2023 Differential Geometry- TD 11

1. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

(a) Compute ω in spherical coordinates (ρ, φ, θ) defined by $(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$.
(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.
(c) Compute the pullback $\iota^*_{\mathbb{S}^2}\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.
(d) Show that $\iota^*_{\mathbb{S}^2}\omega$ is nowhere zero.

2. *Formule d'Archimède*
Soit ω la forme volume $xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ sur la sphère $S^2 \subset \mathbb{R}^3$.
a) Expliciter, à l'aide des coordonnées sphériques, une primitive de ω sur $S^2 \setminus \{S \cup N\}$ qui soit invariante par les rotations d'axe NS (on a désigné par N et S les pôles Nord et Sud).
Application : calculer l'aire du "segment de sphère"

$$\Sigma_{h,k} = \{(x, y, z) \in S^2,\ h \le z \le k\}.$$

* b) Expliciter une primitive de ω sur $S^2 \setminus \{S\}$ invariante par les rotations autour de NS.

3. (Nowhere-vanishing 1-form on the circle S^1)

(a) The vector field $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is tangent to the unit circle S^1. Find a 1-form ω on S^1 such that $\omega(X) = 1$.
(b) Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be the radial vector field and $\alpha = dx \wedge dy$ be the area 2-form on $\mathbb{R}^2$. Compute the contraction $i_X(\alpha)$.
(c) Let $h : \mathbb{R} \to S^1 \subset \mathbb{R}^2$ be given by $h(t) = (x, y) = (\cos t, \sin t)$. Compute the pullback $h^*\omega$.
(d) If $g = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$ and $\ell_g : S^1 \to S^1$ is the left multiplication, show that $\ell_g^*\omega = \omega$ for all $g \in S^1$.

4. ~~19.11~~. A C^∞ nowhere-vanishing form on a smooth hypersurface

(a) Let $f(x,y)$ be a C^∞ function on $\mathbb{R}^2$ and assume that 0 is a regular value of f. By the regular level set theorem, the zero set M of $f(x,y)$ is a one-dimensional submanifold of $\mathbb{R}^2$. Construct a C^∞ nowhere-vanishing 1-form on M.

(b) Let $f(x,y,z)$ be a C^∞ function on $\mathbb{R}^3$ and assume that 0 is a regular value of f. By the regular level set theorem, the zero set M of $f(x,y,z)$ is a two-dimensional submanifold of $\mathbb{R}^3$. Let f_x, f_y, f_z be the partial derivatives of f with respect to x, y, z, respectively. Show that the equalities

$$\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$$

hold on M whenever they make sense, and therefore the three 2-forms piece together to give a C^∞ nowhere-vanishing 2-form on M.

(c) Generalize this problem to a regular level set of $f(x^1,\dots,x^{n+1})$ in $\mathbb{R}^{n+1}$.

5. ~~22.5~~. Boundary orientation

Let M be an oriented manifold with boundary, ω an orientation form for M, and X a C^∞ outward-pointing vector field along ∂M.

(a) If τ is another orientation form on M, then $\tau = f\omega$ for a C^∞ everywhere-positive function f on M. Show that $\iota_X \tau = f \iota_X \omega$ and therefore, $\iota_X \tau \sim \iota_X \omega$ on ∂M. (Here "$\sim$" is the equivalence relation defined in Subsection 21.4.)

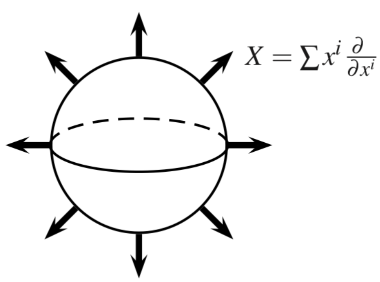
(b) Prove that if Y is another C^∞ outward-pointing vector field along ∂M, then $\iota_X \omega \sim \iota_Y \omega$ on ∂M.

6. ~~22.9~~. Boundary orientation on a sphere

Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Show that an orientation form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1},$$

where the caret $\widehat{\ }$ over dx^i indicates that dx^i is to be omitted. (*Hint*: An outward-pointing vector field on S^n is the radial vector field $X = \sum x^i \partial/\partial x^i$ as in Figure 22.7(b).)



(b) Radial vector field on a sphere.

7. **~~22.10~~. Orientation on the upper hemisphere of a sphere**
Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Let U be the upper hemisphere

$$U = \{x \in S^n \mid x^{n+1} > 0\}.$$

It is a coordinate chart on the sphere with coordinates $x^1, \ldots, x^n$.

(a) Find an orientation form on U in terms of $dx^1, \ldots, dx^n$.
(b) Show that the projection map $\pi \colon U \to \mathbb{R}^n$,

$$\pi(x^1, \ldots, x^n, x^{n+1}) = (x^1, \ldots, x^n),$$

is orientation-preserving if and only if n is even (Figure 22.8).

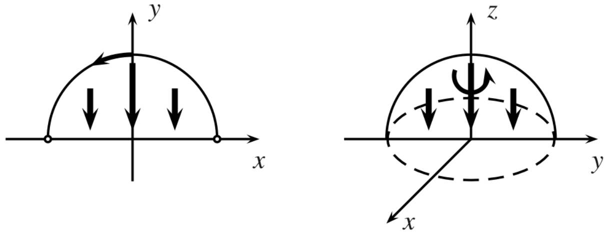


Fig. 22.8. Projection of the upper hemisphere to a disk.