AE 308: Control Theory
AE 775: System Modelling, Dynamics and Control

Lecture 3: Linearization

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Linear vs. Nonlinear Systems

Linear Systems	**Nonlinear Systems**
Simpler to analyze and design	Difficult to analyze and design
Have only one equilibrium point	Can have multiple equilibrium points
No limit cycles	Limit cycles (Self sustained oscillations)
No bifurcations	Bifurcation (No. of equilibrium points and their stability nature can vary with parameter value)
Frequency and amplitude are independent	Frequency and amplitude can be coupled

Linear vs. Nonlinear Systems

Comment on Linearity of both systems

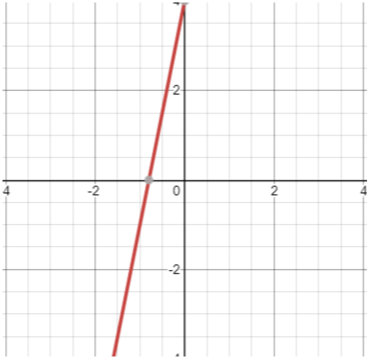

Figure: $y = mx + c$

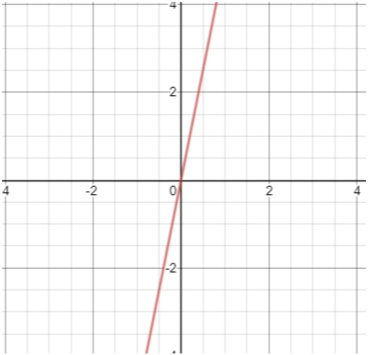

Figure: $y = mx$

Linear vs. Nonlinear Systems

Comment on Linearity of both systems

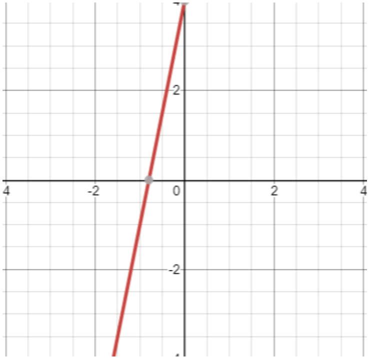

Figure: Nonlinear

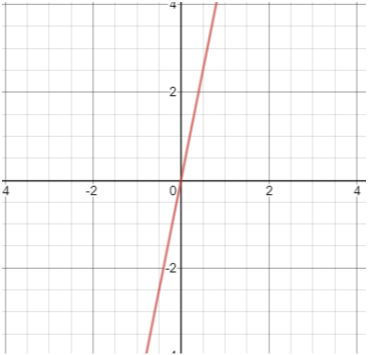

Figure: Linear

Features of Linear systems

- Linear systems greatly simplify the solution procedures as well as allow the extrapolation of results through application of the principle of superposition.
- Linearization is a method to arrive at linear input output relations through a structured process that ignores the higher order terms.
- However, in such cases, the applicability of results gets limited to a small domain over which the linearization is carried out.
- In all such cases, it is important to assess accuracy loss.

Principle of Superposition

Linear Systems follows the **"Principle of Superposition"**

- Multiplying the input(s) by any constant must multiply the output by same constant.

$$F(\alpha x) = \alpha F(x) \qquad \textbf{Homogeneity}$$

- The response to several inputs applied simultaneously must be sum of individual responses to each input applied separately.

$$F(x_1 + x_2) = F(x_1) + F(x_2) \qquad \textbf{Additivity}$$

Principle of Superposition - Examples

Example 1: $\dot{x} = 5x$

1. Homogeneity:

$$\alpha\dot{x} = \alpha(5x) = 5(\alpha x)$$

2. Additivity:

$$\frac{d}{dt}(x_1 + x_2) = \dot{x_1} + \dot{x_2} = 5x_1 + 5x_2 = 5(x_1 + x_2)$$

The system is satisfying both homogeneity and additivity properties, thus the system is linear.

Principle of Superposition - Examples

Example 2: $\dot{x} = 5x + 7$

1. Homogeneity:

$$\alpha\dot{x} = \alpha(5x + 7) \neq 5(\alpha x) + 7$$

2. Additivity:

$$\frac{d}{dt}(x_1 + x_2) = \dot{x_1} + \dot{x_2} = 5(x_1 + 7) + 5(x_2 + 7) \neq 5(x_1 + x_2) + 7$$

The system is NOT satisfying both homogeneity and additivity properties, the system is nonlinear.

Principle of Superposition - Examples

Example 3: $\dot{x} = 5\sin x$

1. Homogeneity:

$$\alpha\dot{x} = \alpha(5\sin x) \neq 5\sin(\alpha x)$$

2. Additivity:

$$\frac{d}{dt}(x_1 + x_2) = \dot{x_1} + \dot{x_2} = 5\sin x_1 + 5\sin x_2 \neq 5\sin(x_1 + x_2)$$

The system is NOT satisfying both homogeneity and additivity properties, the system is nonlinear.

