Topic 5: Vertex Cover Problem: Use of LP

Minimize $w(S) = \sum_{i \in S} w_i$

where S ($S \subset V$) is a vertex cover (i.e., each edge in E has at least one end in S).

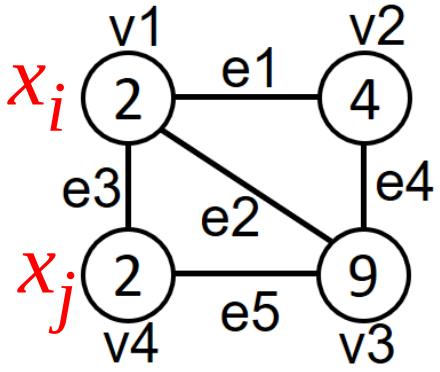
Decision variable for each vertex v_i: $x_i = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$

VC-IP: Vertex Cover as an Integer Program

Minimize $w(S) = \sum_{i \in V} w_i \, x_i$

subject to $x_i + x_j \geq 1$ for $(i, j) \in E$

$x_i \in \{0, 1\}$ for $i \in V$



Matrix Form of VC-IP

Minimize $w(S) = \boldsymbol{w}^t \boldsymbol{x}$

subject to $\mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}$, $A\boldsymbol{x} \geq \mathbf{1}$, and $\boldsymbol{x}$ is an integer vector.

$$\boldsymbol{x} = (x_1, x_2, ..., x_{|V|})^t \qquad \mathbf{1} = (1, 1, ..., 1)^t$$

$$\boldsymbol{w} = (w_1, w_2, ..., w_{|V|})^t \qquad \mathbf{0} = (0, 0, ..., 0)^t$$

Matrix *A*: Rows of *A* correspond to edges in *E*

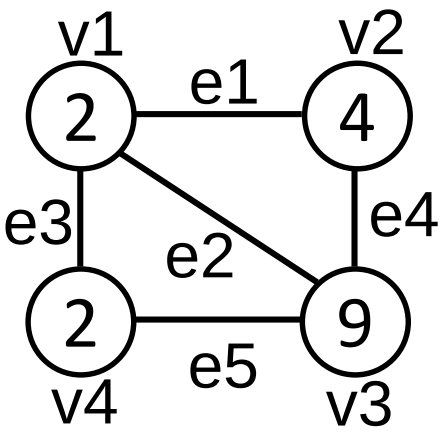
Columns of *A* correspond to vertexes in *V*

$$A[e, i] = \begin{cases} 1 & \text{if vertex } v_i \text{ is an end of edge } e \\ 0 & \text{otherwise} \end{cases}$$

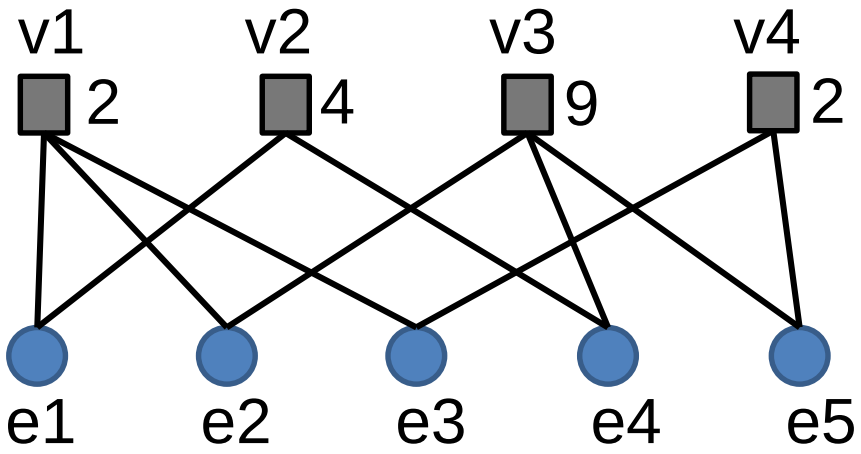
If $\boldsymbol{x}^*$ is the optimal solution of VC-IP, $S = \{v_i \in V: x_i^* = 1\}$ is the optimal vertex cover S^* with the minimum total weight $w(S^*)$.

Exercise (Now):

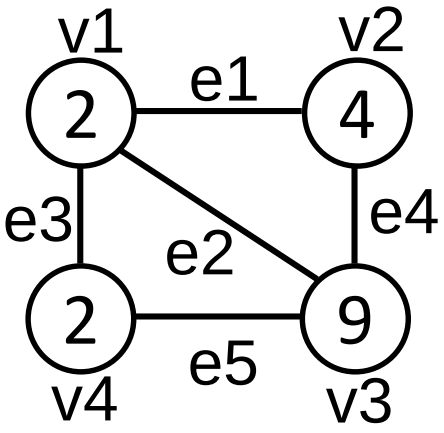
Formulate the following vertex cover problem as a matrix form integer programming problem (i.e., as a VC-IP)

