Calculus: Derivatives

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1 Limit Definition of a Derivative

There are two limit defintions of a derivative. The first one is:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

Which calculates the derivative at one point. On the other hand, the second derivative defintion is written as:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$$

Which computes the derivative as an overall function. Let's take a look at some examples:

1.1 the derivative of $\sqrt{x^2+1}$

Step one: plug into limit definition

$$\lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$

Step two: Simplify. First, We multiply by the conjugate of the numerator:

$$\lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}\times\left(\frac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}\right)$$

Expanding gives

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2+1-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$f'(x)=\lim_{h\to 0}\frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{h(2x + h)}{h\left(\sqrt{(x+h)^2+1} + \sqrt{x^2+1}\right)}$$

$$f'(x) = \lim_{h \to 0} \frac{2x + h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}$$

$$f'(x) = \frac{2x + 0}{\sqrt{(x+0)^2+1} + \sqrt{x^2+1}}$$

$$f'(x) = \frac{2x}{\sqrt{x^2+1} + \sqrt{x^2+1}}$$

$$f'(x) = \frac{2x}{2\sqrt{x^2+1}}$$

$$f'(x) = \frac{x}{\sqrt{x^2+1}}$$

1.2 The derivative of $x^2 + 3x + 2$ at $x = 2$

Plugging into the first definition gives

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

$$\lim_{x \to 2} \frac{(x^2 + 3x + 2) - (2^2 + 3(2) + 2)}{x - 2} = f'(2)$$

$$\lim_{x \to 2} \frac{(x^2 + 3x + 2) - (4 + 6 + 2)}{x - 2} = f'(2)$$

$$\lim_{x \to 2} \frac{x^2 + 3x + 2 - 12}{x - 2} = f'(2)$$

$$\lim_{x \to 2} \frac{x^2 + 3x - 10}{x - 2} = f'(2)$$

$$\lim_{x \to 2} \frac{(x + 5)(x - 2)}{x - 2} = f'(2)$$

$$\lim_{x \to 2} x + 5 = f'(2)$$

$$2 + 5 = f'(2)$$

$$f'(2) = 7$$

2 Differentiation Rules

There are five rules of differetnation, and they allow us to take the derivative of functions without using the limit definition.

2.1 power rule

let n be any real number.

$$\frac{d}{dx}[x^n] = nx^{x-1}$$

2.2 product rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

2.3 quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

2.4 chain rule

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

2.5 constants

let C be a constant.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

2.6 linearity

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

2.7 general formulas

$$\frac{d}{dx}[C] = 0$$

where C is any constant

$$\frac{d}{dx}[x] = 1$$

2.7.1 examples

1.
$$f(x) = \frac{(x-2)}{5x+7}$$

$$f'(x) = \frac{1(5x+7) - 5(x-2)}{(5x+7)^2}$$

$$f'(x) = \frac{5x+7-5x-10}{(5x+7)^2}$$

$$f'(x) = \frac{-3}{(5x+7)^2}$$

2.
$$f(x) = (x+7)\sqrt{x^2-2}$$

$$f(x) = (x+7)\left(x^2-2\right)^{\frac{1}{2}}$$

$$f'(x) = 1(x^2-2)^{\frac{1}{2}} + \frac{1}{2}(x^2-2)^{-\frac{1}{2}}(x+7) \times \frac{d}{dx}[x^2-2]$$

$$f'(x) = (x^2-2)^{\frac{1}{2}} + \frac{(x+7)}{2\sqrt{x^2-2}} \times \frac{d}{dx}[x^2-2]$$

$$f'(x) = (x^2-2)^{\frac{1}{2}} + \frac{(x+7)}{2\sqrt{x^2-2}} \times 2x$$

$$f'(x) = (x^2-2)^{\frac{1}{2}} + \frac{2x(x+7)}{2\sqrt{x^2-2}}$$

$$f'(x) = \sqrt{x^2-2} + \frac{2x(x+7)}{2\sqrt{x^2-2}}$$

3.
$$f(x) = 2x^2 + 3x + 6$$

$$f'(x) = 4x + 3$$

3 trigonometric derivatives

The derivatives of the six main trig functions are:

