2023 Differential Geometry- TD 14

1. Let $\{X_t\}_{t\in I}$ be a smooth family of vector fields on M. On some coordinate neighborhood $(U, x^1, \cdots, x^n)$ for $p \in M$, we write

$$X_t(p) = \sum_{i=1}^{n} a^i(t,p)\frac{\partial}{\partial x^i}, \quad (t,p) \in I \times U$$

where $a^i \in C^\infty(I \times U)$. Define its time derivative by

$$\left(\frac{d}{dt}X_t\right)(p) = \sum_{i=1}^{n} \frac{\partial a^i}{\partial t}(t,p)\frac{\partial}{\partial x^i}$$

Check that this definition is independent of the chart $(U, x^1, \cdots, x^n)$ for $p \in M$.

2.
1. If $\{\alpha_t\}$ and $\{\beta_t\}$ are smooth families of k-forms and ℓ-forms on a manifold M, then
$$\frac{d}{dt}(\alpha_t \wedge \beta_t) = \left(\frac{d}{dt}\alpha_t\right) \wedge \beta_t + \alpha_t \wedge \frac{d}{dt}\beta_t$$
2. If $\{\alpha_t\}$ is a smooth family of differential forms on M, then
$$\frac{d}{dt}d\alpha_t = d\left(\frac{d}{dt}\alpha_t\right).$$
3. As applications, we have
$$\begin{aligned}\mathcal{L}_X(\alpha \wedge \beta) &= (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X\beta \\ \mathcal{L}_X d\alpha &= d\mathcal{L}_X\alpha.\end{aligned}$$

3.

1. The Lie derivative is not $\mathcal{F}$-linear in either variable: Let $\omega \in \Omega^k(M)$, $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then

$$\begin{aligned}\mathcal{L}_X(f\omega) &= (\mathcal{L}_X f)\omega + f\mathcal{L}_X\omega = (Xf)\omega + f\mathcal{L}_X\omega \\ \mathcal{L}_{fX}\omega &= f\mathcal{L}_X\omega + df \wedge \iota_X\omega.\end{aligned}$$

2. Let $\omega = -ydx + xdy$ and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ be the tangent vector on the unit circle S^1. Compute the Lie derivative $\mathcal{L}_X\omega$.
3. Let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Compute the Lie derivative $\mathcal{L}_X\omega$.

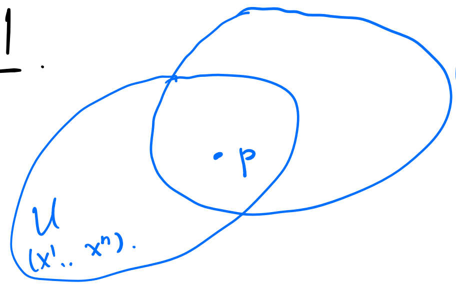
4. Suppose $\varphi : M \to N$ is a smooth map. We say $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ is φ-related if for each $p \in M$, $\varphi_{*,p}(X_p) = Y_{\varphi(p)}$. This is equivalent to $X(f \circ \varphi) = (Y(f)) \circ \varphi$ for any $f \in C^\infty(N)$. When φ is a diffeomorphism, we define the pushforward of X by φ as $(\varphi_* X)_q = \varphi_{*,\varphi^{-1}(q)}\left(X_{\varphi^{-1}(q)}\right)$, which is φ-related to X.

1. Suppose $\varphi : M \to N$ is a smooth map. $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(M)$ such that X_i is φ-related to Y_i for each $i = 1, 2$. Show that $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.
2. Let $S \subset M$ be a submanifold in M. If $Y_1, Y_2 \in \mathfrak{X}(M)$ are tangent to S, show that $[Y_1, Y_2]$ is also tangent to S.
3. Suppose $\varphi : M \to N$ is smooth and surjective, $X_i \in \mathfrak{X}(M)$ is φ-related to $Y_i \in \mathfrak{X}(M)$ for each $i = 1, 2$. Si $[X_1, X_2] = 0$, a t'on $[Y_1, Y_2] = 0$? A t'on la réciproque?

5. Let $M = \{(x, y) \in \mathbb{R}^2,\ x > 0, y > 0\}$ and $\varphi : M \to M$ be the map $\varphi(x, y) = (xy, y/x)$. Show that φ is a diffeomorphism and compute the pushforward $\varphi_* X$ and $\varphi_* Y$, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial x}.$$

1.



