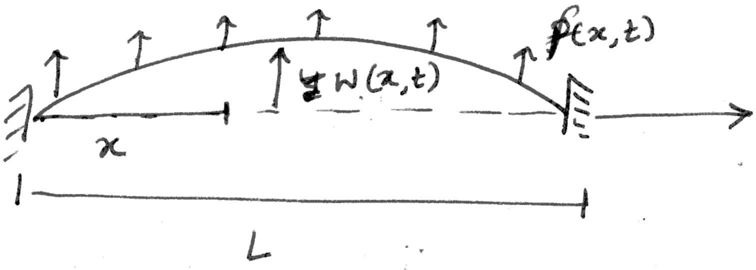
Transverse Vibrations of a string

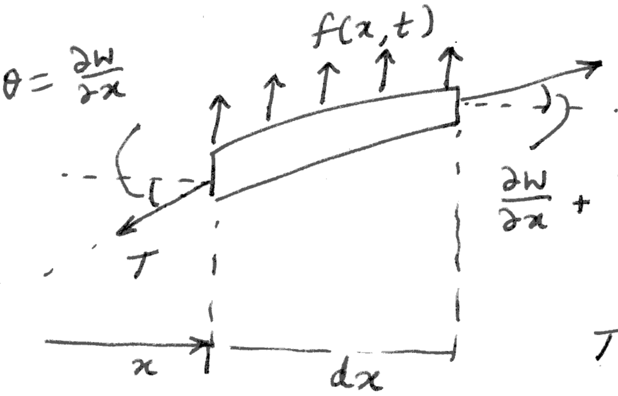


A string is a flexible distributed mass system that can support loads through tension.

$m(x)$: mass per unit length

$f(x,t)$: Applied force, per unit length

$T(x)$: tension within the string.



$$T + \frac{\partial T}{\partial x} dx$$

$$\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \cdot dx = \theta + \frac{\partial \theta}{\partial x} dx$$

The equilibrium equation in the transverse direction is

$$\left(T + \frac{\partial T}{\partial x} \cdot dx\right) \sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right) - T \sin\theta + f\,dx = m\,dx\,\ddot{w}$$

For small deformations, $\sin\theta \approx \theta$., $\sin\left(\theta + \frac{\partial \theta}{\partial x} dx\right) \approx \theta + \frac{\partial \theta}{\partial x} dx$

$$\left(T + \frac{\partial T}{\partial x} \cdot dx\right)\left(\theta + \frac{\partial \theta}{\partial x} \cdot dx\right) - T\theta + f\,dx = m\ddot{w}\,dx$$

$$T \frac{\partial \theta}{\partial x} dx + \frac{\partial T}{\partial x} \cdot \theta\,dx + \frac{\partial T}{\partial x} \cdot \frac{\partial \theta}{\partial x} (dx)^2 + f\,dx = m\ddot{w}\,dx$$

Neglect the term that is second order in dx.

Then ~~the eleem is~~ we obtain

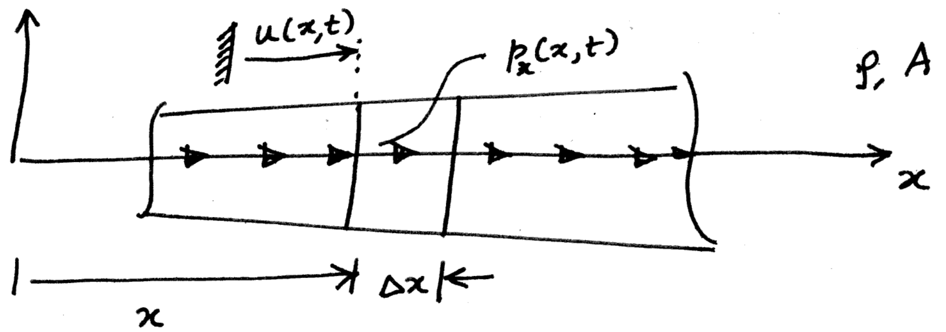
$$\frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + T\frac{\partial^2 w}{\partial x^2} + f = m\ddot{w}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial x}\left(T\frac{\partial w}{\partial x}\right) + f = m\ddot{w}}, \quad 0 < x < L$$

E.O.M for transverse vibrations of a string.

