AE 333/707: QL1. 26/8/22. Total=16. Time 1 hr.

– This is a closed-book exam.
– Answers without detailed explanation/justification will be heavily penalized.

1. (2 points) What are the sources of drag in aerodynamics? Describe each in a few sentences.

   Solution:

   Suggested breakup – 1/4 for mentioning each of the 4 components, 1/4 for describing each of the 4 components.

   There are four sources of drag in aerodynamics:

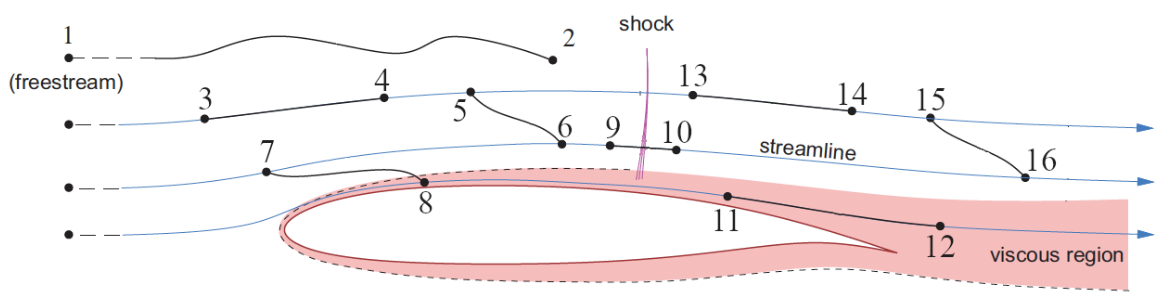
   Skin friction drag: This is caused by viscous interaction of the surrounding flow with the immersed body.

   Form/pressure drag: This is caused by an imbalance of pressure forces on the fore and aft sections of the body, typically due to flow separation.

   Wave drag: This is caused when shocks are formed in the flow over the immersed body due to a transonic or supersonic freestream. Shocks increase the static pressure of the flow, leading to pressure imbalance between the fore and aft of the body. They also thicken the boundary layer, increasing both the skin friction and the chances and/or severity of flow separation.

   Induced drag: Also called lift-induced drag, this is drag due to three-dimensionality of the flow over bodies of finite span. The pressure imbalance – between the pressure and suction surfaces of wings that causes lift – must be equalized at the wing tip due to lack of a physical barrier. This causes outboard flow over the wing and curling around the tips. This constitutes a loss in momentum and energy of the flow that manifests itself as lift-induced drag.

2. (4 points) Consider the steady and two-dimensional flow over an airfoil in the figure below adapted from Drela's textbook. For all the indicated pairs of points – from 1-2 to 15-16 – state whether the vorticity remains unchanged between the points, with brief justification.

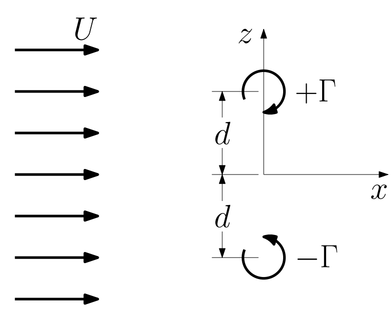


Solution:

Suggested breakup – 1/4 for each of the 8 correct answers, 1/4 for each of the 8 correct justifications.

- 1-2: Since both points are in the freestream, they have the same vorticity, which is zero.
- 3-4: Since both points are outside the boundary layer and upstream of the shock, and they are also incidentally on the same streamline, they have the same vorticity, which is zero.
- 5-6: Since both points are outside the boundary layer and upstream of the shock, they have the same vorticity (which is zero), even though they are not on the same streamline.
- 7-8: One point is outside the boundary layer whereas the other is inside it. The boundary layer is a rotational region whereas the external flow is irrotational. So the two points have different vorticity.
- 9-10: The two points are on the same streamline but they straddle a curved shock. The curved shock introduces vorticity in the otherwise irrotational upstream flow. So, the two points have different vorticity.
- 11-12: The two points are on the same streamline, but both of them are within the boundary layer. In the boundary layer, vorticity is produced so that no two points have the same amount of vorticity, in general. Thus the two points have different vorticity.
- 13-14: The two points are on the same streamline downstream of a shock. There are no sources of vorticity in this region. So, the two points have the same vorticity, and this is possibly nonzero.
- 15-16: The two points are on different streamlines. The curved shock generates vorticity of varying magnitudes along itself. So, the vorticity is not expected to be the same on any pair of streamlines behind it. Thus, the two points have different vorticity.

3. Consider the superposition of a uniform flow (of speed U) and a pair of equal and opposite line irrotational vortices (of circulation $\pm\Gamma$ separated by $2d$). Answer the following questions (in terms of U, Γ and d) starting from first principles:



   (a) (2 points) Derive an expression for the velocity potential as a function of the Cartesian coordinates, i.e., $\phi(x, z)$.

   (b) (2 points) Derive an expression for the x-component of velocity, $u(x, z)$.

(c) (2 points) Derive an expression for the z-component of velocity, $w(x, z)$.

(d) (2 points) Hence find the coordinates of all the stagnation points in the flow.

(e) (1 point) Now you can guess the overall flow pattern; sketch it.

(f) (1 point) What kind of flow do you think will result if you bring the two vortices arbitrarily close to each other while keeping the product Γd $(=: \Lambda)$ constant? What can you say about the **shape** and **size** of the body whose external flow will be thus simulated? Justify.

