

# CO31: Structure of white dwarf stars

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## Abstract

In an experiment, we attempt to study the structure of white dwarf stars. We give a short derivation of the differential equations describing a white dwarf in equilibrium under some assumptions about the matter. Then we introduce the computational approach of our solution and decide on parameters for the calculation. We present the dependence of radius and mass of white dwarfs on central density, as well as the relationship between them, which gives rise to the *Chandrasekhar mass limit*. We discuss some of its implications, after which we present a summary of the experiment and comment on further improvements.

## 1 Introduction

The goal of this experiment is to study the internal structure of white dwarf stars. They are one possible end state in the life cycle of stars. What is particular about white dwarf stars is that the matter they are composed of is different from the matter in a usual star similar to the Sun. While in a usual stable star the equilibrium is largely supported by fusion energy and the pressure of the plasma, a white dwarf star has reached the end of the fusion processes, and the equilibrium is supported by electron degeneracy pressure, making the star extremely dense; for example, a white dwarf of one solar mass is about the size of Earth. In this report, we will first derive the coupled differential equations for mass and density in a spirit similar to [Oxford Physics \(2020\)](#) and [Chandrasekhar \(1984\)](#) in Section 2. Then, in Section 3, we develop the numerical method used to solve the coupled equations. Results and discussion take place in Section 4. Finally, we give a summary of the experiment and possible improvements in Section 5.

## 2 Theory of white dwarf stars

Stars are astronomical objects with ellipsoid shape made up of heated plasma. The force of gravity tries to compress the ball, holding the matter together, while the gaseous plasma opposes gravity with the pressure it exerts. When the two forces balance each other out, the star is in hydrostatic equilibrium. In usual stars, pressure is due to the gas and the radiation. In white dwarf stars, however, the densities are far greater than the ones found in usual stars, and matter behaves differently. All electrons are no longer bound to atoms and are free to roam. The behaviour can approximately be modelled by a Fermi gas of electrons at zero Kelvin (i.e. fully degenerate state). Hence, the pressure ensuring the equilibrium is the degeneracy pressure of the electrons.

### 2.1 Equation of equilibrium

Assume the star is spherically symmetric, in equilibrium, non-rotating, and the effect of magnetic fields is negligible. Therefore, all properties depend only on the distance to the centre of the star,  $r$ . The gravitational force on a small volume of matter with area  $dA$  and radial height  $dr$  is:

$$F_G = -\frac{Gm(r)dm}{r^2}, \quad (1)$$

where  $m(r)$  is the mass of the star contained up to  $r$ ,  $dm = dA dr \rho(r)$  is the mass of the small volume, and the density  $\rho(r)$  was assumed constant (negative sign because it pulls towards the centre). The force due to the pressure is the difference of the forces at  $r$  and  $r + dr$ :

$$F_P = (P(r) - P(r + dr))dA. \quad (2)$$

For a star in equilibrium, the two forces balance each other out, i.e.  $F_G + F_P = 0$ . After rearranging, we get:

$$\frac{dP(r)}{dr} = \frac{P(r + dr) - P(r)}{dr} = -\frac{G\rho(r)m(r)}{r^2}. \quad (3)$$

We can rewrite the hydrostatic equilibrium by using the chain rule:

$$\frac{dP(r)}{dr} = \frac{dP}{d\rho} \frac{d\rho}{dr}, \quad (4)$$

so equation (3) becomes:

$$\frac{d\rho}{dr} = -\frac{dP}{d\rho} \frac{G\rho(r)m(r)}{r^2}. \quad (5)$$

On the other hand, the equation for the inscribed mass is:

$$m(r) = \int_0^r dr' 4\pi r'^2 \rho(r') \Rightarrow \frac{dm}{dr} = 4\pi r^2 \rho(r) \quad (6)$$

We are now left with two coupled differential equations, (5) and (6). The last task before obtaining the full equation is to determine  $\frac{dP}{d\rho}$  which depends on the equation of state.

## 2.2 Equation of state

We assume the star is made up primarily of heavy  $^{56}\text{Fe}$  nuclei and their electrons. The nuclei carry most of the mass and little of the pressure, while the opposite is true for the electrons. Since *very high* densities are considered, we can approximate that the nuclei are stationary while the electrons move freely, not bound to any nuclei. A good model for the freely moving electron gas is the free Fermi gas at  $T = 0\text{ K}$ . This is the fully degenerate state, in which electrons fill all energy (momentum) levels up to the Fermi energy (momentum).

Fermions obey the Pauli exclusion principle, therefore each energy level is  $2S + 1$  degenerate, where  $S = \frac{1}{2}$  in the case of electrons. This degeneracy factor multiplies gives the density function:

$$g(k) = \frac{2S + 1}{2\pi^2} V k^2, \quad (7)$$

where  $k$  is the wave vector of particles and  $V$  is the volume of the system. The number density of particles can therefore be written as:

$$n = \frac{N}{V} = \frac{1}{V} \int_0^{k_F} dk g(k) = \int_0^{k_F} dk \frac{2S + 1}{2\pi^2} k^2. \quad (8)$$

Here we integrate up to  $k_F$  where this is given by  $p_F = \hbar k_F$  and  $p_F$  is the Fermi momentum. Rewrite the above equation in terms of  $p$ :

$$n = \int_0^{p_F} dp \frac{2S + 1}{2\pi^2 \hbar^3} p^2 = \int_0^{p_F} dp \frac{8\pi}{h^3} p^2 = \int_0^{p_F} dp n(p) = \frac{m_e 8\pi}{h^3} p_F^3 \quad (9)$$

where we define  $n(p) dp$  as the number of electrons per unit volume with momentum between  $p$  and  $p + dp$ .

Pressure, on the other hand, is given by the kinetic expression:

$$P = \frac{1}{3} \int_0^{p_F} p v_p n(p) dp. \quad (10)$$

At this point we need to differentiate between non-relativistic and relativistic cases -  $v_p$  has a different value in the two cases:

$$v_p = \begin{cases} \frac{p}{m_e} & \text{in the non-relativistic case} \\ \frac{pc^2}{\sqrt{p^2 c^2 + m_e^2 c^4}} & \text{in the relativistic case} \end{cases}. \quad (11)$$

Let us first work out the answer in the non-relativistic case. The integral becomes  $\text{const} \times \int p^4 dp$ , so the pressure becomes:

$$P_{\text{non}} = \frac{8\pi}{15h^3 m_e} p_F^5. \quad (12)$$

For the relativistic case, rather than do the integral for  $P$  involving hyperbolic functions, differentiate (10) by parts:

$$\frac{dP_{\text{rel}}}{d\rho} = \frac{dP_{\text{rel}}}{dp_F} \frac{dp_F}{d\rho}, \quad \frac{dP_{\text{rel}}}{dp_F} = \frac{8\pi}{3h^3} \frac{d}{dp_F} \int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2 c^2 + m_e^2 c^4}} dp = \frac{8\pi}{3h^3} \frac{p_F^4 c}{\sqrt{p_F^2 c^2 + m_e^2 c^4}}. \quad (13)$$

We can collect the derivative of (12) and the result of (13):

$$\frac{dP}{d\rho} = \frac{dp_F}{d\rho} \frac{8\pi}{3h^3 m_e} p_F^4 \times \begin{cases} 1 & \text{in the non-relativistic case} \\ \left(1 + \frac{p_F^2}{m_e^2 c^2}\right)^{-1/2} & \text{in the relativistic case} \end{cases}. \quad (14)$$

What is left is expressing  $p_F(\rho)$  to complete the equation of state in the form needed by (5). The relation between density and number density can be stated as  $\rho = \mu_e m_p n$ , where  $\mu_e$  is the mean molecular weight per electron and  $m_p$  is the mass of 1 proton. In our case, we can approximate  $\mu_e \approx 2$ . Using this relation and (9), we arrive at:

$$p_F = \left(\frac{h^3}{8\pi} n\right)^{1/3} = \left(\frac{h^3}{16\pi m_p} \rho\right)^{1/3} \Rightarrow \frac{dp_F}{d\rho} = \frac{1}{3} \left(\frac{h^3}{16\pi m_p}\right)^{1/3} \rho^{-2/3} \quad (15)$$

### 2.3 The limiting mass

The limiting case of ultra-relativistic electrons can be obtained by setting  $v_p = c$  (i.e. taking the limit  $pc \gg m_e c^2$ ) in (10) to arrive at the equation of state:

$$P_{\text{ultra}} = \frac{p\pi}{3h^3} \int_0^{p_F} p^3 c dp = \frac{2\pi c}{3h^3} p_F^4. \quad (16)$$

As noted in Chandrasekhar (1984), this leads to a specific equation for the mass with a well-specified value  $M_{\text{lim}} = 5.76\mu_e^{-2} M_{\odot}$ , where  $M_{\odot}$  is one solar mass; this limit is equivalent to infinite mean density and zero radius of the star. In the approximation made above the limiting mass is  $M_{\text{lim}} \approx 1.435 M_{\odot}$ . This will be “experimentally” verified by the computations presented in Section 4. We can draw two important conclusions from this fact: first, there is an upper limit to masses of degenerate stars in the later stages of evolution; and second, we cannot predict the end evolutionary state of stars with mass  $M > M_{\text{lim}}$  in this physical framework (we expand more on this in Section 4).

