AE 326

Vibrations and Structural Dynamics

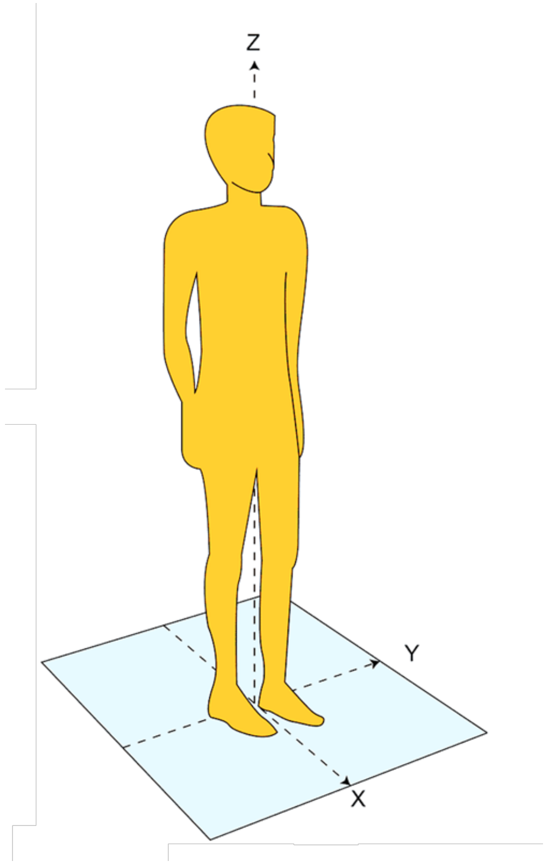
Department of Aerospace Engineering

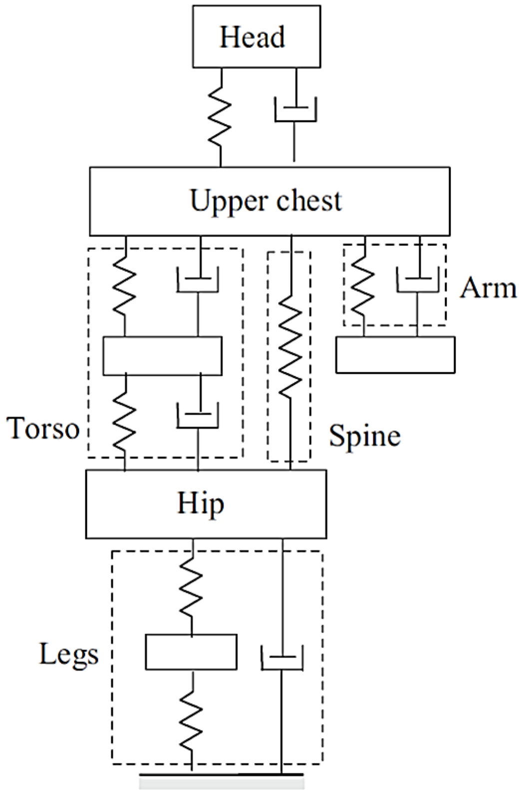
Tutorials:

Abstraction, Usage of Hamilton's principle and Lagrange Equation of Motion

Anshuman Mehta
(anshumanmehta@iitb.ac.in)
PMRF Scholar

Abstraction





Tutorial 1: Cheetah Abstraction

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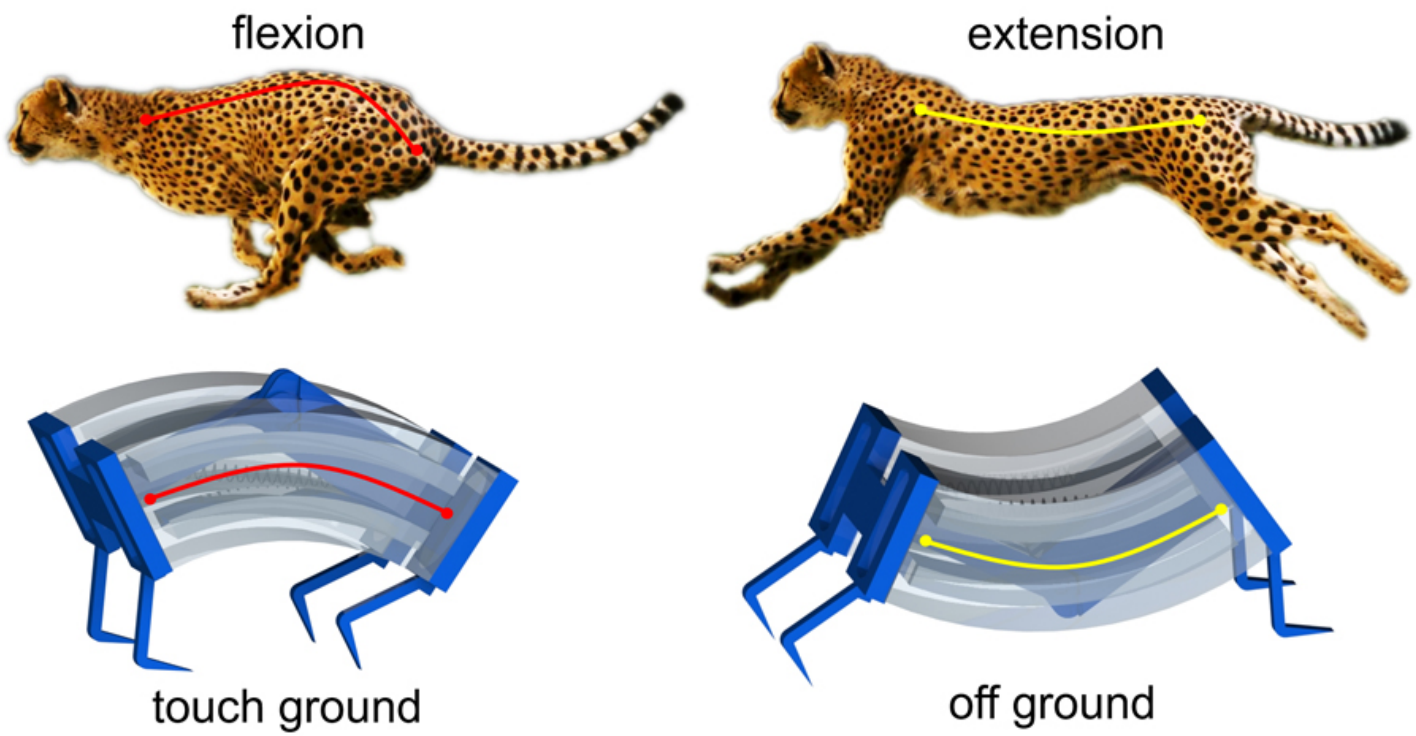
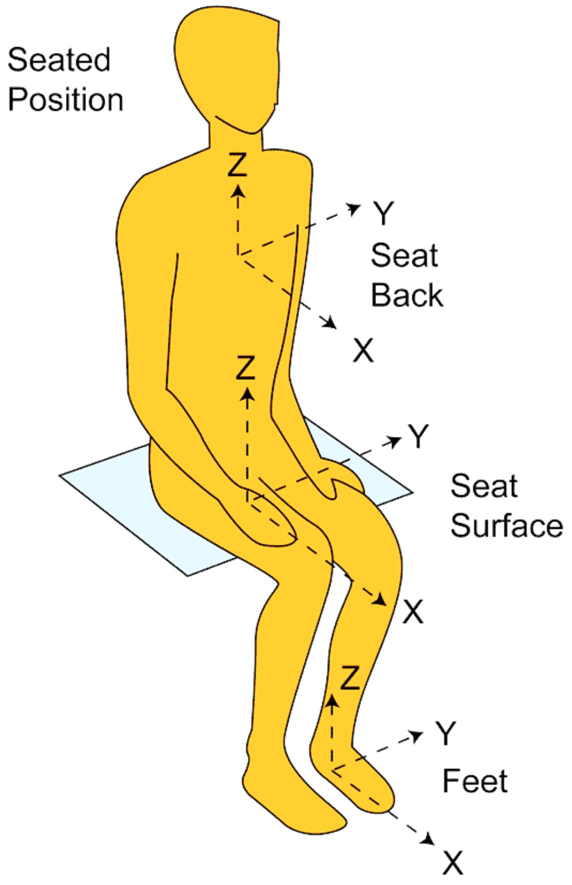
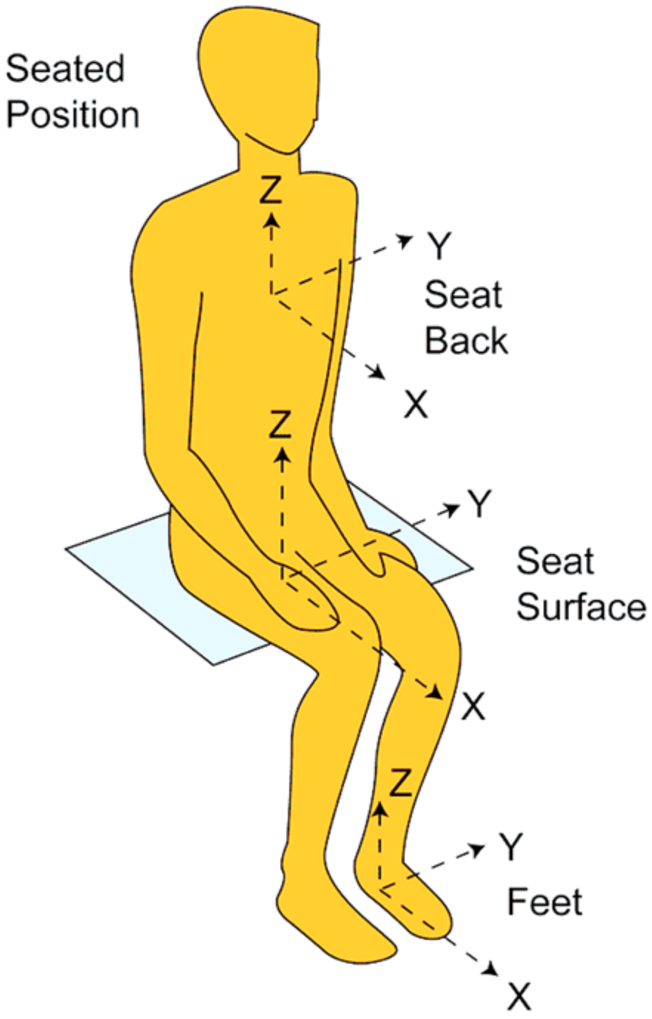


Tutorial 2: Human Body Sitting

A study of the response of human body subjected to vibration/ shock is important in many applications. In the sitting posture, define a model of an occupant in a rear car seat. Develop an abstraction for the human body in this case.



