Topic 5: Vertex Cover Problem: Use of LP

Minimize $w(S) = \sum_{i \in S} w_i$

where S ($S \subset V$) is a vertex cover (i.e., each edge in E has at least one end in S).

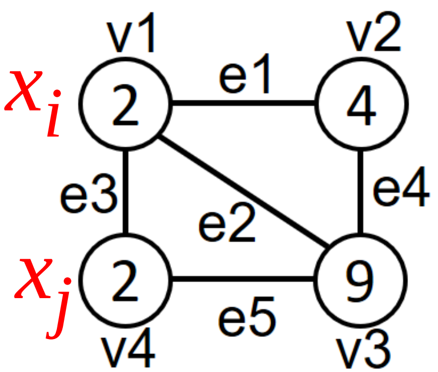
Decision variable for each vertex v_i: $x_i = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$

VC-IP: Vertex Cover as an Integer Program

Minimize $w(S) = \sum_{i \in V} w_i x_i$

subject to $x_i + x_j \geq 1$ for $(i, j) \in E$

$x_i \in \{0, 1\}$ for $i \in V$



Matrix Form of VC-IP

Minimize $w(S) = \boldsymbol{w}^t\ \boldsymbol{x}$

subject to $\mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}$, $A\boldsymbol{x} \geq \mathbf{1}$, and $\boldsymbol{x}$ is an integer vector.

$$\boldsymbol{x} = (x_1,\ x_2, \ldots, x_{|V|})^t \qquad \mathbf{1} = (1, 1, \ldots, 1)^t$$

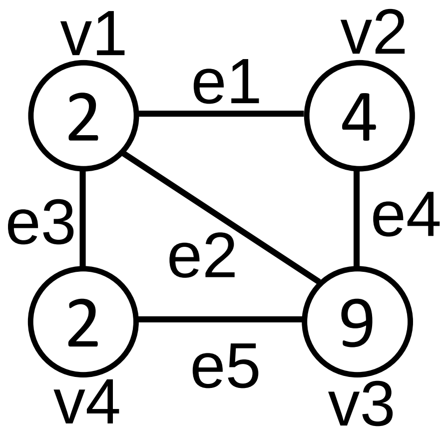
$$\boldsymbol{w} = (w_1,\ w_2, \ldots, w_{|V|})^t \qquad \mathbf{0} = (0,\ 0, \ldots,\ 0)^t$$

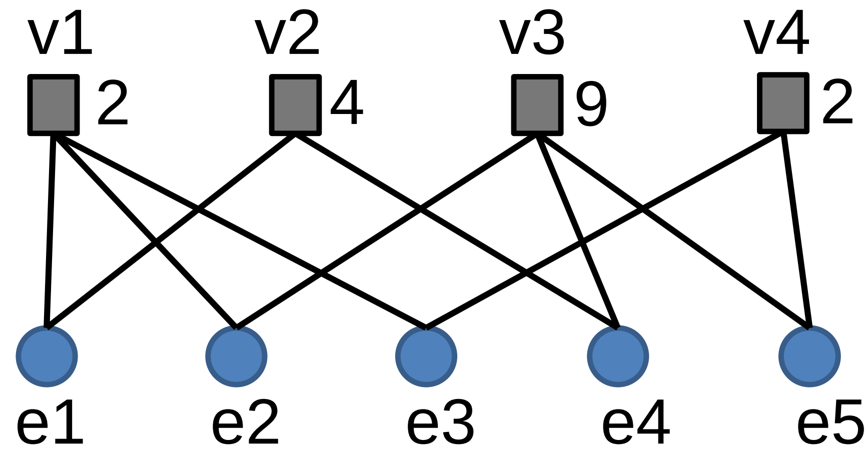
Matrix *A*: Rows of A correspond to edges in E

Columns of A correspond to vertexes in V

$$A[e, i] = \begin{cases} 1 & \text{if vertex } v_i \text{ is an end of edge } e \\ 0 & \text{otherwise} \end{cases}$$

If $\boldsymbol{x}^*$ is the optimal solution of VC-IP, $S = \{v_i \in V: x_i^* = 1\}$ is the optimal vertex cover S^* with the minimum total weight $w(S^*)$.





$$\text{Minimize} \quad \begin{pmatrix} 2 & 4 & 9 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Subject to

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \geq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

x is an integer vector.

VC-LP: Linear Programming Relaxation of VC-IP

Minimize $w(S) = \boldsymbol{w}^t \boldsymbol{x}$

subject to $\mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}$, $A\boldsymbol{x} \geq \mathbf{1}$~~, and $\boldsymbol{x}$ is an integer vector~~.

