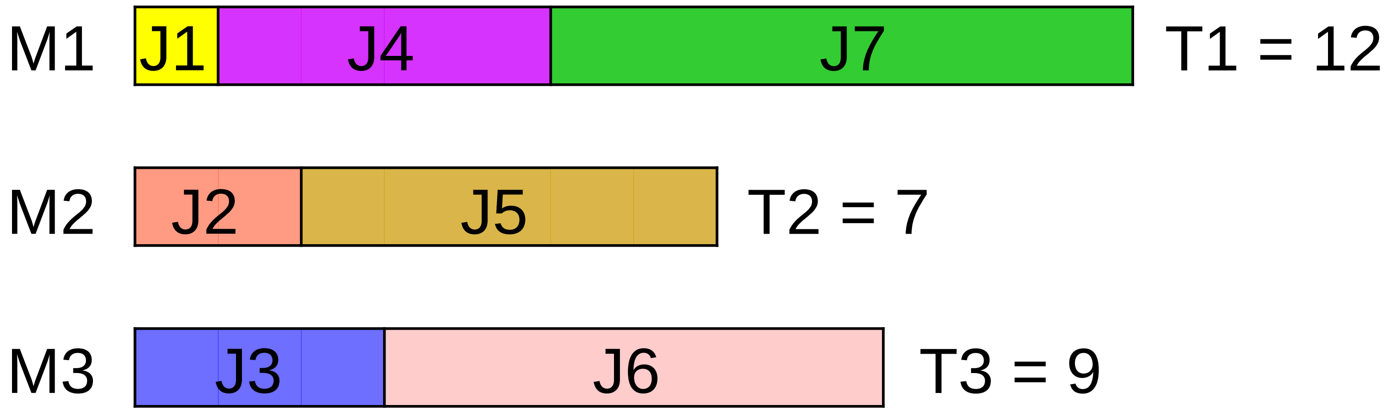
Topic 1: Load Balancing Problem

Input: m identical machines: M1, M2, ..., Mm

n jobs: J1, J2, ..., Jn

Processing time of each job: t_j (j = 1, 2, ..., n)

Example: 3 machines and 7 jobs (t_j = 1, 2, 3, 4, 5, 6, 7)



