AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 16: Nyquist Plot

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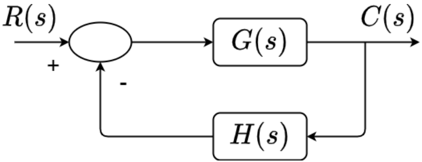
Nyquist Criterion

Nyquist Criterion

- The Nyquist criterion relates stability of a closed loop system to open loop system's frequency response.



- Let us establish four important concepts that will be used during derivation.

1. Relationship between poles of $1 + GH$ and poles of GH.
2. Relationship between zeros of $1 + GH$ and poles of closed loop transfer function.
3. The concept of mapping points.
4. Concept of mapping contours.

Nyquist Criterion

Nyquist Criterion

- Letting

$$G(s) = \frac{N_G}{D_G}, \quad H(s) = \frac{N_H}{D_H}$$

- This results in

$$\begin{aligned} GH =& \frac{N_G N_H}{D_G D_H} \\ 1 + GH =& 1 + \frac{N_G N_H}{D_G D_H} = \frac{N_G N_H + D_G D_H}{D_G D_H} \end{aligned} \tag{1}$$

- The closed loop transfer function, $T(s)$ is given as

$$T(s) = \frac{G}{1+GH} = \frac{N_G D_H}{N_G N_H + D_G D_H} \tag{2}$$

Nyquist Criterion

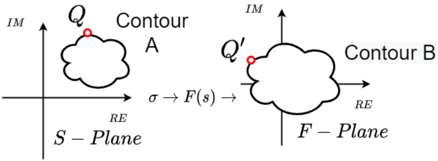
- From the above equations, we can conclude that
  - The poles of $1+GH$ are same as poles of GH, open loop system.
  - The zeros of $1+GH$ are same as poles of T, closed loop system.

Mapping

- Consider a complex number on a s- plane and substitute into function $f(s)$, it results in another complex number.
- This process is called mapping.
- Consider $s = 4 + 3j$ and substitute in $f(s) = s^2 + 2s + 1$. It yeilds $16 + 30j$.
- We can say that $4+3j$ maps into $16+30j$ through $f(s) = s^2+2s+1$.

Nyquist Criterion

Mapping Contours



- Consider a collection of points called contour.
- Assume that $F(s)$ is

$$F(s) = \frac{(s - z_1)(s - z_2)\cdots}{(s - p_1)(s - p_2)\cdots}$$

- Contour A can be mapped through $F(s)$ into a contour B by substitting each point of contour A into a function $F(s)$ and plotting the resulting complex numbers.
- For example Q in s- plane maps into Q' in f-plane through $F(s)$.

Nyquist Criterion

Mapping Contours

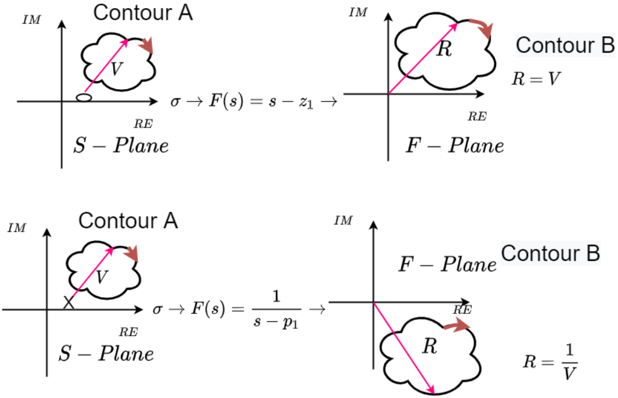
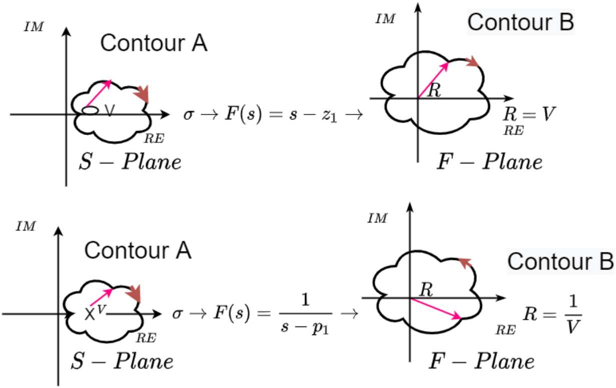


