1. Consider the boundary value problem:

$$u'' = f, \quad u(0) = 0, \quad u(1) = 0,$$

where

$$f(x) = \begin{cases} 1 & 0.4 \le x \le 0.6, \\ 0 & \text{otherwise.} \end{cases}$$

- Find the exact solution of this problem.
- Solve the problem using finite difference method and check the accuracy of your solutions.

Since u'' exists, u' and u are continuous on $[0,1]$.

Let $u(x) = \begin{cases} ax, & 0 \le x \le 0.4 \\ \frac{1}{2}x^2 + cx + d, & 0.4 \le x \le 0.6 \\ b(1-x), & 0.6 \le x \le 1 \end{cases}$.

Then $\begin{cases} 0.4a = u(0.4) = 0.08 + 0.4c + d \\ 0.4b = u(0.6) = 0.18 + 0.06c + d \\ a = u'(0.4) = 0.4 + c \\ -b = 0.6 + c \end{cases} \Rightarrow \begin{cases} a = -0.1 \\ b = -0.1 \\ c = -0.5 \\ d = 0.08 \end{cases}$

2. Consider the boundary value problem:

$$u'' - 2u' + u = 1, \quad u(0) = 0, \quad u'(1) = 1.$$

- Show that the solution is unique by considering the homogeneous problem.
- Develop a 2nd-order finite difference method.
- Solve the problem and check the accuracy of your solutions.

The homogeneous problem : $\begin{cases} u'' - 2u' + u = 0 \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$

Let $u = e^{rx}$. Then $r^2 e^{rx} - 2r e^{rx} + e^{rx} = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow r = 1 \vee 1$.

Thus $u(x) = C_1 e^x + C_2 x e^x$.

Since $0 = u(0) = C_1 + C_2$ and $0 = u'(1) = C_1 e + 2C_2 e$, we obtain $C_1 = C_2 = 0$.

Then the homogeneous problem only has zero solution.

Therefore the solution is unique.

For non-homogeneous one, $u(x) = C_1 e^x + C_2 x e^x + 1$.

Since $0 = u(0) = C_1 + 1$, $C_1 = -1$

Since $1 = u'(1) = C_1 e + C_2 e + C_2 e$, $C_2 = \frac{1+e}{2e}$.

3. Consider the boundary value problem:

$$u'' = \sin(2\pi x), \quad u'(0) = 0, \quad u'(1) = 0.$$

- Show that the consistency condition is satisfied so that the solution of the problem exists.
- Develop a 2nd-order finite difference method.
- Solve the problem and check the accuracy of your solutions.

$$\int_0^1 u''(x)\,dx = \int_0^1 \sin(2\pi x)\,dx = \frac{-1}{2\pi}\cos(2\pi x)\Big|_0^1 = 0.$$

$$\int_0^1 u''(x)\,dx = u'(x)\Big|_0^1 = u'(1) - u'(0) = 0$$

Thus $\int_0^1 \sin(2\pi x)\,dx = u'(1) - u'(0) = 0$.

Therefore the problem satisfies the consistency condition.

Let $u(0) = 0$.

Let $u = A\sin(2\pi x) + C_1 x + C_2$.

Then $-4\pi^2 A\sin(2\pi x) = \sin(2\pi x)$. Thus $A = \frac{-1}{4\pi^2}$

Since $0 = u(0) = C_2$, $C_1 = 0$

Since $0 = u'(1) = \frac{1}{2\pi}\cos(2\pi) + C_1 \Rightarrow C_1 = \frac{1}{2\pi}$.

Therefore $u(x) = \frac{-1}{4\pi^2}\sin(2\pi x) + \frac{1}{2\pi}x$.

4. Consider the boundary value problem:

$$u'' = e^{\sin(x)}, \quad u'(0) = 0, \quad u'(1) = \alpha.$$

- Determine α such that the problem has at least one solution.
- Solve the problem by finding one of its solution.

① Then the problem must satisfy the consistency condition.

$$\int_0^1 u''(x)\,dx = \int_0^1 e^{\sin(x)}\,dx.$$

$$\int_0^1 u''(x)\,dx = u'(1) - u'(0) = \alpha.$$

Thus $\alpha = \int_0^1 e^{\sin(x)}\,dx$

②

$$u''(x) = e^{\sin(x)}$$

$\Rightarrow u'(x) = C_1 + \int_0^x e^{\sin(a)}\,da$, we obtain $0 = u'(0) = C_1$.

$$\Rightarrow u(x) = C_2 + \int_0^x \int_0^a e^{\sin(t)}\,dt\,da$$

Suppose $C_2 = 0$. Then $u(x) = \int_0^x \int_0^a e^{\sin(t)}\,dt\,da$ is one of its solution.

$$\int_0^x \int_0^a e^{\sin(t)}\,dt\,da$$

$$= \int_0^x \int_t^x e^{\sin(t)}\,da\,dt$$

$$= \int_0^x (x-t)\,e^{\sin(t)}\,dt$$

5. Consider the linear boundary value problem:

$$\epsilon u'' + (1+\epsilon)u' + u = 0, \quad u(0) = 0, \quad u(1) = 1.$$

Solve the problem and check the accuracy of your solutions. Choose $\epsilon = 0.01$.

let $u = e^{rx}$.

Then $\varepsilon r^2 + (1+\varepsilon)r + 1 = 0 \Rightarrow r = \dfrac{-1-\varepsilon \pm \sqrt{\varepsilon^2 - 2\varepsilon + 1}}{2\varepsilon}$

$$= \frac{-1-\varepsilon \pm \varepsilon - 1}{2\varepsilon}$$

$$= \frac{-1}{\varepsilon} \cup -1$$

Thus $u(x) = C_1 e^{\frac{-1}{\varepsilon}x} + C_2 e^{-x}$.

Since $0 = u(0) = C_1 + C_2$ and $1 = C_1 e^{\frac{-1}{\varepsilon}} + C_2 e^{-1}$,

$C_1 = \dfrac{-e}{1 - e^{\frac{-1}{\varepsilon}+1}}$ and $C_2 = \dfrac{e}{1 - e^{\frac{-1}{\varepsilon}+1}}$.