Tutorial 2: Human Body Sitting



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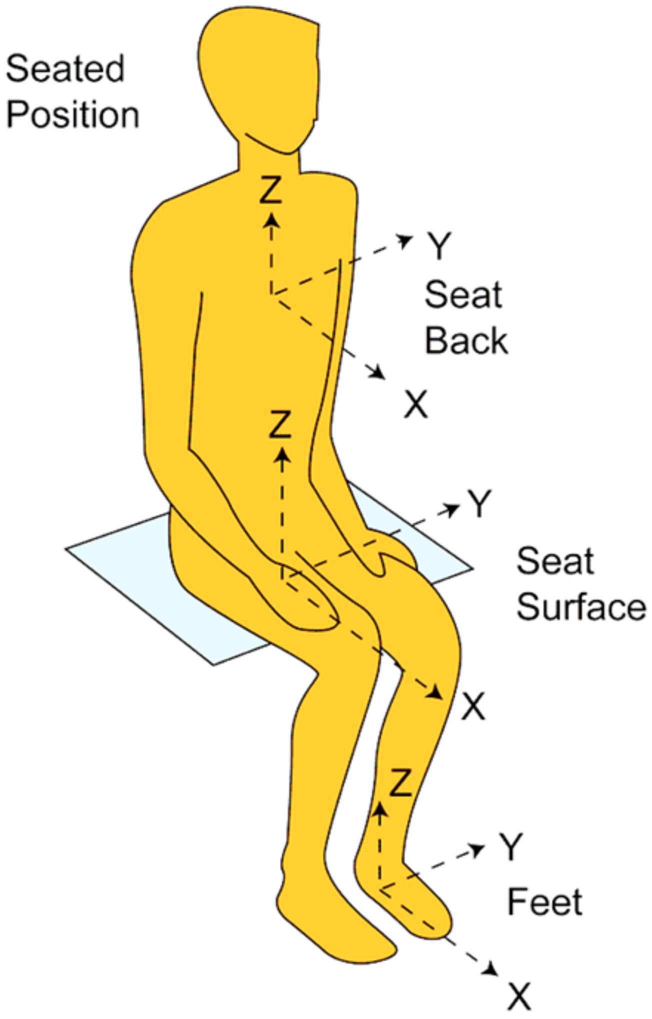
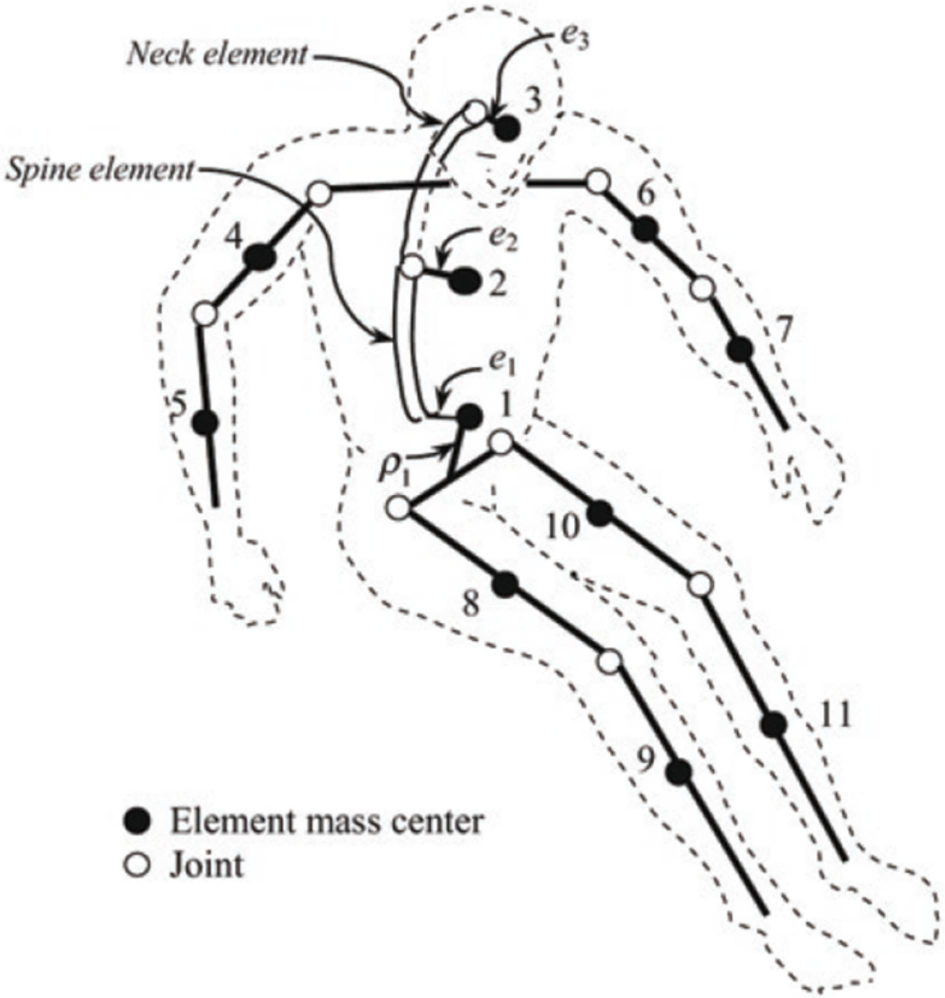
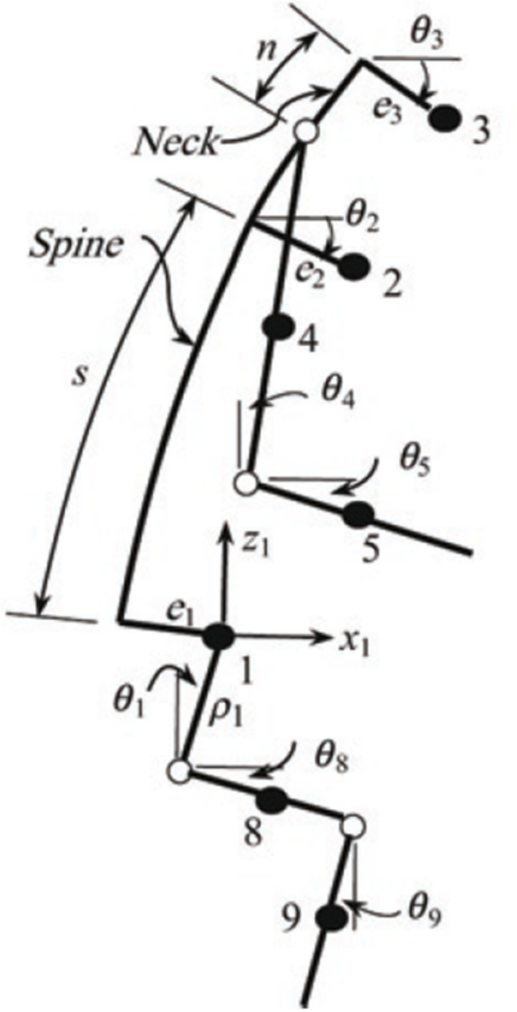
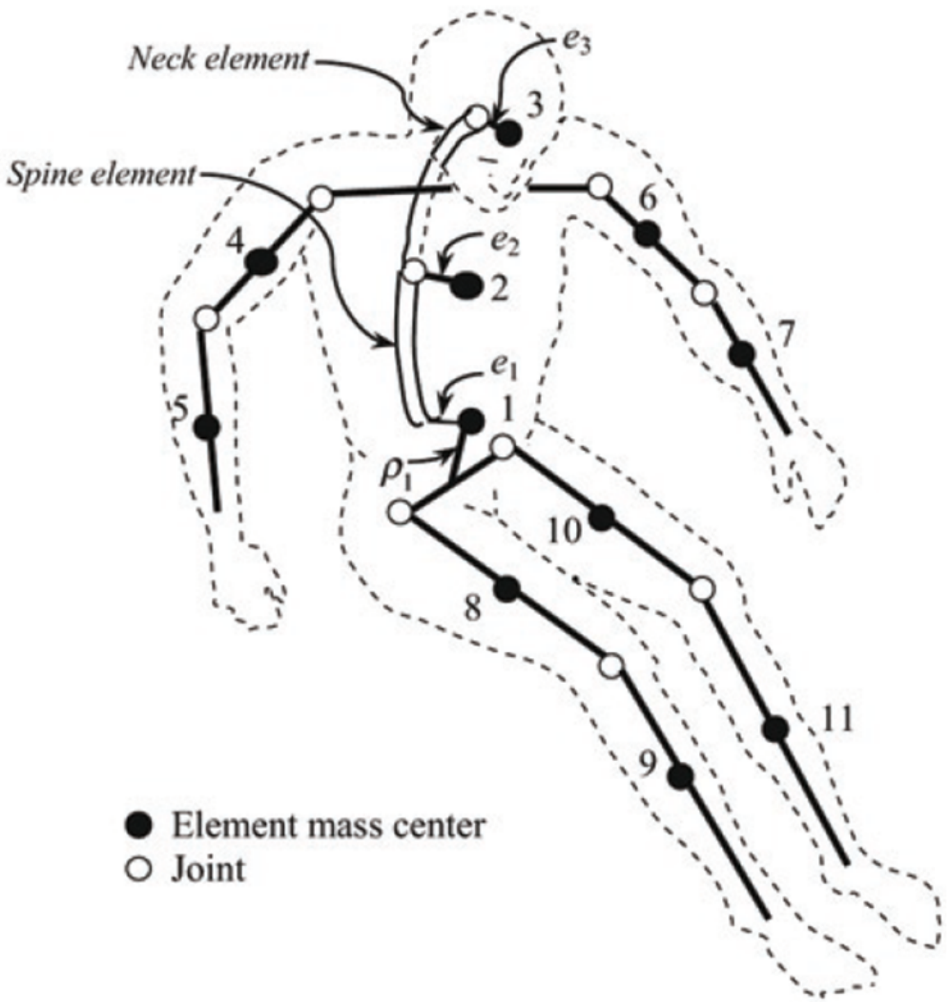


