Newton’s laws for a particle

Interpretation of Newton’s laws

1. Particle at rest or in moving along a straight line with constant velocity preserves that state unless it is acted upon by a force.
   - Concept of mass
   - Inertia is the manifestation of mass for general 3D kinematics

m Mass

2. Particle acted upon by a resultant force moves in such a way that the time rate of change of its linear momentum is equal to the force.
   - Concept of equilibrium
   - Dynamic equations of motion

Only ONE of

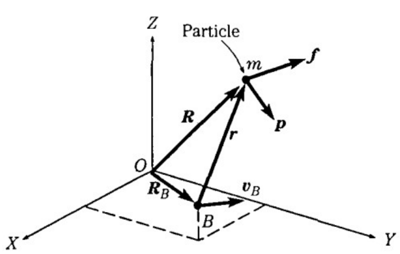
$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \quad \text{or} \quad \boldsymbol{T} = \frac{d\boldsymbol{L}}{dt}$$

Linear momentum   Angular momentum

3. Forces that result from interaction between particles are equal in magnitude and opposite in direction, and lies along the line joining the particles.
   - Constraints and reaction forces

Momentum

Force and linear momentum



$\boldsymbol{v}_B$ is velocity of B in stationary frame O

Motion of particle P with mass 'm' w.r.t observer at point 'B'

Newton's second law

$$\sum \boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

Where, $\boldsymbol{p} = m(\boldsymbol{v} - \boldsymbol{v}_B)$

Where

$\boldsymbol{F} \equiv \boldsymbol{F}(t)$

$\boldsymbol{p} \equiv \boldsymbol{p}(t)$

$\boldsymbol{v} \equiv \boldsymbol{v}(t) = \frac{d\boldsymbol{R}}{dt}$

$\boldsymbol{v}_B \equiv \boldsymbol{v}_B(t) = \frac{d\boldsymbol{R}_B}{dt}$

Note that

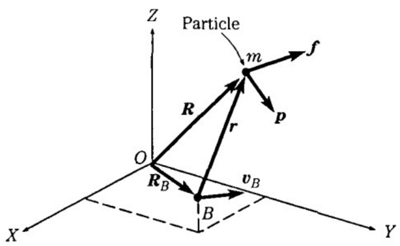
$$\frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(m\boldsymbol{v} - m\boldsymbol{v}_B)$$

If m is constant w.r.t time.

$$= m\boldsymbol{a} - \underbrace{m\boldsymbol{a}_B}_{}$$

= 0 only if point B is at rest or is moving with constant velocity.

Torque and angular momentum



Motion of particle P with mass 'm' w.r.t observer at point 'B'

Newton's second law

$$\boldsymbol{T} = \boldsymbol{r} \times (\Sigma \boldsymbol{F}) = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) - \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p}$$

$\boldsymbol{L}$ is angular momentum

$$= \frac{d\boldsymbol{L}}{dt} - \left(\frac{d\boldsymbol{R}}{dt} - \frac{d\boldsymbol{R}_B}{dt}\right) \times \boldsymbol{p}$$

$\boldsymbol{v}_B$ is velocity of B in stationary frame O

since $\boldsymbol{r} = \boldsymbol{R} - \boldsymbol{R}_B$, it follows that $\frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{R}}{dt} - \frac{d\boldsymbol{R}_B}{dt} = \boldsymbol{v} - \boldsymbol{v}_B$

Also, $\boldsymbol{v} \times \boldsymbol{p} = \boldsymbol{v} \times m\boldsymbol{v} = 0$

Therefore $\boldsymbol{T} = \frac{d\boldsymbol{L}}{dt} + \underbrace{\boldsymbol{v}_B \times \boldsymbol{p}}_{\text{vanishes only if point B is moving with a constant velocity parallel to } \boldsymbol{v}}$

For a particle, torque-angular momentum relation does not give us anything new.

Extension to bodies and systems of particles

Euler extended Newton's laws to systems of particles.

