Some Notes

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bold letters stand for vectors unless otherwise stated

Math preparations

- MathWorld generalized Pauli algebra: https://mathworld.wolfram.com/GeneralizedGell-MannMatrix.html.
- Direct product and partial trace: https://zhuanlan.zhihu.com/p/653816083
- Math fonts: https://typst.app/docs/reference/math/variants/

If $\left\{ \mid \varphi_i^A \rangle \mid i = 1, 2, \cdots, N \right\}$ and $\left\{ |\varphi_i^B\rangle \mid i = 1, 2, \cdots, N \right\}$ are basis sets of space $A \in \mathbb{C}^{\mathbb{N}}$ and $B \in \mathbb{C}^{\mathbb{N}}$, respectively. Let new space $C = A \otimes B \in \mathbb{C}^{\{\mathbb{N} \times \mathbb{N}\}}$, and basis set in this space will be $\left\{ |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle \mid i = 1, 2, \cdots, N; j = 1, 2, \cdots, N \right\}$. Vectors on C can be expanded on this basis set $|f^{AB}\rangle \in C = \sum_{ij} c_{ij}\ |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle$. When we write down $|f^{AB}\rangle$ as state vector, we implicitly specify a pure state, becasue mixed state cannot be expressed as state vector in Hilbert space. The density matrix is

$$\begin{aligned} \rho^{AB} &= |f^{AB}\rangle\langle f^{AB}| \\ &= \left(\sum_{ij} c_{ij} |\varphi_i^A\rangle \otimes |\varphi_j^B\rangle \right) \left(\sum_{kl} c_{kl}^* \langle \varphi_k^A| \otimes \langle \varphi_l^B| \right) \\ &= \sum_{ijkl} c_{ijkl}\ |\varphi_i^A\rangle\langle\varphi_k^A \otimes |\varphi_j^B\rangle\langle\varphi_l^B| \end{aligned} \tag{1}$$

Sometimes in a confusing notation $\rho_{AB} = \sum_{ijkl} |i\rangle_A \langle j| \otimes |k\rangle_B \langle l|$.