Interpolation II

Marcin Kuta

Runge phenomenon

Theorem

Runge (1901)
Polynomial interpolants in equispaced points may diverge exponentially, even if f is analytic

Theorem

Faber (1914)
No polynomial interpolation scheme, no matter how the points are distributed, will converge for all continuous functions

Theorem

Polynomial interpolants in Chebyshev points always converge if f is smooth (e.g. Lipschitz continuous)

Stability of polynomial interpolation

Lebesgue's constant Λ_n plays a role of a condition number of the polynomial interpolation.

Equidistant nodes on $[0, 1]$

$$\Lambda_n = \frac{2^{n+1}}{en\log n}(1 + o(1)) \tag{1}$$

Chebyshev nodes

$$\Lambda_n < \frac{2}{\pi}\log(n+1) + 1 \tag{2}$$

Any set of interpolation nodes

$$\Lambda_n > \frac{2}{\pi}\log(n+1) + \frac{1}{2} \tag{3}$$

Chebyshev nodes

Chebyshev nodes corresponding to extrema of Chebyshev polynomial.

Nodes in interval $[-1, 1]$:

$$t_i = -\cos\left(\frac{i}{n}\pi\right), \quad i = 0, \ldots, n$$

Nodes in interval $[a, b]$:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2} t_i, \quad i = 0, \ldots, n \tag{4}$$

Chebyshev nodes

Chebyshev nodes corresponding to roots of Chebyshev polynomial.

Nodes in interval $[-1, 1]$:

$$t_i = -\cos\left(\frac{2i+1}{2(n+1)}\pi\right), \quad i = 0, \dots, n$$

Nodes in interval $[a, b]$:

$$x_i = \frac{a+b}{2} + \frac{b-a}{2} t_i, \quad i = 0, \dots, n \tag{5}$$

Cubic splines

$$a = t_0 < t_1 < \ldots t_N = b$$

$$S(x) = S_i(x) \text{ for } x \in [t_i, t_{i+1}]$$

- interpolation conditions
  $S_i(t_i) = y_i \quad S_i(t_{i+1}) = y_{i+1}$ for $i = 0, \ldots, N-1$
- continuity of first derivatives
  $S'_{i-1}(t_i) = S'_i(t_i)$ for $i = 1, \ldots, N-1$
- continuity of second derivatives
  $S''_{i-1}(t_i) = S''_i(t_i)$ for $i = 1, \ldots, N-1$

Cubic splines

- natural spline

$$S_0''(a) = S_{N-1}''(b) = 0$$

- clamped spline

$$S_0'(a) = f'(a) \quad S_{N-1}'(b) = f'(b)$$

- not-a-knot spline

$$S_0'''(t_1) = S_1'''(t_1), \; S_{N-2}'''(t_{N-1}) = S_{N-1}'''(t_{N-1})$$

- periodic spline

$$S_0'(a) = S_{N-1}'(b), \; S_0''(a) = S_{N-1}''(b)$$

- quadratic spline
  - S_0 and S_{N-1} are quadratic

Cubic splines

Cubic splines do not preserve monotonicity between neighbouring nodes.
Let h be the largest distance between subsequent knots.

- $|S(x) - f(x)| = O(h^4)$ for clamped cubic splines
- $|S(x) - f(x)| = O(h^2)$ for natural cubic splines and not-a-knot cubic splines

Natural cubic spline minimizes strain energy of curve among functions interpolating given points.

References

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