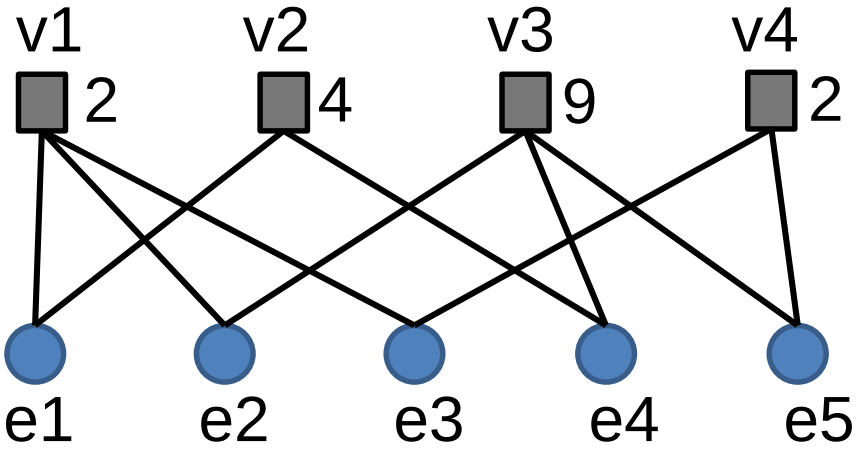


Vertex Cover Problem



Set Cover Problem





Minimize $(?\ ?\ ?\ ?)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

Subject to

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \geq \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} ? \\ ? \\ ? \\ ? \\ ? \end{pmatrix}$$

$\boldsymbol{x}$ is an integer vector.

VC-LP: Linear Programming Relaxation of VC-IP

Minimize $w(S) = \boldsymbol{w}^t\ \boldsymbol{x}$

subject to $\mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}$, $A\boldsymbol{x} \geq \mathbf{1}$ ~~, and $\boldsymbol{x}$ is an integer vector~~.

Optimal value of VC-LP $\leq$ Optimal value of VC-IP

$$\boldsymbol{w}_{\mathbf{LP}} \leq \boldsymbol{w}(\boldsymbol{S}^*)$$

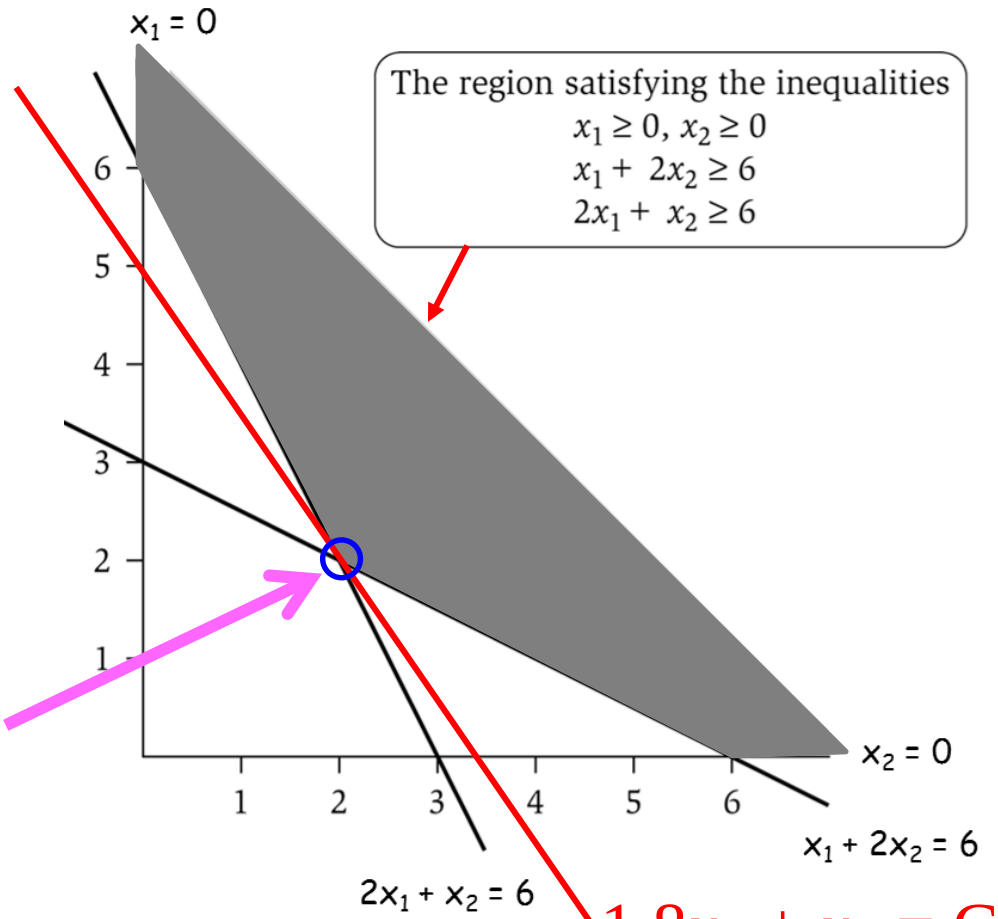
LP: Linear Programming

Most frequently used optimization method
(a number of software packages are available)

Simple Example of a Linear Programming Problem

Minimize $1.8x_1 + x_2$

Subject to $x_1 + 2x_2 \geq 6,\ 2x_2 + x_2 \geq 6,\ x_1 \geq 0,\ x_2 \geq 0$



Optimal solution

Exercise 8-1 (Use of LP):

(1) Find an available LP (Linear Programming) software.

(2) Generate a simple example of the LP problem.

(3) Solve the generated simple example using the LP software. This is to confirm that you are correctly using the software.

(4) Find or generate a larger example of the LP problem.

(5) Solve the large example using the LP software. This is to examine the performance of the software. It is very likely that your LP software can solve LP problems with 1,000,000 variables.

Your presentation will be mainly about the LP software and your simple and larger examples.