by The probability that at least m of the events $(E_i : 1 \le i \le n)$ will occur is

$$S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} + \cdots + (-1)^{n-m} \binom{n-1}{m-1} S_n.$$

$\boxed{\text{Exercise-3}}$ Suppose that we have n distinguishable letters and n distinguishable envelopes. A postman takes each letter and put it into a randomly selected envelope (each envelope can contain at the most one letter and all the empty envelopes are equiprobable). Compute the probability that none of the letters matches the envelopes where ~~a match~~ the i^{th} match ~~is defined to be~~ occurs if i^{th} letter is put in the i^{th} envelop.

$\boxed{\text{Exercise-4}}$ Let Ω be a finite sample space. Show that there exists infinitely many ~~Uncoun~~ (uncountably) probabilites on Ω unless Ω is singletone.

Conditional probability

Consider a sample space Ω and a probability $\mathbb{P}$ on it. $(|\Omega| < \infty)$.

Consider an event $E \in 2^{\Omega}$ such that $\mathbb{P}(E) > 0$.

(there is always an event which has positive probability!)

For any event $A \in 2^{\Omega}$, we define the probability of the event A given E as,

$$\mathbb{P}(A \mid E) = \frac{\mathbb{P}(A \cap E)}{\mathbb{P}(E)}.$$

We call it conditional probability. ~~Show that~~

* $\mathbb{P}(A \mid E) \in [0,1]$ for any $A \in 2^{\Omega}$.
* $\mathbb{P}(\phi \mid E) = 0$ and $\mathbb{P}(\Omega \mid E) = 1$.
* Show that

$$\mathbb{P}(E_1 \cup E_2 \mid E) = \mathbb{P}(E_1 \mid E) + \mathbb{P}(E_2 \mid E) - \mathbb{P}(E_1 \cap E_2 \mid E)$$

(Exercise)

Conclude that if $(E_i : 1 \le i \le n)$ are disjoint events $(E_i \in 2^{\Omega})$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i).$$

Exercise - 6

a) Let $\mathbb{P}(B\cap C) > 0$. Then show that

$$\mathbb{P}(A\cap B \mid C) = \mathbb{P}(B\mid C)\,\mathbb{P}(A\mid B\cap C).$$

b) Let $(E_i : 1\le i\le n)$ be a ~~seq~~ finite collection of disjoint events such that $\bigcup_{i=1}^{n} E_i = E$. Then show that

$$\mathbb{P}(A\cap E) = \sum_{i=1}^{n} \mathbb{P}(E_i)\,\mathbb{P}(A\mid E_i)$$

(The problem is ill-defined find out why).

c) $\boxed{\text{Theorem of total probability}}$

Let $(E_i : 1\le i\le n)$ be a finite collection of disjoint events such that $\bigcup_{i=1}^{n} E_i = \Omega$. Then

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(E_i)\,\mathbb{P}(A\mid E_i) \quad \text{for every } A\in 2^{\Omega}$$

~~and~~ provided $\mathbb{P}(E_i) > 0$ for $i\ge 1$.

d) $\boxed{\text{Bayes' theorem}}$

Let $(E_i : 1\le i\le n)$ be a sequence of exhaustive ($(E_i : 1\le i\le n)$ is a ~~disjoint~~ collection of events such that $\bigcup_{i=1}^{n} E_i = \Omega$) and mutually exclusive (disjoint) events such that $\mathbb{P}(E_i) > 0$ for every $i\ge 1$. Then for any i_0

$$\mathbb{P}(E_i \mid A) = \frac{\mathbb{P}(E_i)\,\mathbb{P}(A \mid E_i)}{\sum_{i=1}^{n} \mathbb{P}(E_i)\,\mathbb{P}(A \mid E_i)}.$$

provided $\mathbb{P}(A) > 0$.

e) (Formula important for computing probabilities)

Let $(E_i : 1 \le i \le n)$ be a finite collection of exhaustive and mutually exclusive events on Ω with probability $\mathbb{P}$. Consider an event A such that

$$\mathbb{P}(A \cap E_i) > 0 \quad \text{for every } i \geq 1.$$

Then for any event $C \in 2^{\Omega}$, we have

$$\mathbb{P}(C \mid A) = \frac{\sum_{i=1}^{n} \mathbb{P}(E_i)\,\mathbb{P}(A \mid E_i)\,\mathbb{P}(C \mid A \cap E_i)}{\sum_{i=1}^{n} \mathbb{P}(E_i)\,\mathbb{P}(A \mid E_i)}.$$

Exercise 7 There are two drawers in each of three boxes that are identical in appearance. The first one contains a gold coin in each drawer, the second contains a silver coin in ~~the other~~ each drawer and the third box contains a gold coin in one drawer and a silver coin in the other. A box is chosen, ~~and~~ one of its drawer is opened and a gold coin is found

What is the probability that the other drawer too will have a gold coin?