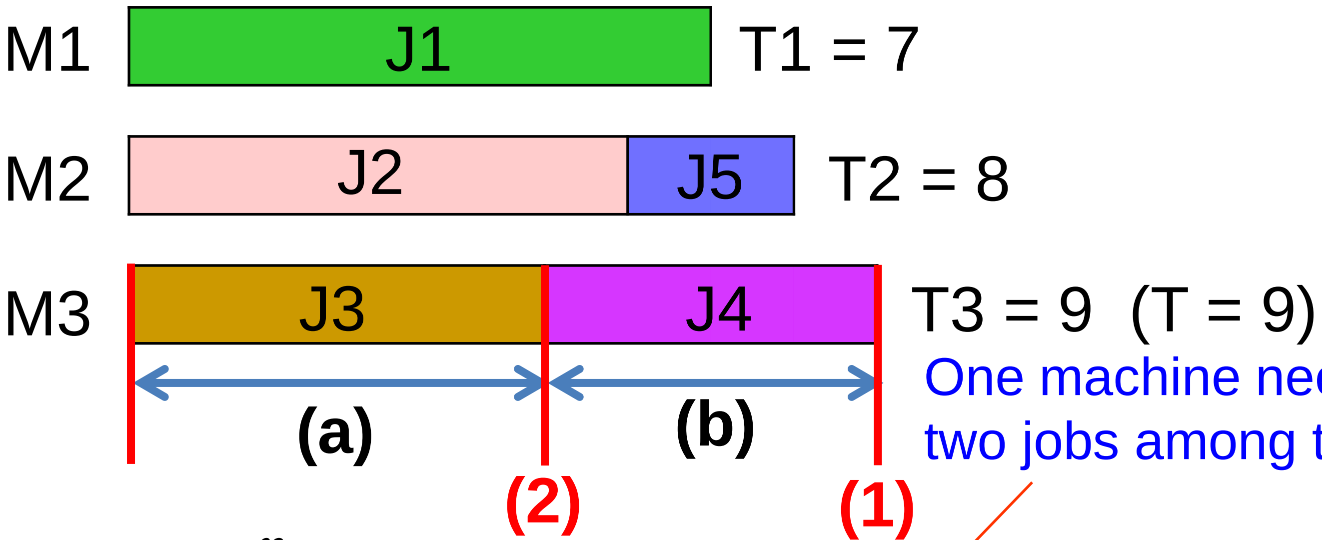
Makespan T = max {T1, T2, T3} = 12

Sorted Greedy Algorithm

Assign a job to the machine with the smallest load in a descending order of jobs (the longest is the first).

Example: 3 machines and 5 jobs (m = 3, n = 5)

$t_1 = 7,\ \ t_2 = 6,\ \ t_3 = 5,\ \ t_4 = 4,\ \ t_5 = 2$



One machine needs to have at least two jobs among the largest (m+1) jobs.

$$\text{(a)} < \frac{1}{m}\sum_{j=1}^{n} t_j \leq T^* \qquad 2t_{m+1} \leq T^* \Rightarrow \text{(b)} \leq t_{m+1} \leq T^*/2$$

In this figure, $t_{m+1} = t_4$ (since m = 3) ==> $2t_4 \leq T^*$

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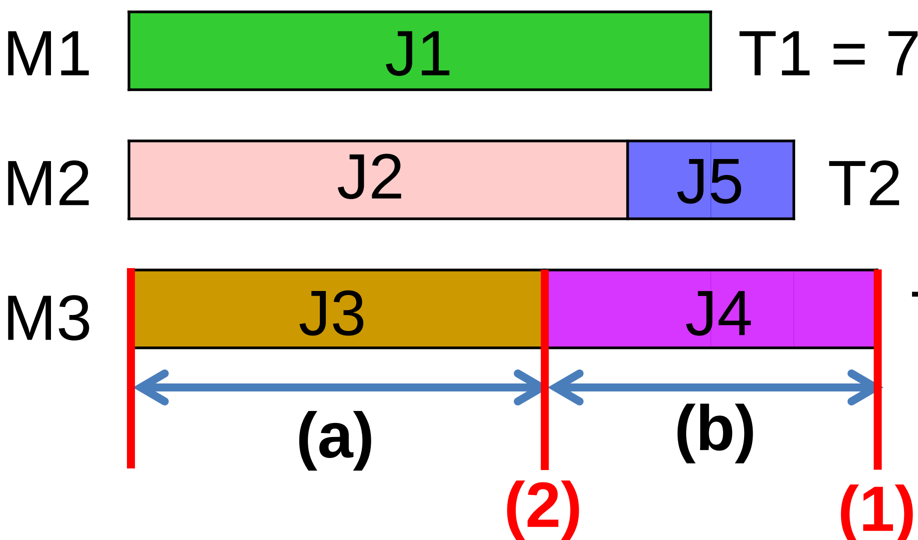


One machine needs to have at least two jobs among the largest (m+1) jobs.

