Lecture – VI. Functions of a random variable.

Theorem

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable.

Let $g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ ~~is~~ be a measurable map (that is, $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for every $B \in \mathcal{B}(\mathbb{R})$).

Then $g(X) : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable map and hence, a random variable.

Corollary Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Then $g : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is also measurable. Therefore, $g(X) : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable whenever $X : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is ~~meas~~ a random variable.

* Any continuous map preserve the open sets. ~~that~~ is, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then $g^{-1}(A)$ is an open ~~subset~~ interval of $\mathbb{R}$ whenever A is an open ~~subset of $\mathbb{R}$.~~ interval of $\mathbb{R}$.

*. The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ contains all the open intervals and generated by open intervals.

More details are beyond the scope of the course.

We shall now see some of the important functions.

Examples

Here we list down some of the important functions.

i) $f(x) = |x|$.

ii) $f(x) = ax+b$.

iii) $f(x) = \max(x,0) = x_+$

iv) $f(x) = \min(x,0) = x_-$.

v) $f(x) = x^\alpha$, $\alpha > 0$.

vi) $f(x) = x^{-\alpha}$, $\alpha > 0$ and $x > 0$. $f: (0,\infty) \to (0,\infty)$.

* Check that all of these functions are continuous.

Hence, if $X: (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a random variable, then $f(X)$ is also a random variable when f is one of the functions listed above.

* Let Y be a function (complicated ~~of~~ and not continuous) of the random variable X. Then Y is a random variable if we can write down the distribution function of the random variable Y explicitly.

* You can think about functions with finite number of points of discontinuities.

Example. (Poisson random variable).

$$P(X=k) = \begin{cases} e^{-\lambda}\dfrac{\lambda^k}{k!}, & k=0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Define $f(x) = x^2+3$. It follows that X^2+3 is a random variable as $x \mapsto x^2+3$ is a continuous function.

Note that X^2+3 takes values in $\mathbb{N}$. to be more specific

$$X^2+3 : \mathbb{N}_0 = \mathbb{N} \cup \{0\} \longrightarrow \{3, 4, 7, 12, 19, \ldots\}$$

For every $y \in \{3, 4, 7, 12, 19, \ldots\}$,

$$P(X^2+3 = y) = P(X = \sqrt{y-3}) = e^{-\lambda}\frac{\lambda^{\sqrt{y-3}}}{(\sqrt{y-3})!}$$

* Here $f(x) = x^2+3 : [0,\infty) \longrightarrow [0,\infty)$ is monotone and hence, has unique inverse.

* We can consider a function $\mathbb{R} \to \mathbb{R}$, which may not have unique inverse but countable number of inverses. By ~~countable~~ countable inverses of a function f, we mean that we get a countable ~~par~~ partition $\mathbb{R}$ into countable subsets $(E_k : k \geq 1)$ such that $f : E_k \to \mathbb{R}$ is one-to-one. We say that restriction of f to E_k

is one-to-one. We denote the restriction of f to E_k by

$$f|_{E_k} : E_k \to \mathbb{R}.$$

We also need E_k's to be Borel subset.

We can then see that for every $A \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
\mathbb{P}(f(X) \in A) &= \mathbb{P}(\{w : f(X)(w) \in A\}) \\
&= \mathbb{P}(\{w : f(X(w)) \in A\}) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{\infty}\left[\{w : f(X(w)) \in A\} \cap \{w : X(w) \in E_k\}\right]\right) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{\infty}\left[\{w : f_k(X(w)) \in A\} \cap \{w : X(w) \in E_k\}\right]\right) \\
&= \sum_{k=1}^{\infty} \mathbb{P}\left(\{w : f_k(X(w)) \in A\} \cap \{w : X(w) \in E_k\}\right).
\end{aligned}$$

Note that f_k is a measurable function that is, $f_k(X) : \mathbb{R} \to \mathbb{R}$ is also measurable.

