From EIM to DEIM (and GEIM): Affinizing the Non-Affine

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October 23, 2025

Learning goals (2h)

- Understand the motivation: offline/online splitting needs affine dependence.
- Derive and implement the Empirical Interpolation Method (EIM).
- Know how DEIM differs (POD basis + same greedy points).
- Glimpse GEIM: replace point samples with general linear functionals.
- Be able to prove: lower-triangularity/invertibility of B_M, basic error bound with Lebesgue constant, and the one-point estimator property.

The problem

Given a parametric function $g(x;\mu)$ (e.g. a source s or coefficient k), we seek a separated approximation

$$I_M^x g(x;\mu) = \sum_{j=1}^{M} \gamma_j(\mu)\, \rho_j(x)$$

so that linear/bilinear forms become affine: $\ell(v;\mu) \approx \sum_{j=1}^{M} \gamma_j(\mu)\, \int_\Omega \rho_j(x) v(x)\, dx$.

EIM in one slide

- Build basis $\{\rho_j\}_{j=1}^M$ and points $\{t_j\}_{j=1}^M$ greedily.
- Interpolation: $I_M^x g(t_i;\mu) = g(t_i;\mu)$, $i = 1,\ldots,M$.
- Linear system $B_M \gamma(\mu) = g_M(\mu)$ with $(B_M)_{ij} = \rho_j(t_i)$.
- B_M is lower triangular with ones on the diagonal $\Rightarrow$ fast $\mathcal{O}(M^2)$ solve.

Greedy EIM algorithm (offline)

1. Choose $\mu_1 = \arg\max_\mu \|g(\cdot;\mu)\|_\infty$, set $\xi_1 = g(\cdot;\mu_1)$; $t_1 = \arg\max_x |\xi_1(x)|$; $\rho_1 = \xi_1/\xi_1(t_1)$.
2. For $m = 1, \dots, M-1$:
    1. $\mu_{m+1} = \arg\max_\mu \|g(\cdot;\mu) - I_m^x g(\cdot;\mu)\|_\infty$; $\xi_{m+1} = g(\cdot;\mu_{m+1})$.
    2. $r_{m+1} = \xi_{m+1} - I_m^x \xi_{m+1}$.
    3. $t_{m+1} = \arg\max_x |r_{m+1}(x)|$.
    4. $\rho_{m+1} = r_{m+1}/r_{m+1}(t_{m+1})$.

Properties and simple proofs

- **Triangularity & invertibility:** B_M is lower triangular with $(B_M)_{ii} = 1$ because t_m is chosen at the maximum of r_m and $\rho_m(r_i) = 0$ for $i < m$.
- **Exactness on X_M:** $I_M^x v = v$ for any $v \in X_M = \operatorname{span}\{\rho_1, \ldots, \rho_M\}$.
- **Error bound:** $\|g - I_M^x g\|_\infty \leq (1 + \Lambda_M) \inf_{z \in X_M} \|g - z\|_\infty$, $\Lambda_M = \sup_x \sum_{i=1}^M |\ell_i^M(x)|$.
- **One-point estimator:** if $g(\cdot;\mu) \in X_{M+1}$ then $\|g - I_M^x g\|_\infty = |g(t_{M+1};\mu) - I_M^x g(t_{M+1};\mu)|$.

Practical discretization

- Discretize Ω by quadrature points $\{x_k\}_{k=1}^{N_q}$.
- Training set $\Xi_{\mathrm{train}} \subset \mathcal{D}$.
- Algebraic form: $g(\mu) \in \mathbb{R}^{N_q}$, $Q = [\rho_1 | \cdots | \rho_M] \in \mathbb{R}^{N_q \times M}$, indices I s.t. $t_j = x_{I_j}$.
- Online: solve $Q_I \gamma(\mu) = g_I(\mu)$ then $g_M(\mu) = Q\,\gamma(\mu)$.

DEIM in two bullets

- Build Q by POD of a snapshot matrix $S = [g(\mu_1)| \cdots |g(\mu_{n_s})]$.
- Select the same interpolation indices greedily; online solve $Q_I \gamma = g_I$.

DEIM error intuition

If σ_{M+1} is the first neglected singular value of S, then

$$\|g(\mu) - g_M(\mu)\|_2 \lesssim \left\|Q_I^{-1}\right\|_2 \sigma_{M+1},$$

assuming the training set is representative.

GEIM: beyond point samples

- Replace evaluations $\delta_{t_i}(g) = g(t_i)$ by linear functionals $L_i(g)$ (e.g. averages, sensor outputs).
- Interpolate: $L_i(I_M^x g) = L_i(g)$, $i = 1, \ldots, M$.
- Same greedy spirit, but maximize residuals in the dual: choose L_{m+1} where the mismatch is largest.
- Useful for inverse problems and data assimilation where only indirect measurements exist.

Affine recovery for forms

Given $k_M = \sum_{j=1}^{M_k} \gamma_j^k(\mu)\rho_j^k(x)$ and $s_M = \sum_{j=1}^{M_s} \gamma_j^s(\mu)\rho_j^s(x)$,

$$a_M(u, v; \mu) = \sum_{j=1}^{M_k} \gamma_j^k(\mu) \int_\Omega \rho_j^k \, \nabla u \cdot \nabla v,$$

$$f_M(v; \mu) = \sum_{j=1}^{M_s} \gamma_j^s(\mu) \int_\Omega \rho_j^s \, v.$$

All parameter dependence is in scalars $\gamma_j^\cdot(\mu)$.

What to prove in class

1. B_M is lower triangular with ones on the diagonal.
2. The Lebesgue-constant error bound.
3. The one-point lower bound and the equality when $g(\cdot;\mu) \in X_{M+1}$.

Mini-exercises

- Implement EIM for $g(x;\mu) = (1+\mu_1)\sin((1+\mu_2)\pi x) + \exp(-20(x-\mu_3)^2)$.
- Compare EIM vs. Chebyshev interpolation on a fixed grid.
- Inspect Λ_M numerically and relate to stability.

Takeaways

- EIM/DEIM recover separability and enable fast online queries.
- Interpolation points are data-driven “magic points”.
- GEIM broadens the game to general measurements.