Force-linear momentum relation when $\boldsymbol{v}_B = \boldsymbol{0}$:

$$\boldsymbol{p}_{sys} = \sum_{i=1}^{n} \boldsymbol{p}_i = \sum_{i=1}^{n} m_i \boldsymbol{v}_i = M \boldsymbol{v}_{cm}$$

where

$$M = \sum_n m_i \quad \text{Total mass}$$

$$\boldsymbol{v}_{cm} = \frac{\sum_n m_j \boldsymbol{v}_j}{\sum_n m_j} \quad \text{Velocity of center of mass}$$

Rate of change of momentum

$$\frac{d\boldsymbol{p}_{sys}}{dt} = \underbrace{\sum_{i=1}^{n} \boldsymbol{F}_i^{ext}}_{\text{Externally applied}} + \underbrace{\sum_{i=1}^{n} \boldsymbol{F}_i^{int}}_{\text{Internal reactions and constraints}}$$

$\sum_{i=1}^{n} \boldsymbol{F}_i^{int} = 0$ since inter-particle constraint and reaction forces disappear due to Newton's 3rd law

Final equation

$$\frac{d\boldsymbol{p}_{sys}}{dt} = \boldsymbol{F}^{ext} = M\boldsymbol{a}_{cm}$$

Extension to bodies and systems of particles

Euler extended Newton's laws to systems of particles.

Torque-angular momentum relation when $v_B = 0$:

$$\boldsymbol{L}_{sys} = \sum_{i=1}^{n} \boldsymbol{L}_i = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{p}_i$$

Therefore, $$\frac{d\boldsymbol{L}_{sys}}{dt} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \frac{d\boldsymbol{p}_i}{dt} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \left(\boldsymbol{F}_i^{ext} + \boldsymbol{F}_i^{int}\right)$$

From Newton's third law, we again have $$\sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i^{int} = 0$$

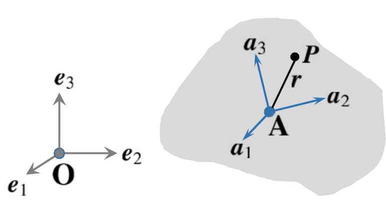
Also, $$\sum_{i=i}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i^{ext} = \sum_{i=i}^{n} \boldsymbol{T}_i^{ext} = \boldsymbol{T}^{ext}$$

Therefore, $$\frac{d\boldsymbol{L}_{sys}}{dt} = \boldsymbol{T}^{ext}$$

General expression describing Newton's second law for angular momentum

Further on angular momentum for a rotating rigid body

- Newton's 2nd law holds in any reference frame.
- Since the rate of change of angular momentum and torque contain a ($\boldsymbol{r}$ x) term, both $\boldsymbol{L}$ and $\boldsymbol{T}$ depend on the specific reference frame chosen to describe the systems of particles.
- It is practically convenient to enforce Newton's second law for rotating bodies using a body fixed reference frame.



- Consider a rigid body rotating in space. Ignore translation for now (can be added in later quite easily).
- Frame **A** is fixed to the body and moves with it. Not necessarily located at the center of mass.
- Angular velocity of the body in frame **A** is $\boldsymbol{\omega} = [\omega_1\ \omega_2\ \omega_3]^T$
- Point $\boldsymbol{P}$ is a particle with mass dm has position vector $\boldsymbol{r}$ in **A**.

Velocity of $\boldsymbol{P}$ with respect to fixed observer (**O**), but referred to **A** is $(\boldsymbol{\omega} \times \boldsymbol{r})$ (transport theorem)

$\boldsymbol{L}$ for a rotating rigid body

Momentum of the particle $\quad d\boldsymbol{p} = dm\,(\omega \times \boldsymbol{r})$

Angular momentum $\quad d\boldsymbol{L} = \boldsymbol{r} \times d\boldsymbol{p} = dm\,[\boldsymbol{r} \times (\omega \times \boldsymbol{r})]$

Vector cross-product $\quad \boldsymbol{r} \times (\omega \times \boldsymbol{r}) = (\boldsymbol{r} \cdot \boldsymbol{r})\,\omega - (\omega \cdot \boldsymbol{r})\,\boldsymbol{r}$

For planar rotations, $\quad \omega \perp \boldsymbol{r} \implies \omega \cdot \boldsymbol{r} = \boldsymbol{0}$ $\qquad$ Not so for a general 3D rotation

Angular momentum of the rigid body is then obtained as an integral

$$\boldsymbol{L} = \int_m d\boldsymbol{L} = \int_m dm\,[\boldsymbol{r} \times (\omega \times \boldsymbol{r})] = \int_m dm\,[(\boldsymbol{r} \cdot \boldsymbol{r})\,\omega - (\omega \cdot \boldsymbol{r})\,\boldsymbol{r}]$$

Let us simplify by switching to an algebraic notation

Simplify…

Let $\quad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$ and $\boldsymbol{r} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad$ be components of vectors expressed in **A**, using triad ($\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}$)

