5. Prove the estimate (12.23).

[*Hint*: for each internal node x_j, $j = 1, \ldots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j)$$

$$= -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2 \, dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2 \, dt\right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \ldots, n-1$. On noting that $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

Remark 12.3 Let $e = u - u_h$ be the *discretization error* grid function. Then,

$$L_h e = L_h u - L_h u_h = L_h u - f_h = \tau_h. \tag{12.22}$$

It can be shown (see Exercise 5) that

$$\|\tau_h\|_h^2 \leq 3\left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2\right) \tag{12.23}$$

from which it follows that the norm of the discrete second-order derivative of the discretization error is bounded, provided that the norms of f at the right-hand side of (12.23) are also bounded. ■

By (12.21), $\tau_h(x_j) = \frac{1}{h^2}\left(R_4(x_j+h) + R_4(x_j-h) \right)$

$$= \frac{1}{h^2}\left(\int_{x_j}^{x_j+h} (u'''(t) - u'''(x_j)) \frac{(x_j+h-t)^2}{2} dt - \int_{x_j-h}^{x_j} (u'''(t) - u'''(x_j)) \frac{(x_j-h-t)^2}{h} dt \right)$$

We apply integration by parts to both integrals.

$$\int_{x_j}^{x_j+h} (u'''(t) - u'''(x_j)) \frac{(x_j+h-t)^2}{2} dt$$

$$= \left(u''(t) - u'''(x_j)\cdot t\right) \frac{(x_j+h-t)^2}{2} \Big|_{x_j}^{x_j+h} + \int_{x_j}^{x_j+h} \left(u''(t) - u'''(x_j)\cdot t\right)(x_j+h-t)\, dt$$

$$= -u''(x_j)\frac{h^2}{2} + u'''(x_j)\, x_j \cdot \frac{h^2}{2} + \int_{x_j}^{x_j+h} u''(t)\,(x_j+h-t)\, dt - \int_{x_j}^{x_j+h} u'''(x_j)\cdot(x_j t + ht - t^2)\, dt$$

$$= -u''(x_j)\frac{h^2}{2} + u'''(x_j)\, x_j \cdot \frac{h^2}{2} + \int_{x_j}^{x_j+h} u''(t)\,(x_j+h-t)\, dt - u'''(x_j)\cdot\left(\frac{t^2}{2}(x_j+h) - \frac{t^3}{3}\right)\Big|_{x_j}^{x_j+h}$$

$$= -u''(x_j)\frac{h^2}{2} + u'''(x_j)\, x_j \cdot \frac{h^2}{2} + \int_{x_j}^{x_j+h} u''(t)\,(x_j+h-t)\, dt - u'''(x_j)\cdot\left(\frac{(x_j+h)^3}{2} - \frac{(x_j+h)^3}{3} - \frac{x_j^2(x_j+h)}{2} + \frac{x_j^3}{3}\right)$$

$$= -u''(x_j)\frac{h^2}{2} + u'''(x_j)\, x_j \cdot \frac{h^2}{2} + \int_{x_j}^{x_j+h} u''(t)\,(x_j+h-t)\, dt - u'''(x_j)\left(\frac{3h^2 x_j + h^3}{6}\right)$$

$$\int_{x_j-h}^{x_j} (u'''(t) - u'''(x_j)) \frac{(x_j-h-t)^2}{2} dt$$

$$= \left(u''(t) - u'''(x_j)\, t\right) \frac{(x_j-h-t)^2}{2} \Big|_{x_j-h}^{x_j} + \int_{x_j-h}^{x_j} \left(u''(t) - u'''(x_j)\, t\right)(x_j-h-t)\, dt$$

$$= \left(u''(x_j) - u'''(x_j)\cdot x_j\right)\frac{h^2}{2} + \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt - \int_{x_j-h}^{x_j} u'''(x_j)\, t\,(x_j-h-t)\, dt$$

$$= \left(u''(x_j) - u'''(x_j)\cdot x_j\right)\frac{h^2}{2} + \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt - \int_{x_j-h}^{x_j} u'''(x_j)(t x_j - th - t^2)\, dt$$

$$= \left(u''(x_j) - u'''(x_j)\cdot x_j\right)\frac{h^2}{2} + \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt - u'''(x_j)\left(\frac{t^2}{2}(x_j-h) - \frac{t^3}{3}\right)\Big|_{x_j-h}^{x_j}$$

$$= \left(u''(x_j) - u'''(x_j)\cdot x_j\right)\frac{h^2}{2} + \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt - u'''(x_j)\,\frac{-3x_j h^2 + h^3}{6}$$

Thus $\tau_h(x_j) = \frac{1}{h^2}\Big(\int_{x_j}^{x_j+h} u''(t)\,(x_j+h-t)\, dt - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt$

$$-u''(x_j)\frac{h^2}{2} + u'''(x_j)\, x_j \cdot \frac{h^2}{2} - u''(x_j)\frac{h^2}{2} + u'''(x_j)\cdot x_j\,\frac{h^2}{2}$$

$$-u'''(x_j)\left(\frac{3h^2 x_j + h^3}{6}\right) + u'''(x_j)\,\frac{-3x_j h^2 + h^3}{6} \Big)$$

$$= -u''(x_j) + \frac{1}{h^2}\left(\int_{x_j}^{x_j+h} u''(t)\,(x_j+h-t)\, dt - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\, dt \right)$$

