1 Problem 1

$$u_t = u_{xx}, \quad 0 \le x \le 1, \quad t \ge 0$$

with initial and boundary conditions

$$u(x,0) = \sin\left(\frac{1}{2}\pi x\right) + \frac{1}{2}\sin(2\pi x), \quad u(0,t) = 0, \quad u(1,t) = e^{-\pi^2 t/4}, \quad t \ge 0.$$

Solve the problem using Forward Euler finite difference method with $\mu = 0.5$ and $\mu = 0.509$.

1.1 Method

We approach to T = 1 in this question.
Let $x_k = \frac{k}{n}$ for all k = $0, \cdots, n$. be the grid points.
Let $\Delta t = \frac{\mu}{n^2}$. Then the total time steps is $S = \lfloor \frac{1}{\Delta t} \rfloor$.
Let $u_k^m = u(x_k, \Delta t \times m)$.

By the Forward Euler finite difference method,

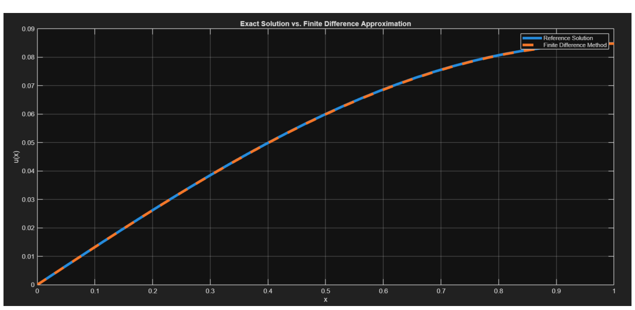
$$u_k^{m+1} = u_k^m + \mu(u_{k+1}^m - 2u_k^m + u_{k-1}^m) \text{ for } k = 1, \cdots, n-1.$$

We use the initial condition to get

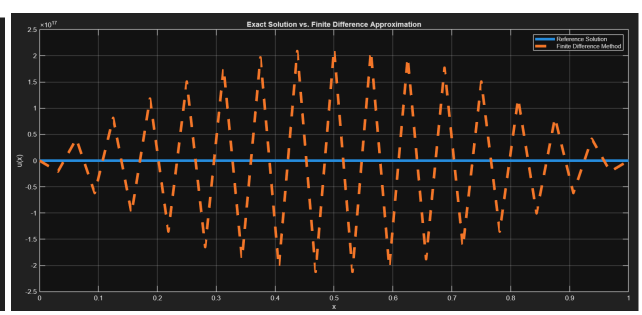
$$u_0^0 = 0, \; u_k^0 = u(x_k, 0) \text{ for } k = 1, \cdots, n-1 \text{ and } u_n^0 = 1.$$

After $S = \lfloor 1/\Delta t \rfloor$ time steps, we obtain the numerical approximation of the solution $(u_1^S, u_2^S, \ldots, u_{n-2}^S, u_{n-1}^S)$ with imposing the boundary conditions at the final time $T = 1$.
Finally, we compare the numerical solution with the exact solution $u(x,t) = e^{\frac{-\pi^2 t}{4}} sin(\frac{\pi x}{2}) + \frac{1}{2}e^{(-4\pi^2 t)} sin(2\pi x)$ to obtain the error.

1.2 Results



(a) $\mu = 0.5$



(b) $\mu = 0.509$

The $||u_h - u_{exact}||_\infty$ of finite difference method is 9.536633e-06 when $\mu = 0.5$, 2.1329×10^{16} when $\mu = 0.509$.

2 Problem 2

$$u_t = u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0$$

with initial and boundary conditions

$$u(x,0) = \sin(2\pi x)\, e^x, \quad u(0,t) = u(1,t) = 0, \quad t \geq 0.$$

With the aid of Fast Since transform, solve the problem using (a) finite difference method and (b) method of line approach to T = 1. Find the order of convergence.

2.1 Method

Let $x_k = \frac{k}{n}$ for all k = $0, \cdots, n$. be the grid points.
Since we can't find the exact solution of this pde, we use the numerical solution with sufficient large grid number to be the exact solution. Here, we use 4096.
To let the 1 / Δt can be an integral, we suppose the

$$\Delta t = \frac{1}{10^{\left(\lfloor \log_{10}(2M^2) \rfloor + 1\right)}}.$$

Then the step of time is M = $10^{\left(\lfloor \log_{10}(2M^2) \rfloor + 1\right)}$
For the finite difference method, we know that

$$u_k^{m+1} = u_k^m + \mu(u_{k+1}^m - 2u_k^m + u_{k-1}^m) \text{ for } k = 1, \cdots, n-1.$$

