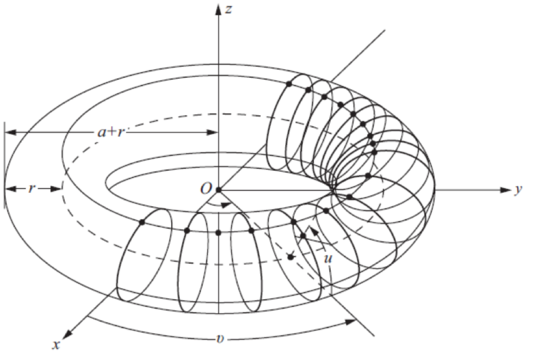
2023 Differential Geometry- TD 9

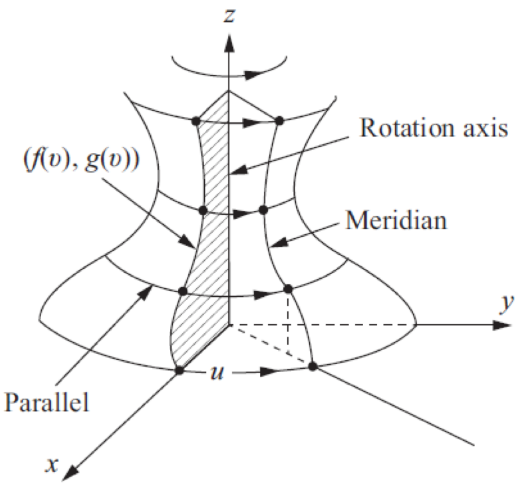
1. A parametrization for the torus T is given by

$$\mathbf{x}(u,v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

where $u \in (0, 2\pi), v \in (0, 2\pi)$. Compute the area of T.



2. Let S be a surface of revolution and C its generating curve. Let s be the arc length of C and denote by $\rho = \rho(s)$ the distance of the rotation axis of the point of C corresponding to s. (*Pappus' Theorem*) Show that the area of S is $2\pi\int_0^\ell \rho(s)ds$, where ℓ is the length of C. Apply this to compute the area of a torus of revolution.



3. Prove that if $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$, then

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})(e_{j_1}, \cdots, e_{j_k}) = \begin{cases} 1, & \text{if } i_1 = j_1, \cdots, i_k = j_k \\ 0, & \text{otherwise} \end{cases}$$

3.7. Transformation rule for a wedge product of covectors

Suppose two sets of covectors on a vector space V, $\beta^1, \dots, \beta^k$ and $\gamma^1, \dots, \gamma^k$, are related by

$$\beta^i = \sum_{j=1}^{k} a^i_j \gamma^j, \quad i = 1, \dots, k,$$

4. for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\, \gamma^1 \wedge \cdots \wedge \gamma^k.$$

3.8. Transformation rule for k-covectors

Let f be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \dots, u_k$ and $v_1, \dots, v_k$ in V are related by

$$u_j = \sum_{i=1}^{k} a^i_j v_i, \quad j = 1, \dots, k,$$

for a $k \times k$ matrix $A = [a^i_j]$. Show that

$$f(u_1, \dots, u_k) = (\det A) f(v_1, \dots, v_k).$$

5. **3.11.* Exterior multiplication**

Let α be a nonzero 1-covector and γ a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector β on V.

4.5. Wedge product

Let α be a 1-form and β a 2-form on $\mathbb{R}^3$. Then

6.

$$\begin{aligned} \alpha &= a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3, \\ \beta &= b_1\, dx^2 \wedge dx^3 + b_2\, dx^3 \wedge dx^1 + b_3\, dx^1 \wedge dx^2. \end{aligned}$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

4.6. Wedge product and cross product

The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\, \alpha^1 + a_2\, \alpha^2 + a_3\, \alpha^3$ on V, we associate the vector $\mathbf{v}_\alpha = \langle a_1, a_2, a_3 \rangle \in \mathbb{R}^3$. To the 2-covector

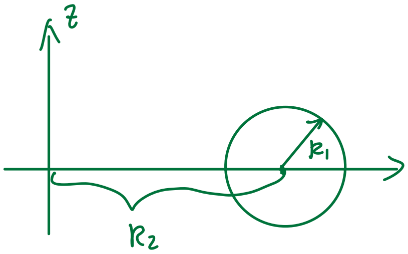
$$\gamma = c_1\, \alpha^2 \wedge \alpha^3 + c_2\, \alpha^3 \wedge \alpha^1 + c_3\, \alpha^1 \wedge \alpha^2$$

on V, we associate the vector $\mathbf{v}_\gamma = \langle c_1, c_2, c_3 \rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\, \alpha^1 + a_2\, \alpha^2 + a_3\, \alpha^3$ and $\beta = b_1\, \alpha^1 + b_2\, \alpha^2 + b_3\, \alpha^3$, then $\mathbf{v}_{\alpha \wedge \beta} = \mathbf{v}_\alpha \times \mathbf{v}_\beta$.

