Lecture - I | Review of probability

Notations:

$\mathbb{P}$ — probability.

Ω — sample space

ω — sample points

Assumptions:

i) Ω is non-empty.

ii) $\mathbb{P}$ is a probability on Ω

To understand the nature, we sometimes conduct an experiment and the result might be non-unique.

i) Tossing a coin or throwing a die

ii) Choosing one or few cards from a deck.

iii) The length of a life of a light bulb.

iv) Consider a pointer that is free to spin about the center of a circle. If the pointer is spun by some impulse, it will finally come to rest at some point. Assume that the mechanism is not rigged in any manner.

v) A rod of length l is thrown onto a flat table which is ruled with parallel lines at the distance $2l$. The experiment is conducted to know whether or not the rod intersects one of the ruled lines.

* The first two experiments ~~are~~ have finite sample space.

* The last three experiments do not have finite sample space.

For each of these experiments, we know all the outcomes in advance. But on any performance of the experiment, however, we do not know what specific outcome will occur. This type of experiments are often referred to as the random or statistical experiments.

Defn A random or statistical experiment is an experiment in which

a) all outcomes of the experiment are known in advance.

b) any performance of the experiments results in an outcome that is not known ~~a~~ in advance.

c) The experiment can be repeated under identical conditions.

Defn. The collection / set of all possible outcomes of a random experiment is called ~~a~~ sample space. Sample space is denoted by Ω.

Example - 1 The outcome of a coin tossing experiment is either HEAD or TAIL.

Def^n The elements of the sample space Ω are called <u>sample points</u>. The sample points are denoted by ω.

In example - 1, $\Omega = \{H, T\}$ where H stands for HEAD and T stands for TAIL.

Exercise Write down the sample space for throwing a die or two dice.

Roughly speaking, an event is a collection of sample points that is, a subset of sample space. The definition of event is rather complex and technical and so, posponed for later.

According to our informal definition of event, A be an event if $A \subset \Omega$. Each sample point $\omega \in A$ is called a <u>sample point favourable</u> to the event A.

Def^n. Two events A and B are called mutually exclusive or disjoint if $A \cap B = \phi$ where ϕ is an empty set. If we have a finite collection of sets $(A_i : 1 \le i \le n)$, we call the sets in the collection to be mutually exclusive if $A_i \cap A_j = \phi$ for every pair (i, j) such that $i \neq j$ and $1 \le i, j \le n$.

Our favourite combinatorial definition of probability

$$\mathbb{P}(A) = \frac{\#\text{ sample points favourable to } A}{\#\text{ ~~samp~~ sample points in } \Omega}$$

$$= \frac{|A|}{|\Omega|} \text{ for every event } A \subset \Omega.$$

($|A|$ ~~for~~ denotes the cardinality of the subset A whenever it makes sense)

Assumptions.

* ~~The Ω~~ There are only finitely many points in Ω.

This formula of probability does not work when $|\Omega| = \infty$. Consider the spinning of pointer experiment. Let A be the event that the pointer ~~stops~~ ~~[illegible]~~ makes ~~an~~ angle at most $\pi/3$ relative to the starting position when it stops.

Question: Is it possible to define probability when the sample space is infinite?
Can we generalize the concept of probability to the case of infinitely many (countable or uncountable) sample points?

* The answer is 'YES' if we define "event" appropriately.

Before we dive into that I want to mention some combinatorial techniques to compute the probability when sample space is finite.

Ball-box arrangement

The combinatorial techniques can be explained using arrangements of balls and box.

Suppose that we have to toss r balls into n boxes. We further assume that the boxes are numbered say $a_1, a_2, \ldots, a_n$. Each toss of a ball corresponds to selection of a box.

Consider an arrangement a_2, a_1, a_1, a_3. This arrangement corresponds to the first ball in the second box, the second ball in the first box, third ball in the first box and fourth ball into third box.

Exclusions / tossing scheme

* drawing sample without replacement

each box is restricted to have at the most one ball.

* drawing sample (arrangement) with replacement

each box can contain all the balls.

* without replacement.

a box can not appear more than once in the arrangement.

Ordered sample.

The balls are numbered.

Unordered sample

The balls are not numbered and identical.

Physics interpretation.

The boxes denote the different energy labels and the balls correspond to the particles.

Ordered sample of size r with replacement.

Maxwell – Boltzman.