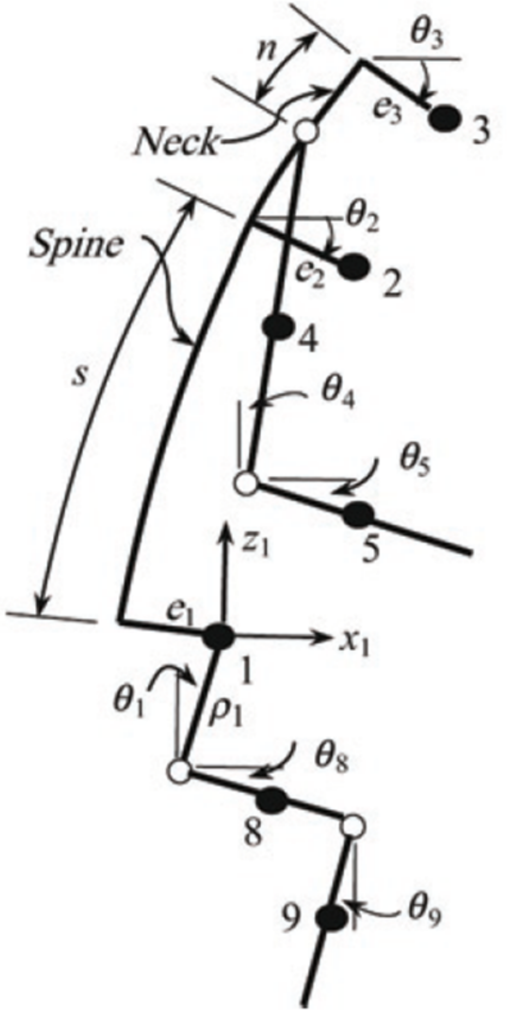


Tutorial 2: Human Body Sitting





Tutorial 2: Human Body Sitting



Segments	Generalized coordinates q_i
1. Lower torso	x_1, z_1, θ_1
2. Upper torso	θ_2
3. Head	θ_3
4, 6. Right and left upper arm	θ_4
5, 7. Right and left forearm	θ_5
8, 10. Right and left thigh	θ_8
9, 11. Right and left lower leg	θ_9
Spine	s
Neck	n
Total	**11**

Usage of Variational Operator

Properties of Variational Operator

If we have "functional" $F(x, y, y')$ where x is independent variable and $y(x)$ is dependent variable. Then the following properties hold true:

- Differentiation Rule:

$$\delta \frac{dF}{dx} = \frac{d\delta F}{dx}$$

- Integration Rule:

$$\delta \int F dx = \int \delta F dx$$

- Distributive Property: If $F(x, y, y') = F_1(x, y, y') \pm F_2(x, y, y')$, then

$$\delta F = \delta F_1(x, y, y') \pm \delta F_2(x, y, y')$$

- Multiplication Rule: If $F(x, y, y') = F_1(x, y, y') \cdot F_2(x, y, y')$, then

$$\delta F = \delta F_1(x, y, y') \cdot F_2(x, y, y') + F_1(x, y, y') \cdot \delta F_2(x, y, y')$$

Properties of Variational Operator

If we have "functional" $F(x, y, y')$ where x is independent variable and $y(x)$ is dependent variable. Then the following properties hold true:

- Division Rule: If $F(x, y, y') = \frac{F_1(x,y,y')}{F_2(x,y,y')}$, then

$$\delta F = \frac{\delta F_1(x, y, y') \cdot F_2(x, y, y') - F_1(x, y, y') \cdot \delta F_2(x, y, y')}{F_2^2}$$

- If $F = y^n$, then

$$\delta F = ny^{n-1}$$

- If $F = f(x)$, then

$$\delta F = 0$$

- If $F = xy^n$, then

$$\delta F = nxy^{n-1}$$

As analogous to differential operator, x is treated as constant while applying delta operator

Question:

$$F[y] = y^3 + \left(y'\right)^3 y^2 + \frac{y}{x} + xy + x^2$$

is a functional of $y = y(x)$, where x is independent variable and y is dependent variable on x. First variation of Functional F can be written as:

Question:

$$F[y] = y^3 + \left(y'\right)^3 y^2 + \frac{y}{x} + xy + x^2$$

is a functional of $y = y(x)$, where x is independent variable and y is dependent variable on x. First variation of Functional F can be written as:

Solution:

$$\delta F = \delta\left(y^3\right) + \delta\left((y')^3 y^2\right) + \delta\left(\frac{y}{x}\right) + \delta(xy) + \delta\left(x^2\right)$$

- $\delta\left(y^3\right) = 3y^2\delta y$
- $\delta\left((y')^3 y^2\right) = 2\left(y'\right)^3 y\delta y + 3\left(y'\right)^2 y^2\delta\left(y'\right)$
- $\delta\left(\frac{y}{x}\right) = \frac{\delta y}{x}$
- $\delta(xy) = x\delta y$
- $\delta\left(x^2\right) = 0$

Hence, the solution:

$$\boxed{\delta F = \left(3y^2 + 3\left(y'\right)^2 y^2 + 2\left(y'\right)^3 y + \frac{1}{x} + x\right)}$$

$$I[u] = \int_a^b \left(p(x)\frac{du}{dx} + q(x)u^2 \right) dx + Pu(a)$$

is a functional of $u = u(x)$, where x is an independent variable; $p(x)$ and $q(x)$ is functions of x; a, b and P are constants. Find its first variation.

Variational Operator Tutorial

$$I[u] = \int_a^b \left(p(x)\frac{du}{dx} + q(x)u^2 \right) dx + Pu(a)$$

is a functional of $u = u(x)$, where x is an independent variable; $p(x)$ and $q(x)$ is functions of x; a, b and P are constants. Find its first variation.

$$\delta I[u] = \int_a^b \left(p(x)\frac{d\delta u}{dx} + 2q(x)u\delta u \right) dx + P\delta u(a)$$

Variational Operator Tutorial

$$I[u,v]=\int_{\Omega}\left(p(x,y)\frac{du}{dx}\frac{dv}{dx}+q(x,y)v\right)dxdy+\int_{\Gamma}Quds$$

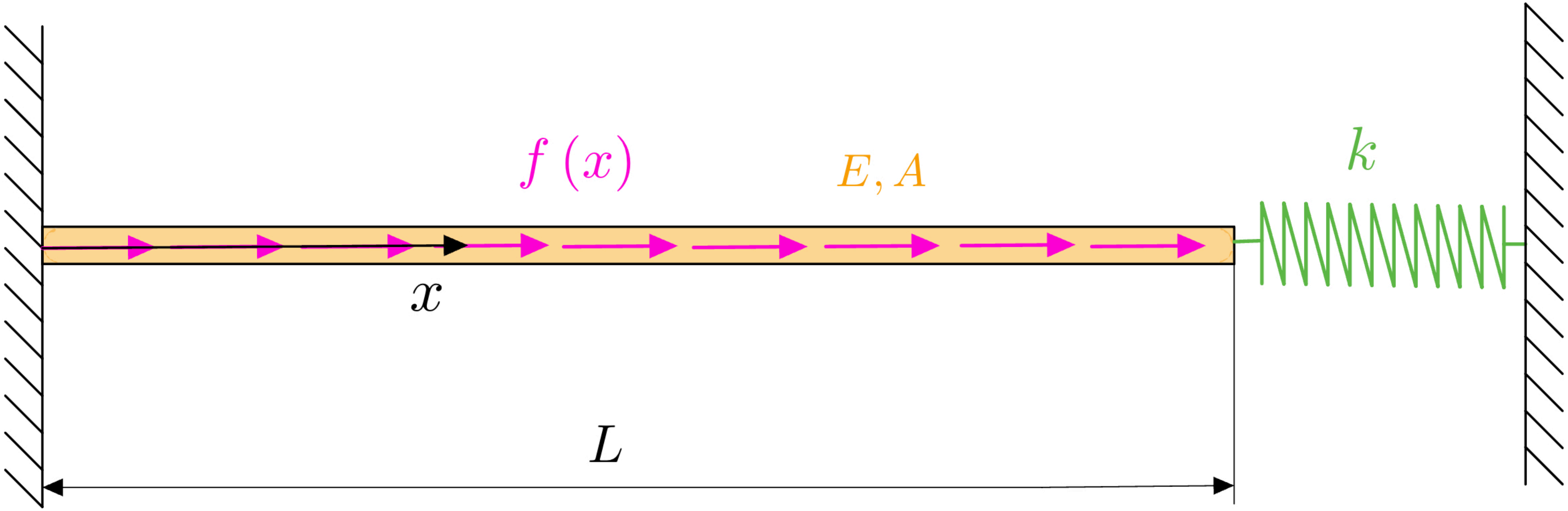
is a functional of $u=u(x,y)$ and $v=v(x,y)$, where x and y are independent variables; $p(x,y)$ and $q(x,y)$ are functions of x and y; Q is a constant. Find its first variation.

$$\delta I[u,v]=\int_{\Omega}\left[p(x,y)\left(\frac{d\delta u}{dx}\frac{dv}{dx}+\frac{d\delta v}{dx}\frac{du}{dx}\right)+q(x,y)\delta v\right]dxdy+\int_{\Gamma}Q\delta uds$$

Hamilton’s Principle

Axial Bar

Consider the axial motion of an elastic bar (as shown in figure) with E : Elastic Modulus, A : Cross-sectional Area, L : Length of bar and ρ : Mass density. The bar is subjected to distributed force f per unit length. It is fixed at one end ($x = 0$) and connected to linear elastic spring with spring constant k at the end ($x = L$). Determine the Equation of Motion (EOM) for the bar.



Axial Bar

Let u be axial displacement of the bar.

