Finite Difference Formulas

By Taylor expansion, we obtain

$$u(x_{i\pm1}) = u(x_i) \pm hu'(x_i) + \frac{h^2}{2}u''(x_i) \pm \frac{h^3}{6}u^{(3)}(x_i) + O(h^4).$$

By combining the expansions, we obtain

$$u'(x_i) = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2), \qquad u''(x_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + O(h^2),$$

1 Problem 1

$$u'' = f, \quad u(0) = 0, \quad u(1) = 0,$$

where

$$f(x) = \begin{cases} 1, & 0.4 \leq x \leq 0.6, \\ 0, & \text{otherwise.} \end{cases}$$

1.1 Method

Let $x_k = \frac{k}{n}$, for $k = 0, \cdots, n$, be the grid points.
We apply the finite difference method to approximate the second derivative:

$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

The local truncation error of this approximation is $O(h^2)$.
After computing, the discretized linear system is

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix},$$

where $u_i = u(x_i), \quad f_i = f(x_i),$ for i $= 1, \cdots, n-1$.
Use the Thomas algorithm to solve the linear system.

1.2 Results

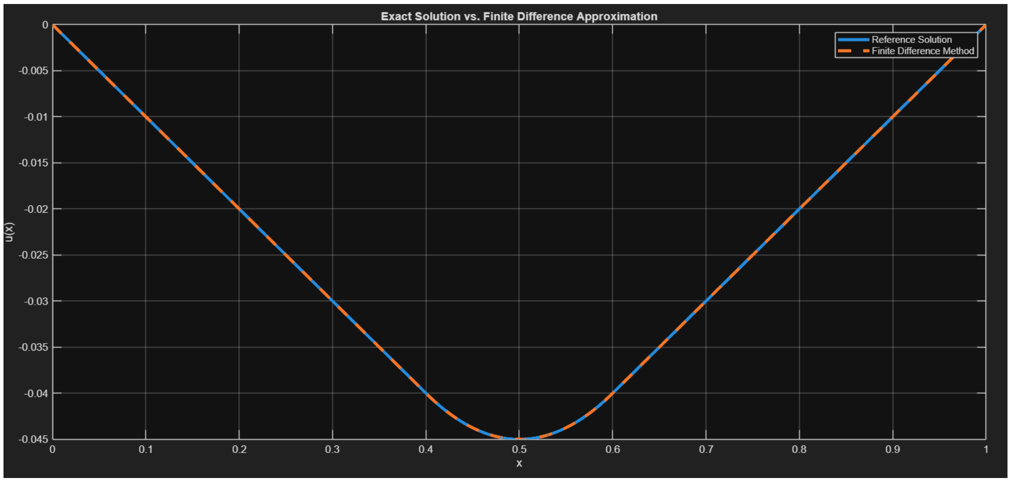
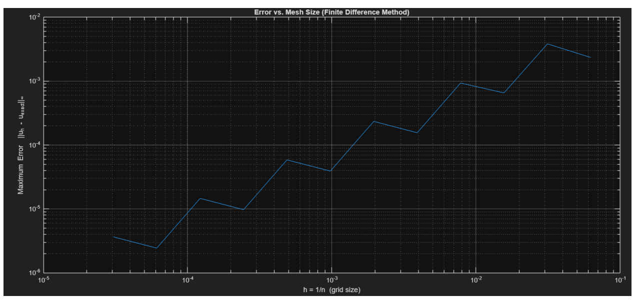
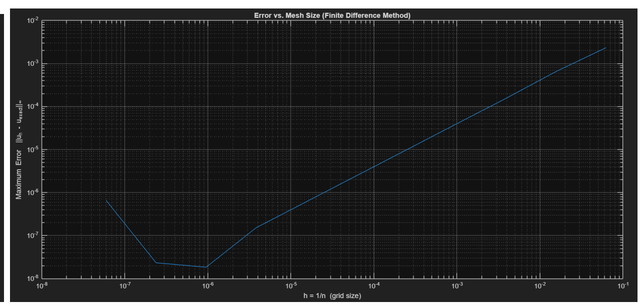


