용어 정리.

linear equation: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ 의 형태
(일차 방정식)

coefficient : a_i (계수)

linear system : 연립 일차 방정식
= system of linear equations

solution : $(s_1, s_2, \cdots, s_n)$
$(x_1, x_2, \cdots, x_n)$ 으로 가능한 답.

solution set : solution 들의 집합.

equivalent : 두 linear system의 solution set이 같다면
두 linear system은 equivalent 하다고 한다.

A system of linear equations has...

1. no solution, or → Inconsistent
2. exactly one solution, or
3. infinitely many solutions.

] → consistent

Matrix Notation.

linear system can be recorded matrix

linear system :

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
5x_1 \quad - 5x_3 &= 10.
\end{aligned}
$$

$\Downarrow$

coefficient matrix. $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$

augmented matrix $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5. & 10 \end{bmatrix}$

Elementary row operations.

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

row equivalent : 일련의 row operations 으로 한 행렬을 다른 행렬로 바꿀 수 있다면. 그 두 행렬을 row equivalent 하다고 한다.

만약 두 augmented matrix가 row equivalent 하다면 두 연립방정식은

같은 solution set을 가진다.

Two fundamental questions about a linear system

1. system이 consistent 한가? → 적어도 하나의 해가 존재하는가?
2. 해가 존재한다면, 정확히 한개인가? → 해가 유일한가?

↳ 이를 판단하기 위해 삼각형 형태의 row equivalent 한 matrix를 찾아본다.

floating point arithmetic : 부동 소수점.

Practice Problem.

1. a. In first step interchange equations 3 and 4.
   And then replace equation 4 by its sum −5 times row 3.
   Then we can get triangular form system.
   After that. we can eliminate x_4 on equations 1, 2 and 3.
   by appropriately adding row 4.
   And we can eliminate x_3 on equations 1, 2 by same method.
   Throught same method we can solve the problem.

   b. It is already triangular form system. So we can solve it easily.
   We can eliminate x_4 on equation 1 by its sum 2times row 4.
   And we can eliminate x_3 on equations 1 and 2 by its

sum -4 times and -2.5 times row 2.

In the same way, we can eliminate x_2 on equation 1

2. The system is consistent.

Its linear system is : $x_1 + 5x_2 + 2x_3 = -6$

$4x_2 - 7x_3 = 2$

$5x_3 = 0$

we can know $x_3 = 0$. And through this fact, we can know x_2 and x_1.

3. equation 1. $5 \cdot 3 - 4 + 2 \cdot (-2)$

$= 15 - 4 - 4 = 7$. (O)

equation 2. $-2 \cdot 3 + 6 \cdot 4 + 9 \cdot (-2)$

$= -6 + 24 - 18 = 0$ (O)

equation 3. $-7 \cdot 3 + 5 \cdot 4 - 3 \cdot (-2)$

$= -21 + 20 + 6 = 5$. (X)

$(3, 4, -2)$ is not a solution.

4. Considering only the left side on equations,

equation 2 $= -3$ equation 1.

So only when $k = -3h$, the system is consistent.

EXERCISES.

1. $$\begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$$

$\Rightarrow x_1 = -8. \quad x_2 = 3.$

2. $$\begin{bmatrix} 2 & 4 & -4 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 5 & 7 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & -7 \end{bmatrix}$$

$\Rightarrow x_1 = 12. \quad x_2 = -7$

3. $$\begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & \frac{9}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{4}{7} \\ 0 & 1 & \frac{9}{7} \end{bmatrix}$$

$(x_1, x_2) = \left(\frac{4}{7}, \frac{9}{7}\right).$

4. $$\begin{bmatrix} 1 & -5 & 1 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 1 \\ 0 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{9}{4} \\ 0 & 1 & \frac{1}{4} \end{bmatrix} \Rightarrow \left(\frac{9}{4}, \frac{1}{4}\right)$$

5. The matrix is already triangular form

We can know $x_3 = 2$ and $x_4 = -5$ through row 3 and 4.

