2023 Differential Geometry- TD 8

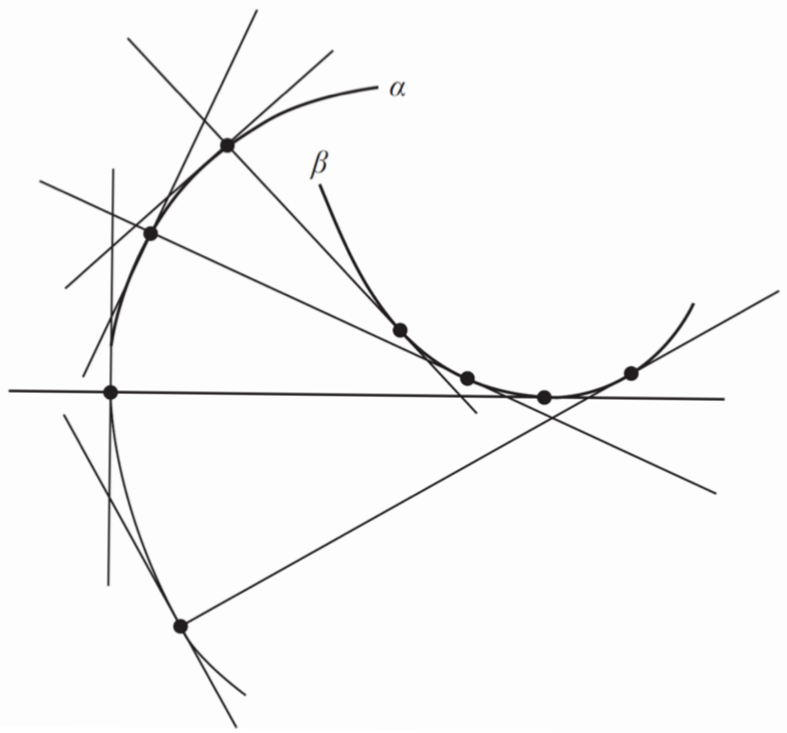
1. Let $\alpha : I \to \mathbb{R}^2$ be a regular parametrized plane curve (arbitrary parameter), and $\mathbf{n}(t), \kappa(t)$ be its unit normal and curvature. Assume that $\kappa(t) \neq 0, t \in I$. The curve

$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)}\mathbf{n}(t), \qquad t \in I$$

is called the *evolute* of α.

(a) Show that the tangent at t of the evolute of α is the normal to α at t;

(b) Consider the normal lines of α at two neighboring points $t_1 \neq t_2$. Let t_1 approach t_2 and show that the intersection points of the normals converge to a point on the trace of the evolute of α.



2. Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized space curve. Assume that its torsion $\tau(s) \neq 0$ and curvature satisfies $\kappa'(s) \neq 0$ for all $s \in I$. Show that $\alpha(I)$ lies on a sphere if and only if

$$\frac{1}{\kappa(s)^2} + \left(\frac{d}{ds}\frac{1}{\kappa(s)}\right)^2 \frac{1}{\tau(s)^2} = \text{const.}$$

3. A space curve α is called a helix if the tangent lines of α make a constant angle with a fixed direction. Assume that $\tau(s) \neq 0$, $s \in I$. Prove that

torsion

(a) α is a helix if and only if $\kappa/\tau =$ constant.

(b) α is a helix if and only if the lines containing the normal $\mathbf{n}(s)$ and passing through $\alpha(s)$ are parallel to a fixed plane.

(c) α is a helix if and only if the lines containing the binormal $\mathbf{b}(s)$ and passing through $\alpha(s)$ make a constant angle with a fixed direction.

(d) The curve

$$\alpha(s) = \left(\frac{a}{c}\int \sin\theta(s)ds, \frac{a}{c}\int \cos\theta(s)ds, \frac{b}{c}s\right),$$

where $c^2 = a^2 + b^2$, is a helix, and that $\kappa/\tau = a/b$.

4. (*Tubular surfaces*) Let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve with nonzero curvature everywhere and arc length as parameter. Let

$$\mathbf{x}(s, v) = \alpha(s) + r\,(n(s)\cos v + b(s)\sin v)\,, \qquad r = \text{const.} \neq 0, \quad s \in I,$$

be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that, when $\mathbf{x}$ is regular, its unit normal vector is

$$N(s, v) = -\,(n(s)\cos v + b(s)\sin v)\,.$$

5. Two regular surfaces S_1 and S_2 intersect *transversally* if whenever $p \in S_1 \cap S_2$ then $T_pS_1 \neq T_pS_2$. Prove that if S_1 intersects S_2 transversally, then $S_1 \cap S_2$ is a regular curve.

6. Prove that if all normal lines to a regular surface S meet a fixed straight line, then S is a piece of a surface of revolution.

7. Show that the area A of a bounded region R of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy,$$

where Q is the normal projection of R onto the xy plane.