1. **Exercice 3** — Soit V un espace vectoriel réel de dimension n, et $(e_1, \dots, e_n)$ une base. Un élément de $\Lambda^k V$ est dit *élémentaire* s'il s'écrit sous la forme $v_1 \wedge \cdots \wedge v_k$, où les v_i appartiennent à V.

1. On suppose $n = 4$. Montrer qu'il n'existe pas de vecteur non nul $v \in V$ tel que
$$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge v = 0.$$
En déduire que $e_1 \wedge e_2 + e_3 \wedge e_4$ n'est pas élémentaire.
2. Montrer que si $n = 2$, tous les éléments de $\Lambda^k V$ sont élémentaires.
3. Étant donnés des réels $\alpha, \beta \neq 0, \gamma$, calculer
$$(e_1 + \frac{\gamma}{\beta} e_2) \wedge (\alpha e_2 + \beta e_3).$$
En déduire que si $n = 3$, tous les éléments de $\Lambda^2 V$ sont élémentaires. Que peut-on dire dans le cas où $n = 3$ et $k \neq 2$?

1: <u>环面的面积</u>



$$f(u,v) = \big((R_2 + R_1 \cos u)\cos v,\ (R_2 + R_1\cos u)\sin v,\ R_1 \sin u \big)$$

$$f_u = (-R_1 \sin u \cos v,\ -R_1 \sin u \sin v,\ R_1 \cos u)$$

$$f_v = (-(R_2 + R_1 \cos u)\sin v,\ (R_2 + R_1 \cos u)\cos v,\ 0)$$

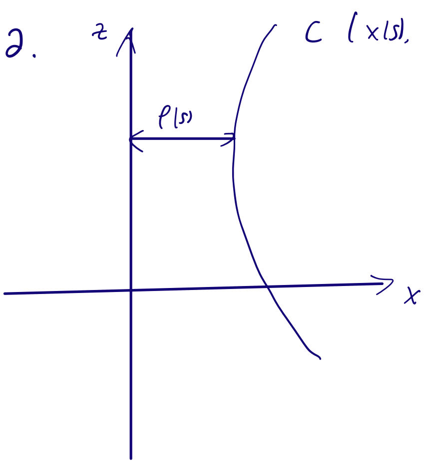
$$\Rightarrow |f_u \wedge f_v| = R_1 (R_2 + R_1 \cos u)$$

$$\text{Area}(T^2) = \iint R_1 (R_2 + R_1 \cos u)\, du\, dv$$

$$= 2\pi R_1 \int_0^{2\pi} (R_2 + R_1 \cos u)\, du$$

$$= 2\pi R_1 (2\pi R_2 + 0) = 4\pi^2 R_1 R_2$$

2.



$$f(s, v) = (x(s)\cos v,\ x(s)\sin v,\ z(s))$$

$$f_s = (\dot{x} \cos v,\ \dot{x} \sin v,\ \dot{z}(s))$$

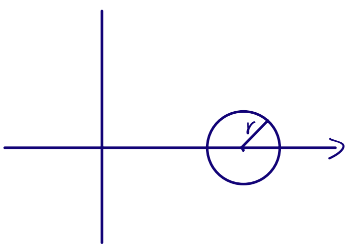
$$f_v = (-x \sin v,\ x \cos v,\ 0)$$

$$|f_s \wedge f_v| = \sqrt{|f_s|^2 |f_v|^2 - (f_s \cdot f_v)^2}$$

$$= \sqrt{1 \cdot x^2 - 0} = |x(s)| = \rho(s)$$

$$\Rightarrow A = \int_0^{l} \int_0^{2\pi} \rho(s)\, ds\, dv = 2\pi \int_0^{l} \rho(s)\, ds$$

For torus



$$x(s) = R + r\cos\frac{s}{2\pi}, \quad z(s) = r \sin\frac{s}{2\pi}$$

$$\rho(s) = x(s)$$

$$\Rightarrow A = 2\pi \int_0^{2\pi} \left(R + r \cos\frac{s}{2\pi} \right) ds = 4\pi^2 R$$

3. $(dx^{i_1} \wedge \cdots \wedge dx^{i_k})(e_{j_1}, \cdots e_{j_k})$

$= A(dx^{i_1} \otimes \cdots \otimes dx^{i_k})(e_{j_1} \cdots e_{j_k})$

$= \sum\limits_{\sigma \in S_k} \operatorname{sgn}\sigma \; dx^{i_1}(e_{\sigma(j_1)}) \cdots dx^{i_k}(e_{\sigma(j_k)})$

