1.2. ROW REDUCTION AND ECHELON FORMS.

nonzero row or column : a row or column that contains at least one nonzero entry.

leading entry : The leftmost nonzero entry.

사다리꼴 행렬 (A rectangular matrix is in echelon form) 조건:

1. 모든 nonzero row들은 모두 0인 row 위에 있어야 한다.
2. 각 행의 leading entry는 윗 행들의 leading entry보다 오른쪽 열에 있어야 한다.
3. 한 열에서 leading entry 아래에 있는 값들은 전부 0이어야 한다.

추가로 아래 2조건을 만족하면 그 matrix는 reduced echelon form 이라고 한다.

4. 각 nonzero row의 leading entry 의 값은 1이다.
5. 각 leading 1은 그 열에서 유일하게 0이 아닌 값이다.

. In fact, Property 3 is a simple consequence of property 2, but we include it for emphasis.

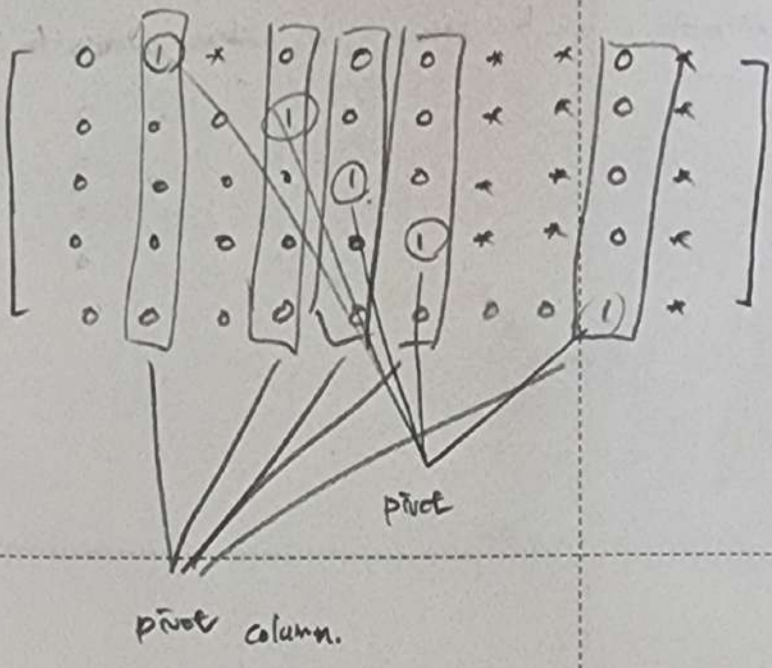
. The "triangular" matrices are in echelon form.

THEOREM 1. Uniqueness of the Reduced Echelon Form.

Each matrix is row equivalent to one and only one reduced echelon matrix.

However, each matrix may be row reduced into more than one matrix in echelon form.

RREF (Reduced Row Echelon Form)



The Row Reduction Algorithm.

STEP1. 가장 왼쪽에 있는 nonzero column을 찾는다. 이 column이 pivot column이 된다.
pivot의 위치는 해당 column의 맨 위로 잡는다.

STEP2. pivot column에서 0이 아닌 값을 선택하고 맨 위의 row와 바꾼다.

STEP3. row replacement operation 작업을 통해 pivot 아래에 모든 값을 0으로 만든다.

STEP4. pivot이 있는 row를 무시한다. step 1-3을 남은 submatrix에 nonzero row가 없어질 때까지 반복한다.

REF algorithm.

STEP5. 가장 오른쪽 pivot 부터 pivot의 위에 위치한 값들을 0으로 바꾼다.
만약 pivot이 1이 아니면 1로 scaling operation을 한다.

RREF algorithm.

NUMERICAL NOTE.

· STEP 2에서, 보통 컴퓨터 프로그램에서는 절대값이 가장 큰 값이 있는 row를 선택한다. 이 전략을 partial pivoting이라고 하는데, 이 경우 roundoff error가 적게 일어난다

In the REF,

basic variables : the variables corresponding to pivot columns.

free variables = the other variables.

RREF를 구하고나면, basic variable들이 다 한번씩만 나오는 linear system으로 바꿀수 있다. free variable들은 어떠한 값이 와도 상관 없으며, free variable들을 모두 결정하고 나서야 basic variable들을 구할 수 있다.