$(V.\ y^1 .. y^n)$. $p \in U \cap V \neq \phi$

$$X_t = \sum_{i=1}^{n} a^i(t.p) \frac{\partial}{\partial x^i}$$

$$X_t = \sum_{j=1}^{n} b^j(t.p) \frac{\partial}{\partial y^j}$$

$$\frac{d}{dt} X_t = \sum_{j=1}^{n} \frac{\partial b^j(t.p)}{\partial t} \cdot \frac{\partial}{\partial y^j} \ \overset{?}{=} \ \frac{d}{dt} X_t = \sum_{i=1}^{n} \frac{\partial a^i(t.p)}{\partial t} \cdot \frac{\partial}{\partial x^i}$$

[在 T_pM. $\left\{\frac{\partial}{\partial x^1} .. \frac{\partial}{\partial x^n}\right\}$. $\left\{\frac{\partial}{\partial y^1} .. \frac{\partial}{\partial y^n}\right\}$.

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} A_i^j \frac{\partial}{\partial y^j}$$

两边对 y^k 作用 $\Rightarrow$ $\frac{\partial y^k}{\partial x^i} = \sum_{j=1}^{n} A_i^j \underbrace{\frac{\partial}{\partial y^j} y^k}_{=\delta_j^k} = A_i^k$

即 $\boxed{\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial}{\partial y^j}}$

已知: $X_t = \sum_{i=1}^{n} a^i(t.p) \frac{\partial}{\partial x^i} = \sum_{j=1}^{n} b^j(t.p) \frac{\partial}{\partial y^j}$

$$\| $$

$$\sum_{i=1}^{n} a^i(t.p) \cdot \sum_{j=1}^{n} \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial}{\partial y^j}$$

$$\Rightarrow \quad b^j(t.p) = \sum_{i=1}^{n} a^i(t.p) \frac{\partial y^j}{\partial x^i}(p)$$

对 t 求导 $\Rightarrow$ $\frac{\partial b^j}{\partial t}(t.p) = \sum_{i=1}^{n} \frac{\partial a^i}{\partial t}(t.p) \cdot \frac{\partial y^j}{\partial x^i}$

$$\Rightarrow \quad \frac{d}{dt} X_t = \sum_{j=1}^{n} \frac{\partial b^j}{\partial t}(t.p) \frac{\partial}{\partial y^j} = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial a^i}{\partial t}(t.p) \cdot \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial}{\partial y^j}$$

$$= \sum_{i=1}^{n} \frac{\partial a^i}{\partial t}(t.p) \frac{\partial}{\partial x^i}$$

2. ① 在 $(U, x^1, \dots, x^n)$ 上，设 $\alpha_t = \sum_{1\le i_1<\dots<i_k\le n} a_{i_1\dots i_k}(t,x)\,dx^{i_1}\wedge\dots\wedge dx^{i_k}$

$$= \sum_{I\in J_k} a_I(t,x)\,dx^I$$

设 $\beta_t = \sum_{J\in J_l} b_J(t,x)\,dx^J$

$J_k = \{1\le i_1<\dots<i_k\le n\}$

$$\alpha_t\wedge\beta_t = \sum_{\substack{I\in J_k\\ J\in J_l}} \underline{a_I(t,x)\,b_J(t,x)}\;\underline{dx^I\wedge dx^J}$$

$$\frac{d}{dt}(\alpha_t\wedge\beta_t) = \sum_{\substack{I\in J_k\\ J\in J_l}}\left(\frac{\partial}{\partial t}a_I(t,x)\cdot b_J + a_I\cdot\frac{\partial}{\partial t}b_J\right)\cdot dx^I\wedge dx^J$$

$$= \left(\frac{d}{dt}\alpha_t\right)\wedge\beta_t + \alpha_t\wedge\frac{d}{dt}\beta_t$$

② $$\frac{d}{dt}(d\alpha) = \frac{d}{dt}\left(\sum_{\substack{I\in J_k\\ 1\le i\le n}}\frac{\partial a_I}{\partial x^i}(t,x)\,dx^i\wedge dx^I\right)$$

$$= \sum_{I,i}\frac{\partial}{\partial x^i}\left(\underline{\frac{\partial}{\partial t}a_I(t,x)}\right)dx^i\wedge dx^I = d\left(\underline{\frac{d}{dt}}\alpha\right)$$