## 3 Numerical approach

To find the radius and mass of the star as a function of its central density, we will numerically integrate the coupled system. The distance  $r$  at which  $\rho(r = R) \approx 0$  is where the star ends; the mass is, therefore,  $m(R)$  where  $m(r)$  is the inscribed mass, as defined in (6). Physically, we can consider solving the equation as an initial value problem. First, write the coupled system in vector form:

$$\mathbf{y} = \begin{pmatrix} \rho \\ m \end{pmatrix}, \quad (17)$$

and the derivatives of its members with respect to  $r$  given by (5) and (6). Then, for a given central density  $\rho_c$  the initial condition for the mass is specified by  $m(\delta r) = \frac{4}{3}\pi(\delta r)^3 \rho_c$  for some small distance from the centre  $\delta r$ : this is done so we can start the calculations ( $r = 0$  would give initial zero mass and the derivative is therefore zero). We will set up the equation solver so that this  $\delta r$  is the step size in radial distance. The source code can be found in Appendix A.

### 3.1 Why integration?

Since we have an initial value problem, the underlying idea is the following: rewrite the derivatives as fractions of finite differences, and get  $\Delta \mathbf{y} = \frac{d\mathbf{y}(x)}{dx} \Delta x$ . This is the change in  $\mathbf{y}$  when we step through by  $\Delta x$ . When the step is very small, the approximation is very good, i.e.  $\lim_{\Delta x \rightarrow dx} \Delta \mathbf{y} = d\mathbf{y}$ .

The basic idea boils down to the most elementary such method, the explicit Euler’s method:

$$\begin{aligned} \mathbf{y}_{n+1} &= \mathbf{y}_n + h \mathbf{f}(x_n, \mathbf{y}_n) \\ x_{n+1} &= x_n + h, \end{aligned} \quad (18)$$

where the vector function is just  $\mathbf{f} = \frac{d\mathbf{y}(x)}{dx}$  i.e. the slope at  $x_n$  and  $h = \Delta x$ . This is a first order method: the local error is  $\sim h^2$  and the global error, therefore, is  $\sim h$ . This method has only one computation of the derivative and is therefore very fast, but also very inaccurate.

A similar explicit second order method is Heun's method (trapezoid rule):

$$\begin{aligned} \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{1}{2}h \left( \mathbf{f}(x_n, \mathbf{y}_n) + \mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{f}(x_n, \mathbf{y}_n)) \right) \\ x_{n+1} &= x_n + h, \end{aligned} \quad (19)$$

Here the local error is  $\sim h^3$  and the global error therefore is  $\sim h^2$ .

The method we will later use to compute properties of white dwarf stars is the 4th order Runge-Kutta (RK4) method. It features four evaluations of the derivative and a global error  $\sim h^4$ , offering a balance between computational time and accuracy. This method takes values for the slope at four different positions and estimates the step size by a weighted average of these four steps. In equations:

$$\begin{aligned} \mathbf{k}_1 &= h\mathbf{f}(x_n, \mathbf{y}_n) \\ \mathbf{k}_2 &= h\mathbf{f}\left(x_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}\mathbf{k}_1\right) \\ \mathbf{k}_3 &= h\mathbf{f}\left(x_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}\mathbf{k}_2\right) \\ \mathbf{k}_4 &= h\mathbf{f}(x_n + h, \mathbf{y}_n + \mathbf{k}_3). \end{aligned}$$

Finally, the new position is calculated as:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4. \quad (20)$$

The RK methods family gives approximate solutions to  $N$ th order by varying the weights in the  $\mathbf{k}$  expressions and the final expression to match coefficients in the Taylor expansion.

We should note that the equation to be solved might prove difficult to solve, even using RK4. These equations are often called *stiff* and in constructing their solutions more precaution needs to be taken, for example by using implicit or semi-implicit methods.

## 3.2 Comparison of methods

In this section, we compare the stability/accuracy of the three methods presented in Section 3.1. For the task, we will try to integrate a solution of the simple harmonic oscillator (SHO), namely:

$$\ddot{x} + \omega^2 x = 0, \quad \text{with solution} \quad x = x_0 \cos(\omega t) \quad (21)$$

where we choose  $x_0 = 1$  and  $\omega = 2\pi$  so the period of oscillations is 1 s. From some initial runs integrating the white dwarf equation, we are aware that the number of intervals relevant to us is on the order of  $10^6$ , so we choose to compare the two methods on this scale (more discussion on the topic in Section 3.4).

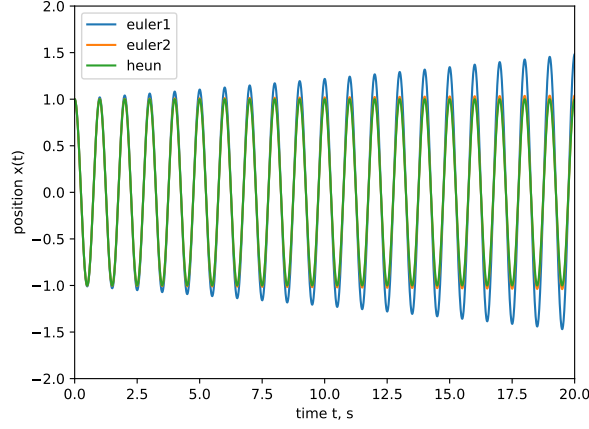
The runs are split into two groups. The first group is with  $10^6$  time intervals of length 1 ms for the three methods; denote the results from these runs by EULER1, HEUN, and RK4, respectively. The second group is  $10^7$  time intervals of length 0.1 ms for the Euler method to cover the same total time interval; denote results by EULER2. Additionally, we will consider the energy in the system:

$$E = \frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2}. \quad (22)$$

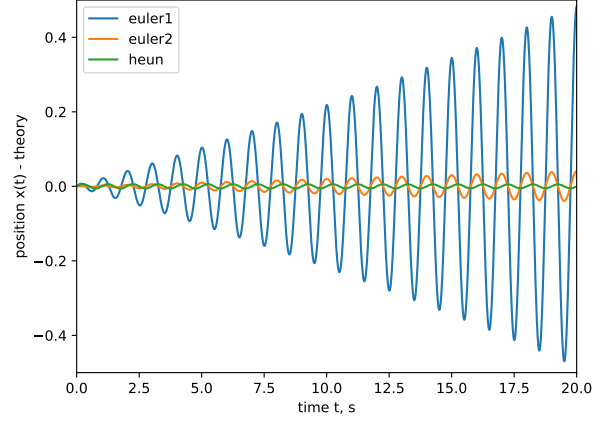
When the method over/underestimates the correction, the effect will build up and change the total energy, which is equivalent to the change in amplitude.