Figure 1: Exact Solution vs. Finite Difference Approximation

Error vs. Mesh Size (Finite Difference Method)



(a) $n\_list = 2^{(4:15)}$



(b) $n\_list = 4^{(2:12)}$

n	h	Max Error
16	6.250×10^{-2}	2.343750×10^{-3}
32	3.125×10^{-2}	3.828125×10^{-3}
64	1.562×10^{-2}	6.542969×10^{-4}
128	7.812×10^{-3}	9.338379×10^{-4}
256	3.906×10^{-3}	1.556396×10^{-4}
512	1.953×10^{-3}	2.346802×10^{-4}
1024	9.766×10^{-4}	3.917694×10^{-5}
2048	4.883×10^{-4}	5.857944×10^{-5}
4096	2.441×10^{-4}	9.763241×10^{-6}
8192	1.221×10^{-4}	1.464963×10^{-5}
16384	6.104×10^{-5}	2.441854×10^{-6}
32768	3.052×10^{-5}	3.662060×10^{-6}

Table 1: Error table

n	h	Max Error
16	6.250×10^{-2}	2.343750×10^{-3}
64	1.562×10^{-2}	6.542969×10^{-4}
256	3.906×10^{-3}	1.556396×10^{-4}
1024	9.766×10^{-4}	3.917694×10^{-5}
4096	2.441×10^{-4}	9.763241×10^{-6}
16384	6.104×10^{-5}	2.441854×10^{-6}
65536	1.526×10^{-5}	6.103585×10^{-7}
262144	3.815×10^{-6}	1.522639×10^{-7}
1048576	9.537×10^{-7}	1.847101×10^{-8}
4194304	2.384×10^{-7}	2.343450×10^{-8}
16777216	5.960×10^{-8}	6.651423×10^{-7}

Table 2: Error table

2 Problem 2

$$u'' - 2u' + u = 1, \quad u(0) = 0, \quad u'(1) = 1.$$

2.1 Method

Let $x_k = \frac{k}{n}$ for all k $= 0, \cdots, n$. be the grid points.
Let $u_i = u(x_i), \quad f_i = f(x_i),$ for i $= 1, \cdots, n-1$.

Applying the finite difference method, we obtain

$$\frac{(1+2h)u_{k+1} + (-2+h^2)u_k + (1-2h)u_{k-1}}{h^2} = 1, \quad k = 1, \cdots, n-1.$$

We want to find coefficients A, B and C such that $Au_{n-2} + Bu_{n-1} + Cu_n = u'_n = 1$.
By Taylor expansion, we obtain

$$u'_n = (A+B+C)u_n - (Bh + 2Ah)u'_n + (\frac{B}{2}h^2 + 2h^2A)u''_n + O(h^3)$$

Thus $\begin{cases} A+B+C=0 \\ Bh+2Ah=1 \\ \frac{B}{2}+2h^2A=0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2h} \\ B = \frac{-2}{h} \\ C = \frac{-3}{2h} \end{cases}$

So $2hu'_n = u_{n-2} - 4u_{n-1} + 3u_n + O(h^2)$.