REF가 답을 구할 때 비효율적일지는 몰라도, Existence and Uniqueness Questions를 답하기에는 충분하다.

THEOREM 2. Existence and Uniqueness Theorem

existence : 맨 오른쪽 열이 pivot column이 아니어야 한다.

$\Leftrightarrow$ $[\,0 \dots 0 \;\; b\,]$ with b nonzero 인 행이 존재하지 않는다.

uniqueness : free variable이 존재하지 않는다.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM.

1. 연립방정식을 argumented matrix로 표현한다.
2. row reduction algorithm을 사용하여 echelon matrix를 만든다.
   system이 consistent한지 결정한다.
   - 만약, 해가 없다면 끝낸다
   - 그렇지 않으면 다음 step을 계속한다.
3. row reduction을 통해 reduced echelon form으로 만든다
4. matrix를 연립방정식으로 표현한다.
5. basic variable들을 free variable들로 나타낸다.

PRACTICE PROBLEMS.

1. $\begin{bmatrix} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

$\begin{cases} x_1 = 8x_3 - 3 \\ x_2 = x_3 - 1 \\ x_3 \text{ is free} \end{cases}$

2. $\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

the equation 3 is $0 = 5$, it is never true.

$\therefore$ the system is inconsistent.

3. The system is consistent and there are infinitely many solutions

(Note that the matrix is a "coefficient" matrix, not a "argumented" matrix)

1.2 EXERCISES.

1. a. reduced echelon form

   b. reduced echelon form.

   c. not in echelon form.

   d. only in echelon form.

2. a. reduced echelon form.

   b. only in echelon form.

   c. not in echelon form.

   d. only in echelon form.

3. $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns are columns 1 and 2.

4. $$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot columns are columns 1, 2 and 4

5. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}$ or $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$

6. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

$$\begin{cases} x_1 = -3x_2 - 5 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

8. $\begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & -1 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

$$\begin{cases} x_1 = -9 \\ x_2 = 4 \\ x_3 \text{ is free} \end{cases}$$

9. $\begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix}$

$$\begin{cases} x_1 = 5x_3 + 4 \\ x_2 = 6x_3 + 5 \\ x_3 \text{ is free} \end{cases}$$

10. $\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 1 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -4 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

$$\begin{cases} x_1 = 2x_2 - 4 \\ x_2 \text{ is free} \\ x_3 = -7. \end{cases}$$

11. $\begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

12. $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{cases} x_1 = 7x_2 - 6x_4 + 5 \\ x_2 \text{ is free} \\ x_3 = 2x_4 - 3 \\ x_4 \text{ is free} \end{cases}$$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = 3x_5 + 5 \\ x_2 = 4x_5 + 1 \\ x_3 \text{ is free} \\ x_4 = -9x_3 + 4 \\ x_5 \text{ is free} \end{cases}$$

14. $$\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = -7x_3 - 9 \\ x_2 = 6x_3 + 3x_4 + 2 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$$

15. a. The system is consistent. and the solution is unique

b. The system is inconsistent.

16. a. The system is consistent and the solution is unique

b. The system is consistent and the solution isn't unique.

17. $$\begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$$

Equation 2 is $0 = 7-2h$.

$\therefore h = \frac{7}{2}$

18. $\begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix}$

equation 2 is $(h+15)x_2 = 3$.

If $h+15=0$, then the equation is $0=3$ and it's never true

So.. $h+15 \neq 0$

$\therefore h \neq -15$

19. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} = \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

(a) To make the system has no solution, there should exist a wrong equation

In order to do that, the equation has no variable and its wrong.

$\Rightarrow 8-4h=0 \quad , \quad k-8 \neq 0$

$\therefore h=2 \quad , \quad k \neq 8$.

(b) To make the system has a unique solution, all of the variables should be basic variables.

$\Rightarrow 8-4h \neq 0 \qquad k$ is all of the real number.

$\therefore h \neq 2 \qquad k \in \mathbb{R}$

(c) To make the system has many solution, there should exist least one free variable. (+ the system should be consistent).

