Solving nonlinear equations

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Convergence rate

Absolute error ε_k:

$$\varepsilon_k = |x_k - x_*|, \tag{1}$$

$$\lim_{k \to \infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^p} = C \tag{2}$$

p – order of convergence (a.k.a. convergence rate)
C – rate of convergence, $C > 0$

Convergence

linear	$p = 1,\ 0 < C < 1$
superlinear	$p > 1$
quadratic	$p = 2$
cubic	$p = 3$

Convergence	Digits gained per iteration
linear	constant, $-\log_\beta C$
superlinear	increasing
quadratic	doubled

Order of convergence

Order of convergence in proximity of simple roots

Method	Convergence	Order of convergence
bisection method	linear	1
Regula falsi	linear	1
Illinois method	superlinear	$\sqrt[3]{3} \approx 1.442$
Pegasus method	superlinear	$\sqrt[4]{\frac{7+\sqrt{57}}{2}} \approx 1.642$
secant method	superlinear	$\frac{1+\sqrt{5}}{2} \approx 1.618$
Newton method	quadratic	2
Steffensen	quadratic	2
inverse quadratic interpolation	superlinear	1.839

Convergence

Global convergence – method finds a root if initial interval $[a, b]$ contains the root and $f(a) \cdot f(b) < 0$, irrespective of x_0.

Local convergence – method finds the root α only if x_0 is close enough to α

Method	Convergence
bisection method	global
Regula falsi	global
Illinois method	global
Pegasus method	global
secant method	local
Newton method	local
Steffensen	local
inverse quadratic interpolation	local

Empirical convergence rate

$$\frac{\varepsilon_{k+1}}{\varepsilon_k^r} = \frac{\varepsilon_k}{\varepsilon_{k-1}^r} \tag{3}$$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = \left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^r \tag{4}$$

$$\log\left(\frac{\varepsilon_k}{\varepsilon_{k-1}}\right)^r = \log\frac{\varepsilon_{k+1}}{\varepsilon_k} \tag{5}$$

$$r = \frac{\log\frac{\varepsilon_{k+1}}{\varepsilon_k}}{\log\frac{\varepsilon_k}{\varepsilon_{k-1}}} \tag{6}$$

Empirical convergence rate

Known x_*

$$r = \frac{\log \frac{|x_{k+1} - x_*|}{|x_k - x_*|}}{\log \frac{|x_k - x_*|}{|x_{k-1} - x_*|}} \tag{7}$$

Unknown x_*

$$r = \frac{\log \frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|}}{\log \frac{|x_k - x_{k-1}|}{|x_{k-1} - x_{k-2}|}} \tag{8}$$

Fixed-point iteration

$$f(x) = 0 \tag{9}$$

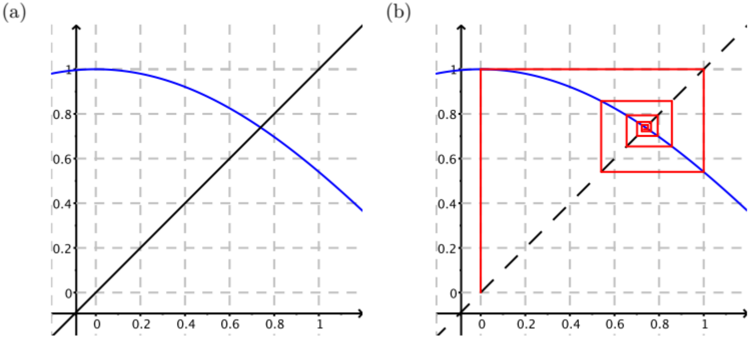
Simple iteration, direct iteration

$$x = \phi(x) \tag{10}$$

$$x_{k+1} = \phi(x_k) \tag{11}$$

Fixed-point iteration

Figure 2.2.1: Finding the fixed point of $\cos(x)$.