Kinetic Energy in the system

$$T = \int_V \frac{\rho}{2} \left(\frac{\partial u}{\partial t} \right)^2 dV = \int_0^L \frac{\rho A}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx$$

Potential Energy due to Strain energy stored in the bar

$$V_1 = \int_V \frac{1}{2} \sigma_{ij} \varepsilon_{ij} dV = \int_0^L \frac{A}{2} \sigma_{xx} \varepsilon_{xx} dx = \int_0^L \frac{EA}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx$$

Potential Energy is stored in the linear spring

$$V_2 = \frac{k}{2} \left(u(L) \right)^2$$

Total Potential Energy

$$V = V_1 + V_2 = \int_0^L \frac{EA}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx + \frac{k}{2} \left(u(L) \right)^2$$

Work done on the bar by the distributed load $f(x)$

$$W = \int_0^L f u dx$$

Axial Bar

Hamilton's Principle

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt$$

First Term:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \delta \left(\int_0^L \frac{\rho A}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx \right) dt = = \int_{t_1}^{t_2} \int_0^L \rho A \frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} dx dt = \int_0^L \left[- \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) dx dt \delta u + \cancel{\left[\rho A \frac{\partial u}{\partial t} \delta u \right]_{t_1}^{t_2}} \right]$$

Second Term:

$$- \int_{t_1}^{t_2} \delta V dt = - \int_{t}^{t_2} \delta \left(\int_0^L \frac{EA}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx + \frac{k}{2} \left(u(L) \right)^2 \right) dt = - \int_{t_1}^{t_2} \int_0^L \left(EA \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} dx + k u(L) \delta u(L) \right) dt$$

$$= \int_{t_1}^{t_2} \int_0^L \left(\frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) dx \delta u - \left[EA \frac{\partial u}{\partial x} \delta u \right]_0^L - k u(L) \delta u(L) \right) dt$$

$$= \int_{t_1}^{t_2} \left[\int_0^L \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) dx \delta u - \left[EA \frac{\partial u}{\partial x} + k u \right]_{x=L} \delta u(L) + \cancel{\delta u(0)} \left[EA \frac{\partial u}{\partial x} \right]_{x=0} \right] dt$$

Axial Bar

Third Term:

$$\int_{t_1}^{t_2} \delta W dt = \int_{t_1}^{t_2} \delta \left(\int_0^L f u dx \right) dt = \int_{t_1}^{t_2} \int_0^L f \delta u dx dt$$

Upon substitution:

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt = \int_{t_1}^{t_2} \left[\int_0^L \left(\left(-\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) \right) + \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) + f \right) dx \delta u - \left[EA \frac{\partial u}{\partial x} + ku \right]_{x=L} \delta u(L) \right] dt$$

Equation of motion:

$$\frac{\partial}{\partial t} \left(\rho A \frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) - f = 0$$

With an additional condition called natural boundary condition:

$$\left[EA \frac{\partial u}{\partial x} + ku \right]_{x=L} = 0$$

Beam with mass attached

Consider the beam (as shown in figure) with

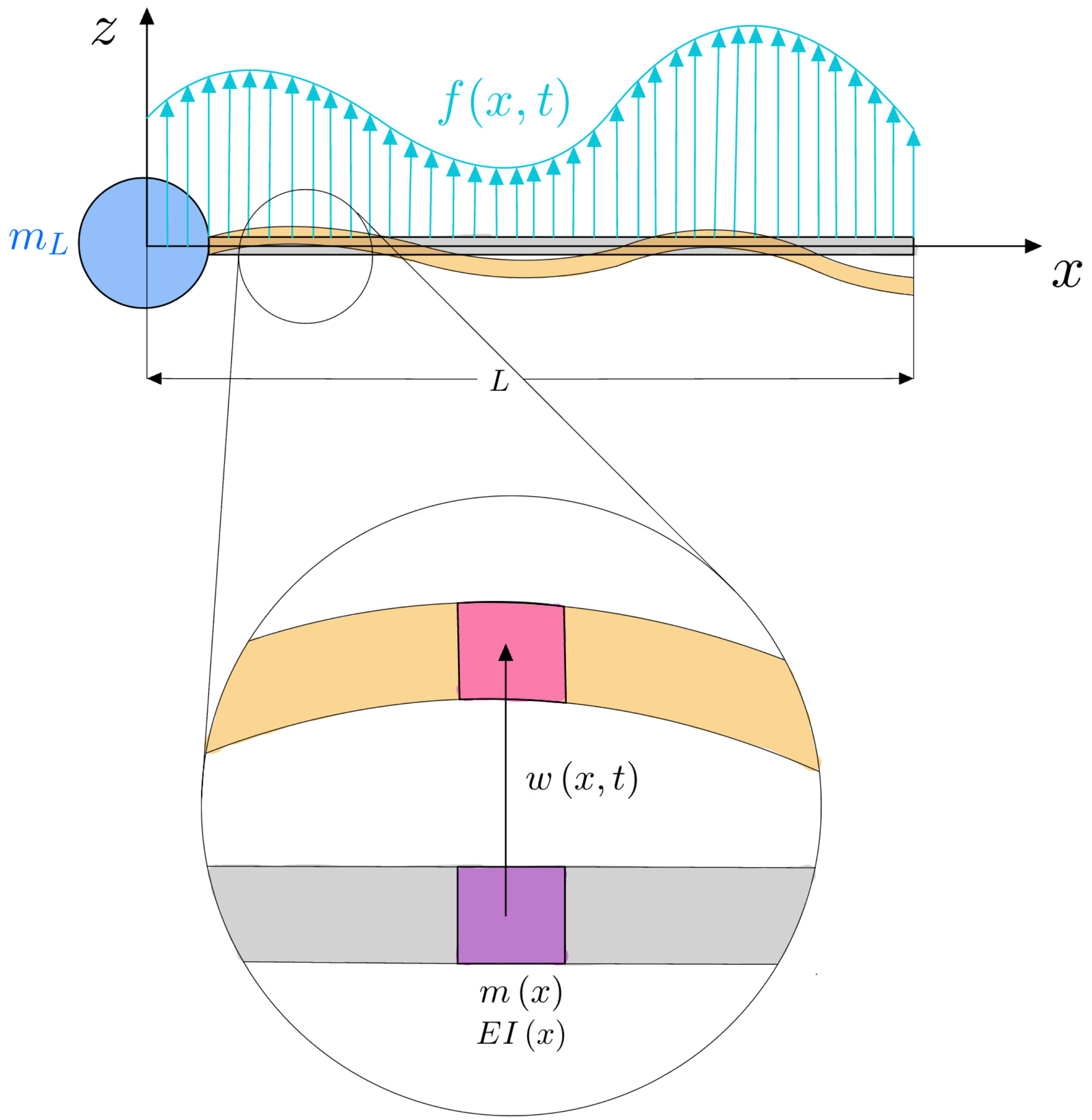
EI : Flexural Rigidity,

L : Length of bar and

m : Mass density.

The bar is subjected to distributed force $f(x,t)$ per unit length. Mass m_L is attached at one end ($x=0$) and beam is free at the other end ($x=L$).

Determine the Equation of Motion (EOM) for the beam.



Beam with mass attached

Kinetic energy:

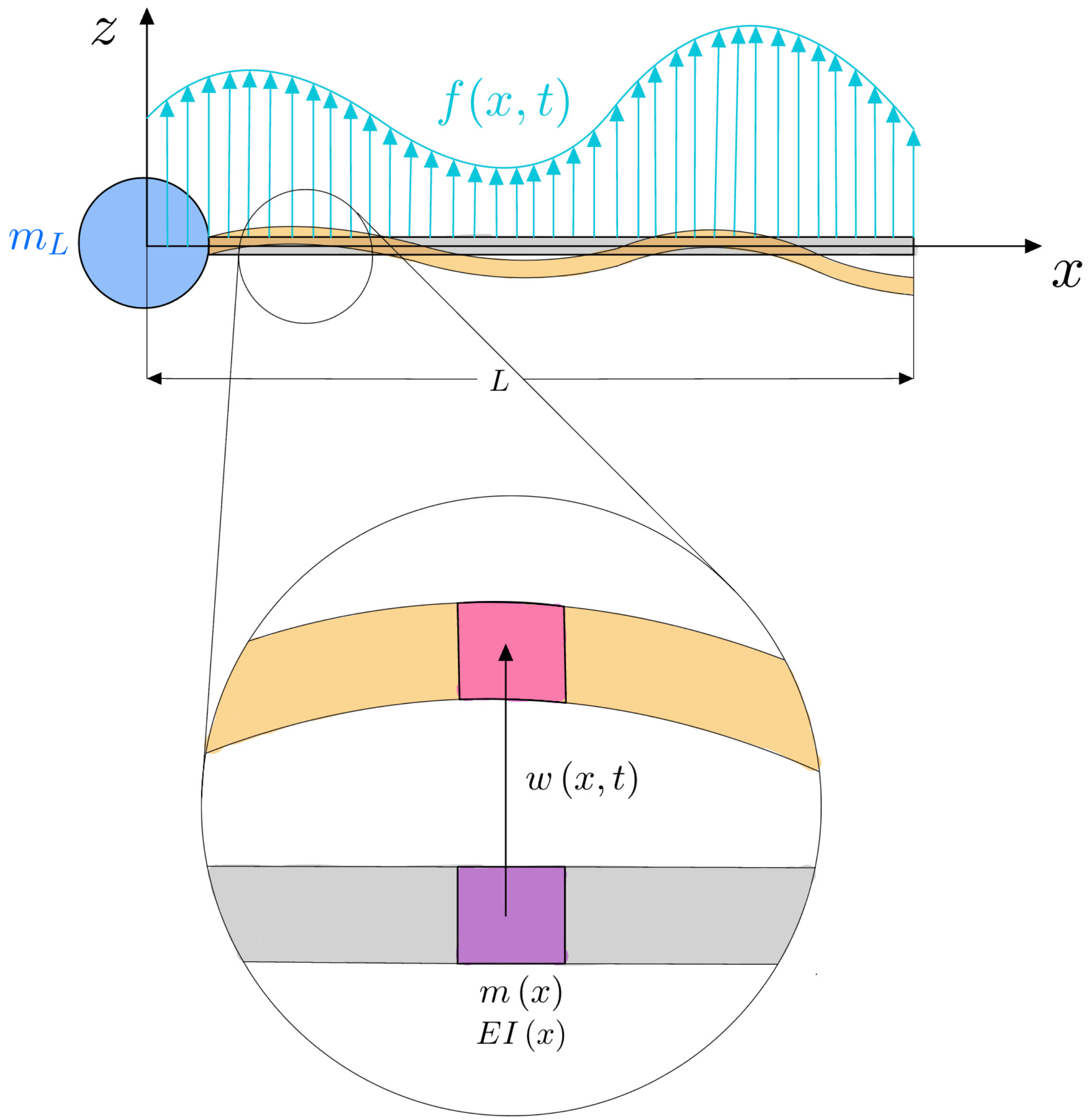
$$T = \int_0^L \frac{1}{2} m\dot{w}^2 dx + \frac{1}{2} m_L(\dot{w}(0))^2$$

Potential energy:

$$V = \int_0^L \frac{1}{2} EI \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx$$

Work Done by external force:

$$W = \int_0^L f(x,t)\, w dx$$



Beam with mass attached

Hamilton's Principle

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt$$

First Term:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \delta \left(\int_0^L \frac{1}{2} m\dot{w}^2 dx + \frac{1}{2} m_L (\dot{w}(0))^2 \right) dt = \int_{t_1}^{t_2} \left[\int_0^L m\dot{w}\delta\dot{w} dx + m_L \dot{w}(0)\delta\dot{w}(0) \right] dt$$

$$= \int_0^L \left[\int_{t_1}^{t_2} m\ddot{w}\delta w dt + \cancel{\left[m\dot{w}\delta\omega \right]_{t_1}^{t_2}} \right] dx - \int_{t_1}^{t_2} m_L \ddot{w}(0)\delta w(0) dt + \cancel{\left[m_L \dot{w}(0)\delta w(0) \right]_{t_1}^{t_2}}$$