Since $(a+b+c)^2 \le 3(a^2+b^2+c^2)$,

$$\tau_h(x_j)^2 = \left(-u''(x_j) + \frac{1}{h^2}\left(\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt\right)\right)^2$$

$$\le 3\left([-u''(x_j)]^2 + \left(\frac{-1}{h^2}\int_{x_j}^{x_j+h} -u''(t)(x_j+h-t)\,dt\right)^2 + \left(\frac{1}{h^2}\int_{x_j-h}^{x_j} -u''(t)(x_j-h-t)\,dt\right)^2 \right)$$

$$= 3\left[f(x_j)^2 + \left(\frac{-1}{h^2}\int_{x_j}^{x_j+h} f(t)(x_j+h-t)\,dt\right)^2 + \left(\frac{1}{h^2}\int_{x_j-h}^{x_j} f(t)(x_j-h-t)\,dt\right)^2 \right] \text{ since } -u''(x) = f(x).$$

Since $\|\tau_h\|_h^2 = h\sum_{j=1}^{n-1} c_j \tau_h(x_j)^2$, where $c_0 = c_n = \frac{1}{2}$ and $c_i = 1$ for $i = 1, \dots, n-1$,

we have $\|\tau_h\|_h^2 = h\sum_{j=1}^{n-1} c_j \tau_h(x_j)^2$

$$\le (3h)\sum_{j=1}^{n-1} c_j \left[f(x_j)^2 + \left(\frac{-1}{h^2}\int_{x_j}^{x_j+h} f(t)(x_j+h-t)\,dt\right)^2 + \left(\frac{1}{h^2}\int_{x_j-h}^{x_j} f(t)(x_j-h-t)\,dt\right)^2 \right] - ①$$

By Cauchy - Schwarz inequality, we know that

$$\left(\int_{x_j}^{x_j+h} f(t)(x_j+h-t)\,dt\right)^2 \le \left(\int_{x_j}^{x_j+h} f(t)^2\,dt\right)\left(\int_{x_j}^{x_j+h} (x_j+h-t)^2\,dt\right) \text{ and}$$

$$\left(\int_{x_j-h}^{x_j} f(t)(x_j-h-t)\,dt\right)^2 \le \left(\int_{x_j-h}^{x_j} f(t)^2\,dt\right)\left(\int_{x_j-h}^{x_j} (x_j-h-t)^2\,dt\right). \quad - ②$$

let $s = x_j + h - t$.

Then $\int_{x_j}^{x_j+h} (x_j+h-t)^2\,dt = -\int_h^0 s^2\,ds = -\frac{s^3}{3}\Big|_h^0 = \frac{h^3}{3}. \quad - ③$

let $w = x_j - h - t$.

Then $\int_{x_j-h}^{x_j} (x_j-h-t)^2\,dt = -\int_0^{-h} w^2\,dw = -\frac{w^3}{3}\Big|_0^{-h} = \frac{h^3}{3}. \quad - ④$

Thus by ①, ②, ③ and ④,

$$\|\tau_h\|_h^2 \le (3h)\sum_{j=1}^{n-1} c_j \left[(f(x_j))^2 + \left(\frac{-1}{h^2}\int_{x_j}^{x_j+h} f(t)(x_j+h-t)\,dt\right)^2 + \left(\frac{1}{h^2}\int_{x_j-h}^{x_j} f(t)(x_j-h-t)\,dt\right)^2 \right]$$

$$\le (3h)\sum_{j=1}^{n-1} c_j \left[(f(x_j))^2 + \frac{1}{h^4}\left(\int_{x_j}^{x_j+h} f(t)^2\,dt \cdot \int_{x_j}^{x_j+h} (x_j+h-t)^2\,dt\right) + \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} f(t)^2\,dt \cdot \int_{x_j-h}^{x_j} (x_j-h-t)^2\,dt\right) \right]$$

$$= (3h)\sum_{j=1}^{n-1} c_j \left[f(x_j)^2 + \frac{1}{h^4}\cdot\frac{h^3}{3}\int_{x_j}^{x_j+h} f(t)^2\,dt + \frac{1}{h^4}\cdot\frac{h^3}{3}\int_{x_j-h}^{x_j} f(t)^2\,dt \right]$$

$$= 3\left[h\sum_{j=1}^{n-1} c_j f(x_j)^2 + \sum_{j=1}^{n-1}\frac{1}{3}\int_{x_j}^{x_j+h} f(t)^2\,dt + \sum_{j=1}^{n-1}\frac{1}{3}\int_{x_j-h}^{x_j} f(t)^2\,dt \right]$$

$$\le 3\left[\|f\|_h^2 + \frac{1}{3}\int_{x_0}^{x_1} f(t)^2\,dt + \frac{2}{3}\int_{x_1}^{x_{n-1}} f(t)^2\,dt + \frac{1}{3}\int_{x_{n-1}}^{x_n} f(t)^2\,dt \right]$$

$$\le 3\left[\|f\|_h^2 + \int_{x_0}^{x_n} f(t)^2\,dt \right]$$

$$= 3\left[\|f\|_h^2 + \|f\|_{L^2(0,1)}^2 \right]$$

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2}x_j(1 - x_j)$.
[*Solution*: use the definition (12.25) with $g(x_k) = 1$, $k = 1, \dots, n-1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k)\right]$$

from which, after straightforward computations, one gets the desired result.]