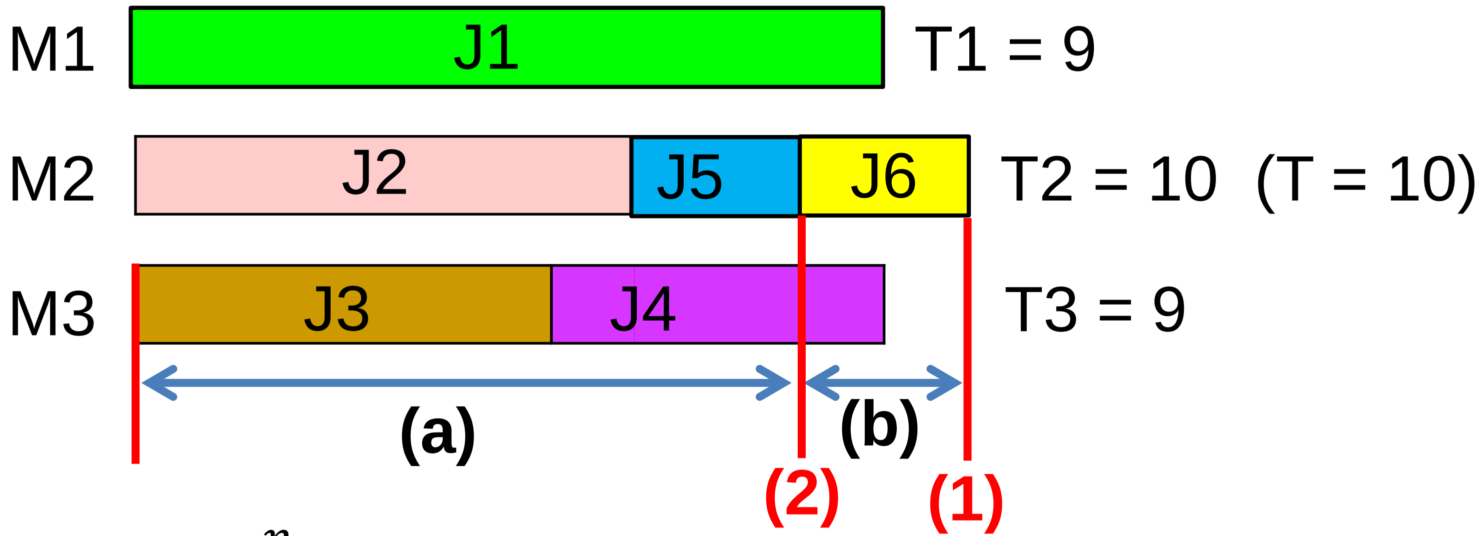
$$2t_{m+1} \le t_m + t_{m+1} \le T^*$$

$$\text{(a)} < \frac{1}{m}\sum_{j=1}^{n} t_j \le T^* \qquad 2t_{m+1} \le T^* \rightarrow \text{(b)} \le t_{m+1} \le T^*/2$$

In this figure, $t_{m+1} = t_4$ (since m = 3) ==> $2t_4 \le T^*$

Another Example: 3 machines and 6 jobs (m = 3, n = 6)