Optimal value of VC-LP $\leq$ Optimal value of VC-IP

$$\boldsymbol{w}_{\mathbf{LP}} \leq \boldsymbol{w}(\boldsymbol{S}^*)$$

LP: Linear Programming

Most frequently used optimization method

(a number of software packages are available)

Creation of an IP solution from the LP optimal solution

Minimize $w(S) = \boldsymbol{w}^t\,\boldsymbol{x}$

subject to $\mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}\,,\quad A\boldsymbol{x} \geq \mathbf{1}$

Let $\boldsymbol{x}^*$ be the optimal solution of VC-LP. A vertex cover S is obtained by $S = \{i \in V : x_i^* \geq 1/2\}$.

[Proof] Consider an edge $(i, j) \in E$. Since $x_i^* + x_j^* \geq 1$, either $x_i^* \geq 1/2$ or $x_j^* \geq 1/2$. Thus the edge (i, j) is covered by S.

[2-approximation] Let S^* be the optimal vertex cover. Then

$$\sum_{i\in S^*} w_i \geq \sum_{i\in V} w_i\, x_i^* \geq \sum_{i\in S} w_i\, x_i^* \geq \frac{1}{2}\sum_{i\in S} w_i \;\Rightarrow\; 2w(S^*) \geq w(S)$$

$\boldsymbol{x}^*$ is the solution of the relaxation problem. $x_i^* \geq 1/2$

General Discussion: Relaxation Problem

0-1 Integer Programming Problem

Minimize $z = f(\boldsymbol{x})$

subject to $g(\boldsymbol{x}) \geq 0, \quad x_i = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots, n$

Relaxation Problem

Minimize $y = f(\boldsymbol{x})$

subject to $g(\boldsymbol{x}) \geq 0, \quad 0 \leq x_i \leq 1 \text{ for } i = 1, 2, \ldots, n$

(1) The optimal value of the relaxation problem is the same as or better than that of the original problem: $y^* \leq z^*$

(2) If the optimal solutions $\boldsymbol{x}^*$ of the relaxation problem is an integer vector, $\boldsymbol{x}^*$ is also the optimal solution of the original problem.

General Discussion: Relaxation Problem

Integer Programming Problem (L_i and U_i are integers)

Minimize $z = f(\boldsymbol{x})$

subject to $g(\boldsymbol{x}) \geq 0, \quad x_i \in \{L_i, L_i + 1, \ldots, U_i\}$ for $i = 1, 2, \ldots, n$

Relaxation Problem

Minimize $y = f(\boldsymbol{x})$

subject to $g(\boldsymbol{x}) \geq 0, \quad L_i \leq x_i \leq U_i$ for $i = 1, 2, \ldots, n$

(1) The optimal value of the relaxation problem is the same as or better than that of the original problem: $y^* \leq z^*$

(2) If the optimal solutions $\boldsymbol{x}^*$ of the relaxation problem is an integer vector, $\boldsymbol{x}^*$ is also the optimal solution of the original problem.

General Discussion: Relaxation Problem

Integer Programming Problem (L_i and U_i are integers)

Minimize $z = f(\boldsymbol{x})$

subject to $g(\boldsymbol{x}) \geq 0, \quad x_i \in \{L_i, L_i + 1, ..., U_i\}$ for $i = 1, 2, ..., n$

Relaxation Problem

Minimize $y = f(\boldsymbol{x})$

subject to $g(\boldsymbol{x}) \geq 0, \quad L_i \leq x_i \leq U_i$ for $i = 1, 2, ..., n$

The relaxation problem is used to evaluated the lower bound of the optimal value z^* (i.e., optimistic estimation: z^* cannot be better than y^*). A greedy algorithm is used to evaluate the upper bound of the optimal value z^* (i.e., pessimistic estimation: z^* cannot be worse than the greedy algorithm result z).

Discussions: How to address the following questions (i.e., how to compare the three algorithms):

Which is the best algorithm among the following three algorithms?

* **Greedy Set Cover:** $H(d)$-approximation algorithm

* **Pricing Method:** 2-approximation algorithm

* **LP-based Method:** 2-approximation algorithm

Can we say that "Greedy set cover is inferior to the other two algorithms because its upper bound is higher than the others when $d \geq 4$"? Can we say that "Greedy set cover is superior to the other two algorithms because its upper bound is lower than the others when $d \leq 3$"?

Exercise 8-2:

Create an example for which the best solution is obtained from the greedy set cover algorithm among the three methods (the greedy set cover algorithm, the pricing method, and the LP-based method) independent of the order of edges in the pricing method. If you cannot create such an example, you can change “independent of the order of edges in the pricing method” to “for some orders of edges”. This can be used in Exercise 8-3 and Exercise 8-4 if needed.

Exercise 8-3:

Create an example for which the best solution is obtained from the pricing method independent of the order of edges.

Exercise 8-4:

Create an example for which the best solution is obtained from the LP-based method independent of the order of edges.

Important: The LP-based result should be obtained by using your LP solver. That is, you need to use the solution from the LP solver.