5. Prove that

$$(n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \le |\omega_{n+1}(x)| \le n!h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = 2/n$.
[$Hint$: let $N = n/2$ and show first that

$$\omega_{n+1}(x) = (x+Nh)(x+(N-1)h)\ldots(x+h)x \\ (x-h)\ldots(x-(N-1)h)(x-Nh). \tag{8.74}$$

Then, take $x = rh$ with $N-1 < r < N$.]

Let $N = \frac{n}{2}$.

Then $W_{n+1}(x) = \prod_{i=0}^{n}(x-x_i)$

$= (x-x_0)(x-x_1)\cdots(x-x_n)$

$= (x+Nh)(x+(N-1)h)\cdots(x+h)x(x-h)\cdots(x-(N-1)h)(x-Nh)$.

Define $x = rh$ with $N-1 < r < N$. Then $x \in (x_{n-1}, x_n)$.

Thus $|W_{n+1}(x)| = |(x+Nh)(x+(N-1)h)\cdots(x-(N-1)h)(x-Nh)|$

$= |(x+Nh)|\cdots|(x-(N-2)h)|\cdot|(x-(N-1)h)(x-Nh)|$

$\ge |(2N-1)h|\cdots|h|\cdot|(x-x_{n-1})(x-x_n)|$

$= (2N-1)!\cdot h^{2N-1}\cdot|(x-x_{n-1})(x-x_n)| = (n-1)!\cdot h^{n-1}|(x-x_{n-1})(x-x_n)|$. — ①

Similarly, $|W_{n+1}(x)| = |(x+Nh)(x+(N-1)h)\cdots(x-(N-1)h)(x-Nh)|$

$= |(x+Nh)|\cdots|(x-(N-2)h)|\cdot|(x-(N-1)h)(x-Nh)|$

$\le |(2Nh)\cdots(2h)|\cdot|(x-x_{n-1})(x-x_n)|$

$= (2N)!\cdot h^{2N-1}\cdot|(x-x_{n-1})(x-x_n)| = n!\cdot h^{n-1}|(x-x_{n-1})(x-x_n)|$. — ②

By ① and ②, we have $(n-1)!\cdot h^{n-1}|(x-x_{n-1})(x-x_n)| \le |W_{n+1}(x)| \le n!\cdot h^{n-1}|(x-x_{n-1})(x-x_n)|$.

6. Under the assumptions of Exercise 5, show that $|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$ (notice that $|\omega_{n+1}|$ is an even function).
[*Hint* : use (8.74) to prove that $|\omega_{n+1}(x+h)/\omega_{n+1}(x)| > 1$ for any $x \in (0, x_{n-1})$ with x not coinciding with any interpolation node.]

Let $x \in (0, x_{n-1}) \setminus \{x_1, x_2, \dots, x_{n-2}\}$.

Then $\left|\frac{\omega_{n+1}(x+h)}{\omega_n(x+h)}\right| = \left|\frac{(x+(N+1)h)(x+Nh)\cdots(x-(N-2)h)(x-(N-1)h)}{(x+Nh)(x+(N-1)h)\cdots(x-(N-1)h)(x-Nh)}\right|$

$$= \left|\frac{(x+(N+1)h)}{(x-Nh)}\right| = \left|\frac{x+(N+1)h}{Nh-x}\right| = \left|1+\frac{2x+h}{Nh-x}\right|.$$

Since $2x+h>0$ and $Nh-x>0$, we have $1 < \left|1+\frac{2x+h}{Nh-x}\right| = \left|\frac{\omega_{n+1}(x+h)}{\omega_{n+1}(x)}\right|$.

Thus $|\omega_{n+1}(x+h)| > |\omega_{n+1}(x)|$.

Since $x \in (0, x_{n-1}) \setminus \{x_1, x_2, \dots, x_{n-2}\}$ is arbitrary,

we know that $|\omega_{n+1}(x+h)| > |\omega_{n+1}(x)|$ for all $x \in (0, x_{n-1}) \setminus \{x_1, x_2, \dots, x_{n-2}\}$.

Also, $|\omega_{n+1}(x)| = 0$ for all $x = x_0, x_1, \dots, x_n$.

Hence $\exists x_p \in (x_{n-1}, x_n)$ s.t. $|\omega_{n+1}(x_p)| = \sup\{|\omega_{n+1}(x)| : x \in [0, x_n]\}$ by EVT.

Since $|\omega_{n+1}|$ is an even function, i.e., $|\omega_{n+1}(x)| = |\omega_{n+1}(-x)|$ for all $x \in [0, x_n]$.

The value $|\omega_{n+1}(x_p)|$ is also attained at $-x_p$ and it's the maximum value on the set $\{|\omega_{n+1}(x)| : x \in [x_0, 0]\}$.

Therefore, $|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$.

8. Determine an interpolating polynomial $Hf \in \mathbb{P}_n$ such that

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \qquad k = 0, \ldots, n,$$

and check that

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j,$$

that is, the Hermite interpolating polynomial on one node coincides with the *Taylor polynomial.*

Let $Hf(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \cdot L_k(x)$ where $L_k(x) \in \mathbb{P}_k$ satisfies

$$\frac{d^p}{dx^p} L_k(x_0) = \begin{cases} 1, & \text{if } k = p \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } k = 0, \ldots, n.$$

Defind $L_k : \mathbb{R} \to \mathbb{R}$ by $L_k(x) = \frac{(x - x_0)^k}{k!}$ for all $k = 0, \ldots, n$.

It is easy to check that L_k satisfies the condition.

Thus $Hf(x) = \sum_{k=0}^{n} f^{(k)}(x_0) \cdot \frac{(x - x_0)^k}{k!}$.