Results are presented in Figure 1. In 1a (first 20 periods of oscillation displayed), we see the calculated position by the three methods EULER1, EULER2, and HEUN. It is immediately clear that EULER1 goes off in the solution pretty quickly. This is confirmed in 1b, where we can have a better look at how the numerical solution deviates from the actual one - EULER1 gains an additional half amplitude in 20 periods, while the other two methods oscillate about zero with HEUN having the smaller error. To compare for the whole time of integration and compare these three runs with RK4, we plot the relative energy deviation compared to the theoretical value of  $2E_0 = \omega^2 \times 1$  on a logarithmic scale. As it can be seen in 1c and 1d, the energy for the first three runs grows exponentially with time after some moment with different exponents, while the fourth run RK4 keeps almost a constant value of energy — the change in energy between  $t_0 = 0$  s and  $t_1 = 100$  s is:

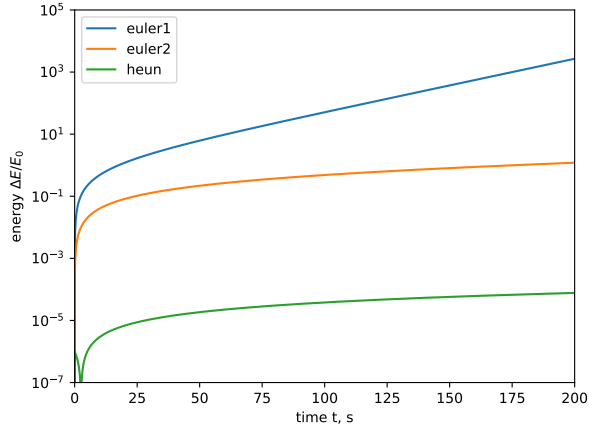
$$\frac{|E(t_1) - E(t_0)|}{E(t_0)} \approx 9.755 \times 10^{-7}. \quad (23)$$



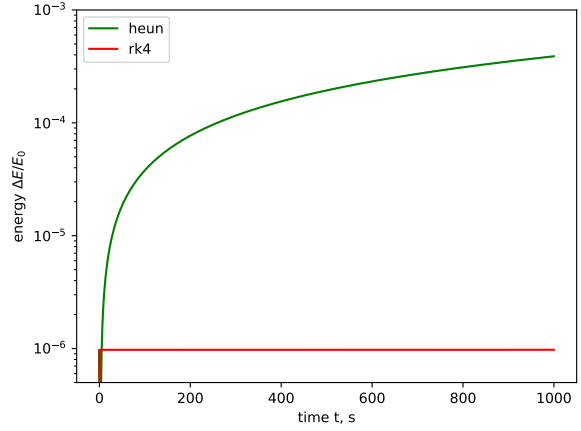
(a) Position as a function of time. Notice how eul1 deviates very fast.



(b) Deviation from the theoretical solution.



(c) Energy deviation  $\Delta E/E_0$  for first three methods.



(d) Energy deviation  $\Delta E/E_0$  for heun and rk4.

Figure 1: Test runs of SHO integrated with different methods. Blue line is for Euler's method integrated with step 1 ms, yellow line for Euler's method integrated with step 0.1 ms, green line is for Heun's method with step 1 ms, and red line is for RK4 method with step 1 ms.

With these results in mind, we make the clear choice for a method and continue forwards using the Runge-Kutta 4th order method in all of the following chapters.

### 3.3 Dimensionless equation for the white dwarf

To calculate values with the computer, we will go to dimensionless quantities in the differential equation. Introduce the following substitutions:

$$\rho = \rho_c \theta, \quad r = \ell \xi, \quad m = \rho_c (1 \text{ m}^3) \mu, \text{ therefore } \mathbf{y}(r) = \begin{pmatrix} \rho(r) \\ m(r) \end{pmatrix} \rightarrow \boldsymbol{\eta}(\xi) = \begin{pmatrix} \theta(\xi) \\ \mu(\xi) \end{pmatrix}. \quad (24)$$

We also specify the length  $\ell$  by the equation  $\frac{4}{3}\pi\ell^3 = 1 \text{ m}^3$ , which gives a value of  $\ell \approx 0.62 \sim 1 \text{ m}$ . This will be useful to make some cancellations in the coefficients in the final form of the differential equation for  $\boldsymbol{\eta}$ . The initial conditions therefore become  $\boldsymbol{\eta}(1) = (\theta(1), \mu(1)) = (1.0, 1.0)$ .

For (6) the conversion is straightforward:

$$\frac{\rho_c d\mu}{\ell d\xi} = 4\pi\ell^2 \xi^2 \rho_c \theta \quad \Rightarrow \quad \frac{d\mu}{d\xi} = 3\xi^2 \theta. \quad (25)$$

For the  $\theta$  equation we have to take into consideration the regime we operate in — whether we solve for a relativistic or non-relativistic equation of state. Firstly, write equations in (15) using the new coordinates we have introduced:

$$p_F = \left( \frac{h^3 \rho_c}{16\pi m_p} \right)^{1/3} \theta^{1/3}, \quad \frac{dp_F}{d\rho} = \frac{1}{3} \left( \frac{h^3}{16\pi m_p \rho_c^2} \right)^{1/3} \theta^{-2/3}. \quad (26)$$

Now substitute (14) in (5) using (26). After some algebra with the constants, the final result is the following:

$$\frac{d\theta}{d\xi} = K_1 \frac{\mu \theta^{1/3}}{\xi^2} \begin{cases} 1 & \text{in the non-relativistic case} \\ (1 + K_2 \theta^{2/3})^{1/2} & \text{in the relativistic case} \end{cases}, \quad (27)$$

where the constants  $K_1$  and  $K_2$  are given by

$$K_1 = -2^{13/3} \pi G m_e m_p^{5/3} \rho_c^{1/3} h^{-2}, \quad K_2 = \frac{1}{m_e^2 c^2} \left( \frac{3h^3 \rho_c}{16\pi m_p} \right)^{2/3}. \quad (28)$$

We have completed the task of bringing the equation to normalized coordinates. This is the setup used in implementing the code for the white dwarf ODE wrapper presented in Appendix A.1.

### 3.4 Convergence and number of intervals

What is left is to choose the coarseness in the grid of  $\xi$ . We know that for some  $\Delta\xi$  small enough, the answers will converge to  $\approx$  a single value. To find the number of intervals (NoI) at which the result converges, we make experimental runs at three different  $\rho_c$  —  $10^6$ ,  $10^{10}$ , and  $10^{14} \text{ kg m}^{-3}$  (this spans the range for which we run the experiment later), and test for an NoI in the range  $\{a \times 10^b \mid a \in \{1, 2, 4, 8\}, b \in \{2, 3, \dots, 8\}\}$ . The maximal radius is set to  $8.1 \times 10^7$  in units of  $\xi$  since this turns out to be about the maximum physical radius a white dwarf can be for the range of  $\rho_c$  we test. Both the non-relativistic and relativistic cases are considered.

The results are presented in Figure 2. We use the relative error in computed value with respect to the converged value. The latter is estimated as the average of the last three values in the calculation, i.e. for NoI  $2 \times 10^8$ ,  $4 \times 10^8$ , and  $8 \times 10^8$ ; the result is  $M_{\text{conv}}, R_{\text{conv}}$ . The relative error is given by:

$$\frac{\Delta M}{M_{\text{conv}}} = \frac{|M - M_{\text{conv}}|}{M_{\text{conv}}}, \quad \frac{\Delta R}{R_{\text{conv}}} = \frac{|R - R_{\text{conv}}|}{R_{\text{conv}}}. \quad (29)$$

What we aim for in our choice for the number of intervals is: on the one side, a low enough number so the computation does not take too long, and on the other side, a high enough number so an acceptable accuracy is achieved. From the logarithmic plots for the mass and the radius determination, we see that the computed value does not really converge to a specific point; it does, however, reduce its error exponentially fast. Let us choose the desired accuracy of below  $10^{-5} = 0.001\%$ , given that our model is not very accurate. From the logarithmic plots again we see that this is satisfied for  $\text{NoI} \geq 2 \times 10^6$ . Therefore, we can use this value as the NoI with a good compromise between speed and accuracy.