Figure: Examples of Contour Mapping

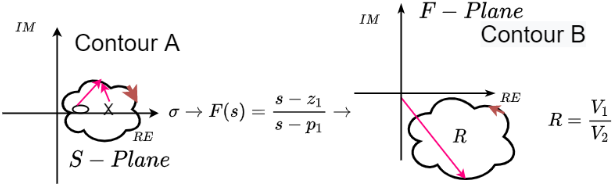
Nyquist Criterion

Mapping Contours



Nyquist Criterion

Mapping Contours



- If we assume a clockwise direction for mapping the points on contour A, then contour B maps in clockwise direction if $F(s)$ has zeros or poles that are not encircled by the contour.
- The contour B maps in anticlockwise direction if $F(s)$ has just poles that are encircled by contour.

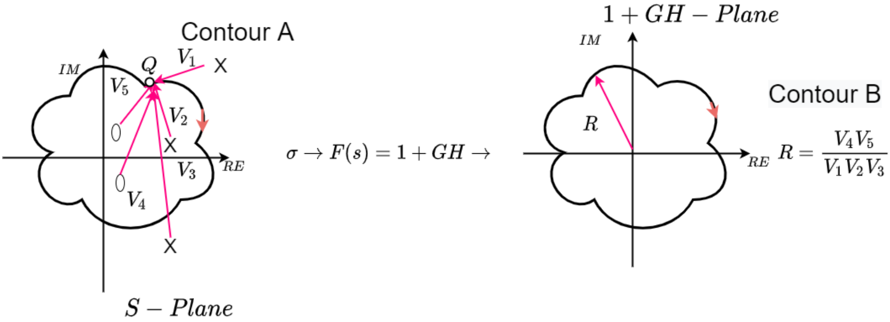
Nyquist Criterion

Nyquist Criterion

- If the pole or zero of $F(s)$ is enclosed by contour A, the mapping encircles origin.
- In the last case, pole and zero rotation cancel, and the mapping does not encircle the origin.
- There exists a unique relationship between the number of poles of $F(s)$, number of zeros of $F(s)$ contained inside a contour A, and number of anticlockwise encirclements of the origin for the mapping of contour B.
- This relationship helps us to determine stability of the closed loop systems.

Nyquist Criterion

Nyquist Criterion



- Assume $F(s) = 1 + GH$, with poles and zeros as shown in the figure. Hence

$$R = \frac{V_4 V_5}{V_1 V_2 V_3}$$

Nyquist Criterion

Nyquist Criterion

- As each point Q of contour is substituted into $1 + GH$, mapped point results on contour B.
- As we move around contour A in clock wise direction, each vector in $1+GH$ that lies inside the contour A will appear to undergo a complete rotation or change in angle of 360.
- Each vector drawn from the poles and zeros of $1+GH$ that are outside the contour A will appear to osillate and return to its position, having a net angular change of 0°.

Nyquist Criterion

Nyquist Criterion

- If we move in clockwise direction along contour A, each zero inside contour A yields a rotation in clockwise direction.
- If we move in clockwise direction along contour A, each pole inside contour A yields a rotation in counterclockwise direction .
- Thus , $N = P - Z$, where

Nyquist Criterion

- $N \rightarrow$ Number of counterclockwise rotations of contour, B
- $P \rightarrow$ Number of poles of $1 + GH$ inside contour A
- $Z \rightarrow$ Number of zeros of $1 + GH$ inside contour A

Nyquist Criterion

Nyquist Criterion

- Recall (1),

$$GH = \frac{N_G N_H}{D_G D_H}$$
$$1 + GH = 1 + \frac{N_G N_H}{D_G D_H} = \frac{N_G N_H + D_G D_H}{D_G D_H}$$

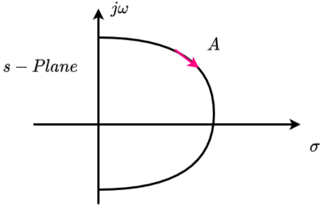
- The poles of $1+GH$ is same as poles of open loop system GH (Known).
- Recall (3),

$$T(s) = \frac{G}{1 + GH} = \frac{N_G D_H}{N_G N_H + D_G D_H} \tag{3}$$

- The zeros of $1 + GH$ is same as poles of closed loop system, $T(s)$ (Unknown).
- Thus P is nothing but open loop poles inside the contour A.
- Z is nothing but closed loop poles inside the contour A.

