Quadratures

Marcin Kuta

Quadratures

Quadrature	Nodes	Cost
Newton-Cotes	Equispaced nodes	$O(1)$
Clenshaw-Curtis	Chebyshev nodes	$O(n \log n)$
Gauss	Legendre nodes	$O(n^2)$

Simple quadratures

Midpoint rectangular formula

$$M(f) = (b-a)f\Big(\frac{a+b}{2}\Big) \tag{1}$$

Trapezoidal formula

$$T(f) = \frac{b-a}{2}\big[f(a) + f(b)\big] \tag{2}$$

Simpson formula

$$S(f) = \frac{b-a}{6}\Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big] \tag{3}$$

Simple quadratures

Quadrature	Degree of exactness	Order of convergence
Midpoint rule	1	3
Trapezoid rule	1	3
Simpson 1/3 (parabolic)	3	5
Simpson 3/8 (cubic)	3	5

Simple quadratures and error bounds

Quadrature	Error bound
Midpoint rule	$\frac{1}{24}f''(\eta)(b-a)^3$
Trapezoid rule	$-\frac{1}{12}f''(\eta)(b-a)^3$
Simpson 1/3 (parabolic)	$-\frac{1}{2880}f^{(4)}(\eta)(b-a)^5$
Simpson 3/8 (cubic)	$-\frac{1}{6480}f^{(4)}(\eta)(b-a)^5$

Composite quadratures

Quadrature	Order of convergence
Midpoint rule	2
Trapezoid rule	2
Simpson 1/3 (parabolic)	4
Simpson 3/8 (cubic)	4

Composite quadratures and error bounds

Let $H = \frac{b-a}{M}$.

Quadrature	Error bound
Midpoint rule	$\frac{b-a}{24} f''(\eta) H^2$
Trapezoid rule	$-\frac{b-a}{12} f''(\eta) H^2$
Simpson 1/3 (parabolic)	$-\frac{b-a}{2880} f^{(4)}(\eta) H^4$
Simpson 3/8 (cubic)	$-\frac{b-a}{6480} f^{(4)}(\eta) H^4$

Hermite quadrature

- Quadratures considered so far are based on Lagrange interpolation
- If we also know values of derivatives at quadrature nodes, we can use quadratures based on Hermite interpolation

Trapezoidal formula

$$T(f) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{4}$$

Corrected trapezoidal formula

$$T(f) = \frac{b-a}{2}\left[f(a) + f(b)\right] + \frac{(b-a)^2}{12}\left[f'(a) - f'(b)\right] \tag{5}$$

Hermite quadrature

Trapezoidal formula

$$T(f) = \frac{b-a}{2}\left[f(a) + f(b)\right] \tag{6}$$

$$E(f) = -\frac{h^3}{12} f''(\xi) \tag{7}$$

Corrected trapezoidal formula

$$T(f) = \frac{b-a}{2}\left[f(a) + f(b)\right] + \frac{(b-a)^2}{12}\left[f'(a) - f'(b)\right] \tag{8}$$

$$E(f) = \frac{h^5}{720} f^{(4)}(\xi) \tag{9}$$

Gaussian quadratures and scaling

$$\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi \approx \sum_{i=0}^{n} A_i f(\xi_i) \tag{10}$$

$$\int_{a}^{b} f(x)\,\mathrm{d}x \approx \frac{b-a}{2} \sum_{i=0}^{n} A_i f(x_i) \tag{11}$$

$$x_i = \frac{b+a}{2} + \frac{b-a}{2}\xi_i \tag{12}$$

- Poor accuracy for functions that are not well behaved at endpoints
- High initialization time

Clenshaw-Curtis quadratures

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{n} A_i f(x_i) \tag{13}$$

$$x_i = \cos(i\pi/n) \tag{14}$$

$$A_i = \int_{-1}^{1} \ell_i(x)\,\mathrm{d}x \tag{15}$$

Empirical order of convergence

$$E(h) \approx Ch^p \tag{16}$$

$$\log E(h) \approx \log(C) + p\log(h) \tag{17}$$

$$p \approx \frac{\log(\frac{E(h_{k+1})}{E(h_k)})}{\log(\frac{h_{k+1}}{h_k})} \tag{18}$$

Tanh-sinh quadratures

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \tag{19}$$

$$x = g(t) = \tanh(\frac{\pi}{2}\sinh t) \tag{20}$$

$$g'(t) = \frac{\frac{\pi}{2}\cosh t}{\cosh^2(\frac{\pi}{2}\sinh t)} \tag{21}$$

$$\int_{-1}^{1} f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(g(t))g'(t)\,\mathrm{d}t \tag{22}$$

Quadratures over infinite interval

$$\int_0^\infty f(t)\,\mathrm{d}t \tag{23}$$

$$s = \frac{1}{t+1} \tag{24}$$

$$t = \frac{1}{s} - 1 = \frac{1-s}{s} \tag{25}$$

$$\mathrm{d}t = -\frac{\mathrm{d}s}{s^2} \tag{26}$$

$$\int_0^\infty f(t)\,\mathrm{d}t = \int_0^1 f\left(\frac{1-s}{s}\right)\frac{\mathrm{d}s}{s^2} \tag{27}$$

References

[1] Michael T. Heath,
Scientific Computing. An Introductory Survey, 2nd Edition,
Chapter 8: Numerical Integration and Differentation
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