# Graph Algorithms

# Representations of graph G with vertices V and edges E

- V x V adjacency-matrix A: $A_{u,v} = 1 \Leftrightarrow (u, v) \in E$

  Size: $|V|^2$

  Better for dense graphs, i.e., $|E| = \Omega(|V|^2)$

- Adjaceny-list, e.g. $(v_1, v_5), (v_1, v_{17}), (v_2, v_3) \ldots$

  Size: O(E)

  Better for sparse graphs, i.e., $|E| = O(|V|)$

## Next we see several algorithms to compute shortest distance

$\delta(u,v)$ := shortest distance from u to v
$\infty$ if v is not reachable from u

Variants include weighted/unweighted, single-source/all-pairs

Algorithms will construct vector/matrix d; we want $d = \delta$

Back pointers $\pi$ can be computed to reconstruct path

# Breadth-first search

Input:

Graph G= (V,E) as adjacency list, and $s \in V$.

Output:

Distance from s to any other vertex

- Discover vertices at distance k before those at distance k+1

Algorithm colors each vertex:

White : not discovered.
Gray : discovered but its neighbors may not be.
Black : discovered and all of its neighbors are too.

```
BFS(G,s)
 For each vertex u ∈ V[G] – {s}
   color[u]:= White; d[u] := ∞; π[u] :=NIL;

 Q:= empty Queue; color[s] := Gray;  d[s] :=0; π[s] :=NIL;

 Enqueue(Q,s)
 While (|Q| > 0) {
   u := Dequeue(Q)       // a vertex with min distance d[u];
   for each v ∈ adj[u]    // checks neighbors
     if color[v] = white {
       color[v] := gray;
       d[v]:= d[u]+1;
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       Enqueue(Q,v)
    }
    color[u]:=Black;
