1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

Let $f \in C^2([a,b])$.

Then $E_0(f) = \frac{H^3}{3} f''(\xi_1)$, where $H = \frac{b-a}{2}$ and $\xi_1 \in (a,b)$ and

$E_1(f) = -\frac{h^3}{12} f''(\xi_2)$, where $h = b-a$ and $\xi_2 \in (a,b)$.

Thus $E_0(f) = \frac{H^3}{3} f''(\xi_1) = \frac{h^3}{24} f''(\xi_1) = E_1(f) \cdot \frac{f''(\xi_1)}{2f''(\xi_2)}$.

Let $\varepsilon > 0$.

Since f'' is conti. on $[a,b]$, which is compact set,

$\exists\, \delta > 0$ s.t. if $\xi_1, \xi_2 \in [a,b]$, $|\xi_1 - \xi_2| < \delta$ then $|f''(\xi_1) - f''(\xi_2)| < \varepsilon$.

Let $b-a = h < \frac{1}{N}$ for all $N \in \mathbb{N}$.

Then for any $\delta > 0$, by Archimedean property, $h < \frac{1}{N} < \delta$.

Thus for any $\varepsilon > 0$, $\exists\, h > 0$ s.t. if $\xi_1, \xi_2 \in [a,b]$, $|\xi_1 - \xi_2| \le h < \frac{1}{N} < \delta$ then $|f''(\xi_1) - f''(\xi_2)| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\lim_{h \to 0} |f''(\xi_1) - f''(\xi_2)| = 0$. Thus $\lim_{h \to 0} \frac{|f''(\xi_1)|}{|f''(\xi_2)|} = 1$.

Hence $\lim_{h \to 0} 2|E_0(f)| = \lim_{h \to 0} |E_1(f)|$. Therefore $|E_1(f)| \simeq 2|E_0(f)|$.

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae:
(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,
(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.
Which is the order of infinitesimal p for (a) and (b)?
[*Solution*: $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

Let $f \in \mathbb{P}_n$. Then $f \in C^\infty([-1,1])$.

Since $I_2(f) = I(f_2) = \int_1^{-1} f_2(x)\,dx$, where $f_2(x) = f(\frac{-1}{2})\,l_0(x) + f(0)\,l_1(x) + f(\frac{1}{2})\,l_2(x)$ for $x \in [-1,1]$.

We have $|E_2(f)| = |I(f) - I_2(f)| = \left| \int_{-1}^{1} f(x) - f_2(x)\,dx \right| \le \int_{-1}^{1} |f(x) - f_2(x)|\,dx$

$= \int_{-1}^{1} \frac{f^3(\xi_x)}{3!}(x+\frac{1}{2})(x)(x-\frac{1}{2})\,dx$

If $n < 3$, it's obvious that $f^3(\xi_x) = 0$ for all $\xi_x \in (a,b)$. Thus $E_2(f) = 0$.

If $n = 3$, then $f^{(3)}(\xi_x) = 6$ is independent of ξ_x.

Thus $\int_{-1}^{1} \frac{f^{(3)}(\xi_x)}{3!}(x+\frac{1}{2})(x)(x-\frac{1}{2})\,dx = \frac{f^{(3)}(\xi_x)}{3!}\int_{-1}^{1}(x+\frac{1}{2})(x)(x-\frac{1}{2})\,dx$

$= \frac{f^{(3)}(\xi_x)}{3!}\left(\frac{x^4}{4} - \frac{x^2}{8}\right)\Big|_{-1}^{1} = \frac{f^{(3)}(\xi_x)}{3!}\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{4} + \frac{1}{8}\right) = 0.$

If $n = 4$, then $f^3(\xi_x)$ is dependent of x.

Thus we don't know if $\int_{-1}^{1} \frac{f^3(\xi_x)}{3!}(x+\frac{1}{2})(x)(x-\frac{1}{2})\,dx$ is equal to 0 or not.