$\Rightarrow 8-4h=0 \qquad k-8=0$

$\therefore h=2 \quad , \quad k=8$

20. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$

(a) $h=9$ and $k \neq 6$

(b) $h \neq 9$

(c) $h=9$ and $k=6$

21. a. False. On page 13,

" Any nonzero matrix may be row reduced into more than one matrix in "echelon form" (not reduced echelon form), using ...,

However, the reduced echelon form one obtains from a matrix is unique."

b. False, On page 12,

" The algorithm (is the row reduction algorithm) applies to "any" matrix, whether or not the matrix is viewed as an augmented matrix for a linear system."

c. True. On page 18,

" The variables corresponding to pivot columns in the matrix are called basic variables."

d. True. On page 19.

" Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty."

e. False,

The row [0 0 0 5 0] means that $5x_4 = 0$.

If we instead had a row [0 0 0 0 5] in an echelon form, then this would make the linear system inconsistent, because it implies that $0 = 5$

21. a. False. On page 13.

"Any nonzero matrix may be row reduced into "more than one" matrix in echelon form. using different sequences of row operations. However, the "reduced" echelon form one obtains from a matrix is unique."

∴

b. False, On page 14.

"The leading entries are always in the same positions in any echelon form obtained from a given matrix"

c. True, On page 17.

d. True. On page 19.

"Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.

e. True.

23. The system is consistent.

To enable the system to be inconsistent, in coefficient matrix, there must exist least one row of all zeros.

But, the matrix has three pivot columns means the matrix has three nonzero rows. And since 3x5 matrix has only 3 rows, the matrix has no zero row.

24. No, the system is inconsistent

In 3x5 augmented matrix, fifth column is the last column. and the fact that the last column is pivot column means that the system is inconsistent. Because in this case, one of the equation is that $0 = x$ (x isn't zero), and it is never true.

25. In this case, the coefficient matrix doesn't have a zero row.

This means that the augmented matrix doesn't have a row of the form

[0 0 0 ... 0 b].

26. Because every columns are pivot columns, every variables are basic variables.

When a matrix is represented in reduced echelon form, It can be

determined that every variables has a unique value.

27. every column in coefficient matrix is pivot column.

28. In order to know that the linear system is consistent,

I have to know that the last column is not pivot column.

And in order to know that the linear system has a unique solution,

I have to know that every column except for the last one is a

pivot column.

29. When the system is represented by a coefficient matrix, the number of rows

is smaller than the number of unknowns. The number of pivots can't

exceed the number of rows. This means that the number of pivot columns

is smaller than the number of unknowns, and the number of basic variables

is also smaller than the number of unknowns i.e. at least one free variable exists.

Summarizing the above, there must be an infinite number of solution.

30. $x_1 + x_2 + x_3 = 1$

$x_1 + x_2 + x_3 = 2.$

31. $x_1 + x_2 = 2$

$x_1 + 2x_2 = 3$

$2x_1 + x_2 = 3.$

This system has a solution : $(x_1, x_2) = (1, 1)$

32 forward phase. (proof of NUMERICAL NOTE on page 20)

- In STEP 3, For $(n-1)$ rows (excluding the first row), it is necessary to perform operations that make the first entry 0.

- For every row, it requires $(2n+1)$ flops.

1 flop : to calculate $\frac{v_{i,1}}{v_{1,1}}$ ($v_{i,j}$ means the value of ith row and jth column)

2n flops : for every n entry (excluding the first entry) in ith row, requires 2 flops. (product operation and minus operation)

we know the first entry must be 0. So, it isn't necessary to perform operations. Just change it 0.

⇒ For every steps, it requires $(n-1)(2n+1)$ flops.

$$\Rightarrow \sum_{k=1}^{n} (k-1)(2k+1) = \sum_{k=1}^{n} 2k^2 - k - 1$$

$$= 2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} - n$$

$$= \frac{2}{3}n^3 + n^2 + \frac{1}{3}n - \frac{1}{2}n^2 - \frac{1}{2}n - n$$

$$= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$$

32. According to the numerical note in Section 1.2, when n is moderately large the reduction to echelon form approximately takes $\frac{2}{3}n^3$ flops. In contrast, further reduction to reduced echelon form needs at most n^2 flops.

Thus the fraction associated with the backward phase is ..

when $n=30$, $\frac{(30)^2}{\frac{2}{3}(30)^3} = \frac{3}{2\cdot 30} = \frac{1}{20} = 0.05 = 5\%$

when $n=300$, $\frac{(300)^2}{\frac{2}{3}(300)^3} = \frac{3}{2\cdot 300} = \frac{1}{200} = 0.005 = 0.5\%$

33. $\begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$\therefore\ p(t) = 7 + 6t - t^2$