Let us take the same example.

$$A_{2m} = \{2\} \quad \text{for all } m \in \mathbb{N} = \{1, 2, 3, \ldots\}$$

$$A_{2m+1} = \{1\} \quad \text{for all } m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

$$\liminf_{n\to\infty} A_n = \phi.$$

$\boxed{\text{Example 4}}$ Consider $A_{2m} = \{1, 2, 3, \ldots\}$ and $A_{2m-1} = \{1, 3, 5, 7, \ldots\}$. Then it follows that

$$\{1, 3, 5, 7, \ldots\} = \liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

$$= \{1. 2, 3, 4, \ldots\} = \mathbb{N}.$$

<u>Theorem</u> (Continuity of probability).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider a non-decreasing sequence of events $(E_n : n \geq 1)$ in $\mathcal{F}$. Then

$$\lim_{n\to\infty} \mathbb{P}(E_n) = \mathbb{P}\left(\lim_{n\to\infty} E_n\right) = \mathbb{P}\left(\bigcup_{n\geq 1} E_n\right).$$

As a consequence, it follows that

$$\lim_{n\to\infty} \mathbb{P}(E_n) = \mathbb{P}\left(\lim_{n\to\infty} E_n\right) = \mathbb{P}\left(\bigcap_{n\geq 1} E_n\right)$$

where $(E_n : n \geq 1)$ is a non-increasing sequence of events in $\mathcal{F}$.

Proof: We shall consider the case where $(E_n : n \geq 1)$ is a sequence of non-decreasing events.

The claim for the non-increasing sequence follows ~~by taking complementation~~ as complementation of a non-decreasing sequence of events leads to a sequence of non increasing events.

We know $E_n \uparrow E = \bigcup_{n\geq 1}^{\infty} E_n$. We have seen $E \in \mathcal{F}$ as $E_n \in \mathcal{F}$ for every $n \geq 1$. Define $\Delta_n = E_{n+1} \setminus E_n$ for every $n \geq 1$. Note that

$$E_n = E_1 \cup \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_{n-1} = E_1 \cup \bigcup_{i\geq 1}^{n-1} \Delta_i \rightarrow (*_1)$$

and E_1 is disjoint of the collection $(\Delta_i : i \geq 1)$ of disjoint events ($\Delta_i = E_{i+1} \setminus E_i = E_{i+1} \cap E_i^c \in \mathcal{F}$ as E_{i+1} and E_i are events). Combining these facts, we get

$$\mathbb{P}(E_n) = \mathbb{P}\left(E_1 \cup \bigcup_{i\geq 1}^{n-1} \Delta_i\right) = \mathbb{P}(E_1) + \sum_{i\geq 1}^{n-1} \mathbb{P}(\Delta_i) \rightarrow (*_2)$$

Now, we can take $\lim_{n\to\infty}$ at both the sides to get

$$\lim_{n\to\infty} \mathbb{P}(E_n) = \mathbb{P}(E_1) + \sum_{i\geq 1}^{\infty} \mathbb{P}(\Delta_i) = \mathbb{P}\left(E_1 \cup \bigcup_{i\geq 1}^{\infty} \Delta_i\right) \rightarrow (*_3)$$

using countable additivity of the probability.

It follows from $(*_1)$ that $E_1 \cup \bigcup_{n=1}^{\infty} \Delta_n = \bigcup_{n=1}^{\infty} E_n = E$.

Hence, the proof is complete. ▨

<u>Continuity from above</u>: Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ $(E_n : n \geq 1)$ be a sequence of events such that

$$E_n \downarrow E \quad \text{then} \quad \lim_{n \to \infty} \mathbb{P}(E_n) = \mathbb{P}(E)$$

<u>Continuity from below</u>: Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(E_n : n \geq 1)$ be a sequence of events such that

$$E_n \uparrow E \quad \text{then} \quad \lim_{n \to \infty} \mathbb{P}(E_n) = \mathbb{P}(E)$$

Lecture III Conditional probability and Baye's theorem.

