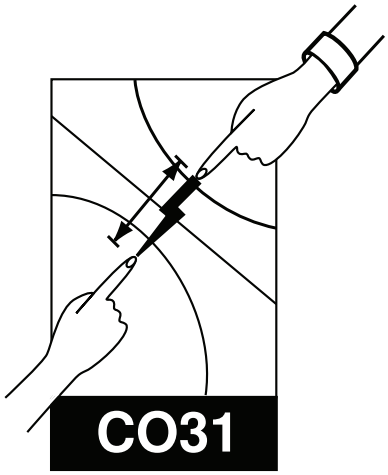
# Structure of white dwarf stars




revised EG, MK, NC October 2020

## Preface

It is common practice in modern software engineering to write programs in a modular and standardised way. To help you get started, appendix A.1 provides you with one or more specific function headers which you **MUST** use to write the functions around which your program should be built and in order to receive a satisfactory mark. Your comments in the header of each function must include:

- author and date,
- purpose (a brief description of what the function does),
- inputs and outputs,
- and, if appropriate, any constraints or limitations of use.

Do not forget that you will also need to keep good records of your progress during this practical in your logbook. If you make plots and/or write any notes electronically, ideally, you would print them and affix them into the pages of your logbook. You should have comments in function and script headers of your code, as well as comments within your code. These aspects may all be considered at marking time.

The template provided is for the default language of the Lab: MATLAB. If you have checked with a demonstrator about using another language and they have authorised you to do so, you must use the functional equivalent of the appendix A.1 function headers in that language.

### Assessment

Please see the Part A Guide in Canvas for Computing Lab deadlines as well as availability (schedule) of Computing Lab Demonstrators who mark your work as well as offer help and advice. Before you meet with a demonstrator for marking, **you must** upload your work (both your code and your report based on AD34 — *the art of scientific report writing*[1]) into WebLearn (also described in Canvas).

## 1 Introduction

White dwarf stars are one possible evolutionary end state of the conventional nuclear processes that occur in stars. They are objects made up of heavy nuclei, mainly $^{56}$Fe, but $^{12}$C may dominate if the nucleosynthesis terminates prematurely. The structure of white dwarf stars is determined by the balance between gravitational forces, which act to compress the star, and electron-degeneracy pressure, which acts to resist the compression. In this practical you will investigate the structure of white dwarf stars by solving the equations defining this balance.

▶ **You cannot do this practical if you have already done CO24 — *chaos*.**

[1] `www-teaching.physics.ox.ac.uk/practical_course/Admin/AD34.pdf`

## 2 Equations of equilibrium

We will simplify our equations by assuming that the star is perfectly spherical, so that the state of matter at any point in the star depends only on the distance to the centre of the star. We also assume that the star is not rotating and that the effects of magnetic fields are not important.

Consider an element of the star in mechanical equilibrium, i.e., the gravitational force acting on the element is balanced by the pressure interior to it.

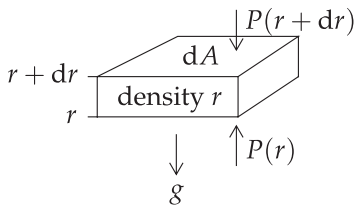



Figure 1: The balance of forces on an element of the star.

If the area of the element is $\mathrm{d}A$ and its density (assumed uniform) is $\rho(r)$ then its mass is $\rho\,\mathrm{d}A\,\mathrm{d}r$ and thus its weight is $(Gm(r)/r^2)\rho(r)\,\mathrm{d}A\,\mathrm{d}r$, where $m(r)$ is the mass of the star interior to $r$ and $G$ is the gravitational constant. The pressure $P(r)$ is greater than the pressure $P(r + dr)$ by just the weight per unit area of the material between $r$ and $r + dr$.

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \frac{P(r + \mathrm{d}r) - P(r)}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2} \tag{1}$$

where the minus sign indicates the pressure increases as $r$ decreases.