Solution:

(a) The velocity potential due to the uniform flow is $\phi_U = Ux$. That due to a clockwise vortex of circulation $+\Gamma$ placed at an arbitrary point $(x_{+\Gamma}, z_{+\Gamma})$ is

$$\phi_{+\Gamma} = -\frac{\Gamma}{2\pi}\theta_{+\Gamma} = -\frac{\Gamma}{2\pi}\operatorname{atan}\frac{z - z_{+\Gamma}}{x - x_{+\Gamma}}.$$

Specializing to our problem, the superposition of the three pieces gives the overall velocity potential function as

$$\begin{aligned}
\phi = \phi_U + \phi_{+\Gamma} + \phi_{-\Gamma} &= Ux - \frac{\Gamma}{2\pi}\operatorname{atan}\frac{z-d}{x} + \frac{\Gamma}{2\pi}\operatorname{atan}\frac{z+d}{x} \\
&= Ux + \frac{\Gamma}{2\pi}\operatorname{atan}\frac{\frac{z+d}{x} - \frac{z-d}{x}}{1 + \frac{z+d}{x}\frac{z-d}{x}} = Ux + \frac{\Gamma}{2\pi}\operatorname{atan}\frac{2dx}{x^2+z^2-d^2}.
\end{aligned}$$

(b) The x-component of velocity may be found from the expression $u = \partial\phi/\partial x$:

$$\begin{aligned}
u &= \frac{\partial}{\partial x}\left(Ux + \frac{\Gamma}{2\pi}\operatorname{atan}\frac{2dx}{x^2+z^2-d^2}\right) \\
&= U + \frac{\Gamma}{2\pi}\frac{1}{1+\left(\frac{2dx}{x^2+z^2-d^2}\right)^2}\left[\frac{2d(x^2+z^2-d^2) - 2dx(2x)}{(x^2+z^2-d^2)^2}\right] \\
&= U - \frac{\Gamma d}{\pi}\frac{x^2 - z^2 + d^2}{(x^2+z^2-d^2)^2 + (2dx)^2} \\
&= U - \frac{\Gamma d}{\pi}\frac{x^2 - z^2 + d^2}{x^4 + z^4 + d^4 + 2x^2z^2 + 2d^2x^2 - 2d^2z^2}.
\end{aligned}$$

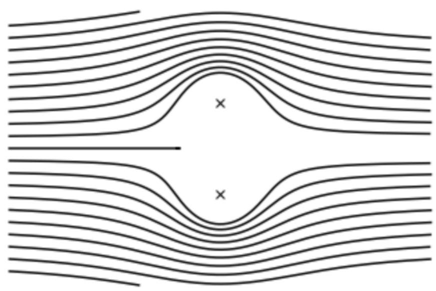
(c) The z-component of velocity may be found from the expression $w = \partial\phi/\partial z$:

$$\begin{aligned}
w &= \frac{\partial}{\partial z}\left(Ux + \frac{\Gamma}{2\pi}\operatorname{atan}\frac{2dx}{x^2+z^2-d^2}\right) \\
&= \frac{\Gamma}{2\pi}\frac{1}{1+\left(\frac{2dx}{x^2+z^2-d^2}\right)^2}\left[-2dx\frac{2z}{(x^2+z^2-d^2)^2}\right] \\
&= -\frac{2\Gamma d}{\pi}\frac{xz}{(x^2+z^2-d^2)^2 + (2dx)^2} \\
&= -\frac{2\Gamma d}{\pi}\frac{xz}{x^4 + z^4 + d^4 + 2x^2z^2 + 2d^2x^2 - 2d^2z^2}.
\end{aligned}$$

(d) From the symmetry of the problem, as well as from the expression for w found above, one can infer that w will vanish everywhere on the x-axis as well as on the z-axis. Also, from the symmetry of the problem, one can conclude that stagnation points, if any, must occur on the x-axis. So, all that is remaining to be done is to examine the expression for u on the x-axis for possible vanishing. Let us suppose that a stagnation point is at $(x_s, 0)$. Then, we must have

$$U - \frac{\Gamma d}{\pi} \frac{x_s^2 + d^2}{x_s^4 + d^4 + 2d^2 x_s^2} = 0 \quad \Longrightarrow \quad \frac{\Gamma d}{\pi(x_s^2 + d^2)} = U \quad \Longrightarrow x_s = \pm\sqrt{\frac{\Gamma d}{\pi U} - d^2}.$$

(e) The flow pattern can be guessed from the behaviour of its individual components. The vortices will tend to slow down the flow in between themselves, and hasten the flow on their outer sides. This, imposed on the uniform flow, along with our experience with the Rankine oval, allows us to sketch the below figure of the flow, visualized by streamlines. Evidently, we are getting the flow over an oval, but with its *minor* axis aligned with the flow. This is called Kelvin's oval.



(f) From the above sketch, intuitively the oval will become a circular cylinder as the vortices are brought closer together. Indeed, a doublet results when two vortices of the opposite sense are brought together keeping the product of the spacing and vortex strength constant. Thus, the flow will come to resemble that over a circular cylinder, albeit an inviscid flow. As d tends to zero keeping $\Lambda := \Gamma d$ finite, the stagnation points come to $x_s = \pm\sqrt{\Lambda/(\pi U)}$. Thus, the radius of the cylinder is $R := \sqrt{\Lambda/(\pi U)}$.