2023 Differential Geometry- TD 12

1. Soit D une domaine régulier de $\mathbb{R}^n$. On appelle *dérivée normale* d'une fonction lisse (ou simplement C^1) sur D et on note $\frac{\partial f}{\partial n}$ la fonction sur ∂D donnée par $\langle \nabla f, N\rangle$, où N est le vecteur normal unitaire sortant.

a) Montrer que si f et g sont des fonctions C^2 sur D, et si $\sigma_{\partial D}$ désigne la forme volume canonique de ∂D, on a

$$\int_D (f\Delta g - g\Delta f) dx^1 \wedge \cdots \wedge dx^n = \int_{\partial D} (f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n})\sigma_{\partial D}$$

(faire apparaître le membre de gauche comme l'intégrale d'une divergence).

b) Montrer que si f est une fonction harmonique sur D (c'est-à-dire si $\Delta f = 0$) qui s'annule sur ∂D, alors $f = 0$ partout. Même question en supposant que $\frac{\partial f}{\partial n}$ s'annule sur ∂D.

2. **Let $\varphi : \mathbf{R}^3 \to \mathbf{R}$ a differentiable function, homogenous of degree k (that is, $\varphi(tx, ty, tz) = t^k\varphi(x, y, z)$). Show that:**

a) If $B = \{p \in \mathbf{R}^3; |p| \leq 1\}$ is the region bounded by the unit sphere S^2, then

$$\int_B \Delta^2\varphi \ dx \wedge dy \wedge dz = \int_{S^2} k\varphi \ \sigma,$$

where σ is the area element of S^2 and $\Delta^2\varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz}$ is the Laplacian of φ.

***Hint*: Notice that by Euler's relation for homogeneous functions (cf. Exercise 18, Chapter 1) $x\varphi_x + y\varphi_y + z\varphi_z = k\varphi$, and use the divergence theorem.**

b) Let $\varphi = a_1x^4 + a_2y^4 + a_3z^4 + 3a_4x^2y^2 + 3a_5y^2z^2 + 3a_6x^2z^2$, then

$$\int_{S^2} \varphi \ \sigma = \frac{4\pi}{5}\sum_{i=1}^{6} a_i.$$

3. Comparaison bord / volume

Soit $vol = dx \wedge dy \wedge dz$ la forme volume canonique de $\mathbb{R}^3$. Soit $S \subseteq \mathbb{R}^3$ une surface compacte. L'intérieur de S est un domaine $N \subseteq \mathbb{R}^3$ dont le bord est $\partial N = S$. Pour $p \in S$, on note $v(p)$ la normale sortante en p à S. Soit la 2-forme d'aire $\sigma \in \Omega^2(S)$ définie par $\sigma(X, Y) = vol(v(p), X, Y)$ si $X, Y \in T_pS$. L'aire de S est $\int_S \sigma$.

(1)- Soit $\alpha = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Calculer $d\alpha$.

(2)- Montrer que si (V_1, V_2) est une base orthonormée directe de T_pS, alors

$$\alpha(V_1, V_2) \leqslant ||p||\sigma(V_1, V_2)$$

(3)- En déduire que si N est contenu dans la boule de centre 0 et de rayon R, alors

$$\text{volume}(N) \leqslant \frac{R}{3}\text{aire}(\partial N)$$

(4). Let $N(x)$ be the unit outerward normal of S at the point $x \in S$. Show that

$$\text{vol}(\Omega) = \frac{1}{3}\int_S \langle x, N(x)\rangle dS.$$

As an application, show that $\text{vol}(\mathbb{B}_r(o)) = \frac{4}{3}\pi r^3$.

4. * *Voisinage tubulaire d'une courbe*

Cet exercice utilise quelques résultats élémentaires sur les courbes paramétrées. Soit $c : \mathbb{R}/L\mathbb{Z} \longrightarrow E$ une courbe plongée de classe C^2 dans un plan euclidien, de longueur L, paramétrée par son abscisse curviligne.

(1) Montrer que l'aire d'un voisinage tubulaire $V_r(c)$ (cf. **III.24**) est égale à $2rL$.

(2) Même question pour une courbe de classe C^1.

