Approximation

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Norms in finite-dimensional spaces

$$||x||_1 = \sum_{i=1}^{n} |x_i| \tag{1}$$

$$||x||_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \tag{2}$$

$$||x||_\infty = \max_{i=1}^{n} |x_i| \tag{3}$$

Norms in finite-dimensional spaces

Norms in finite-dimensional spaces are equivalent

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}\,||x||_\infty \tag{4}$$

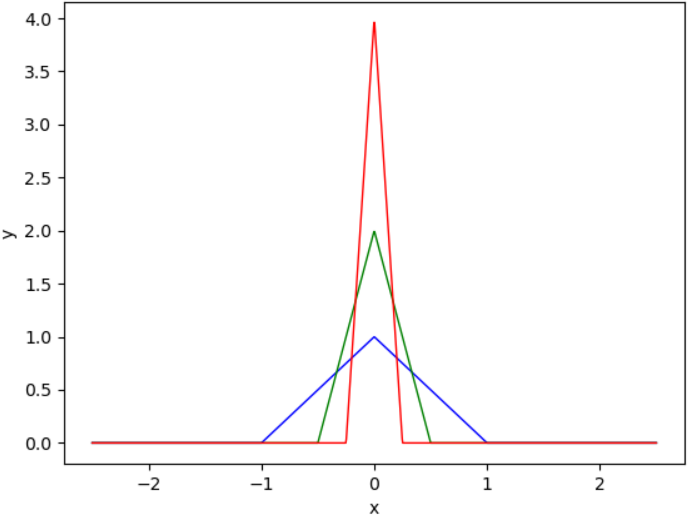
$$||x||_2 \leq ||x||_1 \leq \sqrt{n}\,||x||_2 \tag{5}$$

For any two norms $||\cdot||'$, $||\cdot||''$ defined on finite-dimensional linear space $\mathcal{V}$ there exist $m > 0, M > 0$ such that for all $x \in \mathcal{V}$

$$m||x||' \leq ||x||'' \leq M||x||' \tag{6}$$

Norms in infinite-dimensional spaces

Norms in infinite-dimensional spaces are not equivalent



Approximation

Basis functions: $\{\phi_0(t), \ldots, \phi_n(t)\}$

$$p(x) = \sum_{j=0}^{n} c_j \phi_j(x) \tag{7}$$

$x = [x_0, \ldots, x_m]$ $m > n$ (often $m \gg n$)

$$p(x_j) \approx f(x_j) \tag{8}$$

Approximation error E:

	Discrete	Continuous
L^∞	$\min\limits_{p \in \mathbb{P}_n} \max\limits_{0 \le j \le m} \lvert f(x_j) - p(x_j) \rvert$	$\min\limits_{p \in \mathbb{P}_n} \max\limits_{x \in [a,b]} \lvert f(x) - p(x) \rvert$
L^2	$\min\limits_{p \in \mathbb{P}_n} \sum_{j=0}^{m} \lvert f(x_j) - p(x_j) \rvert^2$	$\min\limits_{p \in \mathbb{P}_n} \int_a^b \lvert f(x) - p(x) \rvert^2$

Discrete least squares approximation

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}, \; c = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix}, \; y = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \dots \\ f(x_m) \end{bmatrix} \tag{9}$$

Normal equation:

$$A^T A c = A^T y \tag{10}$$

$$c = (A^T A)^{-1} A^T y = A^+ y \tag{11}$$

Discrete least squares approximation

$$p(x) = \sum_{j=0}^{n} c_j x^j \tag{12}$$

$$S_k = \sum_{i=0}^{m} x_i^k, \;\; k = 0, 1, \ldots, 2n \tag{13}$$

$$T_k = \sum_{i=0}^{m} x_i^k y_i, \;\; k = 0, 1, \ldots n \tag{14}$$

$$\begin{bmatrix} S_0 & S_1 & \ldots & S_n \\ S_1 & S_2 & \ldots & S_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ S_n & S_{n+1} & \ldots & S_{2n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \ldots \\ c_n \end{bmatrix} = \begin{bmatrix} T_0 \\ T_1 \\ \ldots \\ T_n \end{bmatrix} \tag{15}$$