Thus the discretized linear system is

$$\begin{pmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & 1-2\mu & \mu \\ & & & \mu & 1-2\mu \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} u_1^m \\ u_2^m \\ \vdots \\ u_{n-2}^m \\ u_{n-1}^m \end{pmatrix} = \begin{pmatrix} u_1^{m+1} \\ u_2^{m+1} \\ \vdots \\ u_{n-2}^{m+1} \\ u_{n-1}^{m+1} \end{pmatrix}.$$

We define the system as

$$Au^m = u^{m+1}.$$

Start with $u^0 = (u_0^0, u_1^0, \cdots, u_{n-1}^0, u_n^0)$, after $M = \lfloor 1/\Delta t \rfloor$ time steps, we obtain

$$\begin{pmatrix} 1-2\mu & \mu & & & \\ \mu & 1-2\mu & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & 1-2\mu & \mu \\ & & & \mu & 1-2\mu \end{pmatrix}_{(n-1)\times(n-1)}^{M} \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_{n-2}^0 \\ u_{n-1}^0 \end{pmatrix} = \begin{pmatrix} u_1^M \\ u_2^M \\ \vdots \\ u_{n-2}^M \\ u_{n-1}^M \end{pmatrix}.$$

and $u_0^S,\ u_n^S$ by the boundary conditions.
Here we diagonalize the matrix A, then A $= S\Lambda S^{-1}$, where

$$S = \begin{pmatrix} \sin\frac{\pi}{n} & \sin\frac{2\pi}{n} & \cdots & \sin\frac{(n-1)\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{4\pi}{n} & \cdots & \sin\frac{2(n-1)\pi}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \sin\frac{(n-1)\pi}{n} & \sin\frac{2(n-1)\pi}{n} & \cdots & \sin\frac{(n-1)^2\pi}{n} \end{pmatrix},$$

$$\Lambda = \operatorname{diag}\!\big(1-4\mu\sin^2\tfrac{\pi}{2n},\ 1-4\mu\sin^2\tfrac{2\pi}{2n},\ \ldots,\ 1-4\mu\sin^2\tfrac{(n-1)\pi}{2n}\big).$$

and $S^{-1} = \frac{2}{n}S$. Thus $u^M = A^M u^0 = (S\Lambda S^{-1})^M u^0 = S\Lambda^M S^{-1}u^0$.

Let $S^{-1}u^M = v^M$ and $S^{-1}u^0 = v^0$. Then $v^M = S^{-1}u^M = \Lambda^M S^{-1}u^0 = \Lambda^M v^0$.
Let N = 2n. Then for $l = 1, \cdots, n-1$,

$$\begin{aligned} v_l^0 &= \frac{2}{n}\sum_{j=1}^{n-1} u_j^0 \sin\left(\frac{lj\pi}{n}\right) = \frac{2}{n}\sum_{j=1}^{n-1} u_j^0 \left(\frac{e^{i\frac{lj\pi}{n}} - e^{-i\frac{lj\pi}{n}}}{2i}\right) \\ &= \frac{2}{n}\frac{1}{2i}\left(\sum_{j=1}^{n-1} u_j^0 e^{i\frac{lj\pi}{n}} - \sum_{j=1}^{n-1} u_j^0 e^{-i\frac{lj\pi}{n}}\right) \\ &= \frac{2}{n}\frac{1}{2i}\left(\sum_{j=1}^{n-1} u_j^0 e^{i\frac{2lj\pi}{N}} - \sum_{j=1}^{n-1} u_j^0 e^{-i\frac{2lj\pi}{N}}\right) \\ &= \frac{2}{n}\frac{1}{2i}\left(\sum_{j=1}^{n-1} u_j^0 e^{i\frac{2l(N-j)\pi}{N}} - \sum_{j=1}^{n-1} u_j^0 e^{-i\frac{2lj\pi}{N}}\right) \\ &= \frac{2}{n}\frac{i}{2}\left(\sum_{j=1}^{n-1} u_j^0 e^{-i\frac{2lj\pi}{N}} - \sum_{j=1}^{n-1} u_j^0 e^{i\frac{2l(N-j)\pi}{N}}\right) \\ &= \frac{2}{n}(0,\, u_1^0,\, \ldots,\, u_{n-1}^0,\, 0,\, u_{n-1}^0,\, \ldots,\, u_1^0)\cdot(e^{0},\, e^{-i\frac{2l\pi}{N}},\, \ldots,\, e^{-i\frac{2l(N-1)\pi}{N}}) \\ &= \text{the } l\text{-th component of the DFT of } (0,\, u_1^0,\, \ldots,\, u_{n-1}^0,\, 0,\, u_{n-1}^0,\, \ldots,\, u_1^0). \end{aligned}$$

Thus $v^M = \Lambda^M v^0 = ((1-4\mu\sin^2\frac{\pi}{2n})^M \times v_1^0, \cdots, (1-4\mu\sin^2\frac{(n-1)\pi}{2n})^M \times v_{n-1}^0)$.
Similarly, u_l^M = the l-th component of the DFT of v^M.