Rough idea: $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We say that $A, B \in \mathcal{F}$ are independent events if the occurrence or non-occurrence of A does not influence the 'chance' or "odds" concerning the occurrence of the event B and vice-versa. If A and B are not independent, then occurrence of the event A influences or influenced by occurrence B. "Conditional probability" quantifies or measures the influence.

Example. Let us toss two fair coins

$$\Omega = \{HH, HT, TH, TT\}$$

Define $A = \{\text{both of the coins have same face}\} = \{HH, TT\}$

$B = \{\text{at least one of two faces is } H\} = \{HT, TH, HH\}$

Note that if we know that A has occurred, then the only sample point in favor of B is $\{H, H\}$. Similarly, occurrence of the event B restricts the sample points favoring the event A.

Definition (conditional probability)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $H \in \mathcal{F}$ such that $\mathbb{P}(H) > 0$. For any $A \in \mathcal{F}$, we write

$$\mathbb{P}(A \mid H) = \frac{\mathbb{P}(A \cap H)}{\mathbb{P}(H)} \longrightarrow (*_1)$$

and call $\mathbb{P}(A \mid H)$ the conditional probability of A given H.

> Important note: $\mathbb{P}(A \mid H)$ is undefined when $\mathbb{P}(H) = 0$

It is immediate to check that

$$\mathbb{P}(A \cap H) = \mathbb{P}(A \mid H)\,\mathbb{P}(H) = \mathbb{P}(H \mid A)\,\mathbb{P}(A) \rightarrow (*_2)$$

if $\min(\mathbb{P}(A), \mathbb{P}(H)) > 0$ or $\mathbb{P}(A \cap H) > 0$.

We have a new formula

$$\mathbb{P}(H \mid A) = \mathbb{P}(A \mid H)\,\frac{\mathbb{P}(H)}{\mathbb{P}(A)} \longrightarrow (*_3).$$

if $\mathbb{P}(A \cap H) > 0$.

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $H \in \mathcal{F}$ be such that $\mathbb{P}(H) > 0$. Then define a set function $\mathbb{P}_H$ ~~such that~~ $: \mathcal{F} \longrightarrow [0,1]$ such that

$\mathbb{P}_H(A) = \mathbb{P}(A \mid H)$ for all $A \in \mathcal{F}$.

Then, ~~we have~~ $(\Omega, \mathcal{F}, \mathbb{P}_H)$ is a probability space.

Proof. It is easy to check that

$$0 = \mathbb{P}(\phi \mid H) \le \mathbb{P}(A \mid H) \le \mathbb{P}(\Omega \mid H) = 1.$$

We only have to show that $\mathbb{P}$ satisfies countable additivity. Consider a sequence $(A_n : n \geq 1)$ of disjoint events, then

$$\mathbb{P}_H\left(\bigcup_{i\geq 1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i\geq 1}^{\infty} A_i \mid H\right) = \frac{\mathbb{P}\left(\left(\bigcup_{i\geq 1}^{\infty} A_i\right) \cap H\right)}{\mathbb{P}(H)}$$

$$= \frac{\mathbb{P}\left(\bigcup_{i\geq 1}^{\infty} (A_i \cap H)\right)}{\mathbb{P}(H)}.$$ $(A_i \cap H : i \geq 1)$ are disjoint as $(A_i : i \geq 1)$ are so.

$$= \sum_{i\geq 1}^{\infty} \frac{\mathbb{P}(A_i \cap H)}{\mathbb{P}(H)}$$ [using countable additivity property of probability]

$$= \sum_{i\geq 1}^{\infty} \mathbb{P}(A_i \mid H)$$ (By def.$^{\underline{n}}$ of conditional ~~prob~~ probability)

$$= \sum_{i\geq 1}^{\infty} \mathbb{P}_H(A_i).$$

Therefore, the proof is complete. δ.