③ Recall. $$(L_X\alpha)_p = \lim_{t\to 0}\frac{\varphi_t^*(\alpha_{\varphi_t(p)})-\alpha_p}{t} = \lim_{t\to 0}\frac{(\varphi_t^*\alpha)_p-\alpha_p}{t}$$

$$= \frac{d}{dt}\Big|_{t=0}(\varphi_t^*\alpha)_p$$

$$(L_X(\alpha\wedge\beta))_p = \frac{d}{dt}\Big|_{t=0}\left(\varphi_t^*(\alpha\wedge\beta)\right)_p$$

$$= \frac{d}{dt}\Big|_{t=0}\left((\varphi_t^*\alpha)_p\wedge(\varphi_t^*\beta)_p\right)$$

$$= \underbrace{\left(\frac{d}{dt}\Big|_{t=0}(\varphi_t^*\alpha)_p\right)}_{(L_X\alpha)_p}\wedge\beta_p + \alpha_p\wedge\underbrace{\frac{d}{dt}\Big|_{t=0}(\varphi_t^*\beta)_p}_{(L_X\beta)_p}$$

$$L_X(d\alpha) = \frac{d}{dt}\Big|_{t=0}\varphi_t^*d\alpha = \frac{d}{dt}\Big|_{t=0}d(\varphi_t^*\alpha) = d\Big(\underbrace{\frac{d}{dt}\Big|_{t=0}\varphi_t^*\alpha}_{=L_X\alpha}\Big)$$

3. (1). $L_X(fw) = (L_X f)w + f L_X w$

$= \underbrace{X(f)}\, w + f L_X w$

$\left(\because L_X f = \frac{d}{dt}\Big|_{t=0} \varphi_t^* f = \frac{d}{dt}\Big|_{t=0} f \circ \varphi = X(f) \right)$

Recall: Cartan's magic formula

$$L_X \alpha = d i_X \alpha + i_X(d\alpha)$$

$$L_{fX}(w) = d\left(i_{(fX)}(w)\right) + i_{(fX)}(dw)$$

$$= d(f\, i_X w) + f\, i_X(dw)$$

$$= \underbrace{df \wedge i_X w} + \underbrace{f\, d i_X w + f\, i_X dw}_{f L_X w}$$

(Recall: $(i_X w)(v_1 \dots v_{k-1}) = w(X, v_1 \dots v_{k-1})$

$(i_{(fX)} w)(v_1 \dots v_{k-1}) = w(fX, v_1 \dots v_{k-1}) = f\, w(X, v_1, v_{k-1})$

$= f (i_X w)(v_1 \dots v_{k-1})$))

(2) $L_X w = L_X(-y\,dx + x\,dy) = -X(y)\,dx - y\,L_X(dx)$
$+ X(x)\,dy + x\,L_X(dy)$

$= -X(y)\,dx - y\,d(X(x))$
$+ X(x)\,dy + x\,d(X(y))$

$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \Rightarrow \underbrace{X(x) = -y,\ X(y) = x}$

$\Rightarrow L_X w = -x\,dx - y\,d(-y)$
$-y\,dy + x\,d(x) = 0$

(3)

$$\mathcal{L}_X \omega = \mathcal{L}_X (x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$$

$$= X(x)\, dy \wedge dz + x\, \mathcal{L}_X (dy \wedge dz)$$

$$+ X(y)\, dz \wedge dx + y\, \mathcal{L}_X (dz \wedge dx)$$

$$+ X(z)\, dx \wedge dy + z\, \mathcal{L}_X (dx \wedge dy)$$

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$$

$$\mathcal{L}_X (dy \wedge dz) = (\mathcal{L}_X dy) \wedge dz + dy \wedge \mathcal{L}_X dz$$

$$= d X(y) \wedge dz + dy \wedge d X(z)$$

$$\mathcal{L}_X (dz \wedge dx) = \cancel{d X(z) \wedge d X(x)}$$

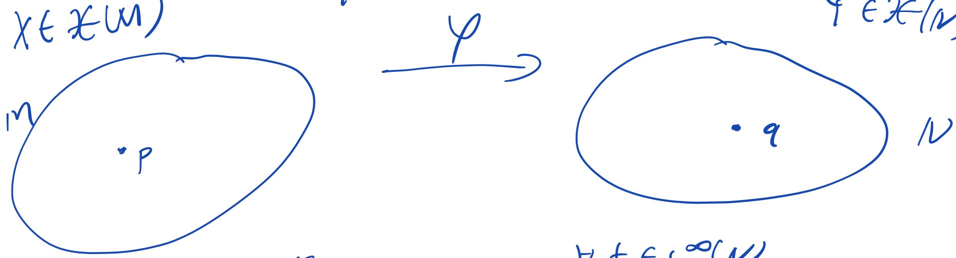
$$\mathcal{L}_X (dx \wedge dy) = \cancel{d X(x) \wedge d X(y)}$$

只需计算 $X(x) = -y$, $X(y) = x$. $X(z) = 0$

$$\Rightarrow \underline{\mathcal{L}_X \omega = 0}.$$

4. X, Y are φ-related

$X \in \mathfrak{X}(M)$ $\quad\quad$ $Y \in \mathfrak{X}(N)$



$f\circ\varphi \in C^\infty(M)$ $\quad\quad$ $\forall f \in C^\infty(N)$

$$X(f\circ\varphi) = Y(f)\circ\varphi$$

if φ is diffeom. $\quad \varphi(p) = q$.