We can eliminate x_3 on equation 2 by its sum 2times row 3.

In the same way, we can get x_1 by adding row 2 and 3 properly to row 1.

6. Replace equation 4 by its sum 3times row 3.

Then we can know x_4 in row 4.

After that we can eliminate x_4 on equations 1, 2 and 3. by appropriately adding row 4.

Then we can see x_3 in row 3.

In the same way, we can eliminate x_3 on equations 1 and 2, and get x_2, and eliminate x_2 on equation 1 and finally, we can get x_1, too.

7. The linear system has no solution.

because equation 3 is ...

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$

$\rightarrow 0 = 1$

the equation $0=1$ is never true so there is no solution.

8. On equation 3, $x_3 = 0$

then on equation 2, $x_2 = 0$

In the same way, on equation 1, $x_1 = 0$

$\Rightarrow$ The linear system has only $(0, 0, 0)$

9. $\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ $\Rightarrow$ $x_1 = 4,\ x_2 = 8,\ x_3 = 5,\ x_4 = 2$

10. given matrix $\Rightarrow$ $\begin{bmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$ ← adding -3 times row 4

← adding 4 times row 4.

$= \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$ $\Rightarrow$ $x_1 = -3,\ x_2 = -5,\ x_3 = 6,\ x_4 = -3$

11. $\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 0 & -2 & -8 & 12 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 0 & -7 & 13 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $\Rightarrow$ equation 3 is never true.

$\Rightarrow$ The system has no solution.

12. $$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 1 & -\frac{5}{2} & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \rightarrow$$ The equation 3 is never true

$\Rightarrow$ the system has no solution

13. $$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 0 & 5 & -5 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

$\Rightarrow x_1 = 5,\ x_2 = 3,\ x_3 = -1.$

14. $$\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 0 & 7 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$\Rightarrow x_1 = 2,\ x_2 = -1,\ x_3 = 1$

15. $\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$

The system is consistent.

16. $\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The equation 4 is always true.

$\Rightarrow$ The system is consistent.

17. A common point of intersection is same with a solution of linear system

So. represent linear system as argument matrix..

$$\begin{bmatrix} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{bmatrix}$$

equation 2 is $7x_2 = -5$. and equation 3 is $-7x_2 = 5$

The equations 2 and 3 are same.

So. It can represent as follows.

$$\begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & -\frac{5}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{13}{7} \\ 0 & 1 & -\frac{5}{7} \end{bmatrix}$$

$\Rightarrow$ commonpoint is $(x_1, x_2) = (-\frac{13}{7}, -\frac{5}{7})$

18. It can represent by argument matrix as follows.

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

The equation 3 is $0 = -5$. and it is never true.

So. the three planes don't have any common point of interection.

19. $$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & h & 4 \\ 0 & 6-3h & -4 \end{bmatrix}$$

The equation 2 is $(6-3h)x_2 = -4$.

To make the equation 2 true, $6-3h \neq 0 \Leftrightarrow 3h \neq 6 \Leftrightarrow h \neq 2$.

$\therefore h \neq 2$.

20. $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & h & -3 \\ 0 & 4+2h & 0 \end{bmatrix}$

The equation 2 is $(4+2h)x_2 = 0$

If $4+2h = 0$, then the equation 2 is $0=0$. It is always true.

So, the linear system is consistent.

If $4+2h \neq 0$, the the equation 2 is $(4+2h)x_2 = 0$, and the solution

is $x_2 = 0$. In this case, the linear system is

also consistent.

So, for any h, the linear system is consistent.