Nyquist Criterion

Stability and Nyquist Criterion



- Include the contour A to entire right half s-plane.
- We can count number of poles in right half s-plane and hence determine system's stability.

- We can obtain the number of open loop poles, P, inside the contour, which are same as the right half plane poles of GH.
- The problem is to find N and mapping.

Nyquist Criterion

Stability and Nyquist Criterion

- Since we know the poles and zeros of GH, we can use mapping function as GH instead of $1 + GH$.
- The resulting contour is same instead that it is translated to left by one unit.
- Hence Nyquist stability criterion is as follows

Stability Criterion

If a contour A, that encircles complete right half s-plane is mapped through GH, then number of closed loop poles Z in right half s -plane is $Z = P - N$.
$N \rightarrow$ number of counterclockwise revloutions around -1
$P \rightarrow$ number of open loop poles in right half s-plane
Mapping is called Nyquist plot

Nyquist Criterion

Solution

- If the system is represented by

$$G(s)H(s) = \frac{k(s+7)}{s(s+3)(s+2)}$$

- The magnitude is $|G(j\omega)H(j\omega)| = \infty$ at $\omega = \infty$

- If pole is added to origin, at $\omega = 0$, the polar plot gets shifted by ?
- If pole is added to origin, at $\omega = 0$, the polar plot gets shifted by -90°

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Nyquist Plot

Nyquist Plot

- The contour that encloses the right half plane can be mapped through the function $G(s)H(s)$ by substituting points along the contour into $G(s)H(s)$.
- Simple sketch of Nyquist diagram is all that is required to comment on the stability of the system.

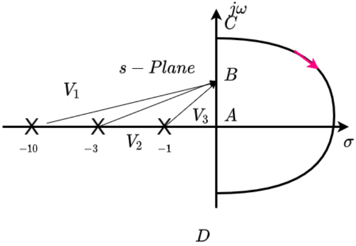
Example

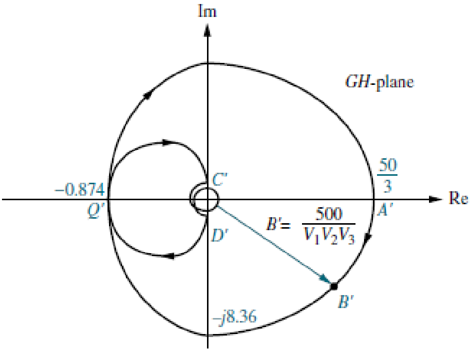
- Sketch the Nyquist plot for the follwing system

$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

Nyquist Plot

Nyquist Plot





Nyquist Plot

Nyquist Plot

- At the point A,

$$G(j\omega) = \left.\frac{500}{(1+j\omega)(3+j\omega)(10+j\omega)}\right|_{\omega=0}$$
$$= \frac{500}{30}\angle 0^\circ$$

- At the point C,

$$G(j\omega) = \left.\frac{500}{(1+j\omega)(3+j\omega)(10+j\omega)}\right|_{\omega=\infty}$$
$$= 0\angle -270^\circ$$

- Thus resultant vector is mapped from A to A' and C to C'.

Nyquist Plot

Nyquist Plot

- Consider frequency response,

$$G(j\omega) = \left.\frac{500}{(s+1)(s+3)(s+10)}\right|_{s\to j\omega} = \frac{500}{(-14\omega^2+30)+j\,(43\omega-\omega^3)}$$

- Simplifying the above function, we obtain

$$G(j\omega) = 500\left(\frac{(-14\omega^2+30)-j(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2}\right)$$

- At $\omega = \sqrt{43}$, the Nyquist diagram crosses negative real axis.
- The point Q' in Nyquist plot is obtained by substituting $\omega = \sqrt{43}$ in $G(j\omega)$. $Q' = -0.874$.

