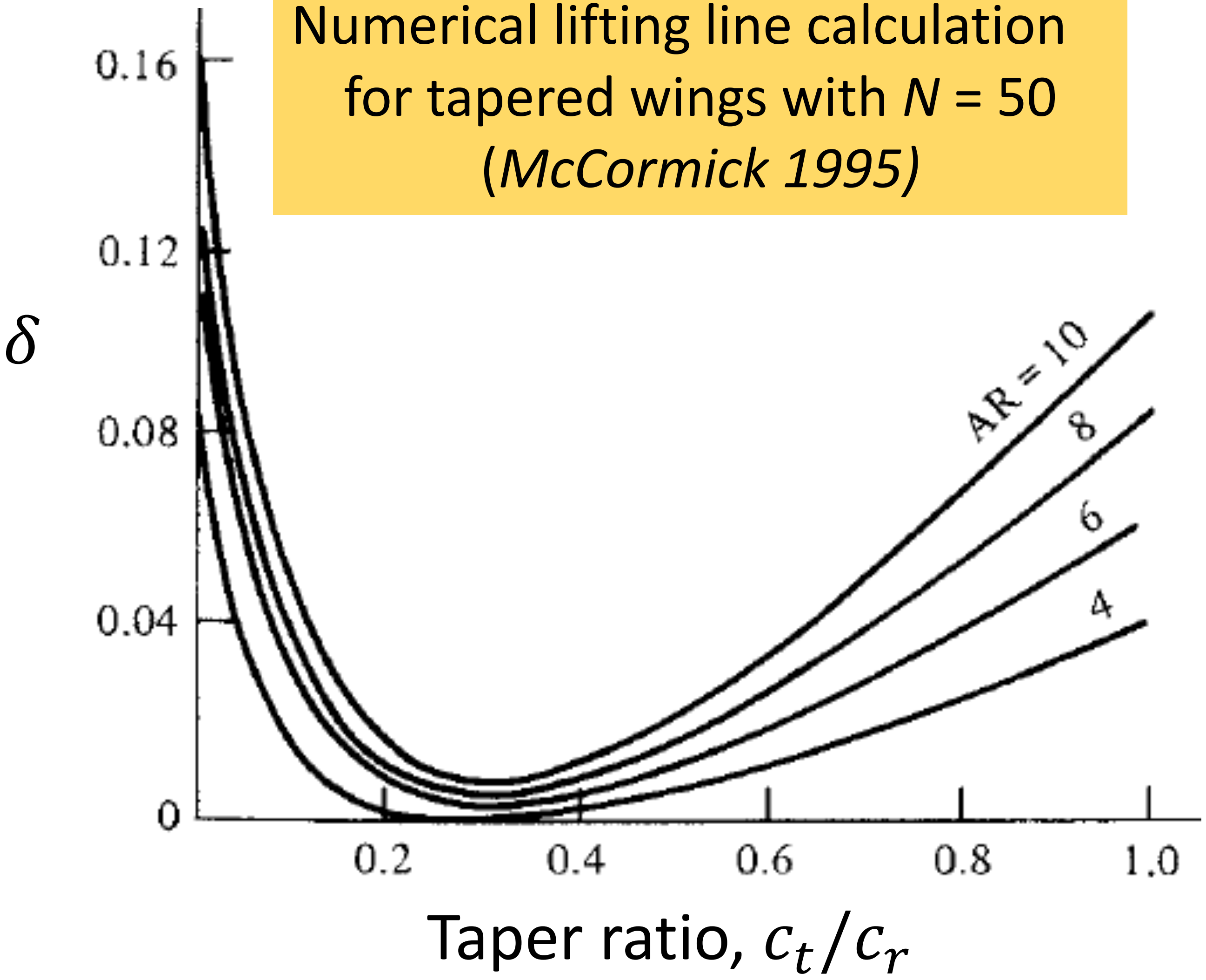
Effect of Taper Ratio & AR on Span Efficiency Factor



