

1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x \log(x)$. This shows that $u \in C^2(0, 1)$ but $u(0)$ is not defined and u' , u'' do not exist at $x = 0$ (\Rightarrow : if $f \in C^0(0, 1)$, but not $f \in C^0([0, 1])$, then u does not belong to $C^0([0, 1])$).

$$\text{By 12.3, } u(x) = \int_0^1 G(x, s) f(s) ds, \text{ where } G(x, s) = \begin{cases} s(1-x) & \text{if } 0 \leq s \leq x \\ x(1-s) & \text{if } x \leq s \leq 1 \end{cases}$$

$$\begin{aligned} \text{Then } u(x) &= \int_0^x s(1-x) \frac{1}{s} ds + \int_x^1 x(1-s) \frac{1}{s} ds \\ &= (1-x)(x) + x \left(\log s - s \right) \Big|_x^1 = x - x^2 + x(-1 - \log x + x) = -x \log x \end{aligned}$$

4. Verify the summation by parts formula

$$\sum_{j=0}^{n-1} (w_{j+1} - w_j) v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} (v_{j+1} - v_j) w_{j+1},$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1} \sum_{j=0}^{n-1} (v_{j+1} - v_j)^2.$$

$$\begin{aligned} \sum_{j=0}^{n-1} (w_{j+1} - w_j) v_j &= w_1 v_0 - w_0 v_0 + w_2 v_1 - w_1 v_1 + \dots + w_n v_{n-1} - w_{n-1} v_{n-1} \\ &= -w_0 v_0 + w_1 v_0 - w_1 v_1 + w_2 v_1 - \dots - w_{n-1} v_{n-1} + w_n v_{n-1} - w_n v_n + w_n v_n \\ &= -w_0 v_0 - w_1 (v_1 - v_0) - w_2 (v_2 - v_1) - \dots - w_{n-1} (v_{n-1} - v_{n-2}) - w_n (v_n - v_{n-1}) + w_n v_n \\ &= w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} (v_{j+1} - v_j) w_{j+1} \end{aligned}$$

$$\begin{aligned} (L_h v_h, v_h)_h &= h \sum_{k=0}^n c_k L_h v_h(x_k) v_k = h \sum_{k=1}^{n-1} L_h v_h(x_k) v_k \quad \text{since } v_h \in V_h^0 \\ &= h \sum_{k=1}^{n-1} -\frac{v_{k+1} - 2v_k + v_{k-1}}{h^2} v_k \\ &= -h \sum_{k=1}^{n-1} (v_{k+1} - v_k) v_k - (v_k - v_{k-1}) v_k \\ &= -h \left[(v_n v_n - v_1 v_1) - \sum_{k=1}^{n-1} (v_{k+1} - v_k) v_{k+1} \right] - (v_{n-1} v_n - v_0 v_1) - \sum_{k=1}^{n-1} (v_{k+1} - v_k) v_k \\ &= h^{-1} \left[\sum_{k=1}^{n-1} (v_{k+1} - v_k) v_{k+1} - \sum_{k=1}^{n-1} (v_{k+1} - v_k) v_k + v_1 v_1 - v_0 v_1 - v_1 v_0 + v_0 v_1 \right] \quad \text{since } v_0, v_n = 0. \\ &= h^{-1} \left[\sum_{k=0}^{n-1} (v_{k+1} - v_k) v_{k+1} - \sum_{k=0}^{n-1} (v_{k+1} - v_k) v_k \right] \\ &= h^{-1} \left[\sum_{k=0}^{n-1} (v_{k+1} - v_k) (v_{k+1} - v_k) \right] = h^{-1} \sum_{k=0}^{n-1} (v_{k+1} - v_k)^2 \end{aligned}$$

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).

[Solution: we prove the result by verifying that $L_h G = h e^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \dots, k-1$ and $l = k+1, \dots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

Let $n \in \mathbb{N}$, $h = \frac{1}{n}$. Let $x_k = \frac{k}{n}$ $\forall k = 0, \dots, n$.

Let a $(n-1) \times (n-1)$ matrix L_h be defined by $\frac{1}{h^2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$.

W.L.O.G., let $a = x_k$ where $k = 1, \dots, n-1$.

Then $G(a, s) = \begin{cases} s(1-a), & 0 \leq s \leq a \\ a(1-s), & a \leq s \leq 1 \end{cases}$

Let $g(a) = [G(a, x_1), \dots, G(a, x_{n-1})]^T$.

Then the i -th component of $L_h g(a)$, denoted by $L_h g(a)^{(i)} = \frac{-G(a, x_{i+1}) + 2G(a, x_i) - G(a, x_{i-1})}{h^2}$.

For $i \leq k-1$, $L_h g(a)^{(i)} = \left(1 - \frac{k}{n}\right) \left(\frac{-\frac{i+1}{n} + 2\frac{i}{n} - \frac{i-1}{n}}{h^2} \right) = 0$

For $i = k$, $L_h g(a)^{(i)} = \frac{1}{h^2} \left(-\frac{k}{n} \left(1 - \frac{k+1}{n}\right) + 2 \frac{k}{n} \left(1 - \frac{k}{n}\right) - \frac{k-1}{n} \left(1 - \frac{k}{n}\right) \right)$
 $= \frac{1}{h^2} \left(\frac{1}{n^2} (-nk + k^2 - k + 2kn - 2k^2 - kn + n + k^2 - k) \right)$
 $= \frac{1}{h^2 n^2} (n) = n = \frac{1}{h}$

For $i \geq k+1$, $L_h g(a)^{(i)} = \frac{1}{h^2} \left(\frac{k}{n} \left[\left(1 - \frac{i+1}{n}\right) + 2 \left(1 - \frac{i}{n}\right) - \left(1 - \frac{i-1}{n}\right) \right] \right)$
 $= \frac{k}{h} \left(\frac{1}{n} (-n + i + 1 + 2n - 2i - n + i - 1) \right)$
 $= 0$

Thus $L_h g(a) = [0, \dots, 0, \frac{1}{h}, 0, \dots, 0]^T$ where $\frac{1}{h}$ is in the k -th position.

By def., $L_h G^k = e^k = [0, \dots, 0, 1, 0, \dots, 0]^T$ where 1 is in the k -th position.

So $L_h G^k = e^k = h [0, \dots, 0, \frac{1}{h}, 0, \dots, 0]^T = h L_h g(a) = L_h h g(a)$.

We have $G^k = (L_h)^{-1} L_h g(a) = (L_h)^{-1} L_h h g(a) = h g(a)$ since L_h is invertible.

Thus $G^k(x_i) = h G(a, x_i) = h G(x_k, x_i) = h G(x_i, x_k)$ since G is symmetric.