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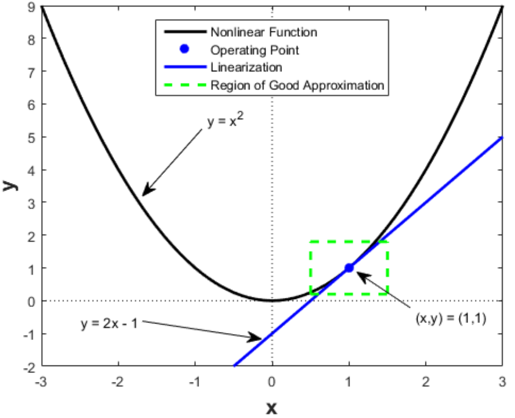
Linearization Process

Consider a general input-output relation as shown below.



Linearization is a linear approximation of a nonlinear system that is valid in a small region around an operating point.

Linearization Process

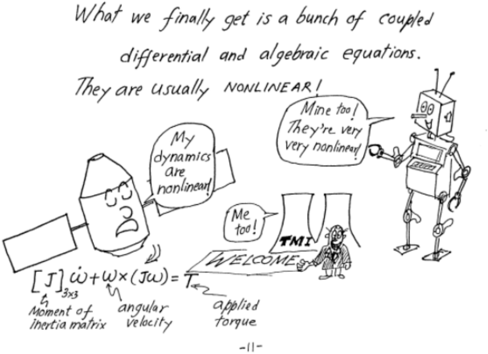


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

- A linear relation is obtained by assuming that variables deviate around the operating condition.
- Under these conditions, it is possible to express nonlinear relation, through a Taylor's series expansion, as follows.

  Given a equation, $y = f(x)$

  At operating point, $y_0 = f(x_0)$

  Therefore,

$$y = y_0 + \frac{df}{dx}|_{x=x_0}(x - x_0) + \frac{1}{2!}\frac{d^2 f}{dx^2}|_{x=x_0}(x - x_0)^2 + \ldots$$

- For small $(x - x_0)$, we can ignore quadratic and higher terms, so that 'y' becomes linear with respect to x.

Linearization Process

Rewriting equations,

$$
\begin{aligned}
& y = y_0 + k(x - x_0), \quad \text{where } k = \frac{df}{dx}|_{x=x_0} \\
\Rightarrow & y - y_0 = k(x - x_0) \\
\Rightarrow & \delta y = k\delta x
\end{aligned}
$$

where, $k = \frac{df}{dx}|_{x=x_0}$, $\delta y = y - y_0$, $\delta x = x - x_0$.

It is the **linearized equation** in terms of new variables $(\delta x, \delta y)$, that are defined as small region around the operating point.

Linearization Process

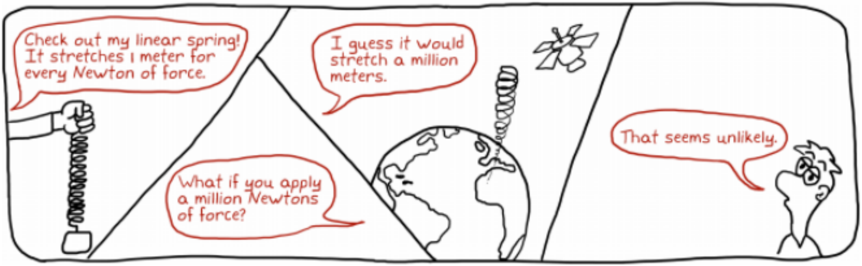


Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

Single Variable Example

Example: Linearize the following equation, around the given operating point $x_0 = 2$, and assess its accuracy for $x = 1.8$:

$$y = 0.2x^3.$$

Solution:

1. Convert it to form:

$$y - y_0 = k(x - x_0)$$

2. Finding the derivative and initial value:

$$k = \frac{dy}{dx}|_{x=2} = 2.4, \quad y_0 = 0.2x_0^3 = 1.6$$

3. Comparing results:

$$y - 1.6 = 2.4(x - 2), \quad y = 2.4x - 3.2$$

$$y(1.8) = 1.12.$$

The exact solution is $y = 1.17$.

Multivariable Linearization

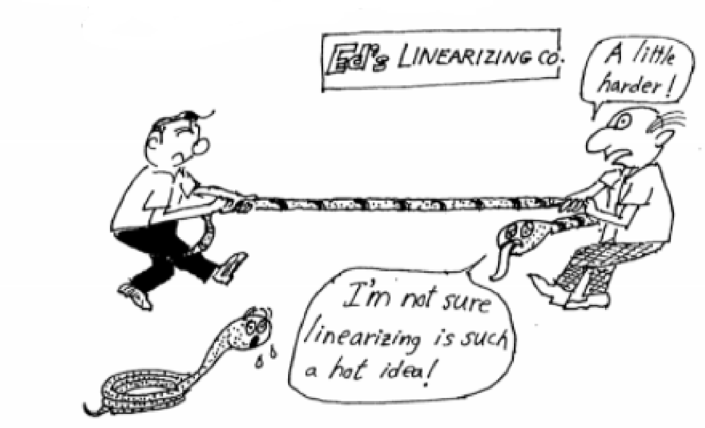


Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

Multivariable Linearization

Partial derivatives are employed in place of total derivatives as

$$
\begin{aligned}
y &= f(x_1, x_2) \\
&= f(x_{10}, x_{20}) + \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} (x_1 - x_{10}) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} (x_2 - x_{20}) \\
&\quad + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x_1^2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} (x_1 - x_{10})^2 + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}} (x_2 - x_{20})^2 + \cdots
\end{aligned}
$$

