AE 308: Control Theory
AE 775: System Modelling, Dynamics & Control

Lecture 19: P and PI Control Designs

Dr. Arnab Maity
Department of Aerospace Engineering
Indian Institute of Technology Bombay
Powai, Mumbai 400076, India

P Control - Introduction

Introduction

- Simplest possible control strategy
- P controller impacts closed loop behavior on two counts
  - Modify overall loop gain and hence steady state behavior
  - Change dominant pole location, and thereby both relative and transient response
- It is quite restrictive as it only provides one design degree of freedom. Thus it is not possible to achieve a wide range of performance in the closed loop

P Control - Root Locus

Root locus design steps

- Convert specifications into the desired dominant poles
- Draw root locus of $G(s)$, for K from 0 to ∞ and establish the existing pole location
- Superimpose closed loop performance parameters onto the root locus
- Use graphical technique to determine the total gain

P Control - Root Locus

Example 1

- Design a P controller using the root locus to achieve the following performance parameters in closed loop

  Ramp error constant ≥ 2.5
  Peak overshoot $\leq 20\%$
  Settling time ≤ 3.0 Seconds

$$G(S) = \frac{40}{s(s+4)(s+10)}$$

- Ramp error constant is given by

$$K_V = \lim_{s \to 0} sG(s)C_{controller}(s)$$

P Control - Root Locus

Example 1 - solution

- Convert the specifications into poles

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \leq 0.20 \rightarrow \zeta \geq 0.456$$

$$T_s \approx \frac{4}{\sigma} \leq 3.0 \rightarrow \sigma \geq 1.33$$

$$\omega_n = \frac{\sigma}{\zeta} = 2.92 \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 2.59$$

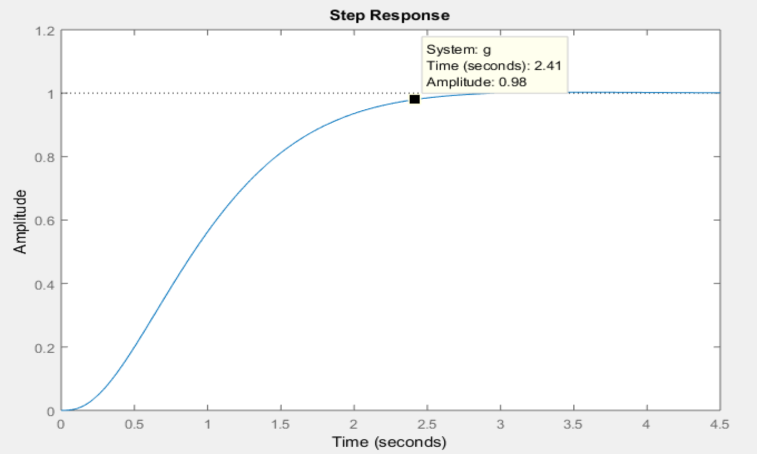
$$\text{pole better than: } -1.33 \pm j2.59$$

- There are no requirements on steady-state response, which indicates that we are required to maintain existing tracking performance, as far as possible

P Control - Root Locus

Example 1 - solution

- Step response of uncompensated system



P Control - Root Locus

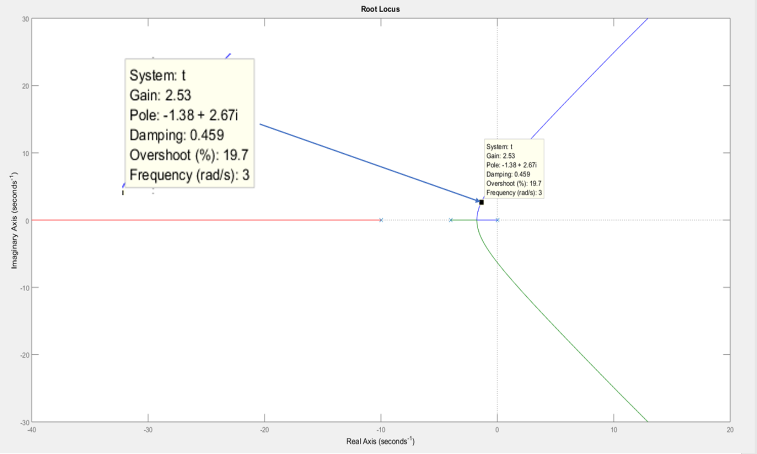
Example 1 - solution

- Basic system has no overshoot and a settling time of much less than 3 sec
- However, K_V is 1.0, as against the requirement of 2.5
- Therefore, a higher gain is feasible, if the desired time response is achieved