Angular momentum $\quad \boldsymbol{L} = L_1\,\boldsymbol{a_1} + L_2\,\boldsymbol{a_2} + L_3\,\boldsymbol{a_3}$

Substitute $\boldsymbol{\omega}$ and $\boldsymbol{r}$ into $\quad \boldsymbol{L} = \int_m dm\,[(\boldsymbol{r} \cdot \boldsymbol{r})\,\omega - (\omega \cdot \boldsymbol{r})\,\boldsymbol{r}]$ $\quad$ and simplify

Equating coefficients of $\boldsymbol{a_i}$ where $i = 1, 2, 3$, on both sides, we obtain

$$L_i = \omega_i \int_m \left(x_1^2 + x_2^2 + x_3^2\right) dm + \int_m x_i\,(\omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3)\,dm$$

Simplify…

Rearranging terms, we obtain
$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$
or, in vector notation $\boldsymbol{L} = \boldsymbol{I}\,\boldsymbol{\omega}$

Here, $\boldsymbol{I}$ is the inertia tensor

Moments of inertia
$$I_{11} = \int_m \left(x_2^2 + x_3^2\right) dm \qquad I_{22} = \int_m \left(x_1^2 + x_3^2\right) dm \qquad I_{33} = \int_m \left(x_1^2 + x_2^2\right) dm$$

Products of inertia
$$I_{12} = I_{21} = \int_m x_1\, x_2 dm \qquad I_{23} = I_{32} = \int_m x_2\, x_3 dm \qquad I_{13} = I_{31} = \int_m x_1\, x_3 dm$$

A measure of imbalance in mass distribution

Note $\boldsymbol{L} \equiv \boldsymbol{L}_{sys}$ therefore $\dfrac{d\boldsymbol{L}_{sys}}{dt} = \dfrac{d}{dt}(\boldsymbol{I} \cdot \boldsymbol{\omega})$

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Some notes

- Position vector of point $\boldsymbol{P}$ in frame **O** is not constant with respect to time. Therefore, the resultant inertia tensor is time dependent.
- Preferred approach is to use a frame is fixed to the rigid body and rotating with it, so that $\boldsymbol{I}$ is a constant matrix.

For **A** moving with the body, Newton's 2nd law becomes
$$\boldsymbol{T}^{ext} = \frac{d\boldsymbol{L}_{sys}}{dt} = \frac{d}{dt}(\boldsymbol{I} \cdot \boldsymbol{\omega}) = \boldsymbol{I} \cdot \boldsymbol{\alpha} + \boldsymbol{\omega} \times (\boldsymbol{I} \cdot \boldsymbol{\omega})$$

$\boldsymbol{\alpha}$ is angular acceleration, same in $\boldsymbol{A}$ and $\boldsymbol{O}$

- In the above, even though $\boldsymbol{I}$ is constant in time, it is measured w.r.t a moving frame **A**. Therefore the transport theorem has to be invoked for the time derivative.
- Origin at the center-of-mass (CM) of the body is a special case of the body fixed frame.
- Expressions for $\boldsymbol{p}$ and $\boldsymbol{L}$ have to re-derived by considering $dm/dt \neq 0$ if mass changes with respect to time (due to fuel burn, for example).

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D’Alembert’s principle

D’Alembert’s principle

Newton’s 2nd law:

Rectilinear motion $\dot{\boldsymbol{p}}_{sys} = \boldsymbol{F}^{ext}$

Angular motion $\dot{\boldsymbol{L}}_{sys} = \boldsymbol{T}^{ext}$

- Newton’s 2nd law quantifies the reaction of a mass to applied forces and moments.
- Let us drop the subscript ‘*sys*’ and superscript ‘*ext*’ for convenience.
- A subtle but important concept due to D’Alembert is the interpretation of $-d\boldsymbol{p}/dt$ and $-d\boldsymbol{L}/dt$ as ‘inertial’ force and moment.
- This enables us to express Newton’s 2nd law as a statement of balance of forces on a massless body.

$$\boldsymbol{F} - \dot{\boldsymbol{p}} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{T} - \dot{\boldsymbol{L}} = \boldsymbol{0}$$

D'Alembert's principle

$$\boldsymbol{F} - \dot{\boldsymbol{p}} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{T} - \dot{\boldsymbol{L}} = \boldsymbol{0}$$

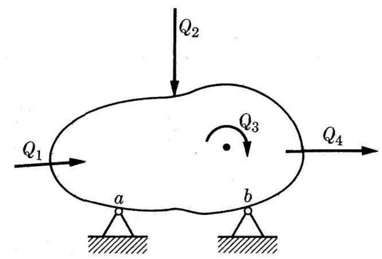
- The above equations are formally called the '*equations of dynamic equilibrium*' or '*equations of motion*'.
- In several situations of practical interest, the terms LHS of the above equations is sometimes called the *effective force* (or *moment*).
- It is convenient to combine the two sets of equations into a single set of generalized forces, momenta, displacements, and velocities.
- Following the change in representation, we can now apply concepts originally developed for analysis of static equilibrium to the dynamic setting.
- We first start with the principle of virtual work.

