2023 Differential Geometry- TD 10

1. Si $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ est C^∞, on definit $f^* : \Omega^p(V) \to \Omega^p(U)$ par:

$$f^*(\alpha)_x(X_1, \cdots, X_p) := \alpha_{f(x)}(df_x(X_1), \cdots, df_x(X_p)).$$

Dans $\mathbb{R}^n$, $dv := dx_1 \wedge \cdots \wedge dx_n = e_1^* \wedge \cdots \wedge e_n^*$.

(a) Si $f : U \to V$, $g : V \to W$ C^∞, $(g \circ f)^* = f^* \circ g^*$;

(b) Si $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$, alors $f^* dv = \mathrm{Jac}(f) dv$.

2. (Le produit intérieur) On definit une application bilineaire $i_X : \Lambda^p(E^*) \to \Lambda^{p-1}(E^*)$ par la formule

$$i_X \alpha(X_1, \cdots, X_{p-1}) := \alpha(X, X_1, \cdots, X_{p-1}).$$

(a) $i_X \circ i_X = 0$.

(b) Si $\alpha^i \in \Lambda^1(E^*), i = 1, \cdots, k$, alors

$$i_X(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(X) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k.$$

(c) Si $\alpha \in \Lambda^p(E^*), \beta \in \Lambda^q(E^*)$, alors

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge i_X \beta.$$

3. **Exercice 4** — Sur $\mathbb{R}^3$ muni des coordonnées canoniques (x, y, z), on définit la 1-forme $\alpha = dz - y dx$.

1. Calculer $d\alpha$ et en déduire qu'il n'existe pas de fonction lisse f telle que $df = \alpha$.
2. Calculer $\alpha \wedge d\alpha$, puis montrer qu'il n'existe pas de fonction lisse f telle que df et α aient même noyau en tout point. *Indication : deux formes linéaires ont même noyau si et seulement si elles sont proportionnelles, avec un coefficient non nul.*

4. **Exercice 5** — Sur $U = \mathbb{R}^2 \setminus \{0\}$ muni des coordonnées canoniques (x, y), on définit la fonction $r = \sqrt{x^2 + y^2}$ et la forme $\delta\theta = -yr^{-2}dx + xr^{-2}dy$.

1. Calculer dr et $dr \wedge \delta\theta$. Montrer que pour tout $p \in \mathbb{R}^2 \setminus \{0\}$, $(dr_p, \delta\theta_p)$ est une base de T_p^*U.
2. Calculer $d(\delta\theta)$.
3. Supposons qu'il existe une fonction lisse f sur U telle que $df = \delta\theta$. Soit $\gamma : \mathbb{R} \to U$ définie par $\gamma(t) = (\cos t, \sin t)$. Calculer la dérivée de $f \circ \gamma$. En déduire qu'une telle fonction f n'existe pas.

5. • *Formes homogènes*
Une forme différentielle ω sur $\mathbb{R}^n$ sera dite *homogène de degré* α si

$$h_t^*(\omega) = t^\alpha \omega,$$

où l'on a désigné par h_t l'homothétie de rapport t $(t > 0)$. Montrer que si ω est de degré k, cela revient à dire que les coefficients sont homogènes de degré $n - \alpha$. Montrer que la différentielle d'une forme homogène est homogène de même degré.

6.

19.1. Pullback of a differential form

Let U be the open set $]0,\infty[\times]0,\pi[\times]0,2\pi[$ in the (ρ,ϕ,θ)-space $\mathbb{R}^3$. Define $F : U \to \mathbb{R}^3$ by

$$F(\rho,\phi,\theta) = (\rho \sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

If x, y, z are the standard coordinates on the target $\mathbb{R}^3$, show that

$$F^*(dx \wedge dy \wedge dz) = \rho^2 \sin\phi \, d\rho \wedge d\phi \wedge d\theta.$$

19.2. Pullback of a differential form

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x,y) = (x^2 + y^2, xy).$$

If u, v are the standard coordinates on the target $\mathbb{R}^2$, compute $F^*(u\,du + v\,dv)$.

7. CARTAN'S LEMMA: Let M be a smooth n-manifold with or without boundary, and let $(\omega^1, \ldots, \omega^k)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subseteq M$ such that $(\omega^1|_p, \ldots, \omega^k|_p)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \ldots, \alpha^k$ on U such that

$$\sum_{i=1}^k \alpha^i \wedge \omega^i = 0,$$

show that each α^i can be written as a linear combination of $\omega^1, \ldots, \omega^k$ with smooth coefficients.