P Control - Root Locus

Example 1 - solution

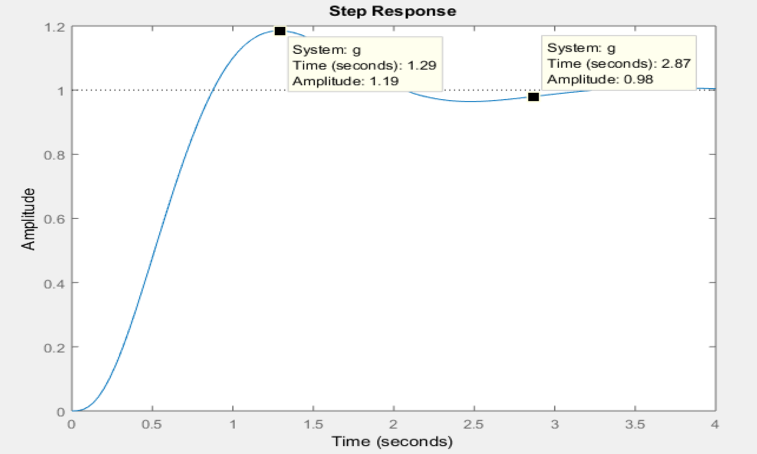
- Root locus of given system



P Control - Root Locus

Example 1 - solution

- Design verification - step response



P Control - Root Locus

Example 1 - Analysis

- We find that by satisfying a single specification of ramp error constant, it is possible to also meet specifications on transient response
- Further, while root locus predicts a 19.8% overshoot, step response shows only 19% overshoot, indicating that we could increase gain marginally

P Control - Root Locus

Example 2

- Consider the following plant

$$G(s) = \frac{1}{(s+1)^3}$$

- Design a P gain to achieve a peak time of around 3.4 s, and 2% settling time of around 7s
- In case the design is not feasible, give the best possible solution

P Control - Root Locus

Example 2 - Solution

- Convert specifications

$$T_s \approx \frac{4}{\sigma} \rightarrow \sigma \approx 0.571$$

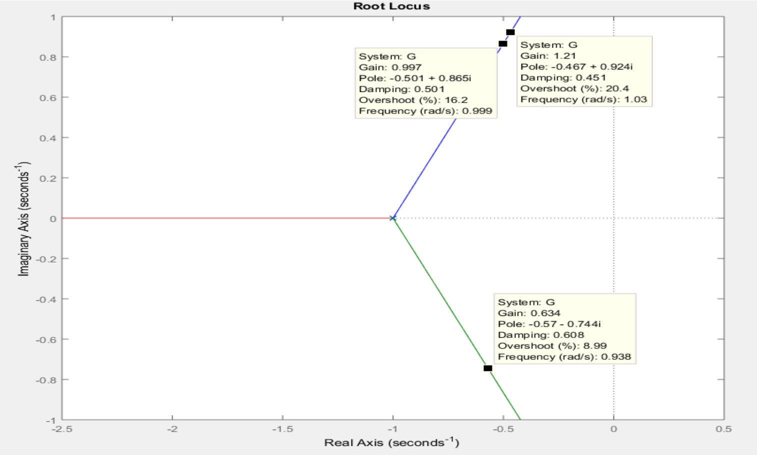
$$T_p = \frac{\pi}{\omega_d} \approx 3.4 \rightarrow \omega_d = 0.924$$

$$\text{Pole:} \quad -0.571 \pm j0.924$$

P Control - Root Locus

Example 2 - Solution

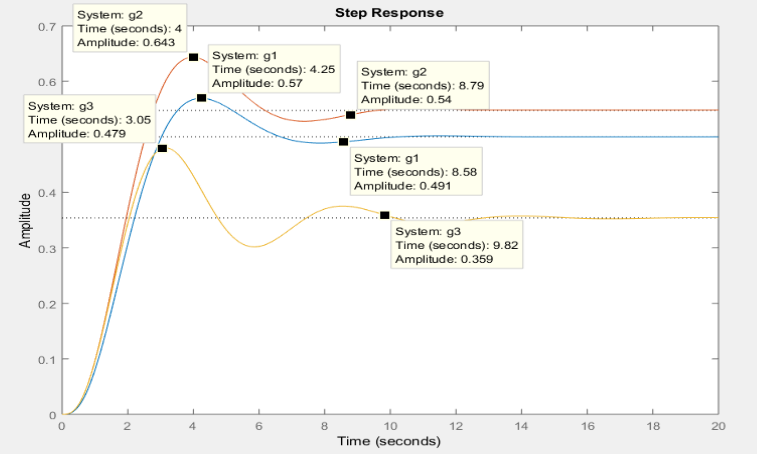
- Root locus design domain



P Control - Root Locus

Example 2 - Solution

- Design visualization - Step response



P Control - Root Locus

Example 2 - Analysis

- We find that P-control is unable to meet the requirements as stated
- In this context, a compromise is necessary, which can be arrived at, by bringing in additional information
- If settling time is treated as a soft requirement, we can use a higher gain to improve tracking
- Similarly, if settling time is a hard requirement, we can reduce gain to also improve the peak overshoot

P Control - Bode plot

Example 3

- Design a P-controller using bode plot to achieve the following performance for the compensated plant