Principle of virtual work

Principle of Virtual Work

"*If a body is in equilibrium and remains in equilibrium while it is subjected to a virtual distortion compatible with the geometric constraints, the virtual work done by the external forces is equal to the virtual work done by the internal stresses.*"

Assume that a body is acted upon by a set of 'n' generalized external forces $\boldsymbol{Q}_m$, $m = 1, 2, \ldots, n$.



The action of these forces produces an internal stress state $\boldsymbol{\sigma}^Q$ in the body.

Now assume that a virtual displacement field $\delta \boldsymbol{u}$ is imposed on the body.

The word 'virtual' implies that this displacement field is not related to the externally applied forces or the internal stress field. It is purely arbitrary and satisfies only the geometric constraints that are imposed on the body.

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Principle of Virtual Work

Let δU be the work done by the internal stresses due to the virtual displacement $\delta \boldsymbol{u}$. Then,

$$\delta U = \int_{\text{vol}} \boldsymbol{\sigma}^Q : \delta \boldsymbol{\epsilon} \, dV$$

$\delta \boldsymbol{\epsilon}$ is the strain field produced by $\delta \boldsymbol{u}$.

Let the virtual displacements, produced by the imposed displacement field $\delta \boldsymbol{u}$, at the locations where forces $\boldsymbol{Q}_m$ act be denoted by $\delta \boldsymbol{q}_m$, $m = 1, 2, \ldots, n$, respectively.

Then, the mathematical statement of the principle of virtual work is expressed as

$$\delta U = \boldsymbol{Q}_m \cdot \delta \boldsymbol{q}_m$$

repeated indices imply summation

It is important to understand here that the <u>above equation has to be satisfied for every admissible virtual displacement field that can be imposed on the body</u>.

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PVW for a single particle with fixed mass

In practice the following interpretation of PVW is used.

The preferred or natural displacement field of a body in equilibrium is such that the internal stresses produced by the deformation field and the externally applied loads satisfy (δU - $\boldsymbol{Q}_m$. $\delta \boldsymbol{q}_m$) = 0 for every arbitrary admissible virtual displacement field $\delta \boldsymbol{u}$.

How does this work for the dynamics of a single particle with fixed mass '*m*'?

Suppose the particle is acted upon by a force **F**(t). Following the PVW, the natural or preferred path **r**(t) is one for which

$$(\boldsymbol{F} - \dot{\boldsymbol{p}}) \cdot \delta \boldsymbol{r} = 0 \qquad \text{for all admissible } \delta \boldsymbol{r}.$$

Note that $\boldsymbol{p} = m\dot{\boldsymbol{r}}$ For m = constant, the above expression becomes

$$(\boldsymbol{F} - m\ddot{\boldsymbol{r}}) \cdot \delta \boldsymbol{r} = 0$$

This expression is also termed as the generalized D'Alembert's principle for a single particle

PVW for a system of N particles, each with fixed mass

Let us consider a system of N particles, each with mass m_i and acted upon by force $\boldsymbol{F}_i$. The preferred path $\mathbf{r}_i$ satisfies the following equation:

$$\sum_{i=1}^{N} (\boldsymbol{F}_i - m_i \ddot{\boldsymbol{r}}_i) \cdot \delta \boldsymbol{r}_i = 0 \qquad \text{for all admissible } \delta \boldsymbol{r}_i$$

Where, it is understood that $\boldsymbol{F}_i$, $\boldsymbol{r}_i$, and $\delta \boldsymbol{r}_i$ are functions of time.

The above expression is the generalized D'Alembert's principle for a system of particles.

The above expression assumes that constraints, if present, are workless in nature, i.e., the virtual work due to constraints is zero.

The above expression provides the starting point for us to step into analytical mechanics and obtain the Lagrange's equations of motion.