8. **19.9.* Vertical planes**

Let x, y, z be the standard coordinates on $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is *vertical* if it is defined by $ax + by = 0$ for some $(a, b) \neq (0, 0) \in \mathbb{R}^2$. Prove that restricted to a vertical plane, $dx \wedge dy = 0$.

1\. (1) 对 $\alpha \in \Omega^p(W)$

$$(g\circ f)^*(\alpha)_x(X_1, \dots X_p) = \alpha_{g\circ f(x)}\left(d(g\circ f)_x(X_1), \dots \quad d(g\circ f)(X_p)\right)$$

$$= \alpha_{g(f(x))}\left(dg|_{f(x)}\circ df|_x(X_1), \dots dg|_{f(x)}\circ df|_x(X_p)\right)$$

$$f^*\circ g^*(\alpha)_x(X_1, \dots X_p) = g^*\alpha|_{f(x)}\left(df_x(X_1), \dots \quad df_x(X_p)\right)$$

$$= \alpha_{g(f(x))}\left(dg|_{f(x)}\cdot df_x(X_1), \dots, dg|_{f(x)}\circ df_x(X_p)\right)$$

So, $(g\circ f)^* = f^*\circ g^*$

(2). $$f^*(dv)\left(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^n}\right) = dv|_{f(x)}\left(df\left(\frac{\partial}{\partial x^1}\right), \dots \quad df\left(\frac{\partial}{\partial x^n}\right)\right)$$

$$= dv|_{f(x)}\left(\sum_{j=1}^n \frac{\partial f^j}{\partial x^1}\frac{\partial}{\partial x^j}, \quad \dots \quad \sum_{j=1}^n \frac{\partial f^j}{\partial x^n}\frac{\partial}{\partial x^j}\right)$$

$$= \sum_{1\le j_1,\dots,j_n\le n} \frac{\partial f^{j_1}}{\partial x^1}\cdots\frac{\partial f^{j_n}}{\partial x^n}\cdot dv|_{f(x)}\left(\frac{\partial}{\partial x^{j_1}}\dots \quad \frac{\partial}{\partial x^{j_n}}\right)$$

$$= \sum_{\sigma\in S_n} \frac{\partial f^{\sigma(1)}}{\partial x^1}\cdots\frac{\partial f^{\sigma(n)}}{\partial x^n}\cdot \operatorname{sgn}(\sigma)\, dv|_{f(x)}\left(\frac{\partial}{\partial x^1}\dots \frac{\partial}{\partial x^n}\right)$$

$$= \det\left(\frac{\partial f^j}{\partial x^i}\right) dx^1\wedge\dots\wedge dx^n\left(\frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^n}\right)$$

2\. (1) $$(i_X\circ i_X(\alpha))(X_1, \dots \quad X_{p-2}) = i_X\alpha\left(X, X_1 \dots \quad X_{p-2}\right)$$

$$= \alpha\left(X, X, X_1, \dots \quad X_{p-2}\right) = 0$$

(2) $$\left(i_X(\alpha^1\wedge\dots\wedge\alpha^k)\right)(X_1, \dots \quad X_{k-1}) = \alpha^1\wedge\dots\wedge\alpha^k\left(X, X_1 \dots \quad X_{k-1}\right)$$

$$= \det\begin{pmatrix} \alpha^1(X), & \alpha^1(X_1) & \dots & \alpha^1(X_{k-1}) \\ \alpha^2(X), & \alpha^2(X_1) & \dots & \alpha^2(X_{k-1}) \\ \vdots & & & \\ \alpha^k(X), & \alpha^k(X_1) & \dots & \alpha^k(X_{k-1}) \end{pmatrix}$$

$$= \sum_{i=1}^k (-1)^{i+1}\alpha^i(X)\det\left(\alpha^l(X_j)\right)_{\substack{1\le j\le k-1 \\ 1\le l\le k,\ l\ne i}}$$

$$= \sum_{i=1}^k (-1)^{i+1}\alpha^i(X)\left(\alpha^1\wedge\dots\wedge\widehat{\alpha^i}\wedge\dots\wedge\alpha^k\right)(X_1, \dots \quad X_{k-1})$$

