AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 13: Frequency Response

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Introduction

- Consider a system acted upon by a **sinusoidal input**, as shown below.

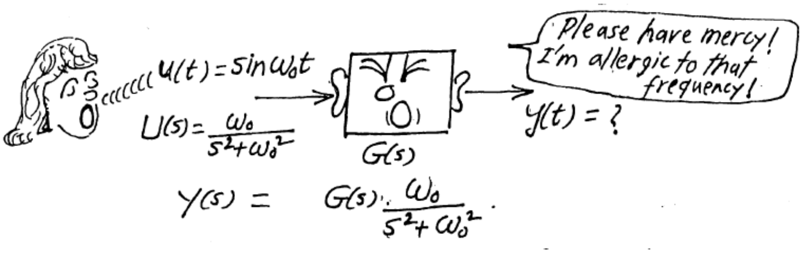

Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- With sinusoidal input we get,

$$Y(s) = G(s)\frac{\omega_0}{s^2 + \omega_0^2}$$

Introduction - Sinusoidal Input

Sinusoidal Input

- Apart from impulse, step, ramp and parabolic inputs, used as test signals, dynamical systems also experience, harmonic inputs quite frequently
- For example, excitations arising from reciprocating engines, rotating machines, ground/airborne vibrations etc. create harmonic forces.
- Frequency response concept aims to characterize the behaviour of LTI systems to such inputs.

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Frequency Response

- We have,

$$Y(s) = G(s)\frac{\omega_0}{s^2+\omega_0^2}, \quad G(s) = \frac{N(s)}{(s-p_1)(s-p_2)...(s-p_n)}$$

- Using partial fraction decomposition,

$$Y(s) = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \sum_{j=1}^{n} \frac{b_j}{s-p_j}$$

- Taking Inverse Laplace, we obtain

$$y(t) = ae^{-jwt} + \bar{a}e^{jwt} + \sum_{j=1}^{n} b_j e^{p_j t}$$

Frequency Response - Example

Example: Determine the response of the following system subjected to the input $u(t) = \sin(10t)$

$$G(s) = \frac{1}{s+1}$$

Frequency Response - Example

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$$G(s) = \frac{1}{s+1}$$

Solution:

- Response is given by,

$$Y(s) = G(s)U(s) = \left(\frac{1}{s+1}\right)\left(\frac{10}{s^2+100}\right)$$

- Apply partial fraction,

$$Y(s) = \frac{\alpha_1}{s+1} + \frac{\alpha_0}{s+j10} + \frac{\alpha_0^*}{s-j10}$$

Frequency Response - Example

- After solving we get, $\alpha_1 = \frac{10}{101}, \alpha_0 = \frac{j}{2(1-j10)}, \alpha_0^* = \frac{-j}{2(1+j10)}$,

$$Y(s) = \frac{\frac{10}{101}}{s+1} + \frac{\frac{j}{2(1-j10)}}{s+j10} + \frac{\frac{-j}{2(1+j10)}}{s-j10}$$

- Taking inverse Laplace,

$$y(t) = y_1(t) + y_2(t) = \frac{10}{101}e^{-t} + \frac{1}{\sqrt{101}}\sin(10t+\phi)$$