AXIAL DEFORMATION

Consider the axial deformation of a long, thin member that is shown in the figure below:



A portion of the structural member is shown above. The ~~xxxx~~ axial deformation at location x is denoted by $u(x,t)$.

$u \equiv u(x,t)$ Axial deformation at any x

$p_x \equiv p_x(x,t)$ Externally applied axial force per unit length.

$\rho \equiv \rho(x)$ Mass density

$A \equiv A(x)$ Area of cross section

The equations of motion are derived for the following assumptions:

1) Axis of the member remains straight

2) Cross-sections remain plane and $\perp^{lar}$ to the axis of the member.

3) Material is linearly elastic.

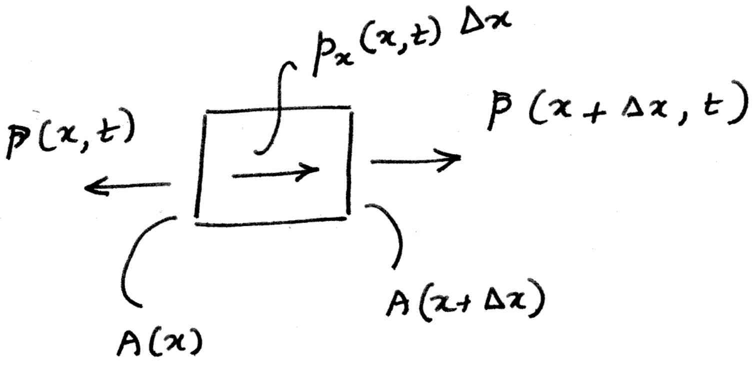
4) Material properties E, ρ, A, are constant at a given section but may vary with 'x'.

Linear strain displacement relations yeild

$$\epsilon \equiv \epsilon(x,t) = \frac{\partial u}{\partial x}$$

The constitutive law : $\sigma = E\,\epsilon$

The axial force on the cross-section



$$p_x \Delta x + P(x+\Delta x, t) - P(x) = \rho A \Delta x \frac{\partial^2 u}{\partial t^2}$$

Let $\rho A = m$, mass per unit length.

$$\Rightarrow \quad p_x + \frac{P(x+\Delta x, t) - P(x)}{\Delta x} = m \frac{\partial^2 u}{\partial t^2} , \quad 0 < x < L$$

Limit as $\Delta x \to 0$ yields

$$p_x + \frac{\partial P}{\partial x} = m \frac{\partial^2 u}{\partial t^2} , \quad 0 < x, L$$

The axial force on the cross-section is

$$P(x,t) = \iint_A \sigma \, dA = \sigma A = AE \frac{\partial u}{\partial x}$$

$\therefore$ E.O.M is $\boxed{\frac{\partial}{\partial x}\left(AE \frac{\partial u}{\partial x}\right) + p_x(x,t) = m \frac{\partial^2 u}{\partial t^2} , \quad 0 < x < L}$

Torsional deformation of circular rods

Assumptions :

1) Axis of the member, the nominal choice for the x-axis, remains straight.

2) Cross-sections remain perpendicular to the axis of the member.

3) Radial lines remain straight & radial as the cross-section rotates through angle θ about the x-axis.

4) Deformations & rotations are small ~~Material is~~

5) Linearly elastic material : $\tau = G\gamma$

shear stress = Shear modulus $\times$ shear strain

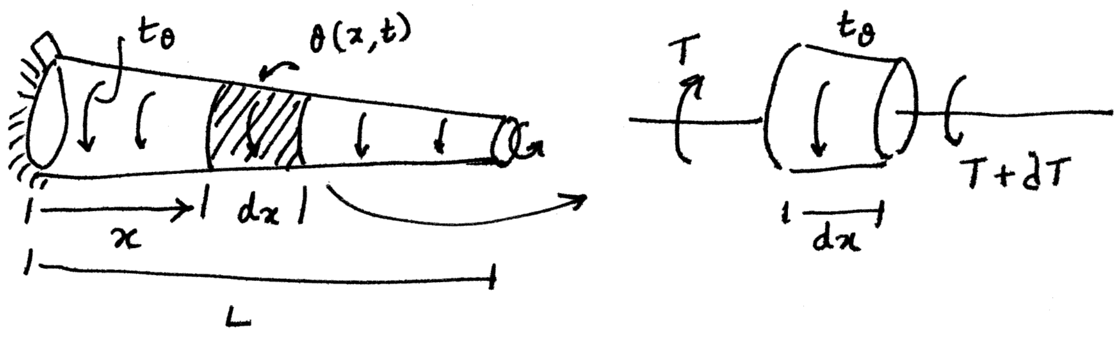
6) $G \equiv G(x)$.

