Testing the limit of Minchev2018 method:

Assumptions:

$$\nabla[Fe/H](\tau) = a\ln(1-\tau/\tau_0) + \nabla[Fe/H](0)$$

$$[Fe/H](R_0,\tau) = b\ln(1-\tau/\tau_0) + [Fe/H](R_0,0)$$

define: $\boxed{\begin{aligned} C_m &:= \nabla[Fe/H](0) \\ C_b &:= [Fe/H](R_0,0) \end{aligned}}$

Let's assume $\tau_0 = 15$ Gyr ($\tau_0 > \tau$ for both fcn. to converge)

$$R_b = \frac{[Fe/H] - [Fe/H](R_b,\tau)}{\nabla[Fe/H](\tau)}$$

$$\because [Fe/H](R_b,\tau) = [Fe/H](R_0,\tau) - \nabla[Fe/H](\tau)R_0$$

$$= C_b - \nabla[Fe/H](\tau)R_0 + b\ln(1-\tau/\tau_0)$$

$$\therefore \underline{R_b} = \frac{[Fe/H] - C_b + \nabla[Fe/H](\tau)R_0 - b\ln(1-\tau/\tau_0)}{a\ln(1-\tau/\tau_0) + C_m}$$

$$= \underline{\frac{[Fe/H]-C_b}{a\ln(1-\tau/\tau_0)+C_m} - \frac{b}{a + C_m/\ln(1-\tau/\tau_0)} + R_0}$$

for $\boxed{\tau \ll \tau_0}$, $\ln(1-\tau/\tau_0) \sim -(\tau/\tau_0) - (\tau/\tau_0)^2\frac{1}{2}$

$$\boxed{R_b} \sim \boxed{\frac{[Fe/H]-C_b}{C_m}\left(1 + \frac{a}{C_m}(\tau/\tau_0) + \left[\left(\frac{a}{C_m}\right)^2 + \left(\frac{a}{2C_m}\right)\right](\tau/\tau_0)^2\right) + \frac{b}{C_m}(\tau/\tau_0)}$$

$$\boxed{\frac{b(2a+C_m)}{2C_m^2}(\tau/\tau_0)^2 + R_0}$$

$$\cong \boxed{\begin{aligned} &\frac{1}{C_m}[Fe/H] + \frac{a}{C_m^2}[Fe/H](\tau/\tau_0) + \left(\frac{b}{C_m} - \frac{C_b a}{C_m^2}\right)(\tau/\tau_0) + \\ &\left[\frac{(2a+C_m)b}{2C_m^2} - \frac{C_b}{C_m}\left[\left(\frac{a}{C_m}\right)^2 + \left(\frac{a}{2C_m}\right)\right]\right](\tau/\tau_0)^2 + R_0 - \frac{C_b}{C_m} \end{aligned}}$$

$\therefore \theta_{[Fe/H]} = 1/C_m$; $\theta_R = 0$; $\theta_\tau = \left(\frac{b}{C_m} - \frac{C_b a}{C_m^2}\right)$, $\theta_{[Fe/H]^2} = 0$,

$\theta_{[Fe/H]R} = 0$; $\theta_{[Fe/H]\tau} = a/C_m^2$; $\theta_{R^2} = 0$; $\theta_{R\tau} = 0$;

$\theta_{\tau^2} = \left[\frac{b(2a+C_m)}{2C_m^2} - \frac{C_b}{C_m}\left[\left(\frac{a}{C_m}\right)^2 + \left(\frac{a}{2C_m}\right)\right]\right]$; $\theta_c = R_0 -+ \frac{C_b}{C_m}$

for $\boxed{\tau \sim \tau_0}$, $\ln(1-\tau/\tau_0) \sim \infty \rightarrow \boxed{R_b \sim R_0 + b/a} \rightarrow$ purterbation from R_0