$$\phi = \tan^{-1} -10 = -84.2^\circ$$

$y_1(t)$ is called transient response and $y_2(t)$ is steady state response

Frequency Response - Example

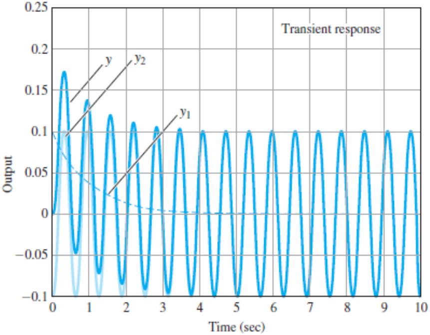


Figure: *Transient Response*

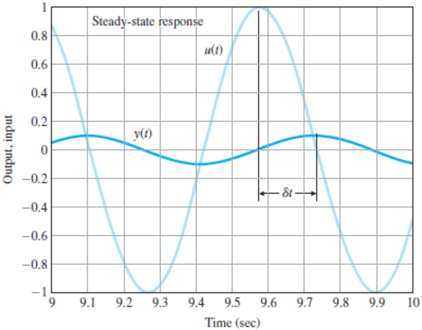
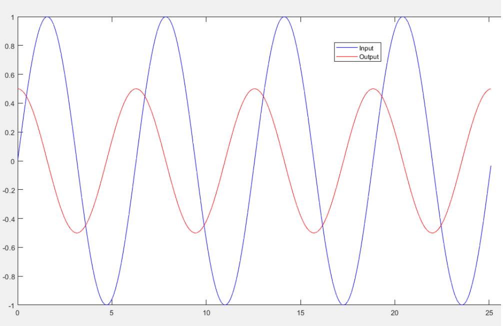


Figure: *Steady State Response*

Source - "Feedback Control of Dynamic Systems" by Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini

Frequency Response - Analysis

Analysis of i/p and o/p



- Input,

$$x(t) = X \sin \omega t$$

- Output,

$$y(t) = Y \sin(\omega t + \phi)$$

- Magnitude and Phase are,

$$Y = X|G(j\omega)|$$

$$\phi = \angle G(j\omega)$$

Frequency Response - Features

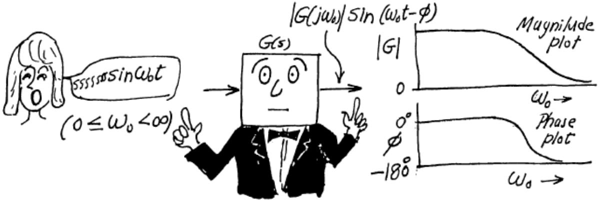
Features

- Ratio of output and input waveform, gives $|G(j\omega)|$
- Measuring the phase difference between output and input waveform, gives $\angle G(j\omega)$.
- These are the magnitude and phase of the system TF, $G(s)$, evaluated at $s = \pm j\omega$.

Frequency Response

$$|G(j\omega_0)| = \text{magnitude of } G(j\omega_0) = \sqrt{(\text{Real part})^2 + (\text{Imag. pt.})^2}$$

$$\phi = \text{phase of } G(j\omega_0) = \tan^{-1}\left[\frac{\text{Imag. part of } G(j\omega_0)}{\text{Real part of } G(j\omega_0)}\right]$$



The plots of $|G(j\omega)|$ and ϕ, versus ω, are called the "frequency response" of the system.

Figure: *Source -"Cartoon Tour Of Control Theory" by S.M. Joshi*

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First Order System - Frequency Response

Example: Consider the following system and obtain its magnitude and phase at $\omega = 0, \pm\infty$

$$G(s) = \frac{K}{s+p}$$

First Order System - Frequency Response

Example: Consider the following system and obtain its magnitude and phase at $\omega = 0, \pm\infty$

$$G(s) = \frac{K}{s+p}$$

Solution:

- Substitute $s = j\omega$,

$$G(j\omega) = \frac{K}{j\omega + p}$$

- Magnitude,

$$|G(j\omega)| = \frac{K}{|j\omega + p|} = \frac{K}{\sqrt{\omega^2 + p^2}}$$

- Phase,

$$\angle G(j\omega) = \angle K - \angle(j\omega + p) = 0 - \tan^{-1}(\omega/p) = -\tan^{-1}(\omega/p)$$

First Order System - Response Analysis

Response Analysis

- It can be seen that,

$$|G(j\omega)| = \begin{Bmatrix} (K/p) & \omega = 0 \\ 0 & \omega = \pm\infty \end{Bmatrix}$$

  Thus, for systems with poles lying on the origin (type≥ 1), the magnitude becomes infinite for $\omega = 0$.
- Also,

$$\angle G(j\omega) = \begin{Bmatrix} 0 & \omega = 0 \\ -90^\circ & \omega = +\infty \\ +90^\circ & \omega = -\infty \end{Bmatrix}$$