5. Volume enclosed by parallel surface

Let S be a compact surface and let $\varepsilon > 0$ be such that $N_\varepsilon(S)$ is a tubular neighborhood. Denote S_t the inner parallel surface at distance $t \in (0, \varepsilon)$ and Ω_t the domain enclosed by S_t. Show that

$$\text{vol}(\Omega) - \text{vol}(\Omega_t) = t\,\text{Area}(S) - t^2\int_S HdA + \frac{t^3}{3}\int_S KdA.$$

6. Gauss curvature

Let $M \subset \mathbb{R}^3$ be a surface and $\mathbf{r} : U \subset \mathbb{R}^2 \to M \subset \mathbb{R}^3$ be an orthogonal parametrization of M such that $\mathbf{r}_u, \mathbf{r}_v$ are orthogonal. Denote $E = |\mathbf{r}_u|^2, G = |\mathbf{r}_v|^2$. Choose an orthonormal frame $e_1 = \frac{\mathbf{r}_u}{\sqrt{E}}, e_2 = \frac{\mathbf{r}_v}{\sqrt{G}}$. Show that

1. The associated dual coframe is given by $\omega_1 = \sqrt{E}du, \omega_2 = \sqrt{G}dv$.
2. Using the formula $d\omega_1 = \omega_{12} \wedge \omega_2$, $d\omega_2 = \omega_{21} \wedge \omega_2$ to show that the connection 1-form $\omega_{12} = -\omega_{21}$ is given by

$$\omega_{12} = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv$$

3. Using the formula $d\omega_{12} = -K\omega_1 \wedge \omega_2$ to show that the Gauss curvature K of M is given by

$$K = -\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right)$$

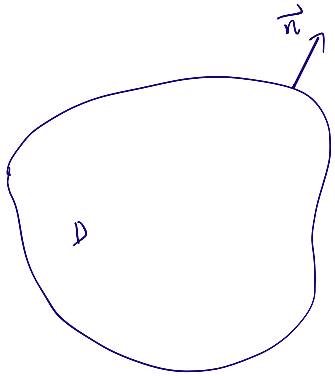
This implies that K depends only on the first fundamental form and is invariant by isometries.

7. **Let $S^2 = \{(x, y, z) \in \mathbf{R}^3; x^2 + y^2 + z^2 = 1\}$. Prove that there exists no differentiable nonzero vector field X on S^2.**
***Hint*: Assume the existence of such a field X. Let $e_1 = X/|X|$ e consider the orthonormal oriented frame $\{e_1.e_2\}$. Then $d\omega_{12} = -K\omega_1 \wedge \omega_2 = -\sigma$, hence**

$$\text{area } S^2 = \int_{S^2} \sigma = -\int_{S^2} d\omega_{12} = -\int_{\partial S^2} \omega_{12} = 0,$$

which is a contradiction.

1. (a)



$$f\Delta g - g\Delta f = \mathrm{div}(f\nabla g - g\nabla f)$$

so. $\int_D (f\Delta g - g\Delta f)\, dx^1\wedge\cdots\wedge dx^n$

$= \int_D \mathrm{div}(f\nabla g - g\nabla f)\, dx^1\wedge\cdots\wedge dx^n$

$= \int_{\partial D} \langle f\nabla g - g\nabla f, \vec{n}\rangle\, \sigma_{\partial D}$

$\sigma_{\partial D} = \vec{n} \lrcorner (dx^1\wedge\cdots\wedge dx^n)$

(b). $\int_D \mathrm{div}(f\nabla f)\, dx^1\wedge\cdots\wedge dx^n = \int_{\partial D} f\langle\nabla f, \vec{n}\rangle\, \sigma_{\partial D}$

$\|$

$\int_D (\langle\nabla f, \nabla f\rangle + f\Delta f)\, dx^1\wedge\cdots\wedge dx^n$

$$\left.\begin{array}{ll}\Delta f = 0 & \text{in } D\\ f = 0 & \text{on } \partial D\end{array}\right\} \Rightarrow \int_D |\nabla f|^2\, dx^1\wedge\cdots\wedge dx^n = 0 \Rightarrow f = 0 \text{ in } D$$

$$\left.\begin{array}{ll}\Delta f = 0 & \text{in } D\\ \frac{\partial f}{\partial n} = 0 & \text{on } \partial D\end{array}\right\} \Rightarrow \int_D |\nabla f|^2\, dx^1\wedge\cdots\wedge dx^n = 0 \Rightarrow f \equiv \text{const in } D$$