Second Term:

$$-\int_{t_1}^{t_2} \delta V dt = -\int_{t}^{t_2} \delta \left[\int_0^L \frac{EI}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx \right] dt = -\int_{t_1}^{t_2} \int_0^L \left(EI \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial^2 w}{\partial x^2} dx \right) dt$$

$$= \int_{t_1}^{t_2} -\int_0^L \left(\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) dx \delta w - \left[EI \frac{\partial^2 w}{\partial x^2} \delta \left(\frac{\partial w}{\partial x} \right) \right]_0^L + \left[\frac{\partial}{\partial x} EI \frac{\partial^2 w}{\partial x^2} \delta w \right]_0^L \right) dt$$

Beam with mass attached

Third Term:

$$\int_{t_1}^{t_2} \delta W dt = \int_{t_1}^{t_2} \delta \left(\int_0^L f(x,t) w dx \right) dt = \int_{t_1}^{t_2} \int_0^L f(x,t) \delta w dx dt$$

Equation of motion:

$$m\ddot{w} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2} \right) - f(x,t) = 0$$

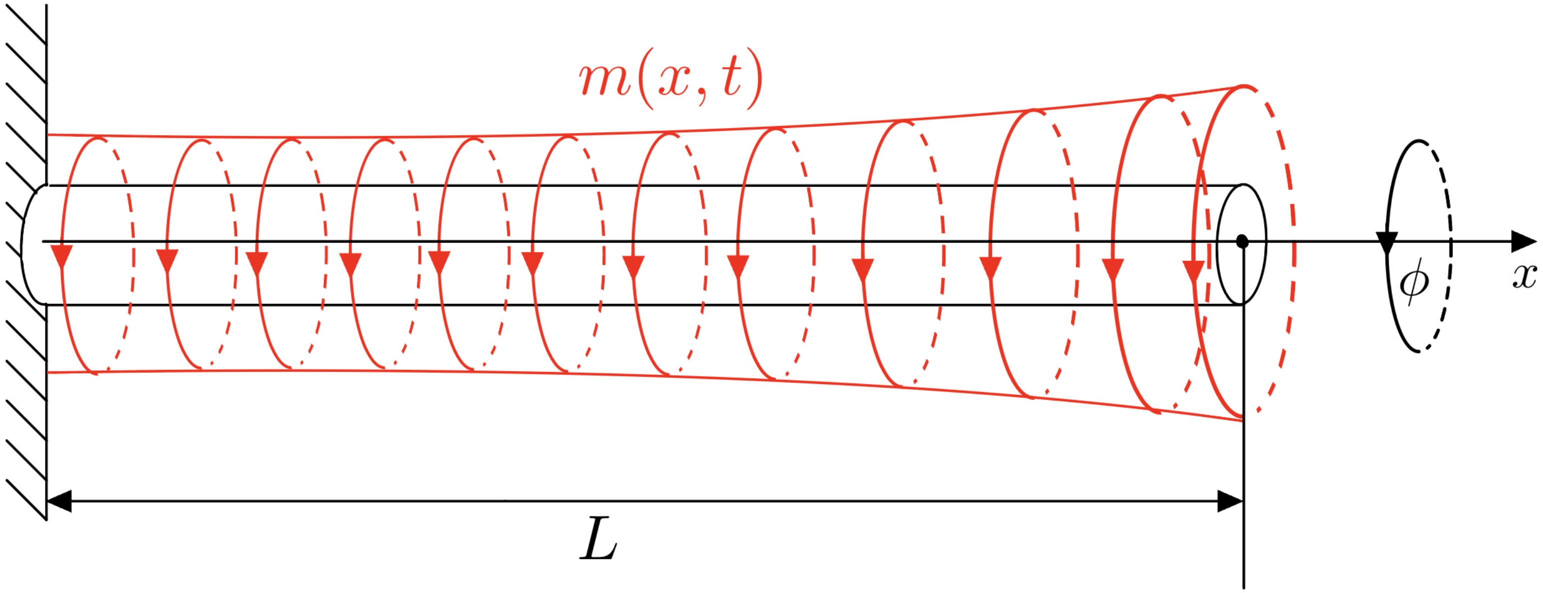
With an additional conditions called natural boundary conditions:

$$\left[EI \frac{\partial^2 w}{\partial x^2} \right]_{x=0} = 0 \qquad \left[EI \frac{\partial^2 w}{\partial x^2} \right]_{x=L} = 0$$

$$\left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2} \right) + m_L \ddot{w} \right]_{x=0} = 0 \qquad \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 w}{\partial x^2} \right) \right]_{x=L} = 0$$

Torsional Vibrations

Consider the torsional motion of a shaft (as shown in figure) with $GJ(x)$: Torsional Rigidity , L : Length of shaft and $I_P(x)$: Mass moment of Inertia/ unit length. The shaft is subjected to distributed twisting moment $m(x, t)$ per unit length. It is fixed at one end ($x = 0$) and free at the other end ($x = L$). Determine the Equation of Motion (EOM) for the shaft.



Torsional Vibrations

Kinetic energy:

$$T = \int_0^L \frac{1}{2} I_P \dot{\phi}^2 dx$$

Potential energy:

$$V = \int_0^L \frac{1}{2} GJ \left(\frac{\partial \phi}{\partial x} \right)^2 dx$$

Work Done by external moment:

$$W = \int_0^L m(x,t)\, \phi dx$$

Hamilton's Principle

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt$$

First Term:

$$\int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2} \delta \left(\int_0^L \frac{I_P}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 dx \right) dt = = \int_{t_1}^{t_2} \int_0^L I_P \frac{\partial \phi}{\partial t} \frac{\partial \delta \phi}{\partial t} dx dt$$

$$= \int_0^L \left[- \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(I_P \frac{\partial \phi}{\partial t} \right) dx dt \delta \phi + \left[I_P \frac{\partial \phi}{\partial t} \delta \phi \right]_{t_1}^{t_2} \right]$$

Second Term:

$$- \int_{t_1}^{t_2} \delta V dt = - \int_{t}^{t_2} \delta \left(\int_0^L \frac{GJ}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 dx \right) dt = - \int_{t_1}^{t_2} \int_0^L \left(GJ \frac{\partial \phi}{\partial x} \frac{\partial \delta \phi}{\partial x} dx \right) dt$$

$$= \int_{t_1}^{t_2} \int_0^L \left(\frac{\partial}{\partial x} \left(GJ \frac{\partial \phi}{\partial x} \right) dx \delta \phi - \left[GJ \frac{\partial \phi}{\partial x} \delta \phi \right]_0^L \right) dt$$

Torsional Vibrations

Third Term:

$$\int_{t_1}^{t_2} \delta W dt = \int_{t_1}^{t_2} \delta\left(\int_0^L m(x,t)\phi dx\right) dt = \int_{t_1}^{t_2}\int_0^L m(x,t)\delta\phi dx dt$$

Upon substitution:

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W)\, dt = \int_{t_1}^{t_2}\left[\int_0^L \left(\left(-\frac{\partial}{\partial t}\left(I_P\frac{\partial\phi}{\partial t}\right)\right) + \frac{\partial}{\partial x}\left(GJ\frac{\partial\phi}{\partial x}\right) + m(x,t)\right) dx\delta\phi - \left[GJ\frac{\partial\phi}{\partial x}\right]_{x=L}\delta\phi(L)\right] dt$$

Equation of motion:

$$\frac{\partial}{\partial t}\left(I_P\frac{\partial\phi}{\partial t}\right) - \frac{\partial}{\partial x}\left(GJ\frac{\partial\phi}{\partial x}\right) - m(x,t) = 0$$

With an additional condition called natural boundary condition:

$$\left[GJ\frac{\partial\phi}{\partial x}\right]_{x=L} = 0$$

Euler Lagrange Equations in Mechanics

Euler Lagrange Equation of Motion

Total Kinetic Energy T
Total Potential Energy U
Lagrangian Function $L = T - U$

To get the equation of motion, we use: Lagrangian Formulation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = C_i + F_i$$

where q_i are Generalised Coordinates, F_i are Non conservative forces and C_i Constraint forces

Potential energy is a function of position, so we can regard it as a function $U(q_1, q_2, \ldots, q_n)$ which depends on q_i but not $\dot{q}_i$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = F_i + C_i$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = F_i + C_i + Q_i$$

Here Q_i are conservative forces

Euler Lagrange Equation of Motion

Taking:

$$F_i = Fr_i + Fn_i$$

Here
Fr_i are friction forces and
Fn_i are non-conservative non-friction forces

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Fr_i + Fn_i + C_i$$

Final Euler Lagrange Equation of Motion becomes:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Fn_i + C_i$$

where
R is Rayleigh Dissipation Function

String under self weight

Consider the string with mass (as shown in figure) with

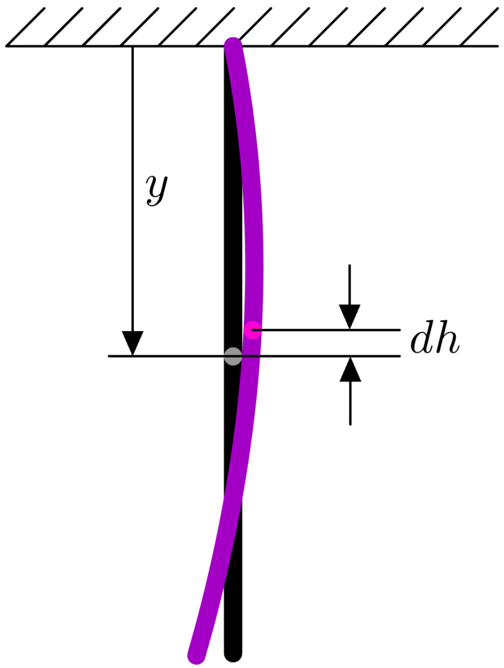
μ : mass per unit length,

L : Length of string and

g : Acceleration due to gravity.

The string is subjected to its own weight and is attached at one end ($y = 0$) and beam is free at the other end ($y = L$).

