9. Given the following set of data

$$\left\{f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1\right\},$$

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

Let $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ s.t.

$H_3(-1) = 1$, $H_3'(-1) = 1$, $H_3'(1) = 2$ and $H_3(2) = 1$.

Then we have $-a_3 + a_2 - a_1 + a_0 = 1$,

$3a_3 - 2a_2 + a_1 + 0 = 1$,

$3a_3 + 2a_2 + a_1 + 0 = 2$ and

$8a_3 + 4a_2 + 2a_1 + a_0 = 1$.

Thus
$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Since $(8,4,2,1) \in \operatorname{span}((-1,1,-1,1), (3,-2,1,0), (3,2,1,0))$.

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$ doesn't have inverse.

Thus we don't have the solution of (a_3, a_2, a_1, a_0).

Therefore H_3 does not exist.

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2 x^2 + a_4 x^4}{1 + b_2 x^2}, \tag{8.75}$$

called the *Padé approximation*. Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10} x^{10} + \ldots.$$

[*Solution*: $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

$$f(x) - r(x) = r_8 x^8 + r_{10} x^{10} + \ldots$$

$$\Rightarrow f(x) - \frac{a_0 + a_2 x^2 + a_4 x^4}{1 + b_2 x^2} = r_8 x^8 + r_{10} x^{10} + \ldots$$

$$\Rightarrow f(x)(1 + b_2 x^2) - a_0 + a_2 x^2 + a_4 x^4 = (r_8 x^8 + r_{10} x^{10} + \ldots)(1 + b_2 x^2)$$

$$\Rightarrow \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)(1 + b_2 x^2) - (a_0 + a_2 x^2 + a_4 x^4) = (r_8 x^8 + r_{10} x^{10} + \ldots)(1 + b_2 x^2)$$

$$\Rightarrow \begin{cases} 1 - a_0 = 0 \\ b_2 - \frac{1}{2} - a_2 = 0 \\ \frac{1}{4!} - \frac{b_2}{2} - a_4 = 0 \\ \frac{-1}{6!} + \frac{b_2}{4!} = 0 \end{cases} \quad \Rightarrow \begin{cases} a_0 = 1 \\ a_2 = \frac{-7}{15} \\ a_4 = \frac{1}{40} \\ b_2 = \frac{1}{30} \end{cases}$$