Testing Limit of Frankle 2018 paper:

from equation 8:

$$R_b = \frac{[Fe/H] - F_m + (F_m + \nabla[Fe/H] R^{now}_{[Fe/H]=0}) f(\tau)}{\nabla[Fe/H]}$$

where $\boxed{\nabla[Fe/H] = C_m, \quad F_m = [Fe/H](0, \tau_0), \quad f(\tau) = (1-\tau/\tau_0)^\gamma}$

not the same τ_0

$$\therefore R_b = \frac{1}{C_m}[Fe/H] - \frac{F_m}{C_m} + \left(\frac{F_m}{C_m} + R_n\right)(1-\tau/\tau_0)^\gamma$$

$R_n = -\frac{C_b}{C_m} + R_0 \rightarrow$ defining same varibles as used before

for $\boxed{\tau \ll \tau_0}$: $(1-\tau/\tau_0)^\gamma \sim [1 - \gamma(\tau/\tau_0) + \frac{1}{2}(\gamma-1)\gamma(\tau/\tau_0)^2]$

$$\boxed{R_b = \frac{1}{C_m}[Fe/H] - \gamma\left(\frac{F_m}{C_m} + R_0 - \frac{C_b}{C_m}\right)(\tau/\tau_0) + \gamma(\gamma-1)\left[\frac{F_m}{2C_m} + \frac{R_0}{2} - \frac{C_b}{2C_m}\right]\left(\frac{\tau}{\tau_0}\right)^2 + R_0 - C_b/C_m}$$

$\theta_{[Fe/H]} = 1/C_m$; $\theta_R = 0$; $\theta_\tau = -\gamma\left(\frac{F_m}{C_m} + R_0 - \frac{C_b}{C_m}\right)$, $\theta_{[Fe/H]^2} = 0$;

$\theta_{[Fe/H]R} = 0$; $\theta_{[Fe/H]\tau} = 0$; $\theta_{R^2} = 0$; $\theta_{R\tau} = 0$;

$\theta_{\tau^2} = \gamma(\gamma-1)\left[\frac{F_m}{2C_m} + \frac{1}{2}\left(R_0 - \frac{C_b}{C_m}\right)\right]$, $\theta_c = R_0 - \frac{C_b}{C_m}$

$\rightarrow \theta_{[Fe/H]}$, θ_c ~~that~~ are the same as Minchev 2018

for $\boxed{\tau \sim \tau_0}$: $(1-\tau/\tau_0)^\gamma \sim 0$ for $\gamma > 0$, also $\gamma \neq 0$

$$\boxed{R_b = \frac{1}{C_m}[Fe/H] - \frac{F_m}{C_m}}$$

Comparing the two:

① for $\tau \ll \tau_0$ case; ① $\theta_{[Fe/H]}$ and θ_c are the same

② Frankle 2018 doesn't have $\theta_{[Fe/H]\tau}$, the rest are the same (there is value)

③ $-\gamma\left(\frac{F_m}{C_m} + \underbrace{R_0 - \frac{C_b}{C_m}}_{R_n}\right)$, $-\left(\frac{b}{C_m} + \frac{aC_b}{C_m^2}\right)$, for θ_τ

↓ Frankle ↓ Minchev

④ $\gamma(\gamma-1)\left[\frac{F_m}{2C_m} + \frac{1}{2}R_n\right]$, $-\frac{C_b}{C_m}\left[\left(\frac{a}{C_m}\right)^2 + \frac{a}{2C_m}\right]$, for θ_{τ^2}

↓ Frankle ↓ Minchev.

for $\tau \sim \tau_0$ case :

① Frankle 2018 is [Fe/H] dependent, Minchev 2018 is const.

OK... So the limit works for both cases, can be reproduced from quadractic solution. what's the point???

Maybe : $\theta_{[Fe/H]}$ represents the slope now for $\tau \ll \tau_0$?
→ obviously...
→ θ_c represents the location where ~~R~~ ([Fe/H]=0)
→ So it is pretty much saying

R_b = perturbation around the location where [Fe/H] = 0, for $\tau \ll \tau_0$