## 4 Results and discussion

We begin by choosing the parameters of the simulation. As discussed in Section 3.4, we use the RK4 method of integration. The range of integration in  $\xi$  will be  $[1, 8.1 \times 10^7]$  with  $2 \times 10^6$  intervals (we implement the white dwarf wrapper in this way to speed up the procedure, see Appendix A.1). This corresponds to physical dimensions of approximately  $r \in [0.6 \text{ m}, 5 \times 10^7 \text{ m}]$  and a step size of  $\Delta r = 25 \text{ m}$ , which makes sure we do not reach the end of the range before the end of the star.

For the central density  $\rho_c$  we are tasked to study the range  $10^6 < \rho_c < 5 \times 10^{14} \text{ kg m}^{-3}$ , so we set up to test using the following values:  $\rho_c \in \{a \times 10^b \mid a = 1, 2, \dots, 9; b = 6, 7, \dots, 14\}$ . The results are presented in Figures 3, 4, and 5. We have chosen logarithmic plots for the first two plots to better highlight the similar behaviour at low densities and the different behaviour at high densities.

In Figure 3 and 4 we can see that at central densities  $\rho_c \approx 10^8$  the results for radius and mass start to deviate. The non-relativistic values follow an exponential relationship, as one might have predicted from (14) and (5). The relativistic case, on the other hand, exhibits more interesting behaviour. At higher densities, the mass tends to a single maximum value — the *Chandrasekhar mass*, which can be seen in Figures 4 and 5. This implies that the theoretical model of a white dwarf cannot properly describe objects with higher mass.

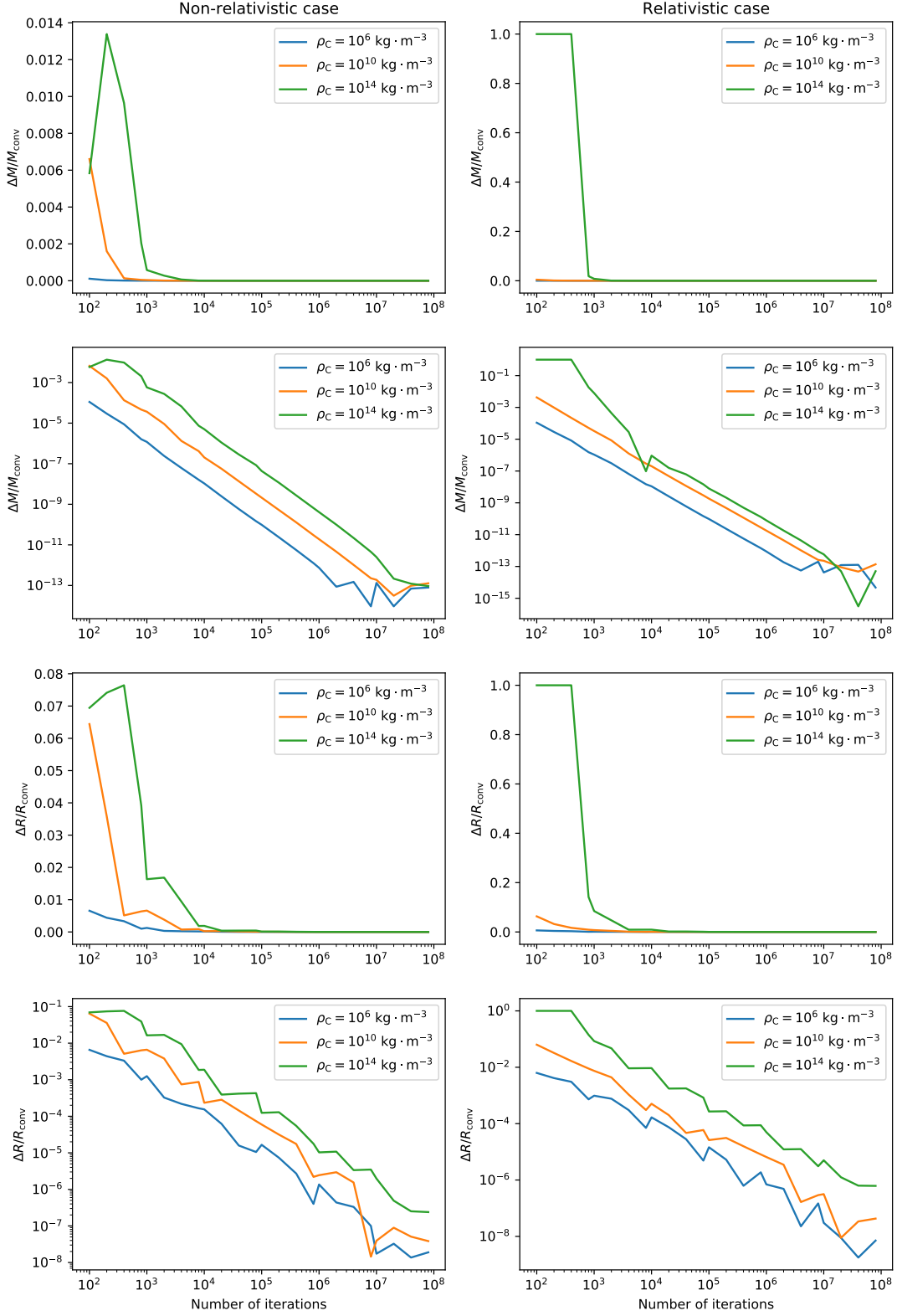


Figure 2: Plot of convergence for three central spanning the tested range of  $\rho_c$  in both non-relativistic (left column) and relativistic (right column) cases. As can be seen, convergence is slower in the relativistic case; after  $10^6$  intervals good accuracy is achieved in both cases for both mass and radius values. The values  $M_{\text{conv}}$ ,  $R_{\text{conv}}$  are computed as the average of the last three values;  $\Delta M = |M - M_{\text{conv}}|$  and  $\Delta R = |R - R_{\text{conv}}|$ . Odd rows have a linear y-axis, even rows are the same plot with a logarithmic y-axis.

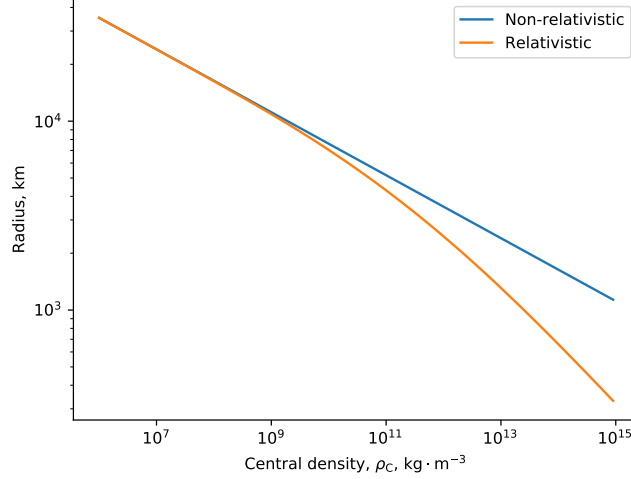


Figure 3: Radius of a white dwarf  $R$  in km as a function of central density  $\rho_c$ .

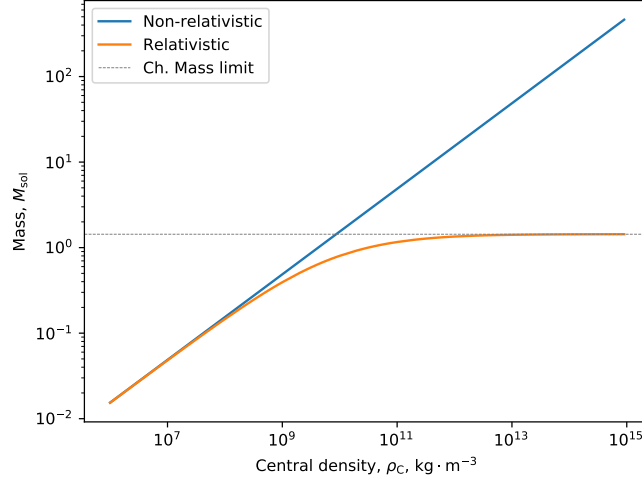


Figure 4: Mass of a white dwarf star  $M$  in solar masses as a function of central density  $\rho_c$ . Note the Chandrasekhar mass limit.

