1 Problem 1: Linear Boundary Value Problem

We consider the boundary value problem

$$-u'' = e^{\sin x}, \qquad u(0) = 0, \quad u(1) = 0.$$

1.1 Background

This is a linear second-order ordinary differential equation with boundary conditions. We apply the finite difference method to approximate the second derivative:

$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

The local truncation error of this approximation is $O(h^2)$,

1.2 Numerical Method

We obtain a tridiagonal linear system

$$A\,u = f,$$

where A is a symmetric tridiagonal matrix. The system can be solved by the Crout factorization,

1.3 Result

We use different grid sizes to compare the error.

N	h	$\|u_h - u\|_\infty$	Richardson
750	1.333×10^{-3}	1.381521×10^{-7}	NAN
1500	6.667×10^{-4}	4.883307×10^{-8}	3.4541×10^{-8}
3000	3.333×10^{-4}	1.723571×10^{-8}	1.2215×10^{-8}
6000	1.667×10^{-4}	6.034371×10^{-9}	4.3229×10^{-9}
12000	8.333×10^{-5}	1.935532×10^{-9}	1.5560×10^{-9}
24000	4.167×10^{-5}	7.702074×10^{-10}	5.1667×10^{-10}
48000	2.083×10^{-5}	8.482267×10^{-9}	1.7615×10^{-9}
96000	1.042×10^{-5}	1.477472×10^{-8}	1.6978×10^{-9}
192000	5.208×10^{-6}	3.254423×10^{-8}	8.1435×10^{-9}

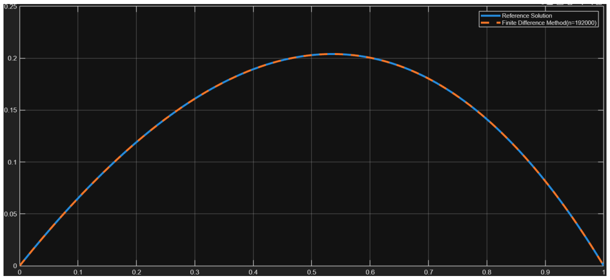

Figure 1

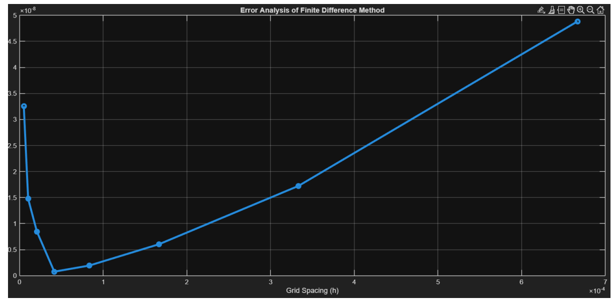

Figure 2: Error Analysis

The total error can be expressed as

$$E(h) \approx O(h^2) + O(h^{-2}),$$

where $O(h^2)$ is truncation error and $O(h^{-2})$ is rounding error.
We can see that for N between 750 and 24000, when N doubles, the error is approximately divided by four. This behavior is consistent with the theoretical truncation error of order $O(h^2)$.
In the table, the smallest error occurs when $N = 24000$. Because of the presence of rounding error $O(h^{-2})$, even though a smaller h reduces the truncation error, the rounding error will increase. Therefore, the total error cannot approach zero.

2 Problem 2: Nonlinear Boundary Value Problem

We consider the nonlinear boundary value problem

$$-u'' + \sin(u) = 0, \qquad u(0) = 1, \quad u(1) = 1.$$

2.1 Background

This is a second-order ordinary differential equation with boundary conditions. We apply the finite difference method to approximate the second derivative:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \sin(u_i), \qquad i = 1, \ldots, N-1,$$

This discretization leads to a nonlinear algebraic system $F(u) = 0$,

2.2 Numerical Method

We solve $F(u) = 0$ by Newton's method and Crout factorization:

$$J(u^{(k)})\, \delta u = -F(u^{(k)}), \qquad u^{(k+1)} = u^{(k)} + \delta u,$$

where the tridiagonal Jacobian is

$$J_{ii} = -\frac{2}{h^2} - \cos(u_i^{(k)}), \qquad J_{i,i\pm 1} = \frac{1}{h^2}.$$

2.3 Result

N	r_1	r_2	r_3	r_4	r_5
10	4.0757×10^{-3}	4.5330×10^{-6}	3.6115×10^{-9}	3.1221×10^{-12}	1.1990×10^{-14}
20	4.0783×10^{-3}	2.1029×10^{-6}	6.3606×10^{-10}	2.6457×10^{-13}	7.5162×10^{-14}
40	4.0789×10^{-3}	9.9668×10^{-7}	1.2908×10^{-10}	2.5591×10^{-13}	3.4084×10^{-13}
80	4.0791×10^{-3}	4.9309×10^{-7}	3.1589×10^{-11}	9.5768×10^{-13}	1.1303×10^{-12}
160	4.0791×10^{-3}	2.6051×10^{-7}	1.3194×10^{-11}	3.9555×10^{-12}	3.5819×10^{-12}
320	4.0791×10^{-3}	1.4307×10^{-7}	2.2029×10^{-11}	2.1300×10^{-11}	2.1300×10^{-11}
640	4.0791×10^{-3}	8.6051×10^{-8}	7.9798×10^{-11}	7.7025×10^{-11}	8.1667×10^{-11}

Coarse N	Fine N	$\max \lvert u_h^{fine} - u_H^{coarse} \rvert$
10	20	3.392973×10^{-5}
20	40	8.474811×10^{-6}
40	80	2.118223×10^{-6}
80	160	5.295258×10^{-7}
160	320	1.323796×10^{-7}
320	640	3.309478×10^{-8}

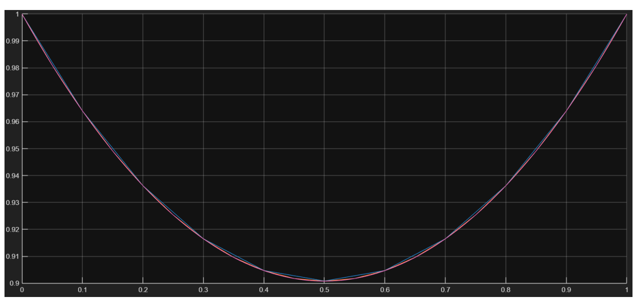

Figure 3: approximate with different grid sizes