  Phase margin $\geq 45.6^0$
  Bandwidth ≥ 3.00

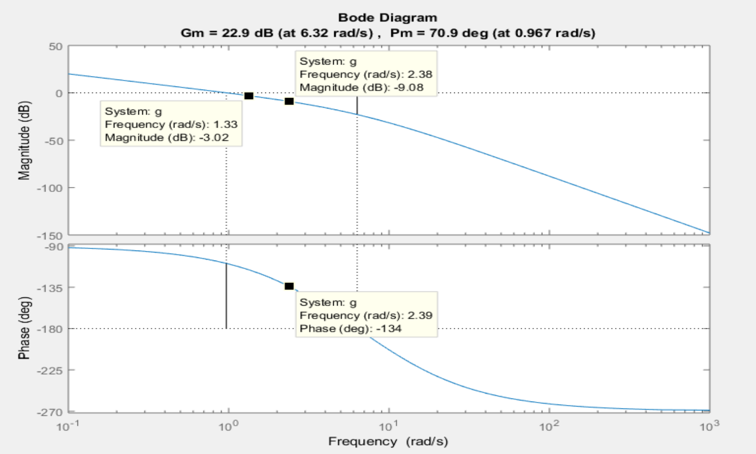
$$G(S) = \frac{40}{s(s+4)(s+10)}$$

- Also comment on change in gain margin

P Control - Bode plot

Example 3 - Solution

- Bode plot of uncompensated plant



P Control - Bode plot

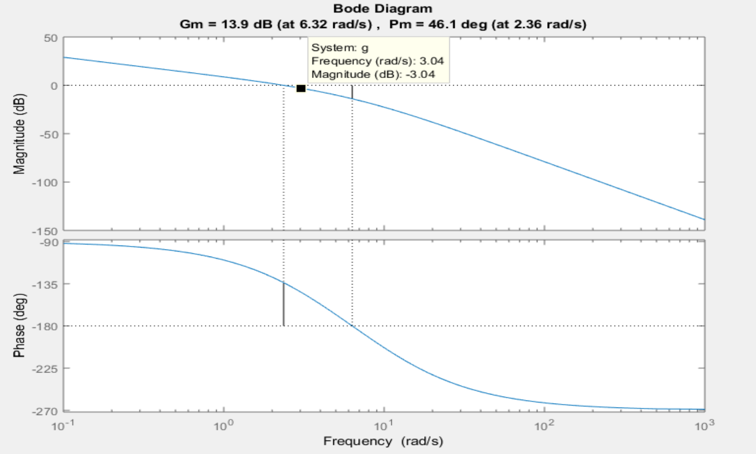
Example 3 - Solution

- PM of uncompensated plant is 70.9^0 and can be reduced by 25.3^0
- This is possible if new GCO is 2.36
- Thus, we can increase gain by 9 dB (2.82) to change GCO, which will also increase BW from current 1.47

P Control - Bode plot

Example 3 - Solution

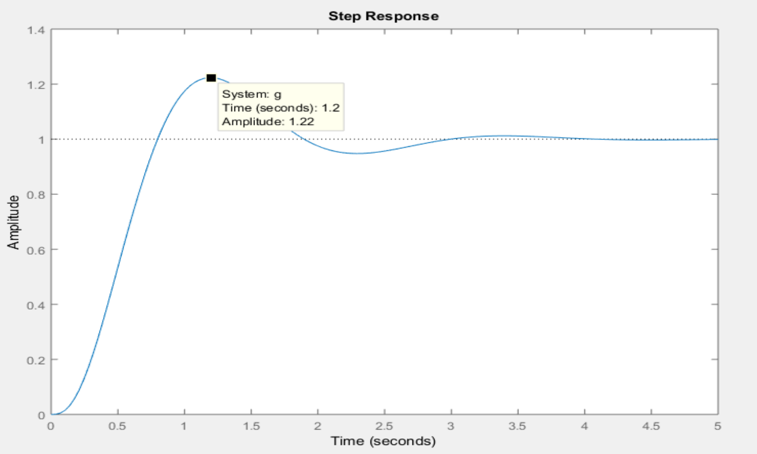
- Bode plot of compensated plant



P Control - Bode plot

Example 3 - Solution

- Step response of compensated plant



P Control - Bode plot

Example 3 - Analysis

- $\omega_b = 3.04$
- $K_V = 2.82$
- Design gives slightly higher M_p but gives much better ω_b, and K_V, along with better T_s

P Control - Bode plot

Example 4

- Consider the following plant

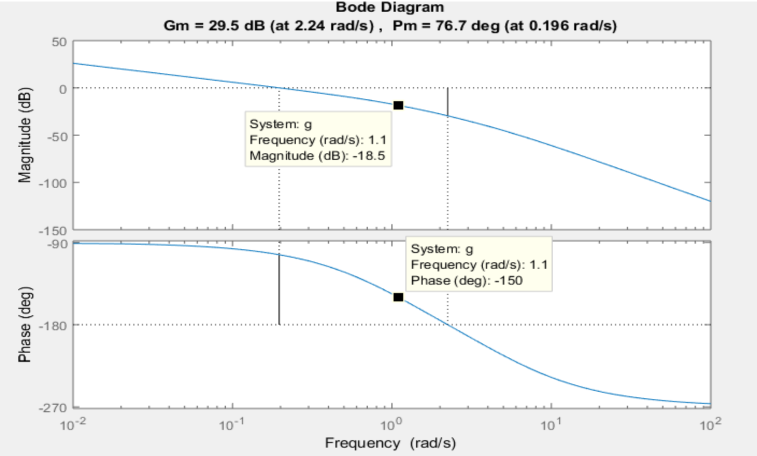
$$G(s) = \frac{1}{s(s+1)(s+5)}$$

- Determine maximum increase possible in ramp error constant to maintain a GM > 6 dB and PM $> 30^0$, and give the dominant pole and closed loop damping ratio