With the boundary condition $u(0) = 0$, the discretized linear system is

$$\begin{pmatrix} -2+h^2 & 1-2h & & & \\ 1+2h & -2+h^2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1+2h & -2+h^2 & 1-2h \\ & & 1 & -4 & 3 \end{pmatrix}_{n\times n} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} h^2 \\ h^2 \\ \vdots \\ h^2 \\ 2h \end{pmatrix}.$$

Since the system is not tridiagonal, we use

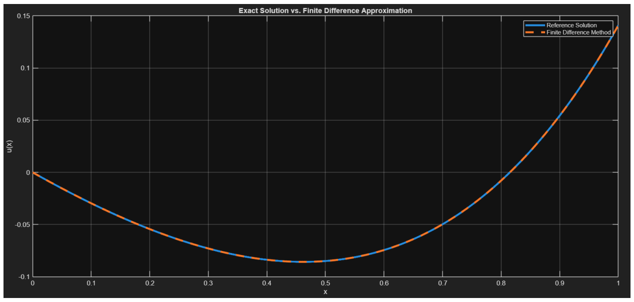
$$u_N = \frac{2h - u_{n-2} + 4u_{n-1}}{3}$$

to rewrite the system and solve u_n separately at the end. Thus the remaining system becomes tridiagonal,

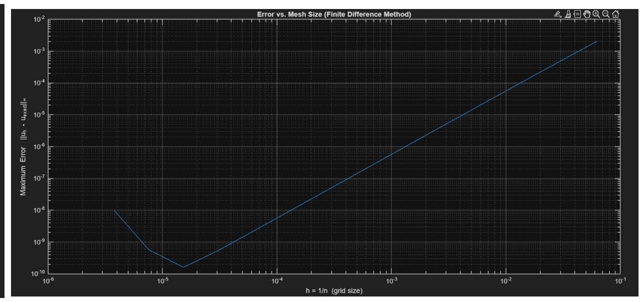
$$\begin{pmatrix} -2+h^2 & 1-2h & & & \\ 1+2h & -2+h^2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1+2h & -2+h^2 & 1-2h \\ & & & \frac{2+4h}{3} & \frac{2-8h+h^2}{3} \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} h^2 \\ h^2 \\ \vdots \\ h^2 \\ \frac{7h^2-2h}{3} \end{pmatrix}.$$

Then we can use the Thomas algorithm.

2.2 Results



(a) Exact Solution vs. Approximation



(b) Error vs. Mesh Size $2.^{(4:18)}$

n	h	Max Error
16	6.250×10^{-2}	2.041343×10^{-3}
32	3.125×10^{-2}	5.318073×10^{-4}
64	1.562×10^{-2}	1.358499×10^{-4}
128	7.812×10^{-3}	3.433920×10^{-5}
256	3.906×10^{-3}	8.632837×10^{-6}
512	1.953×10^{-3}	2.164274×10^{-6}
1024	9.766×10^{-4}	5.418304×10^{-7}
2048	4.883×10^{-4}	1.355528×10^{-7}
4096	2.441×10^{-4}	3.390058×10^{-8}
8192	1.221×10^{-4}	8.481561×10^{-9}
16384	6.104×10^{-5}	2.121165×10^{-9}
32768	3.052×10^{-5}	5.356322×10^{-10}
65536	1.526×10^{-5}	1.614016×10^{-10}
131072	7.629×10^{-6}	5.665124×10^{-10}
262144	3.815×10^{-6}	9.644504×10^{-9}

Table 3: Maximum error for various grid sizes n.

3 Problem 3

$$u'' = \sin(2\pi x), \quad u'(0) = 0, \quad u'(1) = 0.$$

3.1 Method

Let $x_k = \frac{k}{n}$ for all k $= 0, \cdots, n$. be the grid points.
Let $u_i = u(x_i), \quad f_i = f(x_i),$ for i $= 1, \cdots, n-1$.

Apply the finite difference method to approximate the second derivative:

$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

We want to find the A, B and C such that $Au_0 + Bu_1 + Cu_2 = u_0' = 1$.
By the taylor expansion, we obtain

$$u_0' = (A + B + C)u_0 - ((Bh + 2Ch)u_0' + (\frac{B}{2}h^2 + 2h^2C)u_0'' + O(h^3)$$

Thus $\begin{cases} A + B + C = 0 \\ Bh + 2Ch = 1 \\ \frac{B}{2} + 2h^2C = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{-3}{2h} \\ B = \frac{2}{h} \\ C = \frac{-1}{2h} \end{cases}$
So $2hu_0' = -3u_0 + 4u_1 - u_n + O(h^2)$.