2. "$\Rightarrow$" if $|\alpha(s) - p_0|^2 = R^2$

求导 $(\alpha(s) - p_0) \cdot \vec{t}(s) = 0$

求导 $1 + (\alpha(s) - p_0) \cdot \vec{n}(s)\, \kappa(s) = 0$

求导 $(\alpha(s) - p_0) \cdot \left((-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s))\kappa(s) + \vec{n}(s)\dot{\kappa}(s) \right) = 0$

$$\Rightarrow \quad (\alpha(s) - p_0) \cdot \vec{t}(s) = 0$$
$$(\alpha(s) - p_0) \cdot \vec{n}(s) = -\frac{1}{\kappa(s)}$$
$$(\alpha(s) - p_0) \cdot \vec{b}(s) = -\frac{1}{\tau(s)} \frac{d}{ds}\left(\frac{1}{\kappa(s)}\right)$$

$$\Rightarrow \frac{1}{(\kappa(s))^2} + \frac{1}{\tau(s)^2}\left(\frac{d}{ds}\left(\frac{1}{\kappa(s)}\right)\right)^2 = R^2$$

"$\Leftarrow$". 设 $p(s) := \alpha(s) + \frac{1}{\kappa(s)}\vec{n}(s) + \frac{1}{\tau(s)} \frac{d}{ds}\left(\frac{1}{\kappa(s)}\right) \cdot \vec{b}(s)$

求导 $\Rightarrow \frac{d}{ds} p(s) = \left(\frac{\tau(s)}{\kappa(s)} + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right) \right) \cdot \vec{b}(s) = 0$

$$\Rightarrow \quad p(s) \equiv p_0$$
$$|\alpha(s) - p_0|^2 = \frac{1}{\kappa^2} + \left(\frac{1}{\tau}\frac{d}{ds}\frac{1}{\kappa}\right)^2 = \text{const}.$$

3. (a) 设 $\vec{t} \cdot \vec{a} = \text{const} = \cos\theta$

求导 $\Rightarrow \kappa \vec{n} \cdot \vec{a} = 0 \Rightarrow \vec{n} \cdot \vec{a} = 0 \Rightarrow \vec{b} \cdot \vec{a} = \sin\theta$

对 $\vec{n} \cdot \vec{a} = 0$ 求导 $\Rightarrow (-\kappa\vec{t} + \tau\vec{b})\vec{a} = 0$

$$\Rightarrow \frac{\kappa}{\tau} = \frac{\vec{b} \cdot \vec{a}}{\vec{t} \cdot \vec{a}} = \tan\theta$$

反之. if $\frac{\kappa}{\tau} = \text{const} = \tan\theta$

$$\Rightarrow (\vec{t}\cos\theta + \vec{b}\sin\theta)' = \kappa\vec{n}\cos\theta - \tau\vec{n}\sin\theta = 0$$

$$\Rightarrow \vec{t}\cos\theta + \vec{b}\sin\theta \equiv a. \quad \Rightarrow \vec{t} \cdot \vec{a} = \cos\theta = \text{const}$$

(b). (c). (d) 同上.

note: 圆柱螺线 $\alpha(s) = \left(a\cos\frac{s}{c},\ a\sin\frac{s}{c},\ \frac{b}{c}s\right)$. $c^2 = a^2 + b^2$

有 $\kappa(s) \equiv \frac{a}{c^2}$, $\tau(s) \equiv \frac{b}{c^2}$

4. $x_s = (1 - r k(s) \cos v)\vec{t} - r\tau(s)\sin v\,\vec{n} + r\tau(s)\cos v\,\vec{b}$

$x_v = r(\vec{b}(s)\cos v - \vec{n}(s)\sin v)$

$\Rightarrow x_s \wedge x_v = -r(1 - r k(s)\cos v)(\vec{b}(s)\sin v + \vec{n}(s)\cos v)$

$\Rightarrow \vec{N} = \dfrac{x_s \wedge x_v}{|x_s \wedge x_v|} = -(\vec{b}(s)\sin v + \vec{n}(s)\cos v)$

5 在 $p \in S_1 \cap S_2$ 附近, S_1 given by $F_1(x, y, z) = 0$

S_2 given by $F_2(x, y, z) = 0$

$\{0\}$ is regular value of both F_1 and F_2

$S_1 \cap S_2$ is given by

$$F(x,y,z) = (F_1(x,y,z), F_2(x,y,z)) = (0,0)$$

由 $T_pS_1 \neq T_pS_2 \Rightarrow \nabla F_1 \not\parallel \nabla F_2 \Rightarrow \text{rank}(dF) = 2$

即 $(0,0)$ is a regular value of F

so $S_1 \cap S_2$ is a regular curve. $\square$

6. 设 L 为 "fixed straight line".

任给一点 $p \in S$, 记 P_1 为过 L and p 的平面

P_2 为过点 p 与 L 垂直的平面

由曲面 S 在点 p 的法线过 $L \Rightarrow T_pS$ 与 P_2 transversal

$\Rightarrow S \cap P_2$ 为一条平面上的 regular curve, 记为 C

注意到 $P_1 \cap P_2$ 与 C 垂直.

$\Rightarrow C$ 为 circle.

7. $r(x,y) = (x, y, f(x,y))$

$r_x = (1, 0, f_x)$,

$r_y = (0, 1, f_y)$

$\Rightarrow r_x \wedge r_y = (-f_x, -f_y, 1)$

$|r_x \wedge r_y| = \sqrt{1 + f_x^2 + f_y^2}$

so, $\text{Area} = \iint_{\mathcal{U}} \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$