P Control - Bode plot

Example 4 - Solution

- Bode plot of uncompensated plant



P Control - Bode plot

Example 4 - Solution

- For maximum ramp error constant, gain should be maximum
- If we increase gain by more than 18.5 dB PM will be less than 30^0
- Thus, we can add gain by 18.5 dB
- Thus, open loop TF will become

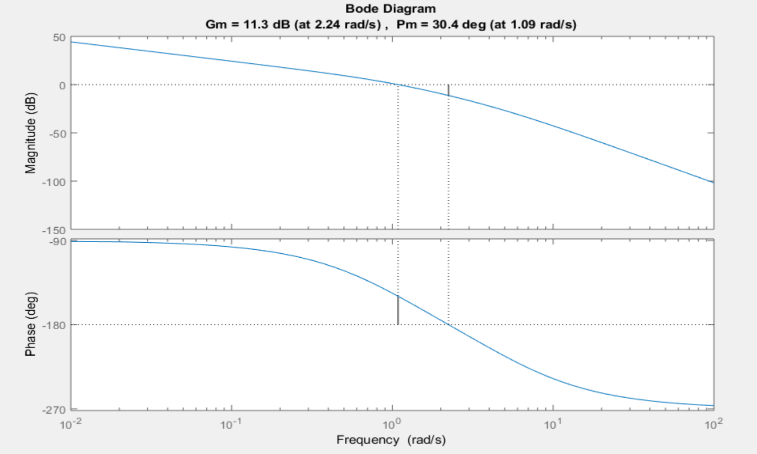
$$KG = \frac{8.2}{s(s+1)(s+5)}$$

- Maximum possible ramp error constant will be 1.64

P Control - Bode plot

Example 4 - Solution

- Bode plot of compensated plant



PI Control - Configuration

Introduction

- Given below is the basic form of the PI controller

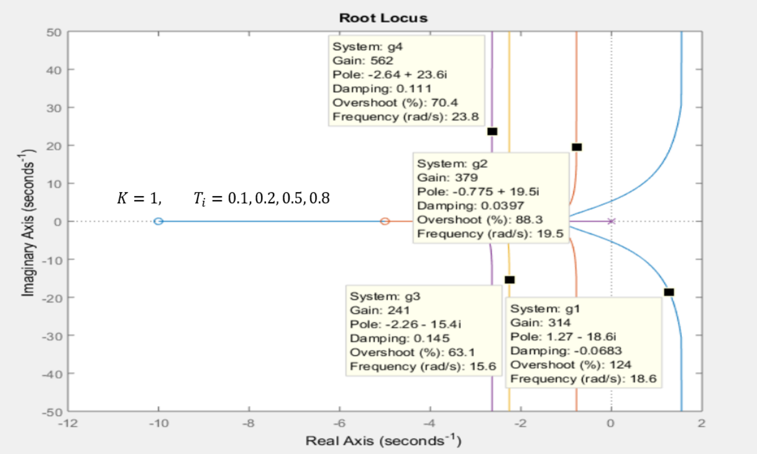
$$G_{PI} = K_p + \frac{K_I}{s} = K\left(1 + \frac{1}{T_i s}\right) = \frac{K(s + z_1)}{s}$$

- PI controller adds a pole at the origin and a zero at $s = -z_1(1/T_i)$, where T_i is the integrator time constant
- When $T_i \rightarrow 0$, PI controller becomes a pure integrator. If it is assumed that pure integral controller adds a zero at infinity, it is same as $T_i \rightarrow 0$
- As we can see, with PI control, system type increases by 1, so that tracking performance improves significantly.