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Second Order System - Frequency Response

Example: Consider the following system and obtain its magnitude and phase at $\omega = 0, \omega_n, \infty$

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Second Order System - Frequency Response

Example: Consider the following system and obtain its magnitude and phase at $\omega = 0, \omega_n, \infty$

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Solution:

- Substitute $s = j\omega$,

$$G(j\omega) = \frac{K}{(j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2}$$

- Magnitude,

$$|G(j\omega)| = \frac{K}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

- Phase,

$$\angle G(j\omega) = -\angle\left((j\omega)^2 + j2\zeta\omega_n\omega + \omega_n^2\right) = -\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}\right)$$

Second Order System - Response Analysis

Response Analysis

- It can be seen that,

$$|G(j\omega)| = \begin{cases} (K/2\zeta\omega_n^2) & \omega = \omega_n \\ (K/\omega_n^2) & \omega = 0 \\ 0 & \omega = \infty \end{cases}$$

- Also,

$$\angle G(j\omega) = \begin{cases} 0 & \omega = 0 \\ -90^\circ & \omega = \omega_n \\ -180^\circ & \omega = \infty \end{cases}$$

Frequency Response - Observations

What do you observe?

Frequency Response - Observations

Observations

- We see from preceding examples that frequency response features depend on $G(s)$, and also show varied characteristics over the applicable frequency range.
- Thus, it appears logical to explore the variation of $G(j\omega)$ over the complete range of 'ω', i.e. from '0' to '∞'.
- In this context, it is worth noting that, while we can get analytical expressions for $|G(jw)|$ and $\angle G(jw)$, these tend to become unwieldy as system order increases.
- In view of the above, **graphical representations** are employed for better overall view of the response.

Frequency Response - Graphical Representation

Graphical Representation

- Among the many possible ways of graphically representing the frequency response, Bode and Nyquist plots are the most commonly employed forms.
- An important added advantage of these representations is the possibility of visualizing the closed loop behaviour and applicable control actions, in a simple and intuitive manner.
- Therefore, the above plots are also useful tools for the design of closed loop control systems.

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Bode Plots - Introduction

Introduction

- Bode diagrams show variation of $|G(j\omega)|$ and $\angle G(j\omega)$ on two separate plots as ω varies from 0 to ∞.
- Both $|G(j\omega)|$ and $\angle G(j\omega)$ plots use a log scale either octave or decade for ω.

Why log scale ???

Bode Plots - Introduction

Why log scale????

- Log scale permits a large frequency to be shown together.
- Log converts multiplication and division into addition and subtraction, respectively, of log magnitudes of the numerator and denominator factors of $G(s)$.
- Similarly, total phase of the frequency response can also be expressed as addition and subtraction of phase angles of the numerator and denominator factors.
- Thus, it is possible to create a complex bode plot through a building block approach by synthesizing it using simple factors, e.g. pure gain, 1^{st} order and 2^{nd} order.

Bode Plots - Construction

Construction

Suppose we have a transfer function, $G(s) = \frac{K(s+z_1)}{s(s+p_1)}$

1. Rewrite it in standard form,

$$G(s) = \frac{Kz_1(s/z_1+1)}{sp_1(s/p_1+1)}$$

2. Substitute $s = j\omega$, take $\log_{10}$ both sides and multiply by 20,

$$20\log_{10}(G(j\omega)) = 20\log_{10}\left(\frac{Kz_1(j\omega/z_1+1)}{j\omega p_1(j\omega/p_1+1)}\right)$$

3. Taking magnitude,

$$\begin{aligned}|20\log_{10}(G(j\omega))| = {} & 20\log_{10}|K| + 20\log_{10}|z_1| \\ & + 20\log_{10}|((j\omega/z_1)+1)| - 20\log_{10}|p_1| \\ & - 20\log_{10}|(j\omega)| - 20\log_{10}|((j\omega/p_1)+1)|\end{aligned}$$

Bode Plots Construction - Constant Term

The entire Bode log magnitude plot is the result of the superposition of all the straight line terms. Each of these individual terms is very easy to show on a logarithmic plot.