The relation between torque & twist is

$$\frac{d\theta}{dx} = \frac{T}{G I_p}$$

$T \equiv T(x,t)$ twisting moment ~~at~~ on cross-section at x.

$I_p \equiv I_p(x)$: polar moment of inertia at x.



According to Newton's law

$$\Sigma M = \rho I_p \frac{\partial^2 \theta}{\partial t^2} dx$$

$$t_\theta\, dx + T + \frac{\partial T}{\partial x} . dx - T = \rho I_p \frac{\partial^2 \theta}{\partial t^2} dx$$

$$\Rightarrow \quad t_\theta + \frac{\partial T}{\partial x} = \rho I_p \frac{\partial^2 \theta}{\partial t^2}$$

Substituting from the balance equation,

$$\boxed{t_\theta + \frac{\partial}{\partial x}\left(G I_p \frac{\partial \theta}{\partial x}\right) = \rho I_p \frac{\partial^2 \theta}{\partial t^2}.} \quad , \; 0 < x < L$$

Here, $t_\theta \equiv t_\theta(x,t)$ $\qquad \theta \equiv \theta(x,t)$

$I_p \equiv G I_p(x)$

$G \equiv G(x)$

$\rho = \rho(x)$

Differential equation for torsional vibration of a linearly elastic rod w/ circular cross-section.

<u>B.Cs</u>

$\theta(x_e, t) = 0$ fixed end,

$x_e = 0$ and/or L.

$T(x_e, t) = T(t)$ end with applied torque.

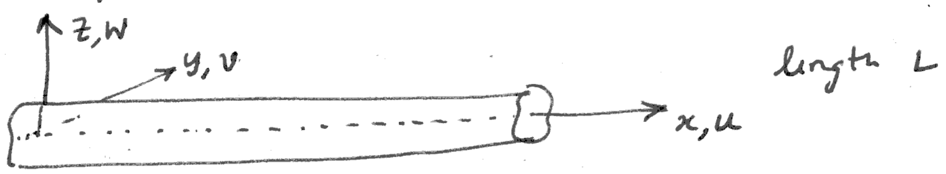
$x_e = 0$ and/or L.

This condition can also be written as

$$\left[G I_p \frac{\partial \theta}{\partial x} \right]_{x = x_e} = T(x_e, t).$$

BENDING DEFORMATION

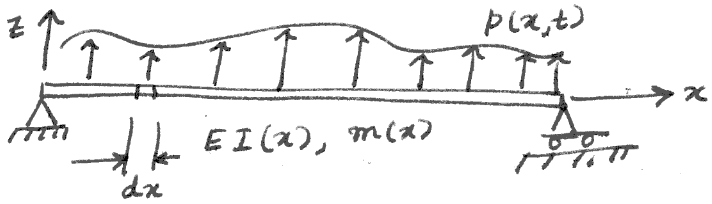
Consider a straight non-uniform beam of stiffness $EI \equiv EI(x)$ and mass per unit length $m \equiv m(x)$.

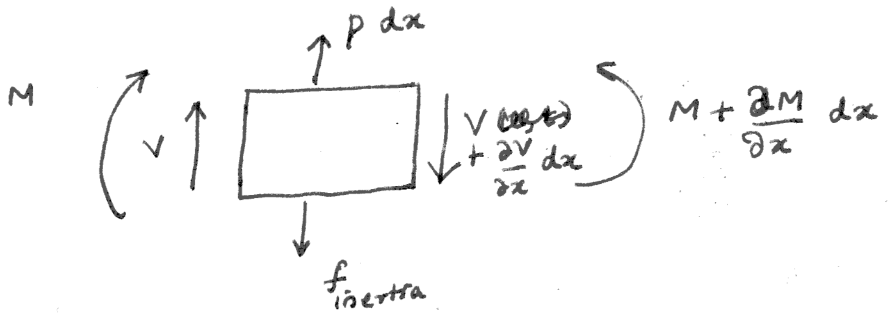


The beam is subjecto to a transverse loading $p(x,t)$.