PI Control - Introduction

Effect on root locus

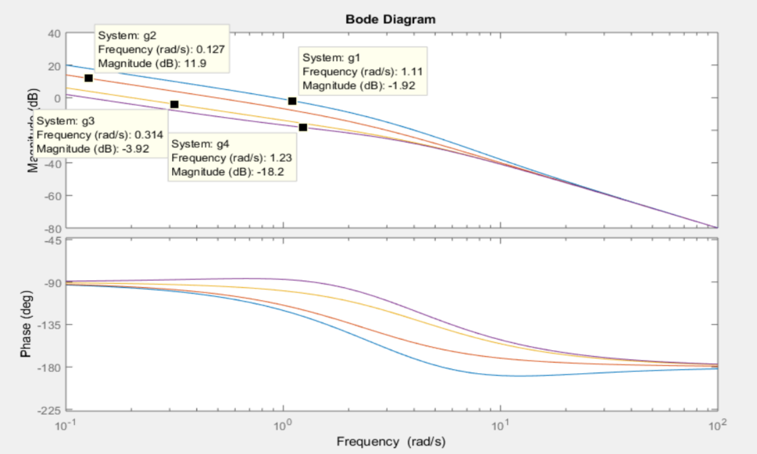
- Effect of integral gain on root locus



PI Control - Introduction

Effect on bode plot

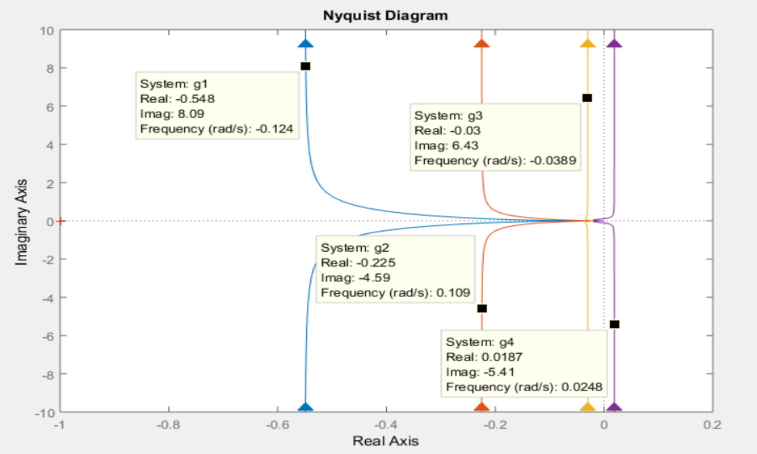
- Effect of integral gain on bode plot



PI Control - Introduction

Effect on nyquist plot

- Effect of integral gain on nyquist plot



PI Control - Introduction

Design problem description

- Problem of PI control design is generally posed in terms of achieving a specified K_V
- This is usually stated along with required stability margins and/or dominant system behaviour
- PI controller is used mainly for improving the tracking of step and ramp inputs

PI Control - Root Locus

Root locus design steps

- Root locus is quite convenient for the design of PI control as dominant poles are explicitly visualized
- This is done by generating uncompensated root locus and noting existing dominant poles and step error constant
- As system type automatically increases, we decide the location of zero on the guideline that the negative phase added at existing dominant poles is $\approx 3 - 5^0$
- Lastly, proportional gain is decided by the amount of increase desired in the K_V.

PI Control - Root Locus

Example 5

- Consider the following open loop transfer function

$$G(s) = \frac{8}{(s+1)(s+5)}$$

- Design a PI controller to achieve a K_v of 8.0, while maintaining the existing dominant closed loop poles and also determine changes, if any
- Existing dominant poles:

$$s^2 + 6s + 13 = 0 \rightarrow s_{1,2} = -3 \pm 2j$$

PI Control - Root Locus

Example 5 - Solution

- In actual practice, it is more convenient to fix K/Ti first, by imposing the KV requirement, as follows.

$$G_{PI} = \frac{K}{T_i s}(T_i s + 1)$$

$$\lim_{s \to 0}(sG_{PI}G(s)) = K_V$$

$$1.6\frac{K}{T_i} = 8 \rightarrow \frac{K}{T_i} = 5$$

$$G_{PI}(s) = \frac{5}{s}(T_i s + 1)$$

- Thus, we see that T_i now fixes the zero location, while leaving K_V unaffected

PI Control - Root Locus

Example 5 - Solution

- T_i is obtained from angle condition, as follows

$$\angle G_{PI} = \angle(T_i s + 1)|_{s=-3\pm 2j} - \angle s|_{s=-3\pm 2j} = -5^0$$

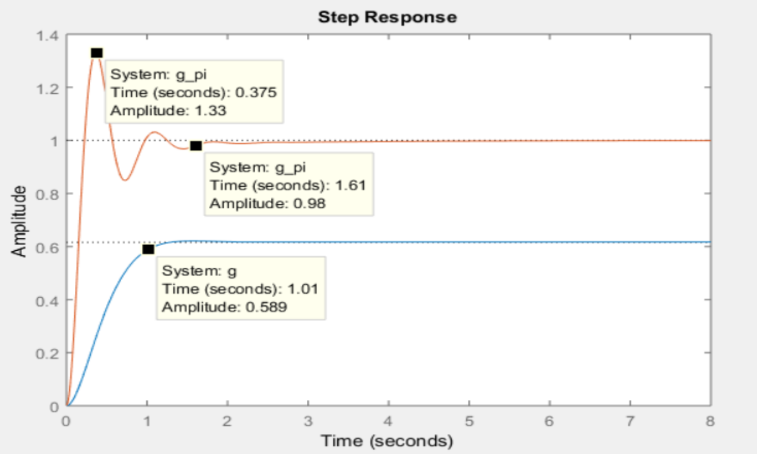
$$\tan^{-1}\left(\frac{2T_i}{1-3T_i}\right) = \tan^{-1}\left(\frac{2}{-3}\right) - 5^0 = 146.3^0$$

$$\frac{2T_i}{1-3T_i} = -8.01 \rightarrow T_i \approx 2 \rightarrow G_{PI}(s) = \frac{5}{s}(2s+1)$$

