2. Show that, for $n+1$ Chebyshev points of the second kind, the barycentric weights are (after rescaling)

$$w_i = (-1)^i, \quad i = 1, \cdots, n-1,$$

and $w_0 = 1/2$, $w_n = (-1)^n/2$.

By HW1, we know that $\omega'_{n+1}(x_i) = \prod_{j=0, j\neq i}^{n} (x_i - x_j) = w_i \quad \forall i = 0, \dots, n$, where $\omega_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$.

Let $x = \cos\theta$ for $\theta \in [0, \pi]$.

Then
$$\omega_{n+1}(x) = \prod_{i=0}^{n} (x - x_i) = (x-1)(x+1)\frac{U_{n-1}(x)}{2^{n-1}} = \frac{1}{2^{n-1}}(x^2-1)\frac{\sin(n\cos^{-1}(x))}{\sin(\cos^{-1}(x))}$$
$$= \frac{-1}{2^{n-1}} \sin(n\cos^{-1}(x)) \cdot \sin(\cos^{-1}(x))$$
$$= \frac{1}{2^n}\left(\cos((n+1)\cos^{-1}(x)) - \cos((n+1)\cos^{-1}(x))\right)$$
$$= \frac{1}{2^n}\left(\cos(n\theta)\cos(\theta) - \sin(n\theta)\sin\theta - \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin\theta\right)$$
$$= \frac{1}{2^n}\left(-2\sin(n\theta)\sin\theta\right).$$

Thus
$$\frac{d\omega_{n+1}(x)}{dx} = \frac{d\,\frac{1}{2^n}(-2\sin(n\theta)\sin\theta)}{d\theta} \cdot \frac{d\theta}{dx}$$
$$= \frac{1}{2^n}\left(-2(n\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta)\right) \cdot \frac{d\theta}{d\cos\theta}$$
$$= \frac{1}{2^n}\left(-2(n\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta)\right) \cdot \frac{-1}{\sin\theta} \quad \text{denoted by } f(\theta)$$

Claim: $\sin(n\theta) = \left(\sum_{i=1}^{n} \cos((n-i)\theta)\cos^{i-1}\theta\sin\theta\right) + \sin\theta\cos^{n-1}\theta \quad \forall n \in \mathbb{N}\setminus\{1\}$.

pf: Suppose $\sin(n\theta) = \left(\sum_{i=1}^{n-1} \cos((n-i)\theta)\cos^{i-1}\theta\sin\theta\right) + \sin\theta\cos^{n-1}\theta$ — (*) $\forall n \in \mathbb{N}\setminus\{1\}$.

For $n=2$, $\sin(2\theta) = \cos\theta\sin\theta + \sin\theta\cos\theta$ holds.

Suppose $n = N$, (*) holds.

For $n = N+1$,
$$\sin((N+1)\theta) = \sin(N\theta)\cos\theta + \cos(N\theta)\sin\theta$$
$$= \left(\sum_{i=1}^{N} \cos((N-i)\theta)\cos^{i-1}\theta\sin\theta\right)\cos\theta + \cos(N\theta)\sin$$
$$= \left(\sum_{i=1}^{N} \cos((N-i)\theta)\cos^{i}\theta\sin\theta\right) + \cos(N\theta)\sin\theta$$
$$= \left(\sum_{i=0}^{N} \cos((N-i)\theta)\cos^{i}\theta\sin\theta\right)$$
$$= \left(\sum_{i=0}^{N-1} \cos((N-i)\theta)\cos^{i}\theta\sin\theta\right) + \cos^N\theta\sin\theta$$
$$= \left(\sum_{i=1}^{N} \cos((N+1-i)\theta)\cdot\cos^{i-1}\theta\cdot\sin\theta\right) + \sin\theta\cos^N\theta \quad \text{holds.}$$

By induction, $\sin(n\theta) = \left(\sum_{i=1}^{n-1} \cos((n-i)\theta)\cos^{i-1}\theta\sin\theta\right) + \sin\theta\cos^{n-1}\theta \quad \forall n \in \mathbb{N}\setminus\{1\}$.

So
$$\frac{1}{2^n}\left(-2(n\cos(n\theta)\sin\theta + \sin(n\theta)\cos\theta)\right) \cdot \frac{-1}{\sin\theta}$$
$$= \frac{1}{2^n}\left(2\left(n\cos(n\theta) + \left(\sum_{i=1}^{n-1}\cos((n-i)\theta)\cdot\cos^{i}\theta\right) + \cos^n\theta\right)\right) \quad \text{for } n \in \mathbb{N}\setminus\{1\}.$$

Then
$$\omega'_{n+1}(x_0) = \omega'_{n+1}(1) = f(0) = \frac{1}{2^n}\left(2\left[n\cos(0) + \left(\sum_{i=1}^{n-1}\cos(0)\cdot\cos^{i}(0)\right) + \cos^n(0)\right]\right)$$
$$= \frac{1}{2^n}\left(2(n + n - 1 + 1)\right) = \frac{2n}{2^{n-1}}.$$

$$W_{n+1}'(x_i) = f\left(\frac{i\pi}{n}\right) = \frac{1}{2^n}\left(-2\left[n\cdot\cos\left(n\cdot\frac{i}{n}\pi\right)\cdot\sin\left(\frac{i}{n}\pi\right) + \sin\left(n\cdot\frac{i}{n}\pi\right)\cos\left(\frac{i}{n}\pi\right)\right]\right)\cdot\frac{-1}{\sin\left(\frac{i}{n}\pi\right)}$$

$$= \frac{1}{2^n}\left(-2\left[-n\cdot(-1)^i\cdot 1 - 0\cdot\cos\left(\frac{i}{n}\pi\right)\right]\right)$$

$$= \frac{1}{2^n}\left(2n(-1)^i\right) = \frac{2n(-1)^i}{2^n}.$$

$$W_{n+1}'(x_n) = W_{n+1}'(-1) = f(\pi) = \frac{1}{2^n}\left(2\left[n\cos(n\pi) + \left(\sum_{i=1}^{n-1}\cos((n-i)\pi)\cdot\cos^i\pi\right) + \cos^n(\pi)\right]\right)$$

$$= \frac{1}{2^n}\left(2\left[n(-1)^n + \left(\sum_{i=1}^{n-1}(-1)^{n-i}\cdot(-1)^i\right) + (-1)^n\right]\right)$$

$$= \frac{1}{2^n}\left(2\left[n(-1)^n + n\cdot(-1)^n\right]\right) = \frac{2n(-1)^n}{2^{n-1}}.$$

Thus the barycentric weight are $W_0 = \dfrac{1}{\prod_{j=1}^{n}(x_0 - x_j)} = \dfrac{1}{W_{n+1}'(x_0)} = \dfrac{2^{n-1}}{2n}$.

$$W_i = \frac{1}{\prod_{j=0,\, j\neq i}^{n}(x_i - x_j)} = \frac{1}{W_{n+1}'(x_i)} = \frac{2^n}{2n}(-1)^i \quad \forall i = 1,\dots,n-1.$$

$$W_n = \frac{1}{\prod_{j=0}^{n-1}(x_n - x_j)} = \frac{1}{W_{n+1}'(x_n)} = \frac{2^{n-1}}{2n}(-1)^n.$$

After rescaling, $W_0 = \frac{1}{2}$, $W_i = (-1)^i$ $\forall i = 1,\dots,n-1$ and $W_n = \frac{1}{2}(-1)^n$.