Remark. Define $\mathcal{F}_H = \{E \cap H : E \in \mathcal{F}\}$. Show that $(H, \mathcal{F}_H, \mathbb{P}_H)$ is a probability space. Note that $\mathcal{F}_H$ is a smaller σ-fied than the original one. (WHY? Think about it).

In $*_2$ we derived a product formula for computing probability of intersection of two events in terms of ~~con~~ conditional probability. We can generalize this product formula for any finite collection events using a induction. The formula is given as follows.

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \left(\prod_{i=2}^{n} \mathbb{P}\left(A_i \,\middle|\, \bigcap_{j=1}^{i-1} A_j\right)\right) \mathbb{P}(A_1) \rightarrow **_4$$

This formula ~~turns~~ turns out to be very important to describe complex random experiment (MARKOV CHAIN!).

We now generalize the total probability rule for an infinite sequence of exhaustive and disjoint events.

<u>Total probability rule</u>

Consider a sequence of disjoint events $(A_i : i \geq 1)$ such that $\Omega = \bigcup_{i=1}^{\infty} A_i$. Assume that

$\mathbb{P}(A_i) > 0$ for every $i \geq 1$.

Then for any $B \in \mathcal{F}$, we have

$$\mathbb{P}(B) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)\, \mathbb{P}(B \mid A_j) \longrightarrow **_5$$

Theorem (Bayes rule)

Let $(A_n : n \geq 1)$ be sequence of disjoint events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We further assume that the sequence of events $(A_n : n \geq 1)$ to be exhaustive that is, $\Omega = \bigcup_{n \geq 1}^{\infty} A_n$. Let $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. Then

$$\mathbb{P}(A_n \mid B) = \frac{\mathbb{P}(A_n)\, \mathbb{P}(B \mid A_n)}{\sum_{n \geq 1}^{\infty} \mathbb{P}(A_n)\, \mathbb{P}(B \mid A_n)} \longrightarrow **6$$

* We generalized the previous Bayes formula for countably many disjoint and exhaustive events.

Independence of events

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. Then we have

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\, \mathbb{P}(A \mid B) \longrightarrow **7$$

Assume now that the occurrence of the event B does not influence the occurrence of the event A. Then we can expect to have $\mathbb{P}(A \mid B) = \mathbb{P}(A)$. Then the formula in **7 reduces to.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\, \mathbb{P}(B).$$

and now, we make it a definition of independent events.

Definition (Independent events).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and A and B are two events. We say A and B to be independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

* Note that we did not define independence of two events through conditional probability as conditional probability is not defined when conditioning event has probability zero.

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A \in \mathcal{F}$ and $B \in \mathcal{F}$, and ~~the~~ be independent. Then

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \text{ if } \mathbb{P}(B) > 0.$$

and $\mathbb{P}(B \mid A) = \mathbb{P}(B)$ if $\mathbb{P}(A) > 0$.

Exercise.

If A and B are independent events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then show that the following pairs of events are independent.

i) A and B^c.

ii) A^c and B

iii) A^c and B^c.

We now would like to generalize the concept of independence of an infinite (countable or uncountable) collection of events. The following definitions are very useful for that.

<u>Definition</u> (Pairwise independent events).

Let $\mathcal{U}$ be a family (countable or uncountable) of events from $\mathcal{F}$. We say that the events $\mathcal{U}$ are pairwise independent if for every pair of distinct events $A, B \in \mathcal{U}$,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

<u>Definition</u> (mutually or completely independent family events).

Let $\mathcal{U}$ be a family of events from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say $\mathcal{U}$ to be mutually or completely independent family if for every finite subcollection $\{A_1, A_2, \ldots, A_k\}$ of $\mathcal{U}$, the following relation holds:

$$\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) = \prod_{i=1}^{k} \mathbb{P}(A_i)$$

<u>Pairwise independent events vs. mutually independent events</u>

mutual independence implies pairwise independence.