Determine the Equation of Motion (EOM) for the string.



$$dl = \left(1 + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2\right) dy$$

$$dh = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 dy$$

String under self weight

Kinetic energy:

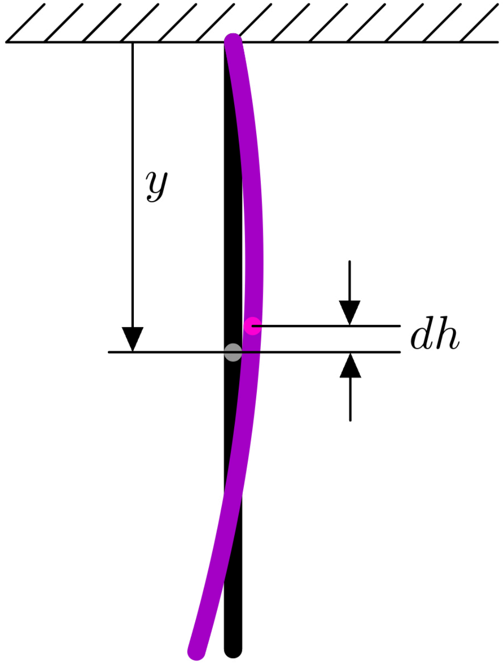
$$T = \int_0^L \frac{1}{2}\mu \dot{u}^2 dy$$

Potential energy:

$$V = \int_0^L \frac{1}{2} g(L-y)\mu \left(\frac{\partial u}{\partial y}\right)^2 dy$$

Lagrangian:

$$\mathbb{L} = T - V = \frac{1}{2}\left[\mu\left(\frac{\partial u}{\partial t}\right)^2 - g(L-y)\mu\left(\frac{\partial u}{\partial y}\right)^2\right]$$



$$dl = \left(1 + \cancel{\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2}\right) dy$$

$$dh = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 dy$$

String under self weight

For Conservative systems

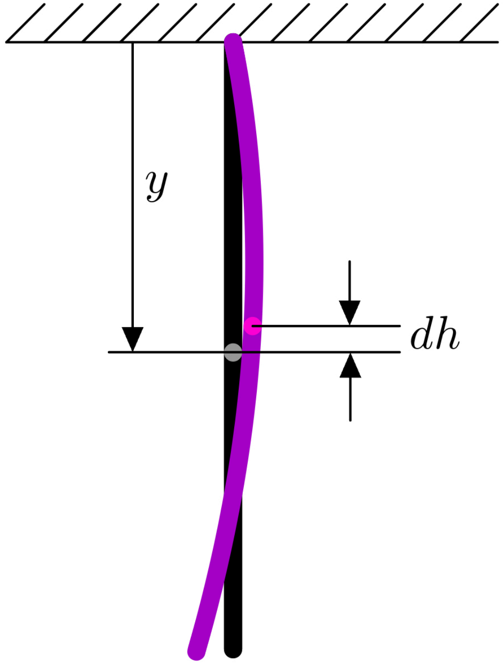
Lagrange Equation of Motion:

$$\frac{\partial \mathbb{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathbb{L}}{\partial \dot{q}_i} = 0$$

Where q_i are degree of freedoms

The EOM:

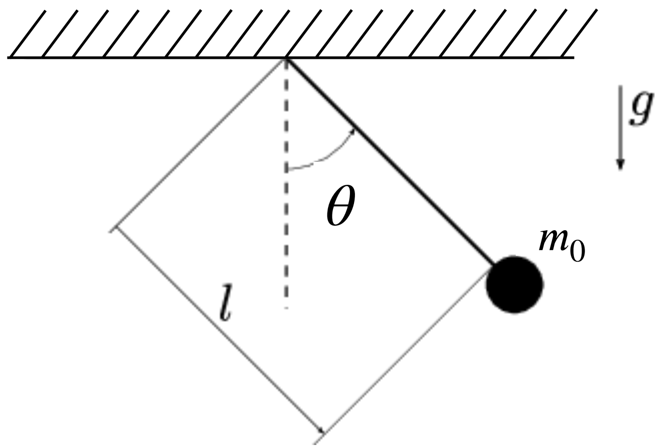
$$\frac{\partial^2 u(y,t)}{\partial t^2} = g(L-y)\frac{\partial^2 u(y,t)}{\partial y^2} - g\frac{\partial u(y,t)}{\partial y}$$



$$dl = \left(1 + \cancel{\frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2}\right) dy$$

$$dh = \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 dy$$

Pendulum 1

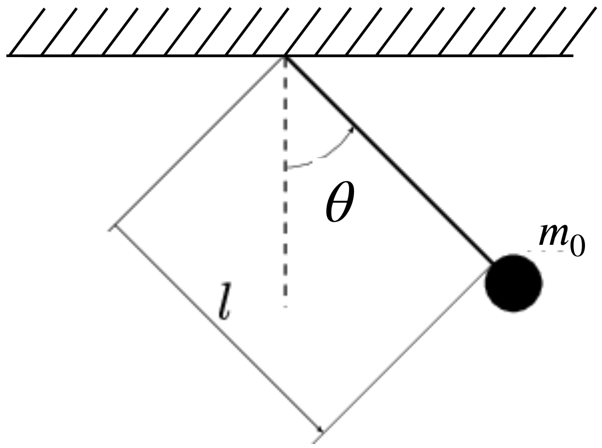


Physical Coordinates (n)		$\theta(t)$
Constants		l, m_0
Constraints (m)		-
Degree of freedom $(n - m)$		1
Generalised Coordinates		$\theta(t)$

$$q_i = \theta$$
$$F_i = 0$$

$$v = l\dot{\theta}$$
$$T = \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 l^2 \dot{\theta}^2$$

$$y = l(1 - cos\theta)$$
$$U = m_0 g y = m_0 g l(1 - cos\theta)$$

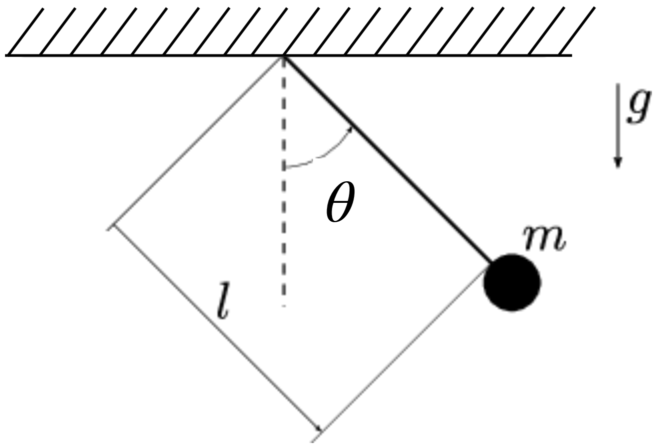


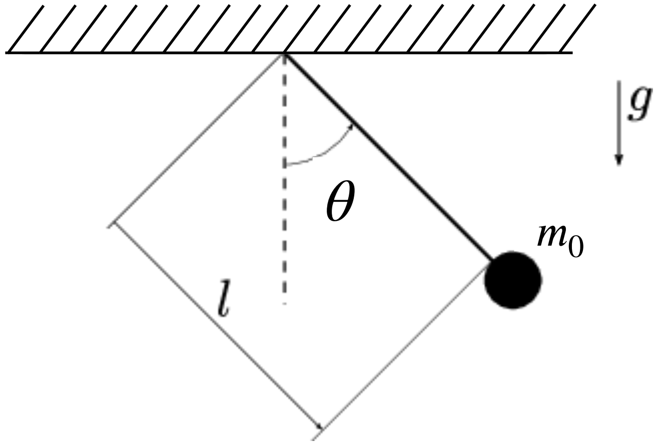
$$L = T - U = \frac{1}{2}m_0 l^2 \dot{\theta}^2 - m_0 g l(1 - cos\theta)$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\ddot{\theta} + \frac{g}{l}sin\theta = 0$$

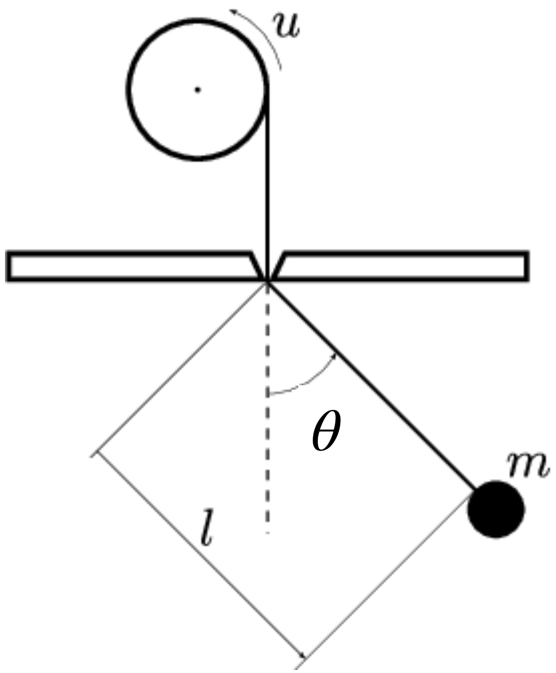
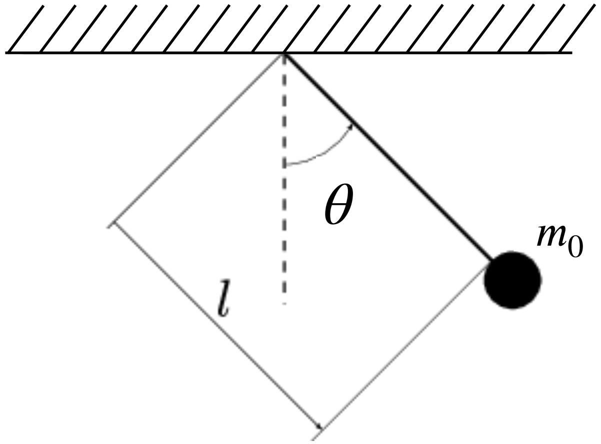
Pendulum 2





Physical Coordinates (n)	$x(t), y(t)$	$\theta(t)$
Constants	l, m	l, m_0
Constraints (m)	$\lambda\colon\ x^2 + y^2 - l^2 = 0$	-
Degree of freedom $(n - m)$	1	1
Generalised Coordinates	$\theta(t)$	$\theta(t)$

Pendulum3

		
Physical Coordinates (n)	$l(t), \theta(t)$	$\theta(t)$
Constants	l_0, u, m	l, m_0
Constraints (m)	$l(t) = l_0 - ut$	-
Degree of freedom $(n - m)$	1	1
Generalised Coordinates	$\theta(t)$	$\theta(t)$

Pendulum Variant 4

$$
\begin{aligned}
x(t) &= \ell(t)\sin\theta(t) & \dot{x}(t) &= \dot{\ell}(t)\sin\theta(t) + \ell(t)\dot{\theta}(t)\cos\theta(t) \\
y(t) &= -\ell(t)\cos\theta(t) & \dot{x}(t) &= -\dot{\ell}(t)\cos\theta(t) + \ell(t)\dot{\theta}(t)\sin\theta(t)
\end{aligned}
$$