$t_1 = 9,\ \ t_2 = 6,\ \ t_3 = 5,\ \ t_4 = 4,\ \ t_5 = 2\ ,\ \ t_6 = 2$

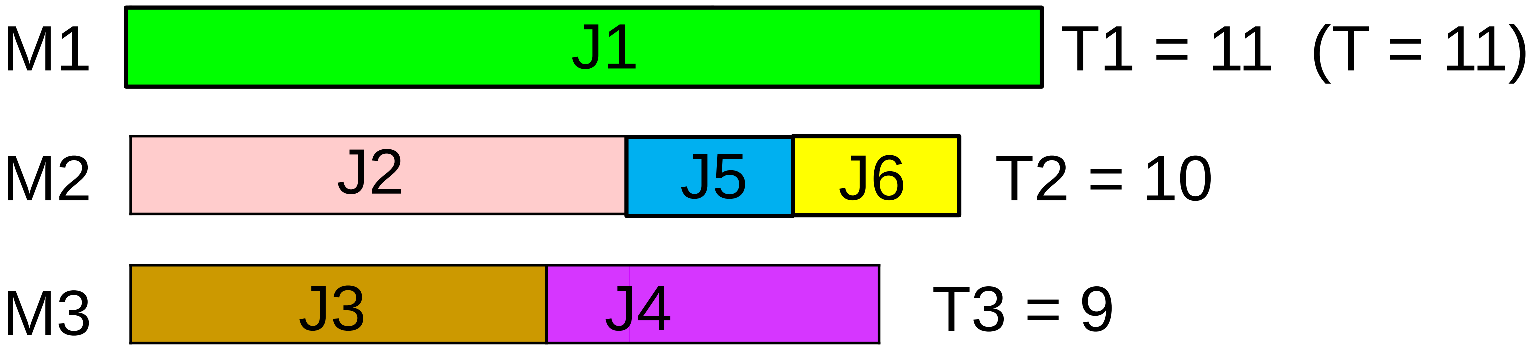


$$\text{(a)} < \frac{1}{m}\sum_{j=1}^{n} t_j \leq T^* \qquad 2t_{m+1} \leq T^* \rightarrow \text{(b)} \leq t_{m+1} \leq T^*/2$$

When the makespan T is the sum of the processing times of at least two jobs, this proof is applicable: T = (a) + (b) < 3T*/2.

Another Example: 3 machines and 6 jobs (m = 3, n = 6)

$t_1 = 11$, $t_2 = 6$, $t_3 = 5$, $t_4 = 4$, $t_5 = 2$, $t_6 = 2$



When the makespan T is the same as the processing time of a single job (i.e., the first job with the longest processing time), it is clear that $T = T^* < 3T^*/2$.

Sorted Greedy Algorithm

Assign a job to the machine with the smallest load in a descending order of jobs (the longest is the first).

Q: How good is this sorted greedy algorithm?

The obtained makespan T is not worse than $(3/2)T^*$ ($T \leq (3/2)T^*$). 3/2-approximation

Q: How tight is this upper bound?

Exercise 2-1 (Homework)

Try to create an example where the obtained makespan T is very close to $(4/3)T^*$.

The sorted greedy algorithm is a 4/3-approximation algorithm.
R. L. Graham. Bounds for multiprocessing timing anomalies. *SIAM J. Applied Mathematics* 17 (1969), 263–269.

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LPT-List-Scheduling(m, n, t1,t2,…,tn) {
   Sort jobs so that t1 ≥ t2 ≥  … ≥ tn

   for i = 1 to m {
      Li ← 0          ← load on machine i
      J(i) ← φ        ← jobs assigned to machine i
   }

   for j = 1 to n {
      i = argmink Lk          ← machine i has smallest load
      J(i) ← J(i) ∪ {j}       ← assign job j to machine i
      Li ← Li + tj            ← update load of machine i
   }
   return J(1), …, J(m)
}
```

procedure SORTED-BALANCE

Sort jobs in descending order of processing time (so $t_1 \geq t_2 \geq \cdots \geq t_n$)

for $j = 1, \ldots, n$ **do**

Let M_i = machine that achieves $\min_k T_k$

Assign job j to machine M_i

Set $A(i) \leftarrow A(i) \cup \{j\}$

Set $T_i \leftarrow T_i + t_j$

end for

end procedure

General Settings of Load Balancing

Some machines are more efficient than the others

Example: M3 needs less processing times than the others.

Three Machines: M1, M2, M3

Ten Jobs: J1, J2, ..., J10

Processing time (t_j): 2, 4, 6, ..., 20 on M1 and M2

1, 2, 3, ..., 10 on M3

Exercise 2-2 (Homework)

Design an algorithm to solve this problem, and explain your algorithm using easy and difficult examples.

General Settings of Load Balancing

Some machines can process only a part of jobs.

Example: M1 can process J1, J2, ..., J7
M2 can process J1, J2, ..., J8
M3 can process all jobs (J1, J2, ..., J10)

Processing time (t_j): 2, 4, 6, 8, 10, 12, 14 on M1
2, 4, 6, 8, 10, 12, 14, 16 on M2
1, 2, 3, 4, 5, 6, 7, 8, 9, 10 on M3