Iteration schemes

$$x_{k+1} = \phi(x_k, x_{k-1}) \tag{12}$$

$$x_{k+1} = x_k - q_k^{-1} f(x_k) \tag{13}$$

Newton's method:

$$q_k = f'(x_k) \tag{14}$$

Secant method

$$q_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \tag{15}$$

Regula falsi $\equiv$ Bracketed secant method

$$q_k = \frac{f(x_k) - f(x_{k'})}{x_k - x_{k'}} \tag{16}$$

$$k' = \arg\max_{i<k} f(x_i) f(x_k) < 0$$

Iteration schemes

Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{17}$$

Secant method

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} \tag{18}$$

Regula falsi

$$x_{k+1} = \begin{cases} \frac{x_k f(a) - a f(x_k)}{f(a) - f(x_k)} & \text{if } f(a)f(x_k) < 0 \\ \frac{x_k f(b) - b f(x_k)}{f(b) - f(x_k)} & \text{if } f(b)f(x_k) < 0 \end{cases} \tag{19}$$

Regula falsi as generalization of bisection method

Bisection method

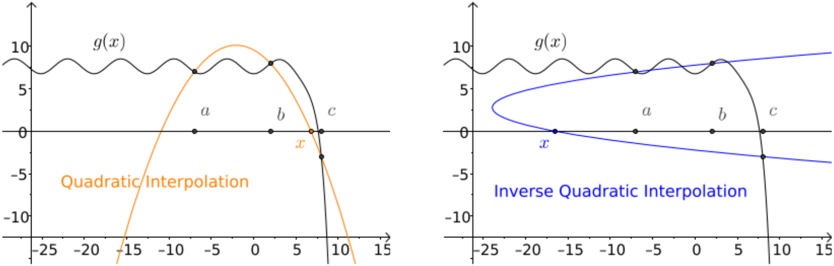
$$c_n = \frac{a_n + b_n}{2} \tag{20}$$

Regula falsi

$$c_n = \frac{|f(b_n)|}{|f(a_n)| + |f(b_n)|} a_n + \frac{|f(a_n)|}{|f(a_n)| + |f(b_n)|} b_n \tag{21}$$

Inverse quadratic interpolation

Figure 2.7.2: Quadratic and inverse quadratic interpolation.



- Quadratic function has two roots or no roots
- Inverse quadratic function has always exaxtly one root

Sufficient conditions for convergence

Sufficient conditions for convergence of bisecton method and regula falsi

(C1) $f \in C[a, b]$
  - f continuous in $[a, b]$

(C2) $f(a) \cdot f(b) < 0$

Sufficient conditions for convergence of Newton method for any initial point

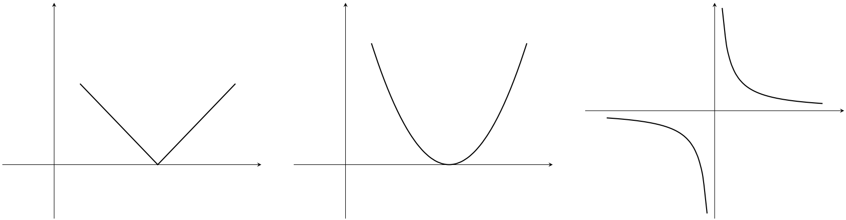
(C3) $f \in C^2[a, b]$
  - f, f', f'' continuous in $[a, b]$

(C4) f' and f'' do not change sign in $[a, b]$
  - no minima, maxima or inflection points