PI Control - Root Locus

Example 5 - Solution

- Design visualization - step response



PI Control - Root Locus

Example 5 - Analysis

- We find that, as expected, compensated system response is significantly affected, even though only -5^0 angle is added to pole
- We find that PI designed using root locus, while ensuring the K_V, significantly increases gain at higher frequencies, and, adversely affects transient response
- Therefore, PI should be employed when the plant has sufficient stability margins

PI Control - Bode Plot

Basics

- PI Controllers can also be designed in frequency domain using GM, PM as the specifications, along with ramp error constant requirement
- Similar to root locus, design of PI with bode aims to ensure that uncompensated margins are maintained
- This results in the need to increase low frequency gain, without affecting the high frequency behaviour
- In that sense, it aims to achieve a better control, but also becomes a bit more complex than the root locus method.

PI Control - Bode Plot

PI design methodology using Bode

- Design of PI in frequency domain is primarily governed by the requirements on low frequency gain, that is to be achieved for compensated system, and is driven by K_V
- In addition, PM is required to be nearly unchanged so that existing transient response is ensured
- However, as increase in K changes GCO, we first compensate for K, and later choose $(1/Ti)$ such that GM is available at ω_{PCO} (1 decade lower than GCO)

PI Control - Bode Plot

Example 6

- Consider the following open loop transfer function

$$G(s) = \frac{8}{(s+1)(s+5)}$$

- Design a PI controller to achieve a K_V of 8.0 and Phase margin of at least 45^0

Example 6 - Solution

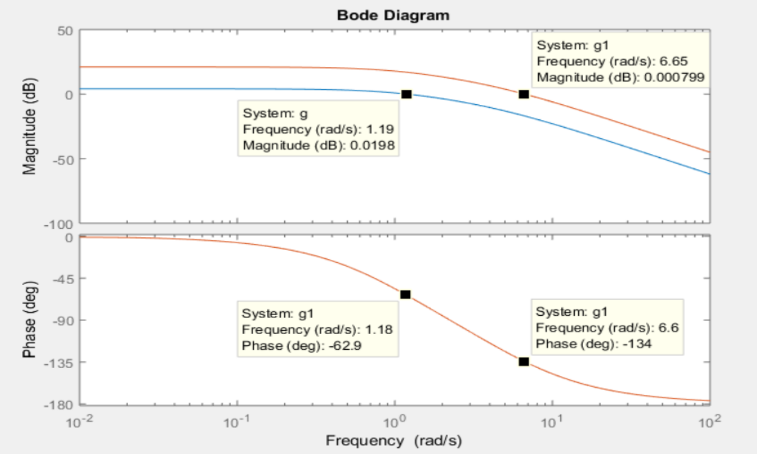
- Let us increase overall gain by a factor of 7 and assume the PI controller of the following form

$$G_{PI}(s) = \frac{7(s+z)}{s}$$

PI Control - Bode Plot

Example 6 - Solution

- Bode plot of assumed system



PI Control - Bode Plot

Example 6 - Solution

- Existing PM is 116.6^0 which reduces to 46^0, when K_p is 7 (≈ 17)dB
- Next, we choose a suitable value of corner frequency, which turns out to be 0.7 (1 decade lower than GCO, where $GCO \approx 6.65$)
- The resulting PI controller is as follows

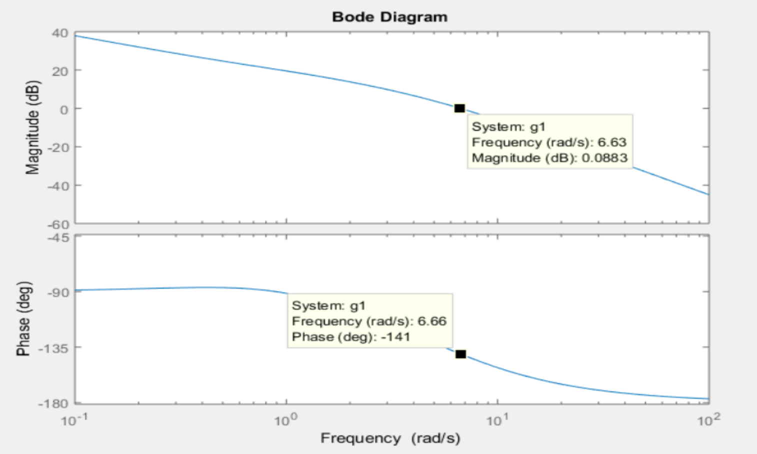
$$G_{PI}(s) = \frac{7(s+0.7)}{s}$$

- It should be noted that the above PI controller changes both the ω_{GCO} and PM, as shown next

