Interpolation

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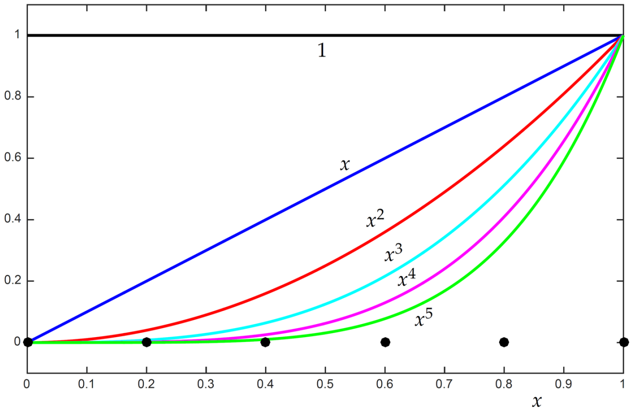
Ill-conditioned base of monomials



Lagrange interpolation

Characteristic polynomials:

$$\ell_j(t) = \frac{\prod_{k=1,k\neq j}^{n}(t - t_k)}{\prod_{k=1,k\neq j}^{n}(t_j - t_k)} \quad j = 1, \ldots, n \tag{1}$$

$$\ell_j(t_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i, j = 1, \ldots, n \tag{2}$$

Lagrange form:

$$p_{n-1}(t) = y_1\ell_1(t) + y_2\ell_2(t) + \cdots + y_n\ell_n(t) \tag{3}$$

Lagrange interpolation

Drawbacks of Lagrange interpolation with (3):

- each evaluation of $p_{n-1}(t)$ requires $O(n^2)$ multiplications
- adding a new point requires recomputation from scratch
- lack of stability

Lagrange interpolation

Barycentric weights

$$w_k = \frac{1}{\prod_{j \neq k}(t_k - t_j)} \tag{4}$$

Barycentric weights do not depend on f.

Second form of the barycentric interpolation formula

$$p_{n-1}(t) = \frac{\displaystyle\sum_{k=1}^{n} \frac{w_k}{t - t_k} y_k}{\displaystyle\sum_{k=1}^{n} \frac{w_k}{t - t_k}} \tag{5}$$

Newton interpolation

$$\pi_j(t) = \prod_{k=1}^{j-1}(t - t_k) \quad j = 1, \dots, n \tag{6}$$

$$\pi_j(t_i) = 0 \quad \text{for } i < j \tag{7}$$

$$\begin{aligned} p_{n-1}(t) =& x_1\pi_1(t) + x_2\pi_2(t) + \cdots + x_n\pi_n(t) \\ =& x_1 + x_2(t - t_1) + x_3(t - t_1)(t - t_2) + \dots \\ & + x_n(t - t_1)(t - t_2)\dots(t - t_{n-1}) \end{aligned} \tag{8}$$

Newton form:

$$p_{n-1}(t) = f[t_1]\pi_1(t) + f[t_1, t_2]\pi_2(t) + \cdots + f[t_1, t_2, \dots, t_n]\pi_n(t)$$

Vandermonde matrix

Points: $x_1, \ldots, x_n$

Vandermonde matrix:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

Vandermonde matrix

Data points: $(x_1, y_1) \dots, (x_n, y_n)$

In the monomial basis $\phi(x) = (x^0, x^1, \dots, x^n)$, coefficients $a_1, \dots, a_n$ of polynomial interpolating points $(x_1, y_1), \dots, (x_n, y_n)$ can be found from equation:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Vandermonde matrix

Data points: $(x_1, y_1) \ldots, (x_n, y_n)$

In the basis $\phi_1(x), \ldots, \phi_n(x)$, coefficients $a_1, \ldots, a_n$ of polynomial interpolating points $(x_1, y_1), \ldots, (x_n, y_n)$ can be found from equation:

$$\begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \phi_3(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \phi_3(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_n) & \phi_2(x_n) & \phi_3(x_n) & \cdots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Determination of polynomial

Determination of coefficients of polynomial $p(t)$ of degree n

Base	cost	
monomials	$O(n^3)$	solving system of linear equations with Vandermonde matrix
Newton polynomials	$O(n^2)$	divided differences
Lagrange polynomials	trivial	y_is are given

Evaluation of polynomial

Evaluation of polynomial $p(t)$ of degree n

Base	number of multiplications
monomials	$n(n+1)/2$ (naive approach) n (Horner's scheme)
Lagrange polynomials	$2n(n+1)$
Lagrange barycentric	$O(n^2)$ (weights w_k) $O(n)$ (each point)
Newton polynomials	$O(n^2)$ (divided differences) $n(n+1)/2$ (naive approach) n (Horner's scheme)
orthogonal polynomials	$O(n)$ (Clenshaw algorithm)

References

[1] Michael T. Heath,
Scientific Computing. An Introductory Survey, 2nd Edition,
Chapter 7: Interpolation
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[2] Michael T. Heath,
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