## 4.1 Compact objects

White dwarfs are part of the more general framework of compact objects as end stages in stellar evolution, which also include neutron stars and black holes. These objects are the final result of stellar evolution. In which one a star settles is almost completely determined by its initial mass during the stable stage on the Main sequence.

Low to medium mass stars ( $0.6M_{\odot} \lesssim M \lesssim 8M_{\odot}$ ) usually evolve by hydrogen burning joined by helium burning in the red giant stage. Since their carbon cores are not massive enough to ignite, the stars contract again and the strong stellar wind expels all surrounding gas to form a planetary nebula with a hot central star, which in turn cools down to a white dwarf. These white dwarfs are usually carbon and oxygen-rich.

High mass stars ( $M \gtrsim 8M_{\odot}$ ) have massive enough cores to ignite further production of heavier elements via the alpha and triple alpha processes. After most of the fuel is burned through and a dense iron core has formed, they undergo a similar process of implosion in which the core rapidly compactifies (in some edge cases about  $8M_{\odot}$  only partial fusion of carbon occurs, they settle to a white dwarf state). In this case, however, the core mass has passed the (effective) Chandrasekhar limit and electron degeneracy pressure cannot support the equilibrium. Thus, the core collapses further to a neutron star or a black hole (the exact mass threshold for neutron stars is estimated to be between  $2M_{\odot}$  and  $3M_{\odot}$ ).

It is the discovery of this limit in [Chandrasekhar \(1931\)](#), however, that started the whole further theoretical study



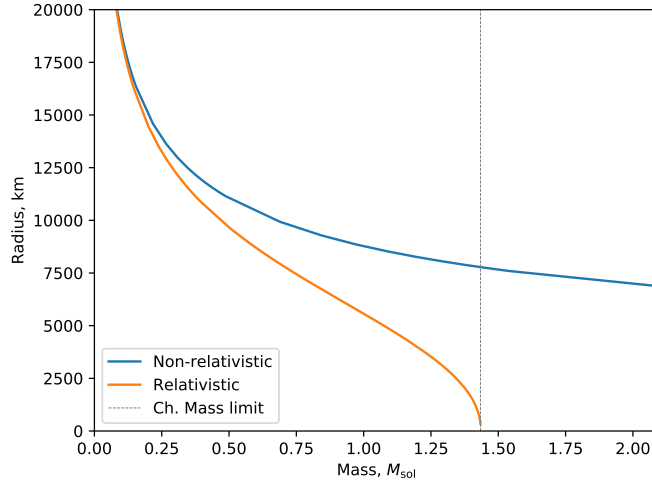


Figure 5: Radius of a white dwarf  $R$  in km as a function of the mass  $M$  in solar masses. Note the Chandrasekhar mass limit.

of the topic of compact objects. In the following year, the neutron was discovered, and Landau published a paper that discussed a more precise calculation of the white dwarf mass limit and pondered the nature of heavier stars. This laid the groundwork for further discoveries in the 20th century.

## 4.2 Binary systems

Additionally, white dwarfs are often part of binary or multiple systems (about a third of all star systems are multiple systems). In the case the white dwarf is part of a very close binary, there is an accretion of mass to the surface of the white dwarf to a thin dense atmosphere layer atop the surface of the star. This atmosphere, usually comprised of hydrogen, is heated up by the star, which causes the ignition of runaway fusion. This is observed as a sudden flash of light - a *nova*.

If there is accretion from a more massive companion to a carbon-oxygen white dwarf (heavier elements support further collapse since more energy is needed for their ignition), this can cause its mass to increase beyond the Chandrasekhar limit and ignite carbon fusion in the centre of the white dwarf due to compressional heating. This results in the destruction of the star and is believed to be the physical model behind Type Ia *supernovae*.

These two phenomena further highlights the critical mass limit of existence for white dwarfs. Without it, the above would not be observed in nature.

## 5 Summary

In this practical, we study the structure of white dwarf stars by numerically solving the equations of equilibrium. For a white dwarf star, the equilibrium is supported by the electron degeneracy pressure. We derive these equations in Section 2, differentiating between the cases of non-relativistic and relativistic electrons. Additionally, we consider the ultrarelativistic limit and anticipate the result of calculations — a limit to white dwarf mass in the case of relativistic electrons.

We choose to solve these equations as an initial value problem, where the starting point is a small region at the centre of the star. In Section 3, we compare the Euler, Heun, and Runge-Kutta 4th order (RK4) methods, which resulted in our choice to use the RK4 for solving the white dwarf equations. Then we introduce a transformation of variables to dimensionless quantities, after which we test the convergence of white dwarf mass and radius for a sample of central density values. We conclude that when the tested interval for radial distance in dimensionless units is  $\xi \in [1, 8.1 \times 10^7]$  with  $2 \times 10^6$  intervals we have a good compromise between accuracy and speed (this corresponds to physical dimensions of approximately  $r \in [0.6 \text{ m}, 5 \times 10^7 \text{ m}]$  and a step size of  $\Delta r = 25 \text{ m}$ ).

We present the solution to the white dwarf equation in Section 4. We note the difference of solutions between the non-relativistic and relativistic equation of state as the central density grows, seen in Figures 3 and 4. There we also discuss the implications of the existence of the *Chandrasekhar limit* and the place of white dwarfs in the evolutionary track of stars as an end stage of medium mass stars  $M \lesssim 8M_{\odot}$ , compared to other compact objects (neutron stars

and black holes). Finally, we note some other astrophysical phenomena which occur due to the special properties of white dwarfs — novae and supernovae Type Ia.

While this project takes steps towards a numerical solution of white dwarf equations, we have made some major assumptions along the way with regards to the physics. Among the effects to be taken into account in a further study are the following:

- The interaction of electrons with their surroundings: we can introduce magnetic fields and the electrostatic interaction in the force balance equations, which modifies the white dwarf equation.
- Chemical composition: since stars form an onion-like structure of elements in their cores, the mean molecular weight per electron  $\mu_e$  varies slightly, which changes the constant factors in the equation for the white dwarf.
- Effects of General Relativity should also be taken into account. White dwarfs are compact objects, so as the density grows larger, the effects are more and more significant.
- We assumed that the star is in a complete stationary equilibrium state. However, we also have to consider the cases where the star might be rotating (a compact object can have a significant rotation speed due to conservation of angular momentum) or pulsating, which can disturb the conditions in the centre.

An introductory account of some of these effects can be found in e.g. [Sagert et al. \(2005\)](#), while a more standard and in-depth textbook exposition to the topic can be found in [Shapiro and Teukolsky \(1991\)](#).

Additionally, there is room for improvement in the computational methods employed as well. The fairly simple RK4 method can be modified to have a varying step size that self-corrects depending on the derivative at a given point. That way the calculations can skip over a region with small a small gradient by increasing the step size, and decrease the step size to get a more accurate value for the change in a region with a big gradient. Another direction of improvement would be to change from RK4 to another, a more sophisticated method that has better accuracy and faster computational time like the Bulirsch-Stoer method. This would be especially helpful if the system of differential equations to be solved was much more complex in its behaviour. A good introductory text on the subject is [Press et al. \(2007\)](#), Chapter 17.

## References

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## A Source code

The structure of the code has been guided by the guidance set in [Oxford Physics \(2020\)](#). Additionally, [Press et al. \(2007\)](#) has had some influence on general code structure. The documentation for the main ODE solver is in the Python code. A copy in an online repository can be found [here](#).