$X \in \mathfrak{X}(M)$. $\quad \varphi_* X \in \mathfrak{X}(N)$. is defined by

$$(\varphi_* X)_q = \varphi_{*,\varphi^{-1}(q)}\left(X_{\varphi^{-1}(q)}\right)$$

(note: if $\exists\ p_1 \neq p_2$. s.t $\varphi(p_1) = \varphi(p_2) = q$

$\varphi_{*,p_1}(X_{p_1}) \neq \varphi_{*,p_2}(X_{p_2})$)

① 由条件. $\quad X_i(f\circ\varphi) = Y_i(f)\circ\varphi$. $\quad \forall f \in C^\infty(N)$

$$\begin{aligned}
[X_1, X_2](f\circ\varphi) &= X_1(X_2(f\circ\varphi)) - X_2(X_1(f\circ\varphi)) \\
&= X_1(Y_2(f)\circ\varphi) - X_2((Y_1 f)\circ\varphi) \\
&= Y_1(Y_2(f))\circ\varphi - Y_2(Y_1(f))\circ\varphi \\
&= [Y_1, Y_2](f)\circ\varphi. \qquad \square
\end{aligned}$$

2. Let $S \subset M$ be a submanifold in M. If $Y_1, Y_2 \in \mathfrak{X}(M)$ are tangent to S, show that $[Y_1, Y_2]$ is also tangent to S.

$S \subset M$. $\quad \forall p \in S$, $\quad T_pS \subset T_pM$, $\quad X \in \mathfrak{X}(M)$ is tangent to S. if $X_p \in T_pS$

$i: S \hookrightarrow M.$ $\quad i_*(Y_i) = Y_i$, $\quad i_* : T_pS \to i_*(T_pS) = T_pS \subset T_pM$

$\Rightarrow [Y_1, Y_2] = [i_*Y_1, i_*Y_2] = i_*[Y_1, Y_2] \in i_*(T_pS) = T_pS$

(3) : 反例 , $\quad \varphi: \mathbb{R}^3 \to \mathbb{R}$

$(x_1, x_2, x_3) \mapsto x_1$

取 $X_1, X_2 \in \mathfrak{X}(\mathbb{R}^3)$. 取值 in $\{0\} \times \mathbb{R}^2$, 且 $[X_1, X_2] \neq 0$

$Y_1 = Y_2 = 0$, $\Rightarrow [Y_1, Y_2] = 0$

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$$\boxed{(\varphi_* X)_q := \varphi_{*,\varphi^{-1}(q)}\left(X_{\varphi^{-1}(q)}\right)} \quad \forall q \in M.$$

$$\varphi : M \longrightarrow M$$
$$(x,y) \longmapsto (u,v).$$

$$\begin{cases} u = xy \\ v = y/x \end{cases} \Rightarrow \begin{cases} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{cases}$$

$$\mathrm{Jac}(\varphi) = \begin{pmatrix} y, & x \\ -\frac{y}{x^2}, & \frac{1}{x} \end{pmatrix} = \begin{pmatrix} \sqrt{uv}, & \sqrt{\frac{u}{v}} \\ -\frac{\sqrt{uv}}{u}\cdot v, & \sqrt{\frac{v}{u}} \end{pmatrix}$$

$$\varphi_{*,p} : T_pM \to T_{\varphi(p)}M$$
$$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}. \qquad \left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\}$$

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

$q = (u,v)$

$\varphi^{-1}(q) = (x, y)$

$= \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right)$

$$(\varphi_* X)_q = \varphi_{*,\varphi^{-1}(q)}\left(X_{\varphi^{-1}(q)}\right)$$

$$= \sqrt{\frac{u}{v}} \cdot \underline{\varphi_{*,\varphi^{-1}(q)}\left(\frac{\partial}{\partial x}\Big|_{\varphi^{-1}(q)}\right)} + \sqrt{uv} \cdot \varphi_{*,\varphi^{-1}(q)}\left(\frac{\partial}{\partial y}\Big|_{\varphi^{-1}(q)}\right)$$

$$= \sqrt{\frac{u}{v}}\left(\sqrt{uv}\,\frac{\partial}{\partial u} - \frac{v^{3/2}}{\sqrt{u}}\cdot\frac{\partial}{\partial v}\right)$$
$$+ \sqrt{uv}\left(\sqrt{\frac{u}{v}}\,\frac{\partial}{\partial u} + \sqrt{\frac{v}{u}}\,\frac{\partial}{\partial v}\right)$$

$$= 2u\frac{\partial}{\partial u} \qquad \square$$

$$\underline{\varphi_* X = 2u\frac{\partial}{\partial u}}$$