$\int_{-1}^{1} x^4\,dx = \frac{-2}{5}$, $I_2(f) = \frac{2}{3}\left(2\cdot\frac{1}{16} - 0 + 2\cdot\frac{1}{16}\right) = \frac{1}{6}$. $I(f) \neq I_2(f)$

Similarly, $|E_4(f)| = |I(f) - I(f_4)| = \left| \int_{-1}^{1} \frac{f^4(\xi_x)}{4!}(x+1)(x+\frac{1}{3})(x-\frac{1}{3})(x-1)\,dx \right|$.

If $n < 4$, it's obvious that $f^4(\xi_x) = 0$ for all $\xi_x \in (a,b)$. Thus $E_2(f) = 0$.

If $n = 4$, then $f^{(4)}(\xi_x) = 24$ is independent of ξ_x.

But $\left| \int_{-1}^{1} \frac{f^{(4)}(\xi_x)}{4}(x+1)(x+\frac{1}{3})(x-\frac{1}{3})(x-1)\,dx \right| = 4\left| \int_{-1}^{1} x^4 - \frac{10}{9}x^2 + \frac{1}{9}\,dx \right| = 4\cdot\left(\frac{x^5}{5} - \frac{10}{27}x^3 + \frac{x}{9}\right)\Big|_{-1}^{1} = \frac{-64}{135}$.

$\int_{-1}^{1} x^4\,dx = \frac{-2}{5}$, $I_4(f) = \frac{1}{4}\left(1 + \frac{1}{27} + \frac{1}{27} + 1\right) = \frac{16}{27}$. $I(f) \neq I_4(f)$.

Therefore, the degree of exactness of (a), (b) are 3.

(a)

Let $f \in C^4[-1,1]$. Let $W(x) = \int_{-1}^{x} \omega_{n+1}(t)\, dt$ for $x \in [-1,1]$, where $\omega_{n+1}(t) = (t+\frac{1}{2})(t)(t-\frac{1}{2})$

Then $E_2(f) = \int_{-1}^{1} \frac{f^{(3)}(\xi_x)}{3!}(x+\frac{1}{2})(x)(x-\frac{1}{2})\, dx = \frac{f^{(3)}(\xi_x)}{3!} W(x)\Big|_{-1}^{1} - \int_{-1}^{1} \frac{f^{(4)}(\xi_x)}{3!} W(x)\, dx$ by integration by parts.

Since $W(1) = 0$ and $W(-1) = 0$, $E_2(f) = -\int_{-1}^{1} \frac{f^{(4)}(\xi_x)}{3!} W(x)\, dx = \int_{-1}^{1} \frac{f^{(4)}(\xi_x)}{3!} - W(x)\, dx$

Since $\frac{f^{(4)}(\xi_x)}{3!}$ is conti and $-W(x) = \frac{-x^4}{4} + \frac{x^2}{8} + \frac{1}{8} \geq 0$ for $x \in [-1,1]$ is conti, by MVT. for integrals, $\exists\, \xi_c \in (a,b)$ s.t. $\int_{-1}^{1} \frac{f^{(4)}(\xi_x)}{3!}(-W(x))\, dx = \frac{f^{(4)}(\xi_c)}{3!} \int_{-1}^{1} -W(x)\, dx$

Let $t = -1 + rh$, where $h = \frac{1}{2}$ and $r \in [0,4]$.

Then $-\frac{f^{(4)}(\xi_c)}{3!} \int_{-1}^{1}\int_{-1}^{x} \omega_{n+1}(t)\, dt\, dx = -\frac{f^{(4)}(\xi_c)}{3!} \int_{-1}^{1}\int_{t}^{1} \omega_{n+1}(t)\, dx\, dt = -\frac{f^{(4)}(\xi_c)}{3!} \int_{-1}^{1} \omega_{n+1}(t)(1-t)\, dt$

$= -\frac{f^{(4)}(\xi_c)}{3!} \int_{0}^{4} (-1+rh-(-1+h))(-1+rh-(-1+2h))(-1+rh-(-1+3h))((-1+4h)-(-1+rh))\, h\, dr$

$= \frac{-f^{(4)}(\xi_c)}{3!} \int_{0}^{4} ((r-1)h)((r-2)h)((r-3)h)((4-r)h)\, h\, dr$

$= \frac{-f^{(4)}(\xi_c)}{3!} h^5 \int_{0}^{4} (r-1)(r-2)(r-3)(4-r)\, dr.$

(b)

Let $f \in C^4[-1,1]$. Let $\omega_{n+1}(t) = (t+1)(t+\frac{1}{3})(t-\frac{1}{3})(t-1)$.