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BFS(G,s)

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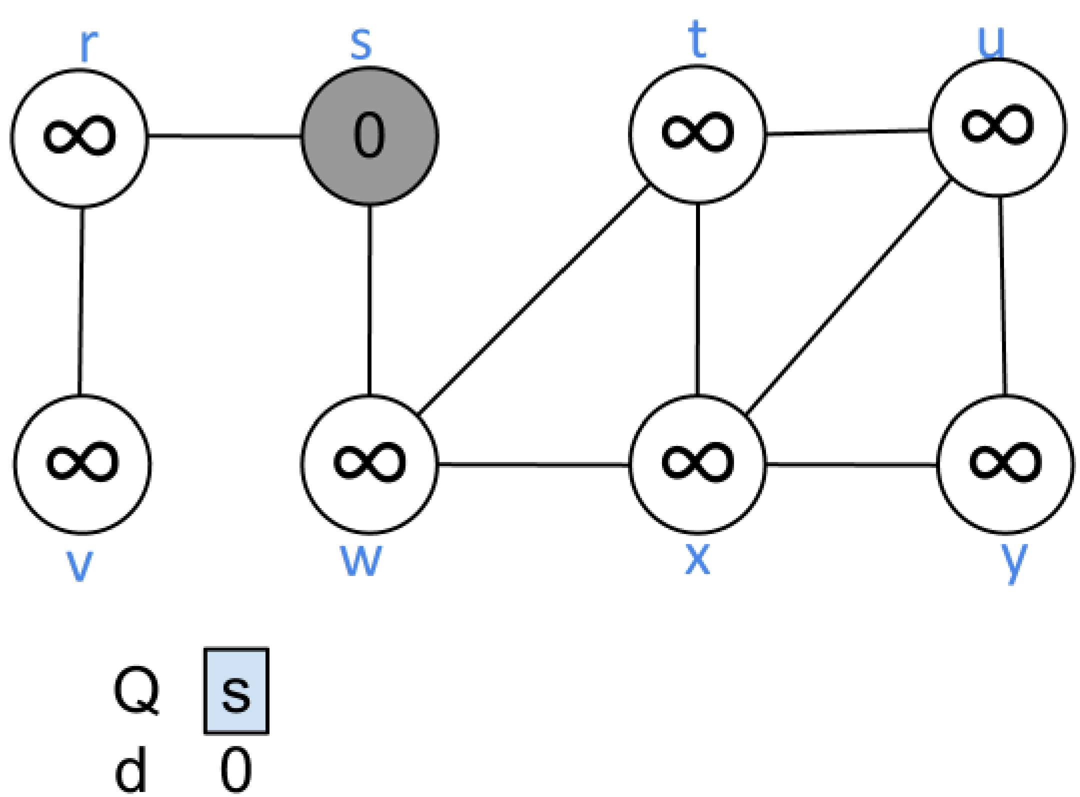

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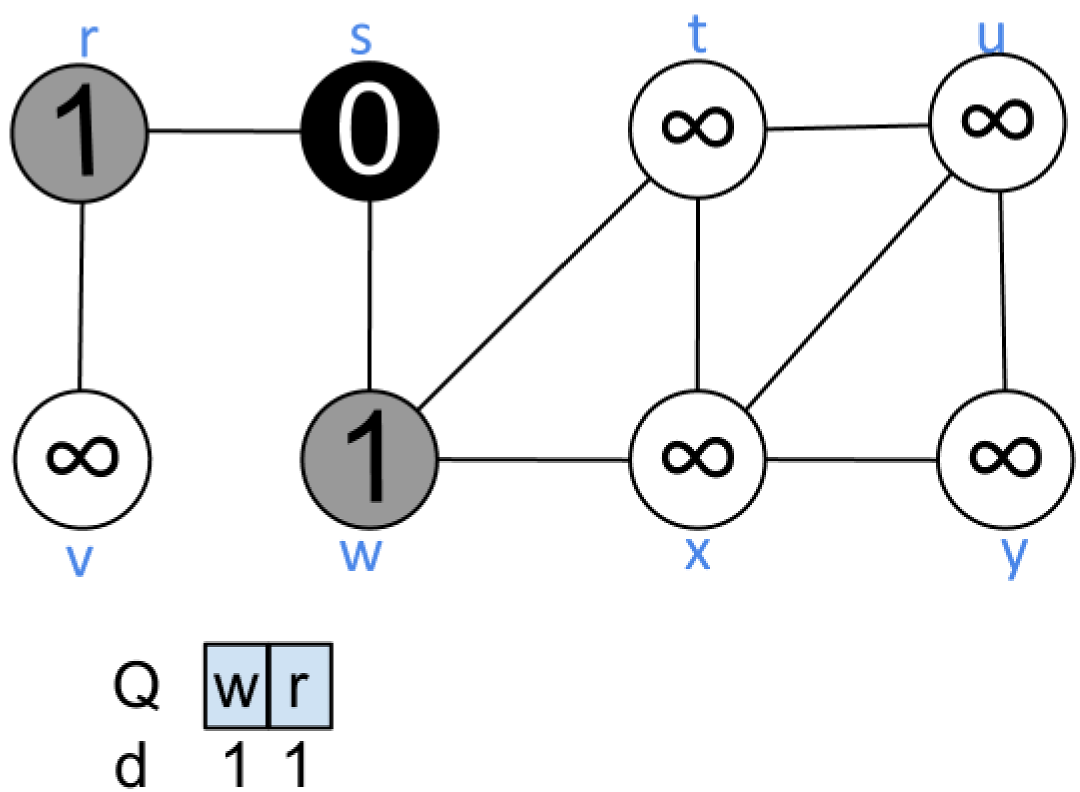

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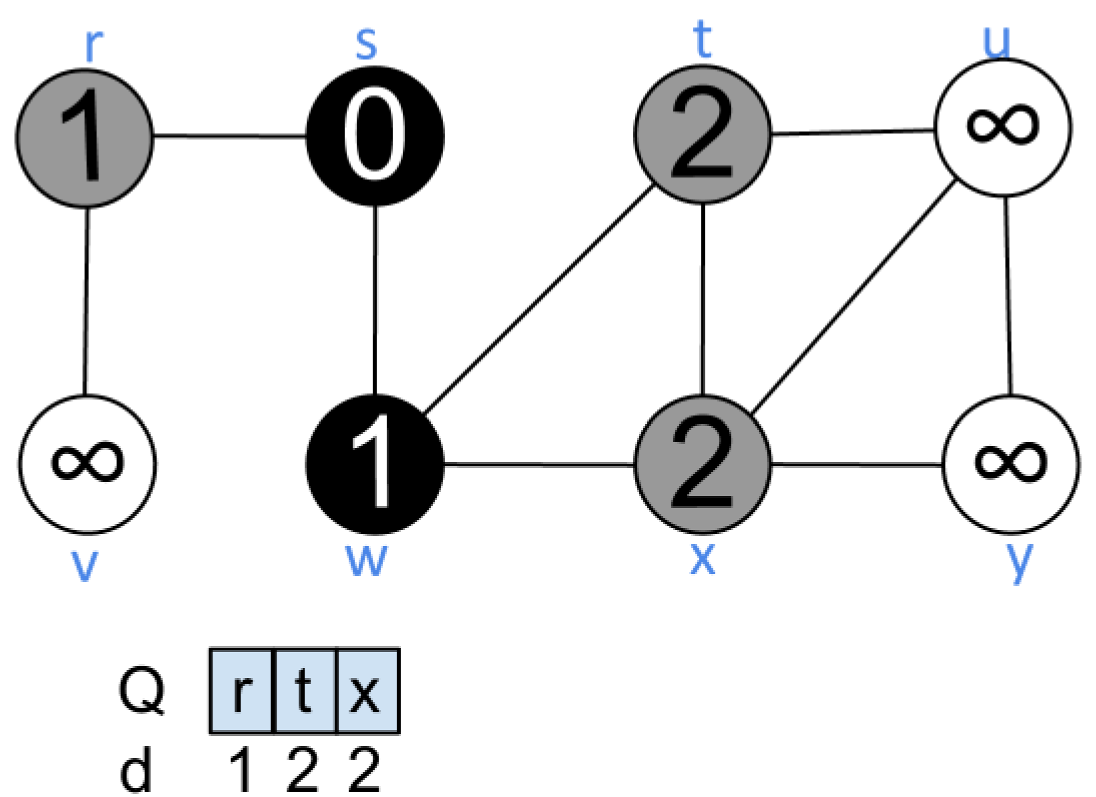

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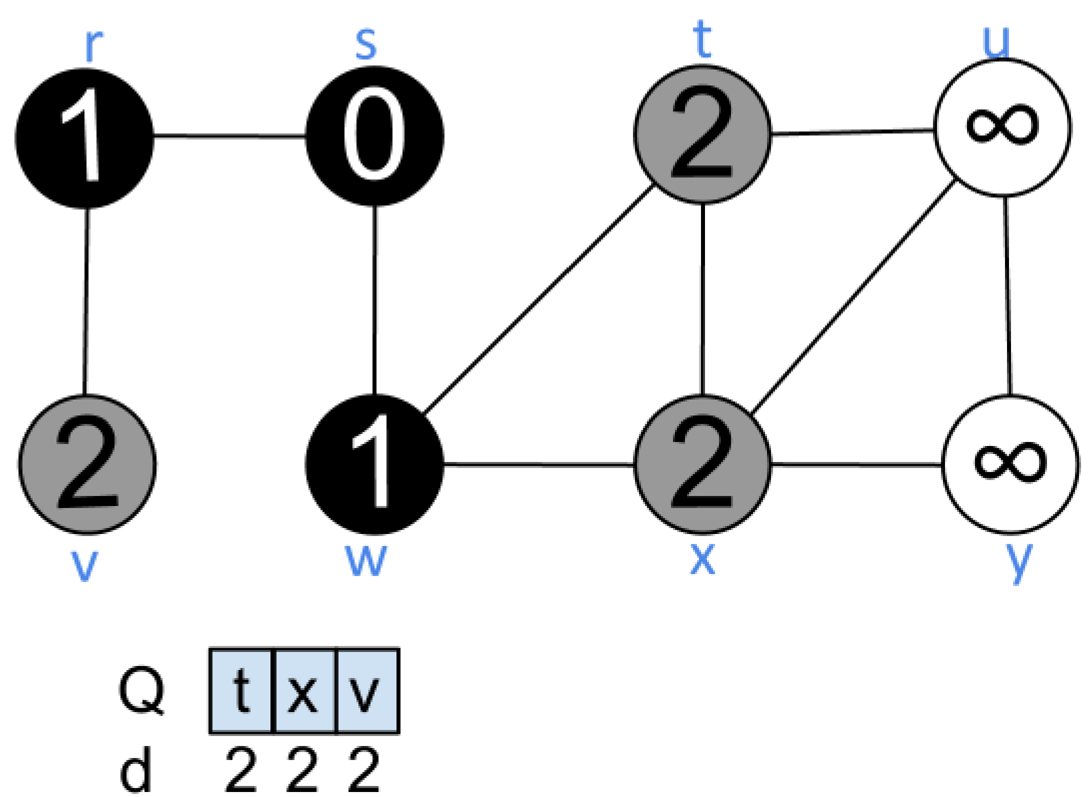

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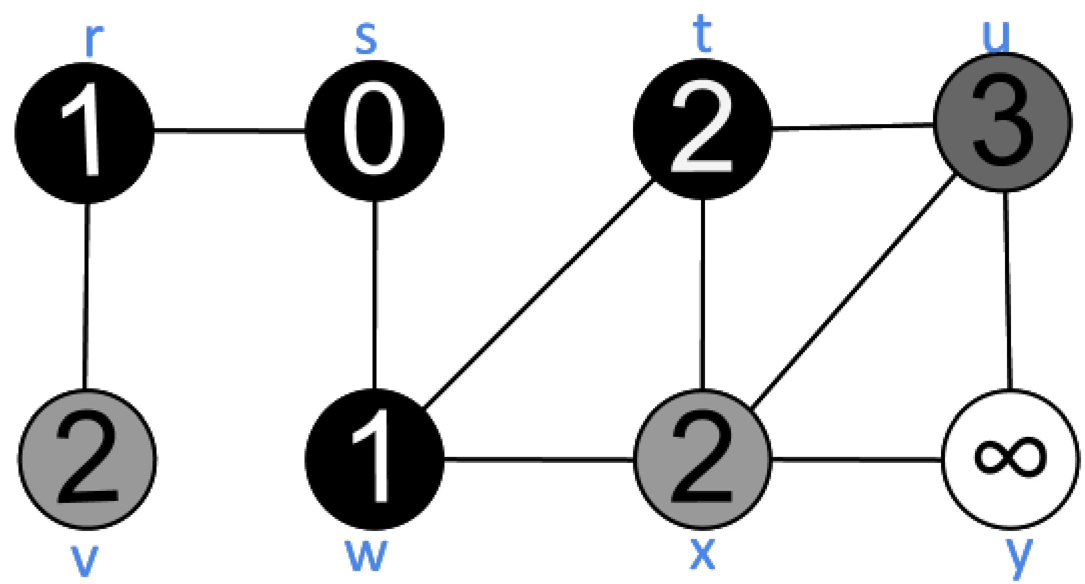


| Q | x | v | u |
|---|---|---|---|
| d | 2 | 2 | 3 |

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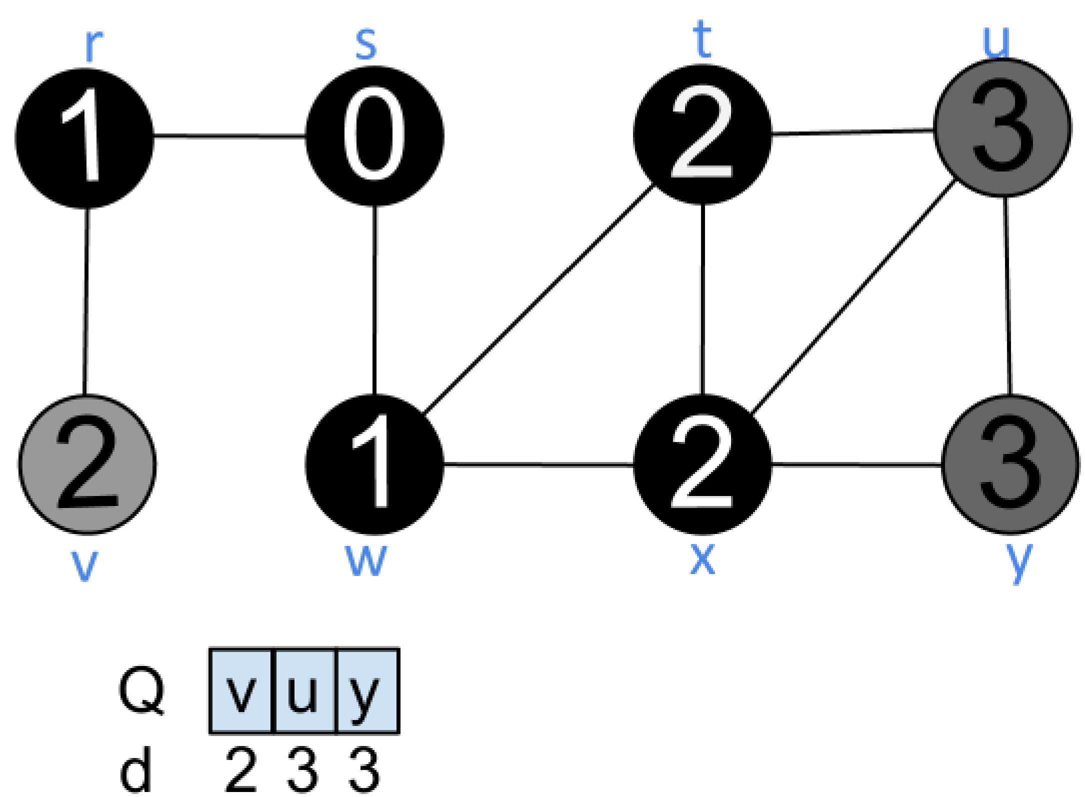

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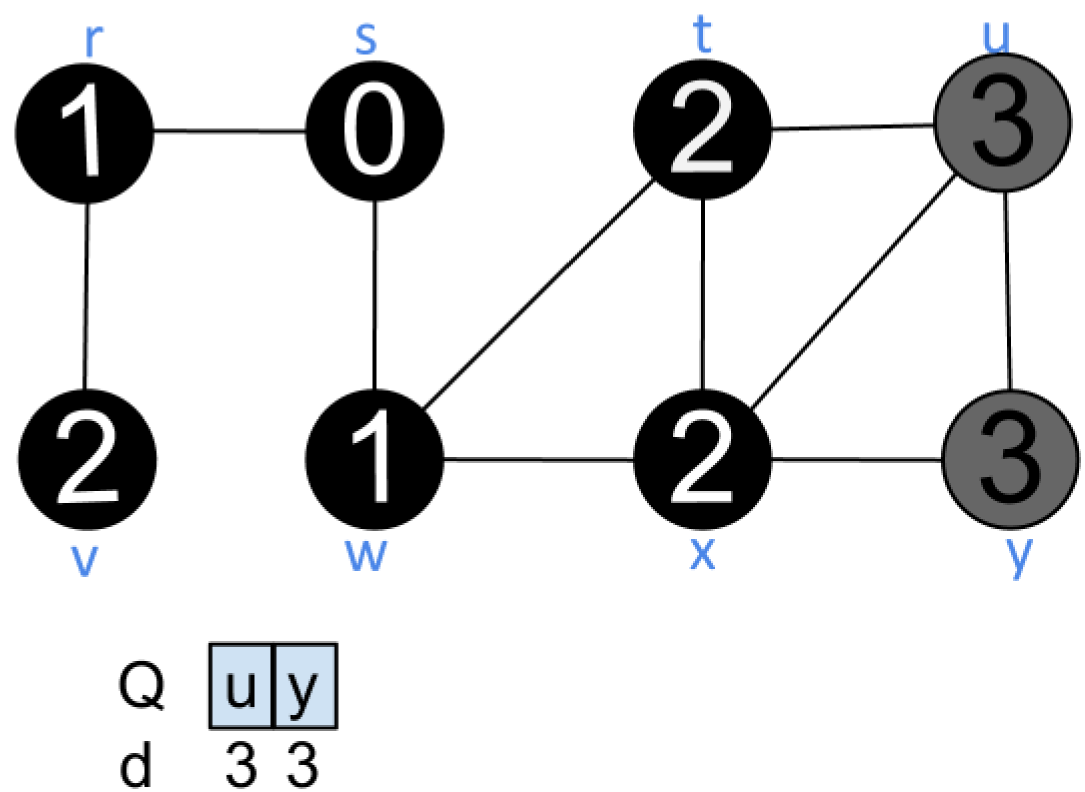

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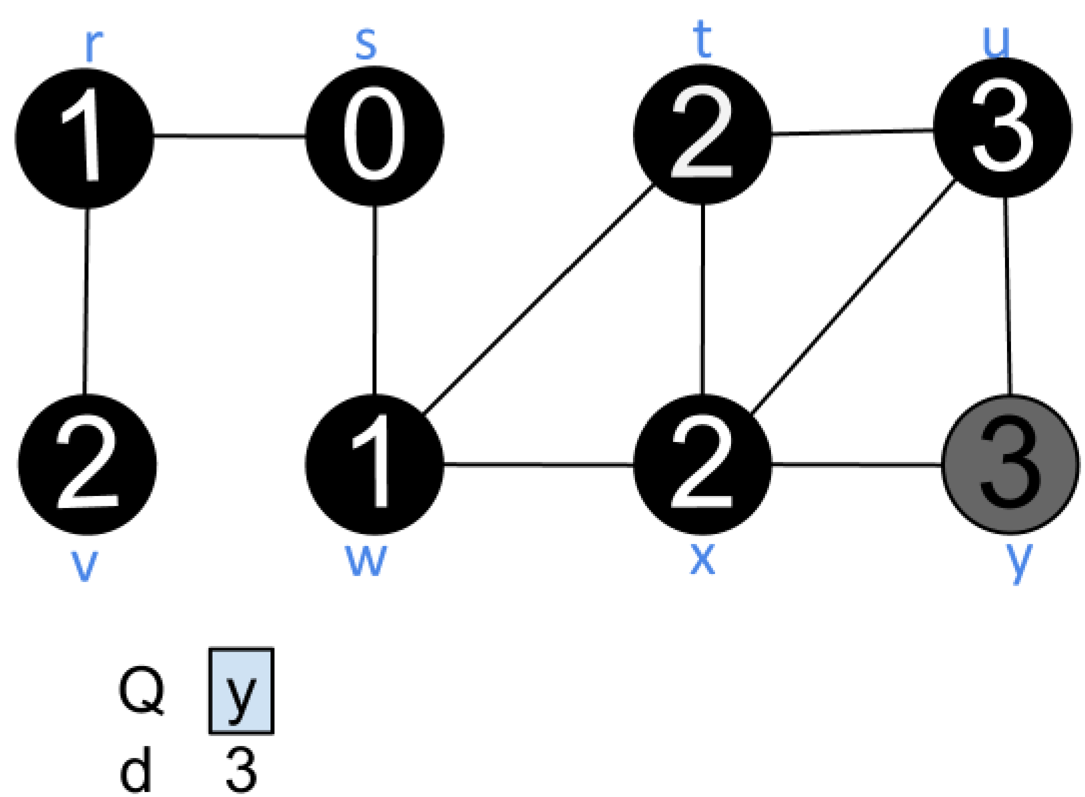

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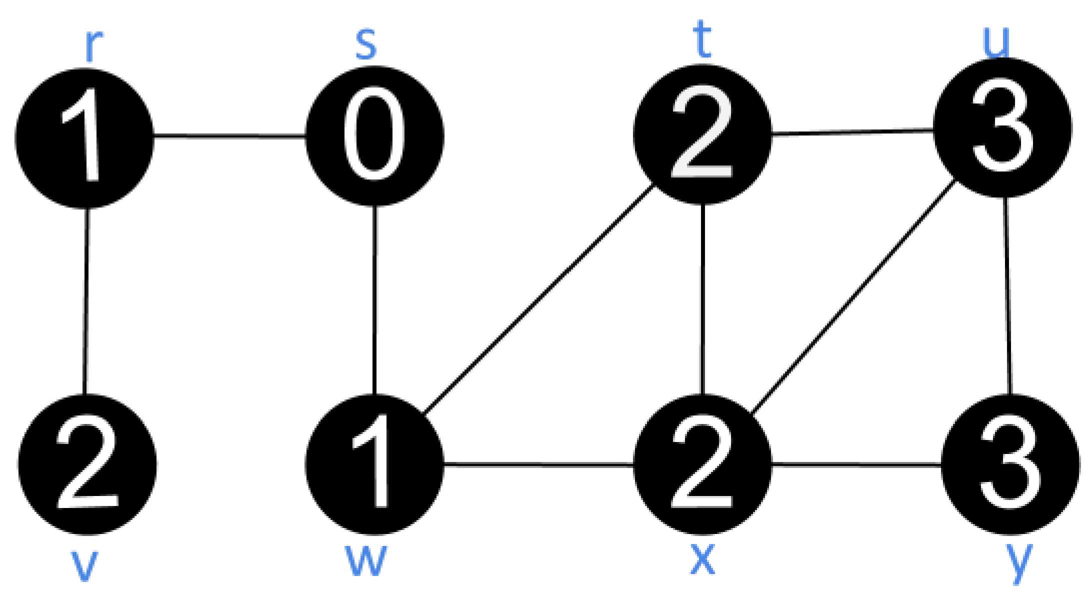


Q

d

Running time of BFS in adjacency-list representation

Recall Enqueue and Dequeue take time ?

Running time of BFS in adjacency-list representation

Recall Enqueue and Dequeue take time O(1)

Each edge visited O(1) times.

Main loop costs O(E).

Initialization step costs O(V)

Running time O(V + E)

What about space?

# Space of BFS

$\Theta(V)$ to mark nodes

Optimal to compute all of d

What if we just want to know if u and v are connected?

**Theorem:** Given a graph with n nodes, can decide if two nodes are connected in space $O(\log^2 n)$

**Proof:**