The transverse displacement is $w(x,t)$.

DOFS u and v are considered to be absent.





The equilibrium of the mass element is

$$V + p\,dx - \left(V + \frac{\partial V}{\partial x}dx\right) - f_{inertia} = 0$$

$$f_{inertia} = m\frac{\partial^2 w}{\partial t^2}\cdot dx$$

$$\Rightarrow \quad \frac{\partial V}{\partial x} = p - m\frac{\partial^2 w}{\partial t^2}$$

Note that in the above expression, the inertia term modifies the standard static relationship between shear force V and transverse loading.

The second equilibrium equation is from balance of moments

$$M + V\,dx - \left[M + \frac{\partial M}{\partial x}\,dx\right] = 0$$

$$\Rightarrow \quad \frac{\partial M}{\partial x} = V$$

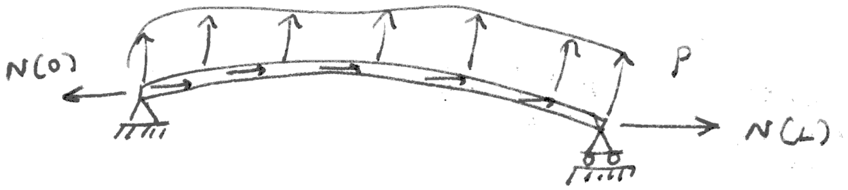
~~Substitute~~ Furthermore, recall that $M = EI\,\frac{\partial^2 w}{\partial x^2}$.

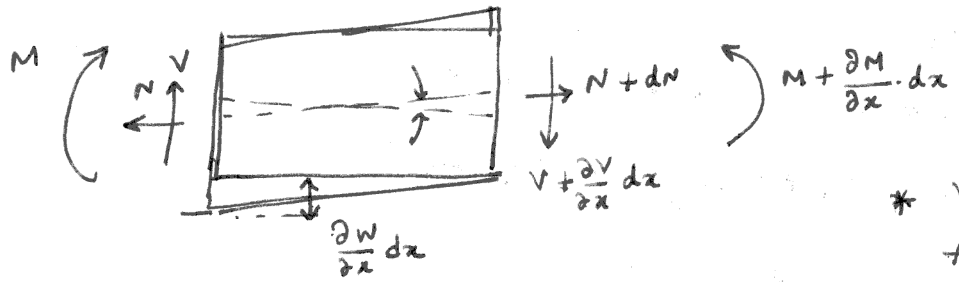
Substituting the two relations into the e.o.m,

$$\boxed{\frac{\partial^2}{\partial x^2}\left[EI\,\frac{\partial^2 w}{\partial x^2}\right] + m\,\frac{\partial^2 w}{\partial t^2} = p}$$

With boundary conditions specified at $x=0$ & $x=L$.

If an axial force is in the horizontal direction





* V is not the shear force as it is not acting normal to the beam axis.

The ~~e.o.m~~ moment equilibrium becomes

$$M + V\,dx + N\,\frac{\partial w}{\partial x}\,dx - \left[M + \frac{\partial M}{\partial x}\,dx\right] = 0$$

$$\Rightarrow \quad V = -N\,\frac{\partial w}{\partial x} + \frac{\partial M}{\partial x}$$

The final e.o.m becomes

$$\frac{\partial^2}{\partial x^2}\left[EI\,\frac{\partial^2 w}{\partial x^2}\right] - \frac{\partial}{\partial x}\left[N\,\frac{\partial w}{\partial x}\right] + m\,\frac{\partial^2 w}{\partial t^2} = p$$