Then $E_3(f) = \int_{-1}^{1} \frac{f^{(4)}(\xi_x)}{4!}(x+1)(x+\frac{1}{3})(x-\frac{1}{3})(x-1)\, dx = \int_{-1}^{\frac{1}{3}} \frac{f^{(4)}(\xi_x)}{4!} \omega_{n+1}(x)\, dx - \int_{\frac{1}{3}}^{1} \frac{f^{(4)}(\xi_x)}{4!}(-\omega_{n+1}(x))\, dx.$

Since $\frac{f^{(4)}(\xi_x)}{4!}$ is conti. on $[\frac{1}{3},1]$ and $-\omega_{n+1}(x) \geq 0$ and is conti. on $[\frac{1}{3},1]$, by MVT. for integrals,

there exists $\xi_c \in (-1,1)$ s.t. $\int_{\frac{1}{3}}^{1} \frac{f^{(4)}(\xi_x)}{4!}(-\omega_{n+1}(x))\, dx = \frac{f^{(4)}(\xi_2)}{4!} \int_{\frac{1}{3}}^{1} -\omega_{n+1}(x)\, dx.$ — ①

Let $\Omega(x) = \int_{-1}^{x} (t+1)(t+\frac{1}{3})(t-\frac{1}{3})\, dt$ for all $x \in [-1,1]$.

Then $\int_{-1}^{\frac{1}{3}} \frac{f^{(4)}(\xi_x)}{4!} \omega_{n+1}(x)\, dx = \int_{-1}^{\frac{1}{3}} \omega_{n+1}(x)\, f[x_0, \dots, x_3, x]\, dx$

$= \int_{-1}^{\frac{1}{3}} \frac{d\Omega(x)}{dx} (f[x, x_0, x_1, x_2] - f[x_0, x_1, x_2, x_3])\, dx$

$= \int_{-1}^{\frac{1}{3}} \frac{d\Omega(x)}{dx} f[x, x_0, x_1, x_2]\, dx - \int_{-1}^{\frac{1}{3}} \frac{d\Omega(x)}{dx} f[x_0, x_1, x_2, x_3]\, dx$ — ②

Since $\int_{-1}^{\frac{1}{3}} \frac{d\Omega(x)}{dx}\, dx = \Omega(\frac{1}{3}) - \Omega(-1) = 0$ and $f[x_0, x_1, x_2, x_3]$ is constant, $\int_{-1}^{\frac{1}{3}} \frac{d\Omega(x)}{dx} f[x_0, x_1, x_2, x_3]\, dx = 0$

Since Ω and $f[x, x_0, x_1, x_2] = \frac{f^{(3)}(\xi_x)}{3!}$ are $C^1([-1,1])$ and $\Omega(x) \geq 0$ $\forall x \in [-1,\frac{1}{3}]$, by MVT. for integral and integration by parts,

$\int_{-1}^{\frac{1}{3}} \frac{d\Omega(x)}{dx} f[x, x_0, x_1, x_2]\, dx = \Omega(x) f[x, x_0, x_1, x_2]\Big|_{-1}^{\frac{1}{3}} - \int_{-1}^{\frac{1}{3}} \Omega(x) \frac{d f[x, x_0, x_1, x_2]}{dx}\, dx$

$= 0 - \int_{-1}^{\frac{1}{3}} \Omega(x) \lim_{h \to 0} \frac{f[x, x_0, x_1, x_2] - f[x_0, x_1, x_2, x+h]}{h}\, dx$