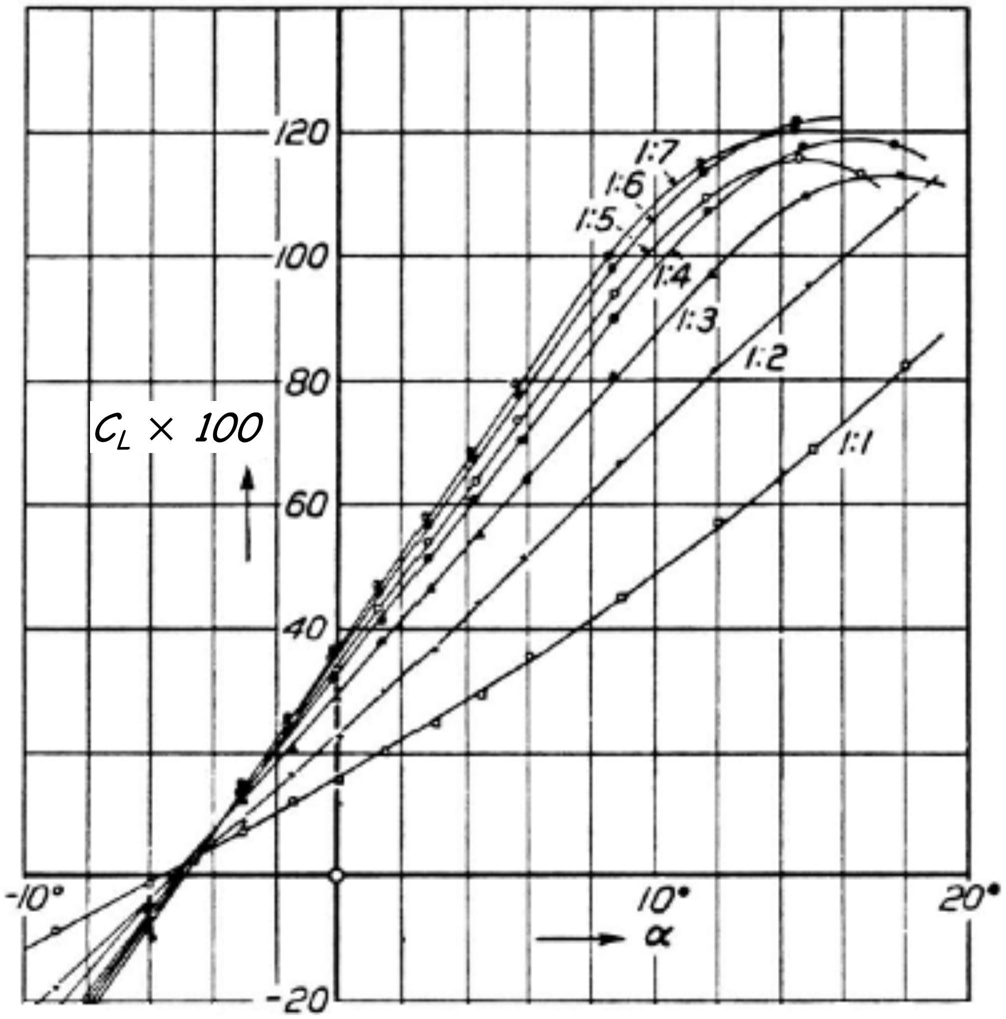
$$C_{D,i} = \frac{C_L^2}{\pi \mathrm{AR}}(1+\delta) = \frac{C_L^2}{\pi e \mathrm{AR}}$$

Anderson, 2011

Numerical lifting line calculation for tapered wings with N = 50 (*McCormick 1995*)



Prandtl’s lift curves of rectangular wings



Another effect of three-dimensionality of flow is on lift curve slope

- Lift slope degrades with decreasing aspect ratio
- All untwisted rectangular wings have same $\alpha_{L=0}$