```
REACH(u, v, n) :=      \\ is v reachable from u in n steps?
  Enumerate all nodes w {
      If REACH(u, w, n/2) and REACH(w, v, n/2) return YES
  }
  Return NO
```

$S(n)$ := space for REACH(u, v, n).

$S(n) := O(\log n) + S(n/2)$. Reuse space for 2 calls to REACH.

$S(n) = O(\log^2 n)$ ■

# Next: weighted single-source shortest path

Input: Directed graph $G= (V,E)$, $s \in V$, $w: E \to Z$

Output: Shortest paths from s to all the other vertces

- Note: Previous case was for $w : E \to \{1\}$

- Note: if weights can be negative, shortest paths exist $\Leftrightarrow$ s cannot reach a cycle with negative weight

```
Bellman-Ford(G,w, s)
  d[s] :=0; Set the others to ∞

 Repeat |V| stages:
   for each edge (u,v) ∈ E[G]
       d[v] := min{ d[v], d[u]+w(u,v); }  //relax(u,v)
```

At the end of the algorithm, can detect negative cycles by:

```
for each edge (u,v) ∈ E[G]
   if d[v] > d[u]+w(u,v)
       Return Negative cycle

return No negative cycle
```

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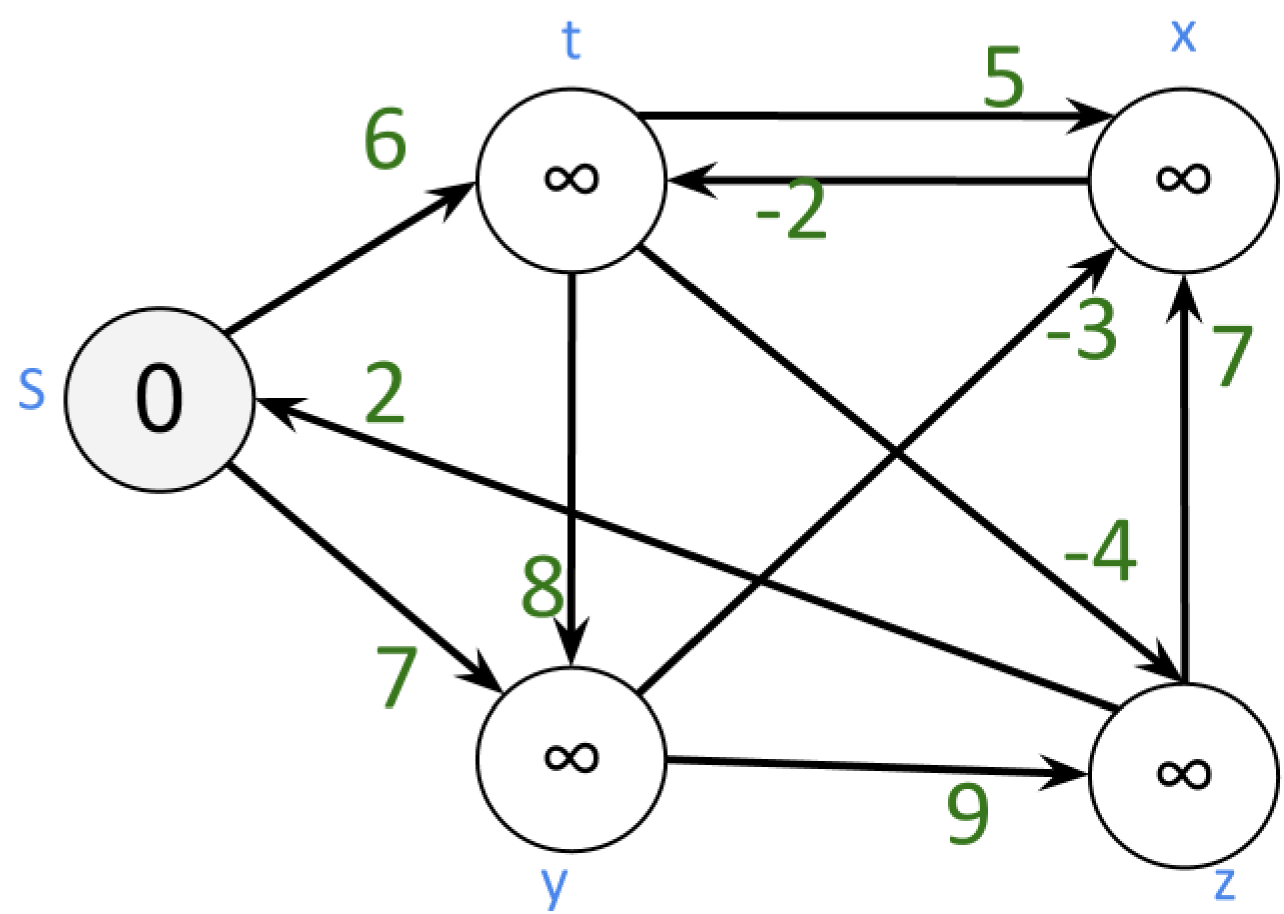

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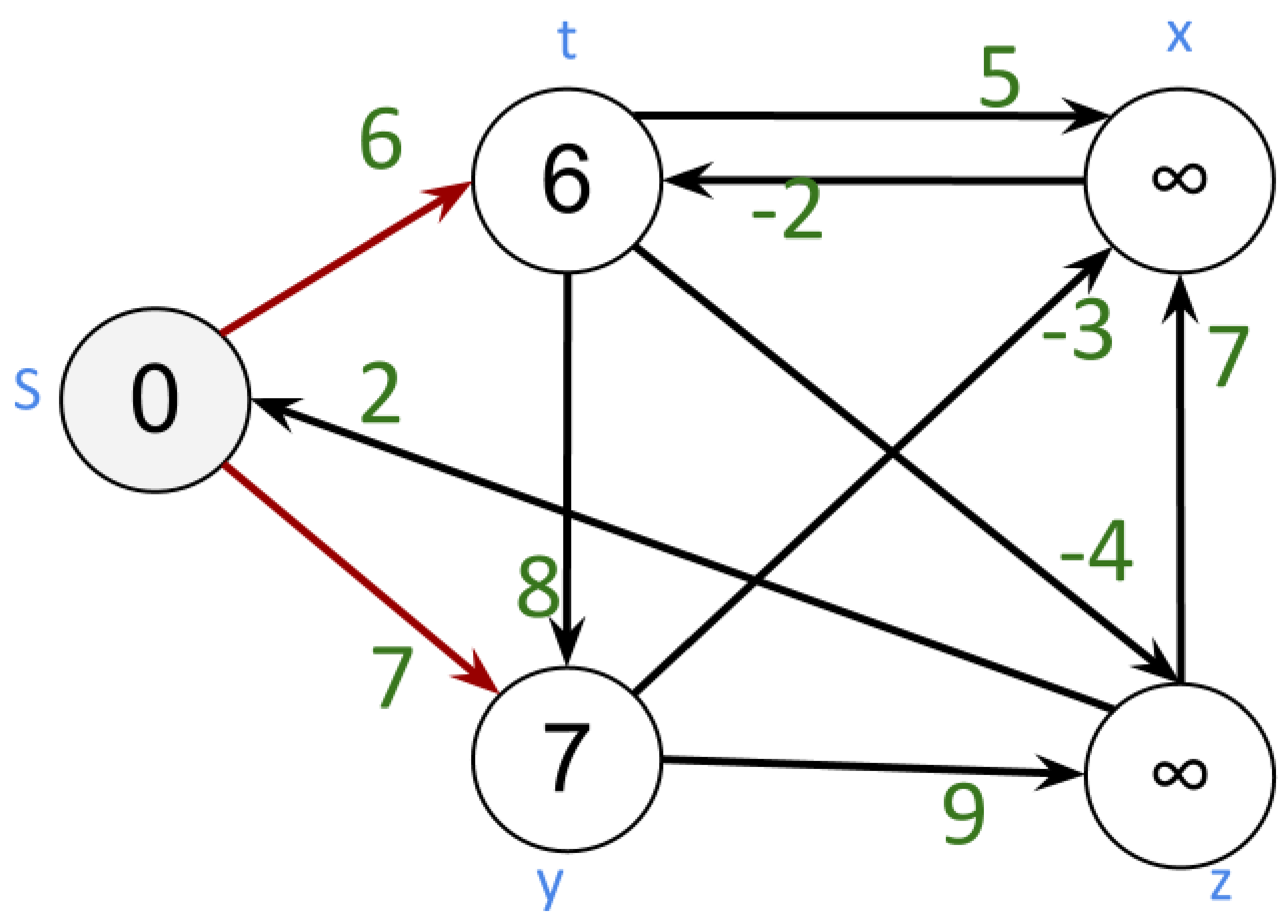

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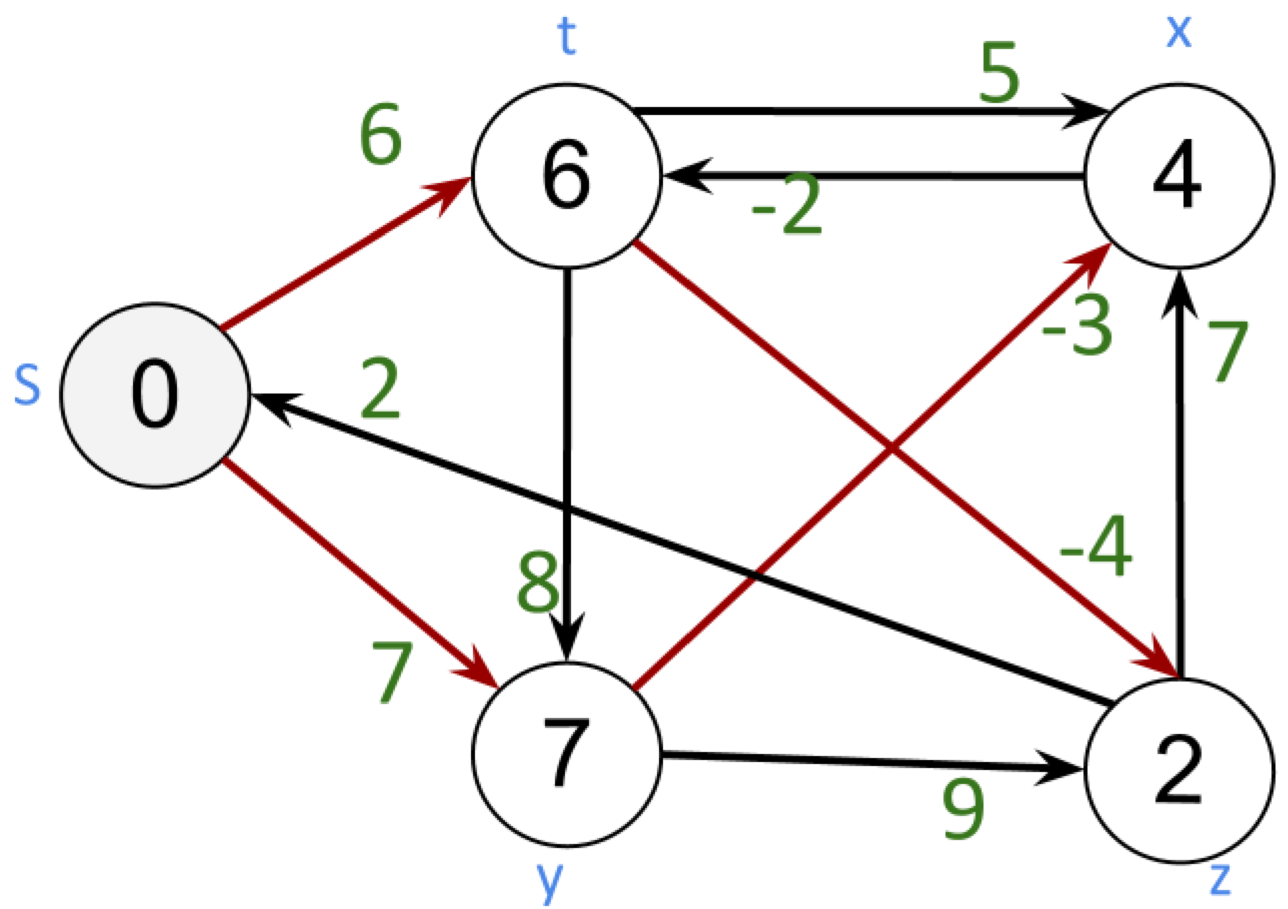

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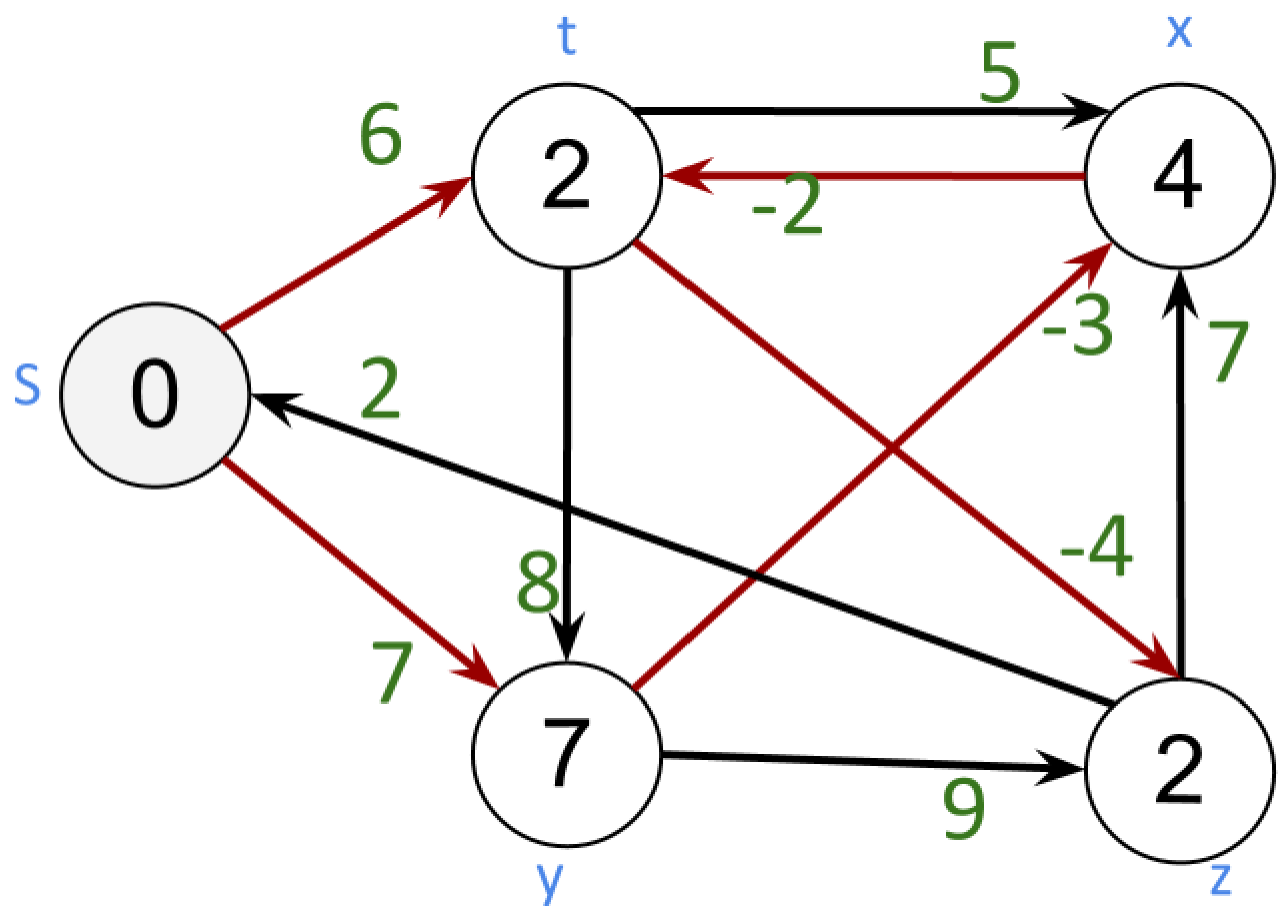

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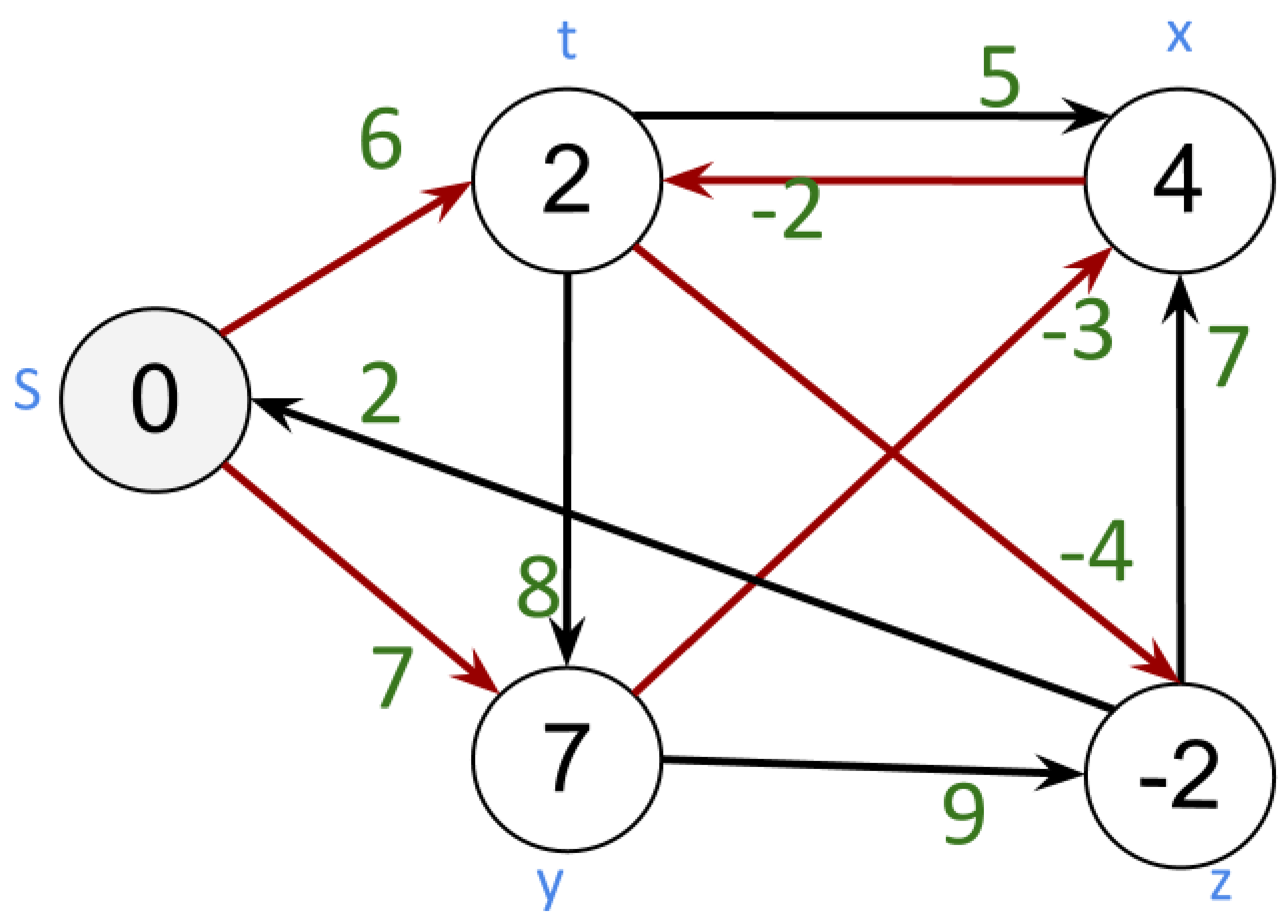

# Running time of Bellman-Ford

Bellman-Ford(G,w, s)
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Time = ??

# Running time of Bellman-Ford

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Time = O(|V|.|E|)

Analysis of Bellman-Ford(G,w, s)

```
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Repeat |V| stages:
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```

- Claim: d = δ if no negative-weight cycle exists.
- Proof: Consider a shortest path $s \to u_1 \to u_2 \to \dots \to u_k$

  $k \le n$ by assumtion.

