Final Exam for Differential Equations

July 5, 2023

ν always stands for the outward unit normal vector on the boundary, and U is always a bounded domain in $\mathbb{R}^n$.

1.(25 Marks)

(a)(10 Marks) Consider $\Delta u = 0, u > 0 \ in \ B_1(0) \subseteq R^n, u \in C^3$. Prove that $\sup\limits_{B_{\frac{1}{2}}(0)} |D \log u| \leq C_n$.

(b)(15 Marks) Consider $u_t = \Delta u, u > 0 \ in \ U_T, u \in C_2^3(U_T), V \subset\subset U$ is conneted. Then for each $0 < t_1 < t_2 \leq T$, we have $\sup\limits_V u(\cdot, t_1) \leq C \inf\limits_V u(\cdot, t_2)$, where C depends on V, t_1, t_2, n.

2.(20 Marks)

(a)(5 Marks) Consider

$$\begin{cases} \Delta u = 1 & in \ U \subseteq R^n \\ u = 0 & on \ \partial U \end{cases}$$

$u \in C^2(U) \cap C(\overline{U})$, denote $d = diam(U)$, prove that: $-\frac{d^2}{2n} \leq u \leq 0$.

(b)(5 Marks) Consider

$$\begin{cases} \Delta u = f & in \ U \subseteq R^n \\ u = \varphi & on \ \partial U \end{cases}$$

Prove that $\max\limits_{\bar{U}} |u(x)| \leq \max\limits_{\partial U} |\varphi| + C \max\limits_{\bar{U}} |f|$.

(c)(10 Marks) Consider $\Delta u = f \ in \ U \subseteq R^n, f \in C^1(\bar{U}), \ u \in C^3(U) \cap C^1(\bar{U})$. Prove that $\sup\limits_{\bar{U}} |Du(x)| \leq C(1 + \sup\limits_{\partial U} |Du|)$, where $C \sim U, f, n$.

3.(15 Marks) Let $u \in H^1(B_1(0)), \ B_1(0) \subseteq R^2, \ f(x_1, x_2) = x_1 + x_2^2$, $W = \{u \in H^1(B_1(0)) : u - f \in H_0^1(B_1(0))\}$. Compute $\inf\limits_{u \in W} \int_{B_1(0)} |Du|^2 dx$.

4.(35 marks) Let $T > 0$, $U_T = U \times (0, T]$. Assume the functions w_k are normalized eigenfunctions for $-\Delta$ in $H_0^1(U)$ such that $\{w_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(U)$. Assume $0 \leq t \leq T, u_m(x, t) = \sum\limits_{k=1}^m d_k(t) w_k(x)$ satisfy $d_k(0) = (g, w_k)$ and

$$\int_U (u_m'(x, t) w_k(x) + \nabla u_m \nabla w_k) dx = \int_U f w_k dx, \ 1 \leq k \leq m$$

where $f \in L^2(0, T; L^2(U)), g \in L^2(U)$.

(a)(10 marks) Prove that:

$$\sup_{0 \leq t \leq T} \|u_m(\cdot, t)\|_{L^2(U)} + \|u_m\|_{L^2(0,T;H_0^1(U))} \leq C \left(\|g\|_{L^2(U)} + \|f\|_{L^2(0,T;L^2(U))} \right)$$

where $C \sim U, T$.
(b)(5 marks) If $g \in H_0^1(U)$, prove that: $\|u_m(\cdot,0)\|_{H_0^1(U)} \leq \|g\|_{H_0^1(U)}$
(c)(10 marks) Suppose $f \in L^2(0,T;H^2(U)), g \in H_0^1(U)$. Assume $u \in L^2(0,T;H_0^1(U)), u' \in L^2(0,T;H^{-1}(U))$ be a weak solution to:

$$\begin{cases} u_t = \Delta u + f(x,t) & (x,t) \in U_T \\ u(x,t) = 0 & (x,t) \in \partial U \times [0,T] \\ u(x,0) = g(x) & x \in U \end{cases}$$

Prove that:

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \|u(\cdot,t)\|_{H_0^1(U)} \leq C\left(\|g\|_{H_0^1(U)} + \|f\|_{L^2(0,T;L^2(U))}\right)$$

where $C \sim U, T$.
(d)(10 marks) If $g \in H^2(U) \cap H_0^1(U), f \in H^1(0,T;L^2(U))$, prove that:

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \|u_m'(\cdot,t)\|^2_{L^2(U)} \leq C\left(\|g\|^2_{H^2(U)} + \|f\|^2_{H^1(0,T;L^2(U))}\right)$$

where $C \sim U, T$.
5.(15 marks) Suppose $a_{ij}(x,t) \in C^1(\overline{U \times \mathbb{R}_+})$, $\lambda|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2$, $c(x,t) \in L^\infty(U \times \mathbb{R}_+)$, $g \in H_0^1(U)$, $h \in L^2(U)$. Assume $u \in C^\infty(U \times \mathbb{R}_+)$ is a solution to:

$$\begin{cases} u_{tt} = \sum\limits_{i,j}(a_{ij}(x,t)u_i)_j + c(x,t)u & in \quad U \times \mathbb{R}_+ \\ u = 0 & on \quad \partial U \\ u = g,\ u_t = h & t = 0 \end{cases}$$

Prove that:

$$\int_U (u^2 + |Du|^2 + u_t^2)dx \leq e^{Ct}\int_U (g^2 + |Dg|^2 + h^2)dx$$

where $C \sim U$, coefficients of L.
6.(10 marks) Assume $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ is a solution of $u_{tt} = \Delta u$ in $\mathbb{R}^n \times \mathbb{R}_+$. If $u = u_t = 0$ in $B_R(0) \times \{t = 0\}$, prove that: $u = 0$ in $\{(x,t) : |x| + t < R\}$