PI Control - Bode Plot

Example 6 - Solution

- Bode plot of compensated system



PI Control - Bode Plot

Example 6 - Solution

- PM reduces to 39^0, while K_V is 7.84
- There is now a need to reduce the gain to recover the PM, which can be done in two ways
- Reduce K to 6 or reduce $1/Ti$ to 0.6, resulting in the following controller options

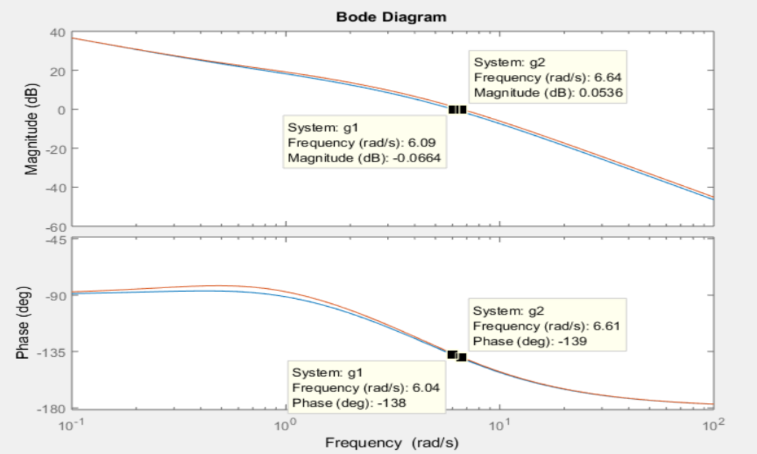
$$G_{PI}(s) = \frac{6(s+0.7)}{s} \quad G_{PI}(s) = \frac{7(s+0.6)}{s}$$

- In both these cases, the ramp error constant reduces to 6.72, so that PM can be maintained

PI Control - Bode Plot

Example 6 - Solution

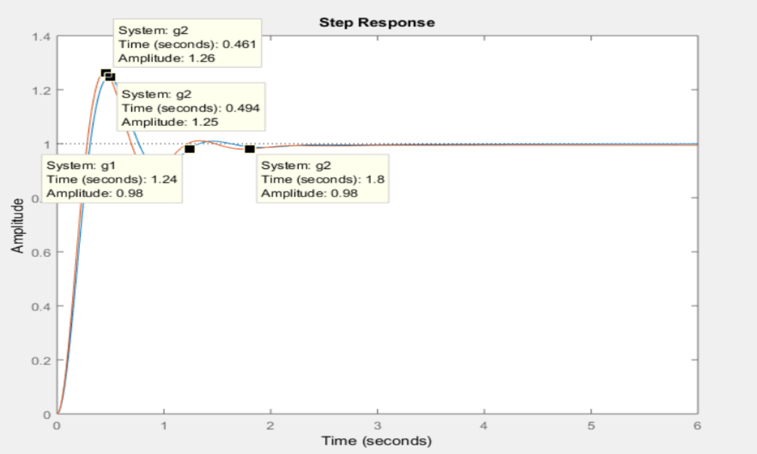
- Bode plot of compensated system



PI Control - Bode Plot

Example 6 - Solution

- Design visualization - step response



Lag Compensator - Introduction

Concept of Lag compensator

- PI controller increases system type, which may not be desired for systems that are already type 1 or higher
- Lag compensator is counterpart of PI, which improves the ramp error constant, without changing system type
- The basic structure of lag compensator is as follows

$$G_{Lag}(s) = K_c \frac{\beta(Ts+1)}{(\beta Ts+1)} = K_c \frac{\left(s+\frac{1}{T}\right)}{\left(s+\frac{1}{\beta T}\right)}$$

Lag Compensator - Introduction

Lag compensator structure

- Here, K_c is compensator gain, T is the compensator time constant and β is a parameter that decides the amount of improvement in ramp error constant
- Lag compensator adds a zero at $s = -1/T$ and a pole at $s = -1/(\beta T)$, to the plant, so that system type is preserved
- Further, as a bonus, we also get additional design degrees of freedom, to better achieve the specifications

Lag Compensator - Introduction

Lag compensator features

- Further, pole is closer to the origin than zero, as $\beta > 1$, so that lag compensator adds a net negative angle at the dominant poles
- We also see that in the limit when $\beta \to \infty$, pole lies at the origin, which results in PI controller
- Similarly, when $T \to 0$ and $\beta \to \infty$, pole lies at origin, while zero lies at infinity, resulting in a pure integral control
- Lastly, lag compensator has one more design variable, in comparison to PI, which is expected to help in better management of performance specifications
- It should be noted here that increase in error constant is $K_c\beta$, so that we can get same tracking performance, while ensuring different transient response