<u>note:</u> $d\sigma = i_{\vec{n}}(dx^1\wedge\cdots\wedge dx^n)$

$$\int_D \Delta\varphi\, dv = \int_D \mathrm{div}(\nabla\varphi)\cdot dv = \int_{\partial D} i_{\nabla\varphi}\cdot dv = \int_{\partial D} \langle\nabla\varphi, \vec{n}\rangle\, \underbrace{i_{\vec{n}}\cdot dv}_{d\sigma}$$

<u>Thm</u> 12.4. $\int_S i_X(dv) = \int_D \mathrm{div}(X)\cdot dv.$

2. (a) $$\int_B \Delta\varphi \, dv = \int_{\partial B} \langle \nabla\varphi, \vec{n} \rangle \, d\sigma$$

$$\vec{n} = (x, y, z)$$

$$\Rightarrow \langle \nabla\varphi, \vec{n} \rangle = x\varphi_x + y\varphi_y + z\varphi_z = k\varphi$$

so. $$\int_B \Delta\varphi \, dv = \int_{S^2} k\varphi \, d\sigma$$

(b). φ is homogeneous of degree $k=4$

由 (a) $\Rightarrow$ $$\int_{S^2} \varphi \, d\sigma = \frac{1}{4}\int_B \Delta\varphi \, dv$$

$$\varphi_x = 4a_1x^3 + 6a_4xy^2 + 6a_6xz^2 \Rightarrow \varphi_{xx} = 12a_1x^2 + 6a_4y^2 + 6a_6z^2$$

$$\varphi_y = 4a_2y^3 + 6a_4x^2y + 6a_5z^2y \Rightarrow \varphi_{yy} = 12a_2y^2 + 6a_4x^2 + 6a_5z^2$$

$$\varphi_z = 4a_3z^3 + 6a_5y^2z + 6a_6x^2z \Rightarrow \varphi_{zz} = 12a_3z^2 + 6a_5y^2 + 6a_6x^2$$

$F: \mathbb{R}^+ \times S^2 \to \mathbb{R}^3$, $F(r, z) = rz$, $z \in S^2$

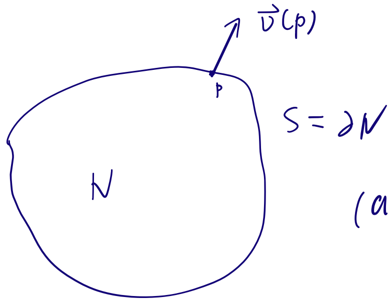
$$F^*(dx\wedge dy\wedge dz) = r^2 \sin\varphi \, dr\wedge d\varphi\wedge d\theta = r^2 dr \wedge d\sigma.$$

$$\int_B x^2 \, dx\wedge dy\wedge dz = \frac{1}{3}\int_B (x^2+y^2+z^2)\, dx\wedge dy\wedge dz = \frac{1}{3}\int_0^1 r^4 \, dr \cdot \int_{S^2} d\sigma = \frac{4\pi}{15}$$

$$\Rightarrow \int_B \Delta\varphi = 12\sum_{i=1}^{6} a_i \int_B x^2 \, dx\wedge dy\wedge dz = \frac{4}{5} \times 4\pi \cdot \sum_{i=1}^{6} a_i$$

$$\Rightarrow \int_{S^2} \varphi \, d\sigma = \frac{1}{4}\int_B \Delta\varphi \, dvol = \frac{4\pi}{5}\sum_{i=1}^{6} a_i$$

3.



$\sigma(X, Y) = vol(v(p), X, Y)$, $X, Y \in T_pS$

(a) $\alpha = x\, dy\wedge dz + y\, dz\wedge dx + z\, dx\wedge dy$

$d\alpha = 3\, dx\wedge dy\wedge dz$

(b) $\sigma(V_1, V_2) = vol(\vec{v}(p), V_1, V_2) = 1$

$\alpha(V_1, V_2) = x\, dy\wedge dz(V_1, V_2) + y\, dz\wedge dx(V_1, V_2) + z\, dx\wedge dy(V_1, V_2)$

$= x\,(dy(V_1)\,dz(V_2) - dy(V_2)\,dz(V_1))$

$+ y\,(dz(V_1)\,dx(V_2) - dz(V_2)\,dx(V_1))$

$+ z\,(dx(V_1)\,dy(V_2) - dx(V_2)\,dy(V_1))$

$$= \begin{vmatrix} x, & y, & z \\ dx(V_1) & dy(V_1) & dz(V_1) \\ dx(V_2) & dy(V_2) & dz(V_2) \end{vmatrix}$$