Exercise 8 In answering a question on a multiple choice test, an examinee either knows the answer (with probability p) or he guesses (with probability $1-p$). Let the probability of answering the question correctly be 1 for an examinee who knows the answer and $1/m$ (m being the number of multiple choice alternatives). Supposing an examinee answers a question correctly, what is the probability that he really ~~known~~ knows the answer?

Exercise 9 r distinguishable objects are distributed among n cells (distinguishable). Each cell can receive any number of objects. What is the probability that exactly m of the cells remain empty?

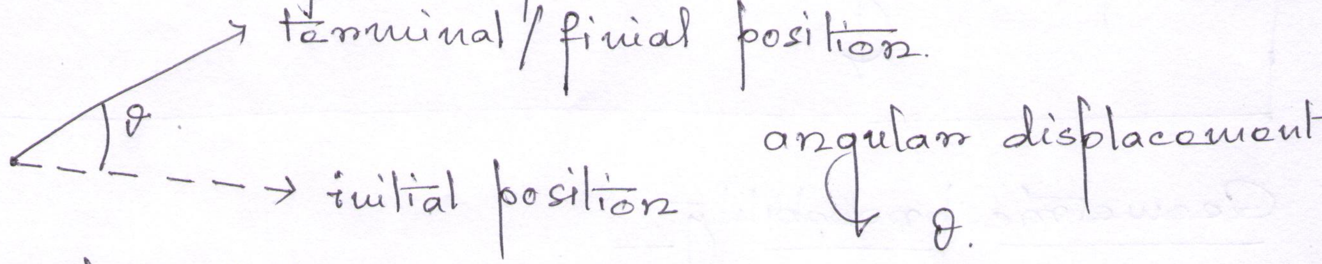
Lecture – II | Foundations of theoretical probability

> Question: Can we generalize the concept of probability when $|\Omega| = \infty$?

Geometric probability

$\boxed{\text{Example – iii}}$ Suppose that we have a light bulb. We are interested in the length of life of the light ~~bulb~~ bulb. The length of life of light bulb can be any positive real number. So $\Omega = (0, \infty)$. If we buy 10 bulbs, then all the bulbs not necessarily have the same length of life (we measure length of life in hours). A very useful question might be to know the probability that ~~long~~ life of a light bulb can take a value in the interval (a, b).

$\boxed{\text{Example – iv}}$ Consider a pointer that is free to spin about the center of a circle. If the pointer is spun by an impulse, it will finally to come to rest at some point. We call it final direction. and the starting direction is called initial direction. We further assume that the mechanism is not rigged in any direction. We are interested in angular displacement of the pointer.



The angular displacement is measured in radian.

$\Omega = [0, 2\pi]$.

We are interested in probability that the angular displacement is at the most $\pi/3$.

$\boxed{\text{New definition of probability}}$

~~probability~~ $\mathbb{P}(A) = \dfrac{\text{length of } A}{\text{length of } \Omega}$

assuming Ω to be an interval (of finite length) and an event to be a sub-interval of Ω.

* Can we generalize this to ~~with~~ higher dimension?

If Ω is a bounded region in $\mathbb{R}^2$ (~~two diment~~ two dimensional Euclidean space, then we define

$$\mathbb{P}(A) = \frac{\text{area of region } A}{\text{area of region } \Omega}.$$

If Ω is a bounded region in the higher dimensional Euclidean spaces

$$\mathbb{P}(A) = \frac{\text{volume of the region } A}{\text{volume of the region } \Omega}.$$

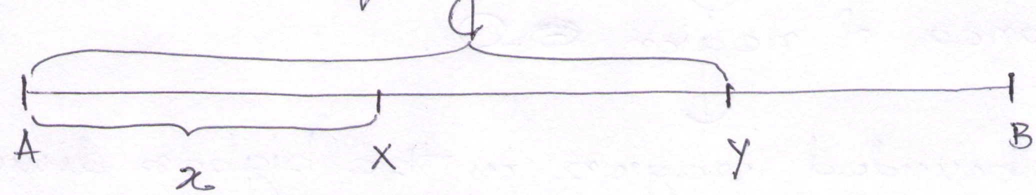
Idea: Probability is interpreted as the proportion of ~~to~~ our favourable ~~area to the~~ space to the sample space.