Multivariable Linearization

For small $(x_1 - x_{10})$ and $(x_2 - x_{20})$, we can ignore quadratic and higher terms, so that 'y' becomes linear with respect to x_1, x_2 as

$$\delta y = k_1 \delta x_1 + k_2 \delta x_2$$

where,

$$\delta y = y - y_0, \quad \delta x_1 = x_1 - x_{10}, \quad \delta x_2 = x_2 - x_{20}$$

$$k_1 = \left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}}, \quad k_2 = \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{10} \\ x_2 = x_{20}}}$$

Multivariable Example I

Example: Linearize the following equation, around the operating point $x_0 = 6, y_0 = 11$, and assess its accuracy for $x = 5, y = 10$:

$$z = xy.$$

Solution:

- Convert it to form:

$$z - z_0 = k_1(x - x_0) + k_2(y - y_0)$$

Multivariable Example I

- Finding the derivative and initial values:

$$k_1 = \left.\frac{\partial z}{\partial x}\right|_{\substack{x_0=6\\y_0=11}} = 11, \quad k_2 = \left.\frac{\partial z}{\partial y}\right|_{\substack{x_0=6\\y_0=11}} = 6$$

$$z_0 = x_0 y_0 = 66$$

- Comparing results

$$z - 66 = 11(x-6) + 6(y-11), \quad z = 11x + 6y - 66$$

$$z(5,10) = 49.$$

The exact solution is $z(5,10) = 50.$

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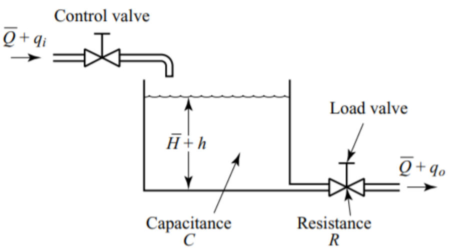
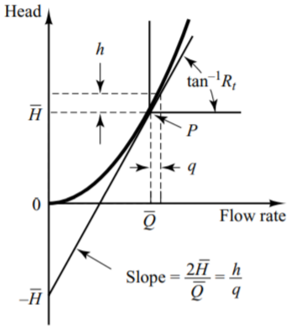
Linearization: Water Tank





The steady state flow rate (for turbulent flow) is given by

$$Q = k\sqrt{H}, \quad \text{where } k = \text{Valve constant}$$

Linearization: Water Tank

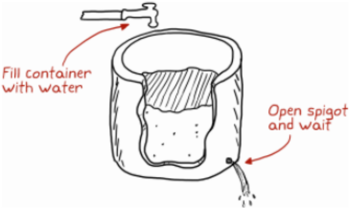


Figure: *Source - "The Fundamentals of Control Theory" by B. Douglas*

1. Convert it to form:

$$Q = \bar{Q} + \frac{dQ}{dH}\Big|_{(\bar{Q},\bar{H})}(H - \bar{H})$$

2. Computing derivatives:

$$\frac{dQ}{dH} = \frac{k}{2\sqrt{\bar{H}}} = \frac{\bar{Q}}{2\bar{H}}$$

3. Linearized equation:

$$H - \bar{H} = h, \;\; Q - \bar{Q} = q$$

$$q = \frac{k}{2\sqrt{\bar{H}}}h = \frac{\bar{Q}}{2\bar{H}}h$$

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Linearization: Pendulum

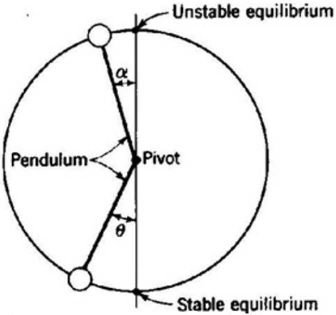

Figure: *Source - "Cartoon Tour Of Control Theory" by S. M. Joshi*

Dynamic equation:
$J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta = 0$
where,
m: Mass, l: Length, J: Moment of inertia,
B: Damping constant

Linearization: Pendulum

Dynamic equation can be linearized as follows

1. Convert it to form:

$$y = J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta = y_0 + \frac{dy}{d\theta}|_{(\theta=\theta_0)}(\theta - \theta_0)$$

2. Computing derivatives:

$$y - y_0 = \delta y = \frac{d[J\ddot{\theta} + B\dot{\theta} + mgl\sin\theta]}{d\theta}|_{(\theta=\theta_0)}(\theta - \theta_0)$$

$$\delta y = [J\frac{d^2}{dt^2} + B\frac{d}{dt} + mgl\cos\theta]|_{(\theta=\theta_0)}\delta\theta$$

3. Linearized equation:

$$\delta y = J\delta\ddot{\theta} + B\delta\dot{\theta} + mgl\cos\theta_0\delta\theta$$

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