Lag Compensator - Analysis

Effect of β on step response

- Let us consider the following plant, augmented with the lag compensator

$$G = \frac{1}{(s+2.5)(s+4)}$$

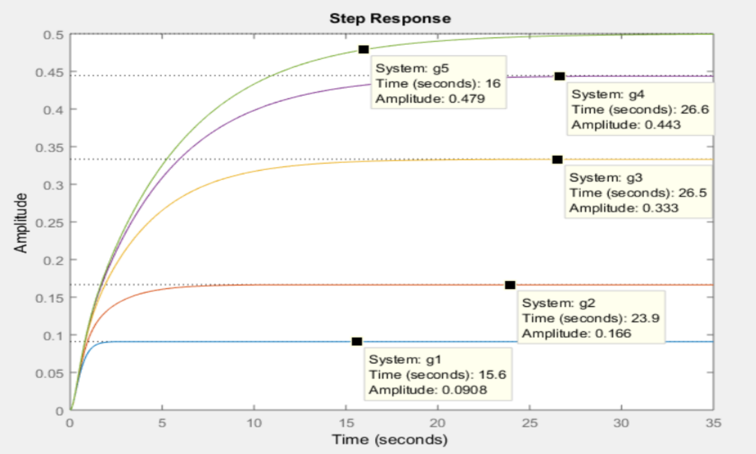
$$G_c = \frac{\beta(s+1)}{(\beta s+1)}$$

$$\beta = 1, 2, 5, 8, 10$$

Lag Compensator - Analysis

Effect of β on step response

- Effect of β on step response



Lag Compensator - Analysis

Effect of T on step response

- Let us consider the following plant, augmented with the lag compensator

$$G = \frac{1}{(s+2.5)(s+4)}$$

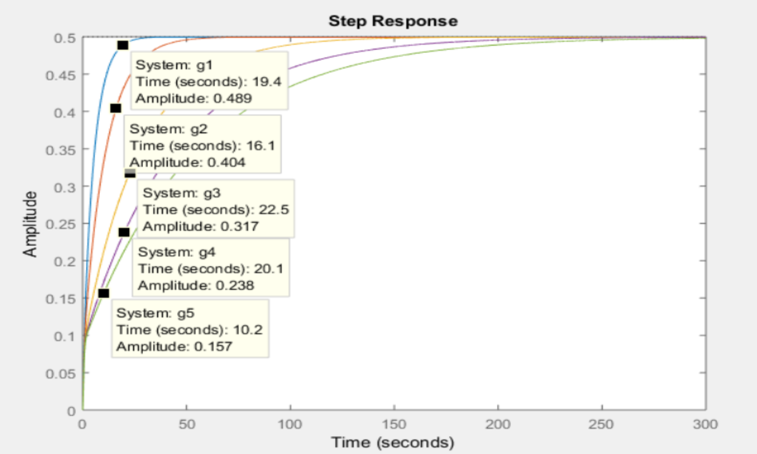
$$G_c = \frac{10(Ts+1)}{(10Ts+1)}$$

$$T = 1, 2, 5, 8, 10$$

Lag Compensator - Analysis

Effect of T on step response

- Effect of T on step response



Lag Compensator - Analysis

Generic bode plot of lag compensator

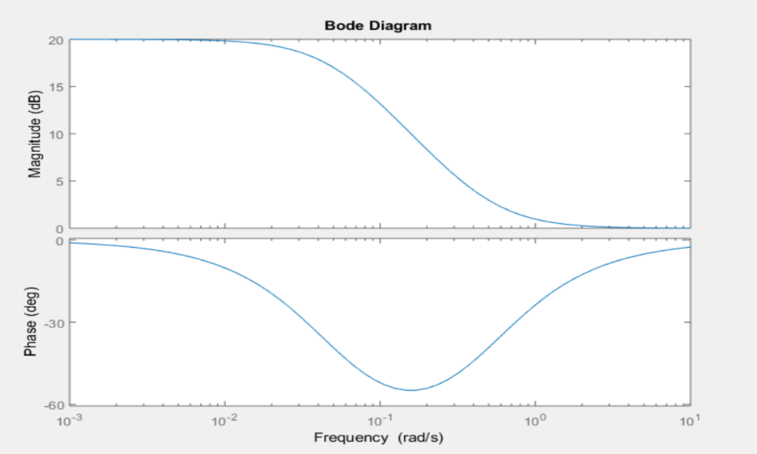
- Consider a lag compensator

$$G_{Lag}(s) = \frac{\left(s + \frac{1}{2}\right)}{\left(s + \frac{1}{20}\right)}$$

Lag Compensator - Analysis

Generic bode plot of lag compensator

- Generic bode plot of lag compensator



References I

- Gene F. Franklin, J. David Powell, and Abbas Emami-Naeini: "*Feedback Control of Dynamic Systems*", Pearson Education, Inc., Upper Saddle River, New Jersey, Seventh Edition, 2015.
- Katsuhiko Ogata: "*Modern Control Engineering*", Pearson Education, Inc., Upper Saddle River, New Jersey, Fifth Edition, 2010.
- Farid Golnaraghi and Benjamin C. Kuo: "*Automatic Control Systems*", John Wiley & Sons, Inc., New Jersey, Ninth Edition, 2010.
- Norman S. Nise: "*Control Systems Engineering*", John Wiley & Sons, Inc., New Jersey, Sixth Edition, 2011.
- Ashok Joshi: "*System Modeling Dynamics and Control*", Lecture Notes, IIT Bombay, Mumbai, 2019.