Example – iv Two points are chosen independently and at random on a rectalinear segment of length l. What is the probability that the three segments form a traingle?

(Here independence means that choice of one point does not influence the choice of the other)

Let AB be the original line segment. X and Y denote the two randomly chosen points on the segment AB. Let the lengths of the segments AX and AY be denoted by x and y.

Case - I : $x \le y$

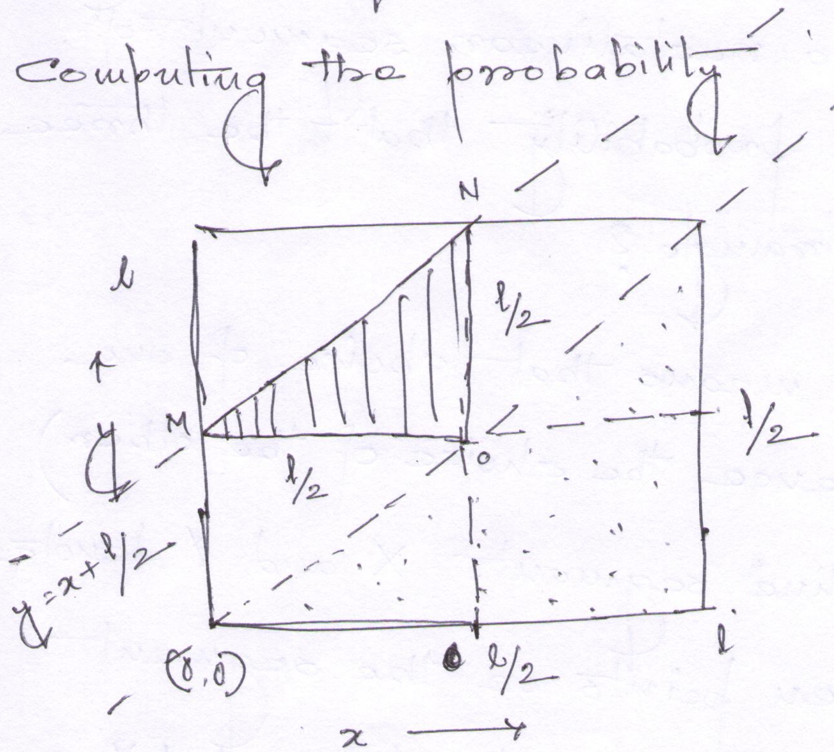


We want to compute the probability that segments AX, XY and YB form a triangle.

Then we must have

$$\left.\begin{array}{ll} x + (y-x) \geq l-y & \text{i.e. } y \geq \frac{l}{2}, \\ (y-x) + (l-y) \geq x & \text{i.e. } x \leq \frac{l}{2} \\ \text{and } (l-y) + x \geq y-x & \text{i.e. } y-x \leq \frac{l}{2} \end{array}\right\} (*)_1$$

Computing the probability



RATIONALE: The points X and Y are chosen independently

$0 \le x \le l$

$0 \le y \le l.$

So we have square on the (x,y) plane with length of side l. The area of the square l^2.
We are interested in the area of the region

which ~~favour~~ favors our event.

If we use the inequalities $(*_1)$ on the plane, then we obtain a right-angled triangle MNO favoring our event. The area of the right angled triangle is

$$\frac{1}{2} \times \frac{l}{2} \times \frac{l}{2} = \frac{l^2}{8}.$$

~~So the~~

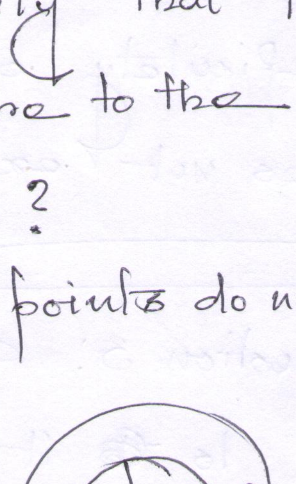
<u>Case-II</u> : $y < x$

Left as an exercise. The area of the favorable region is $l^2/8$. So combining these computations we get

$$P(AX,\ XY \text{ and } YB \text{ form a triangle}) = \frac{l^2/8 + l^2/8}{l^2} = \frac{1}{4}.$$

Example-iii) Let n points be taken at random and <u>independently</u> of one another inside a sphere of radius R. What is the probability that the distance from the center of sphere to the nearest point is not less than r ?

(Independence means that choice of points do not influence each other.)



Fix a point. Then, the probability that the fixed point ~~is covered~~ lies inside or on the sphere of radius r is