  We claim at stage $i = 1..|V|$, $d[u_i] = \delta(s, u_i)$

  This holds by induction, because:

  $d[u_i] = \delta(s, u_i)$ and relax $u_i \to u_{i+1}$ ➔ $d[u_{i+1}] = \delta(s, u_{i+1})$.

  d is never increased

  d is never set below δ (exercise next)

■

Exercise: Consider an algorithm that starts with $d[s] = 0$ and $\infty$ otherwise, and only does edge relaxations.

Prove that $d \geq \delta$ throughout

Analysis of negative-cycle detection at the end of algorithm:

```
for each edge (u,v) ∈ E[G]
   if d[v] > d[u]+w(u,v)
      Return Negative cycle

return No negative cycle
```

- Proof of correctness:

If not ∃ neg-cycle, $d = \delta$, tests pass (triangle inequality).

O.w. let $u_1 \to u_2 \to \ldots \to u_k = u_1$ so that $\sum_{i < k} w(u_i , u_{i+1}) < 0$

We know $\forall\ i<k : d[u_{i+1}] \leq d[u_i] + w(u_i , u_{i+1})$

➔ now what?

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$\Rightarrow \sum_{i<k} d[u_{i+1}] \leq \sum_{i<k} d[u_i] + \sum_{i<k} w(u_i, u_{i+1}) \Rightarrow 0 < 0$ ■

# Dijkstra's algorithm

Input: Directed graph $G=(V,E)$, $s \in V$, non-negative $w: E \rightarrow N$

Output: Shortest paths from $s$ to all the other vertices.

```
Dijkstra(G,w,s)
  d[s] :=0; Set others d to ∞; Q := V

  While (|Q|> 0) {
    u:= extract-remove-min(Q) // vertex with min distance d[u];
     for each v ∈ adj[u]
         d[v] := min{ d[v], d[u]+w(u,v)}    //relax(u,v)
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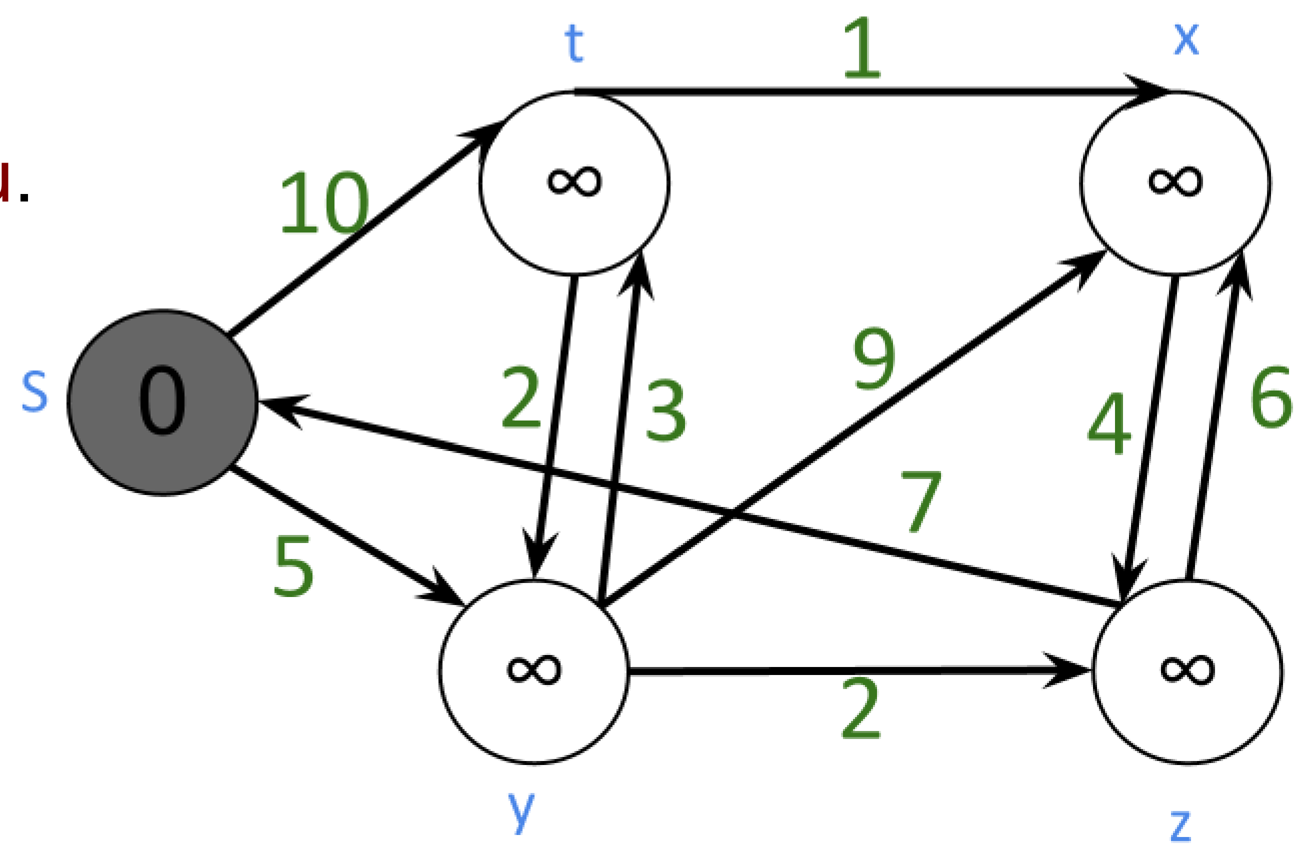
```

Gray: the extracted u.
Black: not in Q



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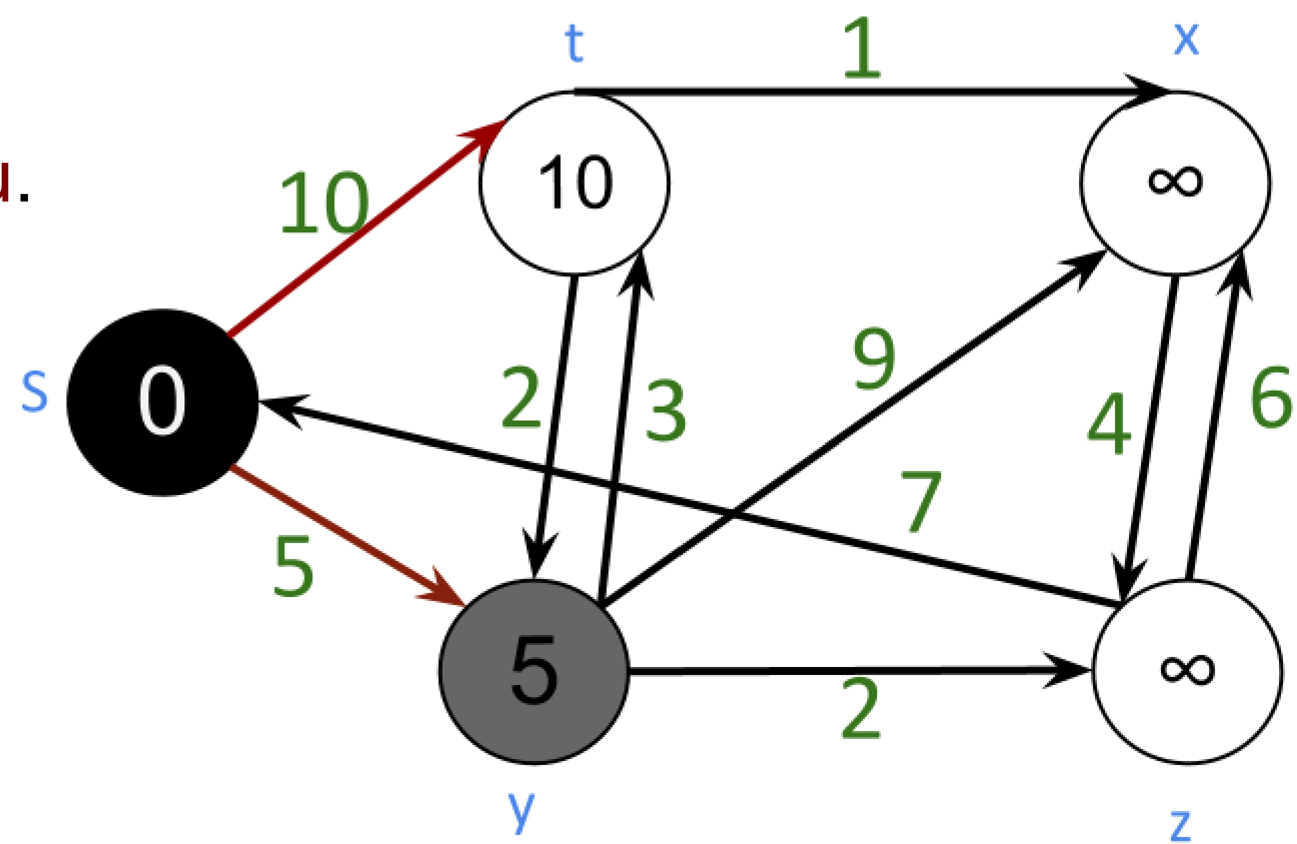

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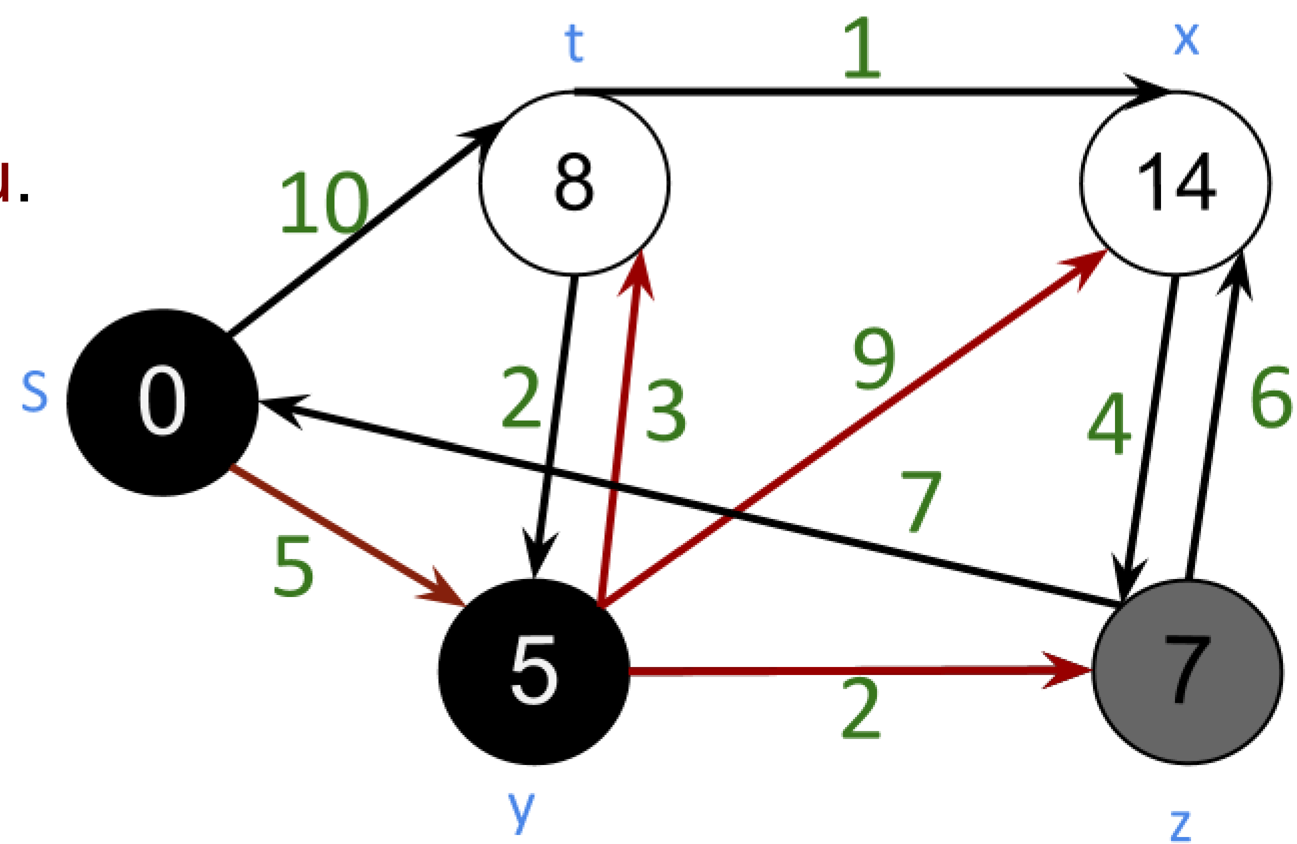