STRAIN ENERGY CONTRIBUTIONS

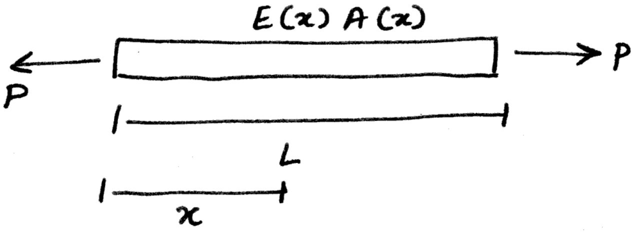
The strain energy of a flexible system is

$$U = \iiint_{Volume} \left[\int_0^{\epsilon_{ij}} \sigma_{ij} \, d\epsilon_{ij} \right] dVol.$$

where

$$u_d = \text{strain energy density} = \int_0^{\epsilon_{ij}} \sigma_{ij} \, d\epsilon_{ij}$$

Consider a bar in axial deformation:



At any location 'x', the stress is

$$\sigma_{xx} = \frac{P}{A(x)} = E(x)\, \epsilon_{xx}$$

All other stresses are zero.

$$\epsilon_{xx} = \frac{du}{dx} \quad \text{(strain-displacement relations)}$$

where $u \equiv u(x,t)$ is the axial deformation.

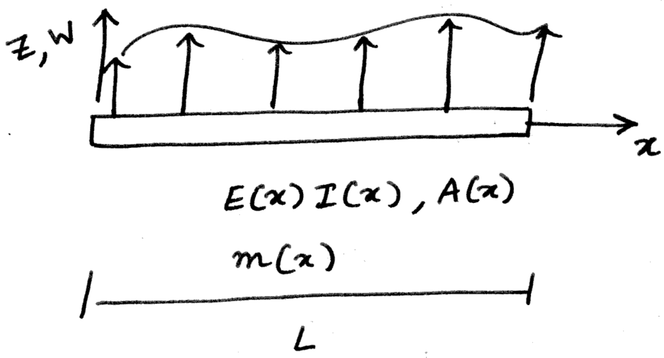
$$\Rightarrow \quad \sigma_{xx} = E \frac{\partial u}{\partial x}$$

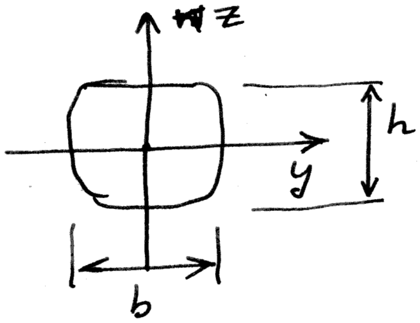
$$\Rightarrow \quad U = \int_0^L \int_0^{\epsilon_{11}} \sigma_{11} \, d\epsilon_{11} \, A \, dx$$

$$= \int_0^L E \frac{\epsilon_{11}^2}{2} A \, dx$$

$$\boxed{U = \frac{1}{2} \int_0^L EA \left(\frac{\partial u}{\partial x}\right)^2 dx}$$

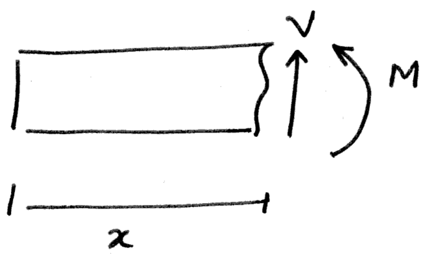
Consider a beam in bending:





~~The neutral axis passes through the centroid of the cross section.~~

The x-axis ~~x~~ is the neutral axis of the beam.



The bending moment and shear force acting at any location x are $M(x,t)$, $V(x,t)$.

From beam theory, following the Euler-Bernoulli assumptions,

$$\sigma_{xx} = -\frac{Mz}{I} \quad \text{and} \quad \sigma_{xz} = \frac{VQ}{Ib}$$

where $I = \int_A z^2 \, dA$

All other stress components are assumed to be zero.

Q is the first moment of area about the y-axis.

In terms of stress components,

$$U = \int_{Vol} \left(\frac{1}{2E} \sigma_{xx}^2 + \frac{1}{2G} \sigma_{xz}^2 \right) dV$$

$$= \int \left(\frac{1}{2E} \frac{M^2 z^2}{I^2} + \frac{1}{2G} \frac{V^2 Q^2}{I^2 b^2} \right) dV$$

M and V are functions of x alone.

$$\therefore \quad U = \underbrace{\int_0^L \frac{M^2}{2EI} dx}_{\text{strain energy due to bending}} + \underbrace{\int_0^L \frac{V^2}{2GI^2} \int_A \frac{Q^2}{b^2} dA}_{\text{strain energy due to shear}}$$

For typical beam geometries, the strain energy in shear is negligible compared to the strain energy in bending. Therefore, in most structural dynamics applications, the strain energy in shear is neglected when dealing with slender beam-like structures.