Nyquist Plot

Nyquist Plot

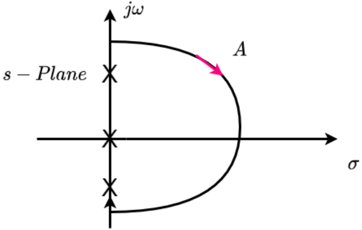
- Traversing the contour from C to D, at point D,

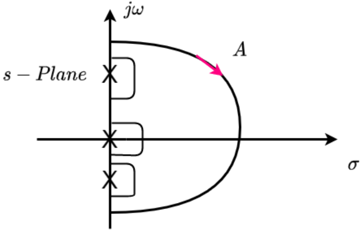
$$G(j\omega) = \left.\frac{500}{(1 + j\omega)(3 + j\omega)(10 + j\omega)}\right|_{\omega=-\infty} = 0\angle + 270^\circ$$

- The resulting vector is mapped from C to C' and D to D' with the change in angle of $270 + 270 = 540^\circ$.
- The mapping of negative imaginary axis (A to D) is a mirror image of mapping of the positive imaginary axis.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot

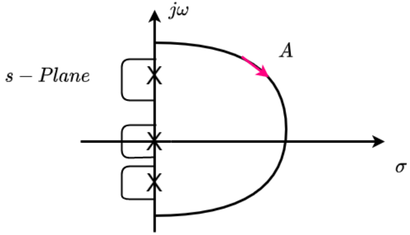




- Assume $G(s)H(s) = \frac{N(s)}{D(s)}$, where $D(s)$ has imaginary roots.
- To sketch Nyquist plot, the contour must detour around each pole lying on imaginary axis.
- The detour can be to the right of the pole, which makes it clear that each poles's vector rotates through $+180°$.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot



- We can also detour to the left of open loop poles .
- In this case, each pole rotates through an angle of $-180°$.
- Detour must be small, or else we might include some left half plane poles in the count.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot

- Sketch the Nyquist diagram of unity feedback system : $G(s) = \frac{(s+2)}{s^2}$

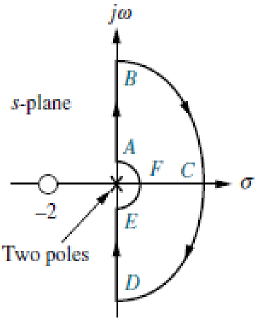


Figure: Source - "Control Systems Engineering" by Norman S Nise

- The system's poles are located at origin, hence contour must bypass origin.
- The mapping starts from points A, B, C, D, E and F.
- Consider the contour mapping from A to B.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot

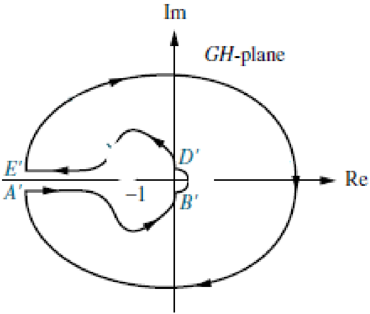


Figure: Source - "Control Systems Engineering" by Norman S Nise

- At point A,

$$G(s) = \frac{2\angle 0}{\epsilon\angle 90^\circ \epsilon\angle 90^\circ} = \infty\angle -180^\circ$$

- At point B,

$$G(s) = \frac{\infty\angle 90^\circ}{\epsilon\angle 90^\circ \epsilon\angle 90^\circ} = 0\angle -90^\circ$$

- Thus mapped vector moves from -180° at A' to -90° at B.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot

- The mapped vector goes from -180° at A to -90° at B.
- The magnitude changes from infinity to zero since at point B, there is one infinite length from zero divided by two infinite length from two poles.
- As we travel BCD, function magnitude remains zero (one infinite length of zeros divided over two infinite length of two poles).
- The zero's vector and poles's vector undergo changes of -180° each.
- Thus mapped vector undergoes a net change of 180°.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot

- Consider transfer function $G(s)$,

$$G(s) = \frac{R_{-2}\angle\theta_{-2}}{R_0\angle\theta_0 R_0\angle\theta_0}$$

- At a point B, $R_{-i} = \infty$, all the angles are 90°. Hence resultant vector at B is $\infty\angle 90 - ((\infty\angle 90)\,(\infty\angle 90)) = 0\angle -90^\circ$.
- At point C, $R_{-i} = \infty$ and angles $\theta_{-i} = 0$. Hence resultant vector is $0\angle 0$.
- At point D, $R_{-i} = \infty$, $\theta_{-i} = -90$. Hence resultant vector at D is $\infty\angle -90 - ((\infty\angle -90)\,(\infty\angle -90)) = 0\angle 90^\circ$.
- Hence change in resultant vector from B to D is $0\angle 90^\circ$-$0\angle -90^\circ$= $0\angle 180^\circ$.