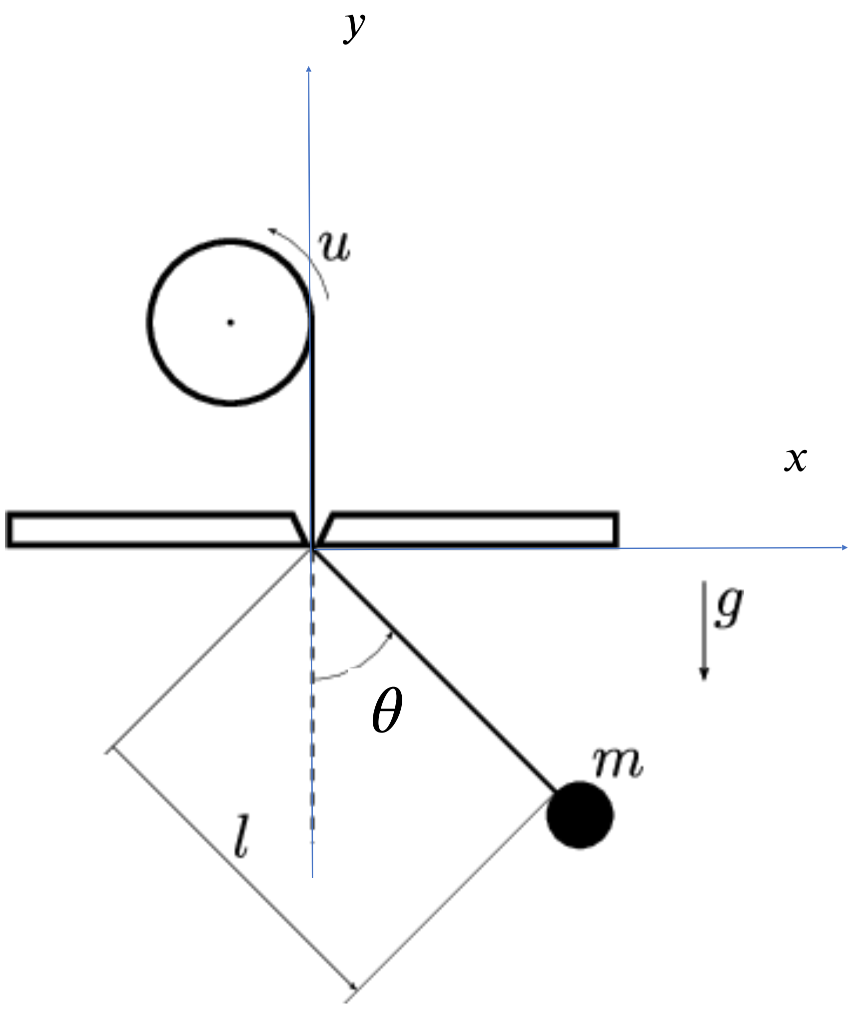
So the kinetic and potential energies of the mass are

$$
T = \frac{1}{2}m\left[\dot{x}(t)^2 + \dot{y}(t)^2\right] = \frac{1}{2}m\left[\dot{\ell}(t)^2 + \ell(t)^2\dot{\theta}(t)^2\right]
$$

$$
U = mgy(t) = -mg\ell(t)\cos\theta(t)
$$

The corresponding Lagrangian is

$$
\begin{aligned}
\mathcal{L}(\theta, \dot{\theta}, t) = T - U &= \frac{1}{2}m\left[\dot{\ell}(t)^2 + \ell(t)^2\dot{\theta}^2\right] + mg\ell(t)\cos\theta \\
&= m\left[\frac{1}{2}\ell(t)^2\dot{\theta}^2 + g\ell(t)\cos\theta + \frac{1}{2}\dot{\ell}(t)^2\right]
\end{aligned}
$$



Pendulum Variant 4

and Lagrange's equation of motion is

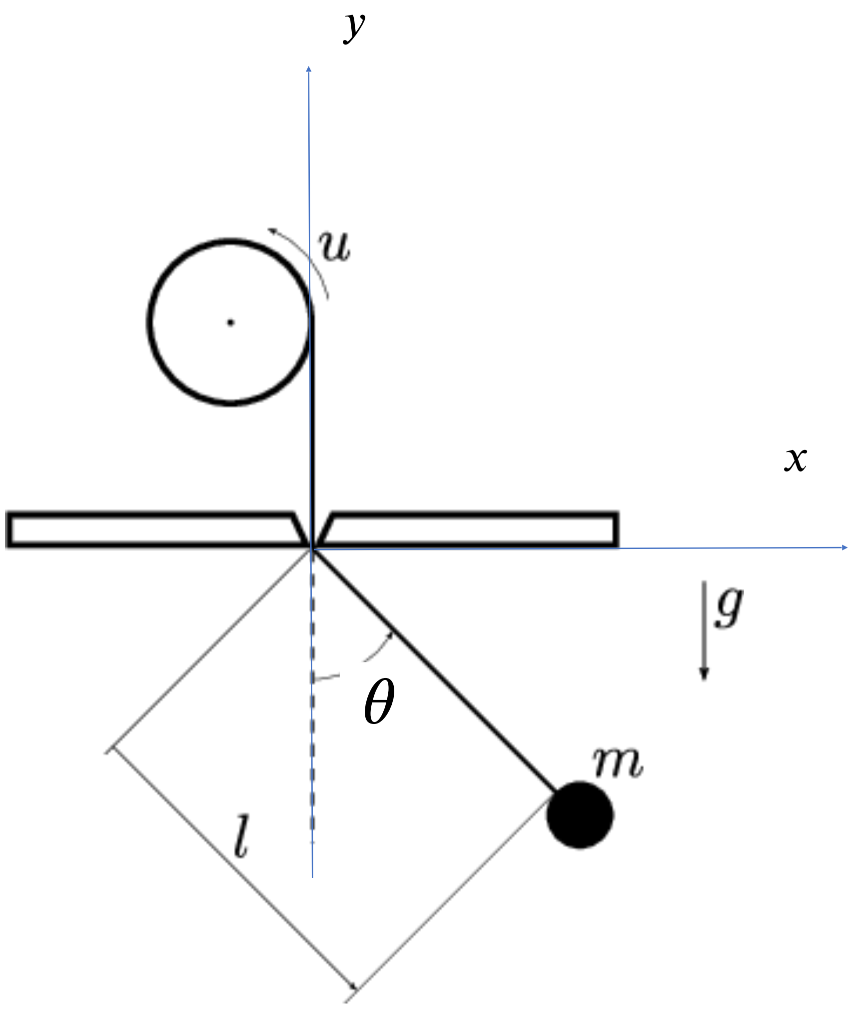
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \theta}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt} m\left[\ell(t)^2 \dot{\theta}\right] - m[-g\ell(t)\sin\theta] = 0$$

$$m\left[\ell(t)^2 \ddot{\theta}(t) + 2\ell(t)\dot{\ell}(t)\dot{\theta}(t) + g\ell(t)\sin\theta(t)\right]$$

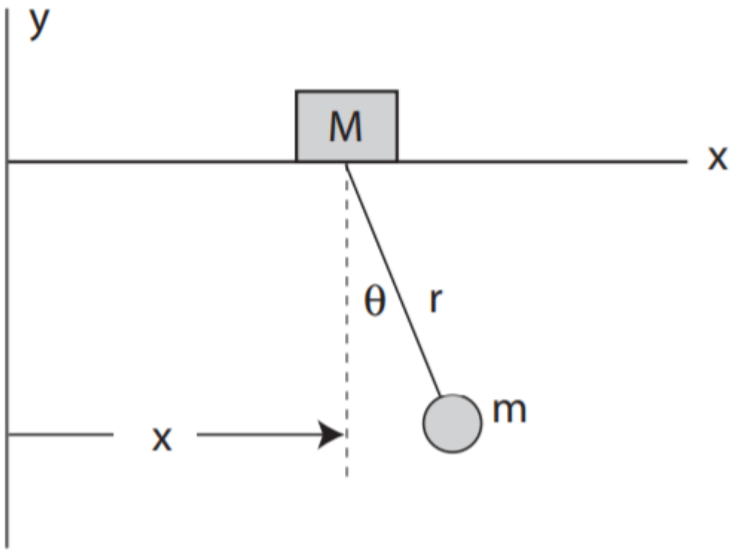
$$\frac{d^2\theta}{dt^2}(t) + 2\frac{\dot{\ell}(t)}{\ell(t)}\dot{\theta}(t) + \frac{g}{\ell(t)}\sin\theta(t) = 0$$

$$\ddot{\theta} - \frac{u}{l_0 - ut}\dot{\theta} + \frac{g}{l_0 - ut}\sin\theta = 0$$



Pendulum Variant 5

Figure shows a simple pendulum consisting of a string of length r and a bob of mass m that is attached to a support of mass M. The support moves without friction on the horizontal plane. Find equation of motion using EL Equations.



Pendulum Variant 5

Figure shows a simple pendulum consisting of a string of length r and a bob of mass m that is attached to a support of mass M. The support moves without friction on the horizontal plane. Find equation of motion using EL Equations.

The x component of the velocity of the bob is given by $\dot{x} + r\dot{\theta}\cos\theta$ and the y component by $r\dot{\theta}\sin\theta$. So the overall kinetic energy of the system is given by:

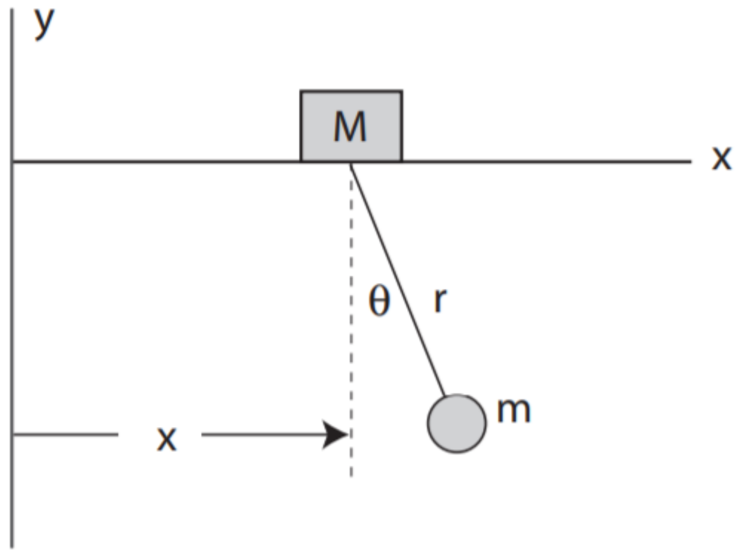
$$T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left[\dot{x}^2 + r^2\dot{\theta}^2 + 2\dot{x}r\dot{\theta}\cos\theta\right]$$