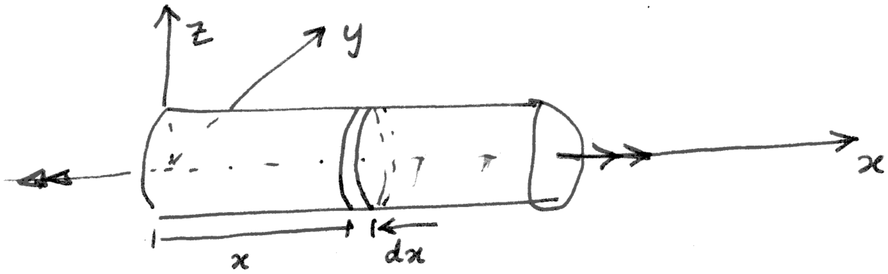
From the problem of pure bending, we have

$$\frac{M}{EI} = \frac{\partial^2 w}{\partial x^2}.$$

$$\therefore \quad \boxed{U = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx.}$$

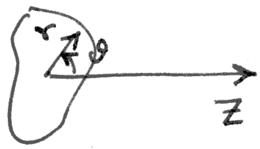
Strain energy in torsion

Consider a circular shaft of length L in a state of pure torsion.



For this case, the non-zero stresses are σ_{xz} and σ_{xy} respectively.

The problem can be reposed in a cylindrical coordinate system (r, θ, z) as shown in the figure.



In this coordinate system, the non zero stress is $\sigma_{\theta z}$.

The strain energy is then

$$U = \frac{1}{2} \int_{\text{Vol.}} \sigma_{\theta z} \gamma_{\theta z} \, dV$$

Note that $\sigma_{\theta z} = G \gamma_{\theta z}$, where $\gamma_{\theta z} = r \dfrac{\partial \varphi}{\partial x}$, where

φ is the twist angle. ~~per unit length~~.

$$\therefore \quad U = \frac{1}{2} \int_{\text{Vol}} G \int_A r^2 \left(\frac{\partial \varphi}{\partial x}\right)^2 dA \, . \, dx$$

The twist angle is a function of x alone. (z in the cylindrical coordinate system). Also $G \equiv G(x)$.

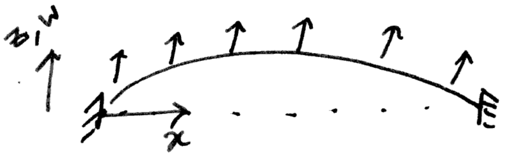
$$\therefore \quad U = \frac{1}{2} \int_0^L G \left(\frac{\partial \varphi}{\partial x}\right)^2 dx \, . \int_A r^2 \, dA$$

Note that $\int_A r^2\, dA = J.$

$$\Rightarrow \quad \boxed{U = \frac{1}{2} \int_0^L G J \left(\frac{\partial \varphi}{\partial x} \right)^2 dx}$$

For shafts with non-circular cross-section, the application of torque produces warping of the cross-section and the axial deformation is no longer zero. The treatment of such structural members requires a more rigorous analysis that is not considered within the scope of the current course. Consequently, the problems of torsion dealt with here will be based ~~on~~ shafts with circular cross-section.

· Consider the transverse vibration of string:



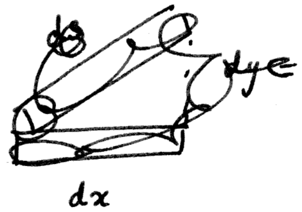
The shape (deformation) of the string is given by $w(x,t)$.

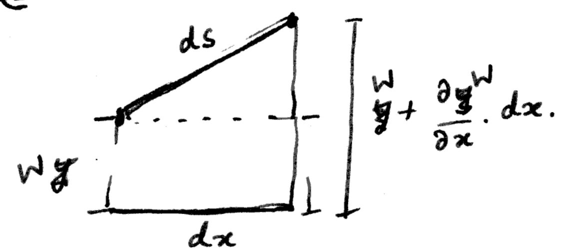
The potential energy term arises due to the tendency of the tension to restore the string to its undeformed state.

$$U = \int_0^L T(x) \cdot [\, ds(x,t) - dx(x) \,]$$

where ds is the length of the element after stretching.