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

and the follow trig inverse derivatives are:

$$\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\cos^{-1}(x)] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\csc^{-1}(x)] = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\cot^{-1}(x)] = -\frac{1}{1+x^2}$$

3.1 examples

1.

$$f(x) = \sin(x) + \sin^{-1}(x^2)$$

$$f'(x) = \cos(x) + \frac{1}{\sqrt{1-(x^2)^2}} \times \frac{d}{dx}[x^2]$$

$$f'(x) = \cos(x) + \frac{1}{\sqrt{1-x^4}} \times [2x]$$

$$f'(x) = \cos(x) + \frac{2x}{\sqrt{1-x^4}}$$

2.
$$f(x) = \cos(3x) - \tan(x^5)$$

$$f'(x) = -\sin(3x) \times \frac{d}{dx}[3x] - \sec^2(x^5) \times \frac{d}{dx}[x^5]$$

$$f'(x) = -3\sin(3x) - 5x^4 \sec^2(x^5)$$

4 derivative of expoenentials and logarithms

Derivatives of expoenential functions and logarithmic functions can be generalizedi into two formulas:

$$\frac{d}{dx}[b^x] = b^x \ln(b)$$

and

$$\frac{d}{dx}[\log_b(x)] = \frac{1}{x\ln(b)}$$

This can lead us two the derivative of two common functions: e^x and $\ln(x)$, where

$$\frac{d}{dx}[e^x] = e^x$$

and

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

4.1 examples

1.
$$f(x) = \ln(\sin(x)) + e^{\tan(x)}$$

$$f'(x) = \frac{1}{\sin(x)} \times \frac{d}{dx}[\sin(x)] + e^{\tan(x)} \times \frac{d}{dx}[\tan(x)]$$

$$f'(x) = \frac{1}{\sin(x)} \times \cos(x) + e^{\tan(x)} \times \sec^2(x)$$

$$f'(x) = \frac{\cos(x)}{\sin(x)} + \sec^2(x)e^{\tan(x)}$$

$$f'(x) = \cot(x) + \sec^2(x)e^{\tan(x)}$$

2.
$$f(x) = \log_4(x) + 3^{3x^3-5}$$

$$f'(x) = \frac{1}{x\ln(4)} + 3^{3x^3-5}\ln(3) \times \frac{d}{dx}[3x^3 - 5]$$

$$f'(x) = \frac{1}{x\ln(4)} + 3^{3x^3-5}\ln(3)(9x^2)$$

5 implicit differentation

The examples shown before express variables as a function of x. However, when we see expressions such as

$$x^2 + y^2 = 6$$

Where x and y are dependent on eachtoher, we have to find a different way to differentiate. This said method is called implicit differentiation. The normal rules of differentiation still apply, however we treat y as a function of x. This means that when we take the derivative for a function y, we have to apply the chain rule. In order to find the derivative, we need to solve for $\frac{dy}{dx}$

5.1 example

1.
$$4x^3 + \sin(y) = 6y^2$$

$$12x^2 + \cos(y)\frac{dy}{dx} = 12y\frac{dy}{dx}$$

$$12x^2 = 12y\frac{dy}{dx} - \cos(y)\frac{dy}{dx}$$

$$12x^2 = (12y - \cos(y))\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{12x^2}{12y - \cos(y)}$$

2.

$$ln(y) + 3x = 7$$

$$\frac{1}{y}\frac{dy}{dx} + 3 = 0$$

$$3 = -\frac{1}{y}\frac{dy}{dx}$$

$$\frac{3}{-\frac{1}{y}} = \frac{dy}{dx}$$

$$-3y = \frac{dy}{dx}$$