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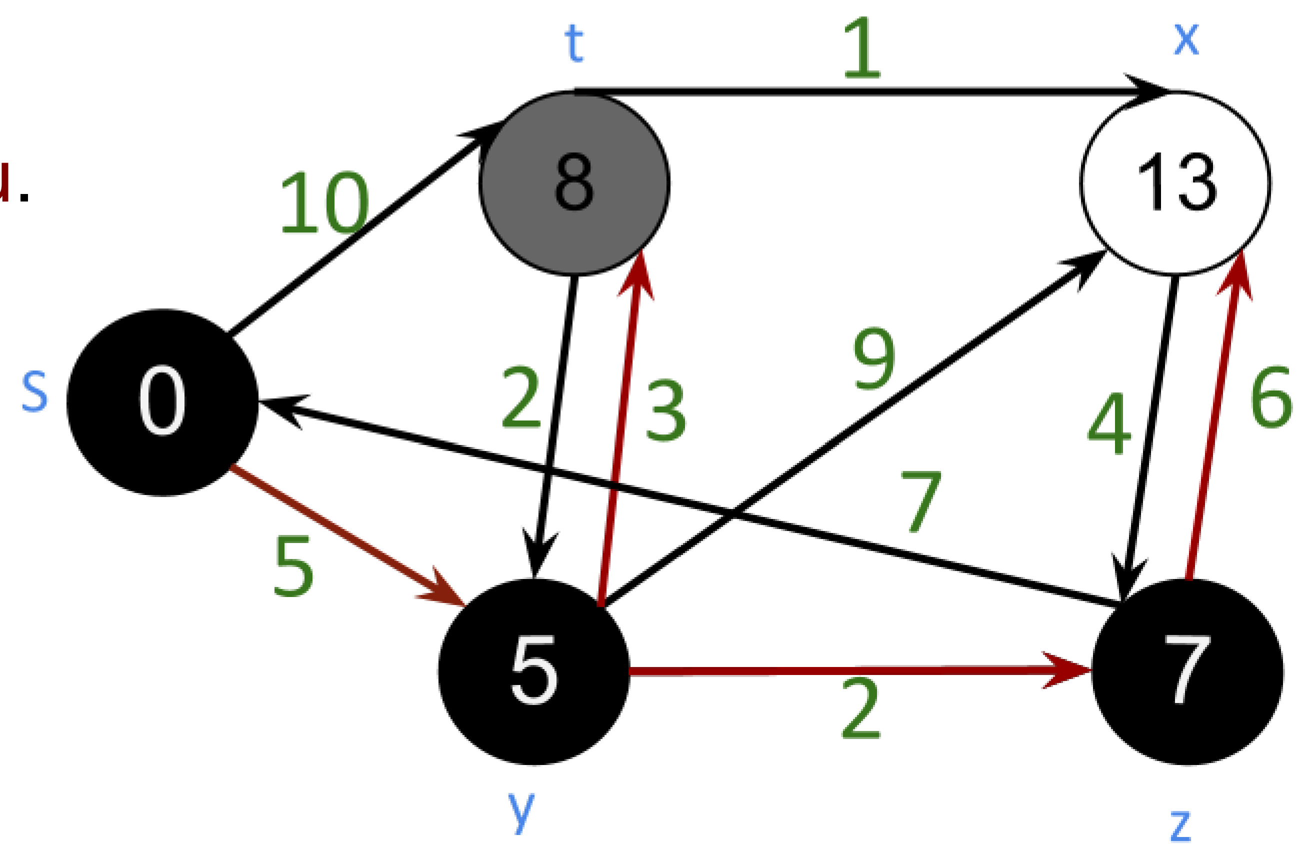

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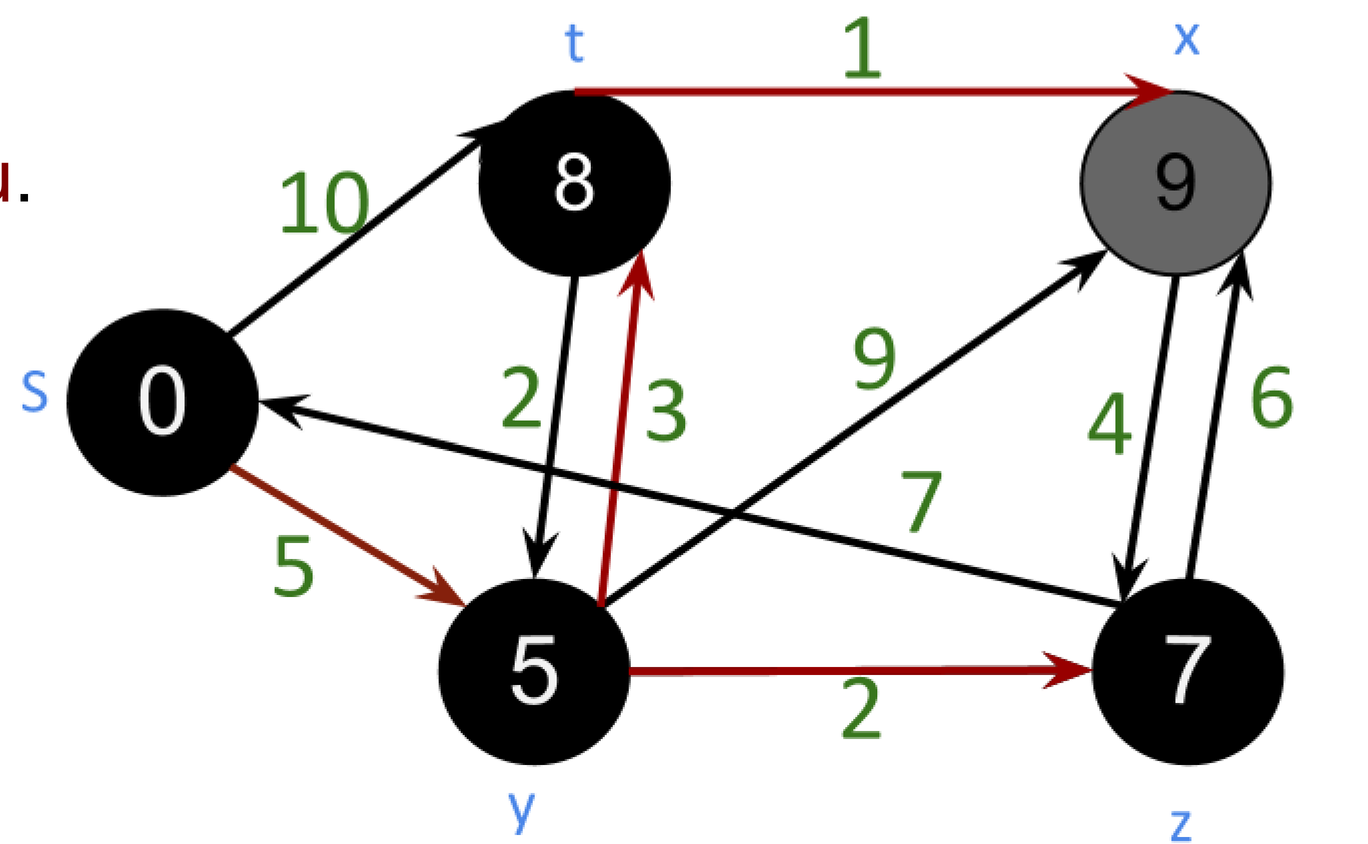

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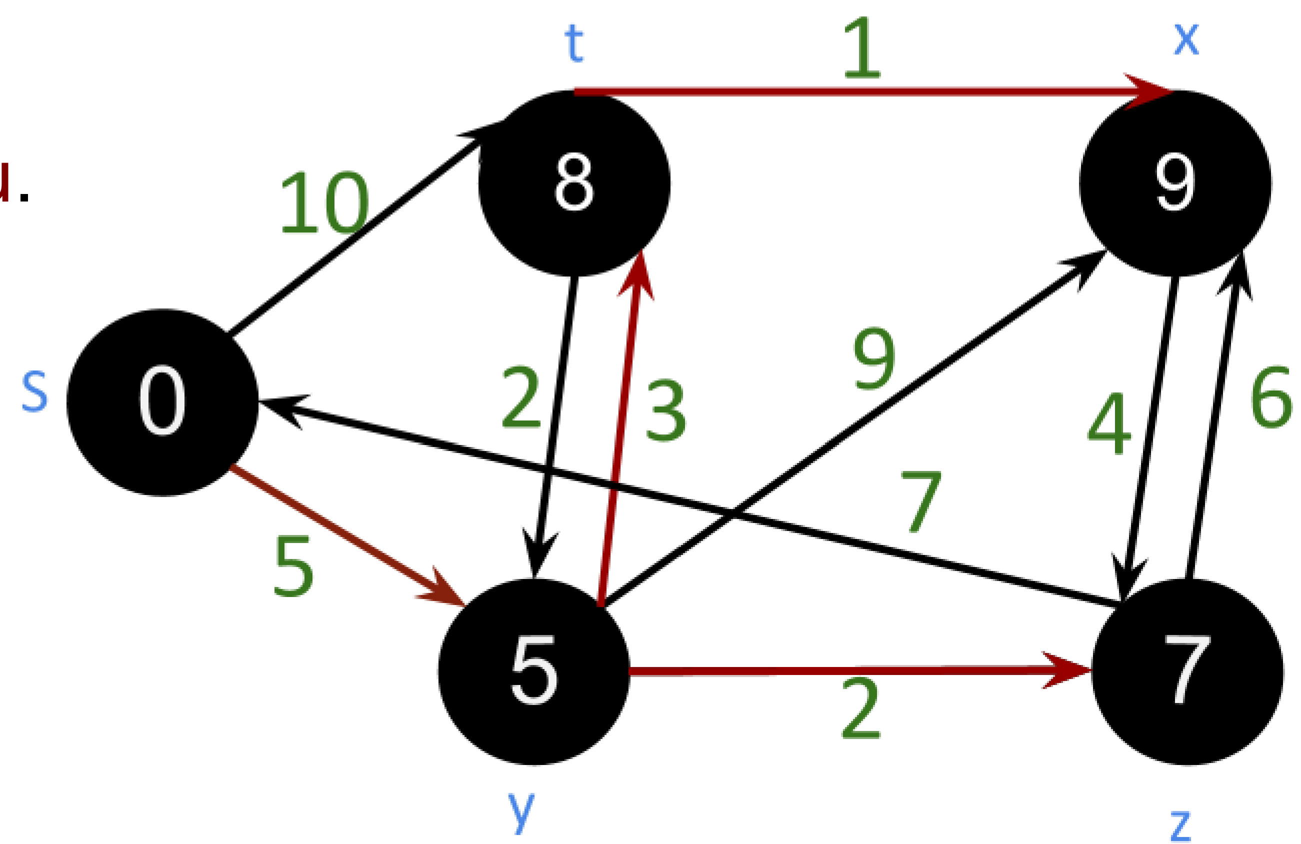

# Running time of Dijkstra(G,w,s)

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Running time depends on data structure for Q

Naive implementation, array: Extract-min in time ?

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➔ running time = ?

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Naive implementation, array: Extract-min in time |V|

➔ running time = $O(V^2 + E)$

Can we do better?

## Running time of Dijkstra(G,w,s)

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Running time depends on data structure for Q

Naive implementation, array: Extract-min in time |V|

➔ running time = $O(V^2 + E)$

Implement Q with min-heap. Extract-min in time ?

# Running time of Dijkstra(G,w,s)

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d[s] :=0; Set others d to ∞; Q := V

While (|Q|> 0) {
  u:= extract-remove-min(Q) // vertex with min distance d[u];
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 }
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Running time depends on data structure for Q

Naive implementation, array: Extract-min in time |V|

➔ running time = $O(V^2 + E)$

Implement Q with min-heap. Extract-min in time O(log V)

➔ running time = ?

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➔ running time = O((V+E) log V)

Note: Can be improved to V log V + E

## Analysis of Dijkstra

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Proof: Let $u \neq s$ be first violation, and Q right before extract

Let $s \to \ldots \to x \to y \to \ldots \to u$ be a shortest path,

where $s \notin Q$ and y is first $\in Q$

Note d[x] = ?

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$d[y] = ?$

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$d[y] = \delta(s,y)$ (since $x \to y$ was relaxed)

Then d[u] ? d[y] How do they compare?

# Analysis of Dijkstra

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$d[y] = \delta(s,y)$ (since $x \to y$ was relaxed)

Then $d[u] \leq d[y]$ (because d[u] is minimum)
$= \delta(s,y) \leq \delta(s,u)$ ∎

# All-pairs shortest paths

Input:

- Directed graph $G= (V,E)$, and $w: E \rightarrow R$

Output:

- The shortest paths between all pairs of vertices.

- Run Dijkstra $|V|$ times: $O(V^2 \log V + E V)$ if $w \geq 0$
- Run Bellman-Ford $|V|$ times: $O(V^2 E)$

- Next, simple algorithms achieving time about $|V|^3$ for any $w$

# All-pairs shortest paths

Dynamic programming approach:

$d_{i,j}^{(m)}$ = shortest paths of lengths $\leq m$

$d_{i,j}^{(m)} = \min_k \{ d_{i,k}^{(m-1)} + w(k,j) \}$

(Includes $k = j$, $w(j,j) = 0$)

Compute $|V|$ x $|V|$ matrix $d^{(m)}$ from $d^{(m-1)}$ in time $|V|^3$.

➔ $d^{|V|}$ computables in time $|V|^4$

How to speed up?

# All-pairs shortest paths

Note:

$d_{i,j}^{(m)} = \min_k \{ d_{i,k}^{(m-1)} + w(k,j) \}$

Is just like matrix multiplication: $d^{(m)} = d^{(m-1)} W$,
except $+ \rightarrow \min$
$\quad\quad\quad x \rightarrow +$

Like matrix multiplication, this is associative. So, instead of doing $d^{|V|} = (...)W)W)W$ can do ?

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Compute $d^{(2)} = W^2$
$\quad\quad d^{(4)} = ?$

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$\quad\quad ...$

To get $d^{|V|}$ need ?