Using the relation

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \frac{\mathrm{d}\rho}{\mathrm{d}r}\frac{\mathrm{d}P}{\mathrm{d}\rho}$$

we can rewrite the 'hydrostatic equilibrium' equation 1 as

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{\mathrm{d}\rho}{\mathrm{d}P}\frac{G\rho(r)m(r)}{r^2} \tag{2}$$

The mass of the star interior to $r$ is just the integral of the density in spherical co-ordinates:

$$m(r) = 4\pi \int_0^r \rho(r')r'^2\,\mathrm{d}r'$$

This is usually written as a differential equation:

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2\rho(r) \tag{3}$$

Equations 2 and 3 are two coupled first order differential equations that determine the structure of the star for a given equation of state. The equation of state is an intrinsic property of the matter giving the pressure $P(\rho)$ required to maintain it at a given density. If we can find an algebraic expression for the relationship between the pressure and density then we would be in a position to solve equilibrium equations.

The values of the dependent variables at a small displacement $\delta r$ from the stellar centre are $\rho = \rho_c$, the central density, and $m = \frac{4}{3}\pi(\delta r)^3\rho_c$. Integrating outward in $r$ gives the density profile, i.e., $\rho(r)$. The

radius $R$ is determined by the point at which the density is effectively zero. The total mass of the star is then $M = m(R)$. Since both the mass and the radius depend on the central density $\rho_c$, variation of this parameter allows stars of different mass to be studied.

## 3 Equations of state

We assume that the star is made up of large $^{56}$Fe nuclei and their electrons. The nuclei contribute nearly all of the mass but contribute very little to the pressure, whereas the electrons contribute virtually all of the pressure but, being lighter, very little of the mass. We will consider densities far greater than ordinary matter, where the electrons are no longer bound to the individual nuclei but move freely through the material. In this case a good model is the free Fermi gas of electrons at zero temperature.

In the free Fermi sea, all electrons occupy the lowest energy states with momentum $p \leq p_F$, the Fermi momentum. Each of these states is 2-fold degenerate with 2 electrons in each *cell* of volume $h^3$, where $h$ is Planck's constant.

If $n(p)\,\mathrm{d}p$ denotes the number of electrons per unit volume with momentum between $p$ and $p + \mathrm{d}p$ then the assumption of degeneracy is equivalent to (see Podsiadlowski[4] for a derivation)

$$n = \begin{cases} \left(\frac{8\pi}{h^3}\right) p^2, & p \leq p_F \\ 0, & p > p_F \end{cases}$$

and the total number of electrons per unit volume is

$$n = \int_0^{p_F} n(p)\,\mathrm{d}p = \tfrac{8\pi}{3h^3} {p_F}^3 \tag{4}$$

For the distribution $n(p)$ the pressure is given by

$$P = \tfrac{8\pi}{3h^3} \int_0^{p_F} p^3 v_p\,\mathrm{d}p \tag{5}$$

If we use the non-relativistic $v_p = p/m_e$, we get a relationship for the pressure due to the electrons

$$\begin{aligned} P &= \frac{8\pi}{15h^3 m_e} {p_F}^5 \\ &= \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e} n^{5/3} \end{aligned} \tag{6}$$

We now need to relate $n(p)$ to the density $\rho$. Except for hydrogen (which is not present in white dwarves anyway) each electron is accompanied by a proton and very nearly one neutron in a electrically neutral atom, so

$$\rho = n_e(m_e + m_p + m_n)$$

Now we can write our state equation, $P = P(\rho)$, by substituting $n_e \sim \rho/2mp$ (ignoring the electron mass) in equation 6 for the pressure in a white dwarf:

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e}\left(\frac{\rho}{2m_p}\right)^{5/3}, \tag{7}$$

and thus

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = \frac{1}{48}\frac{h^2}{m_e}\frac{2^{1/3}}{{m_p}^{5/3}}\left(\frac{3\rho}{\pi}\right)^{2/3}. \tag{8}$$

## 4 Numerical approach

There is extensive literature on the numerical solution of sets of first order differential equations, see e.g. [3]. In that book the authors remark that Runge-Kutta methods nearly always work, but are not usually fastest; they will be the first choice for a new problem or where one has no reason to expend much time on efficiency.

One well-known Runge-Kutta algorithm [2], derived in full in [3], can be explained as follows.

We can express equations 2 and 3 as

$$\frac{\mathrm{d}y_i}{\mathrm{d}r} = f_i(y_1, y_2)$$

where $i = 1, 2$.

Starting from the values of $y_i$ at some radius $r$ we compute the values of $y_i$ at a slightly larger radius $(r + \delta r)$. By repeated application of this process we obtain solutions at radii $(r + m\delta r)$ ($m$ an integer). Each step is identical and includes several intermediate steps. We let $y_{i,0}$ be the values of $y_i$ at the beginning of the step, say at radius $r$, and $y_{i,4}$ be the values at the end of the step, i.e., at radius $(r + \delta r)$.