Continuous least squares approximation

$p_\star(t)$ to be optimal requires approximation error $E = E(c_0, \ldots, c_n)$ to satisfy

$$\frac{\partial E}{\partial c_0} = \ldots \frac{\partial E}{\partial c_n} = 0 \tag{16}$$

Continuous least squares approximation

$$\langle f, g \rangle = \int_{-1}^{1} w(x) f(x) g(x) dx, \tag{17}$$

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \dots & \langle \phi_0, \phi_n \rangle \\ \langle \phi_1, \phi_0 \rangle & \dots & \langle \phi_1, \phi_n \rangle \\ \dots & \dots & \dots \\ \langle \phi_n, \phi_0 \rangle & \dots & \langle \phi_n, \phi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \dots \\ \langle f, \phi_n \rangle \end{bmatrix} \tag{18}$$

Continuous least squares approximation

For orthogonal polynomials $\phi_0, \ldots, \phi_n$ we have $\langle \phi_i, \phi_j \rangle = 0$ for $i \neq j$.

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & 0 & \ldots & 0 \\ 0 & \langle \phi_1, \phi_1 \rangle & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \langle \phi_n, \phi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \ldots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \ldots \\ \langle f, \phi_n \rangle \end{bmatrix} \tag{19}$$

$$c_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \tag{20}$$

$$p_\star = \sum_{k=0}^{n} c_k \phi_k = \sum_{k=0}^{n} \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k \tag{21}$$

Orthogonal polynomials

Name	Symbol	Interval	Weight function
Legendre	P_n	$[-1,1]$	1
Chebyshev, 1st kind	T_n	$[-1,1]$	$(1-t^2)^{-1/2}$
Chebyshev, 2st kind	U_n	$[-1,1]$	$(1-t^2)^{1/2}$
Jacobi	J_n	$[-1,1]$	$(1-t)^{\alpha}(1+t)^{\beta}$
Lagurerre	L_n	$[0,\infty)$	e^{-t}
Hermite	H_n	$(-\infty,\infty)$	e^{-t^2}

Chebyshev polynomials

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

$$T_k(x) = \cos(k \arccos(x))$$

T_0	1
T_1	x
T_2	$2x^2 - 1$
T_3	$4x^3 - 3x$
T_4	$8x^4 - 8x^2 + 1$
T_5	$16x^5 - 20x^3 + 5x$

$$\langle T_j, T_j \rangle = \begin{cases} \pi & \text{if } j = 0 \\ \pi/2 & \text{if } j > 0 \end{cases}$$

Legendre polynomials

$$P_k(x) = \tfrac{2k-1}{k} P_{k-1}(x) - \tfrac{k-1}{k} P_{k-2}(x)$$

P_0	1
P_1	x
P_2	$(3x^2 - 1)/2$
P_3	$(5x^3 - 3x)/2$
P_4	$(35x^4 - 30x^2 + 3)/8$
P_5	$(63x^5 - 70x^3 + 15x)/8$

$$\langle P_j, P_j \rangle = \tfrac{2}{j+1}$$

Orthogonal trigonometric functions

Functions	Interval	Weight
$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin(nx), \frac{1}{\sqrt{\pi}}\cos(nx), n = 1, 2\ldots$	$[-\pi, \pi]$	1
$\frac{2}{\sqrt{\pi}}\sin(nx), n = 1, 2\ldots$	$[0, \pi]$	1
$\frac{1}{\sqrt{\pi}}, \frac{2}{\sqrt{\pi}}\cos(nx), n = 1, 2\ldots$	$[0, \pi]$	1

References

[1] Michael T. Heath,
Scientific Computing. An Introductory Survey, 2nd Edition,
Chapter 7: Interpolation
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