(C5) $f(a) \cdot f(b) < 0$

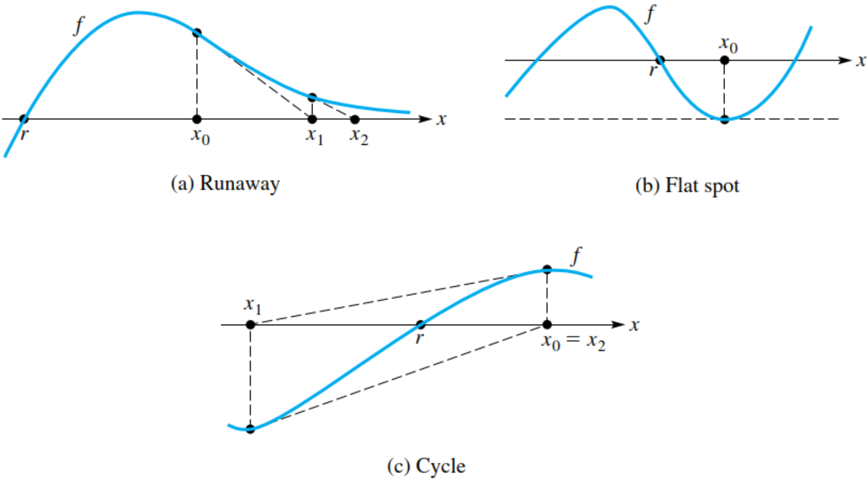
Bisection method

Cases for which bisection method fails:



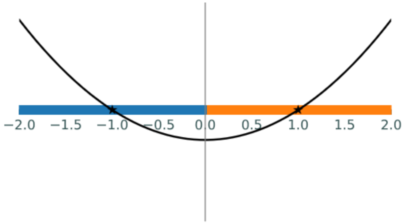
Newton method

Cases for which Newton method fails:

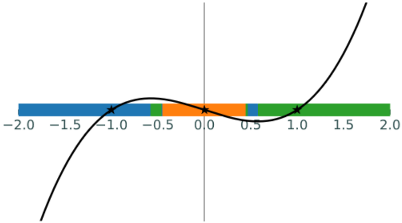


Newton method

Sensitivity of Newton method to starting point:



(a) Basins of attraction for $f(x) = x^2 - 1$.



(b) Basins of attraction for $f(x) = x^3 - x$.

Aitken's Acceleration with Δ^2 method

Let sequence $\{x_k\}_{i=0}^{\infty}$ be a linearly convergent sequence to a simple root arising from fixed point iterations (10).

Aitken's extrapolation formula:

$$\hat{x}_k = x_k - \frac{(x_k - x_{k-1})^2}{(x_k - x_{k-1}) - (x_{k-1} - x_{k-2})}, \; k \geq 2 \tag{22}$$

Denoting $\Delta x_k = x_k - x_{k-1}$ and $\Delta^2 x_k = \Delta(\Delta x_k) = \Delta x_k - \Delta x_{k-1}$ we can rewrite (22) as Δ^2 method:

$$\hat{x}_k = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k}, \; k \geq 2 \tag{23}$$

Sequence $\{\hat{x}_k\}_{i=0}^{\infty}$ has quadratic convergence.

Stopping critera

(S1) increment size

$$|x_k - x_{k-1}| < \epsilon$$

for $\phi'(\alpha) \approx 1$ unrealiable
suitable for Newton method

(S2) relative increment size

$$\frac{|x_k - x_{k-1}|}{|x_k|} < \epsilon$$

(S3) residuum

$$|f(x_k)| < \epsilon$$

for $|f'(\alpha)| \simeq 1$ satisfactory
for $|f'(\alpha)| \ll 1$ not reliable
for $|f'(\alpha)| \gg 1$ too restrictive

(S4) number of iterations

Implementation details

Bisection method

$$c_n = \frac{a_n + b_n}{2}$$

$$c_n = a_n + \frac{b_n - a_n}{2}$$

Condition (C2)

$$f(a)f(b) < 0$$

$$\mathrm{sign}(f(a)) \neq \mathrm{sign}(f(b))$$

Housholder's method

$$x_{k+1} = x_k + d\frac{(1/f)^{(d-1)}(x_k)}{(1/f)^{(d)}(x_k)} \tag{24}$$

$d = 1$ – Netwon's method

$d = 2$ – Halley's method

$$x_{k+1} = x_k + 2\frac{(1/f)'(x_k)}{(1/f)''(x_k)} = x_k - \frac{2f(x_k)f'(x_k)}{2[f'(x_k)]^2 - f(x_k)f''(x_k)} \tag{25}$$

References

[1] http://heath.cs.illinois.edu/scicomp/notes/cs450_chapt05.pdf

[2] Alfio Quarteroni, Riccardo Sacco, Fausto Salieri,
Numerical Mathematics, Second Edition,
2007