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To get $d^{|V|}$ need log |V| multiplications only ➔ ? time

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To get $d^{|V|}$ need log |V| multiplications only ➔ $|V|^3 \log |V|$ time

# The Floyd-Warshall algorithm

## A more clever dynamic programming algorithm

Before, $d_{i,j}^{(m)}$ = shortest paths of lengths ≤ m

Next: $d_{i,j}^{(m)}$ = shortest paths from i to j such that
all INTERMEDIATE vertices are ≤ m

$d^{(0)} = W$

$d^{(m)} = ???$

# The Floyd-Warshall algorithm

## A more clever dynamic programming algorithm

Before, $d_{i,j}^{(m)}$ = shortest paths of lengths $\leq m$

Next: $d_{i,j}^{(m)}$ = shortest paths from i to j such that
all INTERMEDIATE vertices are $\leq m$

$d^{(0)} = W$

$$d^{(m)}_{i,j} = \begin{cases} w(i,j) \\ \min \left( d^{(m-1)}_{i,j} \; , \; d^{(m-1)}_{i,m} + d^{(m-1)}_{m,j} \right) & \text{if } m \geq 1. \end{cases}$$

Floyd-Warshall(W)

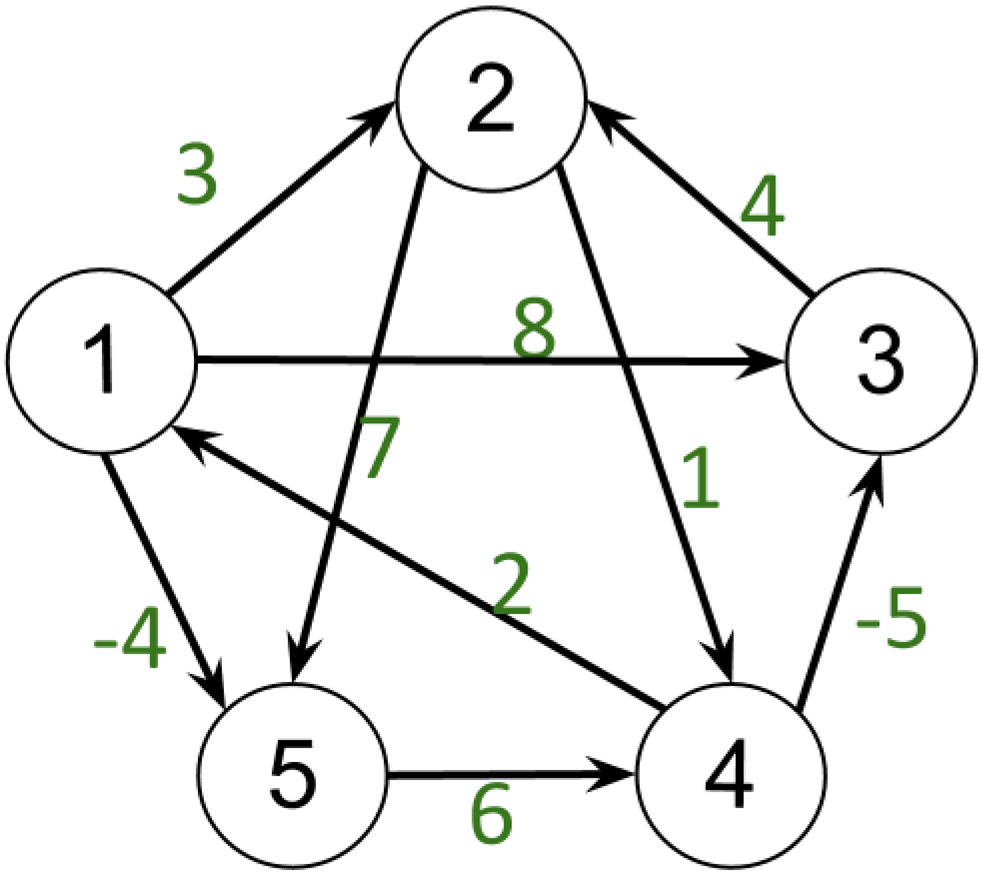
$D^{(0)} := W;$

for $m = 1$ to $n$
    for every $i,j$ :
        $d^{(m)}_{i,j} = \min \left(d^{(m-1)}_{i,j}\ ,\ d^{(m-1)}_{i,m} + d^{(m-1)}_{m,j}\right)$

Return $D^{(n)}$

Time $\Theta(|V|^3)$

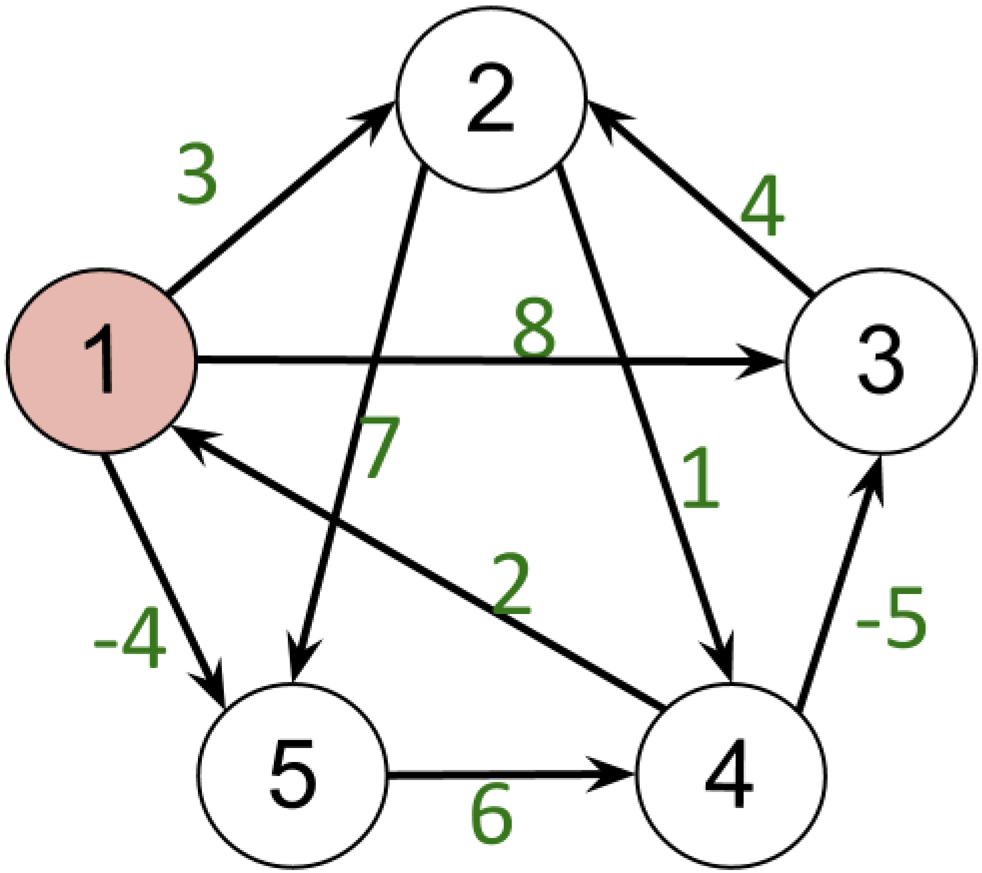
# The Floyd-Warshall Example




$$d^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$d^{(0)}$ = adjacency matrix with diagonal 0

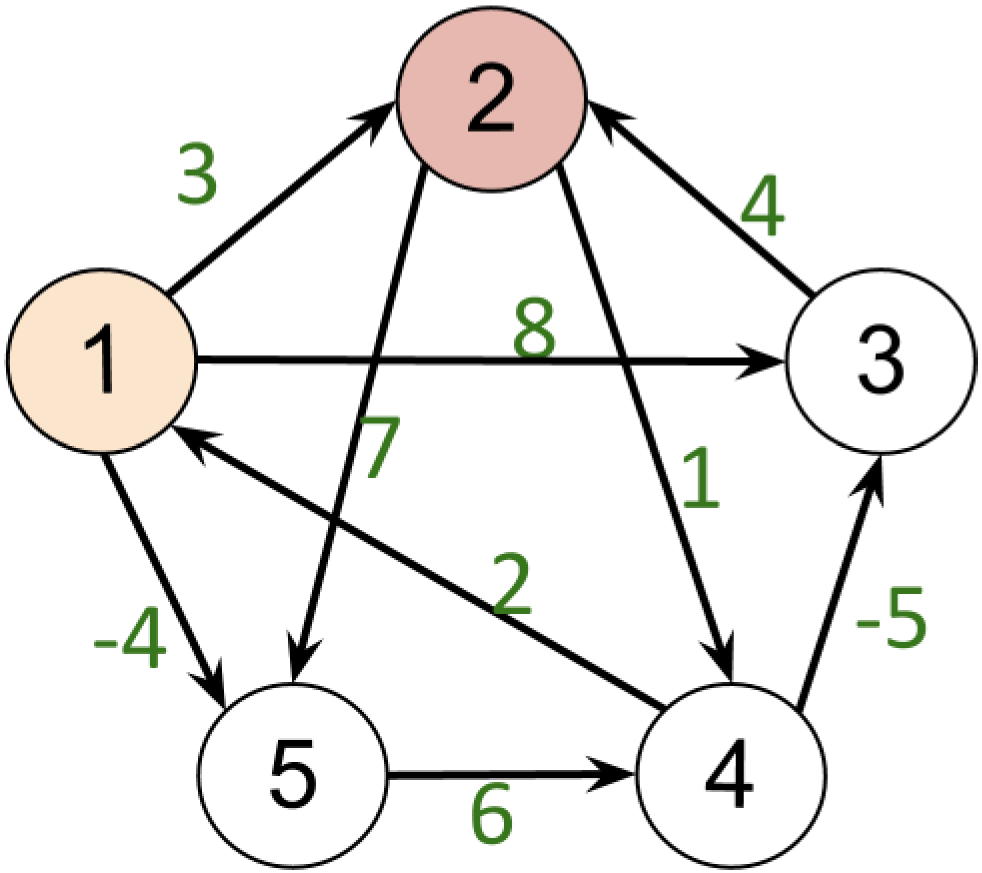
# The Floyd-Warshall Example




$$d^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

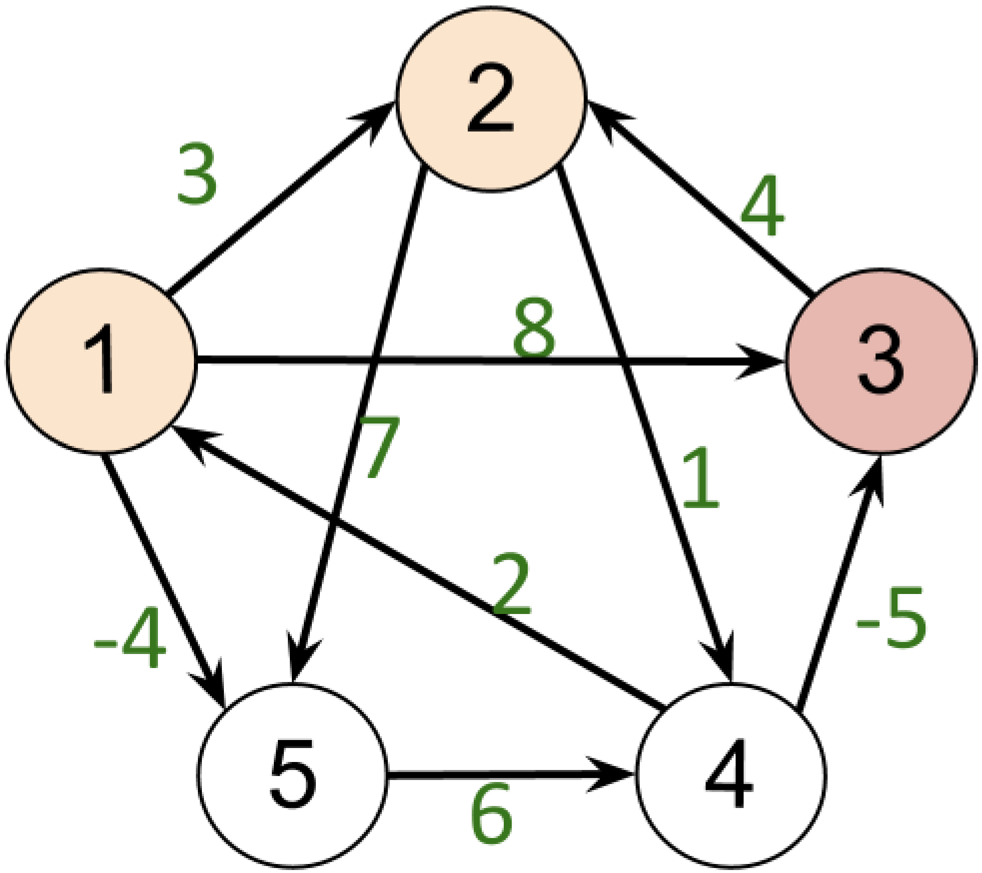
Entries $d_{(4,2)}$ and $d_{(4,5)}$ updated

