1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

$$h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h)$$

$$= \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds - \left[\frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right)\right]$$

$$= \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] + \frac{h}{2}[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]$$

let $f(s, y(s)) = g(s)$.

let $g(s) = g(t_n) + g'(t_n)(t - t_n) + \frac{g''(t_n)}{2!}(t - t_n)^2 + \frac{g'''(\xi_n)}{3!}(t - t_n)^3$, where $\xi_n \in (t_n, t_{n+1})$.

Then $\int_{t_n}^{t_{n+1}} f(s, y(s))ds = \int_{t_n}^{t_{n+1}} g(s)\,ds$

$$= \int_{t_n}^{t_{n+1}} g(t_n) + g'(t_n)(s - t_n) + \frac{g''(\xi_n)}{2!}(s - t_n)^2\,ds$$

$$= g(t_n)\cdot s + g'(t_n)\frac{(s - t_n)^2}{2} + g''(\xi_n)\frac{(s - \xi_n)^3}{6}\Bigg|_{t_n}^{t_{n+1}}$$

$$= g(t_n)\cdot h + g'(t_n)\cdot\frac{h^2}{2} + g''(\xi_n)\frac{h^3}{6}$$

Since $f(t_{n+1}, y_{n+1}) = g(t_{n+1}) = g(t_n) + g'(t_n)\cdot h + g''(\xi_1)\cdot\frac{h^2}{2}$,

we have $\int_{t_n}^{t_{n+1}} f(s, y(s))\,ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$

$$= g(t_n)\cdot h + g'(t_n)\cdot\frac{h^2}{2} + g''(\xi_n)\frac{h^3}{6} - \frac{h}{2}g(t_n) - \frac{h}{2}\cdot\left(g(t_n) + g'(t_n)\cdot h + g''(\xi_1)\cdot\frac{h^2}{2}\right)$$

$$= g''(\xi_n)\cdot\frac{h^3}{6} - g''(\xi_1)\cdot\frac{h^3}{2} = O(h^3)$$

Since $|f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))| \le L|y_{n+1} - y_n - hf(t_n, y_n)|$,

by forward Euler method, $|y_{n+1} - y_n - hf(t_n, y_n)| = O(h^2)$

Thus $E_2 = \frac{h}{2}\{|f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n)| = O(h^3)$

Since $h\tau_{n+1} = E_1 + E_2 = O(h^3)$, $\tau_{n+1} = O(h^2)$.

By the def. of global truncation error, $\tau = \max_{0 \le n \le N_h - 1} |\tau_{n+1}|$.

τ is also equal to $O(h^2)$. Therefore, Heuns method has order 2 with respect to h.

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

let $h\tau_{n+1}(h) = y(t_{n+1}) - y(t_n) - \frac{h}{2}\left(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\right)$

let $f(s, y(s)) = g(s)$.

Then $h\tau_{n+1}(h) = y(t_{n+1}) - y(t_n) - \frac{h}{2}\left(g(t_n) + g(t_{n+1})\right)$

$$= \int_{t_n}^{t_{n+1}} g(s)\,ds - \frac{h}{2}\left(g(t_n) + g(t_{n+1})\right)$$

$$= -\frac{h^3}{12} g''(\xi_n) = \frac{-h^3}{12} f''(\xi_n, y(\xi_n))$$

Thus $\tau_{n+1}(h) = O(h^2)$.

By the def. of global truncation error, $\tau = \max_{0 \le n \le N_h - 1} |\tau_{n+1}|$.

τ is also equal to $O(h^2)$. Therefore, Crank-Nicoloson method has order 2 with respect to h.