Therefore,

$$\{w : f_k(X(w)) \in A\} \in \mathcal{F}.$$

and $\{w : X(w) \in E_k\} \in \mathcal{F}$ as $E_k \in \mathcal{B}(\mathbb{R})$ and X is a random variable.

<u>Example</u> Suppose X is a Poisson random variable. We are interested in the distribution of $\sin X$.

Functions of continuous random variables

Recall that X is called a continuous random variable if the distribution function of X is continuous.

Example. X be a uniform random variable on $(-1,1)$.

i) Distribution of X^2

ii) Distribution of X_+.

$$F_X(x) = \begin{cases} 0 & \text{if } x \le -1 \\ \frac{x+1}{2} & \text{if } x \in (-1,1] \\ 1 & \text{if } x \geq 1. \end{cases}$$

i) If $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \left((-1,1), \mathcal{B}((-1,1))\right)$, then it is clear that

$$X^2 : (\Omega, \mathcal{F}, \mathbb{P}) \to \left([0,1), \mathcal{B}([0,1))\right).$$

As $x \mapsto x^2$ is continuous, it is immediate that X^2 is a random variable if X is so.

To derive the distribution of X^2, we can see that

$$\mathbb{P}(X^2 \le x) = \mathbb{P}(-\sqrt{x} \le X \le \sqrt{x}) = F_X(\sqrt{x}) - F_X(-\sqrt{x}-).$$

$$= F_X(\sqrt{x}) - F_X(-\sqrt{x}) \quad (\text{as } F \text{ is continuous})$$

$$= \frac{\sqrt{x}+1}{2} - \frac{1-\sqrt{x}}{2} = \sqrt{x}. \quad \text{for all } x \geq 0.$$

$$F_{X^2}(x) = \begin{cases} 0 & \text{if } x \le 0. \\ \sqrt{x} & \text{if } 0 \le x < 1. \\ 1 & \text{if } x \geq 1. \end{cases}$$

ii) If $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \left((-1,1), \mathcal{B}((-1,1))\right)$, then it is clear that $X_+ : (\Omega, \mathcal{F}, \mathbb{P}) \to \left([0,1), \mathcal{B}([0,1))\right)$.

As $x \longmapsto \max(x,0)$ is continuous, it follows that

$$X_+ : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow ([0,1), \mathcal{B}([0,1))).$$

is a random variable ~~given X is~~ if X is a random variable.

We are interested in the distribution function of X_+. We denote it by F_{X_+}.

$$F_{X_+}(0-) = \cancel{\mathbb{P}(X<0)} = \mathbb{P}(\max(X,0)<0) = 0.$$

$$F_{X_+}(0) = \mathbb{P}(\max(X,0) \le 0) = \mathbb{P}(\{\max(X,0)=0\} \cup \{\max(X,0)<0\})$$

$$= \mathbb{P}(\{\max(X,0)=0\}) + \mathbb{P}(\max(X,0)<0)$$

$$= \mathbb{P}(X \in (-1,0])$$

$$= F_X(0) - F_X(-1) = \frac{1}{2}$$

$$F_{X_+}(x) = \mathbb{P}(\max(X,0) \le x)$$

$$= \mathbb{P}(\{\max(X,0) = 0\} \cup \{0 < \max(X,0) \le x\})$$

$$= \mathbb{P}(\{\max(X,0)=0\}) + \mathbb{P}(0 < \max(X,0) \le x)$$

$$= \mathbb{P}(X \in (-1,0]) + \mathbb{P}(X \in (0,x])$$

$$= \frac{1}{2} + \frac{x}{2} = \frac{x+1}{2} \quad \text{for } x > 0.$$

$$F_{X+}(x) = \begin{cases} 0 & \text{if } x<0. \\ \frac{1}{2} & \text{if } x=0. \\ \frac{1+x}{2} & \text{if } 0<x\leq 1. \\ 1 & \text{if } x \geq 1. \end{cases}$$

* IMPORTANT: X_+ has a mass at the sample point 0. that is, 0 is a point of discontinuity of the dist$^{\underline{n}}$ function F.