(3). 由线性性，只需对 $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^p$, $\beta = \alpha^{p+1} \wedge \cdots \wedge \alpha^{p+q}$ 证

$$i_X(\alpha \wedge \beta) = i_X(\alpha^1 \wedge \cdots \wedge \alpha^p \wedge \alpha^{p+1} \wedge \cdots \wedge \alpha^{p+q})$$

$$= \left(\sum_{i=1}^{p} (-1)^{i-1} \alpha^i(X) \cdot \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^p \right) \wedge \alpha^{p+1} \wedge \cdots \wedge \alpha^{p+q}$$

$$+ \sum_{j=1}^{q} (-1)^{p+j-1} \alpha^{p+j}(X) \alpha^1 \wedge \cdots \wedge \alpha^p \wedge \alpha^{p+1} \wedge \cdots \wedge \widehat{\alpha^{p+j}} \wedge \cdots \wedge \alpha^{p+q}$$

$$= i_X \alpha \wedge \beta + (-1)^p \alpha^1 \wedge \cdots \wedge \alpha^p \wedge \left(\sum_{j=1}^{q} (-1)^{j-1} \alpha^{p+j}(X) \alpha^{p+1} \wedge \cdots \wedge \widehat{\alpha^{p+j}} \wedge \cdots \wedge \alpha^{p+q} \right)$$

$$= i_X \alpha \wedge \beta + (-1)^p \alpha \wedge i_X \beta$$

3. ① $d\alpha = -dy \wedge dx = dx \wedge dy \neq 0$

if $\alpha = df$, then $d\alpha = d^2 f = 0$ 矛盾

② $\alpha \wedge d\alpha = (dz - y\,dx) \wedge dx \wedge dy = dz \wedge dx \wedge dy \neq 0$

if $\alpha = \lambda\, df$, $d\alpha = d\lambda \wedge df$

$\Rightarrow \alpha \wedge d\alpha = \lambda\, df \wedge d\lambda \wedge df = 0$ 矛盾

4. $U = \mathbb{R}^2 \setminus \{0\}$. $r = \sqrt{x^2 + y^2}$ $\delta\theta = \dfrac{x\,dy - y\,dx}{r^2}$

(1). $dr = \dfrac{x\,dx + y\,dy}{r}$, $dr \wedge \delta\theta = \dfrac{1}{r^3}(x\,dx + y\,dy) \wedge (x\,dy - y\,dx)$

$$= \frac{1}{r^3}(x^2 + y^2)\, dx \wedge dy$$

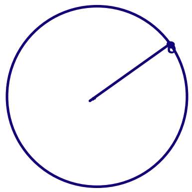
$$= \frac{1}{r}\, dx \wedge dy$$

$\Rightarrow$ for $\forall\, p \in \mathbb{R}^2 \setminus \{0\}$, $dr \wedge \delta\theta \neq 0$. so $\{dr, \delta\theta\}$ is a basis

(2). $d(\delta\theta) = \dfrac{1}{r^2}(dx \wedge dy - dy \wedge dx) - 2\dfrac{1}{r^3}\, dr \wedge (x\,dy - y\,dx)$

$$= \frac{2\, dx \wedge dy}{r^2} - 2\frac{dr}{r} \wedge \delta\theta = 0$$

(3)



$\gamma(t)$

$$\frac{d}{dt} f(\gamma(t)) = df(\gamma'(t))$$

$$\gamma(t) = (\cos t, \sin t)$$

$$\gamma'(t) = -\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}$$

$$\Rightarrow \quad df(\gamma'(t)) = \delta\theta(\gamma'(t))$$

$$= \frac{x\,dy - y\,dx}{r^2}\left(-\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}\right)$$

$$= \frac{1}{r^2}\left(x \cos t + y \sin t\right) = \frac{x^2+y^2}{r^2} = 1$$

定义: $\gamma: I \to U \subset \mathbb{R}^2$ is a curve, ω is a 1-form on U

$$\int_\gamma \omega := \int_I \omega_{\gamma(t)}(\gamma'(t))\, dt$$

now: for $\omega = \delta\theta$. if $\delta\theta = df$ for some function f.

since γ is a closed curve

设 $\gamma(t) = (x(t), y(t))$

$$\int_\gamma \delta\theta = \int_\gamma df = \int_0^{2\pi} df\big|_{\gamma(t)}(\gamma'(t))\, dt$$

$$= \int_0^{2\pi} \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)\Big|_{\gamma(t)} \left(\dot{x}(t)\, \partial x + \dot{y}(t)\, \partial y\right) dt$$

$$= \int_0^{2\pi} \left(\frac{\partial f}{\partial x} \cdot \dot{x}(t) + \frac{\partial f}{\partial y} \cdot \dot{y}(t)\right) dt$$

$$= \int_0^{2\pi} \frac{d}{dt} f(\gamma(t)) \cdot dt = f(\gamma(2\pi)) - f(\gamma(0)) = 0$$

$$\int_C df = \int_{\partial C} f = 0$$

5. ① $\omega = \sum_I a_I \, dx^I$ $\quad I = (1 \le i_1 < i_2 < \cdots < i_k \le n)$