The potential energy is:

$$U = -mgr\cos\theta$$

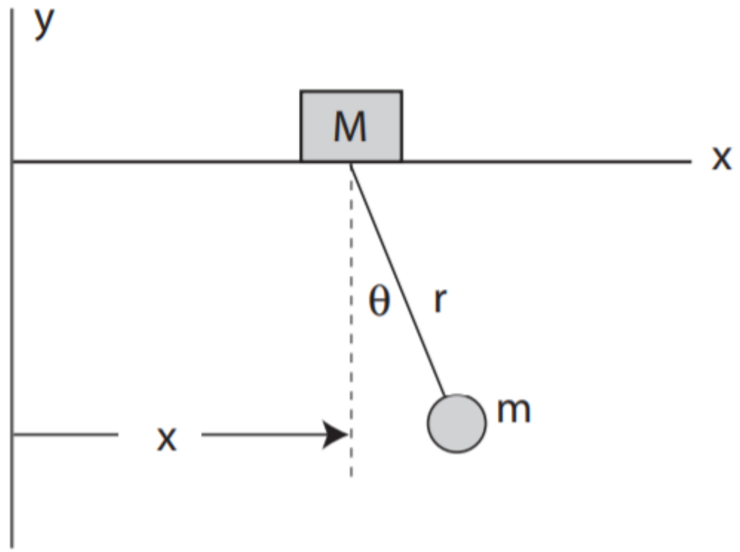
and the Lagragian:

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}(m + M)\hat{x}^2 + \frac{1}{2}\left(r^2\dot{\theta}^2 + 2\,\mathrm{xr}\,\dot{\theta}\cos\theta\right) + mgr\cos\theta$$



Pendulum Variant 5

Figure shows a simple pendulum consisting of a string of length r and a bob of mass m that is attached to a support of mass M. The support moves without friction on the horizontal plane. Find equation of motion using EL Equations.



Apply Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}[(m+M)\dot{x} + mr\dot{\theta}\cos\theta] = 0$$

and

$$\frac{d}{dt}\left[m\left(r^2\dot{\theta} + \dot{x}r\cos\theta\right)\right] = -m(\dot{x}r\dot{\theta} + gr)\sin\theta$$

Pendulum Variant 5

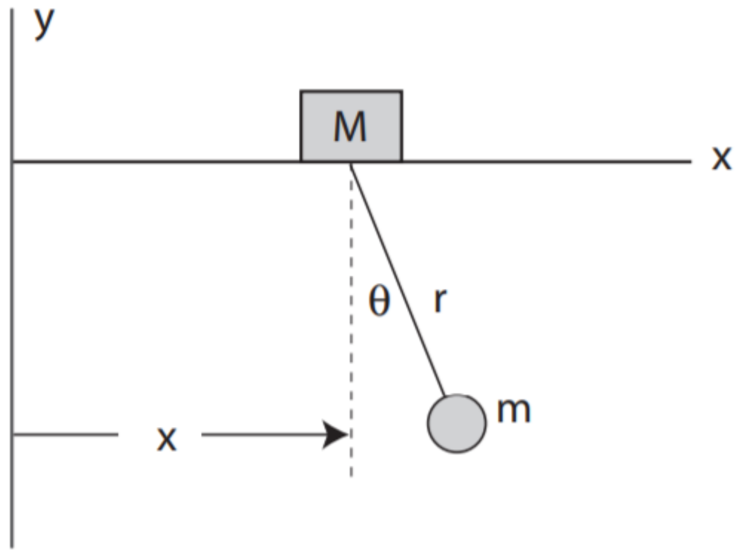
Figure shows a simple pendulum consisting of a string of length r and a bob of mass m that is attached to a support of mass M. The support moves without friction on the horizontal plane. Find equation of motion using EL Equations.

$$\ddot{\theta} + \frac{\ddot{x}}{r}\cos\theta + \frac{g}{r}\sin\theta = 0$$

Note that when the support is moving with constant velocity ($\ddot{x} = 0$) we just have the equation of motion for a pendulum. Suppose we set $\ddot{\theta} = 0$. Then:

$$\tan\theta = -\frac{\ddot{x}}{g}$$

If we accelerate the support to the right then the pendulum hangs motionless at the angle given by the above equation.

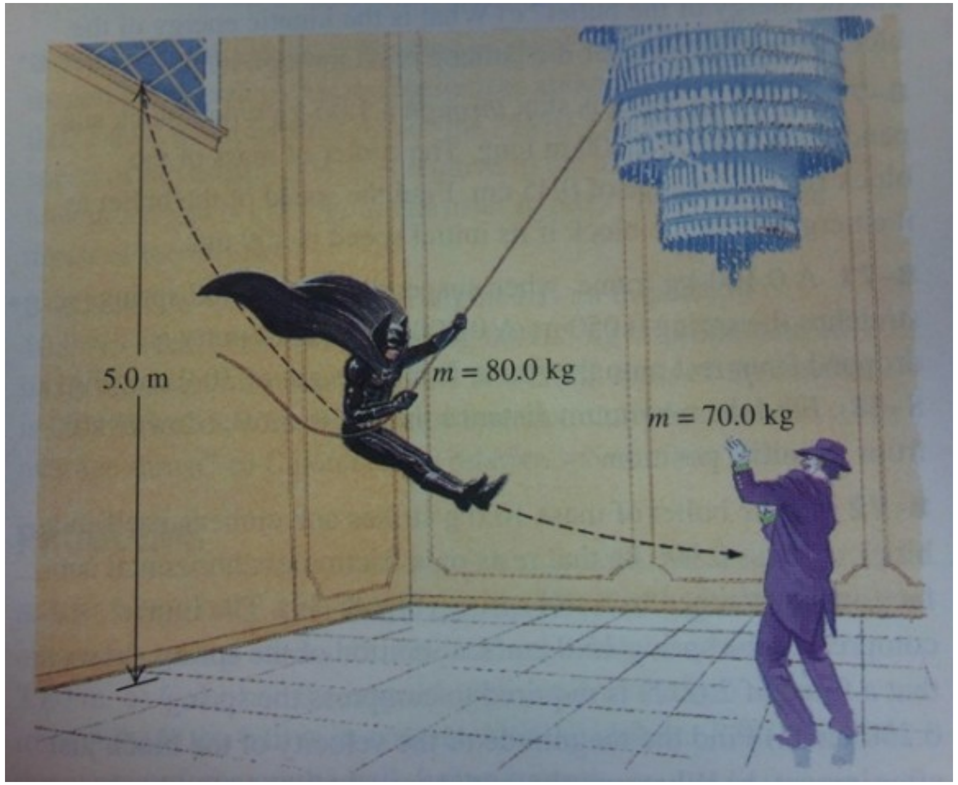


Warning

Heavy In-Syllabus Memes

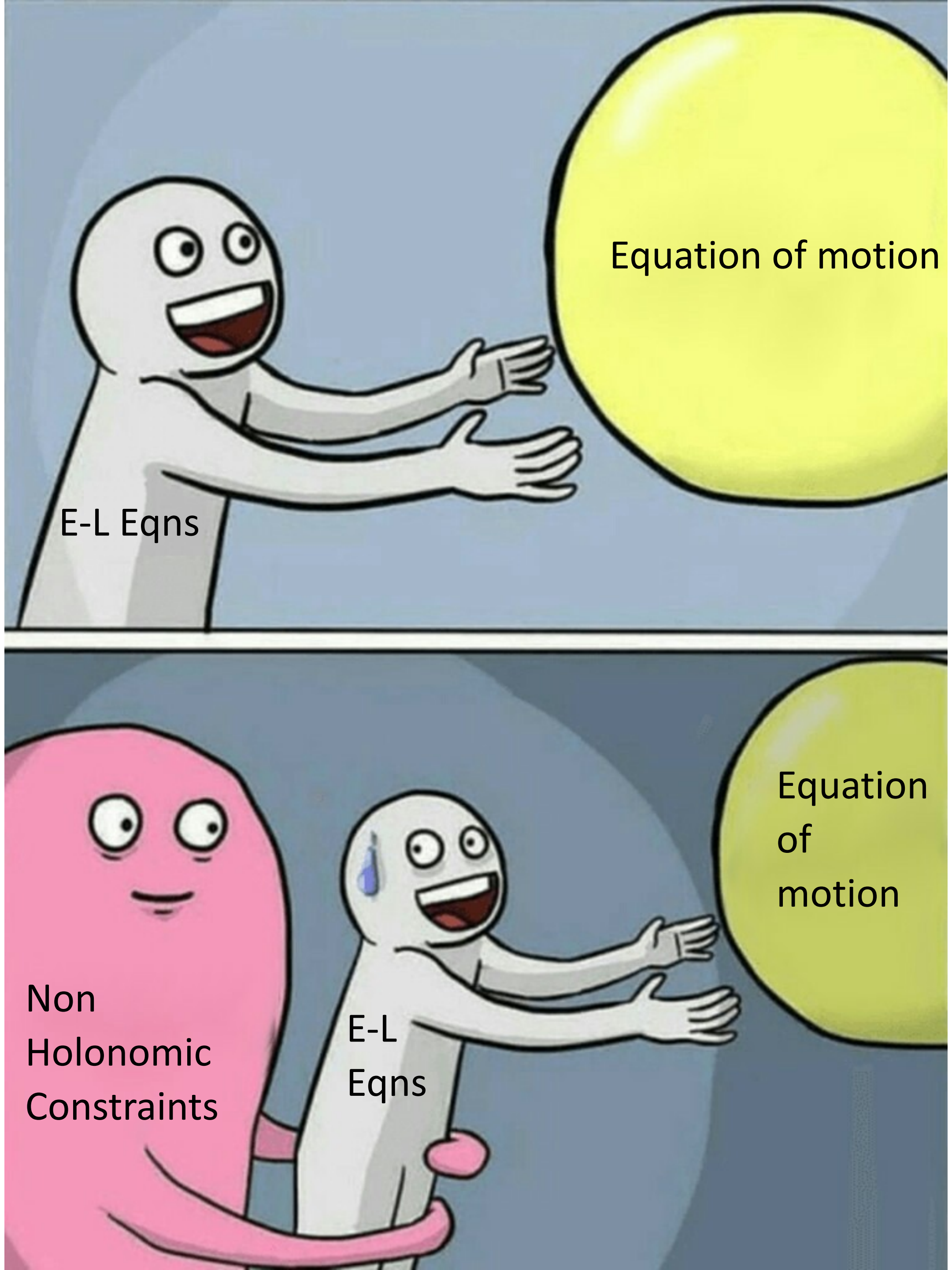
Pay attention

Simple Pendulum



Experiment.

Starting Lagrangian Mechanics...


Equation of motion
E-L Eqns
Equation of motion
Non Holonomic Constraints
E-L Eqns


C 1
C 2
C 3
Lagrange Multipliers

END