Thus the discretized linear system is

$$\begin{pmatrix} -3 & 4 & -1 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 1 & -4 & 3 \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} 0 \\ f_1 \\ \vdots \\ f_n \\ 0 \end{pmatrix}.$$

Since this system has infinitely many solutions(A$(1, \cdots, 1)^T = 0$), we need to impose an additional condition to make the solution unique. Here, we impose u(0) = 0. Therefore, the system becomes

$$\begin{pmatrix} 1 & 0 & 0 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 1 & -4 & 3 \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} 0 \\ f_1 \\ \vdots \\ f_n \\ 0 \end{pmatrix}.$$

Similar to Problem 2, we first transform the system into a tridiagonal one. Then we solve it with the Thomas algorithm, and finally compute u_n .

$$\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & \frac{2}{3} & \frac{-2}{3} \end{pmatrix}_{(n-1)\times(n-1)} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix}$$

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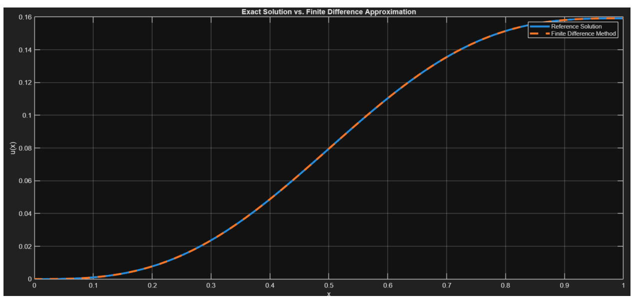
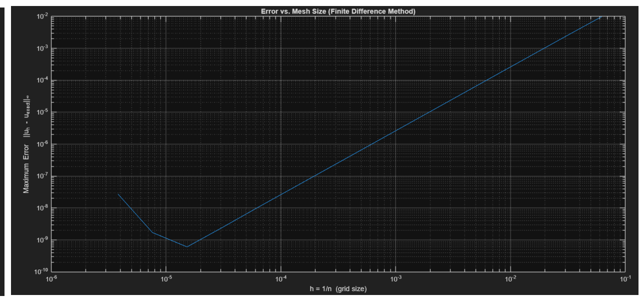
3.2 Results



(a) Exact Solution vs. Approximation



(b) Error vs. Mesh Size 2.$^{(4:18)}$

n	h	Max Error
16	6.250×10^{-2}	9.908273×10^{-3}
32	3.125×10^{-2}	2.536630×10^{-3}
64	1.562×10^{-2}	6.379066×10^{-4}
128	7.812×10^{-3}	1.597114×10^{-4}
256	3.906×10^{-3}	3.994252×10^{-5}
512	1.953×10^{-3}	9.986548×10^{-6}
1024	9.766×10^{-4}	2.496695×10^{-6}
2048	4.883×10^{-4}	6.241778×10^{-7}
4096	2.441×10^{-4}	1.560447×10^{-7}
8192	1.221×10^{-4}	3.901588×10^{-8}
16384	6.104×10^{-5}	9.771988×10^{-9}
32768	3.052×10^{-5}	2.391487×10^{-9}
65536	1.526×10^{-5}	6.070796×10^{-10}
131072	7.629×10^{-6}	1.706178×10^{-9}
262144	3.815×10^{-6}	2.751095×10^{-8}

Table 4: Maximum error for various grid sizes n.

4 Problem 4

$$u'' = e^{\sin(x)}, \quad u'(0) = 0, \quad u'(1) = \alpha.$$

4.1 Method

Let $x_k = \frac{k}{n}$ for all k $= 0, \cdots, n$. be the grid points.
Let $u_i = u(x_i), \quad f_i = f(x_i),$ for i $= 1, \cdots, n-1$.