### A.1 Main ODE facilities

```
1  """
2  Implementation of a generic ODE integrator.
3  """
4
5  import numpy as np
6  from . import integrators
7
8  class ODEinit(object):
9      """
10     Base class for a general ODE to be numerically integrated. It provides
11     facilities to compute a value of some linear system given the ODE form as
12     `y' = f(x, y)` for some interval of values of `x` and initial conditions of
13     the vector `y`.
14
15     Parameters
16     -----
17     y0 : np.ndarray
18         Initial conditions of the vector `y` at position `x = span[0]`
19     deriv
20         The function `f` in a functional form. Should accept two parameters at
21         which to compute the derivative:
22             x: float, the value of the independent variable;
23             y: np.ndarray, the value of the dependent variable vector `y`.
24     span : np.ndarray
25         The interval values over which to compute the value of the vector `y`.
26
27     Attributes
28     -----
29     y0 : np.ndarray
30         The initial state of the system. Passed at initialization.
31     deriv
32         The function `f` in a functional form. Passed at initialization.
33     span : np.ndarray
34         The interval of values to be integrated over. Passed at initialization.
35         **Assumed to be equidistant!!**
36     interval : float
37         The size of the interval step; equal to `span[1] - span[0]`.
38     yOut : np.ndarray
39         The computed values of `y` after integration. Initialized as a matrix
40         of zeros; if integration terminates before the end of `span`, the rest
41         of the points are left as zeros.
42     flagIntegrated : bool
43         A flag to indicate whether the integration has been done (and `yOut`
44         filled) to be used by methods depending on having computed values.
45
46     Methods
47     -----
48     integrate
49         Integrates the differential equation.
```

```

50 """
51
52 def __init__(self, y0: np.ndarray, deriv, span: np.ndarray) -> None:
53     self.y0 = y0
54     self.deriv = deriv
55     self.interval = span[1] - span[0]
56     self.span = span
57
58     # size of this is number of values to be computed x dimensions of y
59     self.yOut = np.zeros((self.span.size, self.y0.size))
60
61     # a flag taking note if the equation has been integrated
62     self.flagIntegrated = False
63
64 def integrate(self, integrator: int=3) -> None:
65     """
66     Integrates the equation using the function of the derivative `deriv`.
67     User has the option to choose an integrator from the library, the
68     default is a RK4 integrator. As it computes, it fills the `yOut` with
69     the computed values. Can terminate prematurely if the value passed to
70     `deriv` is inappropriate; then the rest of the values are left as zero.
71
72     Parameters
73     -----
74     integrator : {1, 2, 3}, optional
75         Choice of the integrator to be used given by the number:
76         1: the Euler first order method;
77         2: the Heun method, a second order method;
78         3: the Runge-Kutta 4th order method.
79
80     See Also
81     -----
82     modules.integrators
83     """
84     # choose an integrator
85     self.integrator = integrators.index[integrator]
86
87     # fill in the initial value of y
88     self.yOut[0,:] = self.y0[:]
89
90     #cycle does the integration
91     for i in range(self.span.size-1):
92         # the try except handles the case where integration is impossible,
93         # i.e. when the value of an element of y is nonphysical. deriv
94         # decides when this is the case.
95         try:
96             # calculate the new value
97             adder = self.integrator(
98                 self.span[i], self.yOut[i,:], self.deriv, self.interval)
99             # append new value
100             self.yOut[i+1,:] = adder
101         except ValueError:
102             # ends the integration if it is impossible to integrate - Value
103             # Error concerns the y-values
104             break
105
106     # to assert that the ODE has been integrated in functions that use yOut.

```

```

107         self.flagIntegrated = True
108
109     if __name__=="__main__":
110         print(
111             "This file contains declaration of a generic ODE class.")

```

---

```

1     """
2     White dwarf class wrapper.
3     """
4     import numpy as np
5     from scipy import constants as cs # constants for the derivative equation
6
7     from . import derivatives, ode
8
9     class whiteDwarf(ode.ODEinit):
10         """
11         Provides a wrapper of the ODEinit class in the case of white dwarfs with
12         some equation of state. Initializes an ODE with appropriate initial
13         conditions while taking into account a normalization which leaves the ODE
14         working with dimensionless quantities.
15
16         Parameters
17         -----
18         rhoC : float
19             The density of matter in the small region around the center. In units
20             of kg * m-3
21         span : np.ndarray
22             The range of values of r to be calculated at. In normalized units to
23             cut down on computational time and improve calculation accuracy.
24             Expects an array with first member = 1 for the starting central mass to
25             be correctly defined. Unit of normalization given by 1 (see below).
26         regime : {1, 2, 3}, optional
27             The regime indicates the choice of equation of state -> derivative, by
28             default 1 - non-relativistic.
29
30         Attributes
31         -----
32         rhoC : float
33             The density of matter in the small region around the center
34         const1 : float
35             Constant to be used in the derivative, depends on the `rhoC` value.
36             Value after the normalization.
37         const2 : float
38             Second constant to be used in the derivative for the relativistic case,
39             depends on `rhoC`. Value after the normalization.
40         Radius : float
41             Physical radius of the star. Has a value only after `getRadius()` has
42             been called.
43         Mass : float
44             Mass of the star. Has a value only after `getRadius()` has been called.
45
46         Methods
47         -----
48         getRadius
49             Returns the tuple `(Radius, Mass)` of the star if the equation has been
50             integrated.
51

```

## Background

The differential equation is a coupled system of the density  $\rho$  and mass  $m$  as functions of the distance from the centre  $r$ . These have some dimensions in powers of kg and m. We transform the variables with some scaling constants so that the ODE solver works with dimensionless variables. The following choice of constants is made:

```
rho = rhoC * theta,
m = rhoC * mu,
r = l * ksi,
```

where  $\rho_C$  is given as input by the user and  $l^3 4\pi/3 = 1$ . The choice of  $l$  gives it a value on the order of 1, and we will choose it as our step size in  $r$  i.e. the step size in  $\text{ksi}$  is just 1. This is a convenient choice for normalization.

```
def __init__(self, rhoC: float, span: np.ndarray, regime: int=1) -> None:
    self.rhoC = rhoC
    # vector to be given as initial conditions, [rho, mass]
    init_condit = np.array([1.0, 1.0*(span[0]**3)])

    # constants in the derivative; computed here so that they are computed
    # only once rather than every time the derivative func is called
    self.const1 = -(2**(13.0/3))*cs.pi*cs.gravitational_constant *\
        cs.electron_mass*(cs.proton_mass**(5.0/3)) * \
        (rhoC**(1.0/3))*(cs.h**(-2.0))
    self.const2 = (3*rhoC/16/cs.pi/cs.proton_mass *
        (cs.h/cs.c/cs.electron_mass)**3)**(2.0/3)

    # choose the equation of state and pass constants, prepares a function
    # to be passed to the constructor of ODEinit accepting (x, y) params
    if regime == 1:
        func = lambda a, b: derivatives.nonRelativGas(a, b, self.const1)
    elif regime == 2:
        func = lambda a, b: derivatives.relativGas(a, b, self.const1, self.const2)
    elif regime == 3:
        func = lambda a, b: derivatives.ultraRel(a, b, self.const1)

    super(whiteDwarf, self).__init__(init_condit, func, span)

def getRadiusMass(self) -> tuple:
    """
    Finds the radius and mass of the white dwarf, and returns them. Takes
    into account the normalization of the ODE.

    The function scans `yOut[:,0]` column containing `rho(r)` values. We
    know that this is a decreasing function; the first instance where it
    has a value <= 0 it where the star "ends".

    Returns
    -----
    tuple
    Returns a tuple `(Radius, Mass)`. Radius and Mass are calculated
    from `yOut` with the appropriate normalization constants. For
    Radius it is  $l = (3/(4\pi))^{(1/3)}$ , result is in m. For Mass it is
     $\rho_C$ , result is in kg.
    """
```

```

109
110     # find index of rho = 0 (changing signs)
111     if self.flagIntegrated:
112         index = np.searchsorted(-self.yOut[:, 0], 0)
113         # index is the position where we have yOut[index,:] ~ [0,0]
114
115         l = (3/(4*np.pi))*(1.0/3.0) # normalization const
116         self.Radius = (index-1)*self.interval*l # in m
117         self.Mass = self.yOut[index-1, 1]*self.rhoC # in kg*m-3
118         return self.Radius, self.Mass
119     else:
120         print("Integrate the eqn first")
121         return None, None
122
123 if __name__=="__main__":
124     print(
125         "This file contains declaration of a white dwarf class.")