<u>Remark</u>: Suppose $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a continuous random variable with density f. Although, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function $g(X)$ may not be a continuous random variable.

Q$^{\underline{n}}$. Can we put any additional restriction on the continuous function g so that $g(X)$ is a continuous function ?

The following theorem provides the answer.

<u>Theorem</u>

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a continuous random variable with probability density function f. Consider a function $g : \mathbb{R} \to \mathbb{R}$ which is differentiable for all $x \in \mathbb{R}$ such that

$$g'(x) = \frac{d}{dx} g(x) > 0 \text{ or } g'(x) < 0 \text{ for all } x \in \mathbb{R}.$$

Then $Y = g(X) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is also a continuous random variable with probability density function given by

$$h(y) = \begin{cases} f[g^{-1}(y)] \left| \dfrac{d}{dy} g^{-1}(y) \right| & \min(g(\infty), g(-\infty)) < y < \max(g(\infty), g(-\infty)) \\ 0 & \text{otherwise.} \end{cases}$$

Remark. We need the monotonicity of the ~~differentiable~~ differentiable function g for the inverse f^n g^{-1} to be unique.

Remark. Suppose that the density function of X vanishes outside an interval $[a, b]$. Then we must assume that g is differentiable in (a, b) and either $g'(x) > 0$ or $g'(x) < 0$ for all $x \in (a, b)$.

Proof. We assume that g is differentiable and $g'(x) > 0$ for all $x \in \mathbb{R}$. Then it follows that g is continuous and strictly increasing which implies that g^{-1} is continuous, increasing and differentiable.

Also note that

$$\alpha = \min(g(-\infty), g(\infty)) \text{ and } \beta = \max(g(-\infty), g(\infty))$$

exist but might be infinite.

We take the distribution function approach, that is write down the distribution function of $Y = g(X)$ explicitly. Note that

$$\mathbb{P}(Y \le y) = \mathbb{P}(g(X) \le y) = \mathbb{P}(X \le g^{-1}(y))$$

$$= \int_{-\infty}^{g^{-1}(y)} dx\, f(x).$$

$$h(y) = \frac{d}{dy}\left[\mathbb{P}(Y \le y)\right] = \frac{d}{dy}\left(\int_{-\infty}^{g^{-1}(y)} dx\, f(x)\right)$$

$$= f(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

A similar computation can be used to derive the probability density function of Y when $g'(x) < 0$ for all $x \in \mathbb{R}$. (Left as an exercise!) $\blacksquare$

Example.

Suppose X is ~~an~~ a uniformly distributed random variable on $(-1, 1)$. Then the p.d.f of X can be written as

$$f(x) = \begin{cases} 0 & \text{if } x < -1. \\ \frac{x+1}{2} & \text{if } -1 \le x < 1. \\ 0 & \text{if } x \geq 1. \end{cases}$$

~~Note that~~ $g : (-1, 1) \to [0, 1)$ such that $g(x) = x^2$ ~~does n~~ is not monotone and hence, does not have a unique inverse. So we can not use the theorem to derive ~~dom~~ density of the random variable X^2.

NOT A GREAT NEWS!

Example.

Let $X \sim$ Uniform $(0,1)$ that is, X has density

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 0 \le x < 1. \\ 0 & \text{if } x \geq 1. \end{cases}$$

Define $Y_1 = g_1(X)$ where $g_1(x) = e^x$.

Define $Y_2 = g_2(X)$ where $g_2(x) = x^2$.

We shall focus on the random variable Y_1.

~~the proof~~ Note that the function $g_1 : (0,1) \to (1, e)$ is continuous, differentiable and $g_1'(x) \neq 0$. Also note that $g_1^{-1}(x) = \log x$. According to the theorem probability density function of Y_1 is given by

$$f_{Y_1}(y) = \begin{cases} 1 \cdot \frac{1}{y} & \text{for } 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

Exercise! Derive the density function of $g_2(Y)$.