Apply the finite difference method to approximate the second derivative:

$$u''(x_i) \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

We already know that

$$2hu_n' = u_{n-2} - 4u_{n-1} + 3u_n + O(h^2)$$

and

$$2hu_0' = -3u_0 + 4u_1 - u_n + O(h^2)$$

in Problem 2 and 3.
Thus the discretized linear system is

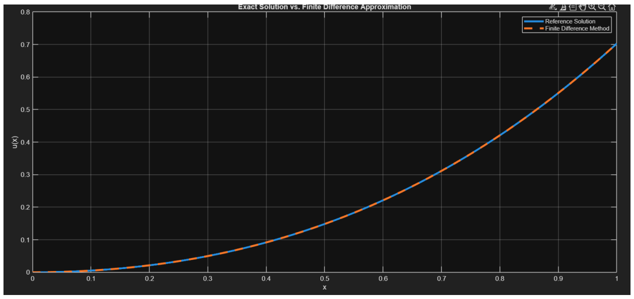
$$\begin{pmatrix} -3 & 4 & -1 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 1 & -4 & 3 \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} 0 \\ f_1 \times h^2 \\ \vdots \\ f_n \times h^2 \\ \alpha \times 2h \end{pmatrix}.$$

Since this system has infinitely many solutions(A$(1, \cdots, 1)^T = 0$), we need to impose an additional condition to make the solution unique. Here, we impose u(0) = 0. Therefore, the system becomes

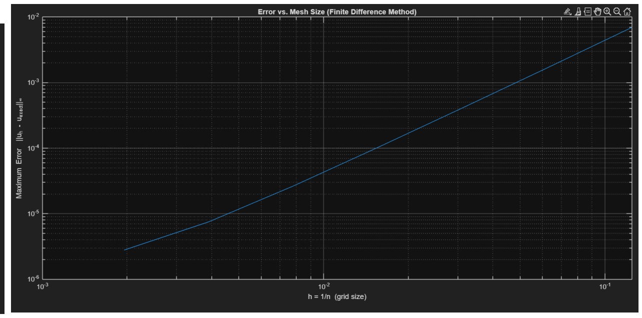
$$\begin{pmatrix} 1 & 0 & 0 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 1 & -4 & 3 \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ f_1 \times h^2 \\ \vdots \\ f_n \times h^2 \\ \alpha \times 2h \end{pmatrix}.$$

Then we solve u by $u = A^{-1}rhs$.

4.2 Results



(a) Exact Solution vs. Approximation



(b) Error vs. Mesh Size 2.$^{(4:18)}$

n	h	Max Error
8	1.250×10^{-1}	6.978901×10^{-3}
16	6.250×10^{-2}	1.684021×10^{-3}
32	3.125×10^{-2}	4.132019×10^{-4}
64	1.562×10^{-2}	1.030588×10^{-4}
128	7.812×10^{-3}	2.652754×10^{-5}
256	3.906×10^{-3}	7.524004×10^{-6}
512	1.953×10^{-3}	2.789511×10^{-6}

Table 5: Maximum error for various grid sizes n.

5 Problem 5

$$\epsilon u'' + (1+\epsilon)u' + u = 0, \quad u(0) = 0, \quad u(1) = 1.$$

5.1 Method

Let $x_k = \frac{k}{n}$ for all k $= 0, \cdots, n$. be the grid points.
Let $u_i = u(x_i), \quad f_i = f(x_i),$ for i $= 1, \cdots, n-1$.

