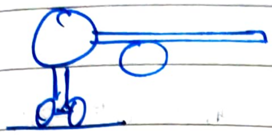
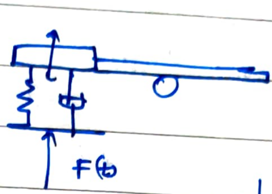
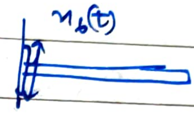
→ 2-DoF systems





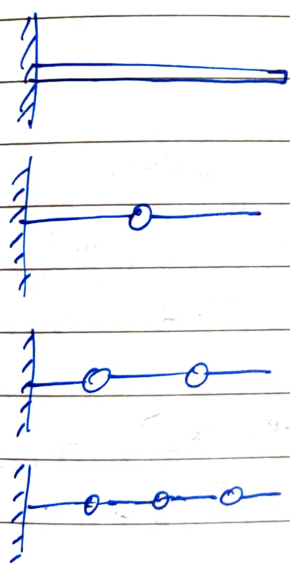


$$T = \frac{1}{2} m(\dot{x}_b + \dot{x})^2$$

1-DoF ⟶ 2-DoF big conceptual jump

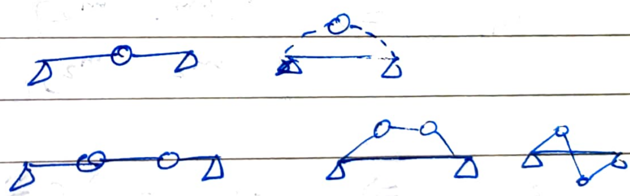
2-DoF → N-DoF only numerical jump

→ we started by ∞ DoF system



← $\sin \frac{n\pi x}{L}$, ω_n

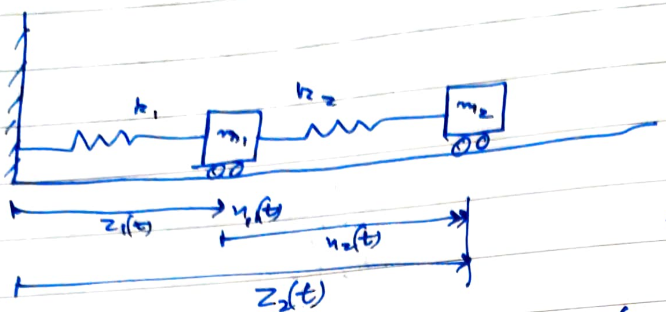
shapes are DoF



→ Synchronous shapes or arrangements

↳ mode shapes

coordinate transformation relevant here



$x_1(t) = z_1(t)$

2 ways $(x_1(t)\ \ x_2(t))$ $\quad$ $(z_1(t)\ \ z_2(t))$

$$T = \frac{1}{2} m_1 (\dot{x}_1(t))^2 + \frac{1}{2} m_2 \cancel{(\dot{x}_2(t))^2} (\dot{x}_1(t) + \dot{x}_2(t))^2$$

$$= \frac{1}{2} m_1 (\dot{z}_1(t))^2 + \frac{1}{2} m_2 (\dot{z}_2(t))^2$$

$$V = \frac{1}{2} k_1 (x_1^2) + \frac{1}{2} k_2 (x_2^2) = \frac{1}{2} k_1 z_1^2 + \frac{1}{2} k_2 (z_2 - z_1)^2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

diag

$$(x_1, x_2) \Rightarrow \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$(z_1, z_2) \Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & +k_2 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

diag

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

coord transform

$\exists$ some q_1 q_2 such we get only diag matrix

$$\begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{0\}$$

$$w(x,t) = f(x)\, q(t)$$

$q_1\ q_2$ called principle coordinates.

$u = Tv$ $\quad$ u, v coord sets

$$w(x,t) = \sum f_j(t) \sin j\frac{\pi x}{2}$$

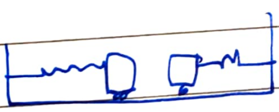
→ generalised eq^n

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

subj to $\{x(0)\} = \{x_0\}$

$\{\dot{x}(0)\} = \{\dot{x}_0\}$

$$\{X(t)\} = \{X_0\} e^{st}$$

 ← 2 DoF)

$T = \frac{1}{2} m_1 \dot{u}_1^2 + \frac{1}{2} m_2 \dot{u}_2^2$

$V = \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 u_2^2$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\{\ddot{\ }\} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\{\ \}^2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

we get 2 freq, $\omega_1 = \sqrt{k_1/m_1}$

$\omega_2 = \sqrt{k_2/m_2}$

assumption some not imply ~~th~~ vibration together.

char shape $\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i\omega_{n_1} t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i\omega_{n_2} t}$

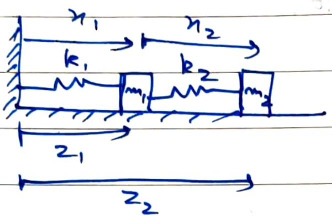
—

 will capture

7/11/22

→ 2 DoF system

- wrap up coord transform
- look for "special coords (natural modes)
- A note about damping
- course wrap up.