$h_t^* \omega = \sum_I (a_I \circ h_t) h_t^* dx^I = t^k \sum_I (a_I \circ h_t) \, dx^I$

由 $h_t^* \omega = t^\alpha \omega = t^\alpha \sum_I a_I \, dx^I$

$\Rightarrow a_I \circ h_t = t^{\alpha - k} a_I$

即 a_I is homog. of degree $\alpha - k$

② $h_t^* (d\omega) = d(h_t^* \omega) = d(t^\alpha \omega) = t^\alpha d\omega$

6. ① $F^*(dx \wedge dy \wedge dz) = F^* dx \wedge F^* dy \wedge F^* dz$

$F^* dx = d(\rho \sin\phi \cos\theta) = \sin\phi \cos\theta \, d\rho + \rho \cos\phi \cos\theta \, d\phi - \rho \sin\phi \sin\theta \cdot d\theta$

$F^* dy = d(\rho \sin\phi \sin\theta) = \sin\phi \sin\theta \, d\rho + \rho \cos\phi \sin\theta \, d\phi + \rho \sin\phi \cos\theta \, d\theta$

$F^* dz = d(z \circ F) = d(\rho \cos\phi) = \cos\phi \, d\rho - \rho \sin\phi \, d\phi$

$\Rightarrow F^* dx \wedge F^* dy \wedge F^* dz$

$= (\rho \cos\phi \cos\theta \, d\phi - \rho \sin\phi \sin\theta \, d\theta) \wedge (\rho \cos\phi \sin\theta \, d\phi + \rho \sin\phi \cos\theta \, d\theta) \wedge \cos\phi \, d\rho$

$+ (\rho \sin\phi \cos\theta \, d\rho - \rho \sin\phi \sin\theta \, d\theta) \wedge (\sin\phi \sin\theta \, d\rho + \rho \sin\phi \cos\theta \, d\theta) \wedge (-\rho \sin\phi \, d\phi)$

$= \rho^2 \cos^2\phi \sin\phi \cos^2\theta \, d\phi \wedge d\theta \wedge d\rho + \rho^2 \cos^2\phi \sin\phi \sin^2\theta \, d\phi \wedge d\theta \wedge d\rho$

$+ \rho^2 \sin^3\phi \cos^2\theta \, d\rho \wedge d\phi \wedge d\theta + \rho^2 \sin^3\phi \sin^2\theta \, d\theta \wedge d\rho \wedge d\phi$

$= \rho^2 \sin\phi \, d\rho \wedge d\phi \wedge d\theta$

② $F(x, y) = (x^2 + y^2, \; xy)$

$F^*(u \, du + F^*(v \, dv)$

$= F^* u \cdot F^* du + F^* v \cdot F^* dv$

$$= u\circ F \cdot d(u\circ F) + v\circ F \, d(v\circ F)$$

$$= (x^2+y^2)\cdot(2x\,dx + 2y\,dy) + xy\,(dx\cdot y + x\cdot dy)$$

$$= \left(2x(x^2+y^2) + xy^2\right) dx + \left(2y(x^2+y^2) + x^2y\right) dy$$

7. Cartan's lem :

由 $\sum_{i=1}^{k} \alpha^i \wedge w^i = 0$

take wedge with $w^1 \wedge \dots \wedge w^{k-1}$

$$\Rightarrow \alpha^k \wedge w^k \wedge w^1 \wedge \dots \wedge w^{k-1} = 0$$

$$\Rightarrow \alpha^k \in \operatorname{span}\{w^1 \dots w^k\}.$$

8. $F: \mathbb{R}^3 \to \mathbb{R}$ $\quad F(x, y, z) = ax + by$

$S = \{ F(x, y, z) = ax + by = 0 \}$

由 $(a, b) \neq (0, 0)$. $\{0\}$ is a regular value of F.

$\nabla F = (a, b, 0) \perp T_pS$. $\forall p \in S$

$\Rightarrow$ S 的切平面由 $v_1 = (-b, a, 0)$, $v_2 = (0, 0, 1)$ 生成

设 $T_p\mathbb{R}^3$ 的基为 $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

$T_p^*\mathbb{R}^3$ 基为 dx, dy, dz

$$\Rightarrow dx \wedge dy\,(v_1, v_2) = dx(v_1)\,\underbrace{dy(v_2)}_{=0} - \underbrace{dx(v_2)}_{=0}\,dy(v_1) = 0$$