7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \text{Re} z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = e^{-t} \frac{t^z}{z} \Big|_0^\infty - \int_0^\infty -e^{-t} \frac{t^z}{z} dt$$

$$= 0 + \frac{1}{z} \int_0^\infty e^{-t} t^z dt = \frac{\Gamma(z+1)}{z}$$

Since $z \in \mathbb{C}$ and $\text{Re} z > 0$, $z\Gamma(z) = \Gamma(z+1)$.

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1 - \frac{\alpha}{2}) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

Let $y_{n+1} = y_n + h[(1 - \frac{\alpha}{2}) f(x_n, y_n) + \frac{\alpha}{2} f(x_{n+1}, y_{n+1})] + h\,T_{n+1}(h)$.

Since $y_{n+1} = y(x_{n+1}) = y(x_n) + hy'(x_n) + O(h^2)$, $y_{n+1} - y_n = hf(x_n, y_n) + O(h^2)$.

Since $f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + h f_x(x_n, y_n) + (y_{n+1} - y_n) f_y(x_n, y_n) + O(h^2)$

$= f(x_n, y_n) + h f_x(x_n, y_n) + h f(x_n, y_n) f_y(x_n, y_n) + O(h^2)$,

$y_{n+1} = y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) + O(h^3)$

$= y(x_n) + h f(x_n, y_n) + \frac{h}{2} f'(x_n, y_n) + O(h^3)$

$= y(x_n) + h f(x_n, y_n) + \frac{h}{2} (h f_x(x_n, y_n) + (hf(x_n, y_n) + O(h^2)) \cdot f_y(x_n, y_n)) + O(h^3)$

$= y(x_n) + h f(x_n, y_n) + \frac{h^2}{2} f_x(x_n, y_n) + \frac{h^2}{2} f(x_n, y_n) f_y(x_n, y_n) + O(h^3)$

Then $h\,T_{n+1}(h) = y_{n+1} - y_n - h[(1 - \frac{\alpha}{2}) f(x_n, y_n) + \frac{\alpha}{2} f(x_{n+1}, y_{n+1})]$

$= h f(x_n, y_n) + \frac{h^2}{2} f_x(x_n, y_n) + \frac{h^2}{2} f(x_n, y_n) f_y(x_n, y_n) -$

$h[(1 - \frac{\alpha}{2}) f(x_n, y_n) + \frac{\alpha}{2}(f(x_n, y_n) + h f_x(x_n, y_n) + h f(x_n, y_n) f_y(x_n, y_n))] + O(h^3)$

$= h f(x_n, y_n) - h f(x_n, y_n) + \frac{h^2}{2} f_x(x_n, y_n) - \frac{\alpha h^2}{2} f_x(x_n, y_n)$

$+ \frac{h^2}{2} f(x_n, y_n) f_y(x_n, y_n) - \frac{\alpha h^2}{2} f(x_n, y_n) f_y(x_n, y_n) + O(h^3)$

$= \frac{h^2}{2}(1 - \alpha) f_x(x_n, y_n) + \frac{h^2}{2}(1 - \alpha) f(x_n, y_n) f_y(x_n, y_n) + O(h^3)$.

Thus, if $\alpha = 1$, then $T_{n+1}(h) = O(h^2)$.

If $\alpha \neq 1$, then $T_{n+1}(h) = O(h)$.

Therefore the method is consistent for any value of α since $T(h) \to 0$ as $h \to 0$.

For $\alpha = 1$, $u_{n+1} = u_n + h\left[\frac{1}{2} f(x_n, u_n) + \frac{1}{2} f(x_{n+1}, u_{n+1})\right]$.

Then $u_{n+1} = u_n + h(-5u_n - 5u_{n+1})$

$\Rightarrow (1+5h)u_{n+1} = (1-5h)u_n$

$\Rightarrow u_{n+1} = \frac{1-5h}{1+5h} u_n$

Thus $u_{n+1} = \left(\frac{1-5h}{1+5h}\right)^n$

If $\left|\frac{1-5h}{1+5h}\right| < 1$, then the method is absolutely stable.

For $h > \frac{1}{5}$, $\left|\frac{1-5h}{1+5h}\right| = \frac{5h-1}{1+5h} = 1 - \frac{2}{1+5h} < 1$.

For $h = \frac{1}{5}$, $\left|\frac{1-5h}{1+5h}\right| = 0 < 1$.

For $0 < h < \frac{1}{5}$, $\left|\frac{1-5h}{1+5h}\right| = \frac{1-5h}{1+5h} = -1 + \frac{2}{1+5h} < 1$ since $\frac{2}{1+5h} < 2$.

Therefore for all $h > 0$, the method is absolutely stable.