T, V should be ~~[illegible]~~ conserved

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \cancel{\text{in one system}} \quad \& \quad k = \begin{bmatrix} \neq 0 & \neq 0 \\ \neq 0 & \neq 0 \end{bmatrix} \quad \text{in one system}$$

$$\tilde{M} = \begin{bmatrix} \neq 0 & \neq 0 \\ \neq 0 & \neq 0 \end{bmatrix} \quad \& \quad \tilde{k} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \quad \text{in other system}$$

$$\{x\} = [T]\{z\}$$

$$T = \frac{1}{2} \dot{x}^T M \dot{x} = \frac{1}{2} \dot{z}^T \tilde{M} \dot{z}$$

$$V = \frac{1}{2} x^T k x = \frac{1}{2} z^T \tilde{k} z$$

substn $x = Tz$ in above

$$\frac{1}{2} \dot{z}^T T^T M T \dot{z} = \frac{1}{2} \dot{z}^T \tilde{M} \dot{z}$$

$$\tilde{M} = \underbrace{T^T M T}_{\tilde{M}} \checkmark$$

similarly $\tilde{k} = T^T k T$ ✓

- "special coords" that we can identify variable seperable solⁿ.

$x(t) = f(t)\,\underline{u}$ $\quad$ $f \rightarrow$ scalar funcⁿ

$M\ddot{x} + kx = 0$

$M\ddot{f}\underline{u} + kf\underline{u} = 0$

left multiply by $\underline{u}^T$

$\ddot{f}(t)\,\underbrace{\underline{u}^T M \underline{u}}_{\text{scalar}} + f(t)\,\underbrace{\underline{u}^T k \underline{u}}_{\text{scalar}} = 0$

define $\lambda = \dfrac{\underline{u}^T k \underline{u}}{\underline{u}^T M \underline{u}}$

$\ddot{f}(t) + \lambda f(t) = 0$ $\quad$ eqⁿ of $f(t)$

for vibratory soluⁿ, $\lambda > 0$

call $\lambda = \omega^2$

$\ddot{f}(t) = -\omega^2 f(t)$

$-M\omega^2 f(t)\underline{u} + kf(t)\underline{u} = 0$

$(-M\omega^2 + k)f(t)\underline{u} = 0$

for non-trivial solⁿ for $f(t)$

$M\omega^2\underline{u} = k\underline{u}$

$Av = \lambda v$ $\quad$ $(M^{-1}k)\underline{u} = \omega^2\underline{u}$ $\quad$ $\underline{u}$ need to satisfy this eqⁿ

$\omega^2 \rightarrow$ eigenvalue

$\underline{u} \rightarrow$ eigenvector

- M is +ve definite
- k is +ve definite for our prob

u are eigenvectors of $M^{-1}k$
ω^2 are eigenvalues of $M^{-1}k$

2 DoF $\quad \dfrac{u_1}{\omega_1}, \dfrac{u_2}{\omega_2}$

N-DoF $\quad u_1 \quad u_2 \; - - - - - \; u_n$
$\quad \omega_1 \quad \omega_2 \; - - - - - \; \omega_N$

- K matrix becomes singular when deformation doesn't give strain energy

$$x(t) = A u_1 e^{(i(\omega_1 t - \phi_1))} + B u_2 e^{(i(\omega_2 t - \phi_2))}$$

or $x(t) = A u_1 e^{i\omega_1 t} + B u_2 e^{i\omega_2 t}$

A, B are real.

u_1, u_2 are called mode shapes

ω_1, ω_2 " natural frequency

We can prove, $\underline{u_1}^T M \underline{u_2} = 0 \qquad \underline{u_1}^T k \underline{u_2} = 0$
$\underline{u_2}^T M \underline{u_1} = 0 \qquad \underline{u_2}^T k \underline{u_2} = 0$

$$M\ddot{x} + kx = 0$$

$$I\ddot{x} + M^{-1}kx = 0$$

$$x = [\underline{u_1} \; \underline{u_2}] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$

$$= \cancel{[U]\{q\}} \; [U]\{q(t)\} \approx \underline{u_1} q_1 + \underline{u_2} q_2$$

$$[U]\{\ddot{q}\} + [M^{-1}k][U]\{q\} = 0$$

$$[U]^T[U]\{\ddot{q}\} + [U]^T[M^{-1}k][U]\{q\} = 0$$

$I\{\ddot{q}\} + [\Lambda]\{q\} = 0$

find [M], [K] solve $[M^{-1}K]$ eigenvalue prob find ω.

→ Damping in N-DoF system.

- not easy to diagonalise $\qquad M\ddot{x} + C\dot{x} + Kx = 0$

- Imp: does the [C] matrix allow diagonalization?

- If possible, choose [C] to enable diagonalization

$C = aM + bK$, choose $a, b \in \mathbb{R}$ to fit damping.

proportional damping ↳ Rayleigh damping - widely used model

$M\ddot{x} + (aM + bK)\dot{x} + Kx = 0$

in software, a & b are needed as input.

$$\zeta_j = \frac{1}{2}\left(\frac{a}{\omega_j} + b\omega_j\right)$$

if $b = 0$ $\zeta \downarrow$ as $\omega \uparrow$

if $a = 0$ $\zeta \uparrow$ as $\omega \uparrow$

if a is incorrect no prob, b needs to be very low otherwise activity in high ω suppressed.

$o(b) = o\left(\frac{1}{\omega_j^2}\right) \qquad o(a) = o(\omega_j)$

$KM^{-1}C = (KM^{-1}C)^T$ necessary ~~cond~~ & sufficient cond^n for diagonalization to happen.

then C is proportional.

$$C = M^{1/2}\underbrace{\left[\qquad\qquad\right]}_{\sum_{i=0}^{n-1} a_{i,j}\left[M^{-1/2} K M^{1/2}\right]^{i/j}} M^{1/2}$$

↳ Caughy damping

generalized rayleigh damping.