```

```

1  """
2  Integrators Library(`integrators.py`)
3  =====
4
5  This is a home to the different integrators used alongside the target of the
6  exercise, namely Runge-Kutta 4th order (rk4) method. We implement two other
7  methods to evaluate the advantages of the rk4 method, both in its accuracy and
8  precision.
9
10 Integration Methods implemented
11 -----
12
13     euler    Euler method - 1st order.
14     heun     Heun method - a variant of a 2nd order.
15     rk4      Runge-Kutta 4th order method.
16
17 Implementation notes
18 -----
19 All of the functions defined are to act on np.ndarray structures, therefore
20 should be element-wise functions; all of the magic is done for us by numpy.
21 The differential equation which they iterate is set up as `dy/dx = f(x, y)`,
22 where `y` and `f` are N-dimensional vectors. Hence, all of the functions
23 receive four parameters:
24
25     x : float, the point at which we start computing
26     y : array_like, the values of the y-vector at x
27     deriv : function, the derivative of y to be computed at some (x, y) coords
28     h : float, the interval size
29
30 All of the functions return the value of `y` for the next place in the
31 integration range. How we determine its value, or more precisely the increment
32 from the previous value, is the difference between these methods.
33
34 Error handling is implemented in `rk4` (the one used for actual calculations),
35 where evaluating the `deriv` function might not be possible.
36
37 Choice of an integrator is aided by the dictionary `index`, mapping integer
38 values to the integrator functions.
39 """

```

```

40
41 import numpy as np
42
43 def euler(x: float, y: np.ndarray, deriv, h: float) -> np.ndarray:
44     """
45     Compute an iteration using Euler's method: a 1st order method.
46
47     This function computes an iteration in the numerical integration of an ODE
48     of the form `dy/dx = f(x, y)` using the first order Euler method:
49     `y(x + h) = y(x) + h*f(x, y(x))`, where h is the size of the step in the
50     x-coordinate.
51
52     Parameters
53     -----
54     x : float
55         Starting coordinate in x.
56     y : ndarray
57         Starting coordinate in y. In general can be an N-dimensional array.
58     deriv : function
59         A function which computes the derivative of the y-vector given the
60         x and y coords.
61     h : float
62         The step size in x.
63
64     Returns
65     -----
66     ndarray
67         The returned value is the approximation of y(x + h) using the Euler
68         algorithm.
69     """
70     return y + h*deriv(x,y)
71
72 def heun(x: float, y: np.ndarray, deriv, h: float) -> np.ndarray:
73     """
74     Compute an iteration using Heun's method, a kind of 2nd order method.
75
76     This function computes an iteration in the numerical integration of an ODE
77     of the form `dy/dx = f(x, y)` using the second order Heun method:
78     `y(x + h) = y(x) + 0.5*h*(f(x, y(x)) + f(x + h, y + h*f(x, y)))`, where h
79     is the size of the step in the x-coordinate. This is a predictor-corrector
80     method, and uses the prediction of the slope at `x + h` to correct the
81     final calculation of the slope to be added.
82
83     Parameters
84     -----
85     x : float
86         Starting coordinate in x.
87     y : ndarray
88         Starting coordinate in y. In general can be an N-dimensional array.
89     deriv : function
90         A function which computes the derivative of the y-vector given the
91         x and y coords.
92     h : float
93         The step size in x.
94
95     Returns
96     -----

```



```

97     np.ndarray
98     The returned value is the approximation of  $y(x + h)$  using the Heun
99     algorithm.
100     """
101     return y + 0.5*h*(deriv(x, y) + deriv(x + h, y + h*deriv(x,y)))
102
103 def rk4(x: float, y: np.ndarray, deriv, h: float) -> np.ndarray:
104     """
105     Compute an iteration using a 4th order Runge-Kutta method.
106
107     This function computes an iteration in the numerical integration of an ODE
108     of the form  $\text{`dy/dx = f(x, y)}`$  using a fourth order Runge-Kutta method.
109
110     Parameters
111     -----
112     x : float
113         Starting coordinate in x.
114     y : ndarray
115         Starting coordinate in y. In general can be an N-dimensional array.
116     deriv : function
117         A function which computes the derivative of the y-vector given the
118         x and y coords.
119     h : float
120         The step size in x.
121
122     Returns
123     -----
124     np.ndarray
125         The returned value is the approximation of  $y(x + h)$  using the 4th order
126         Runge-Kutta algorithm.
127
128     Raises
129     -----
130     ValueError
131         If an error is raised by the deriv function indicating impossible
132         evaluation, raise an error to indicate the termination of integration.
133
134     Notes
135     -----
136     Note that in the case deriv cannot be computed for any of the steps, the
137     error is handled by raising a value error back to the integrator method of
138     the ODE class where this is handled. Additionally, an improvement in the
139     calculation order is implemented as per [1]_, where calculation cycle
140     number is reduced, as well as memory usage.
141
142     Sources
143     -----
144     ..[1] Press, William H., Saul A. Teukolsky, William T. Vetterling, and
145     Brian P. Flannery. "Numerical Recipes : The Art of Scientific Computing."
146     Third ed. Cambridge, 2007.
147     """
148     # TODO think of fix
149     try:
150         # USE OLD implementation, uses more cycles and memory (allegedly)
151         k1 = h*deriv(x,y)
152         k2 = h*deriv(x + 0.5*h, y + 0.5*k1)
153         k3 = h*deriv(x + 0.5*h, y + 0.5*k2)

```

```

154         k4 = h*deriv(x + h, y + k3)
155
156         # 'FASTER' implementation - produced an error while running tests on
157         # the SHO for the report, revert to the more abstract version - some
158         # cycles but sure it is correct
159
160         # hh = 0.5*h
161         # h6 = h/6.0
162         # xh = x + hh
163         # k1 = deriv(x, y)
164         # yTemp = y + hh*k1
165         # k2 = deriv(xh, yTemp)
166         # yTemp = y + hh*k2
167         # k3 = deriv(xh, yTemp)
168         # yTemp = y + k3
169         # k3 += k2
170         # k2 = deriv(x+h, yTemp)
171     except ValueError:
172         raise ValueError("Invalid integration, terminate integration!")
173
174     return y + (k1 + k4)/6.0 + (k2 + k3)/3.0
175     # return y + h6*(k1 + k2 + 2*k3)
176
177 # dictionary to choose integrator; for neater code in modules.ode
178 index = {1: euler, 2: heun, 3: rk4}
179
180 if __name__=="__main__":
181     print(
182         "This is a module home to several integrators to be used in solving \
183         ODEs: Euler, Heun, Runge-Kutta4.")

```

```

1  """
2  Derivatives Library(`derivatives.py`)
3  =====
4
5  This is a home to several functions which give the derivative `f` in a system
6  of linear ODEs `y'(x) = f(x, y)` in functional form.
7
8  White Dwarf Functions
9  -----
10 nonRelativGas      Implements the eqn. of state for non-relativistic electrons
11 relaitvGas         Implements the eqn. of state for relativistic electrons.
12
13 Testing Functions
14 -----
15 SHO                Implements the Simple Harmonic Oscillator.
16
17 Implementation Notes
18 -----
19 Choice of an equation of state for the white dwarf is aided by the dictionary
20 `index`, mapping integer values to the integrator functions.
21 """
22
23 import numpy as np
24
25 def SHO(x: float, y: np.ndarray) -> np.ndarray:
26     """

```

```

27     Function of the derivative for the simple harmonic oscillator written as a
28     system of linear ODEs.
29
30     Parameters
31     -----
32     x : float
33         Value of the independent variable in the ODE.
34     y : np.ndarray
35         Value of the dependent variable in the ODE. Two element vector
36         `[z', z]`, where the SHO ODE is `z''(x) + z(x) = 0`.
37
38     Returns
39     -----
40     np.ndarray
41         The value of the derivative at (x, y)
42
43     Notes
44     -----
45     The equation `z''(x) + z(x) = 0` is written using `p = z'` in the form:
46     `y' = [p, z]' = [-z, p]`. The function returns the latter vector.
47     """
48     return np.array([-4*np.pi*np.pi*y[1], y[0]])
49
50 def nonRelativGas(x: float, y: np.ndarray, const: float) -> np.ndarray:
51     """
52     Function for the coupled ODE using a non-relativistic equation of state.
53
54     Function of the derivative for the system of coupled equations
55     `y = [rho, m](x)` in the non-relativistic case. An additional argument
56     `const` is passed to save computational time - it is calculated only once
57     in the ODE definition, and called as value to be used in the computations.
58
59     Parameters
60     -----
61     x : float
62         Value of the independent variable in the ODE.
63     y : np.ndarray
64         Value of the dependent variable in the ODE. *Two elements only.*
65     const: float
66         The scaling constant used in the definition.
67
68     Returns
69     -----
70     np.ndarray
71         The value of the derivative at (x, y). Look at Notes for info on the
72         functional form.
73
74     Raises
75     -----
76     ValueError
77         If the value `y[0]` < 0, the computation is discarded on physical
78         grounds - cannot have negative matter density. Raises an exception
79         propagated by the integrator to be handled in the .integration method.
80
81     Notes
82     -----
83     The functional form is determined by the equation of state in the