The following diagram applies.





$$ds^2 = dx^2 + \left(\frac{\partial w}{\partial x} \cdot dx\right)^2 = dx^2 \left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\Rightarrow \quad ds = dx \left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]^{1/2}$$

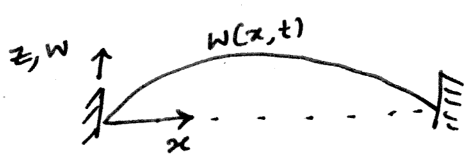
For small $\partial w/\partial x$, $\quad ds \cong dx \left[1 + \frac{1}{2}\left(\partial w/\partial x\right)^2\right]$

$$\Rightarrow \quad (ds - dx) = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx$$

$$\therefore \quad \boxed{U = \frac{1}{2} \int T \left(\partial w/\partial x\right)^2 dx}$$

KINETIC ENERGY CONTRIBUTIONS

Transverse vibrations of a string:

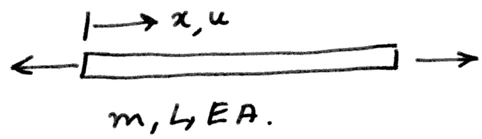


$$T = \frac{1}{2}\int_0^L m\left(\frac{\partial w}{\partial t}\right)^2 dx$$

$m \equiv m(x) =$ mass per unit length of the string.

Axial deformations

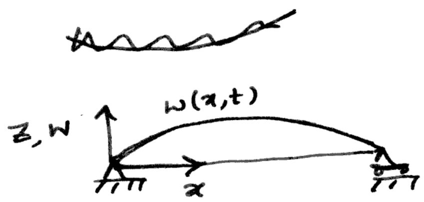
$$T = \frac{1}{2}\int_0^L m\left(\frac{\partial u}{\partial t}\right)^2 dx$$



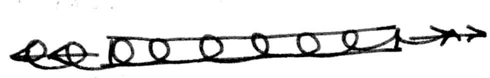
$m \equiv m(x)$.

Bending

$$T = \frac{1}{2}\int_{vol} \rho\left(\frac{\partial w}{\partial t}\right)^2 d\,vol.$$



$$+ \frac{1}{2}\int_{vol} \rho\left(\frac{\partial u}{\partial t}\right)^2 d\,vol$$



For an Euler-Bernoulli beam,

$$u(x,t) = u_0(x,t) - z\frac{\partial w}{\partial x}\ ;$$

u_0: extension of the neutral axis

For an inextensible beam, the axial deformation of the neutral axis is zero.

$$\Rightarrow \quad u(x,t) = -z\frac{\partial w}{\partial x}$$

$$\therefore \quad T = \frac{1}{2}\int_{Vol} \rho \left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}\int_{Vol} \rho \left(\frac{\partial u}{\partial t}\right)^2 dx$$

$$= \frac{1}{2}\int_0^L \int_A \rho \left(\frac{\partial w}{\partial t}\right)^2 dA \, . \, dx + \frac{1}{2}\int_0^L \int_A \rho z^2 \left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 dA \, dx$$

$$= \underbrace{\frac{1}{2}\int_0^L m . \left(\frac{\partial w}{\partial t}\right)^2 dx}_{\text{translational}} + \underbrace{\frac{1}{2}\int_0^L \rho I \left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 dx}_{\text{rotational}}$$

For slender beam, the rotational component is negligible compared to the translational component.

$$\therefore \quad \boxed{T = \frac{1}{2}\int_0^L m \left(\frac{\partial w}{\partial t}\right)^2 dx.}$$

where $\quad m = \int_A \rho \, dA \quad$ = mass per unit length
$\equiv m(x)$.

~~The expression~~

<u>Torsion</u>.

For a circular shaft, the kinetic energy is

$$\boxed{T = \frac{1}{2}\int_0^L \rho J \left(\frac{\partial \phi}{\partial t}\right)^2 dx}$$

, ρ is density
J is polar (area) moment of inertia
ϕ is twist $\equiv \phi(x,t)$