# The Floyd-Warshall Example



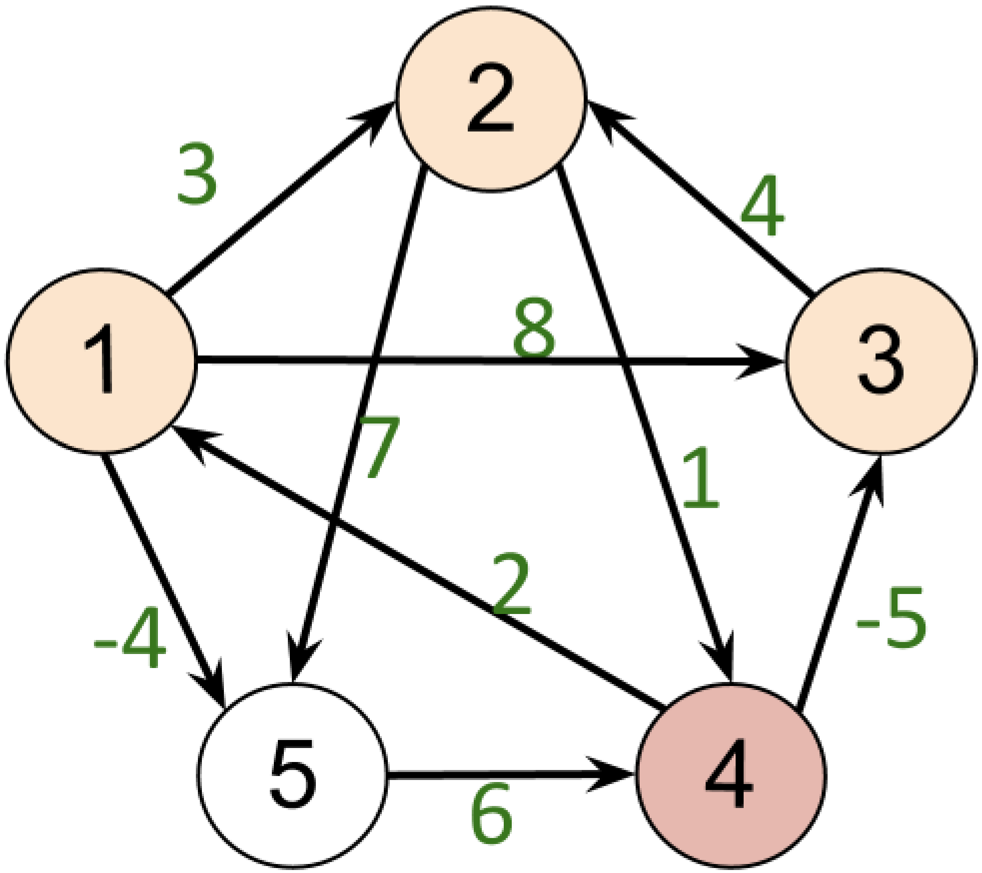

$$d^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

# The Floyd-Warshall Example



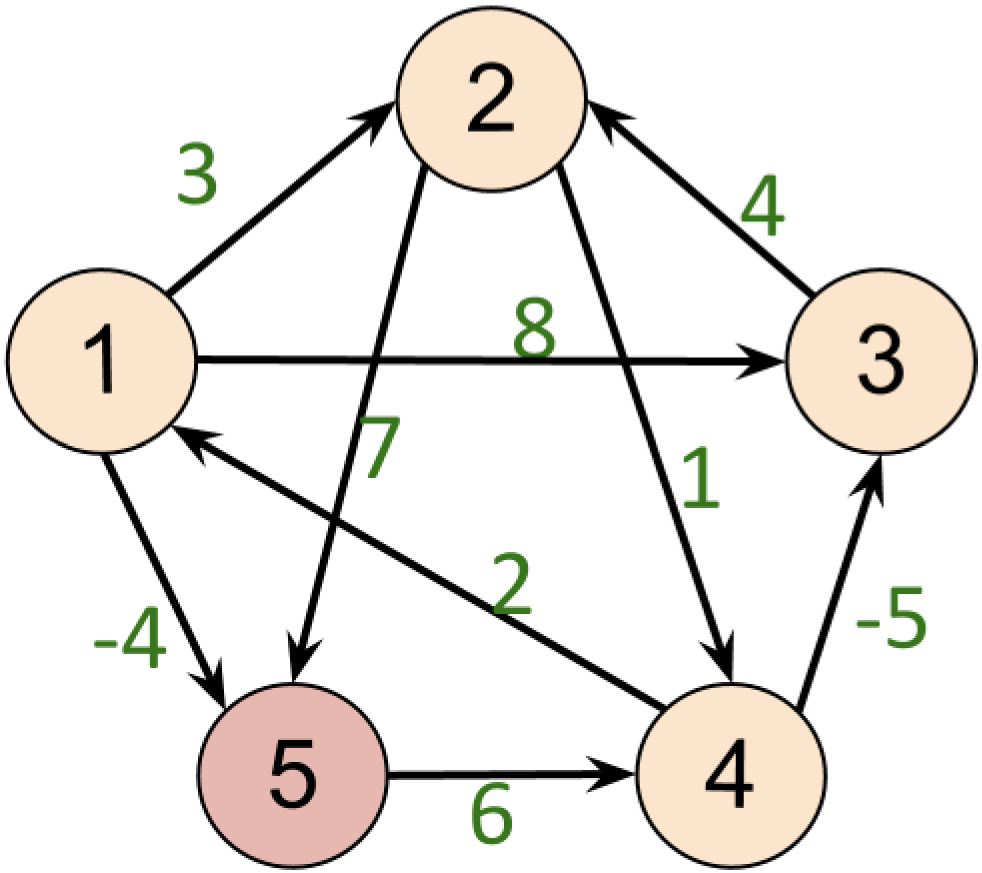

$$d^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

# The Floyd-Warshall Example




$$d^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

# The Floyd-Warshall Example




$$d^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Note: Matrix multiplication/ Floyd Warshall allow for $w < 0$

If $w \geq 0$, can repeat Dijkstra. Time: $O(V^2 \log V + VE) = O(|V|^3)$

Floyd Warshall is easier and has better constants

Johnson algorithm matches Dijkstra but allows for $w < 0$.

Johnson:

Idea: Reweigh so that shortest paths don't change, but $w \geq 0$

Add new node s, with zero weight edges to all previous nodes

Run Bellman-Ford to get minimum distances from s (only)

Use Bellman-Ford distances bf(s,x) to reweigh:

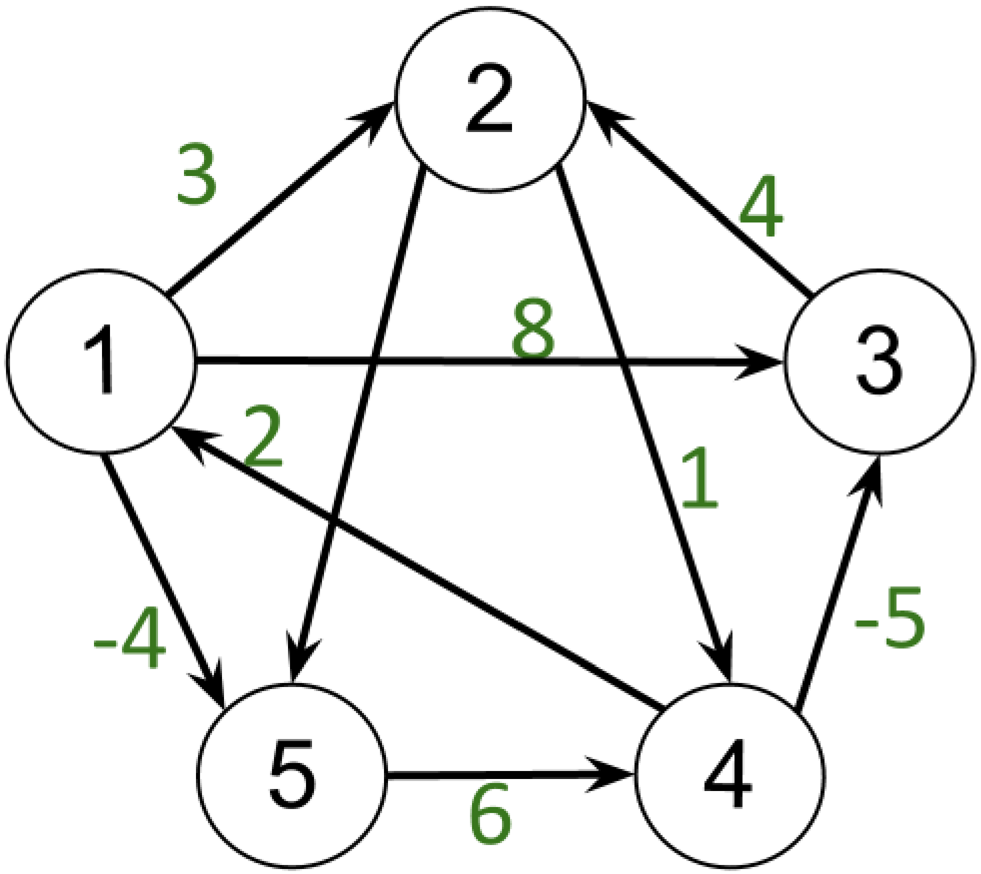
$w'(u,v) := w(u,v) + bf(u) - bf(v)$

(Can show this preserves shortest paths, and $w' \geq 0$)

Now run Dijkstra |V| times

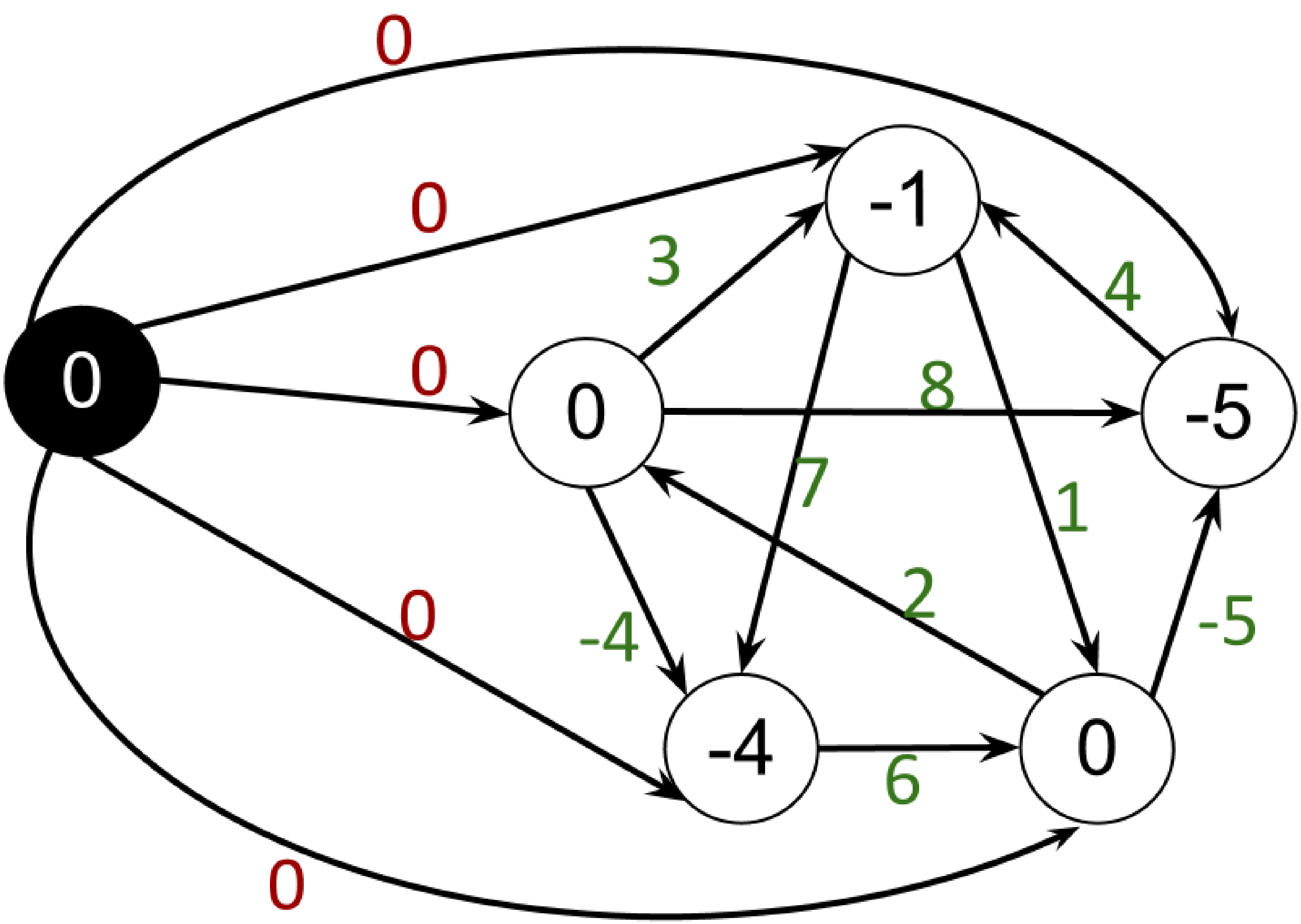
Time: $O(V E + V^2 \log V + VE) = O(V^2 \log V + VE)$.

# Jonhson's Algorithm Example



# Jonhson's Algorithm Example

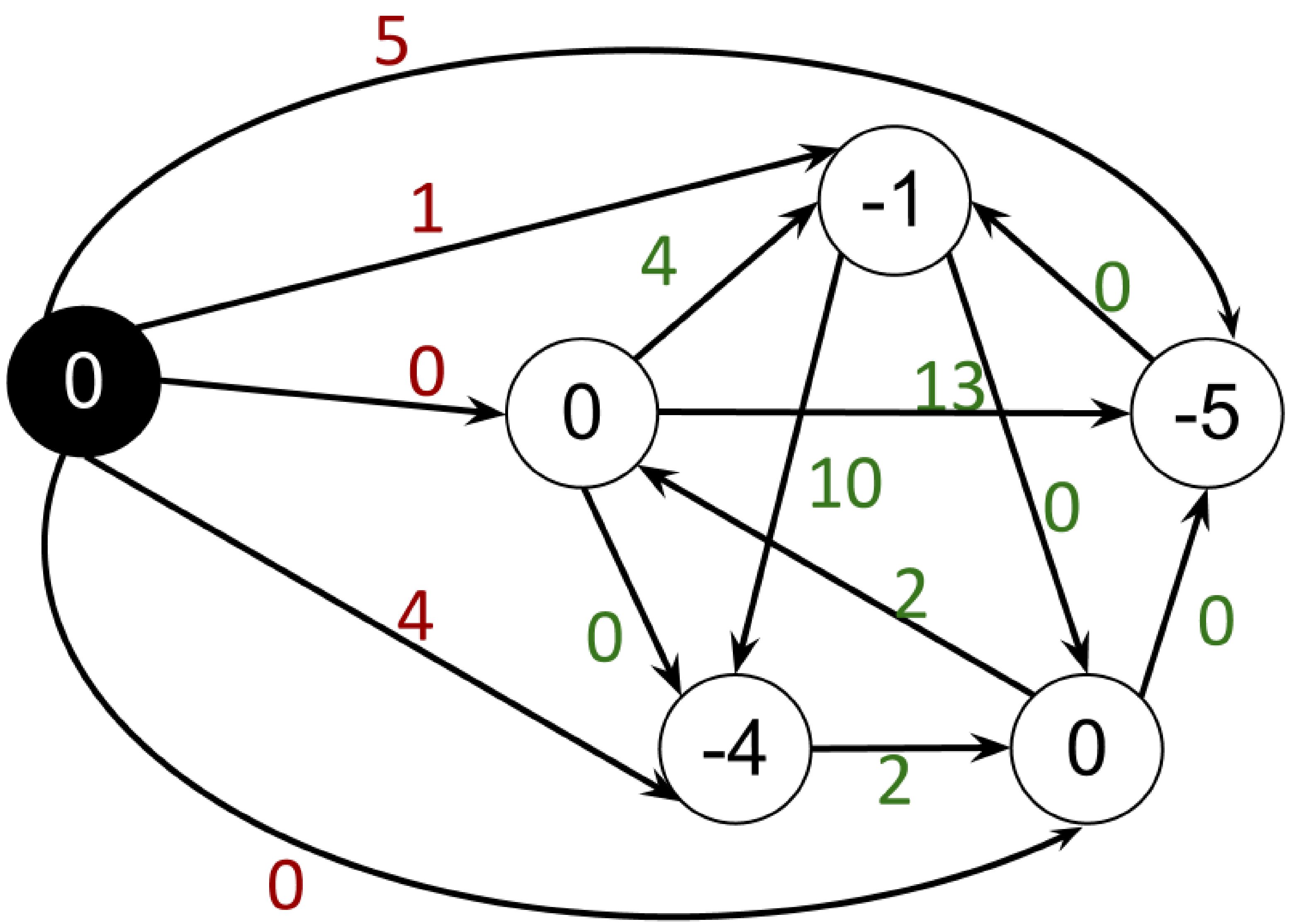
Add new node s, with weight-0 edges to all previous nodes.
Compute Bellman-Ford distance bf(s,x) from s to all nodes x
(distance shown inside the nodes)



