**Written assignments**

1. Let $u(x,t) = 1$ be the exact solution of (1). ($f \equiv 0$.) Explain why the energy estimate (2) does not hold here.

The heat equation is given by

$$u_t = \nu u_{xx} + f(x,t). \tag{1}$$

$$E(t) \le e^{-\gamma t}E(0) + \frac{1}{\gamma}\int_0^t e^{\gamma(s-t)}F(s)\,ds, \tag{2}$$

where $F(t) = \int_0^1 f^2(x,t)\,dx$.

To prove (2), we use the Poincare inequality : $\int_0^1 u^2 dx \le C_p^2 \int_0^1 (u_x)^2 dx$.

If $u(x,t) = 1$, then $1 = \int_0^1 1^2 dx \le C_p^2 \int_0^1 0\,dx = 0$, which does not hold.

It can be regarded as the discrete counterpart in $[0,1]$ of the following *Poincaré inequality*: for every interval $[a,b]$ there exists a constant $C_P > 0$ such that

$$\|v\|_{L^2(a,b)} \le C_P \|v^{(1)}\|_{L^2(a,b)} \tag{12.16}$$

for all $v \in C^1([a,b])$ such that $v(a) = v(b) = 0$ and where $\|\cdot\|_{L^2(a,b)}$ is the norm in $L^2(a,b)$ (see (8.29)). ■ book P.544

Since $u(x,t) = 1$, $u(0,t) = 1$ and $u(1,t) = 1$.

But the boundary conditions require $u(0,t) = u(1,t) = 0$, a contradiction.

Therefore, $u(x,t) = 1$ can't be a solution of any heat equation with homogeneous Dirichlet boundary conditions.