Prandtl, L., Applications of modern hydrodynamics to aeronautics, NACA Report No. 116, 1921

Prandtl's lift curves of rectangular wings – Explained

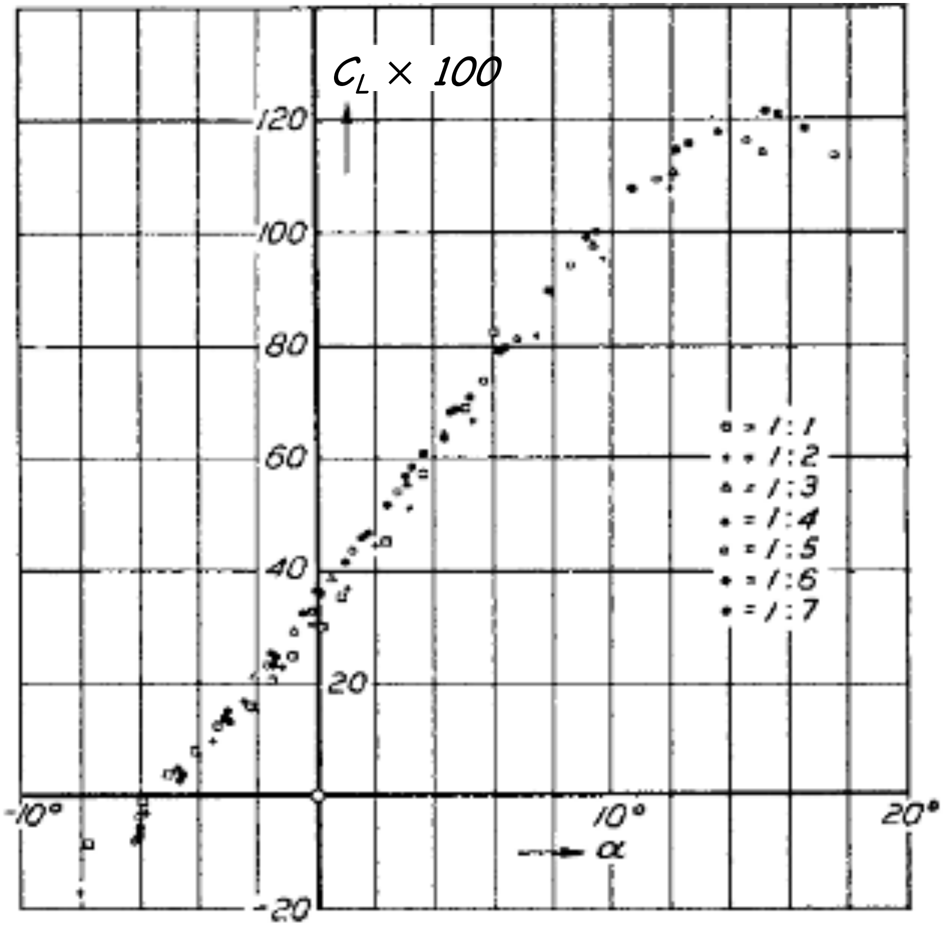
For untwisted elliptic wing with same shape at all sections, lift slope a is related to the airfoil lift slope a_0 as

$$a := \frac{dC_L}{d\alpha} = \frac{a_0}{1 + a_0/\pi \mathrm{AR}} < a_0$$

Extending this to arbitrary wings,

$$a = \frac{a_0}{1 + a_0/\pi \mathrm{AR}(1+\tau)}$$

τ is a function of A_n's; it usually ranges from 0.05 to 0.25



Prandtl found τ from data at $\mathrm{AR} = 5$ and rescaled all data with it

Prandtl, L., Applications of modern hydrodynamics to aeronautics, NACA Report No. 116, 1921

End of Topic

Finite Wing Aerodynamics