We require several intermediate values of $f_i$ between $r$ and $(r + \delta r)$. All components of $y_i$ and $f_i$ must be calculated at each intermediate point. We will speak of $y_{i,n}$ and $f_{i,n}$, where at the $n$th intermediate point in our calculation of one step we evaluate the set of $f_{i,n}$ from the set of $y_{i,n}$.

Each step involves the following intermediate calculations:

(a) $f_{i,0}$ using the initial values $y_{j,0}$ with $j = 1, 2$;

(b) $f_{i,1}$ using $y_{j,1} = y_{j,0} + f_{j,0}\delta r/2$ with $j = 1, 2$;

(c) $f_{i,2}$ using $y_{j,2} = y_{j,0} + f_{j,1}\delta r/2$ with $j = 1, 2$;

(d) $f_{i,3}$ using $y_{j,3} = y_{j,0} + f_{j,2}\delta r$ with $j = 1, 2$ N.B. no factor of $1/2$ !

The step is then completed using:

$$y_{i,4} = y_{i,0} + (f_{i,0} + 2f_{i,1} + 2f_{i,2} + f_{i,3})\delta r/6.$$

The values $y_{i,4}$ then become $y_{i,0}$ in the next step.

## 5 Non-relativistic case

Write an ODE solver using the template given in appendix A and implement the non-relativistic density and mass solver using the template in appendix A.2. With the implemented function find the radius of the white dwarf given a central density value (solve the ODEs for $r$ ranging from 1 to $10^4$ metres). Estimate the outer radius of the white dwarf to be where the density is sufficiently small compared to the central density. The total radius should be below $\sim 5 \times 10^7$ m. Note that for radii larger than the radius of the star, results will become nonphysical.

Plot the radius and mass as a function of the central density, $\rho_c$, for $10^6 < \rho_c < 10^{14}$ kg/m$^3$. Ensure that you sample enough $\rho_c$ values across the whole of this range.

## 6 Relativistic case

If the electron gas in a white dwarf is so compressed that the Fermi momentum $p_F$ approaches $m_e c$ (that is, the electron velocity approaches $c$), the above assumption of $v_p = p/m_e$ starts to break down.

[2] The above is one way to look at the RK method. An alternative is to look into this method and use in vector form.

Derive the relativistic form of the state equation, equation 7, using: $v_p = pc^2/\sqrt{p^2c^2+m_e{}^2c^4}$, and differentiate it with respect to $\rho$ to find a relativistic form of equation 8.

Add the relativistic case to your density and mass solver based on the template in appendix A. Solve the coupled differential equations as in the last section but with the new equation of state and plot the radius and mass of the white dwarf as a function of the central density in the range $10^6 < \rho_c < 5 \times 10^{14}\,\mathrm{kg/m^3}$. Compare the plot with the non-relativistic version and note the critical mass (the so-called Chandrasekhar mass).

Plot the radius as a function of mass for both the relativistic and non-relativistic cases. Compare the plots. **Hint:** the integral

$$\int_0^{p_F} \frac{p^4}{\sqrt{p^2 + m_e{}^2c^2}}\,\mathrm{d}p$$

can be solved using the substitution $p = m_e c \sinh\theta$ and the relationship

$$\sinh^4\theta = \frac{\cosh 4\theta}{8} - \frac{\cosh 2\theta}{2} + \frac{3}{8}.$$

Or, alternatively, the extreme relativistic case $v_p = c$ will give a limiting value for $m(r)$.

Some useful constants are:

$m_e = 9.11 \times 10^{-31}\,\mathrm{kg}$
$m_p = 1.672 \times 10^{-27}\,\mathrm{kg}$
$G = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$

**▶ You may opt at any point to discuss results or plots with a demonstrator before proceeding.**

## 7 Final considerations

Discuss some of the following in your write-up:

The relative advantages and disadvantages of the RK4 method, its mathematical origin, its associated global and local errors and its suitability for solving a system of a large number of coupled equations. How might such a system be solved? You may wish to consult our local short option on numerical methods [5].

The evolutionary cycle of a star, from the nuclear fusion phase to the end phase, highlighting further possible end states beyond the white dwarf state. Discuss the implications of the Chandrasekhar mass. For a detailed discussion on these topics, see [4]. You can also analytically derive the mass-radius relations for white dwarves, both in the non-relativistic and relativistic case.