$= \langle (x, y, z), V_1 \times V_2 \rangle \leq \|P\|\,\|V_1\times V_2\| = |P|\cdot\sigma(V_1, V_2)$

(c). $vol(N) = \int_N vol = \frac{1}{3}\int_N d\alpha = \frac{1}{3}\int_{\partial N}\alpha$

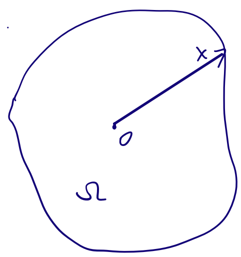
$Area(\partial N) = \int_{\partial N}\sigma$

由于 $N \subset B(0, R)$, so. $|P| \leq R$ $\forall\, p\in\partial N$

so. $\alpha(V_1, V_2) \leq |P|\,\sigma(V_1, V_2) \leq R\,\sigma(V_1, V_2)$, $\forall\, V_1, V_2 \in T_p\partial N$

$\Rightarrow Area(\partial N) = \int_{\partial N}\sigma \geq \frac{1}{R}\int_{\partial N}\alpha = \frac{3}{R} vol(N)$

(d).



$x = (x_1, x_2, x_3)$ 为位置向量

$\operatorname{div} X = 3$

divergence thm $\Rightarrow$

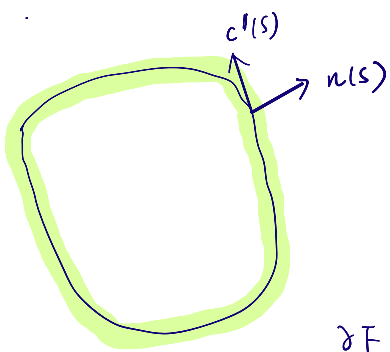
$$\int_\Omega \operatorname{div} X \, dv = \int_{\partial\Omega} \langle x, \vec{n} \rangle \, dS$$

$$\Rightarrow \operatorname{vol}(\Omega) = \frac{1}{3} \int_{\partial\Omega} \langle x, \vec{n} \rangle \, dS ,$$ $\vec{n}$ 为单位外法向

对半径为 r 的球 S_r^2 , $x = r\vec{n}$

so. $\operatorname{vol}(B(o,r)) = \frac{1}{3} \int_{S(o,r)} r \, dS = \frac{r}{3} \cdot \operatorname{Area}(S(o,r)) = \frac{4\pi r^3}{3}$

4.



Tubular neighborhood is given by parametrization

$F : (s,t) \longmapsto c(s) + t\,n(s).$ $\quad s \in [0, L]$, $t \in (-r, r)$

$k(s)$. curvature of $c(s)$.

$$\frac{\partial F}{\partial s} = c'(s) + t\,n'(s) = (1 - t\,k(s)) \cdot c'(s)$$

$$\frac{\partial F}{\partial t} = n(s)$$

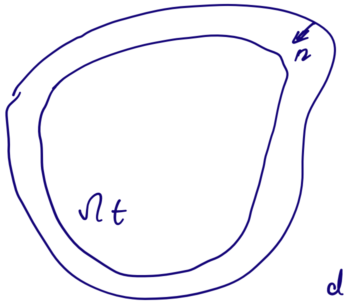
therefore $F^*(dx \wedge dy) = dx \circ F \wedge dy \circ F$

$$= \det\left(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right) ds \wedge dt = (1 - t\,k(s))\, ds \wedge dt$$

$$\Rightarrow \operatorname{area}(V_r(c)) = \int_{V_r(c)} F^*(dx \wedge dy) = \int_{[0,L]\times[-r,r]} (1 - t\,k(s))\, ds\, dt$$

$$= \int_0^L \left(\int_{-r}^{r} (1 - t\,k(s)) \cdot dt\right) \cdot ds = 2rL .$$ $\square$

5.



$S = \partial\Omega$ $\qquad F: S\times\mathbb{R}\to\mathbb{R}^3$

$F(x,t) = x + t\vec{n}(x)$

$x\in S$. $\vec{n}(x)$ 为单位内法向.