Nyquist Plot - Poles on Imaginary Axis

Nyquist Plot

- The mapping of the section of the contour from D to E is the mirror image of mapping of A to B.
- Consider the section, EFA. At point E, $G(s)$ is

$$G(s) = \frac{2\angle 0}{\epsilon\angle-90\epsilon\angle-90} = \infty\angle 180^\circ$$

- At F,

$$G(s) = \frac{2\angle 0}{\epsilon\angle 0\epsilon\angle 0} = \infty\angle 0$$

- At A,

$$G(s) = \frac{2\angle 0}{\epsilon\angle 90\epsilon\angle 90} = \infty\angle-180^\circ$$

- Hence, the resultant mapped vector is $\infty\angle-360^\circ$.

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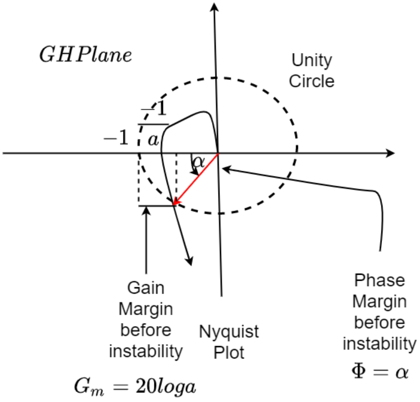
Stability and Nyquist Criterion

Stability and Nyquist Criterion



- Using Nyquist diagram, let us define two quantitative measures regarding the stability of the system.
- These two quantities are called Gain Margin and Phase Margin.
- Systems with greater margins can withstand greater change in system parameters before becoming unstable.

Stability and Nyquist Criterion

Stability and Nyquist Criterion

Definitions

Gain Margin, G_M: The gain margin is change in open loop gain, expressed in decibels (dB), required at 180° of phase shift to make closed loop system unstable.

Phase Margin, Φ_M: The phase margin is change in open loop phase shift required at unity gain to make the closed loop system unstable.

Stability and Nyquist Criterion

Example

- Find the gain and phase margin for the loop transfer function

$$G(s)H(s) = \frac{6}{(s^2+2s+2)(s+2)}$$

Stability and Nyquist Criterion

Solution

- Find the gain and phase margin for the follwing system

$$G(s)H(s) = \frac{6}{(s^2+2s+2)(s+2)}$$

- To find the gain margin, first find the frequency where the Nyquist plot crosses the negative real axis. Finding $G(j\omega)H(j\omega)$, results in

$$G(j\omega)H(j\omega) = \left.\frac{6}{(s^2+2s+2)(s+2)}\right|_{s\to j\omega}$$
$$= \frac{6\left[4(1-\omega^2) - j\omega(6-\omega^2)\right]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

- The Nyquist diagram crosses the real axis at a frequency of $\sqrt{6} rad/s$ and real part is -0.3.

Stability and Nyquist Criterion

Solution

- Hence the gain margin is

$$G_M = 20 \log \frac{1}{a} = 10.45dB$$

- To find the phase margin, find the frequency for which magnitude is unity (calculated through Matlab)
- The magnitude of $G(j\omega)H(j\omega)$ is

$$|G(j\omega)H(j\omega)| = \frac{6}{16} \frac{\left[(4(1-\omega^2))^2 + (\omega(6-\omega^2))^2\right]^{\frac{1}{2}}}{\left[(16(1-\omega^2))^2 + (\omega^2(6-\omega^2)^2)^2\right]^{\frac{1}{2}}} = 1$$

- System has unity gain at the frequency of $1.253 rad/s$.
- At this frequency, the phase angle is $-112.3°$.
- Hence phase margin is $\Phi_M = 180 - 112.3 = 67.7°$