$= \det\left(dx^{i_k}(e_{j_l}) \right)$

if $i_1 = j_1, \cdots i_k = j_k$, then $\cdots = 1$ ✓

if not. 设 $i_1 = j_1 \cdots i_{l-1} = j_{l-1}$, $i_l \neq j_l$, $\cdots$

不妨设 $i_l < j_l$, 由于 $j_l < j_{l+1} \cdots < j_k$

$\Rightarrow i_l \neq j_1 \cdots j_k$

$\Rightarrow$ 第 l 列 of $(dx^{i_m}(e_{j_n}))$ 为零 $\Rightarrow \det = 0$

4. (1) $\alpha = a_1 dx^1 + a_2 dx^2 + a_3 dx^3$

$\beta = b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2$

so. $\alpha \wedge \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3)\, dx^1 \wedge dx^2 \wedge dx^3$

(2). $\alpha = a_1 \alpha^1 + a_2 \alpha^2 + a_3 \alpha^3$

$\beta = b_1 \alpha^1 + b_2 \alpha^2 + b_3 \alpha^3$

$\alpha \wedge \beta = (a_1 b_2 - a_2 b_1)\alpha^1 \wedge \alpha_2 + (a_1 b_3 - a_3 b_1)\alpha^1 \wedge \alpha^3 + (a_2 b_3 - a_3 b_2)\alpha^2 \wedge \alpha^3$

$V_{\alpha\wedge\beta} = (a_2 b_3 - a_3 b_2,\; a_3 b_1 - a_1 b_3,\; a_1 b_2 - a_2 b_1)$

$\left.\begin{array}{l} V_\alpha = (a_1, a_2, a_3) \\ V_\beta = (b_1, b_2, b_3) \end{array}\right\} \Rightarrow V_\alpha \times V_\beta = (a_2 b_3 - a_3 b_2,\; a_3 b_1 - a_1 b_3,\; a_1 b_2 - a_2 b_1)$

5. $\Leftarrow$ if $\gamma = \alpha \wedge \beta$.

$\Rightarrow \alpha \wedge \gamma = \alpha \wedge \alpha \wedge \beta = 0$

$\Rightarrow$. if $\alpha \wedge \gamma = 0$.

extend α to a basis $\alpha^1 = \alpha, \alpha^2, \cdots \alpha^n$ of V^*.

write $\gamma = \sum C_I \alpha^I$.

where $I \in \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$

For, $\alpha \wedge \gamma = \sum C_I \, \alpha \wedge \alpha^I$. if I contains $i_1 = 1$. then $\alpha \wedge \alpha^I = 0$

So. $0 = \alpha \wedge \gamma = \sum_{i_1 > 1} C_I \, \alpha \wedge \alpha^I$

由于 $\{\alpha \wedge \alpha^I\}_{i_1 > 1}$ is a basis for $\Lambda^{k+1}(V)$, it is linearly independent.

So, $C_I = 0$. if $i_1 \neq 1$

$$\Rightarrow \quad \alpha = \sum_{i_1 = 1} C_I \alpha^I = \alpha \wedge \underbrace{\left(\sum_{i_1 = 1} C_I \, \alpha^{i_2} \wedge \cdots \wedge \alpha^{i_k} \right)}_{\in \Lambda^{k-2}(V)}$$

6. (1) $\beta^1 \wedge \cdots \wedge \beta^k = \left(\sum_{j=1}^{k} a_j^1 \gamma^j \right) \wedge \cdots \wedge \left(\sum_{j=1}^{k} a_j^k \gamma^j \right)$.

$$= \sum_{1 \le j_1, \cdots j_k \le k} a_{j_1}^1 \cdots a_{j_k}^k \, \gamma^{j_1} \wedge \cdots \wedge \gamma^{j_k}$$

$$= \sum_{\sigma \in S_k} a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k \, \gamma^{\sigma(1)} \wedge \cdots \wedge \gamma^{\sigma(k)}$$

$$= \underbrace{\sum_{\sigma \in S_k} a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k \operatorname{sgn}(\sigma)}_{= \det A} \gamma^1 \wedge \cdots \wedge \gamma^k$$

(2). $f(u_1, \ldots u_k) = f\left(\sum_{i=1}^{k} a_1^i v_i, \ \cdots \ \sum_{i=1}^{k} a_k^i v_i \right)$

$$= \sum_{1 \le i_1, \cdots i_k \le n} a_{i_1}^1 \cdots a_{i_k}^k f(v_{i_1}, \cdots v_{i_k})$$

$$= \sum_{\sigma \in S_k} a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k f(v_{\sigma(1)}, \ldots v_{\sigma(k)})$$

$$= \underbrace{\sum_{\sigma \in S_k} a_{\sigma(1)}^1 \cdots a_{\sigma(k)}^k \operatorname{sgn}(\sigma)}_{\det A} f(v_1, \ldots v_k)$$

7. — .