$= -\int_{-1}^{\frac{1}{3}} \Omega(x) \lim_{h \to 0} f[x_0, x_1, x_2, x, x+h]\, dx = -\int_{-1}^{\frac{1}{3}} \Omega(x) f[x_0, x_1, x_2, x, x]\, dx$

$= -\int_{-1}^{\frac{1}{3}} \Omega(x) \frac{f^{(4)}(\xi_x)}{4!}\, dx = \frac{f^{(4)}(\xi_1)}{4!} \int_{-1}^{\frac{1}{3}} -\Omega(x)\, dx$ for $\xi_1 \in (-1, \frac{1}{3})$. — ③

By ①, ② and ③, $E_3(f) = \int_{-1}^{\frac{1}{3}} \frac{f^{(4)}(\xi_x)}{4!} w_{n+1}(x)\,dx - \int_{\frac{1}{3}}^{1} \frac{f^{(4)}(\xi_x)}{4!} (-w_{n+1}(x))\,dx$.

$$= \frac{f^{(4)}(\xi_1)}{4!}\int_{-1}^{\frac{1}{3}} -\Omega(x)\,dx - \frac{f^{(4)}(\xi_2)}{4!}\int_{\frac{1}{3}}^{1} -w_{n+1}(x)\,dx = -\left(\frac{f^{(4)}(\xi_1)}{4!}\int_{-1}^{\frac{1}{3}} \Omega(x)\,dx + \frac{f^{(4)}(\xi_2)}{4!}\int_{\frac{1}{3}}^{1} -w_{n+1}(x)\,dx.\right)$$

Since $\int_{-1}^{\frac{1}{3}} \Omega(x)\,dx \geq 0$ and $\int_{\frac{1}{3}}^{1} -w_{n+1}(x)\,dx \geq 0$ and $f^{(4)}$ is conti. on $[-1,1]$, by discrete MVT.,

there exists $\xi \in (a,b)$ s.t. $-\left(\frac{f^{(4)}(\xi_1)}{4!}\int_{-1}^{\frac{1}{3}} \Omega(x)\,dx + \frac{f^{(4)}(\xi_2)}{4!}\int_{\frac{1}{3}}^{1} -w_{n+1}(x)\,dx.\right) = -\left(\int_{-1}^{\frac{1}{3}} \Omega(x)\,dx + \int_{\frac{1}{3}}^{1} -w_{n+1}(x)\,dx\right)\left(\frac{f^{(4)}(\xi)}{4!}\right)$.

Since $w_{n+1}(x) = \frac{d\Omega(x)}{dx}(x+1)$, $\Omega(x)$ and $(x+1) \in C^1([-1,1])$, by integration by parts,

$\int_{-1}^{\frac{1}{3}} w_{n+1}(x)\,dx = \Omega(x)(x+1)\Big|_{-1}^{\frac{1}{3}} - \int_{-1}^{\frac{1}{3}} \Omega(x)\,dx = -\int_{-1}^{\frac{1}{3}} \Omega(x)\,dx$.

Thus $E_3(f) = -\left(\int_{-1}^{\frac{1}{3}} \Omega(x)\,dx + \int_{\frac{1}{3}}^{1} -w_{n+1}(x)\,dx\right)\left(\frac{f^{(4)}(\xi)}{4!}\right) = \int_{-1}^{1} w_{n+1}(x)\,dx\left(\frac{f^{(4)}(\xi)}{4!}\right)$

$$= \int_0^3 (-1+rh-(-1))(-1+rh-(-1+h))(-1+rh-(-1+2h))(-1+rh-(-1+3h))\,h\,dr \left(\frac{f^{(4)}(\xi)}{4!}\right)$$

$$= h^5\int_0^3 r(r-1)(r-2)(r-3)\,dr\ \frac{f^{(4)}(\xi)}{4!},\ \text{where } x = -1+rh,\ h=\tfrac{2}{3},\ r \in [0,3]$$

Therefore the order of infinitesimal of (a), (b) are 5.

5. Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find a and x_1 in such a way that Q has maximum degree of exactness r.
[*Solution*: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

let $f(x) = a_0 \in \mathbb{P}_0$.

Let $I_w(f) - Q(f) = 0$.

Then $\int_0^1 \sqrt{x}\, a_0\, dx - a \cdot a_0 = 0 \Rightarrow a_0 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1 - a \cdot a_0 = 0 \Rightarrow \frac{2}{3} a_0 - a \cdot a_0 = 0 \Rightarrow a = \frac{2}{3}$.

Let $f(x) = b_1 x + a_1 \in \mathbb{P}_1$.

let $I_w(f) - Q(f) = 0$.

Then $\int_0^1 \sqrt{x}\,(b_1 x + a_1)\, dx - \frac{2}{3}(b_1 x_1 + a_1) = 0 \Rightarrow b_1 \frac{2}{5} x^{\frac{5}{2}} + a_1 \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 - \frac{2}{3}(b_1 x_1 + a_1) = 0$

$\Rightarrow b_1 \frac{2}{5} + a_1 \frac{2}{3} - \frac{2}{3} b_1 x_1 - \frac{2}{3} a_1 = 0 \Rightarrow x_1 = \frac{3}{5}$.

Let $f(x) = c_2 x^2 + b_2 x + a_2 \in \mathbb{P}_2$.

Let $I_w(f) - Q(f) = 0$

Then $\int_0^1 (c_2 x^2 + b_2 x + a_2)\, dx - \frac{2}{3}\left(c_2 \cdot \frac{9}{25} + b_2 \cdot \frac{3}{5} + a_2\right) = 0$

$\Rightarrow \frac{x^3}{3} c_2 + \frac{x^2}{2} b_2 + a_2 x \Big|_0^1 - \frac{6}{25} c_2 - \frac{2}{5} b_2 - \frac{2}{3} a_2 = 0$

$\Rightarrow \frac{7}{75} c_2 + \frac{1}{10} b_2 + \frac{1}{3} a_2 = 0$

Thus not all $f \in \mathbb{P}_2$ s.t. $I_w(f) - Q(f) = 0$.

Therefore the maximum degree of exactness is 2.

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1,2,3$ in such a way that Q has degree of exactness $r = 2$.
[*Solution*: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

Since the degree of exactness is 2,

for each $f \in \mathbb{P}_0 \cup \mathbb{P}_1 \cup \mathbb{P}_2$, $I(f) - Q(f) = 0$.

Let $f = a_0 \in \mathbb{P}_0$.

Then $I(f) - Q(f) = a_0 - \alpha_1 a_0 - \alpha_2 a_0 = 0 \Rightarrow \alpha_1 + \alpha_2 = 1$.

Let $f = a_1 x + b_1 \in \mathbb{P}_1$.

Then $I(f) - Q(f) = \int_0^1 a_1 x + b_1 \, dx - (\alpha_1 b_1 + \alpha_2 (a_1 + b_1) + \alpha_3 a_1)$

$= \frac{1}{2} a_1 + b_1 - (\alpha_2 + \alpha_3) a_1 - (\alpha_1 + \alpha_2) b_1 = 0$

$\Rightarrow \alpha_2 + \alpha_3 = \frac{1}{2}$ and $\alpha_1 + \alpha_2 = 1$.

Let $f = a_2 x^2 + b_2 x + c_2 \in \mathbb{P}_2$.

Then $I(f) - Q(f) = \int_0^1 a_2 x^2 + b_2 x + c_2 \, dx - (\alpha_1 c_2 + \alpha_2 (a_2 + b_2 + c_2) + \alpha_3 b_2)$

$= \frac{a_2}{3} + \frac{b_2}{2} + c_2 - \alpha_2 a_2 - (\alpha_2 + \alpha_3) b_2 - (\alpha_1 + \alpha_2) c_2 = 0$

$\Rightarrow \alpha_2 = \frac{1}{3}$, $\alpha_2 + \alpha_3 = \frac{1}{2}$, $\alpha_1 + \alpha_2 = 1$.

Thus $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{6}$.