By 12.25, $T_h g = \sum_{k=1}^{n-1} g(x_k) G^k = \sum_{k=1}^{n-1} g(x_k) \begin{bmatrix} G^k(x_1) \\ \vdots \\ G^k(x_{n-1}) \end{bmatrix}$

Thus $T_h g(x_j) = \sum_{k=1}^{n-1} g(x_k) G^k(x_j) = \sum_{k=1}^{n-1} h g(x_k) G(x_j, x_k) = \sum_{k=1}^{n-1} h G(x_j, x_k)$

$= h\left(\sum_{k=1}^{j} x_k(1 - x_j) + \sum_{k=j+1}^{n-1} x_j(1 - x_k) \right)$

$= h\left(\sum_{k=1}^{j} kh(1 - jh) + \sum_{k=j+1}^{n-1} jh(1 - kh) \right)$

$= h\left(\frac{j(j+1)}{2} h(1 - jh) + (n-j-1)jh - \frac{(n+j)(n-j-1)}{2} jh^2 \right)$

$= \frac{j^2h^2 + jh^2}{2} - \frac{j^3h^3 - j^2h^3}{2} + (n-j-1)jh^2 - \frac{n^2jh^3 - nj^2h^3 - njh^3 + j^2nh^3 - j^3h^3 - j^2h^3}{2}$

$= \frac{j^2h^2 + jh^2}{2} - \frac{j^3h^3 - j^2h^3}{2} + jh - j^2h^2 - jh^2 - \frac{jh - jh^2 - jh^2 + jh^2 - j^3h^3 - j^2h^3}{2}$

$= \frac{-j^2h^2 - jh^2}{2} + \frac{jh}{2} + \frac{jh^2}{2} = \frac{-j^2h^2}{2} + \frac{jh}{2} = \frac{1}{2}(x_j)(1 - x_j)$.

8. Prove Young's inequality (12.40).

We recall now the following Young's inequality (see Exercise 8)

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad \forall a, b \in \mathbb{R}, \quad \forall \varepsilon > 0. \qquad (12.40)$$

Since $a^2, b^2 \ge 0$ and $\varepsilon > 0$, $\varepsilon a^2 \ge 0$ and $\frac{1}{4\varepsilon} b^2 \ge 0$.

Thus by AM-GM inequality, $\frac{\varepsilon a^2 + \frac{1}{4\varepsilon} b^2}{2} \ge \sqrt{\varepsilon a^2 \cdot \frac{1}{4\varepsilon} b^2} = \frac{1}{2} ab$

$\Rightarrow \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \ge ab$.

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty}\ \forall v_h \in V_h$.

W.L.O.G., let $V_h(x_0) \ge V_h(x_n)$ and $V_h(x_m)^2 = \max\{V_h(x_0)^2, V_h(x_1)^2, \dots, V_h(x_{n-1})^2\}$

Let $V_h \in V_h$.

Then $\|V_h\|_h = \langle V_h, V_h\rangle^{\frac{1}{2}} = \left(h \sum_{k=0}^{n} c_k V_h(x_k)^2\right)^{\frac{1}{2}} \le \left(h \sum_{k=0}^{n-1} (V_h(x_k))^2\right)^{\frac{1}{2}}$

$$\le \left(h \sum_{k=0}^{n-1} V_h(x_m)^2\right)^{\frac{1}{2}} = \left(h \cdot n\, V_h(x_m)^2\right)^{\frac{1}{2}}$$

$$= \left(V_h(x_m)^2\right)^{\frac{1}{2}} = |V_h(x_m)| \le \max_{0\le j\le n} |V_h(x_j)| = \|V_h\|_{h,\infty}.$$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.
[*Solution*: apply twice the second order centered finite difference operator L_h defined in (12.9).]

For all $j = 1, \dots, n-1$, $-u''(x_j) = L_h w_h(x_j)$.

Thus $-u^{(iv)}(x_j) = Lu(x_j) = L_h(L_h w_h)(x_j)$

$$= -\frac{-w_j + 2w_{j-1} - w_{j-2}}{h^2} + 2\,\frac{-w_{j+1} + 2w_j - w_{j-1}}{h^2} - \frac{-w_{j+2} + 2w_{j+1} - w_j}{h^2}$$

$$= \frac{w_{j+2} - (2+2)w_{j+1} + (1+4+1)w_j - (2+2)w_{j-1} + w_{j-2}}{h^2}$$

$$= \frac{w_{j+2} - 4w_{j+1} + 6w_j - 4w_{j-1} + w_{j-2}}{h^2} \quad \text{for all } j = 1, \dots, n-1.$$