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Frequency Domain Specifications

Definitions

- **Resonant Peak, M_r** : The resonant peak, M_r is the maximum value of $|M(j\omega)|$
- **Resonant Frequency, ω_r** : The resonant frequency, ω_r is the frequency at which peak resonance occurs
- **Bandwidth, BW** : The bandwidth BW is the frequency at $|M(j\omega)|$ drops to $\frac{1}{\sqrt{2}}$ from its zero frequency value

Frequency Domain Specifications

Resonant Peak and Resonant Frequency

- Consider closed loop second order system,

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The frequency response is given as

$$\begin{aligned} M(j\omega) =& \frac{\omega_n^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2} \\ =& \frac{1}{1 + 2\zeta\frac{\omega}{\omega_n}j - \frac{\omega^2}{\omega_n^2}} \end{aligned}$$

- Let $u = \frac{\omega}{\omega_n}$, the above equation becomes

$$M(ju) = \frac{1}{1 + 2\zeta uj - u^2}$$

Frequency Domain Specifications

Resonant Peak and Resonant Frequency

- The magnitude of $M(ju)$ is given by

$$|M(ju)| = \frac{1}{\left[(1-u^2)^2 + 4\zeta^2 u^2\right]^{\frac{1}{2}}}$$

- The resonant frequency is obtained by setting the derivative of $|M(ju)|$ to zero

$$\frac{d|M(ju)|}{du} = -\frac{1}{\left[(1-u^2)^2 + 4\zeta^2 u^2\right]^{\frac{3}{2}}}\left(2(1-u^2)2u + 8\zeta^2 u\right) = 0$$
$$= \left(-1 + u^2 + 2\zeta^2\right)u = 0 \implies u = \sqrt{1-2\zeta^2}$$

Frequency Domain Specifications

Resonant Peak and Resonant Frequency

- Substitue u in the magnitude expression,

$$M_r = \frac{1}{\sqrt{4\zeta^4 + 4\zeta^2(1-2\zeta^2)}}$$
$$= \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

- Hence, the resonant peak is given by

$$\boxed{M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

- Resubstituting for u, we have

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

Frequency Domain Specifications

Bandwidth

- In accordance with the definition of bandwidth, we set the value of $|M(ju)| = \frac{1}{\sqrt{2}}$

$$\frac{1}{\left[(1-u^2)^2 + 4\zeta^2 u^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

$$\left[\left(1-u^2\right)^2 + 4\zeta^2 u^2\right]^{\frac{1}{2}} = \sqrt{2}$$

- Simplifying the above equation,

$$1 + u^4 - 2u^2 + 4\zeta^2 u^2 - 2 = 0$$

$$u^4 - 2u^2 + 4\zeta^2 u^2 - 1 = 0$$

$$u^4 - 2\left(1 - 2\zeta^2\right) u^2 - 1 = 0$$

Frequency Domain Specifications

Bandwidth

- Solving the quadratic equation in terms of u^4, results in

$$u^2 = \frac{1}{2}\left(2(1-2\zeta^2) \pm \sqrt{4(1+4\zeta^4-4\zeta^2)+4}\right)$$
$$= 1-2\zeta^2 \pm \sqrt{4\zeta^4-4\zeta^2+2}$$

- Plus sign should be considered as u should be positive. Hence the bandwidth is given by

$$\boxed{BW = \omega_n\left(1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}\right)^{\frac{1}{2}}}$$

Frequency Domain Specifications

Bandwidth

- To relate bandwidth with settling time, substitute $\omega_n = \frac{4}{\zeta T_s}$

$$\boxed{BW = \frac{4}{\zeta T_s}\left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{\frac{1}{2}}}$$

Example

- Find the closed loop bandwidth required for 20% overshoot and 2-seconds setling time

Frequency Domain Specifications

Solution

- Find the closed loop bandwidth required for 20% overshoot and 2-seconds setling time
- Step 1 : Find damping ratio from

$$\boxed{M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \implies \zeta = 0.455}$$

- Step 2: Bandwidth from the following

$$\boxed{BW = \frac{4}{\zeta T_s}\left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{\frac{1}{2}} \implies BW = 5.79 rad/s}$$

References I

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