```

```

84 non-relativistic case, where `pressure = someConst * rho^(2/3)`.The final
85 form: `[rho, m]' = [const1*m*rho^(1/3)/x^2, 3*x^2*rho]`.
86 """
87
88 # need y[0] >= 0 to be exponentiated apart from physical grounds
89 # this catches the moment y[0] becomes too small - only it is on the
90 # computation of the following value
91 if y[0] < 1e-10:
92     raise ValueError
93 return np.array([const*y[1]*(y[0]**(1.0/3))*(x**(-2.0)),
94                 3*(x**2)*y[0]])
95
96 def relativGas(x: float, y: np.ndarray, const1: float ,const2: float) -> np.ndarray:
97     """
98     Function for the coupled ODE using a relativistic equation of state.
99
100    Function of the derivative for the system of coupled equations
101    `y = [rho, m](x)` in the relativistic case. Two additional arguments
102    `const1` and `const2` are passed to save on computational time - they are
103    calculated only once in the ODE definition, and called as value to be used
104    in the computations.
105
106    Parameters
107    -----
108    x : float
109        Value of the independent variable in the ODE.
110    y : np.ndarray
111        Value of the dependent variable in the ODE. *Two elements only.*
112    const1: float
113        One of the scaling constants, used in the derivative of rho.
114    const2 : float
115        The second scaling constant, used in the relativistic correction.
116
117    Returns
118    -----
119    np.ndarray
120        The value of the derivative at (x, y). Look at Notes for info on the
121        functional form.
122
123    Raises
124    -----
125    ValueError
126        If the value `y[0]` < 0, the computation is discarded on physical
127        grounds - cannot have negative matter density. Raises an exception
128        propagated by the integrator to be handled in the .integration method.
129
130    Notes
131    -----
132    The functional form is determined by the equation of state in the
133    relativistic case, where `pressure = someConst*rho^(2/3)*relCorr`.The
134    relativistic correction is `relCorr = sqrt(1+const2*rho^(2/3))`^(-1)`.
135    The final form: `[rho, m]' = [const1*m*rho^(1/3)*relCorr/x^2, 3*x^2*rho]`.
136    """
137
138    # need y[0] >= 0 to be exponentiated apart from physical grounds
139    # this catches the moment y[0] becomes too small - only it is on the
140    # computation of the following value

```

```

141     if y[0] < 1e-10:
142         raise ValueError
143
144     return np.array([const1*y[1]*(y[0]**(1.0/3))*(x**(-2.0))*
145                     (1+const2*(y[0]**(2.0/3))**(0.5), 3*(x**2)*y[0]])
146
147 index = {1: nonRelativGas, 2: relativGas}
148
149 if __name__=="__main__":
150     print(
151         "This is a home to several functions which give the derivative in a \
152 system of linear ODEs.")

```

## A.2 Code to run the experiment and testing procedures

```

1  """main.py
2  The main script in this experiment. It calculates the values of the white dwarf
3  radius and mass for different central densities (initial conditions to solve
4  the ODE) by using our purpose built library and outputs them to a .csv file.
5  """
6
7  # parallel computing library
8  from multiprocessing import Pool, cpu_count
9
10 import numpy as np
11 # for saving files to csv (easier than the numpy implementation)
12 import pandas as pd
13 # solar mass for output scale
14 from astropy.constants import M_sun
15 # for setup of integration grid
16 from numpy.core.function_base import linspace
17
18 # our white dwarf integrator module
19 import modules
20
21
22 # Initialize a white dwarf with non-relativistic equation of state
23 def starInitNonRelativ(rhoC):
24     """This function initializes a white dwarf with a non-relativistic equation
25     of state and calculates its radius and mass, then returns the values.
26
27     Parameters
28     -----
29     rhoC : float
30         The density of the center in kg*m-3, used to set the initial
31         conditions for integration.
32
33     Returns
34     -----
35     list
36         Returns a three element list of the density in in kg*m-3, the radius
37         in km/1000, and the mass in solar mass in this order.
38     """
39     # Integration grid in normalized units, optimal step size determined on
40     # convergence ground for more info look at convergence.py
41     span = linspace(1,8.1e7,num=2000000)

```

```

42     # Initialize the white dwarf and integrate
43     star = modules.whiteDwarf(rhoC, span, 1)
44     star.integrate(3)
45
46     radius, mass = star.getRadiusMass()
47     # return the three values rho in kg*m-3, radius in km/1000, mass in solar mass
48     return [rhoC, radius/1000, mass/M_sun.value]
49
50 # Initialize a white dwarf with relativistic equation of state
51 def starInitRelativ(rhoC):
52     """This function initializes a white dwarf with a relativistic equation of
53     state and calculates its radius and mass, then returns the values.
54
55     Parameters
56     -----
57     rhoC : float
58         The density of the center in kg*m-3, used to set the initial
59         conditions for integration.
60
61     Returns
62     -----
63     list
64         Returns a three element list of the density in in kg*m-3, the radius
65         in km/1000, and the mass in solar mass in this order.
66     """
67     # Integration grid, optimal step size determined on convergence ground
68     # for more info look at convergence.py
69     span = linspace(1, 8.1e7, num=2000000)
70     # Initialize the white dwarf and integrate
71     star = modules.whiteDwarf(rhoC, span, 2)
72     star.integrate(3)
73
74     radius, mass = star.getRadiusMass()
75     # return the three values rho in kg*m-3, radius in km/1000, mass in solar mass
76     return [rhoC, radius/1000, mass/M_sun.value]
77
78
79 def main():
80     # the rho values to test in the range, change the list accordingly
81     rhoVal = np.array([float(a)*10**b for b in range(6,15) for a in range(1,10)])
82
83     # Implements parallel computation of the ODEs using multiple processes.
84     # Computation is done over the range of central density values rhoVal.
85
86     # Computation for non-relativistic gas. This process sets up the
87     # multiprocessing and cleans up after the calculations are over.
88     with Pool(cpu_count(), maxtasksperchild=1) as executor:
89         # The executor.map(f, s) applies the function f to each member of the
90         # iterable s and collects the results in a list following a First In
91         # First Out procedure, so the results collected are ordered by the
92         # order in which they were in s, i.e. by central density values.
93         resNonRelativ = executor.map(starInitNonRelativ, rhoVal)
94
95     # Save results for non-relativistic case
96     dfNonRel = pd.DataFrame(resNonRelativ, columns=["rhoC", "radiusKM", "massSolMass"])
97     dfNonRel.to_csv("nonRelativRes.csv", index=None)
98

```

```

99     # Computation for non-relativistic gas. This process sets up the
100     # multiprocessing and cleans up after the calculations are over.
101     with Pool(cpu_count()-6, maxtasksperchild=1) as executor:
102         # The executor.map(f, s) applies the function f to each member of the
103         # iterable s and collects the results in a list following a First In
104         # First Out procedure, so the results collected are ordered by the
105         # order in which they were in s, i.e. by central density values.
106         resRelativ = executor.map(starInitRelativ, rhoVal)
107
108     # Save results for relativistic case
109     dfRel = pd.DataFrame(resRelativ, columns=["rhoC", "radiusKM", "massSolMass"])
110     dfRel.to_csv("relativRes.csv", index=None)
111
112
113 if __name__ == "__main__":
114     main()

```