For Method of line,

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \frac{1}{\Delta x^2}\begin{pmatrix} -2\mu & \mu & & & \\ \mu & -2\mu & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & -2\mu & \mu \\ & & & \mu & -2\mu \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}$$

We define the system as

$$\frac{d}{dt}\vec{u} = \frac{1}{\Delta x^2}A\vec{u}.$$

we diagonalize the matrix A, then A $= S\Lambda S^{-1}$, where S and S^{-1} same as above, and

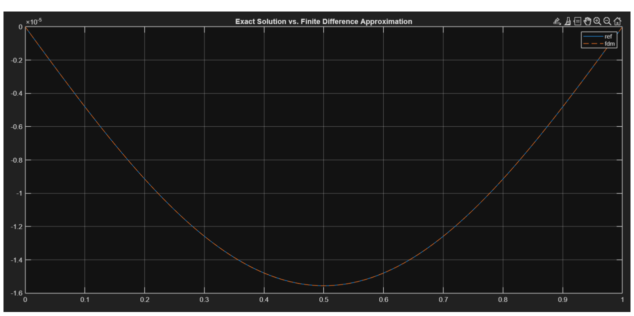
$$\Lambda = \operatorname{diag}\Big(-4\mu\sin^2\tfrac{\pi}{2n},\ -4\mu\sin^2\tfrac{2\pi}{2n},\ \ldots,\ -4\mu\sin^2\tfrac{(n-1)\pi}{2n}\Big).$$

Let $\vec{u} = (u_1, \cdots, u_{n-1})^T$ and $S^{-1}\vec{u} = \vec{v}$,$S^{-1}\vec{u} = \vec{v}$.
Then

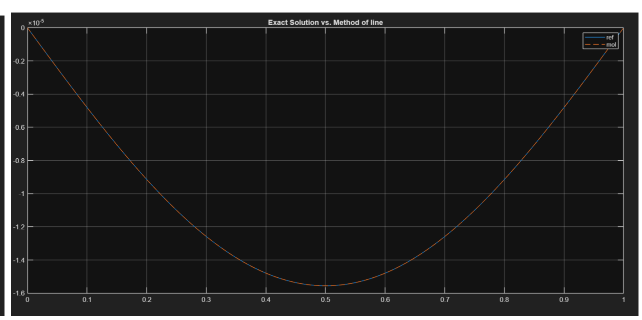
$$\begin{aligned} &\frac{d}{dt}\vec{v} = \frac{1}{\Delta x^2}\Lambda\vec{v} \\ &\Rightarrow \vec{v}(t) = e^{\frac{1}{\Delta x^2}\Lambda t}\vec{v}(0) \\ &\Rightarrow v_l(t) = e^{\frac{1}{\Delta x^2}\lambda_l t}v_l(0) \quad \text{where } \lambda_l = -4\mu\sin^2\tfrac{l\pi}{2n} \text{ for } l = 1, \cdots, n-1. \end{aligned}$$

Thus $\vec{u}(1) = S\vec{v}(1)$.

2.2 Results



(a) Exact Solution vs. FDM of u(1)



(b) Exact Solution vs. MOL of u(1)

n	$\text{error}_{\text{FDM}}$	order	$\text{error}_{\text{MOL}}$	order
2^3	2.073040×10^{-6}	–	2.073048×10^{-6}	–
2^4	4.993772×10^{-7}	2.0535	4.993849×10^{-7}	2.0535
2^5	1.236685×10^{-7}	2.0137	1.236761×10^{-7}	2.0136
2^6	3.083852×10^{-8}	2.0037	3.084604×10^{-8}	2.0034
2^7	7.699342×10^{-9}	2.0019	7.706930×10^{-9}	2.0009
2^8	1.918781×10^{-9}	2.0045	1.926423×10^{-9}	2.0002
2^9	4.739776×10^{-10}	2.0173	4.815631×10^{-10}	2.0001

Table 1: Comparison of FDM and MOL errors and observed orders of convergence.