21. $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 \\ 0 & h+12 & 0 \end{bmatrix}$

For the same reason as Exercise 20, for any h, the linear system is consistent

22. $\begin{bmatrix} 2 & -3 & h \\ -6 & 9 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & h \\ 0 & 0 & 5+3h \end{bmatrix}$

The equation 2 is $0 = 5+3h$

To make the equation 2 true h can be only $-\frac{5}{3}$

$\therefore h = -\frac{5}{3}$

23. a. True. On page 6. "It is important to note that row operations are reversible."

b. False, On page 4, "An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)"

c. False, On page 3. "A solution (is not solution set) of the system is a list $(s_1, s_2, \dots, s_n)$ ..."

d. True. On page 7. refer "Two Fundamental questions about a Linear System"

24. a. True On page 7. "If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set."

b. False On page 6. "Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other."

c. False, On page 4, "A system is inconsistent if it has no solution"

d. True, On page 3, "Two linear systems are called equivalent if they have the same solution set."

25.

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} = \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & 2g+k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & 2g+h+k \end{bmatrix}$$

To make the equation 3 true, $0 = 2g + h + k$.

$\therefore 2g + h + k = 0$

26. $\left[\begin{array}{ccc:c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \cdot \left[\begin{array}{ccc:c} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \cdot \left[\begin{array}{ccc:c} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

27. $\left[\begin{array}{cc:c} 1 & 3 & f \\ c & d & g \end{array}\right] \cdot \left[\begin{array}{cc:c} 1 & 3 & f \\ 0 & d-3c & g-c\cdot f \end{array}\right]$

The equation 2 is $(d-3c)x_2 = g-c\cdot f$

if $d-3c=0$ then, $0=g-c\cdot f \Leftrightarrow g=c\cdot f$

It isn't always true.

if $d-3c \neq 0$ then. $x_2 = \dfrac{g-c\cdot f}{d-3c}$

It always has one solution.

$\rightarrow d \neq 3c$.

28. $\left[\begin{array}{cc:c} a & b & f \\ c & d & g \end{array}\right] = \left[\begin{array}{cc:c} a & b & f \\ 0 & d-\frac{c}{a}\cdot b & g-\frac{c}{a}\cdot f \end{array}\right]$

The equation 2 is $(d-\frac{c}{a}\cdot b)\,x_2 = g-\frac{c}{a}\cdot f$

$d-\frac{c}{a}\cdot b$ must be nonzero, since f and g are arbitrary.

$d-\frac{c}{a}\cdot b \neq 0$

$d \neq \frac{c}{a}\cdot b$

$ad \neq c\cdot b$

$ad-bc \neq 0$

29. Interchange row 1 and 2. / Interchange row 1 and 2.

30. Multiply row 2 by $-\frac{1}{2}$ / Multiply row 2 by -2.

31. Replace row 3 by its sum -4 times row 1.

Replace row 3 by its sum 4 times row 1.

32. Replace row 3 by its sum 3 times row 2.

Replace row 3 by its sum -3 times row 2.

33. $4T_1 - T_2 - T_4 = 30$ — ①

$T_2 = (T_1 + 20 + 40 + T_3)/4$

$\leftrightarrow -T_1 + 4T_2 - T_3 = 60$ — ②

$T_4 = (10 + T_1 + T_3 + 30)/4$

$\leftrightarrow -T_1 - T_3 + 4T_4 = 40$ — ③

$T_3 = (T_4 + T_2 + 40 + 30)/4$

$\leftrightarrow -T_2 + 4T_3 - T_4 = 70$ — ④

34.
$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 4 & 0 & -4 & 20 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & -4 & 14 & 195 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{75}{4} \\ 0 & 0 & 0 & 12 & 270 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{75}{4} \\ 0 & 0 & 0 & 1 & \frac{45}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & \frac{55}{2} \\ 0 & 0 & 1 & 0 & \frac{120}{4} \\ 0 & 0 & 0 & 1 & \frac{45}{2} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & \frac{55}{2} \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & \frac{45}{2} \end{bmatrix} \rightarrow
$$

$T_1 = 20$

$T_2 = \frac{55}{2}$

$T_3 = 30$

$T_4 = \frac{45}{2}$