Total number of arrangements $= n^r$.

Ordered sample of size r without replacement

$\boxed{r \leq n}$ total number of arrangements possible $= n(n-1)(n-2) \cdots \cdots (n-r+1) = \dfrac{n!}{(n-r)!} = (n)_r$

~~Maxwell – Boltzmann~~

Unordered sample of size r without replacement

$\boxed{r \leq n}$ total number of arrangements possible

$$= \frac{n!}{(n-r)!\, r!} = \frac{1}{r!}(n)_r = \binom{n}{r}$$

$\boxed{\text{Fermi – Dirac}}$

Unordered sample of size r with replacement

$\boxed{\text{Bose – Einstein}}$.

The situation is better understood in the following way. Say we have $r = 3$ and $n = 2$.

$*$ — ball
$|$ — box

$* * *$ $\lfloor\ \ |\ \ \rfloor$

To seperate 3 balls into box we used a separator and we call it a bar.

$**|*$, $***|$, $*|**$, $|***$ (4 arrangements)

all possible arrangements of 3 $*$s and 1 bar.

$$\binom{3+1}{3} = \binom{4}{3} = 4.$$

In general, we need $(n-1)$ bars (separators) and r $*$'s or balls. Total number of arrangements possible for r $*$s and $(n-1)$ bars is

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}$$

* Some exercises are provided to practice these concepts and formulae.

Some useful formulae and facts

ASSUME $|\Omega| < \infty$. So we can use the definition of probability given earlier.

* Any subset $E \subset \Omega$ is an event and

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

* The collection of all events is 2^{Ω} (the power set of Ω containing all subsets of Ω). This means
  * ~~E is an event~~ ϕ and Ω are events.
  * E_1 and E_2 are events. Then $E_1 \cup E_2$ and $E_1 \cap E_2$ are events.
  * If E is an event, then E^c is also an event.

* i) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$.

ii) $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$

iii) $\mathbb{P}(E^c) = \mathbb{P}(\Omega \setminus E) = 1 - \mathbb{P}(E)$

where $E, E_1, E_2 \in 2^{\Omega}$.

* $\mathbb{P}(E) \in [0,1]$ for any event E.

$\boxed{\text{Exercise} - 1.}$ a) Suppose that $(E_i : 1 \le i \le n)$ is a finite collection events such that $E_i \subseteq \Omega$ for every $i \geq 1$. Let $\mathbb{P}$ be a probability on the sample space Ω. Then show that

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbb{P}(E_i) - \sum_{i<j}\sum \mathbb{P}(E_i \cap E_j)$$

$$+ \sum_{i}\sum_{j: j>i}\sum_{k: k>j} \mathbb{P}(E_i \cap E_j \cap E_k) - \cdots$$

$$+ (-1)^n \mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right).$$

b) Show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} \mathbb{P}(E_i).$$

c) Show that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} \mathbb{P}(E_i) - \sum_{i}\sum_{j: j>i} \mathbb{P}(E_i \cap E_j).$$

d) Show that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = 1 - \sum_{i=1}^{n} \mathbb{P}(E_i^c) + \sum_{i}\sum_{j:j>i} \mathbb{P}(E_i^c \cap E_j^c) - \cdots + (-1)^n \mathbb{P}\left(\bigcap_{i=1}^{n} E_i^c\right).$$

$\boxed{\text{Exercise-2}}$ Consider a sample space Ω such that $|\Omega| < \infty$. Let $\mathbb{P}$ be a probability on Ω. Consider a collection $(E_i : 1 \le i \le n)$ such that $E_i \in 2^{\Omega}$. Define S_k to be the sum of ~~all events~~ probabilities of intersections of n events taken k at a time. That is,

$$S_k = \sum_{i_1=1}^{n} \sum_{1\le i_2 \le n:\, i_2 > i_1} \sum_{1 \le i_3 \le n:\, i_3 > i_2} \cdots \sum_{1 \le i_k \le n:\, i_k > i_{k-1}} \mathbb{P}\left(\bigcap_{j=1}^{k} E_{i_j}\right).$$

a) The probability that exactly m events among the collection $(E_i : 1 \le i \le n)$ will occur is

$$\mathbb{P}(\text{exactly } m \text{ events among } (E_i : 1 \le i \le n) \text{ will occur})$$

$$= S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} + \cdots + (-1)^{n-m}\binom{n}{m} S_n.$$