# Jonhson's Algorithm Example

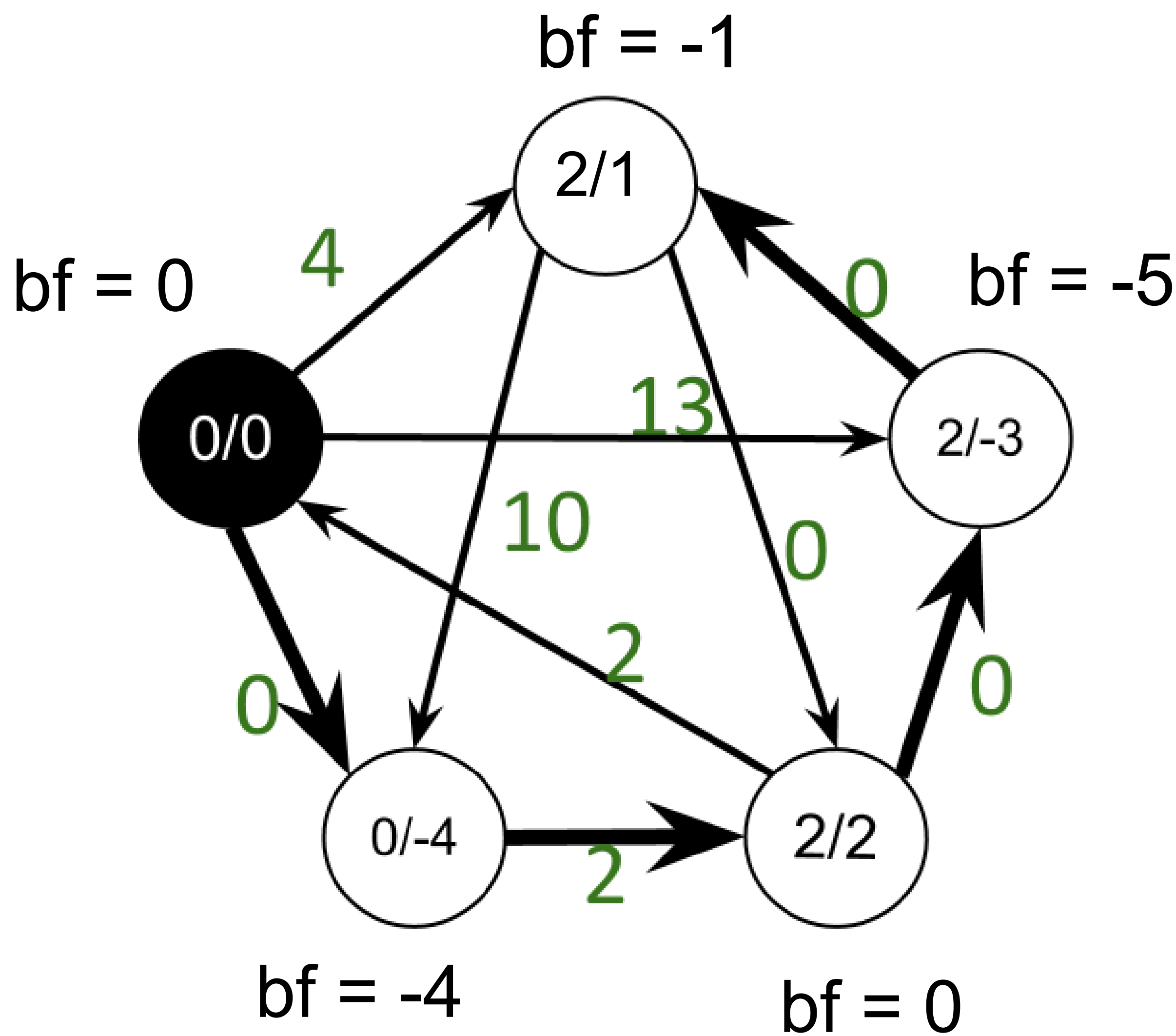
Use Bellman-Ford distances bf(s,x) to reweight:

w'(u,v) = w(u,v) + bf(u) – bf(v)



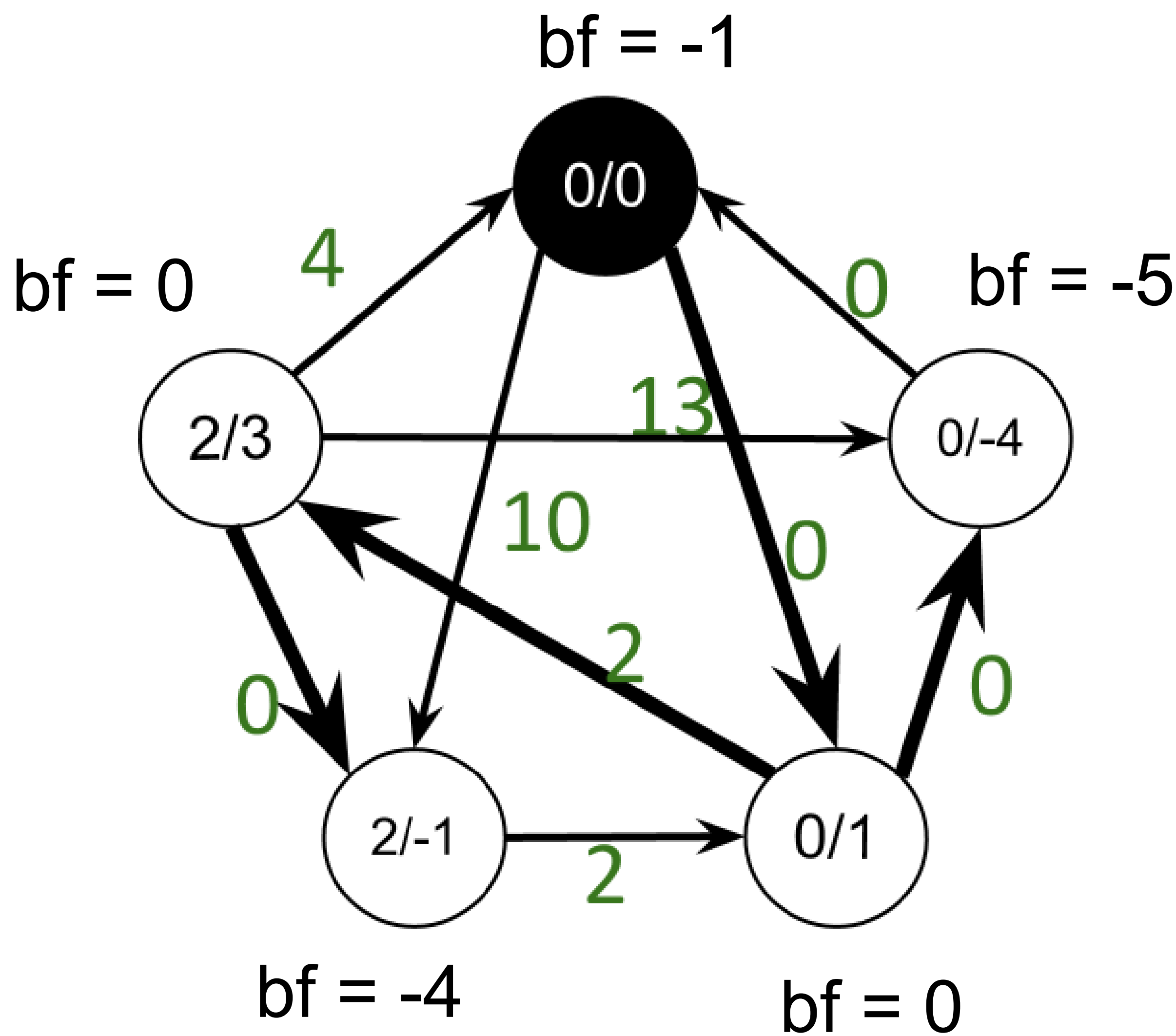
# Jonhson's Algorithm Example

Run Dijkstra algorithm from each node.
Inside each node are minimum distances d'/d w.r.t. w' and w
$d(u,v) = d'(u,v) + bf(v) - bf(u) \geq 0$



# Jonhson's Algorithm Example

Run Dijkstra algorithm on each node.



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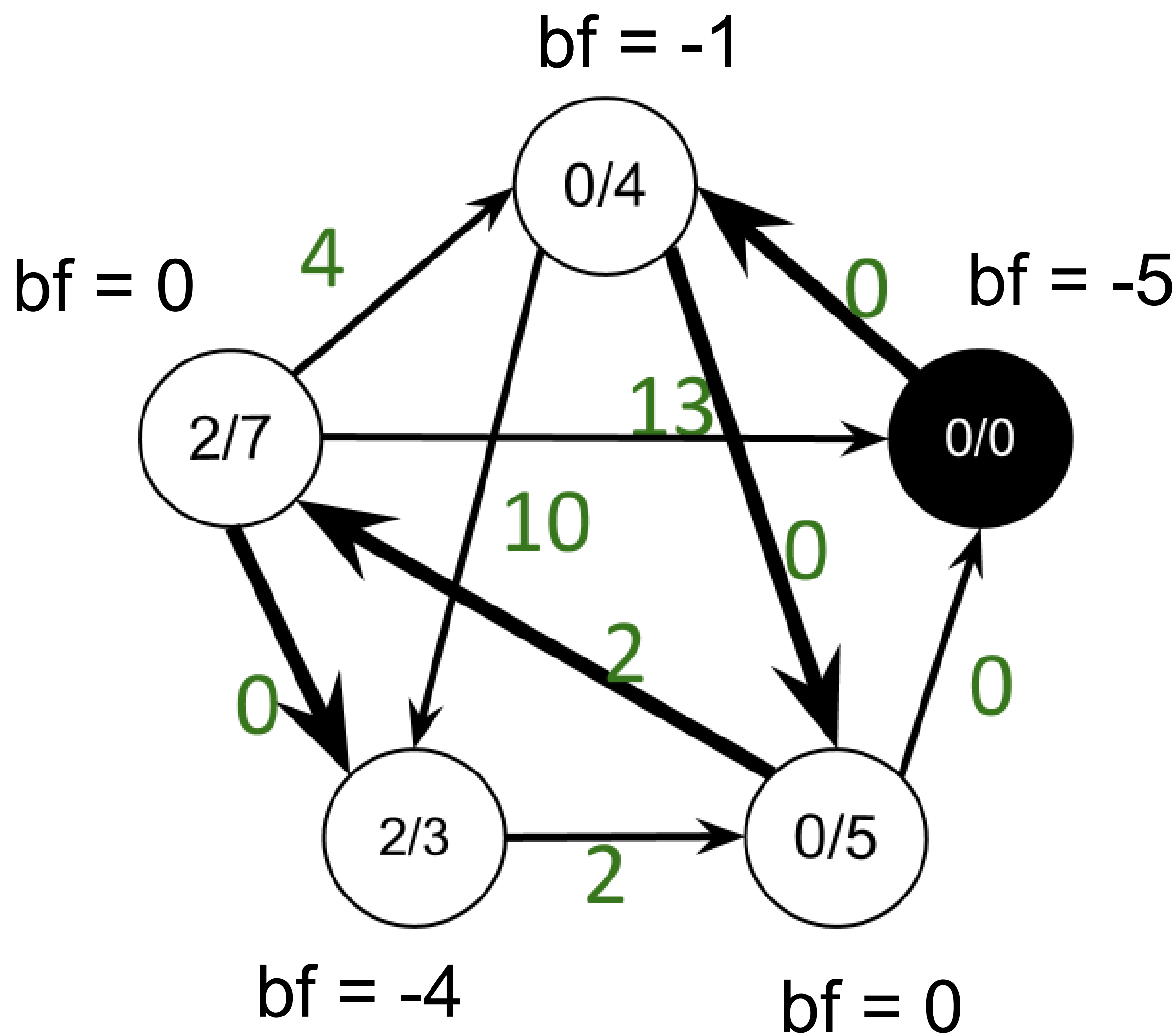

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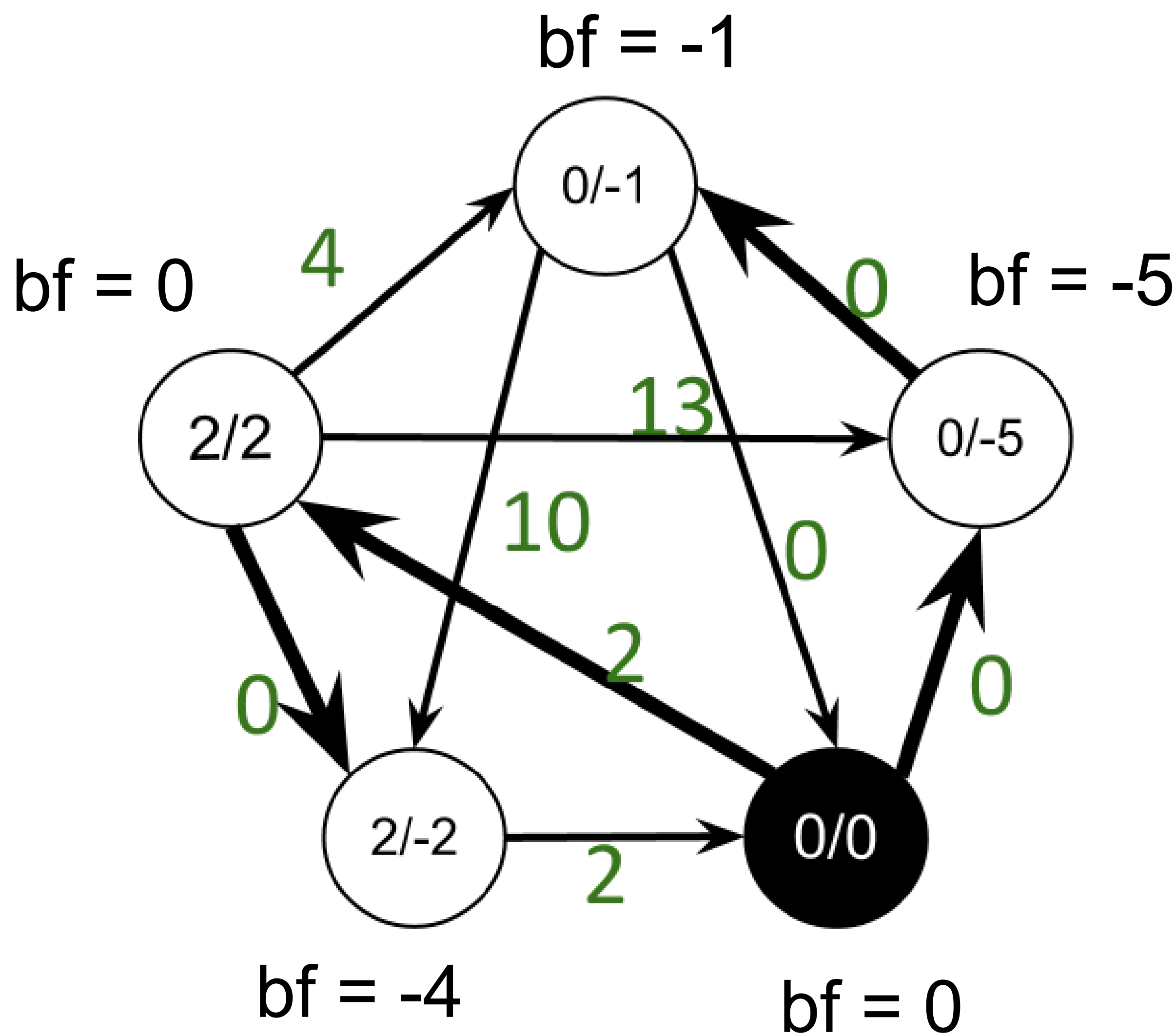

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