Going from Newtonian to analytical mechanics

The goal of analytical mechanics

- **Force and momentum** are the fundamental quantities in Newtonian mechanics, also called vectorial mechanics. The analysis of forces and moments is the basic concern.
- Euler and Lagrange are the original creators of what is known today as analytical mechanics. The name stems from the philosophy that it is the analysis of mechanics or the analytical viewpoint of mechanics.
- A particle is no longer treated as an isolated unit but as part of a ‘system’. Thus, we now consider a mechanical system as an assembly of particles and aim for a more unified treatment.
- The goal is to find a **unifying principle** that describes the behavior of the entire set of particles rather than each particle individually. As a consequence, we avoid the need to describe all of the inter-particle interactions, which can get cumbersome and complicated, and focus on only those that contribute in some manner to the overall behavior of the system.
- The statement of the principle is independent of any special set of coordinates. Therefore, the analytical equations of motion are also invariant with respect to any coordinate transformations.

Principal differences between Newtonian and analytical mechanics

- When we move to analytical mechanics, force and momentum are replaced with scalars – **work function** and **kinetic energy** (originally due to Leibnitz).
- This change of fundamental quantities from vectorial to scalars quantities is quite important and non-intuitive since motion inherently involves direction.
- As we shall see, the fundamental scalars contain the complete dynamics of even extremely complicated systems. They are used as the basis of a principle rather than an equation.
- Unlike vectorial mechanics, which is geometric and intuitive in nature, analytical mechanics is abstract and purely mathematical.
- While coordinates are required to establish a one-to-one correspondence between the physical world and abstract algebraic quantities, all manipulations and operations are performed on the algebraic quantities without reference to their physical meaning.

The kinetic energy state function

The kinetic energy is defined as the word done on a particle to increase its momentum from 0 to ***p*.**

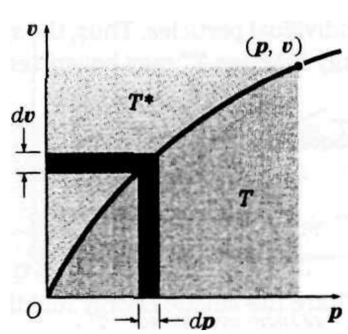
$$T(\boldsymbol{p}) = \int_0^{\boldsymbol{p}} \boldsymbol{v} \cdot d\boldsymbol{p}$$

Where ***v*** and ***p*** are the velocity vector and momentum, respectively

The complementary kinetic energy state function, also called the kinetic co-energy function, is defined as

$$T^*(\boldsymbol{v}) = \int_0^{\boldsymbol{v}} \boldsymbol{p} \cdot d\boldsymbol{v}$$

The above definitions are independent of the relation between ***p*** and ***v***.



The kinetic energy state function for specific cases: Newtonian particles

For Newtonian particles, $\boldsymbol{p} = m\boldsymbol{v}$, where the mass is independent of velocity and momentum. For this case,

$$T(\boldsymbol{p}) = \int_0^{\boldsymbol{p}} \frac{1}{m}\boldsymbol{p}\cdot d\boldsymbol{p} = \frac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} = \frac{1}{2}m\,\boldsymbol{v}\cdot\boldsymbol{v}$$

The complementary kinetic energy state function is identical to the kinetic energy state function:

$$T^*(\boldsymbol{v}) = \int_0^{\boldsymbol{v}} m\,\boldsymbol{v}\cdot d\boldsymbol{v} = \frac{1}{2}m\,\boldsymbol{v}\cdot\boldsymbol{v}$$

For distributed masses, we define the kinetic energy state function as a density

$$T = \int_V \frac{1}{2}dm\,\boldsymbol{v}\cdot\boldsymbol{v}\,dV$$

Where, V is the volume

Work function

The work function is defined as the work done on a body by a force field $\boldsymbol{f}$ as the body moves from $\boldsymbol{r_0}$ to $\boldsymbol{r}$

$$W(\boldsymbol{r}) = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{r})\cdot d\boldsymbol{r}$$

This definition includes all types of forces acting on the body.

The force field is characterized as conservative if the <u>work done on a body along any closed path is zero</u>. For such cases, the work function is termed as ‘potential energy’ and is defined as

$$V(\boldsymbol{r}) = -\int_{\boldsymbol{r}_0}^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{r})\cdot d\boldsymbol{r}$$

The negative sign indicates that potential energy decreases if the field does work on the body.

The potential energy is a state function. The right hand side is integrable and produces an explicit form that is called as the *potential*.

END