1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

By 12.3, $u(x) = \int_0^1 G(x,s) f(s)\,ds$, where $G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s \le x \\ x(1-s) & \text{if } x \le s \le 1 \end{cases}$

Then $u(x) = \int_0^x s(1-x)\frac{1}{s}\,ds + \int_x^1 x(1-s)\frac{1}{s}\,ds$

$= (1-x)(x) + x(\log s - s)\Big|_x^1 = x - x^2 + x(-1 - \log x + x) = -x\log x$

4. Cerify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_h)w_{j+1},$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2.$$

$$\sum_{j=0}^{n-1}(W_{j+1}-W_j)V_j = W_1V_0 - W_0V_0 + W_2V_1 - W_1V_1 + \dots + W_nV_{n-1} - W_{n-1}V_{n-1}$$

$$= -W_0V_0 + W_1V_0 - W_1V_1 + W_2V_1 - \dots - W_{n-1}V_{n-1} + W_nV_{n-1} - W_nV_n + W_nV_n$$

$$= -W_0V_0 - W_1(V_1-V_0) - W_2(V_2-V_1) - \dots - W_{n-1}(V_{n-1}-V_{n-2}) - W_n(V_n-V_{n-1}) + W_nV_n$$

$$= W_nV_n - W_0V_0 - \sum_{j=0}^{n-1}(V_{j+1}-V_j)W_{j+1}$$

$$(L_hV_h, V_h)_h = h\sum_{k=0}^{n} c_k L_hV_h(x_k)V_k = h\sum_{k=1}^{n-1} L_hV_h(x_k)V_k \quad \text{since } V_h \in V_h^0$$

$$= h\sum_{k=1}^{n-1} -\frac{V_{k+1}-2V_k+V_{k-1}}{h^2}V_k$$

$$= -h^{-1}\sum_{k=1}^{n-1}(V_{k+1}-V_k)V_k - (V_k-V_{k-1})V_k$$

$$= -h^{-1}\Big[\Big(V_nV_n - V_1V_1 - \sum_{k=1}^{n-1}(V_{k+1}-V_k)V_{k+1}\Big) - \Big(V_{n-1}V_n - V_0V_1 - \sum_{k=1}^{n-1}(V_{k+1}-V_k)V_k\Big)\Big]$$

$$= h^{-1}\Big[\sum_{k=1}^{n-1}(V_{k+1}-V_k)V_{k+1} - \sum_{k=1}^{n-1}(V_{k+1}-V_k)V_k + V_1V_1 - V_0V_1 - V_1V_0 + V_0V_0\Big] \quad \text{since } V_0,\ V_n = 0.$$

$$= h^{-1}\Big[\sum_{k=0}^{n-1}(V_{k+1}-V_k)V_{k+1} - \sum_{k=0}^{n-1}(V_{k+1}-V_k)V_k\Big]$$

$$= h^{-1}\Big[\sum_{k=0}^{n-1}(V_{k+1}-V_k)(V_{k+1}-V_k)\Big] = h^{-1}\sum_{k=0}^{n-1}(V_{k+1}-V_k)^2$$

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).
[*Solution*: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \ldots, k-1$ and $l = k+1, \ldots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

let $n \in \mathbb{N}$, $h = \frac{1}{n}$. let $x_k = \frac{k}{n}$ $\forall k = 0, \ldots, n$.

let a $(n-1)\times(n-1)$ matrix L_h be defined by $\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$.

W.L.O.G., let $a = x_k$ where $k = 1, \ldots, n-1$.

Then $G(a,s) = \begin{cases} s(1-a), & 0 \le s \le a \\ a(1-s), & a \le s \le 1 \end{cases}$

let $g(a) = [G(a, x_1), \ldots, G(a, x_{n-1})]^T$.

Then the i-th component of $L_h g(a)$, denoted by $L_h g(a)^{(i)} = \dfrac{-G(a, x_{i+1}) + 2G(a, x_i) - G(a, x_{i-1})}{h^2}$.

For $i \le k-1$, $L_h g(a)^{(i)} = (1 - \frac{k}{n})\left(\dfrac{-\frac{i+1}{n} + 2\frac{i}{n} - \frac{i-1}{n}}{h^2}\right) = 0$

For $i = k$, $L_h g(a)^{(i)} = \frac{1}{h^2}\left(-\frac{k}{n}(1 - \frac{k+1}{n}) + 2\frac{k}{n}(1 - \frac{k}{n}) - \frac{k-1}{n}(1 - \frac{k}{n})\right)$

$= \frac{1}{h^2}\left(\frac{1}{n^2}(-nk + k^2 - k + 2kn - 2k^2 - kn + n + k^2 - k)\right)$

$= \frac{1}{h^2 n^2}(n) = n = \frac{1}{h}$

For $i \ge k+1$, $L_h g(a)^{(i)} = \frac{1}{h^2}\left(\frac{k}{n}\left[-(1 - \frac{i+1}{n}) + 2(1 - \frac{i}{n}) - (1 - \frac{i-1}{n})\right]\right)$

$= \frac{k}{h}\left(\frac{1}{n}(-n + i + 1 + 2n - 2i - n + i - 1)\right)$

$= 0$

Thus $L_h g(a) = [0, \ldots, 0, \frac{1}{h}, 0, \ldots, 0]^T$ where $\frac{1}{h}$ is in the k-th position.

By def., $L_h G^k = e^k = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ where 1 is in the k-th position.

So $L_h G^k = e^k = h[0, \ldots, 0, \frac{1}{h}, 0, \ldots, 0]^T = h L_h g(a) = L_h h g(a)$.

We have $G^k = (L_h)^{-1} L_h G^k = (L_h)^{-1} L_h h g(a) = h g(a)$ since L_h is invertible.

Thus $G^k(x_i) = h G(a, x_i) = h G(x_k, x_i) = h G(x_i, x_k)$ since G is symmetric.