Weingarten map $W = -d\vec{n}: T_xS\to T_xS$

$dF_{(x,t)}(v,0) = v + tW(v)$ $\quad$ 对 $\forall\, v\in T_xS$

$dF_{(x,t)}(0,1) = \vec{n}(x)$

故 $dF_{(x,0)}$ is a linear isomorphism from $T_xS\times\mathbb{R}\to\mathbb{R}^3$

Inverse function theorem $\Rightarrow$ $\exists$ x 的邻域 $x\in V\subset S$. $\delta>0$

s.t. $F\big|_{V\times(-\delta,\delta)}: V\times(-\delta,\delta)\to\mathbb{R}^3$ is diffeom.

由于 S is compact. 用有限个 $V_1,\cdots,V_m$ 覆盖 S.

取 $\varepsilon = \min\limits_{1\le i\le m}\delta_i$ 则 $N_\varepsilon(S) = F(S,(-\varepsilon,\varepsilon))$ 为 S 的 tubular 邻域

$S_t = F(S,t)$ 为 S 的平行曲面.

$$\mathrm{Vol}(\Omega\setminus\Omega_\varepsilon) = \int_0^\varepsilon\int_S |\mathrm{Jac}\,F(x,t)|\,d\sigma\,dt$$

为计算 $\mathrm{Jac}\,F$: 在 $x\in S$. 取 S 主方向 e_1, e_2. 对应主曲率 k_1, k_2

则 $dF_{(x,t)}(e_1,0) = (1-tk_1(x))e_1$

$\quad dF_{(x,t)}(e_2,0) = (1-tk_2(x))e_2$

$$\begin{aligned}
|\mathrm{Jac}\,F(x,t)| &= \det\big(\underline{dF_{(x,t)}(e_1,0)},\ dF_{(x,t)}(e_2,0),\ dF_{(x,t)}(0,1)\big)\\
&= \det\big(\underline{(1-tk_1(x))e_1},\ (1-tk_2(x))e_2,\ \vec{n}(x)\big)\\
&= (1-tk_1(x))(1-tk_2(x))\\
&= 1 - t(k_1+k_2) + t^2k_1k_2 = 1-2tH+t^2K
\end{aligned}$$

$$\Rightarrow \mathrm{Vol}(\Omega\setminus\Omega_\varepsilon) = \int_0^\varepsilon\Big(\mathrm{Area}(S) - 2t\int_S H\,ds + t^2\int_S K\,ds\Big)dt$$

$$= \varepsilon\,\mathrm{Area}(S) - \varepsilon^2\int_S H\,ds + \frac{\varepsilon^3}{3}\int_S K\,ds$$

6.

$$I = E\,du\,du + G\,dv\,dv$$

$$e_1 = \frac{r_u}{\sqrt{E}}, \quad e_2 = \frac{r_v}{\sqrt{G}},$$

$$w_1 = \sqrt{E}\,du, \quad w_2 = \sqrt{G}\,dv \quad \text{dual forms}$$

$$dw_1 = w_{12} \wedge w_2 = (\sqrt{E})_v\,dv \wedge du = -\frac{(\sqrt{E})_v}{\sqrt{G}}\,du \wedge w_2$$

$$dw_2 = w_{21} \wedge w_1 = (\sqrt{G})_u\,du \wedge dv = \frac{(\sqrt{G})_u}{\sqrt{E}}\,w_1 \wedge dv$$

$$\Rightarrow \quad w_{12} = -w_{21} = -\frac{(\sqrt{E})_v}{\sqrt{G}}\,du + \frac{(\sqrt{G})_u}{\sqrt{E}}\,dv$$

$$dw_{12} = -k\,w_1 \wedge w_2$$

$$dw_{12} = \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v du \wedge dv + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u du \wedge dv$$

$$\Rightarrow \quad K = -\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right)$$

Gauss curvature, depends only on first fundamental form

7.

on S^2. 若有切向量场 X. s.t $X \neq 0$ $\forall x \in S^2$

令 $e_1 = \frac{X}{|X|}$, e_2 与 e_1 垂直.

则 $\{e_1, e_2\}$, o.n. frame.

$\Rightarrow$ dual frame $\{w_1, w_2\}$

$$dw_{12} = -K\,w_1 \wedge w_2 = -\,w_1 \wedge w_2$$

$$\Rightarrow \text{Area}(S^2) = \int_{S^2} w_1 \wedge w_2 = -\int_{S^2} dw_{12} = -\int_{\partial S^2} w_{12} = 0.$$ 矛盾