Effect of Constant Term

- Constant term, K contribute a straight horizontal line of magnitude $20\log_{10}(K)$

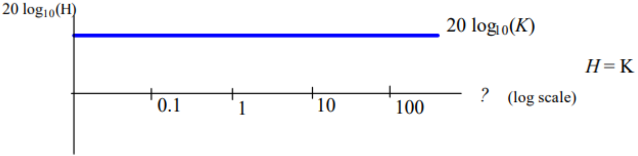


Figure: *Source - “https://my.ece.utah.edu/ ee3110/bodeplot.pdf"*

Bode Plots Construction - Poles and Zeros

Effect of Individual Zeros and Poles not at the Origin

- Zeros and Poles not at the origin are indicated by the $(1 + j\omega/z_i)$ and $(1 + j\omega/p_i)$.
- The values of z_i and p_i in each of these expressions are called a critical frequency (or break frequency).
- Below their critical frequency, these terms do not contribute to the $\log$ magnitude of the overall plot.
- Above the critical frequency, they represent a ramp function of 20 dB per decade.

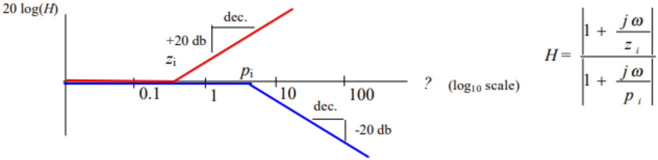


Figure: *Source - "https://my.ece.utah.edu/ ee3110/bodeplot.pdf"*

Bode Plots Construction - Origin Zeros

Effect of Individual Zeros at the origin

- A zero at the origin occurs, when there is an s or $j\omega$ multiplying the numerator.
- Each occurrence of this causes a positively sloped line passing through $\omega = 1$ with a rise of 20 dB over a decade.

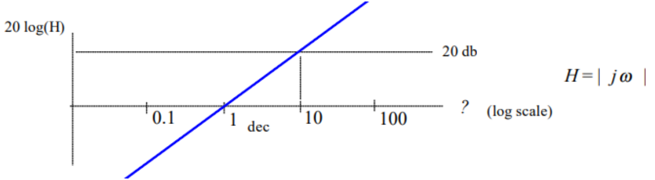


Figure: *Source - "https://my.ece.utah.edu/ ee3110/bodeplot.pdf"*

Bode Plots Construction - Origin Poles

Effect of Individual Poles at the origin

- A pole at the origin occurs when there is an s or $j\omega$ multiplying the denominator.
- Each occurrence of this causes a negatively sloped line passing through $\omega = 1$ with a drop of 20 dB over a decade.

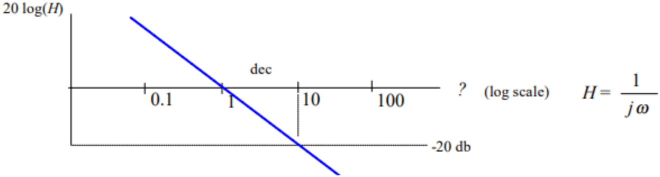


Figure: *Source - "https://my.ece.utah.edu/ ee3110/bodeplot.pdf"*

Bode Plot - Example

Question: Draw Bode magnitude plot for,

$$G(s) = \frac{10(s+1)}{(s+10)}$$

Solution:

- Rewrite it in standard form,

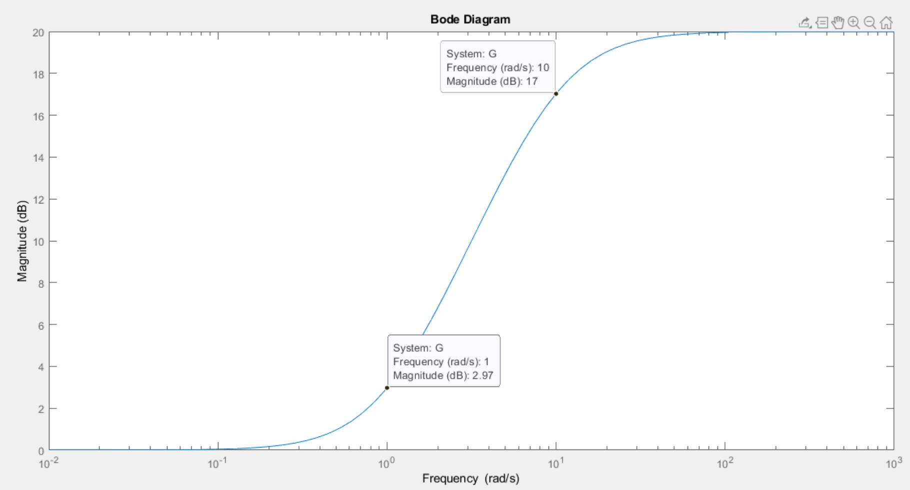
$$G(s) = \frac{(s+1)}{(\frac{s}{10}+1)}$$

- **Critical frequencies** are $1, 10$ as $z=-1, p=-10$
- **Contant Term**(K) is 1 therefore,

$$20\log_{10} 1 = 0$$

Thus the plot starts with 0 dB magnitude

Bode Plot - Example



Bode Plot - Application

Question: Identify the **filters** from the Bode magnitude plot

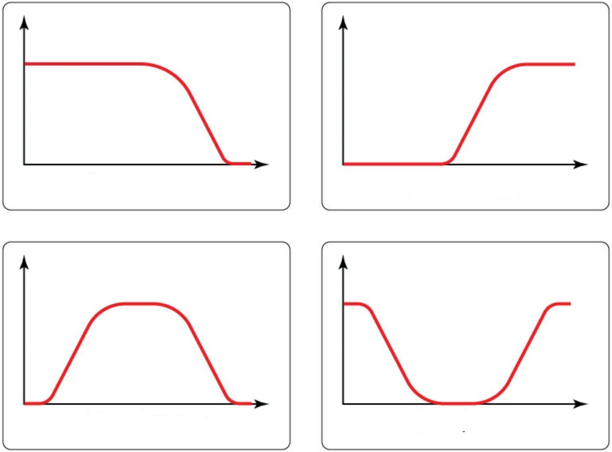


Figure: *Source: "https://www.allaboutcircuits.com"*

Bode Plot - Application

Question: Identify the **filters** from the Bode magnitude plot

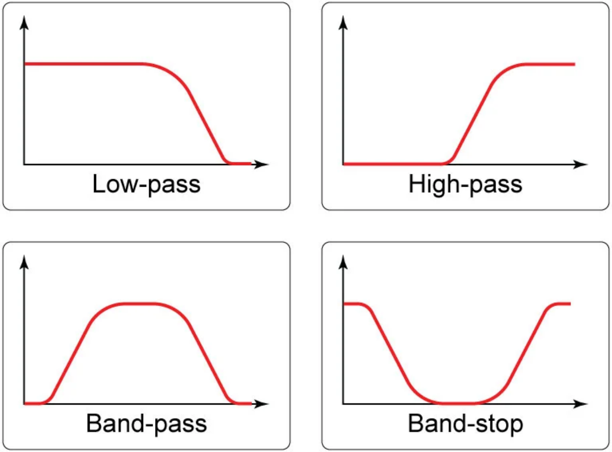


Figure: *Source: "https://www.allaboutcircuits.com"*

Bode Plot - Application

Question: Comment on the **filtering nature** of the following transfer functions

- $G(s) = \frac{10(s+1)}{(s+10)}$
- $G(s) = \frac{100}{(s+5)}$
- $G(s) = \frac{10(s+1)(s+15)}{(s+2)(s+10)}$

References I

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