Applying the finite difference method, we obtain

$$\frac{(\epsilon + \frac{h}{2} + \frac{h\epsilon}{2})u_{k+1} + (h^2 - 2\epsilon)u_k + (\epsilon - \frac{h}{2} - \frac{h\epsilon}{2})u_{k-1}}{h^2} = 1, \quad k = 1, \cdots, n-1.$$

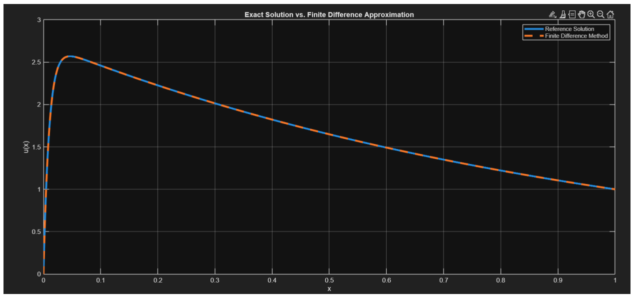
Thus the discretized linear system is

$$\begin{pmatrix} 1 & 0 & 0 & & \\ \epsilon - \frac{h}{2} - \frac{h\epsilon}{2} & h^2 - 2\epsilon & \epsilon + \frac{h}{2} + \frac{h\epsilon}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \epsilon - \frac{h}{2} - \frac{h\epsilon}{2} & h^2 - 2\epsilon & \epsilon + \frac{h}{2} + \frac{h\epsilon}{2} \\ & & 0 & 0 & 1 \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = h^2 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

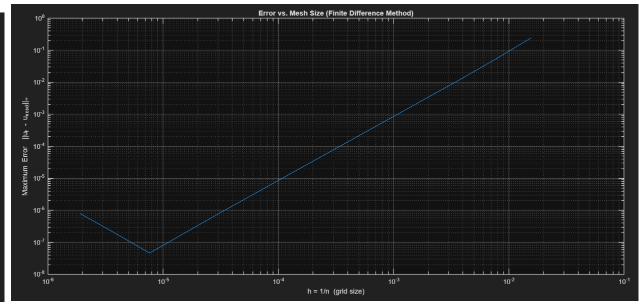
Similarly to Problem 2, we use the boundary condition to eliminate u_n and modify the right-hand side, thus transforming the system into a tridiagonal one and solving it with the Thomas algorithm. Therefore, the system becomes

$$\begin{pmatrix} h^2 - 2\epsilon & \epsilon + \frac{h}{2} + \frac{h\epsilon}{2} & & & \\ \epsilon - \frac{h}{2} - \frac{h\epsilon}{2} & h^2 - 2\epsilon & \epsilon + \frac{h}{2} + \frac{h\epsilon}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \epsilon - \frac{h}{2} - \frac{h\epsilon}{2} & h^2 - 2\epsilon & \epsilon + \frac{h}{2} + \frac{h\epsilon}{2} \\ & & & \epsilon - \frac{h}{2} - \frac{h\epsilon}{2} & h^2 - 2\epsilon \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -(\epsilon + h + h\epsilon) \end{pmatrix}.$$

5.2 Results



(a) Exact Solution vs. Approximation



(b) Error vs. Mesh Size 2.$^{(6:19)}$

n	h	Max Error
64	1.562×10^{-2}	2.441445×10^{-1}
128	7.812×10^{-3}	5.504198×10^{-2}
256	3.906×10^{-3}	1.313055×10^{-2}
512	1.953×10^{-3}	3.287223×10^{-3}
1024	9.766×10^{-4}	8.191862×10^{-4}
2048	4.883×10^{-4}	2.046337×10^{-4}
4096	2.441×10^{-4}	5.116222×10^{-5}
8192	1.221×10^{-4}	1.278982×10^{-5}
16384	6.104×10^{-5}	3.196996×10^{-6}
32768	3.052×10^{-5}	7.975343×10^{-7}
65536	1.526×10^{-5}	1.924982×10^{-7}
131072	7.629×10^{-6}	4.638914×10^{-8}
262144	3.815×10^{-6}	1.961152×10^{-7}
524288	1.907×10^{-6}	7.959186×10^{-7}

Table 6: Maximum error for various grid sizes n.