# 8 Preparing for assessment

Part A Computing Practicals are an exercise in computer programming as well as in scientific report writing. Your report must be written in the style described in AD34 — *the art of scientific report writing*[a]. To write your report, you have a few options; Microsoft Word or LaTeX typesetting are most common. Whatever your choice, the system you use must be able to produce text-readable PDF format (not PDF created from a scanned image on a printer).

When you have finished assembling your code (and have tested it completely) and your report is complete, you must upload your work (both the code and the report) electronically to WebLearn (described in the Part A Guide in Canvas). Be sure to complete these uploads well in advance of meeting with a demonstrator for marking. Deadlines and Computing Demonstrator schedules are also found in Canvas.

**At marking time, be prepared** with a copy of your report and your logbook (where you wrote extra notes during your program development). Be prepared to describe your code and demonstrate its execution.

[a] `www-teaching.physics.ox.ac.uk/practical_course/Admin/AD34.pdf`

# A Functions to be implemented

## A.1 ODE Solver

```
function [y] = ode_solve_rk(f, y0, t)
% Author: ??? , Date: ??/??/????
% Solve ODE problem, i.e. dy/dt = f(y,t), using Runge-Kutta algorithm.
% Input:
% * f: a function that receives the current state, y, and the current position/time, t,
%      and returns the derivative value of the state, dy/dt.
% * y0: the initial state of the system, given in a column matrix (M x 1).
% * t: vector of position/time steps with length N where the values of y will be returned.
%
% Output:
% * y: (M x N) matrix that contains the values of y at every position/time step
%      columns correspond to the position/time and rows to the element of y.
%
% Constraints:
% * Do not use the built-in ODE solvers of MATLAB
%
% Example use:
% >> % Example of solving the harmonic oscillation equation, d2x/dt2 = -x
% >> % The ODE problem is posed with 2 element vector: y = [dx/dt; x]
% >> % Therefore, dy/dt = [d2x/dt2; dx/dt] = [-y(2); y(1)]
% >> f = @(y, t) [-y(2); y(1)];
% >> y0 = [0; 1];
% >> t = linspace(0, 10, 1000);
% >> y = ode_solve_rk(f, y0, t);
% >> plot(t, y(1,:)); % plot x vs t
end
```

If there are $M$ coupled variables with $M$ coupled equations, then `y` in the function above is a vector that contains $M$ variables and `f` is a function with input `y` and output a vector with $M$ elements.

## A.2 Density and mass solver

```
function [rho, mass] = get_density(rho0, r, rel)
% Author: ??? , Date: ??/??/????
% Obtain the density, rho, as function of the radial distance, r, using the implemented
% ODE solver and the non-relativistic or relativistic equation.
% Input:
% * rho0: the central density at r = 0.
% * r: the grid points of radial distance where the density is calculated in form of
%      a vector with N elements.
% * rel: boolean distinguishing relativistic and non-relativistic cases.
%
% Output:
% * rho: an N-element vector that contains the density at the radial grid points given
%         in r.
% * mass: the cumulative mass of the white dwarf from r=0 to the radial grid point given
%          in r (a vector with N elements).
%
% Example use:
% >> rho0 = 1e13;
% >> r = linspace(0, 5e7, 1000);
% >> [rho, mass] = get_density(rho0, r, 0);  % non-relativistic
% >> plot(r, rho); % plot density vs radial distance
end
```

# Bibliography


[1] S. Chandrasekhar, Reviews of Modern Physics **56**, 137–147 (1984). doi: 10.1103/revmodphys.56.137

[2] D. Koester, G. Chanmugam, Rep. Prog. Phys. **53**, 837–915 (1990). doi: 10.1088/0034-4885/53/7/001

[3] W. H. Press et al. *Numerical Recipes: the Art of Scientific Computing*, Cambridge Univ. Press (2007).

[4] P. Podsiadlowski, *B3 Astrophysics Lecture Series*, Oxford Physics, 2006[3].

[5] A. O'Hare, *Numerical Methods for Physicists*, Oxford Physics, 2005[4].


[3] https://www-astro.physics.ox.ac.uk/~podsi/lec_mm03.html

[4] https://www-teaching.physics.ox.ac.uk/computing/NumericalMethods/nummethods.html